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CTMT preparatory module

In order to give teachers and students a chance to get familiar with Geogebra and also introduce some CT skills, this activity can be used as a practice or warm-up session.

It would be intended for a 40-60 minute class. Have a look at this introduction video.

This whole document can then be used to record your progress through the algorithms and activities. Perhaps record some new skills you have learned or new ideas. This document will be referred to as your CTMT portfolio.

First entry into your CTMT portfolio …

0. Functional Algorithms Introduction

The following exercises will take you from finding the equation of a line through two points through to exploring how functions in general behave.

It will be done using computational thinking (CT) techniques and skills. This includes thinking in terms of an algorithm, decomposing problems into smaller problems, generating and spotting patterns, making general equations that will work in all scenarios, and testing and debugging your algorithms.

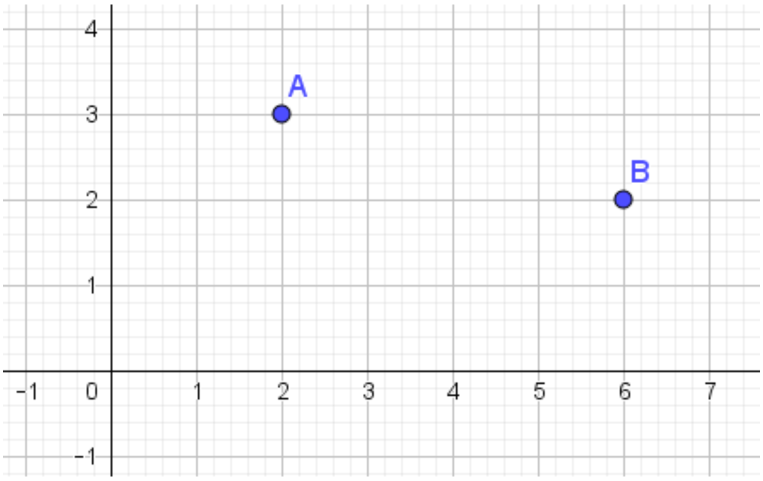
There is one Algorithm called Algorithm X. It is an algorithm to discover patterns in how graphs change, using computational thinking techniques. Your teacher will inform you whether to do this algorithm between Algorithms 3 and 4 or to do this algorithm after you have completed the other algorithms.

You can use this document in tandem with Geogebra. For example, you can record your solutions in your copy of this word document, or you can use it to explain some of the reasons for solving problems your way. (This will be of great help to the European project)

You can also click the link in the titles of Algorithms 1-4 and Algorithm X. This will give you access to a single resource within Geogebra that allows you to plot, graph and program all in one place.

Best of luck.

1. Line Through Two Given Points



Let's begin with the points A(2,3) and B(6,2). In GeoGebra we want to draw a line through these two points, but with the help of your own commands or algorithm. In GeoGebra there is a button  with which you can draw a line through two points.

The button was created by the GeoGebra programmers.

We will not use that button, but we will consider what kind of calculation is hidden behind the button.

Here are some Tips that might help you. There are more **T**ips at the end of these tasks.

You can plot a point, say A(2,3), by going to the Input Box and entering A = (2,3)

You can plot a line, say y = 3x -1, by going to the Input Box and entering y = 3x -1.

1.1

Use GeoGebra to enter points *A* and *B*. See below for tips **(T)**

1.2

Write down the steps you take when you work out the equation of the line *l*, through *A* and *B*. The steps below might help as a guide.

- **Given the points A(2,3) and B(6,2).**
- **A straight line can be drawn through these points. Let's call the line *l*. You can represent it by $y = mx + c$ or use $y - y_1 = m(x - x_1)$ to find the line.**
- **This is how I calculate the slope, *m* :**
- **This is how I find the equation of the line *l* :**
- **Here is my equation of line *l* : *y* = ...**
-
-
- **Enter your equation into Geogebra.**
-

Generalising the Equation

1.3
In Geogebra, if you move points *A* or *B*, the line you entered does not change. We want to make an equation that depends on the coordinates of *A* and *B*, so that if you drag points *A* or *B*, the equation changes along with it.
1.4
Write down below the steps you take to define the equation of a line *l* through points A and B.

Steps to form the equation of a line

Given the points A(x_A, y_A) and B(x_B, y_B).

(You may be used to using A(x_1, y_1) and B(x_2, y_2). This is fine also)

The line is given as: *y = mx + c.*

Or you can use $y-y_1 = m(x-x_1)$ to find the equation.

This is how I represent the slope, m :

$m = \frac{y_B - ?}{? \ - ?}$ (1)

This is how I find the equation of the line :

$y-y_A = m(x-x_A)$ ..... and tidy up into the form $y = mx + c$ (2)

OR

$y_A = m \cdot x_A + c \quad \therefore \quad c =$ ______________ (2)

Therefore, the equation of line *l* is: *y* = ...(1)..............*x* + ...(2)...............

1.5

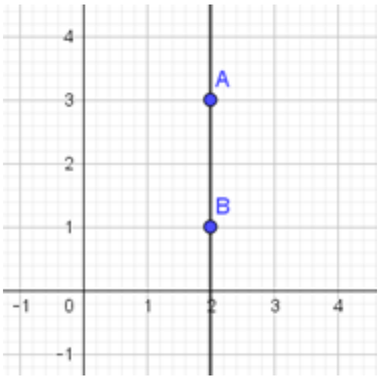
Enter this new equation in GeoGebra, so that it will change when you drag the points *A* and *B*.

Test your solution.

The x and y coordinates of A and B can be accessed by entering x(A), y(A), x(B) and y(B). See tips below **(T)**.

Exceptions to the Algorithm

There is one more difficult case left: a vertical line. If *A* and *B* are directly above each other, it has no slope. Move point A directly above point B so they have the same x-coordinates. What do you notice in your equation?



