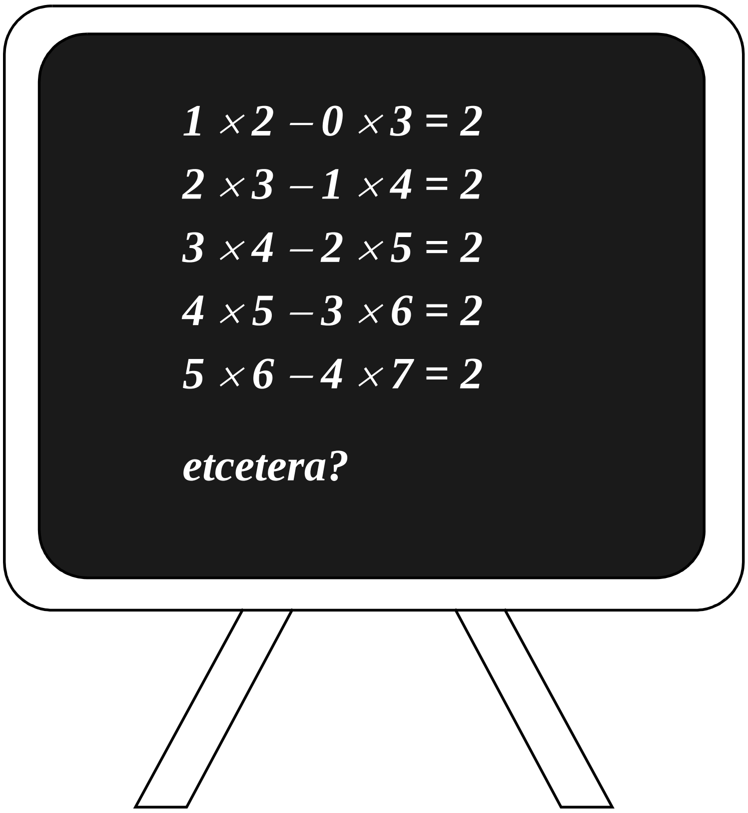
Positive Algebra



A collection of productive exercises

Freudenthal Instituut, University of Utrecht.

Positive Algebra

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composed by

Martin Kindt

Preface

Exercises in algebra mostly have a *reproductive* character. After one or more examples the students have to make series of algebraic calculations: expanding, factorizing, etc. This way of exercising can only be effective if it is practiced with frequency during a long time. In current mathematics education there is no time for this. The math program for the 12 - 16 age group is too broad and the number of lessons too small to practice algebra weekly over a number of years.

In connection with the project 'Long Term Algebra', and inspired by ideas for primary education which are developed in the Freudenthal Institute, I have tried to design a collection of examples of so called *productive exercises.*
These are exercise not meant for training automatisms, but for challenging the students to think and to reason. In many of the exercises I restricted myself to the domain of natural numbers, because the students are familiair with the 'world' of these numbers.
In other exercises the variables represent positive (rational or real) numbers; negative numbers are avoided in this collection.
In the article 'Algebra kan ook natuurlijk zijn' (Algebra can also be natural; published earlier as part of an article I wrote with Aad Goddijn under the title *Knelpunten en toekomstmogelijkheden voor de wiskunde in het VO*, Tijdschrift voor Didactiek van de β–wetenschappen, year 18, nr.1, 2001) the reader can find some thoughts on the possibility of postponing the infiltration of negative numbers in algebra.

Some ideas from the collection were part of the inspiration for designing Algebra Applets on Wisweb; for instance: 'Number strips with labels', 'Spotting Numbers' en 'Geometric Algebra 2D'. In the WELP project (*Mathematics applets and teaching practice*) these and other applets are integrated into algebra chapters.

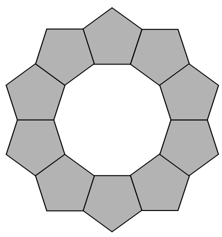
Martin Kindt
Utrecht, June 2004

Algebra can also be natural

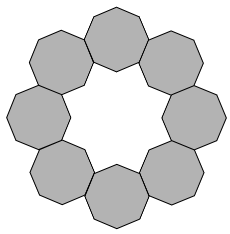
Complaints about algebraic ability are nothing new. Their vehemence and number have increased recently though. Is it a sign of the times that the average student has too much on his mind and can no longer concentrate enough or is not meticulous enough to learn algebra?
Or is there something wrong with the way mathematics is taught? Some people blame advanced calculating machines, such as the graphic calculator. On the other hand, others claim that the very same machines make algebra almost redundant
We believe that the complaints about the lack of elementary algebraic ability are not unfounded. Even stronger, we regularly find a lack of confidence in students when using algebra. This deficit can partly be blamed on the reproduction-oriented algebra-didactics in the various methods. Without pretending to have a ready-made solution, we want to launch some ideas here that could involve the student more actively in the teaching process and ultimately would give him more opportunity to adequately use algebra in approproate situations.

It said so in the book

We presented a group of students (15/16 years old, pre-university) with a problem that included among other things the question of whether you can make a circle with a series of congruent regular polygons, and if so, which numbers of polygons are required. The painter Albrecht Dürer pursued this problem and thought that it was impossible using for example a pentagon. Apparently, his drawing was imprecise; but then, he didn't have a computer available and couldn't use for example Cabri to perform precise constructions. With this program the students can make circles of regular polygons almost in an instant.



circle of 10 pentagons



circle of 8 octagons

Apparently, Dürer, who had a lot of affinity for geometry, didn't see that arithmetic and algebra can sometimes help to gain insight into a geometry problem. For this problem, the mathematical core consists of calculating angles and determining divisibility. After all, making a circle involves rotating the polygon a number of times around a certain point outside the polygon until it returns to the original position. An introductory question involved the link between the size of the angle in a regular polygon and its number of sides. One of the students asked the observer whether the following was correct:

$$\text{size of angle} = \frac{(\text{number of angles} - 2) \times 180}{\text{number of angles}}$$

The observer confirmed this, but also asked for an explanation. That was too much to ask for; the answer, though not without some embarrassment, was: 'It said so in the book last year'. To the question of whether he could write down the formula shorter (with the intention of replacing 'number of angles' with one letter), he eagerly answered: you can cross out this (=number of angles) against this. 'Yes, what would that leave you with?' The proposal was rapidly withdrawn. When asked explicitly, he was willing to replace 'number of angles' with *N*. Partly because of what was to come after, the observer was steering towards the 'walking around' strategy: if you make one complete circle around the *N*-angle you will make the same turn *N* times. Result: the number of degrees for an exterior angle is 360/*N* and therefore for an interior angle 180 - 360/*N*. The student understood this immediately, but also wanted to see if this matched the earlier formula, in other words whether

$$\frac{(N-2) \times 180}{N} = 180 - \frac{360}{N}$$

is an identity.

This turned out to be insurmountably difficult. Once the observer returned from looking at what the other students were doing, he decided to put the student out of his misery. So if you want to make the *N* disappear from the denominator... Okay, he multiplied left and right with *N*, made another mistake, arrived at a very simple identity, which after some hard thinking was understood. We point out that this was a very bright group of students and that this was far from the least in the group.

Automatisms exit?

We also had this kind of experiences in the Profi project in which a new math program for pre-university (Nature and Technology profile) was tested: intelligent students with a good approach to problems, who get stuck on elementary algebra. An obvious reaction would be: 'see, they no longer learn algebra these day, they don't have the automatisms'.

The content of the Dutch algebra program for basic secondary education was strongly influenced by the needs of the majority of 12 to 16 yr-olds.

In the justification of the Dutch 12-16 mathematics curriculum it says literally: *Algebra concerns working on problems in which relations between variables play a part. Preferably these are problems arising from realistic situation. The relations can be presented or described in a number of ways, namely 'tables', 'graphs' and 'formulas'.* And a bit further on it says*: In the proposed curriculum, algebraic techniques are not a goal in themselves, but they are dedicated to problems surrounding relations between variables.*

In educational practice this means that from the beginning there is a lot of attention in the algebra chapters for single variable functions, often related to a 'daily-life-context' and the connection between the various representations.

The width of the total math curriculum on the one hand and the objective formulated above on the other have resulted in the number of algebraic-technical exercises in the math textbooks being somewhat limited. Moreover, it is noticeable that the authors' groups are didactically at a loss for what we here a bit facetiously call 'the last remains of classical algebra'. Boring rows of exercises where the student is expected to get rid of or insert brackets on command, do not fit in a book which constantly offers contextually rich mathematics. That didactical style rupture is paid for in practice, because exercising can only have an effect when done patiently over a long period of time, and there is no time available for that in the current situation; so it isn't really surprising that students don't acquire automatisms.

In the most recent editions of most textbooks we can see as a reaction a certain increase in stereotypical rows of exercises. The question is whether this offers structural help. In fact, we're fairly certain it hardly does. Günther Malle says (in Didaktische Probleme der elementaren Algebra, Vieweg 1993) that every teacher finds out sooner or later that the 'exercise ideology' yields relatively little result *(...) Ich erinnere mich an zahlreiche Klagen von Lehrern, die nicht verstehen konnten, warum ihre Schüler trotz 'hunderter' Übungsaufgaben immer noch Fehler beim Termumformen oder Gleichunglösen machen.*

Yet we have to conclude that long ago students knew, at least up to their final exams, many more techniques, especially where it concerned fractions with variables. Old exams show problems of a complexity that is now seen as bewildering. Of course people also made stupid mistakes in the 'good old days' and ... algebraic skills would fade away quickly after leaving school. Furthermore, there were the often insurmountable difficulties with the so-called word problems which can be solved using equations, and what good are those algebraic skills when a student cannot use them to draw up simple models? Transfer to other school subjects such as science or economics also often left much to be desired

In current math education there is rightly more attention from the beginning to building formulas and formulating algebraic models. In our opinion this does have some effect. Modern students dare to translate a problem situation into algebra. On the other hand they are often quite helpless when faced with having to reduce or transform algebraic expressions

We recall one example in a pre-university group (16-18), where students had to calculate the difference of x^3 over the interval $[k, k + 1]$: *sir, how do you do* $(k + 1)^3$*?* That they don't immediately call upon the binomium formula for the third power doesn't bother us at all, one might even say on the contrary. What does bother us is the lack of ability to cope on their own. Note that from the kind of question ('calculate the difference'), the tempting $(k + 1)^3 = k^3 + 1$ had to be rejected: it is impossible for a third-degree function to have a constant difference. This rather bare problem evidently already contains more context than the classical algebra exercises intended by Günther Malle and which call for no scrutiny or reflection.

Beginners' algebra in three methods

When comparing the latest edition of the three leading Dutch math methods (say A, B and C) for the first year of secondary education, it's conspicuous that graphs and ratio tables are at the beginning of the algebra strand. The beginning of the book also gives attention to what could be called advanced arithmetic. In the first year, A gives the most attention to 'letter arithmetic', followed by C, and B (almost nothing).

Overview of first year algebra (chapter titles) in the three methods:

A	B	C
numbers	*ratios*	*ratios*
graphs	*fractions*	*graphs and tables*
negative numbers and formulas	*graphs*	*calculations*
operating with formulas	*discovering rules*	*formulas*
powers and formulas	*negative numbers*	*variables*
	formulas	*powers*
	comparing	
	equations	

We direct our attention towards traditional aspects of school algebra, namely 'arithmetic with symbols', 'manipulating expressions, 'reducing', etc.

A starts with symbolic arithmetic in chapter 8: operating with formulas.

The chapter starts with 'machines' which are used to describe (linear) relations.

Then the translation to a formula is made and the accompanying graphs are looked at. Then 'true-to-life' contexts are abandoned. There is a significant comment in the margin: *Up to now, formulas emerged from stories. But you can also invent formulas without stories. Mathematicians prefer to use the letters x and y in that case.*

After a few examples of the type:

$$4x + 7x = \underbrace{x + x + x + x}_{4x} + \underbrace{x + x + x + x + x + x + x}_{7x} = 11x$$

it says: you *have to* write down this kind of reduction *without an intermediate step*. Under the topic 'removing brackets' the so-called 'parrot beak' is introduced straightaway without further ado as a visual mnemonic; what it comes down to is that students have to keep practicing procedures in a prescribed way and without an orientation base worth mentioning.

$$(x + 2)(y + 3) = xy + 3x + 2y + 6$$

Four chapters later 'powers and formulas' are treated. The first three paragraphs contain some nice examples of quadratic relations, then we get on to the bare calculating work with powers. No more stories, but rows of sums. One exception is the 'extra' problem: *make the largest possible number with three threes.*

Sometimes, it seems as if the authors are incapable of, or refuse to, apply didactical principles from realistic maths to the field of traditional algebra, such as '*productive (or constructive) practicing*' and '*building a net of relations*'; principles that have proved their worth in math didactics. It should be possible to apply them in algebra practice.