In this case, you can use the 'if-then-else' statement from the world of programming. This statement works as follows:
IF *(the x coordinates are equal)* THEN *(the line is vertical…)* ELSE *(the equation of the line is every other case i.e y=mx +c…).*

1.5
In your own words, see if you can write the if-then-else algorithm below.

IF ……………….

THEN …………………..

ELSE ………………..…………

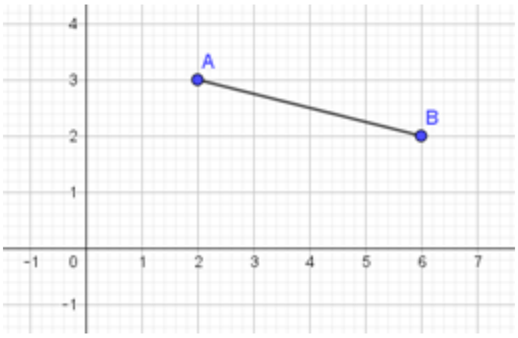
1.6
Edit your GeoGebra commands, so that the line through *A* and *B* is also drawn in this exceptional case.
Use the IF statement in GeoGebra.
And test your solution. **(T)**

Tips (T)

What do you want to do?	**How to do this in GeoGebra**
Enter the coordinate A(2,3)	Enter in entry line: A = (2,3)
Enter an equation, for example y = 3x	Enter in entry line: y = 3x
Enter $y = \frac{1}{4}x + 3\ \frac{1}{2}$	Enter in entry line: y=1/4 → +7/2 (with the "→" key you can get out of the denominator of $\frac{1}{4}$)
Use the x coordinate of point A	Enter in entry line: x(A)
Enter a conditional definition, for example if x(A)=x(B) then the outcome is P and else Q	Enter in entry line: IF(x(A)==x(B), P, Q)
Draw a vertical line, for example x=3	Enter in entry line: x=3

2. Perpendicular Bisector

Finding the equation of the perpendicular bisector between two points



Above you see the line segment through the two points A(2,3) and B(6,2). In GeoGebra we want to draw the perpendicular bisector of line segment *AB*, but with the help of your own algorithm.

In GeoGebra there is a button , that can draw a perpendicular bisector. We will not use that button, but we will look at what kind of calculation is hidden behind the button.

2.1

Enter in GeoGebra the points *A* and *B* and draw line segment *AB.*

The tips below might help **(T)**.

2.2

Write down the steps you take to define the equation of the perpendicular bisector of line segment *AB.* The steps below might help as a guide.

- **Given the points A(2,3) and B(6,2).**
- **Let's call it *l*, the perpendicular bisector of line segment [AB].**
- **Line *l* can be represented by : *y = mx + c***
- **The middle of line segment [AB] is P and has coordinates (........ , ...)**
-
- **To calculate *m*, I must first calculate the slope of AB. Here is how I calculate the slope of AB:**

$$slopeAB = \frac{2 - ?}{? \ - \ ?}$$

-

- **This is then how I calculate the perpendicular slope, *m*:**

$m \cdot slopeAB = -1 \quad \therefore \quad m = -$ **(1)**

- **This is how I calculate the y-axis intercept, c :**

$y_P = m \cdot x_P + c \quad \therefore \quad c =$ ________________ **(2)**

OR

I use the formula $y - y_1 = m(x - x_1)$ and get it in the form $y = mx + c$ (2)

- **Therefore, the equation of line *l*, is:**

***y* = ...(1).............*x* + ...(2)...............**

2.3

Enter into GeoGebra the equation of this line, to test your solution.

What happens when you move point *A* or *B*?

Does the equation change?

Generalising the Equation

You have drawn the perpendicular bisector in GeoGebra of the segment [AB] with A(2,3) and B(6,2). Now we want to make a general equation that will change as the points A and B are moved.

To do this you have to work x(A), x(B), y(A) and y(B) into the formulas you entered in GeoGebra. Below are some steps to help you create an algorithm.

2.4
Write down the steps you take to define an equation of the perpendicular bisector of line segment *AB,* using the general points (x_A, y_A) and (x_B, y_B). The steps below might help as a guide.

Let's call the perpendicular bisector *l*. And represent it by *y=mx+c*.

Let's say P is the midpoint of line segment [AB].

It has the general coordinates $\left(\frac{x_A+x_B}{2}, \frac{\quad + \quad}{\quad}\right)$

To find *m*, you must first find the slope of AB.

Here is how I represent the slope of AB:

$$\text{slopeAB} = \frac{y_B - \quad}{\quad - \quad}$$

This is then how to represent the perpendicular slope, *m*:

$$\text{m} \cdot \text{slopeAB} = -1 \qquad \therefore \qquad \text{m} = - \qquad \textbf{(1)}$$

This is how I represent the y-axis intercept, c :

$$y_P = m \cdot x_P + c \qquad \therefore \qquad c = \underline{\qquad\qquad\qquad} \qquad \textbf{(2)}$$

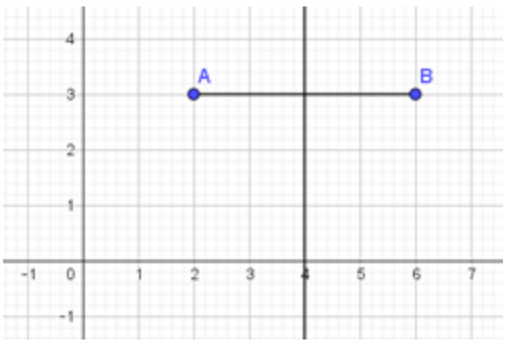
OR

I use $y - y_P = m(x - x_P)$ and get it in the form $y = mx + c$ (2)

Therefore, the equation of line *l* is: $y =$ ....(1)........$x +$ ....(2)...........

2.5
Enter in GeoGebra the equation of this line, so that it will change when you drag the points *A* and *B*. Remember that x_A is entered as x(A), y_A as y(A) and so on. You can use a fresh Geogebra app below, or edit the one above.
Test your solution.

CTMT portfolio : Record your thoughts or progress



As before, to make sure we include this case, we use the "(if,then,else)" statement.

2.6
Fill in the correct equations below if the y-coordinates of A and B are equal.

IF the line segment is horizontal

THEN the perpendicular bisector is vertical

ELSE use the general equation from the last section for the perpendicular bisector

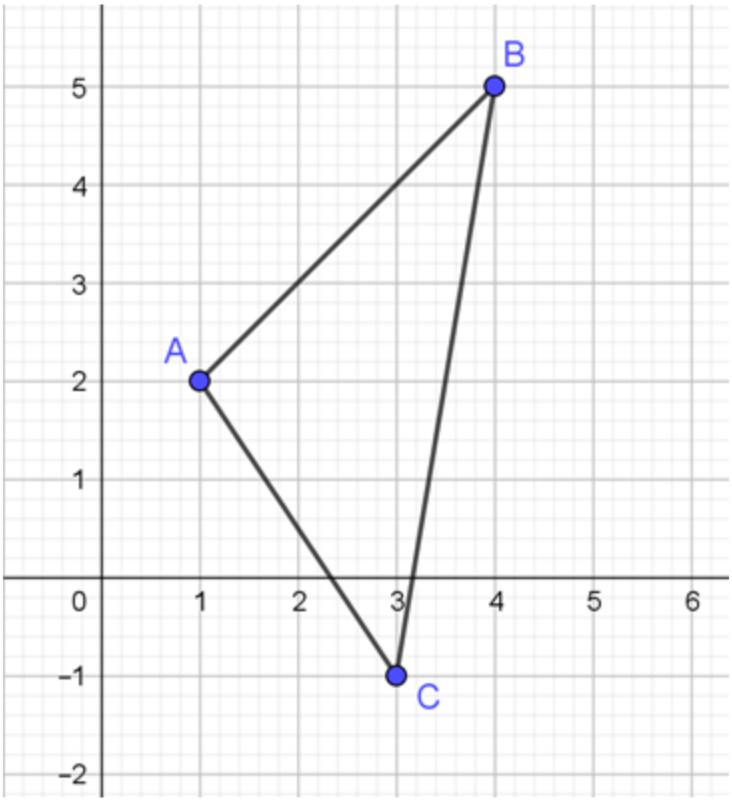
2.7
Edit your GeoGebra file, so that the perpendicular bisector of line segment *AB* is drawn also in this exceptional case.
And test your solution.

Tips (T)

What do you want to do?	**How to do this in GeoGebra**
Draw line segment *AB*	Click the point icon, and click A and B on the graph. Then click the line segment button and select A and B. OR Enter in the input box: Segment(A,B)

3. Centre of a Triangle - Circumcentre

The circumcentre of a triangle is where the perpendicular bisectors of each side intersect.



Above you see the triangle ABC with A(1,2), B(4,5) and C(3,-1). In GeoGebra we want to find the circumcentre of triangle *ABC* by using the perpendicular bisectors.

We will investigate which calculations need to be done to discover your own algorithm.

3.1
Enter into GeoGebra the points *A*, *B* and *C*, and draw triangle *ABC.* **(T)**

3.2
Write down below the steps you take to find the perpendicular bisector of [AB]. These are exactly the same steps as you used in Algorithm 2. You can re-use any of this work at any point in the following steps.
Write down the steps you take to define the equation of the perpendicular bisector of line segment *AB.* The steps below might help as a guide

- **Given the points A(2,3) and B(6,2).**
- **Let's call it *l*, the perpendicular bisector of line segment [AB].**
- **Line *l* can be represented by :** ***y = mx + c***
- **The middle of line segment [AB] is P and has coordinates (......, ...)**
- **To calculate *m*, I must first calculate the slope of AB. Here is how I calculate the slope of AB:**

$$slopeAB = \frac{y_B - ?}{? \ - \ ?}$$

- **This is then how I calculate the perpendicular slope, *m*:**

$$m \cdot slopeAB = -1 \quad \therefore \quad m = \frac{????}{????} \qquad \textbf{(1)}$$

- **This is how I calculate the y-axis intercept, c :**

$$y_P = m \cdot x_P + c \quad \therefore \quad c = \underline{\qquad\qquad\qquad} \qquad \textbf{(2)}$$

OR

I use the formula $y - y_1 = m(x - x_1)$ and change to the form *y = mx + c* (2)

- **Therefore, the equation of line *l*, is:**

***y* = ...(1).............*x* + ...(2)...............**

3.3

Enter into GeoGebra the equation of this line, to test the first step of your solution. What happens when you move point *A* or *B or C?*

3.4

Now repeat the steps for line segments [AC] and [BC].

- **Let's call the perpendicular bisector of [AC] line *h*.**
  It might be a good idea to call the slope of line *h* something like slopeh. Can you think why?

- **Let's call the perpendicular bisector of [BC] line *j*.**
  It might be a good idea to call the slope of line *j* something like slopej. Can you think why?

- Test each equation as you go along in Geogebra.
- Find the point of intersection of lines *h, k, l,* using the intersection button in Geogebra.
- Let's call it point G. (you can verify this yourself by finding the point of intersection of any two of the lines *h, k, l*)

CTMT portfolio : Record your thoughts or progress

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Generalising how we find the circumcentre

You have found the circumcentre of triangle ABC, and verified it is correct.

Try using the circle button in Geogebra to draw the circumscribed circle through A, B, C. Does it go through A,B,C, even when you move the points A,B or C?

Now we want to create a general solution that will change as the points A, B, C are moved. To do this you have to work x(A), x(B), y(A), y(B) and x(C), y(C) into the formulas you entered in GeoGebra.
The steps are identical to those you used in Algorithm 2. You are simply repeating them 3 times. Perhaps give unique names to each variable such as the lines, the slopes, and so on.

3.5
Write down the steps you take to define an equation of the perpendicular bisector of line segment *AB,* using the general points (x_A, y_A) and (x_B, y_B). The steps below might help as a guide.