In B the algebra list contains more chapters, but they are shorter. Symbolic calculation has clearly been moved further back in this method. Working with machines and word formulas is kept up for a long time. We don't see a letter formula until page 258.

The very careful build-up continues in the chapter *Comparing* to finally end in *Equations,* which goes no further than equations of the type $ax + b = c$. There's hardly any symbolic calculation in the first year. A teacher who uses this method recently said: every time you get the feeling you're only taking very long run-ups. We really get the idea that B takes anticipation much too far.

C only looks at (linear) word formulas in chapter 8, called *Formulas.* Chapter 11 (*Variables*) is the first with letter calculations. The build-up looks a bit better thought out than A, and the paragraphs with sums show some more variation. But we get the parrot beak for multiplications such as $7 \cdot (t + 10)$ here as well. One noticeable thing is that the multiplication point is not left out, so no 3*a* yet. All terms are of the first degree.
Chapter 13, *Powers*, starts with a meaningful introduction. In the third paragraaf (*calculating with powers*), reducing expressions like $q^4 \cdot q \cdot q^9$ and $2 \cdot k^3 + 9 \cdot k^3$ is practiced.
Calculating with letters clearly doesn't go as far as in A; for example, there are no terms with more than one variable. Although it looks a bit more friendly than A, there is also not much room for flexibility here and constructive exercises and problems are as good as absent.

American algebra
Together with a team from the University of Madison, the Freudenthal Institute has developed in the mid-nineties a curriculum ('Mathematics in Context') for the American Middleschool (age category 10 to 14). Three main strands were distinguished in formulating the algebra strand:
–*Processes*
–*Restrictions*
–*Patterns*

The strand 'Processes' can more or less be compared with what is called 'Relationships' in the Dutch program. This mainly relates to kinds of growth and their representation. The strand doesn't go as far as the Dutch program, because of the age limit of 14, but the spirit is similar.

The strand 'Restrictions' contains solving equations or systems of equations, first in a pre-formal way and later on more formally, and using linear conditions for optimization problems (an onset for linear programming). All of this as much as possible in a meaningful context.
In Dutch education there is less attention for systems of equations and optimization problems, and, despite all good intentions, solving equations degenerates into all kinds of (badly understood) tricks. It is the gradual and natural structure, starting with systems of equations that has turned out to be remarkably successful in 'Mathematics in Context'. It turns out that young children spontaneously make linear combinations to work towards expressions that lead to the solution (see the unit 'Comparing Quantities'). Linear optimization methods prove to be challenging for students in practice and provide a good introduction of the concept of variables (the unit 'Decision making').

Finally, there is the '*Pattern-line*', which is mainly concerned with the continuation of discovered regularity, building formulas for number patterns and figural patterns, as well as developing insight into the structure of algebraic expressionss. Some examples of this can also be found in the Dutch methods, but in our view these are not used enough in practicing 'letter arithmetic'.

Experience with algebra in Mathematics in Context has taught us that it is possible to have young children invent algebraic procedures on their own and apply them in situations they experience as meaningful.

Algebra down the drain?
In his inaugural lecture *'Math education down the drain' (Wiskundeonderwijs naar de knoppen)* professor Frans Keune (University of Nijmegen) sketched the degeneration of school mathematics as observed by him. The Dutch title refers to the use of calculators in school. Without a doubt, there are some math teachers who feel that the introduction of graphic calculators has ushered in the further decline of algebra, something which will become even worse once calculators equipped with computer algebra are allowed.
In the meantime the graphic calculator (GC) has become established in the higher grades of pre-university schools, and it is to be expected that it will become common in for example the lower grades of these school types.
Nation-wide introduction of the GC was preceded by developmental research from the Freudenthal Institute. In the research report it says among other things

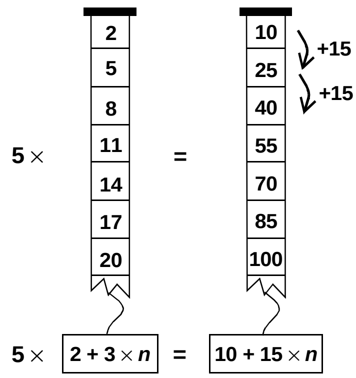
(...) *The use of the graphic calculator incites the students to set themselves new problems and generalize problems. This means a widening of the student's mathematical range and a change in attitude towards math from 'passive-performing' to 'active-investigating'.*
Whether or not this ideal can be achieved, depends on many factors. For instance, one textbook demands much more explicit intervention from the GC than another. It's understandable that textbooks are brand-neutral, but it is unfortunate that this makes it difficult for them to anticipate on a particular machine's specific properties. The teacher can do much to compensate this lack, but it remains to be seen how much of that actually happens.
An example of didactic use of the GC is what is called 'graphical algebra'. Graphically seen, multiplying two linear functions yields a parabola. A good question would be whether it is possible to make all parabolic graphs in that way. Related to this is linear factorizing for polynomic functions. Based on the graph (of for example a third degree function) on the screen, zeros are seen. Division by the linear factor results in a polynom of a lower degree, which in turn can be 'seen' graphically. If this polynom has a zero, it is possible to divide again, and thus an algebraic rule (the factor theorem) can be discovered experimentally.
A strength of the GC is that transformations of graphs can be performed quickly, with the student giving the correct algebraic instruction. The machine provides immediate feedback, and it's easy to correct mistakes. For instance, students of 15 - 16 yearsdiscovered using the GC that inversion of the base a in $y = a^x$ results in a reflection of the graph in the y-axis. For the teacher's suggestion of trying to reflect for instance the graph of $y = \sin x + 1$ in the y-axis, some students, as expected, first tried 1/y. After this 'error'-experience, there was some more 'trial', with other functions as well, until it was established that replacing x with $-x$ always gives the desired effect. Linking this feedback to the first example then yielded the identity $(\frac{1}{2})^x = 2^{-x}$, for which the students then looked for a more precise algebraic explanation.

Generally speaking it is possible to say that on the one hand graphically establishing and/or verifying of algebraic equivalences and on the other hand using suitable formulas to create desired graphical effects, would contribute strongly towards developing algebraic insights in the middle grades of pre-university education.

Before too long we can expect the introduction of the symbolic calculator (SC). This will mean that students will no longer have to perform reductions, certainly of more complicated expressions, themselves. At the very least, there will have to be research into which skills are needed to use the SC successfully. It's unthinkable that a student without basic skills in algebra would be able to use an SC, just as it is unthinkable that someone who has never done any calculations on their own would be able to use an ordinary calculator in a meaningful way. For instance, students will have to develop a kind of symbol sense, and that requires practical exercises. And, just as attention in arithmetic didactics has shifted over the years from complicated calculations to 'simple mental arithmetic' and 'estimating arithmetic', it could be that learning skills in the area of 'simple mental algebra' and 'estimating algebra are to become import aspects of algebra teaching. It's clear that the use of the GC and eventually at a later stage the SC, in secondary education will have drastic effects on algebra education. However, with careful didactic deliberations and use of the rich opportunities offered us by technology, it need not be a negative effect.

Starting algebra with 'natural numbers'.
Let us return to the beginning of algebra education. In the first year of secondary education it is possible to make a careful start with letter calculation even earlier than is the case now, especially if there has been some level differentiation. The world of natural numbers (non-negative integers) is probably most suitable. This is a concrete world for the students, maybe even more so than some of the stories currently used in textbooks.
A few simple examples. The teacher can ask the students to think of two natural numbers that together make 20 and then calculate the product of their chosen duo. Next is inventorying the results using natural questions such as what the lowest and highest possible outcomes are. After methodically writing down the list of all possible outcomes, it becomes then possible to discover a regular pattern, etc. In a simple way, and following on from elementary arithmetic, the concept of 'variable outcome' emerges here. Furthermore, there is natural progression towards a classical problem situation, namely an optimization problem. The problem can also be made 'continuous' (something that should certainly be done as well), for example by asking what the rectangle with the largest surface area that you can set off using a given length of rope is. The advantage of starting with the discrete version is

that a pattern quickly emerges and that there is a finite number of possibilities, at least when the problem is limited to natural numbers.
The problem is also richer than it appears at first sight; it's possible to hang the entire theory of second degree equations on this problem.
Another simple example is the linear expression. Working in the domain of non-negative integers an expression like 2 + 3 × *n* easily brings to mind a number pattern, namely the arithmetic sequence 2, 5, 8, ... Multiplying such a sequence with a certain number, or adding two sequences, gives another arithmetic sequence, because the jump between consecutive numbers remains constant. That makes it simple to gain the insight that 5 × (2 + 3 × *n*) and 10 + 15 × *n* are equivalent and also that (3 + 4 × *n*) + (1 + 5 × *n*) = (4 + 9 × *n*).

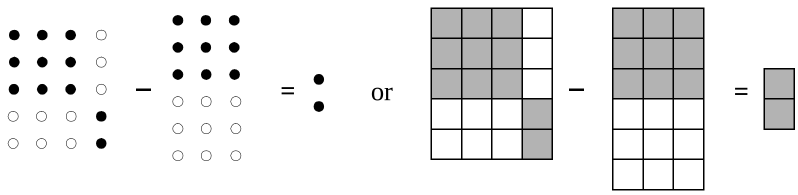


Of course there is much that needs to precede this, and much also depends on presentation. In the MiC material and in the original W 12-16 material a choice was made for number strips to represent sequences. Unfortunately this idea wasn't picked up in the textbooks. In the meantime, the Freudenthal Institute has been working on the development of a Java-applet ('ribbon-algebra') which allows the student to work interactively on the computer; the experiences so far look promising. For instance, simple algebraic calculations can be performed in an early stage, while the student can fall back at need on the visible regularity in the number strip or the sequence. A linear expression where the variable is limited to (positive) integers, is given an arithmetical face in a way! A 'number chart' is needed for a linear expression in two variables - since a dimension is added - and this representation also has its attractions.
A linear expression in one real variable acquires a (geometrical) face once the graph has been discussed, and from then on it is very well possible to interpret and understand the above operations graphically.
An expression in two variables is (probably) too much for young students to look at the graphical interpretation.

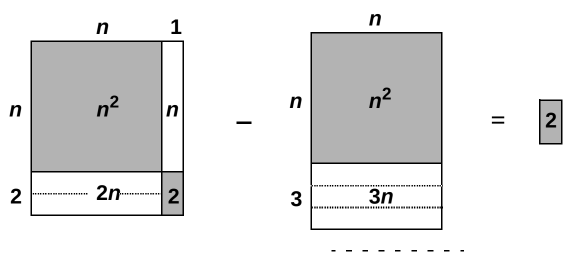
In relation with this we'd like to mention a nice example, from W.W. Sawyer's work 'Vision in Elementary Mathematics'.
Look at the adjoining row of problems:
The student is asked to continue the row.
Can he count on 2 continuing to be the outcome?
And if 'yes', then the answer requires a 'proof'.

$$2 \times 3 - 1 \times 4 = 2$$
$$3 \times 4 - 2 \times 5 = 2$$
$$4 \times 5 - 3 \times 6 = 2$$
$$5 \times 6 - 4 \times 7 = 2$$
....

An initial suggestion could be to represent the products with a dot pattern
(as already used by the Pythagorians) or a rectangular pattern and show the difference through clever use of color.
For instance:



Another couple of examples, and the insight into why it is always true, will start to grow. The generalisation is presented in this illustration:



Teachers shouldn't underestimate the problems students can have with that last representation. The *n* in the picture is often interpreted as a fixed length, while it should represent a variable! Computer animation could offer relief here. A representation where *n* can be stretched or shrunk at need, without the essence of the picture changing, is much more suggestive and can evoke the generalization aspect. An applet to achieve this has been developed.
Finally, it's possible to work towards the formula:

$$\mathbf{(n + 1) \times (n + 2) - n \times (n + 3) = 2}$$

It's possible to choose a more mature version for slightly older students.
Choose four consecutive natural numbers; take the product of the middle two and that of the outside two and note the difference between the outcomes. Investigate whether that difference changes if you take another row of four successive natural numbers. Provide an explanation for your discovery.
Th student will have to get the idea of representing the four numbers by for example n, $n + 1$, $n + 2$, $n + 3$ on his own, and then find his own way using elementary algebra.
A third option is to present the problem using number strips. The translation to the above formula will be almost automatic. The strip method can also evoke an explanation of the type:

$$\mathbf{(n + 1)\,(n + 2) - n\,(n + 3) = n\,(n + 2) + (n + 2) - n\,(n + 2) - n = 2}$$

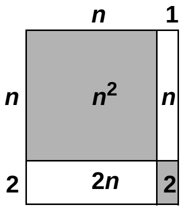
This type of activities, where the student experiences the 'strength' of algebra in revealing arithmetical patterns, is unfortunately rare in textbooks.
Another activity which is stimulated rarely in the teaching methods is the making of so-called 'own productions', where students construct their own examples and/or problems. The above example offers an ideal opportunity for this. One could for example ask the students to come up with a row of analogous problems that all have the same result. Of course the student has to provide an explanation as well.