Let's call the perpendicular bisector *l*. And represent it by *y=mx+c*.

Let's say P is the midpoint of line segment [AB].

P has the general coordinates $\left(\frac{x_A+x_B}{2}, \frac{\quad + \quad}{\qquad}\right)$

To find *m*, you must first find the slope of AB.

Here is how I represent the slope of AB:

$$slopeAB = \frac{y_B - \qquad}{\quad - \quad}$$

This is then how to represent the perpendicular slope, *m*:

$$m \cdot slopeAB = -1 \qquad \therefore \qquad m = \qquad\qquad\qquad\qquad \text{(1)}$$

This is how I represent the y-axis intercept, c :

$$y_P = m \cdot x_P + c \qquad \therefore \qquad c = \underline{\qquad\qquad\qquad\qquad} \qquad \text{(2)}$$

OR

I use $y - y_P = m(x - x_P)$ and get it in the form $y = mx + c$ (2)

Therefore, the equation of line *l* is: $y =$ ....(1)........$x +$ ....(2)...........

3.6

Enter in GeoGebra the equation of this line, so that it will change when you drag the points *A* and *B*. Remember that x_A is entered as x(A), y_A as y(A) and so on.

Repeat this as before for line segments [AC] and [BC].

Here is my general equation of line *h*, the perpendicular bisector of [AC]:

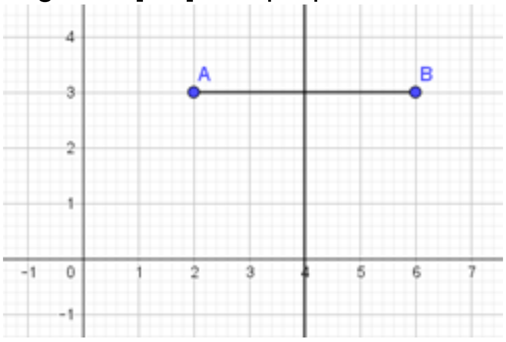
Here is my general equation of line *j*, the perpendicular bisector of [BC]:

Test your solution.

Find the point of intersection of the lines. This is the circumcentre. Verify it changes correctly as the points A,B,C are changed.

Exceptions to the Algorithm

As in Algorithms 1 and 2, vertical lines can pose a problem. With a horizontal segment [AB], the perpendicular bisector has no defined slope.



As before, to make sure we include this case, we use the "(if,then,else)" statement.

3.7

Fill in the correct mathematics below. You can repeat it 3 times. Each new IF command will replace or override the equations you derived for lines *h, j, k*.

IF the side of the triangle is horizontal

THEN the perpendicular bisector is vertical

ELSE use the general equation from the last section for the perpendicular bisector

3.8

Edit your GeoGebra inputs, so that the perpendicular bisectors of all 3 sides of the triangle are drawn for every scenario.

And test your solution. For example, using the circle button, does the circle with your circumcentre as its centre, go through all 3 points A,B,C?

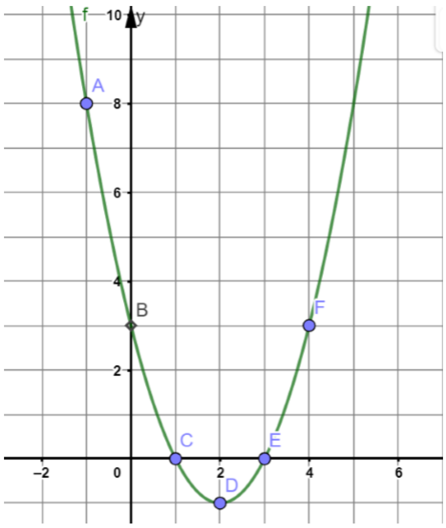
Tips (T)

What do you want to do?	***How do you do that in GeoGebra?***
To draw a triangle ABC	Enter in the entry line: Polygon (A, B, C) OR create three separate line segments

X. Discovering how functions behave

This is a Discovery Algorithm (DA) to explore how functions behave. It is optional whether to do it at this point before moving to Algorithm 4 or to move from Algorithm 4 and return to this algorithm before finishing the exercises. Your teacher will inform you when to explore this algorithm.

As we track a graph from left to right, it changes at a different pace depending on the value of x. How can we figure out this rate of change?



Above you see a graph of the function $f(x) = x^2 - 4x + 3$. There are six points on the graph, labelled A,B,C,D,E,F, going from x=-1 through to x=4. We want to investigate if there are patterns in how the graph changes as we move consecutively from points A through to F.

X.1

Enter into GeoGebra the function $f(x)= x^2-4x+3$, and plot each of the six points {A,B,C,D,E,F}. **(T)** Record the y-coordinates at each of the points.

X.2

Now we want to find the slope at each of the six points to see how quickly or slowly the graph is changing at each point. To do this we will use the  button to find the equation of the tangent at each point.

Repeat for points A-F :

1. Click the button.
2. Select the quadratic function and the point. (Geogebra creates a tangent)
3. Find the slope at the point, using the equation of the tangent.
4. Record the slope at that point.

X.3

Using the following questions as a guide, can you figure out a pattern for how the graph is changing.

(a) Using the y-coordinates of each point, can you figure out the pattern? For example, is it linear, quadratic, exponential.

(b) Using the slopes at each point, can you verify the sequence is a linear pattern?

(c) A linear pattern can be represented by $g(x) = ax + b$. From your knowledge of linear patterns, can you figure out a value for a and b?

Once you have figured out the patterns, investigate what happens with some other quadratic functions. For example, repeat the process with $f(x)= 2x^2-3x+1$. And with $f(x)= 3x^2-2x-2$. Or one of your own.

X.4

Using the following questions as a guide, see if you can find a link between the function $f(x)$ and the slope $g(x)$.

(d) Comparing $f(x) = x^2- 4x+3$ to what you have for $g(x)$, can you see a pattern for how to derive $g(x)$ from $f(x)$? For example, looking at $g(x)$, where might the value for a be coming from or the value for b? Does it work for other functions you investigated?

(e) Make a Prediction. Let's say the point G is (5,8). According to the pattern you have derived, what would be the slope of the function $f(x)$ at the point G?

(f) Test your prediction. Go back to Geogebra, and plot the point G. Use Geogebra to find the slope. Is your pattern for deriving the slope at any point correct? What do you notice?

For the function *f(x)*, here is how I would use Computational Thinking techniques to derive the function *g(x)* that gives me the slope at any point.

f(x) = x^2 -4x + 3 **g(x) = ...**

CTMT portfolio : Outline the method and the CT skills you used to find *g(x)* ..

Generalising how to predict the slope of a graph

In the previous section, we used the specific function *f(x)* = x^2-*4x+3* for the pattern.

Now we want to generalise the algorithm for finding the slope for any quadratic function.

Let's call this general function *f(x)* = *a*x^2*+bx+c.*

We can still use six points on the graph, labelled A,B,C,D,E,F, going from x=-1 through to x=4. We want to investigate if there are general patterns in how the graph changes as we move consecutively from points A through to F.

X.5

In Geogebra, we will create the three variables *a, b, c*, that define the function, and we will give them initial values which can be changed at any time to test the algorithm.

Enter:

a=2

b=-4

c=3

Then enter into GeoGebra the function *f(x)*= *a*x^2+*bx*+*c,* and plot each of the six points {A,B,C,D,E,F}. **(T)**

X.6

We want to find the slope at each of the six points to see how quickly or slowly the graph is changing at each point. To do this we will use the  button to find the equation of the tangent at each point.

Repeat for points A-F :

1. Click the  button.

2. Select the quadratic function and the point. (Geogebra creates a tangent)

3. Find the slope at the point, using the equation of the tangent.

4. Record the slope at that point.

X.7

Using the following questions as a guide, can you figure out a pattern for how the graph is changing.

(a) Using the y-coordinates of each point, can you verify it is a quadratic pattern?

(b) Using the slopes at each point, can you verify it is a linear pattern?

(c) A linear pattern can be represented by *g(x) = ? x + ?*

From your knowledge of linear patterns, can you figure out the missing terms?

(d) Comparing *f(x)* = ax^2 + *bx+c* to what you have for *g(x)*, can you see a pattern for how to derive *g(x)* from *f(x)*? For example, looking at *g(x)*, where is the term in front of x coming from? Does the pattern for *g(x)* include the variable *c* from the *f(x)* function?

(e) Make a Prediction. Let's say the point G is (5,??). According to the pattern you have derived, what would be the slope of the function *f(x)* at the point G?

(f) Test your prediction. Go back to the Geogebra, and plot the point G. Use Geogebra to find the slope. Is your pattern for deriving the slope at any point correct? What do you notice?

X.8

For the function *f(x)*, here is how I would find the function that gives me the slope at any point.

f(x) = $ax^2 + bx + c$ ***g(x) = ...***

Test your solution in Geogebra.

Exceptions to the Algorithm

In previous algorithms, there were exceptions. We had to use the IF-THEN-ELSE technique. Do you think for the type of shape we have been looking at (called a parabola), that there are exceptions?

X.9

Let's look at more functions to see if your generalised algorithm still works.

Using $f(x) = ax^2 + bx + c$, you can vary the values of *a, b* and *c* and change the function.

Does your pattern work in these cases?

X.10

Perhaps there are other functions for which your algorithm will work.

If you wish, you could try a cubic function $f(x)=ax^3+bx^2+cx+d$. Can you derive the function, $g(x)$, to give you the slope at any value of x?

Use similar computational thinking techniques to see if you can spot the pattern for these cubic functions. Try other curves if you wish such as the square root of x or f(x)=1/x. And test your patterns and predictions.

CTMT portfolio : Record some of the things you have tried, or some new ideas or skills you have learned.
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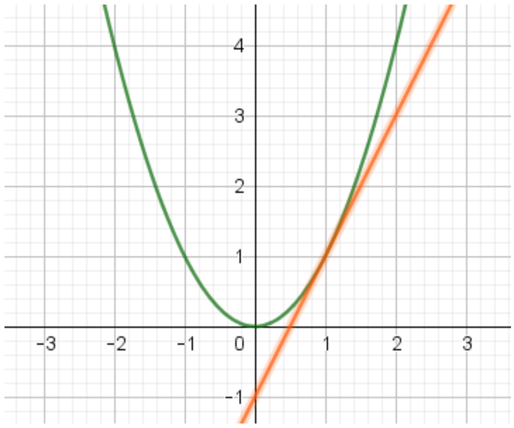
Tips (T)

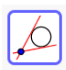
What do you want to do?	***How do you do that in GeoGebra?***
To enter a function	Use the Input Bar to enter any function. For example, a quadratic function: enter h(x)=-5x^2+10x+3 The ^ symbol is above 6 on the keyboard.

	To enter x^3, type x^3 or just type xxx. To examine the square root of x, you can enter h(x)=SQRT(x)

4. Tangent to a parabola



Above you see the parabola with equation $y = x^2$ and the tangent to it at $x = 1$.

We want to draw this tangent in GeoGebra, but without using the built-in commands such as the  button or the GeoGebra Derivative command.

From previous algorithms, you may or may not have a pattern for finding the slope of a function at any point on the function. Either way, this is called finding the derivative and is represented below by *y'(x).*

4.1. Write below the steps you will take to create an equation of the tangent line.

Suppose *y=mx + c* represents the equation of the tangent at point A, where (xA, yA) = (1, 1).
Let's call it line *l*.

***y'(x)* = ____________________**

***m = y'(1)* = _____________**

$y_A = m.x_A + c$, therefore *c* = _____________

So, line *l* is : *y* = ____m___*.x* + _____c______

4.2. Enter your result in GeoGebra and verify the line is the tangent at x=1.

This algorithm works for a specific function ($y=x^2$) at a specific point on the function, where x=1.