The world of whole numbers contains some beautiful opportunities for problem oriented algebra teaching, certainly if concepts such as divisibility and prime numbers were treated. Unfortunately, in a wave of democratization, these subjects have been removed from the math curriculum; a decision we feel should reversed straightaway. In '*33550336: a perfect hit'* the teacher N. Brokamp describes in the *Nieuwe Wiskrant* (volume 14, nr.3) a wonderful experience in an upper grade class investigating what were already called *perfect numbers* in antiquity, that is numbers, like 6 and 28, that are equal to the sum of their 'real' divisors (including 1). Ostensibly a useless topic and a typical example of recreational math. This topic has a rich history and the funny thing is that the class, in a way, went through that history at an accelerated speed. As a matter of fact, the solution leads towards the socalled Mersenne-numbers, that is prime numbers of the expression $2^p - 1$ (p is itself a prime number, since otherwise the number is certainly divisible); these are the large prime number that now and again make it into the papers for breaking a record. These numbers are of prime importance in cryptography. Number theory, for so long the plaything of pure mathematicians, these days also has societal relevance. That students can be gripped by problems from number theory and the magic they sometimes have, is very clear from Brokamp's report.

Negative numbers and geometrical algebra

In one of the previous example we already made use of the so-called 'rectangular model' for multiplying expressions. These rectangular models occur in all textbooks. What is remarkable is that, after a brief introduction, they are then quickly abandoned. The effect is that there will be only a few students for whom the rectangular model will become a concrete foundation for orientation. While the methods continuously anticipate other material, here this phase is hurried through at top speed. Admittedly, this isn't totally incomprehensible, once you think of the difficulties with negative numbers in this model. It's precisely the negative numbers that can create so many problems in reducing algebraic expressions. Our observations over the year in a middle grade class constantly strengthen this opinion. The question arises whether it wouldn't make more sense to first practice algebra with positive numbers for a while. Negative numbers hardly play a part in applications, so it wouldn't be that much trouble. Specifically the rules for multiplying negative and positive numbers, which are now dealt with in the first year, could very easily be postponed to for example the third year. This would have a number of benefits. Algebraic content could be rehearsed in a kind of second round at a more abstract level (principle of telescoped reteaching) and the surface model could be in use for some time, with the students themselves determining when they want to let go of it (principle of progressive schematizing).

An intermediate step, created by the previously mentioned W.W.Sawyer, is the multiplication table, which can be transferred to the negative world.



$\times$	n	1
n	n^2	n
2	$2n$	2

$$(n+1)\times(n+2) = n^2 + 3n + 2$$

The idea of the table has been picked up very well in some textbooks and is also used for the reverse operation, factorizing. However, it is clearly more abstract than the rectangular model and because it is introduced too early, and imposed, most students will use the model reproductively and forget its basis. And then it will become no more than a handy multiplication scheme that you might as well replace with the classical way of doing multiplication
In fact this last scheme is more useful, because it allows moving on to a product of three or more factors, something which is less straightforward in the table scheme.

$$\begin{array}{rl} n+2 & \\ n+1 & \times \\ \hline n+2 & \\ n^2+2n & \\ \hline n^2+3n+2 & + \end{array}$$

Productive practice and structures
In the previously mentioned textbook A there is the following problem in a paragraph where almost all problems start with the imperative 'reduce'.
A class has to make ten reductions for a test. Every reductions has 12ab as a result. Think of ten problems that have 12*ab as a result. Have your problems checked by another student.*
This is an example of what is called 'own production'. The learning effect of such problems can be considerable. Unfortunately, this kind of problem is only too rare in the various methods.
A slightly stronger example of 'productive practice' fits in with a theme we named 'the price of algebra'.

Algebra takes time , and time is money. Here is a detailed price list:

Price list:	
operations +, –, ×, :, /	1 point each time
squaring	2 points each time
taking the 3rd power	3 points each time
taking the 4th power	4 points each time
etc.	etc.
using variables	1 point each time
parentheses and numbers	free

Example 1: what is the price of $3n + m$?

3	**number**	**free**
n	**variable**	**1 point**
$3 \times n$	**multiplication**	**1 point**
m	**variable**	**1 point**
$(3 \times n) + (m)$	**addition**	**1 point**
	total price	**4 points**

One can for example establish that $\mathbf{(3}\boldsymbol{n}\mathbf{ + }\boldsymbol{m}\mathbf{)^2}$ has a price of 6 points; after all, we are careful to note that the implied multiplication also costs 1 point. The equivalent trinomial $\mathbf{9}\boldsymbol{n}^\mathbf{2}\mathbf{ + 6}\boldsymbol{mn}\mathbf{ + }\boldsymbol{m}^\mathbf{2}$ costs the grand sum of 12 points. Having established the cost of several expressions (a kind of exercise in algebraic parsing) it can be established that equivalent expressions do not necessarily have the same value and the cheapest expression can be determined for given expressions. A nice game, which stimulates flexible behavior and where the student sets himself a lot of reduction problems over the course of the game.
For that matter, the game isn't without practival value, because the 'points' can be seen as a measure for 'calculation time'. For computerprograms that have to perform the same complex calculations over and over, it can be important which expressions are selected for the calculations.
Economic calculating has already been put into practice by Newton.
He discovered for example that calculating values of the polynom

$$y^4 - 4y^3 + 5y^2 - 12y + 17$$

takes much more time for substitution in this expression, than for substitution in the equivalent expression:

$$y\,(y\,(y\,(y - 4) + 5) - 12) + 17$$

or

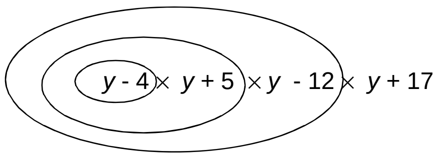
$$(((y - 4)\,y + 5)\,y - 12)\,y + 17$$

In Newton's notation:

$$\overline{\overline{\overline{y - 4}\;y + 5}\;y - 12}\;y + 17$$

In our points game the first polynom expression has 20 points, while the second expression (known as the Horner expression) only scores 11 points.
Aside: it wouldn't be a bad idea to confront the students with Newton's notation. In the first place, there is the historical aspect and secondly it puts our conventions into some perspective.

The most terse way to clarify the structure of a complex algebraic expression is perhaps the use of 'circles'.



In the light of later use of graphic or symbolic calculators it is of the utmost importance that students can work with this kind of schemes..
Other expressions to analyze these structures are 'operational trees' and 'arrow chains', but we won't go into those here.

Testing formulas
Polya writes in Mathematical Discovery (volume 1) off the importance of testing formulas by paying attention to special cases, and where possible to dimensions. We will take the classical formula ascribed to Heron, but which Archimedes must already have known, as an example. If a triangle has for instance the sides 13, 14 and 15, its surface can be calculated as follows: first calculate half the circumference, here 21. Subtract the three sides in turn from half the circumference: 8, 7 and 6. Multiply these results and half the circumference with each other, which produces: 7056. Extract the square root from this number and you will get the surface: 84. One could show students Heron's original text in translation and ask them to convert his description of this algorithm into a formula. The expression as it was in old textbooks, is:

$$O = \sqrt{s(s-a)(s-b)(s-c)}$$

In this expression, *a, b* and *c* represent the sides of the triangle and *s* the half circumference $s = \frac{1}{2}(a + b + c)$.

Next, the formula could be tested for several aspects.
–What for instance happens when $s - a = 0$?
–Can the expression under the root sign be negative?
–What about ‘symmetry’ in the formula?
–What is the effect on O when all sides are multiplied by for example 10?
–What is the result of the formula for the case $a = b = c$? Is that correct?
–And what about $a = b \neq c$?
–For a rectangular triangle with catheti a and b the surface can be expressed directly in a and b; is that expression also in agreement with Heron’s formula?

One formula can yield a cornucopia of meaninful algebra exercises like this. We picked a rather complex example here, but there are other, simpler, surface and volume formulas that give rise to this kind of reflection. Formulas with a more science oriented background are often well suited for this kind of activities. And for students who will be working with computer algebra later, it will be an essential skill to be able to judge a formula on its merit.
We remark that the formula is still not proven even after the described research. The pure geometrical proof given by Heron is elegant, but difficult. A demonstration based on Pythagoras’ Theorem, which involves some fairly elaborate algebra, on the other hand could be explained to the students. The formula in itself isn’t really important, although it is still included in many mathematical handbooks, but as a subject for research it can be interesting.

Points of attention
Looking at the textbooks, there is plenty to do in the coming years to improve the teaching of algebra. The activities around functions of just one variable are too one-sided, and often also too microscopic. It’s noticeable how little problem oriented the methods are. The approach will have to be much wider to give students some familiarity with symbolic calculations.
Introducing formulas to solve challenging problems, is the only thing that can give meaning to algebra. Skills for solving complicated reductions seem to become redundant with the introduction of electroning aids. That means that exercises in algebra techniques can be relatively simple; it isn’t about gaining automatisms and routines, but about developing insight into the structure of formulas and of a certain algebraic confidence.
We explicitly ask for more attention for sequences and number problems in secondary education. Not just because of the mathematical beauty they can have, but mainly for didactical reasons: discrete variables are more concrete than continuous variables. Working with negative numbers is often a factor for interference in early algebra teaching, which causes many errors and makes students feel insecure. If the introduction of negative numbers can be more gradual and the aim of immediate completeness is let go (for instance, solving first degree equations where the parameters have to be solved in all possible combinations of negative and positive), a lot could already be gained. Of course there will be a moment where ‘manipulating with minuses’ is unavoidable as a result of the algebraic-geometrical permanence principle, but it should not interfere with the elementary structural rules of algebra.
In our opinion the following list of points of attentions is important for developers of algebra education in pre-university schools:

–*developing a symbolic language sense*
–*understanding and testing formulas*
–*explaining reductions from the meaning of standard operations*
–*constructing and generalizing*
–*using geometrical models*
–*,making use of the history of algebra*
–*flexible and productive practice*
–*translating into and reasoning with algebra*
–*didactal use of ICT*
–*being resistant to the urge to be complete*

Students should be made to experience that the potency of algebra goes further than concisely describing relations between variables and solving equations. The need for the latter skill has as good as disappeared with the symbolic calculator. Algebra is also a tool for solving problems, generalizing, and proving. Having the student experience this and giving him some skills in it, appears to us to be the most important challenge in curriculum development in algebra in the coming years.

Sum and product(I)

$$? + ? = 12$$

$$? \times ? = ?$$

Two natural numbers, you don't know which ones.
However, not everything is unknown!
This is what you know: if you **add** both numbers,
the result will be **12**.

Now both numbers are **multiplied**
♦ What can be the result ? Show all results you found.

Sum and product (II)

$$6 + 14 = 20 \qquad 6 \times 14 = 84$$

sum
of 6 and 14

product
of 6 and 14

From two natural numbers you only know the **sum,** this is **20**.
If the numbers were 6 and 14 their **product** should be 84.
But there are other pairs of natural numbers which sum is equal to **20**.
Their product will differ from 84.

- ♦ Below you see a chart with the numbers 1, 2, ..., 100.
  Colour all cells that correspond with a possible product.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

- ♦ Write the results in a sequence from large to small.
  Do you discover any pattern in this sequence? Which one?

Sum and product (III)

Two natural numbers have a **product** equal to **24**.
♦ What can be their **sum**?

Three natural numbers have a **som** equal to **10**.
♦ What can be their **product**?

Sum and product (IV)

A and ***B*** represent natural numbers

$A + B = 12$

↓

$A \times B = \ldots$ or $\ldots$ or $\ldots$ or $\ldots$ or $\ldots$ or $\ldots$ or $\ldots$

$A \times B = 12$

↓

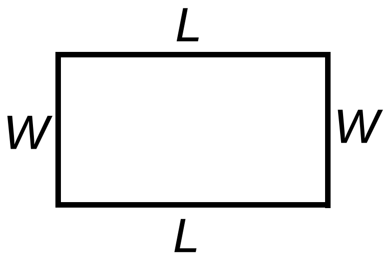
$A + B = \ldots$ or $\ldots$ or $\ldots$

$A \times B \times C = 12$

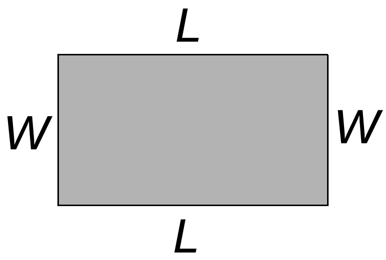
↓

$A + B + C = \ldots$ or $\ldots$ or $\ldots$ or $\ldots$

Perimeter and area (I)



$$perimeter = L + W + L + W$$
$$= 2 \times (L + W)$$
$$= 2 \times L + 2 \times W$$



$$area = L \times W$$

The length and width of a rectangle are a whole number of centimeters, but you don't know how many. The **perimeter** is equal to 18 centimeters.