What happens if you change the function in the input bar? Try out lots of different parabola, such as $x^2 +1$, x^2-3, x^2+x, and so on. Are there limitations to your model?

Now we also want this procedure to work for any parabola with equation $y= ax^2 + bx + c$, *at* $x=1$.

4.3

Adjust your step-by-step plan below so that the tangent line to the parabola at x=1 is always drawn, even if you change the values of *a, b* and *c*.

Suppose $y=mx + c$ represents the equation of the tangent to $y = ax^2 +bx + c$ at point A, where $x_A = 1$. Let's call the tangent line *l*.

$y'(x) =$ ___________________

$m = y'(1) =$ _____________

$y_A = a(1)^2 + b(1) + c$, *so* $y_A =$ ___________

$y_A = m.x_A + c$, therefore $c =$ _____________

So, line *l* is : $y =$ ____m___.x + _____c______

4.4

Enter your result in GeoGebra and let the tangent line be drawn for self-selected values of a, b and c.

Test if your solution works. Save your geogebra file or screenshot your work! You will use it in later algorithms.

CTMT portfolio : Record some of the things you have tried, or some new ideas or skills you have learned.

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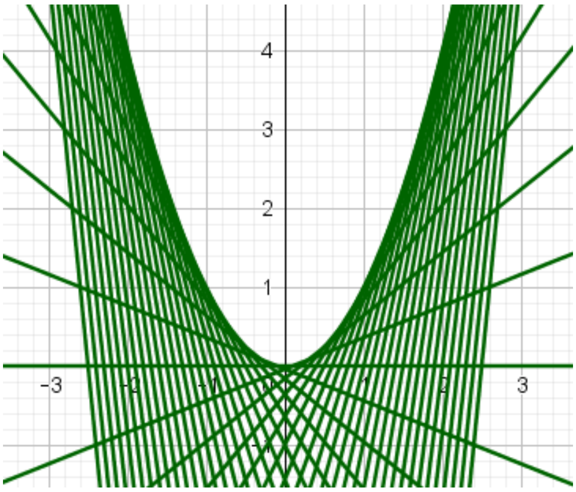
5. Bundles of tangents to a parabola



In the previous assignment you made GeoGebra draw a tangent to a parabola. The other way around, you can also visualize a parabola by drawing a whole bundle of tangents. Above you see that for the parabola with equation $y = x^2$. The intention is now again to do this in GeoGebra. Your work on the previous assignment will come in handy here.

5.1. In the box below, make a step-by-step plan for establishing the equation of the tangent line to the parabola $y = x^2$ in the point with coordinates (p, p^2). Your result will be a formula containing the parameter p.

Suppose $y = mx + c$ is the tangent line to the graph $y = x^2$ at the point (p, p^2).

Let's call it line l. We need to find expressions for m and c, using the point at p.

(In the previous algorithm, we did these steps for (1,1). Now we are using the general form (p, p^2).)

$y'(x) =$ ....

$m = y'(p) =$ ......

$p^2 = m.p + c$, therefore $c = p^2 - m.p =$ ....

So, line l is y = ... x + ... (1)

Be sure to record your work and your progress in this section.

Change the value of p. Can you explain what happens?

5.2. Enter your result in GeoGebra and let the tangent line be drawn for some values of *p*.

Try different values of p from -5 to +5.

Record in your CTMT portfolio what you observe, what worked, what did not work.

To now draw a bundle of tangents, we can use the Sequence command. This allows you to go through lots of values of p, in any sequence of consecutive numbers you want

The syntax of this is:

Sequence(<Expression>, <Variable k>, <Start Value a>, <End Value b>, <Increment>)

In our case, this command creates a "row" of tangent lines to the graph, at different tangent points each time.

<Expression>: here you enter the "general" equation (1) for the tangent line as found in the previous section.

<Variable>: since you want a different value of *p* tangents for each time, the variable here is *p*.

<Start Value> / <End Value> / <Increment>: here you indicate at which point *p* you want to start and at which point you want to end. The increment is the step size (like jumps) to go from the start to the end.

For example, starting at 1, ending at 9 and incrementing by 2, will yield a total of five tangents at the points 1, 3, 5, 7, 9.

5.3. Enter in the box below which elements of the Sequence command you should use to draw a bundle of 50 tangents to the parabola from $p = -5$ to $p = 5$:

Expression: …………………………..
Variable: …………………….
Start Value: …………………….
End Value: ………………………
Increment: ………………………

5.4. Enter the Sequence command in GeoGebra to draw a bundle of tangents. Save your work!

CTMT portfolio : Record some of the things you have tried, or some new ideas or skills you have learned.

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The next challenge is to make this approach work for other parabolas as well. The format for exercise 5.5 will not change, but the algorithms need to work for a general quadratic function..

5.5. Adjust the steps in exercise 5.1 for a parabola with equation $y = ax^2 + bx + c$.

Suppose l: $y = mx + c$ is the tangent line to the graph $y = ax^2 + bx + c$ at the point $(p,\ ap^2 + bp + c)$.

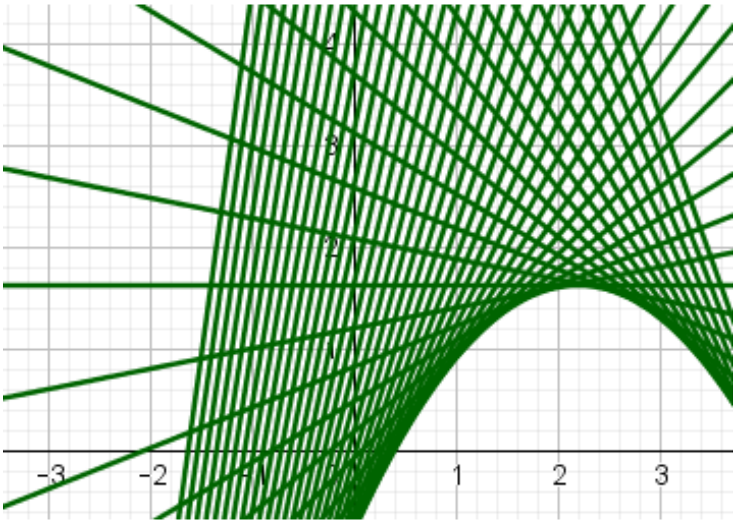
$y'(x) =$……………………..