♦ Draw all possible rectangles.
How many square centimeters can be the **area**?

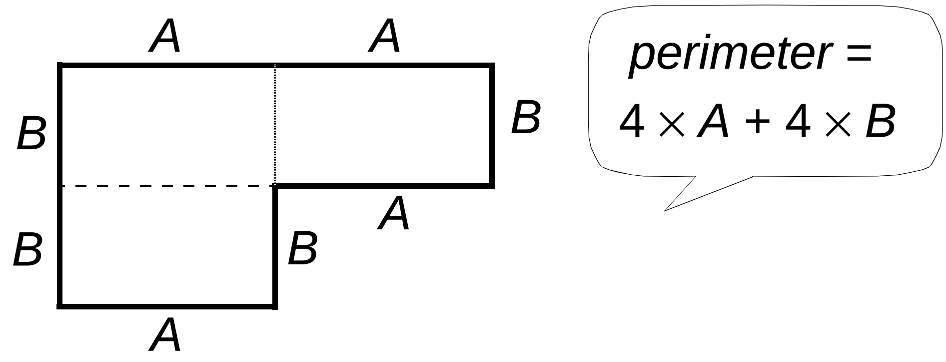
The length and width of a rectangle are a whole number of centimeters.
You have to know that the **area** is equal to 18 cm.

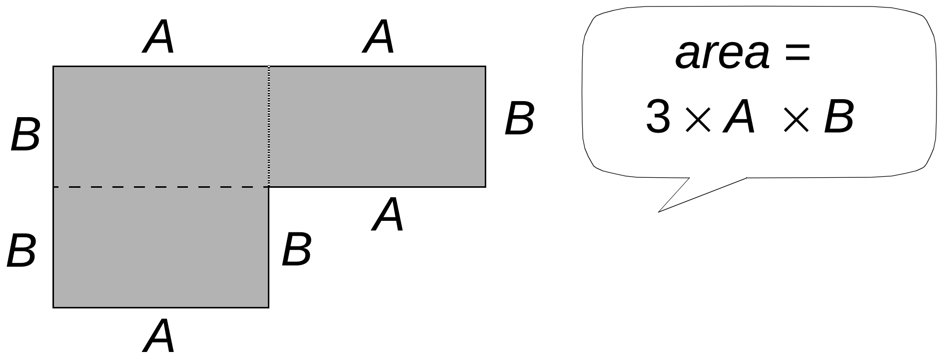
♦ Draw all possible rectangles.
How many centimeters can be the **perimete**r?

A rectangle has **perimeter** of 22 cm and **area** 28 cm^2.

♦ How long and how wide is the rectangle?

Perimeter and area (II)





A and *B* represent whole numbers.

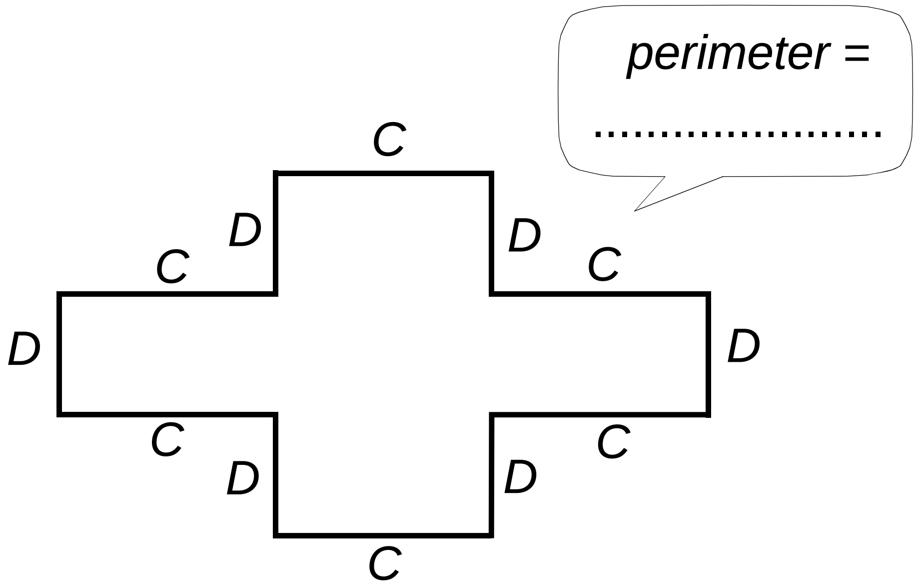
Suppose the **perimeter** of the shape above is 52 cm.

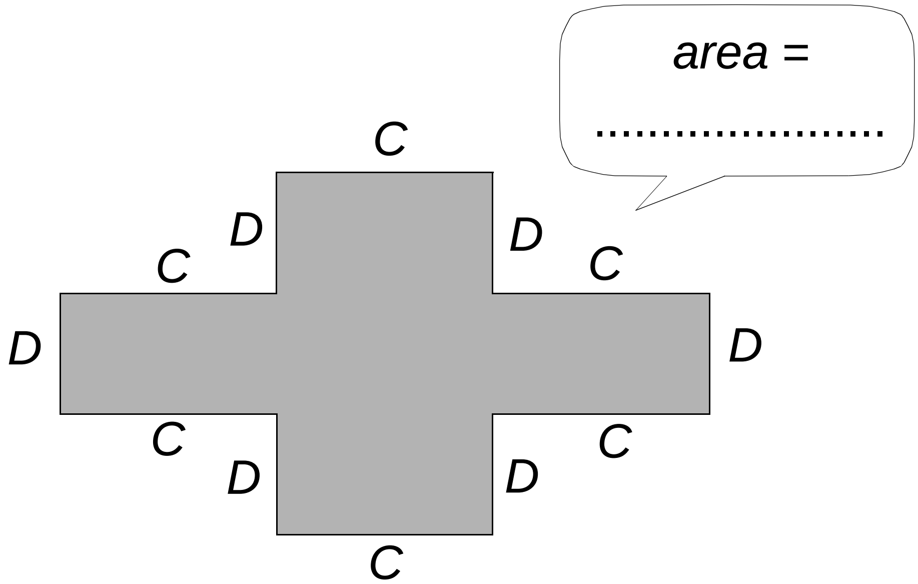
♦ What can be the **area**?

Suppose the **area** of the shape above is 42 cm^2.

♦ What can be the **perimeter**?

Perimeter and area (III)

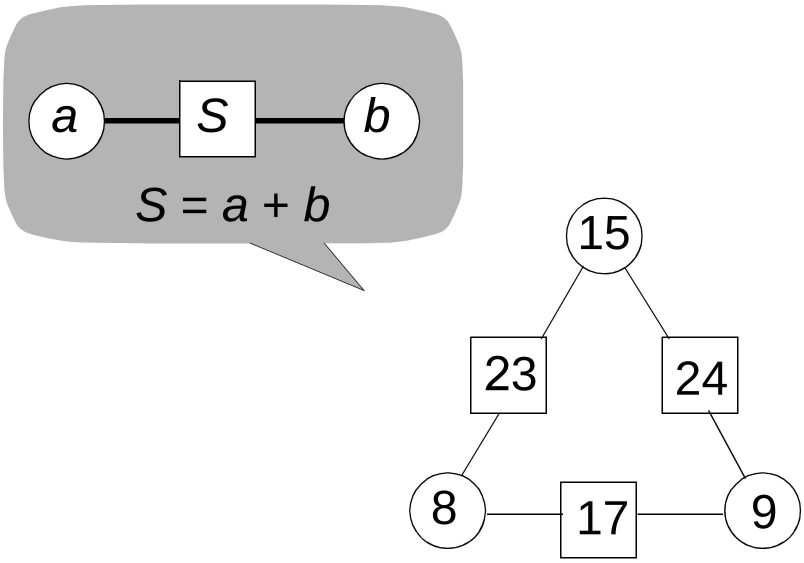




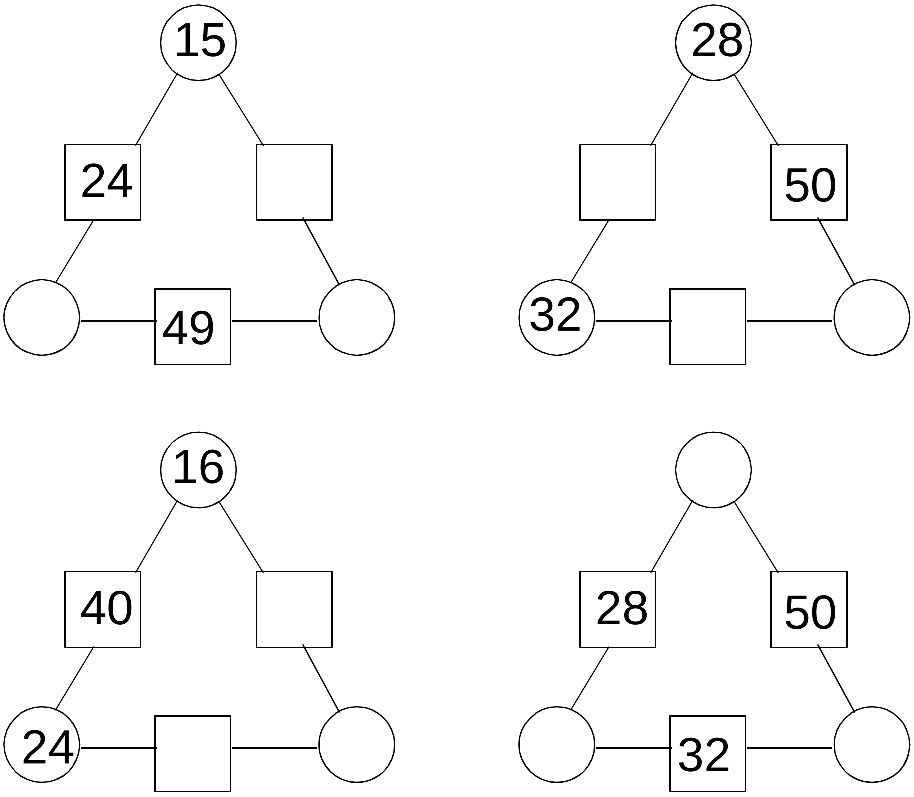
♦ Fill in expressions for **perimeter** and **area**.

♦ Design a problem about perimeter and area of such a cross figure.