$m = y'(p) =$……………………..

Filling in $y = mx + c$ with the coordinates above and the slope gives: $c =$ ……………

So: the general equation of line l is: $y =$ ....x + ...

5.6. Adjust the Sequence command in GeoGebra for the general case to draw a bundle of tangents to another parabola. Will it continue to work if you change the values of a, b, c?



CTMT portfolio : Record some of the things you have tried, or some new ideas or skills you have learned.

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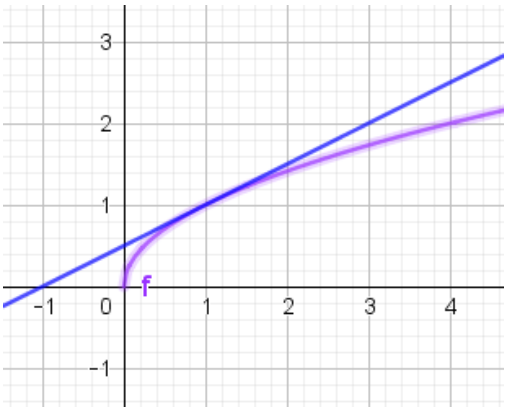
6. Tangents to other graphs



In the previous algorithms, you drew bundles of tangents to parabolas. Now we are going to do the same for other curves, such as the graph of the square root function above. In doing so, we enter and now we allow ourselves to use the full power of GeoGebra.

To use the full power of Geogebra If you enter f(x) = SQRT(x), it will plot the square root function. We can use Geogebra to find the derivative of the square root function. In the input bar, enter g(x) = f'(x). You will see the what the derivative is in the input bar.

6.1. Adjust the steps from the previous command for the graph of the function $f(x) = \sqrt{x}$.

Suppose $y = mx + c$ is the tangent line to the graph at some point $(p, \sqrt{p})$. Let's call it line l.

We need to find expressions for m and c, using the point at x = p.

$y'(x)$ = ___________________

$m = y'(p)$ = ________________

$\sqrt{p} = m.p + c$= therefore c = _____________

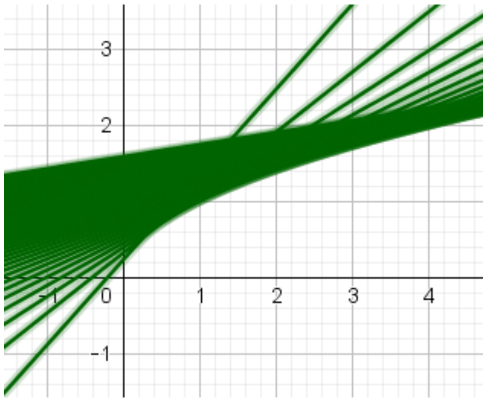
So, line l is : y = ____m___.x + _____c______

Change the value of p. Can you explain what happens? Make some notes in your CTMT portfolio.

6.2. What does the Sequence command look like in this case? Fill in the box below.

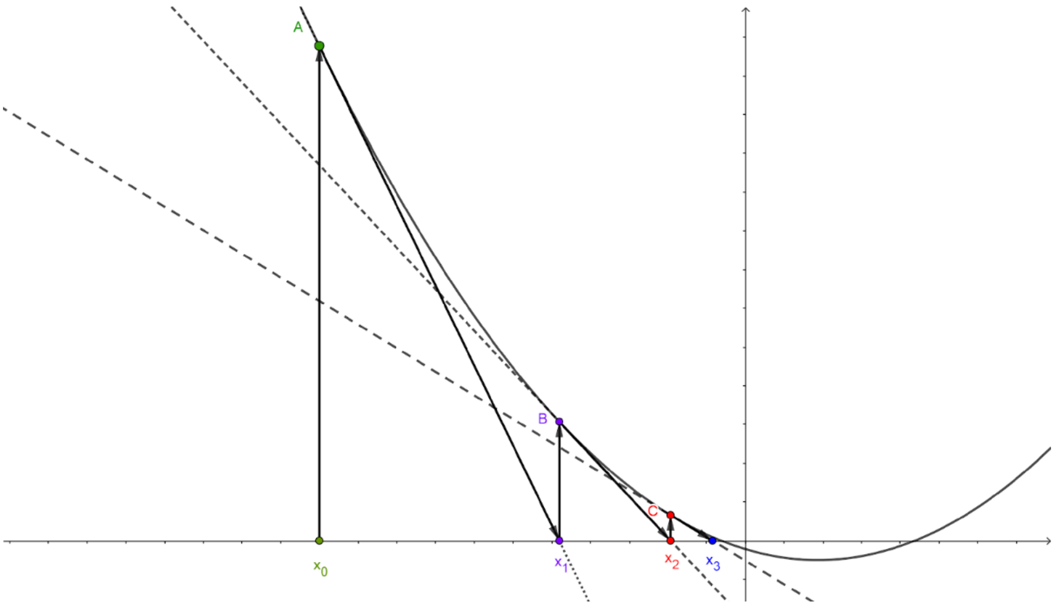
Expression: …………………	Variable: …………………….
Start Value: …………………….	End Value: ………………………
Increment: ………………………	

6.3. Enter the above in GeoGebra and let the graph of the root function "wrap" in a bundle of tangents. Save your file!



6.4. Choose some other functions yourself, like a cubic function for example, and make similar pictures for them.

7. Zeros with tangents: Newton-Raphson model

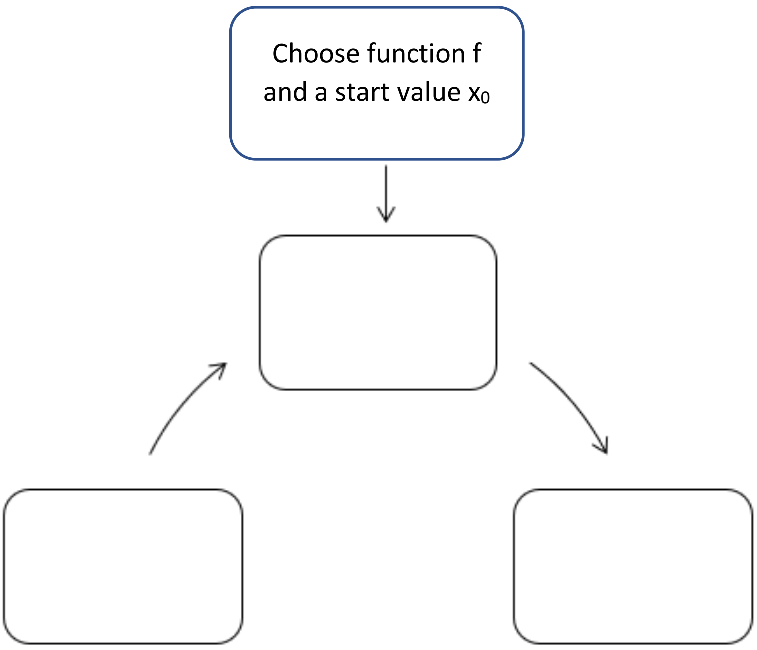


In a previous assignment you had GeoGebra draw a tangent to a parabola. Tangent drawing is used when approximating zeros in the way you see above in the graph of

$f(x)=\left(\frac{1}{5}x-\frac{3}{4}\right)^2-1$. The method is known under the name Newton-Raphson. In many cases it works very well. In a number of steps - you see three above - the zero point is approximated. The intersections of tangents to the x axis should approximate the zero point more and more accurately by then constructing x_4, x_5, x_6 and so on.

You will construct a model for the Newton-Raphson method using the full classic Geogebra application at https://www.geogebra.org/classic.

7.1. Describe how the Newton-Raphson method works. Use the diagram below for this.



In the first box: Choose function f a start value x_0

With this method you could for example approximate the zeros of the parabola $f(x) = x^2 - 2$. So you approach $\sqrt{2}$ or $-\sqrt{2}$. This is fairly laborious with pen and paper. Therefore, we limit ourselves to the first two steps to approach $\sqrt{2}$. If we follow the notation from the graph above, we will therefore look for x_1 and x_2 and use the points *A* and *B*. We may choose the x_0 starting value.

7.2. Take the following steps:

a) Take $f(x) = x^2 - 2$ and $x_0 = 4$ as a starting value. *A* is the point of the graph of *f* with *x* coordinate $x_A = 4$. Give the coordinates of point *A*.

b) Draw up an equation of the tangent to the graph of *f* at point *A,* we call this l_A.

c) Find the intersection of l_A with the *x*-axis and name the *x*-coordinate of that intersection x_1.

d) *B* is the point of the graph of *f* with *x* coordinate $x_B = x_1$. Enter the coordinates of point *B*.

e) Draw up an equation of the tangent l_B to the graph of *f*.

f) Find the intersection of l_B with the *x*-axis and name the *x*-coordinate of that intersection x_2

.

We should repeat the process more often to find a better approach to $\sqrt{2}$. Time to enlist the help of GeoGebra.

7.3. In algorithm 4, we used found the equations of tangents to the general function $f(x) = ax^2 + bx + c$. Now adjust the parabola by choosing $a = 1$, $b = 0$ and $c = -2$. Enter the starting value $x_0 = 4$.

7.4. Starting point A on the graph of f then has coordinates $(x_0, f(x_0))$.
Put in $A = (x_0, f(x_0))$.

7.5. To draw the tangent line to the graph of f at point A, adjust the tangent line formula in the file by using x_0 instead of 1 as the x coordinate of A.

7.6. The x coordinate of A plays the role of x_0. You find x_1 by determining the intersection point B of the tangent line and the x-axis. Check that $x_1 = \frac{-f(x_0)}{f'(x_0)} + x_0$. Enter point B.

With point B in hand, steps 7.4, 7.5 and 7.6 should actually be performed again where the role of A is taken over by B in order to find point C. Repetition always finds a new point.

In GeoGebra you can take steps 7.4, 7.5 and 7.6 together in a so-called “macro”. You then create a new command (let's call it “NR”), which returns the point B if you enter NR (f, A), point C if you enter NR (f, B), and so on. In general: NR (f, P) indicates the intersection of the tangent line at P to the graph of f with the x axis.

7.7. Choose “Buttons” in the drop-down menu (or in the menu for “Macros”, depending on your GeoGebra version) and then “Create new macro”.

Choose x_1 as end object and x_0 as start object and f. Name the macro “NR”.

You can also choose your own image to display the NR button in your toolbar.

Click on “Finish” to create your macro.

If it doesn't work the first time, discard the wrong macro and start over.

7.8. With the command Iteration list you can repeat or iterate the macro “NR”. You do this with the command Iteration list (NR (X, f), X, {x_0}, 5). The command executes NR 5 times in a row with the graph of f and it starts with x_0 as x coordinate. The capital letter X is the variable where the newly found x coordinate is always entered.

Do you really get an approximation of $\sqrt{2}$ like this?

7.9. Can you also approach the other zero point with a small change?

7.10. Can you also approximate the golden ratio by adjusting a, b, c and? First, consider (or look up) which equation the golden ratio is a zero point for.

7.11. The Newton-Raphson method does not work in all cases. Whether you find a zero point and which one depends on your function *f* and especially on the starting value you choose. Investigate the consequences if your starting value changes and try to find an example where the method ends terribly wrong.