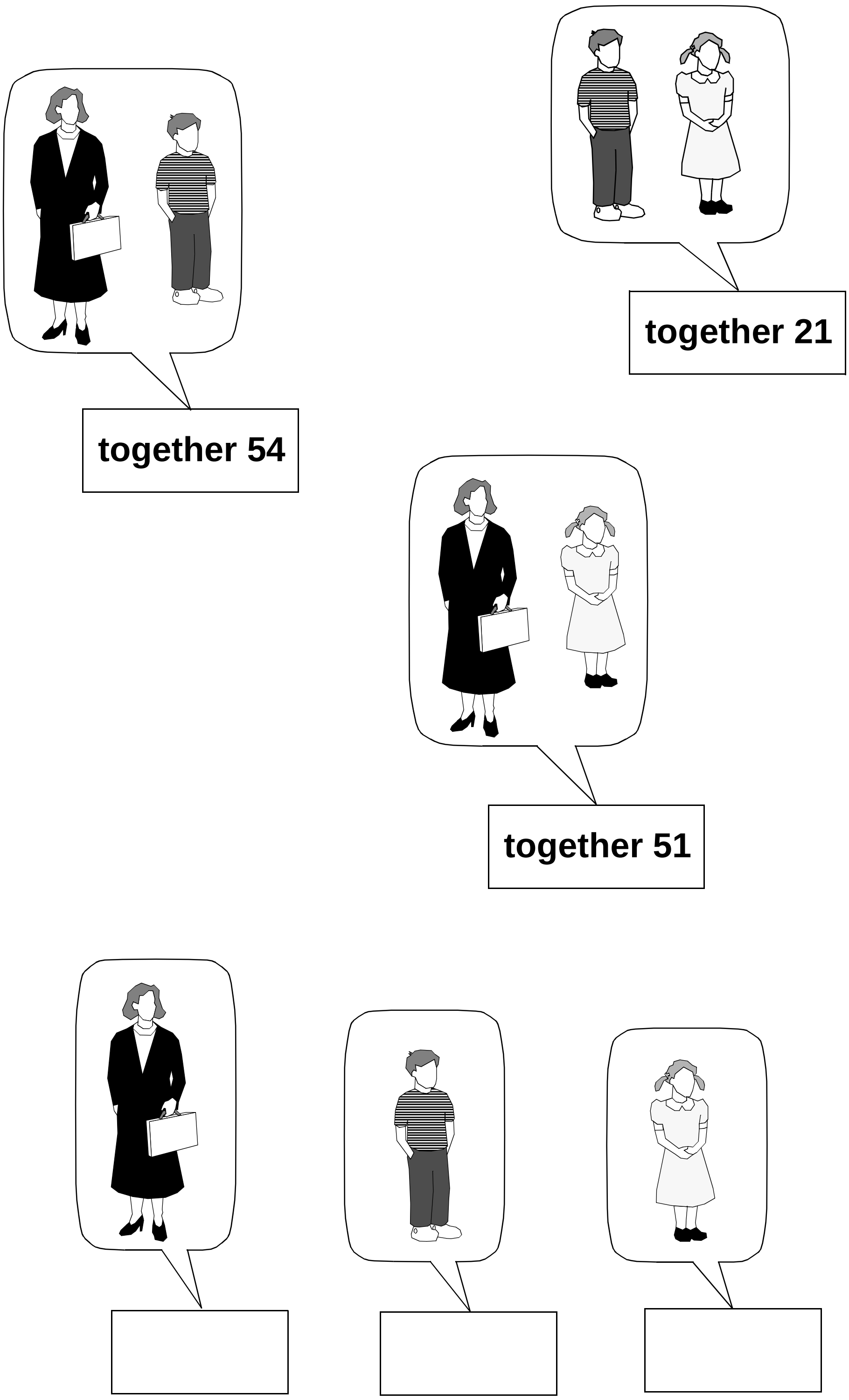
Find the three numbers



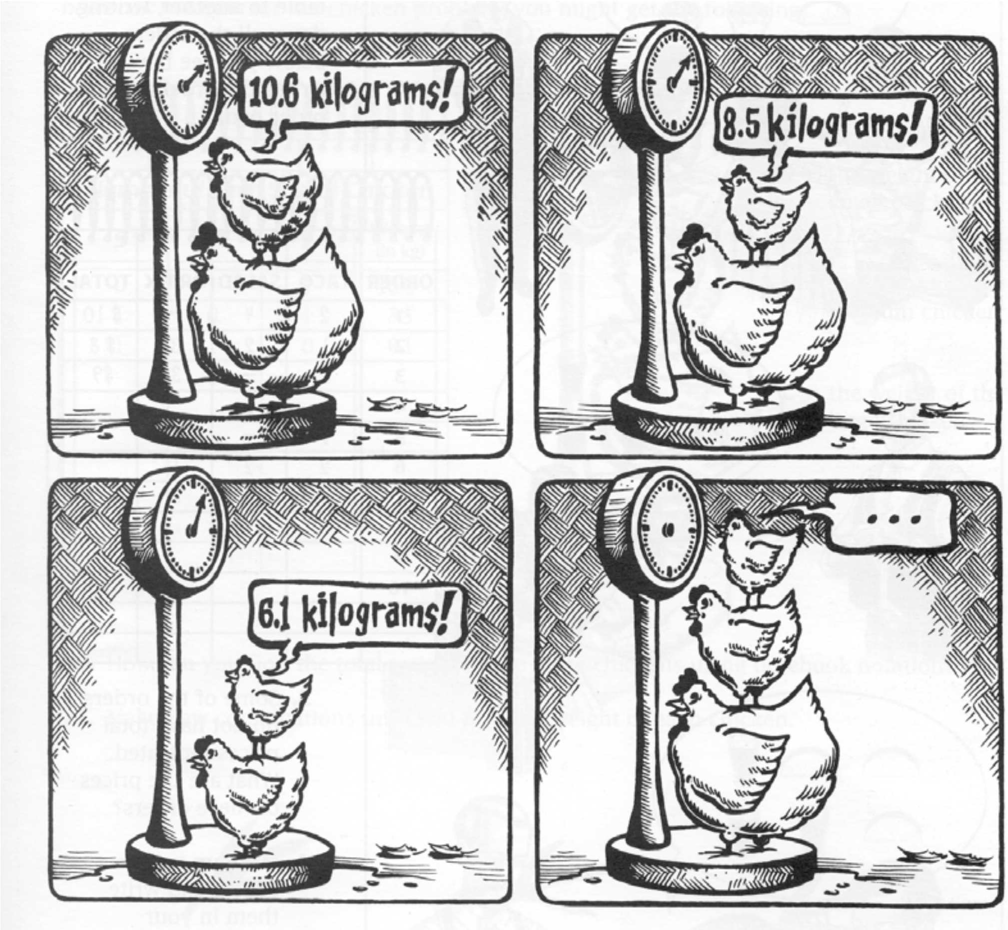
♦ Fill in the missing numbers:



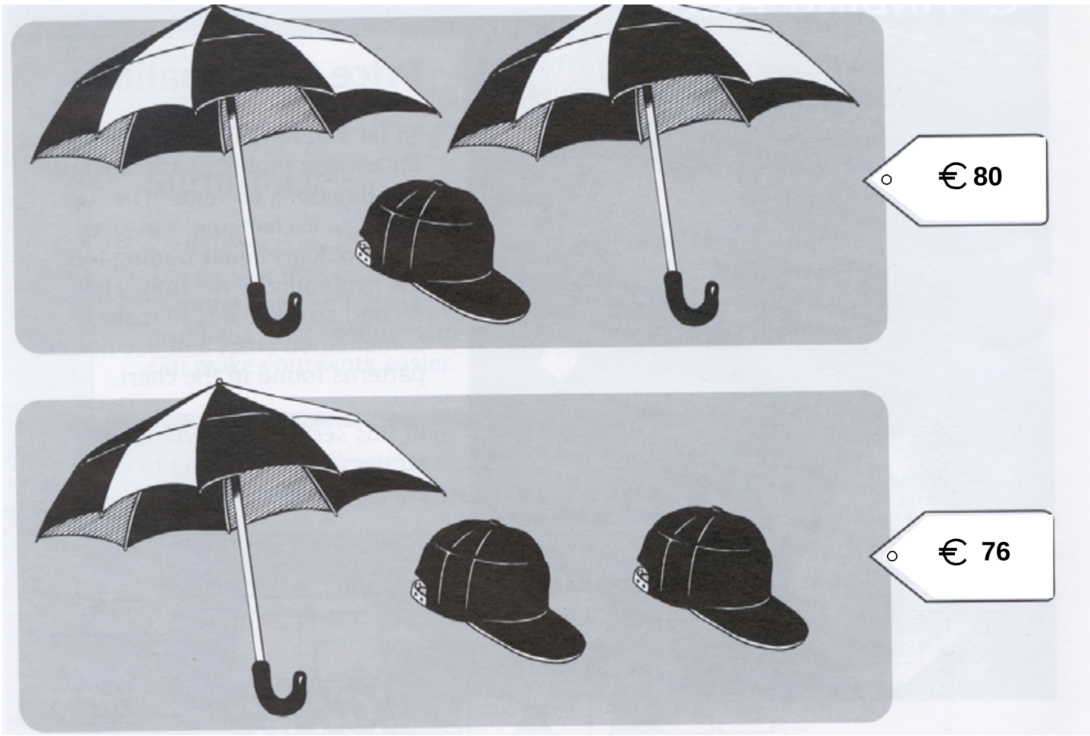
How old?

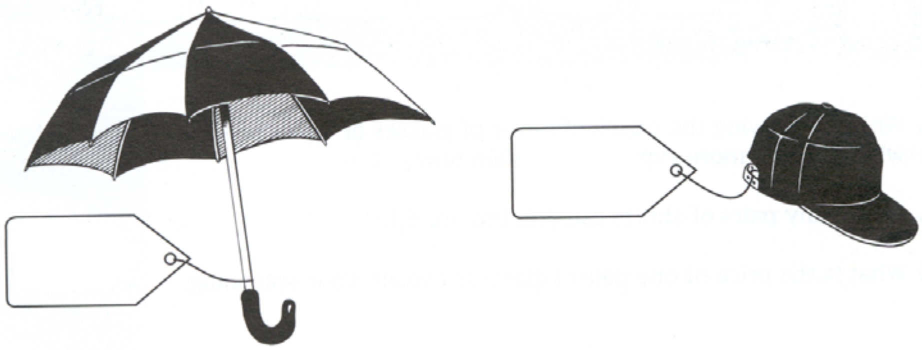


How heavy?



How expensive?

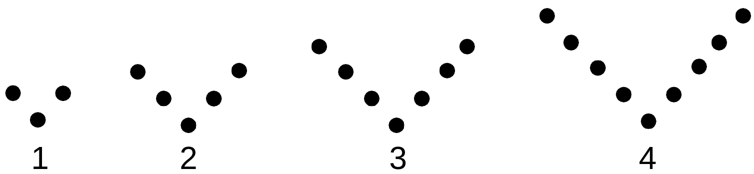




Dot patterns (I)

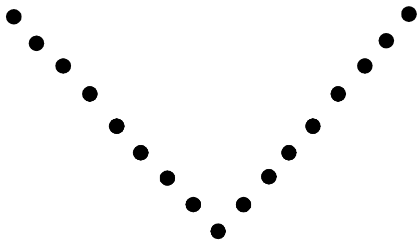
Groups of birds sometimes fly in a ***V-pattern***.

V-pattern with dots:



The figure shows the first four V-patterns. Each pattern has a number in the sequence. Below you see a V-pattern with 17 dots.

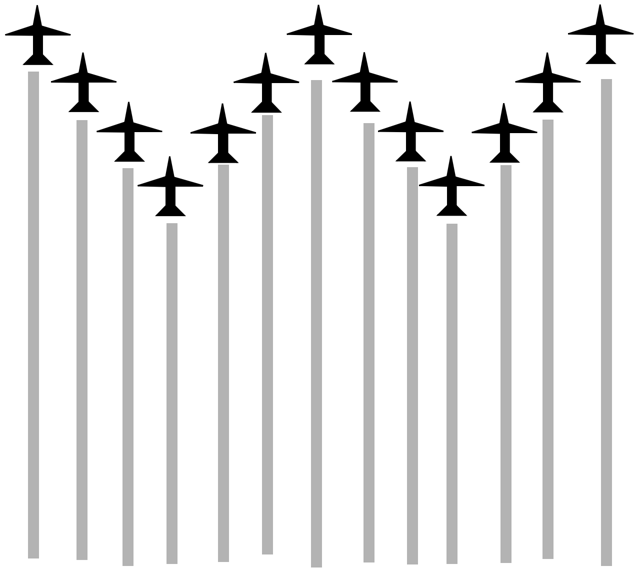
- ♦ Which number in the sequence has this V-pattern?



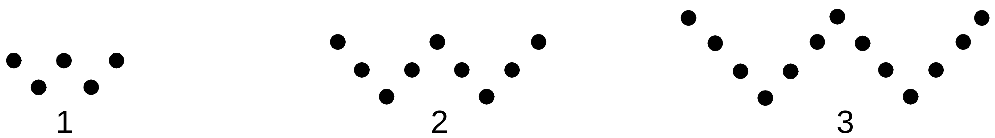
- ♦ How many dots has the V-pattern with number 85 in the sequence?
- ♦ Does exist a V-pattern with 35778 dots? Why or why not?
- ♦ Give a rule to find the number of dots of a V-pattern, knowing the number in the sequence.
- ♦ Represent this rule by a direct formula; use the letters ***n*** and ***V***
  (***n*** = number in the sequence, ***V*** = number of dots)

Dot patterns (II)

During a show a squadron of airplanes flied in a W-formation.



Look at the begin of a sequence of W-patterns:



♦ Fill in the table:

pattern number	1	2	3	4	5	6	
number of dots	5	...	...	...	...	...	

♦ How many dots are in the W-pattern number 25?

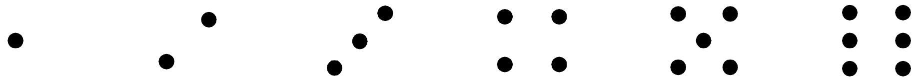
♦ Find a direct formula to describe the number of dots in any W-pattern.
(***n*** = number in the sequence, ***W*** = number of dots)

♦ What is the relationship between a W-number and a V-number corresponding with the same number ***n***?
Is this $\boldsymbol{W = 2 \times V}$ or not? Explain your answer

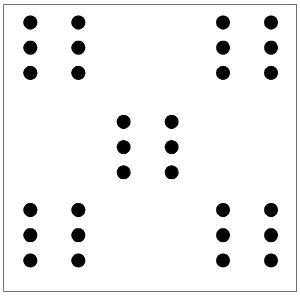
♦ Choose another letter-pattern by yourself and find a corresponding formula.

Dice patterns

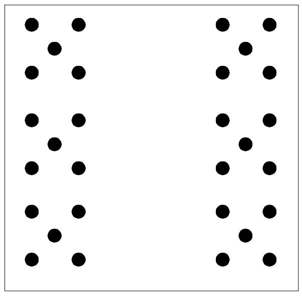
Of course you know the six dot patterns on a dice:



♦ Which pattern, **a** or **b**, has the biggest number of dots?
You can answer this question without counting the dots!

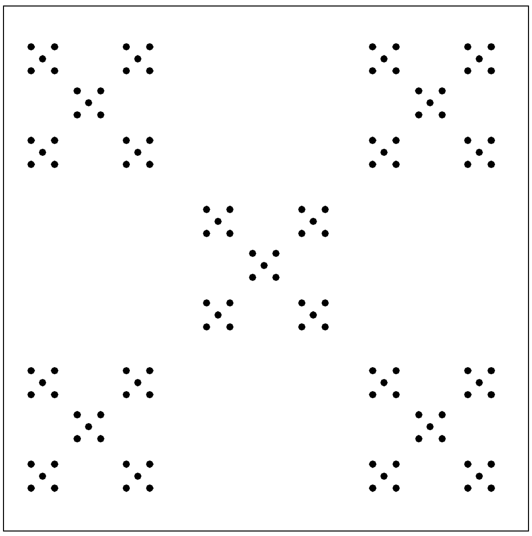


a

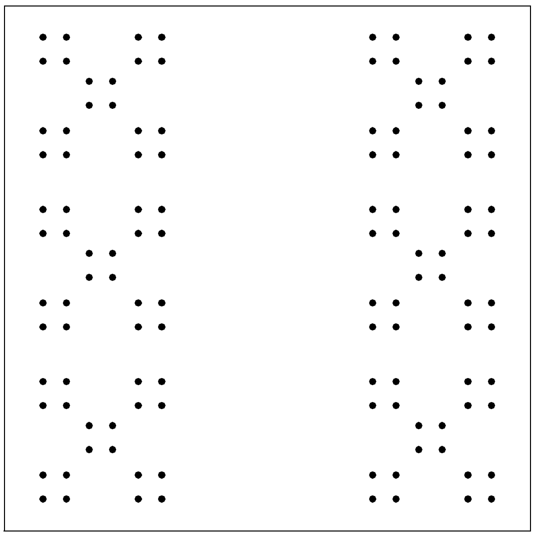


b

♦ Which pattern, **c** or **d**, has the biggest number of dots?:

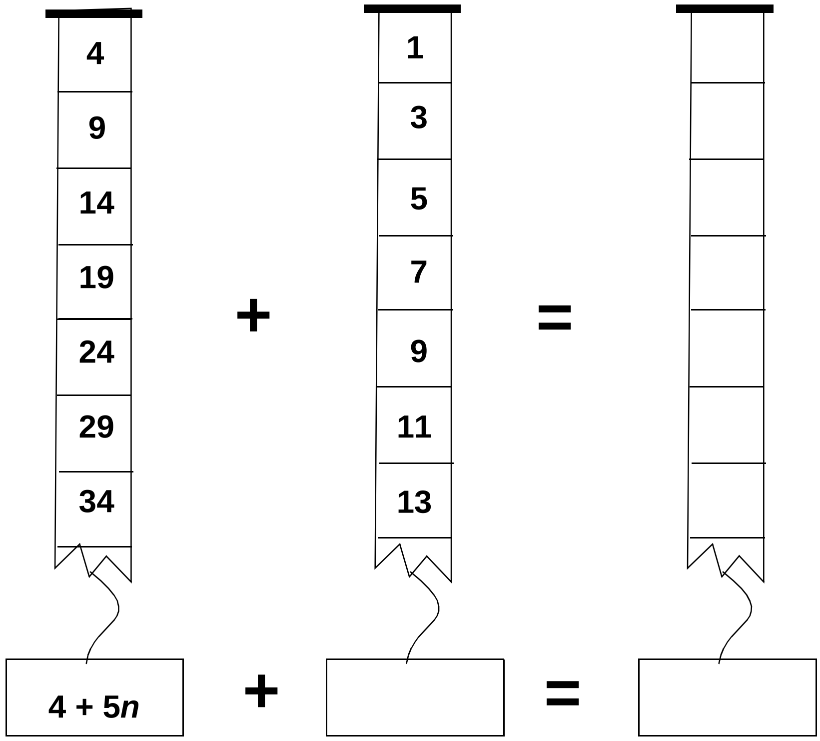


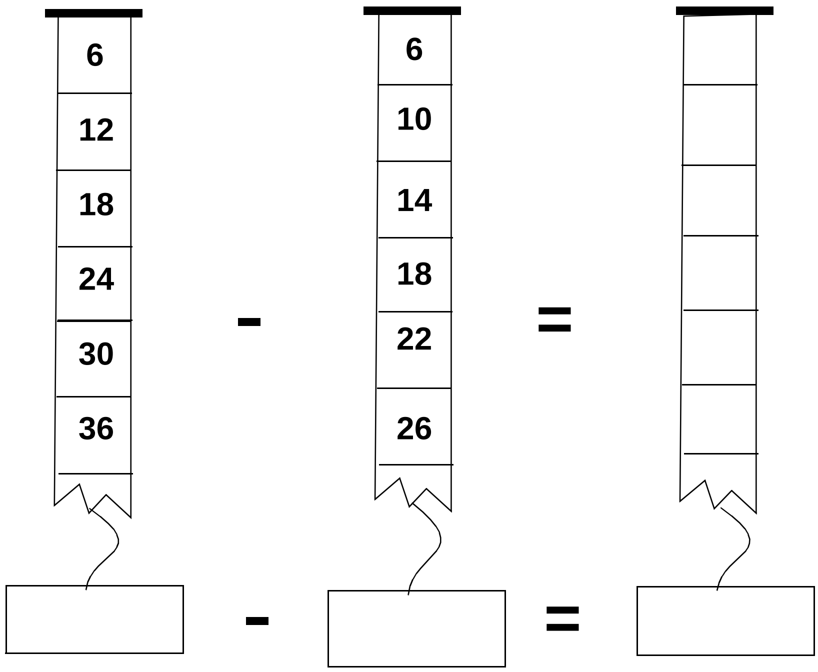
c



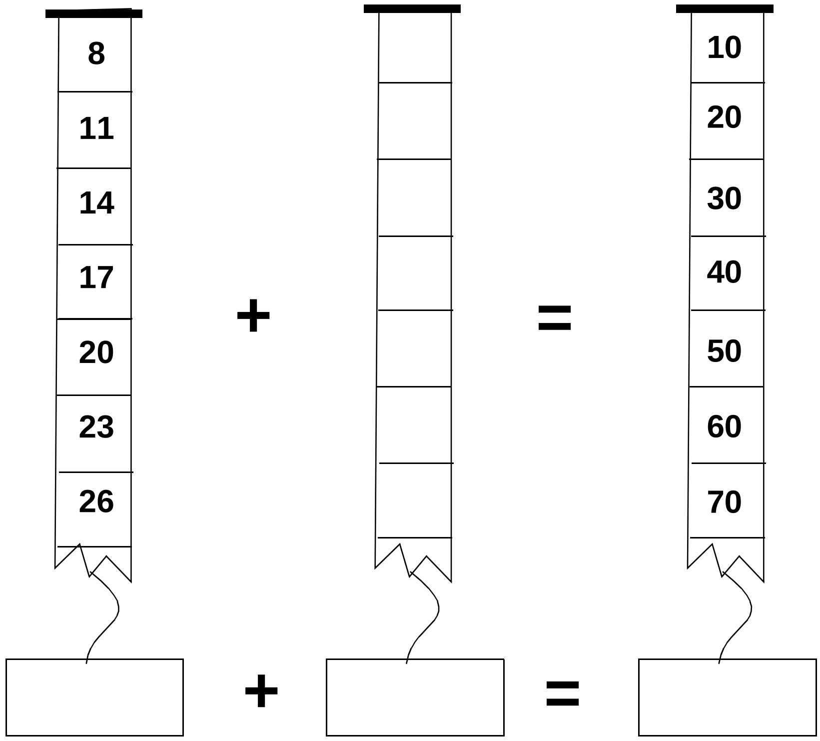
d

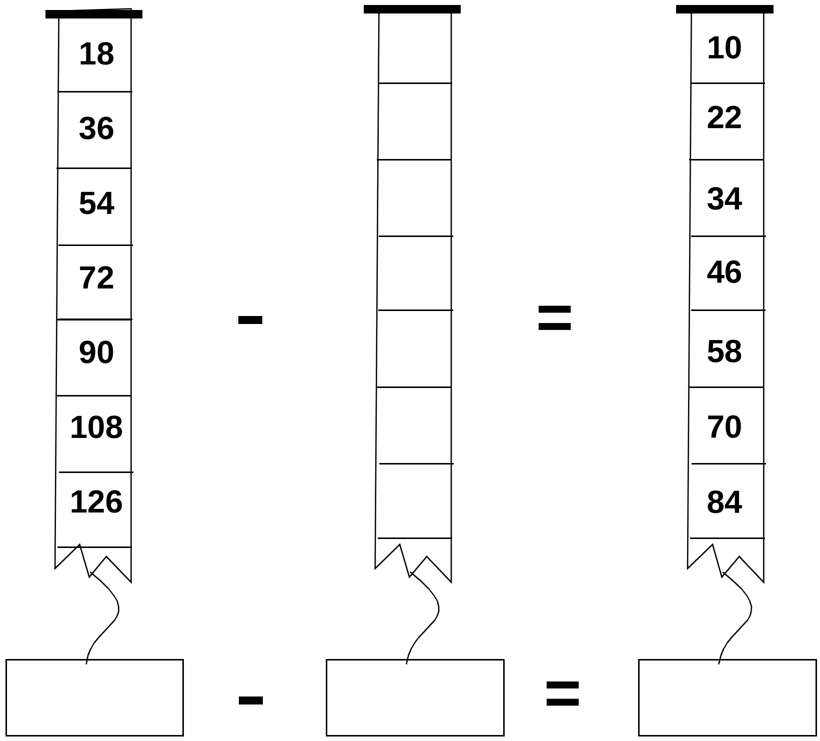
Operating with number strips (I)



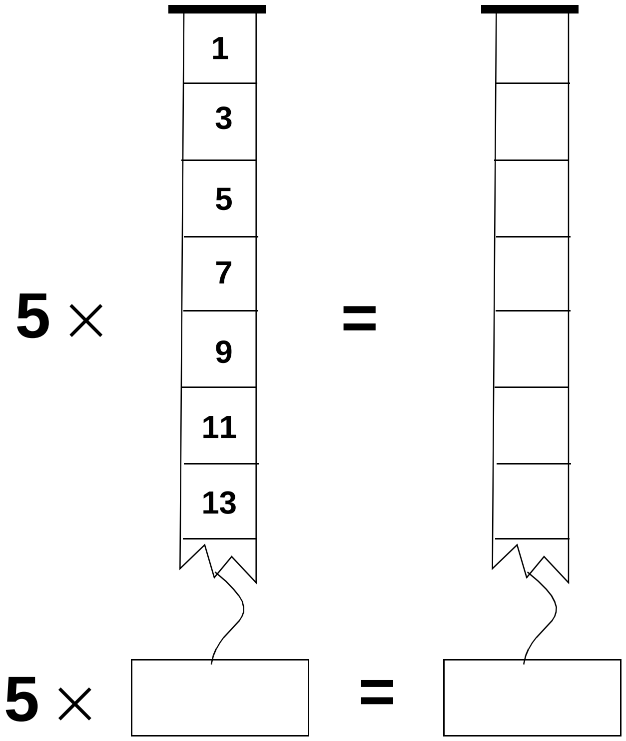


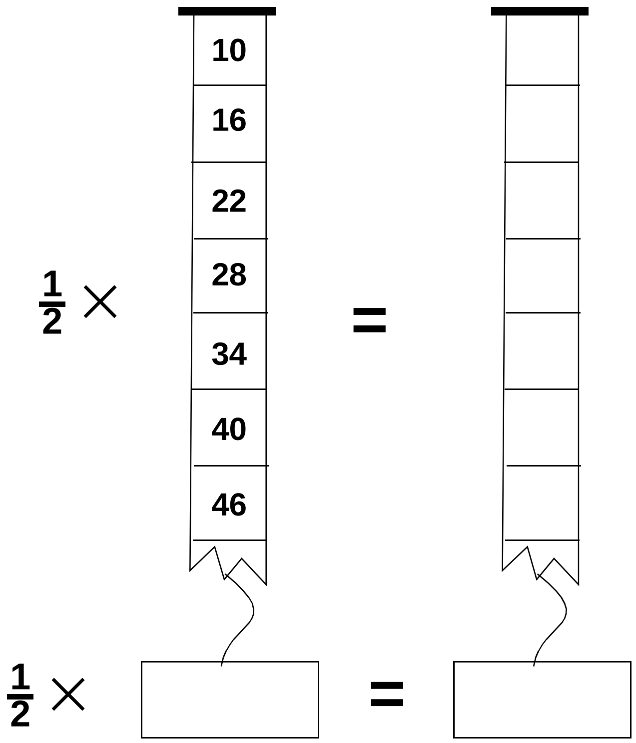
Operating with number strips (II)





Operating with number strips (III)





Operating with number strips (IV)

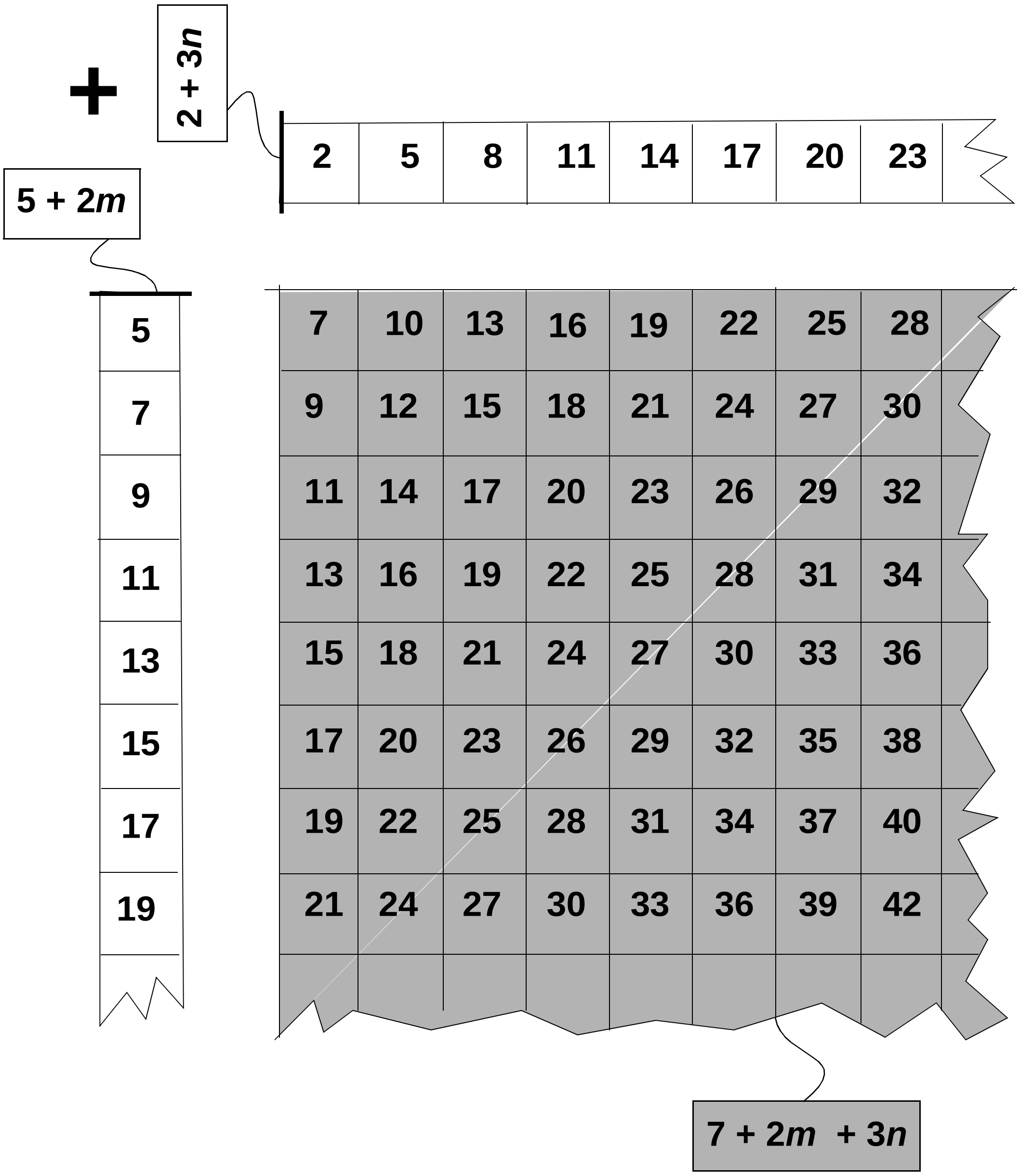
3 ×	+ 2 ×	=
2	1	
3	3	
4	5	
5	7	
6	9	
7	11	
8	13	

3 × ☐ **+ 2 ×** ☐ **=** ☐

2 ×	– 5 ×	=
3	1	
8	3	
13	5	
18	7	
23	9	
28	11	
33	13	

2 × ☐ **– 5 ×** ☐ **=** ☐

Strips and charts (I)

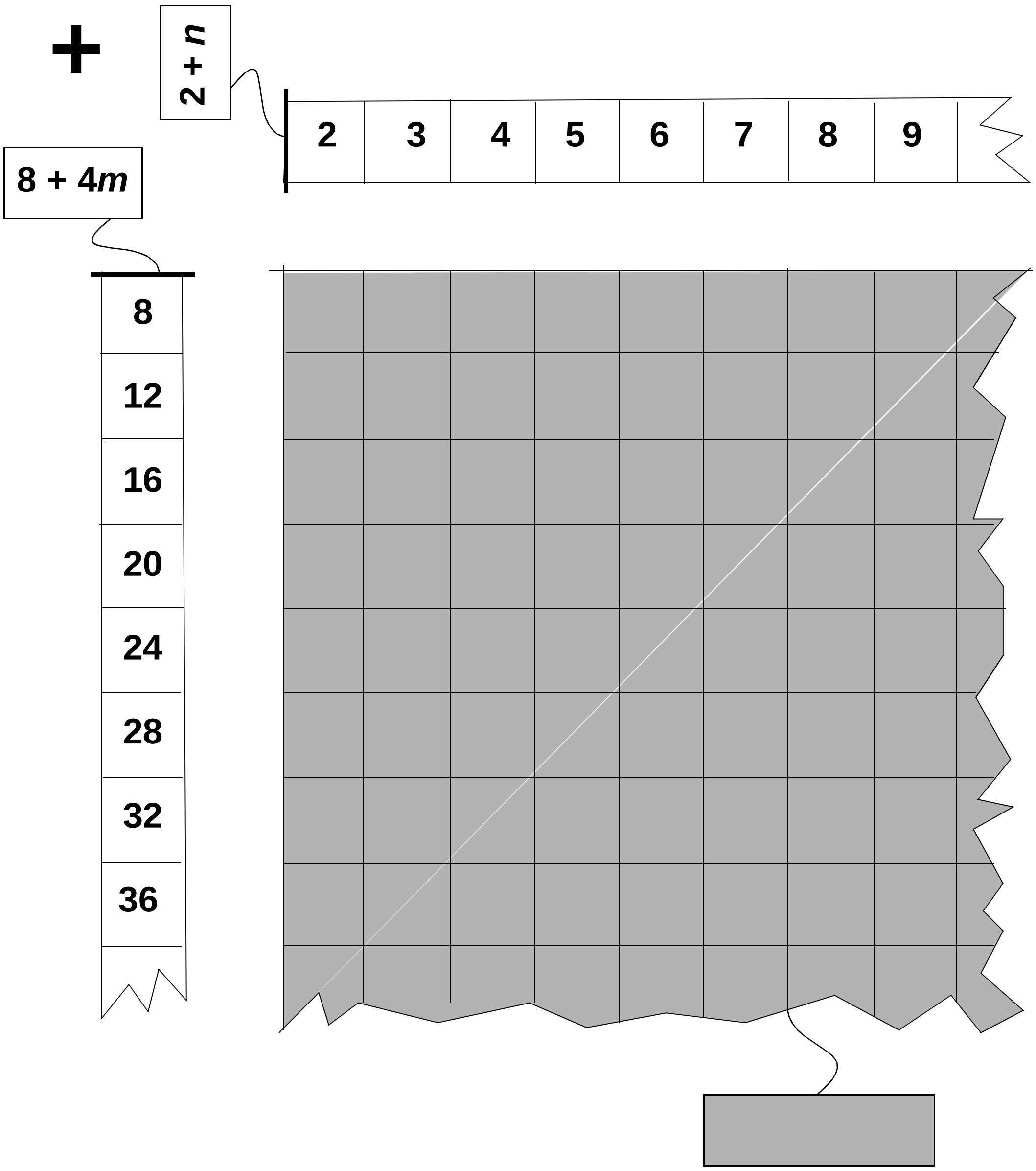


- Find the number that corresponds with m = 3 and n = 2
- Also for m = 3 and n = 5

- A horizontal row in the chart corresponds to m = 4. Which one?
- A ....row in the chart corresponds to n = 5. Which one?

- Which numbers from the chart corespond to m = n?
- Make a number strip for this numbers with the corresponding expression.

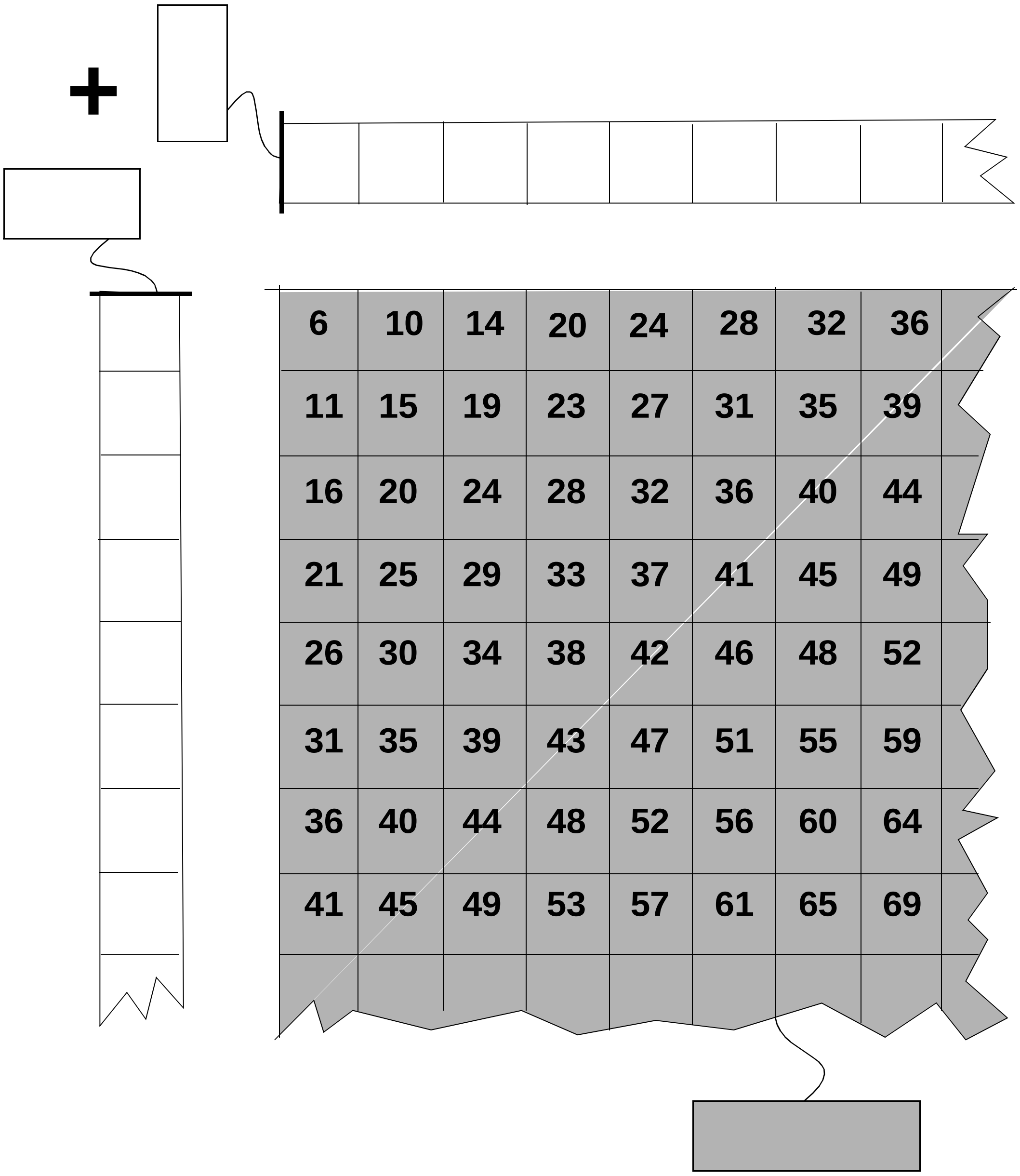
Strips and charts (II)



- Fill in the chart. Which expression fits with the chart?

- Which strip fits with $m = 3$? What is the corresponding expression?
- Same questions for $n = 0$.
- Also for $m = n$.
- Also for $m = n + 1$.

Strips and charts (III)

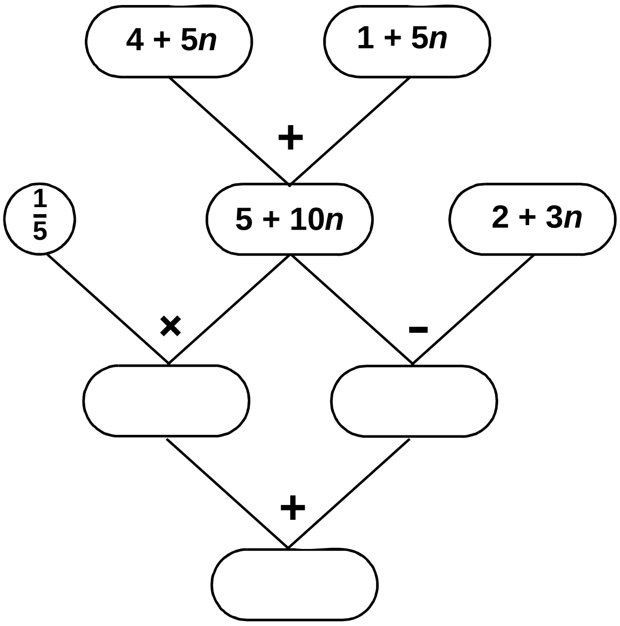


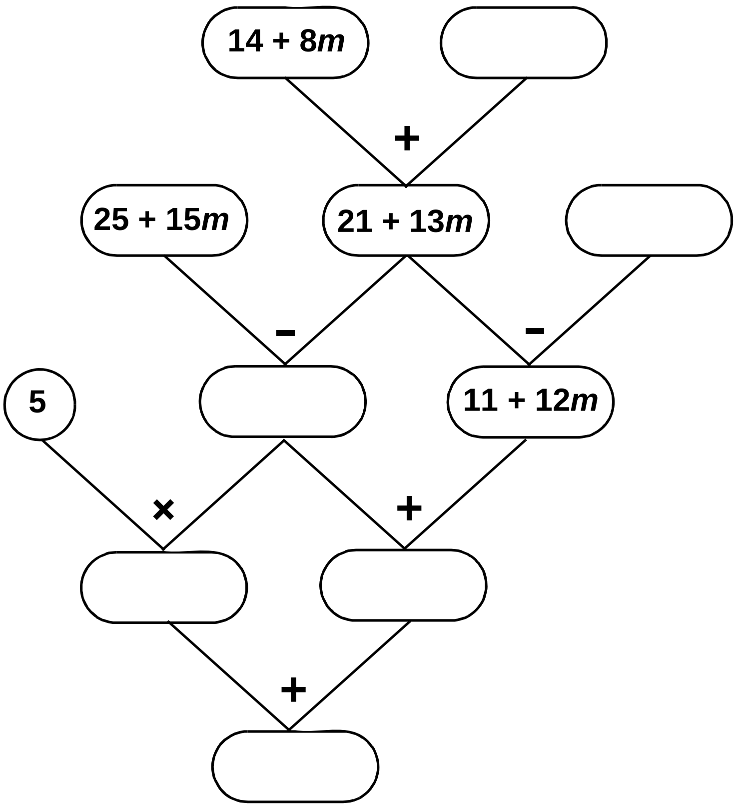
6	10	14	20	24	28	32	36
11	15	19	23	27	31	35	39
16	20	24	28	32	36	40	44
21	25	29	33	37	41	45	49
26	30	34	38	42	46	48	52
31	35	39	43	47	51	55	59
36	40	44	48	52	56	60	64
41	45	49	53	57	61	65	69

♦ Which expression fits with the chart?

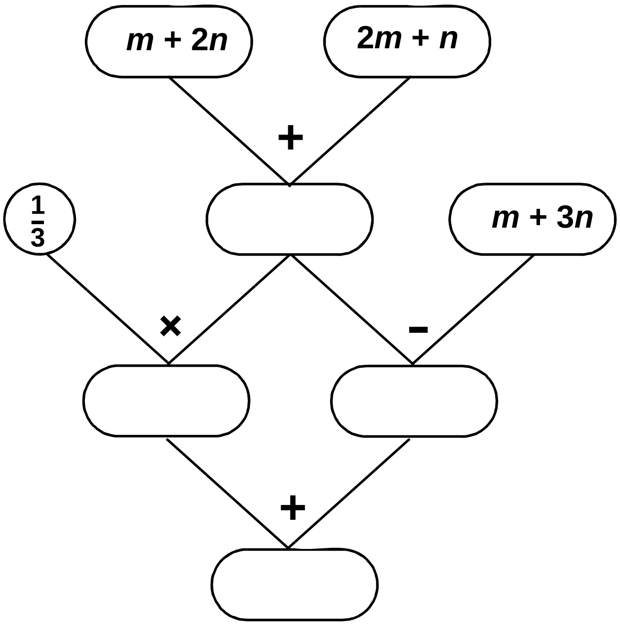
♦ From which strips can the chart be made?

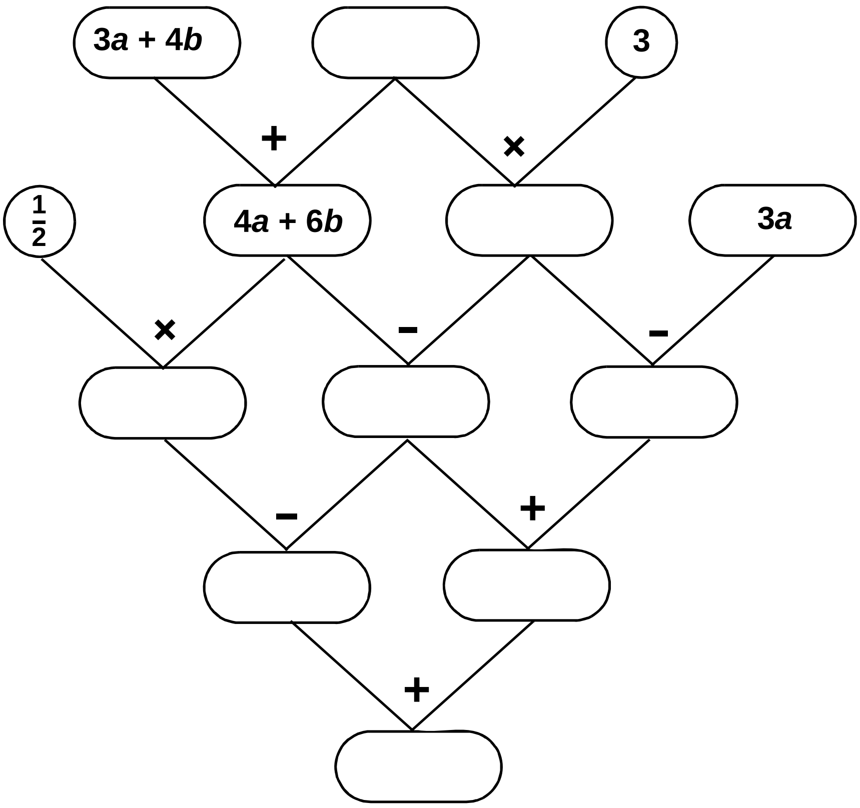
Operating with expressions (I)





Operating with expressions (II)

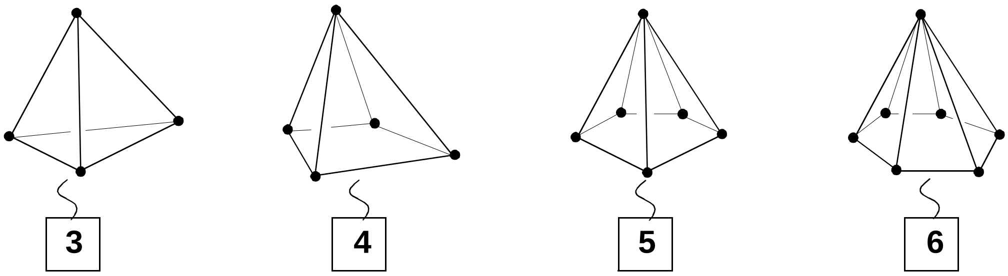




Vertices, edges, faces (I)

The beginning of a sequence of **pyramids**.

- Explain the numbers below the pyramids.



For any pyramid we call:

V = the number of vertices,

E = the number of edges ,

F = the number of faces.

Example: for a quadrilateral pyramid you have ***V*** = 5, ***E*** = 8, ***F*** = 5

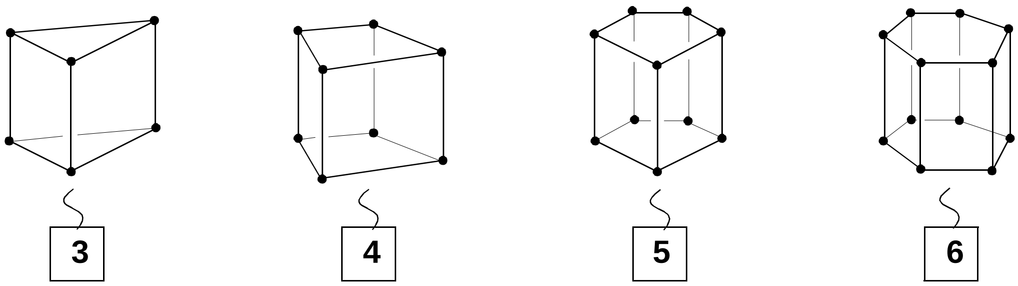
- Fill in the missing numbers:

n	***V***	***E***	***F***
3			
4	5	8	5
5			
6			
7			

- Find direct formulas for ***V*** , ***E*** and ***F*** related to ***n***.
- The four pyramids above satisfy the relation: ***V*** + ***F*** = ***E*** + 2.
  Check this.
- Explain that ***V*** + ***F*** = ***E*** + 2 is true for every pyramid, using the direct formulas for ***V*** , ***E*** and ***F***.

Vertices, edges, faces (II)

Here is a sequence of **prisms**.



Consider the numbers of vertices, edges and faces.

- Fill in the missing numbers:

n	*H*	*R*	*V*
3			
4	8	12	6
5			
6			
7			

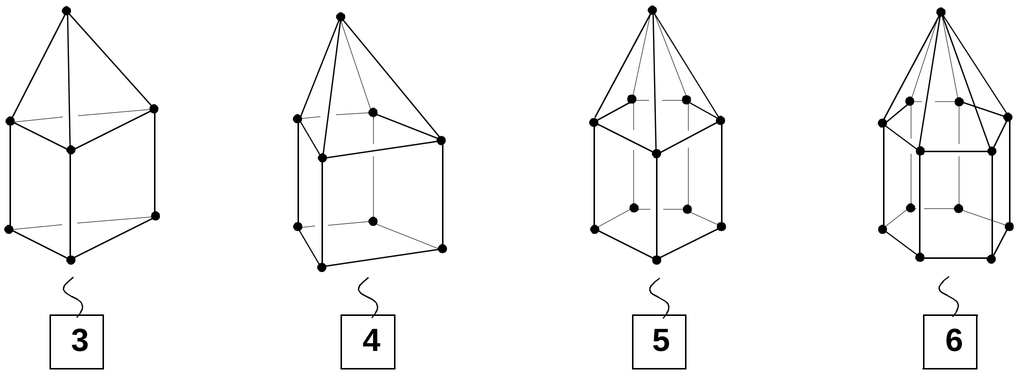
- Now find direct formulas for ***V*** , ***E*** and ***F*** related to ***n***:

  H = ... ***R*** = ... ***V*** = ...

- Does the equality ***V*** + ***F*** = ***E*** + 2 valid for each prism?
  Use the direct formules for ***V*** , ***E*** and ***F*** to investigate this.

Vertices, edges, faces (III)

Here is a sequence of **towers**:



Consider again the numbers of vertices, edges and faces .

- Fill in:

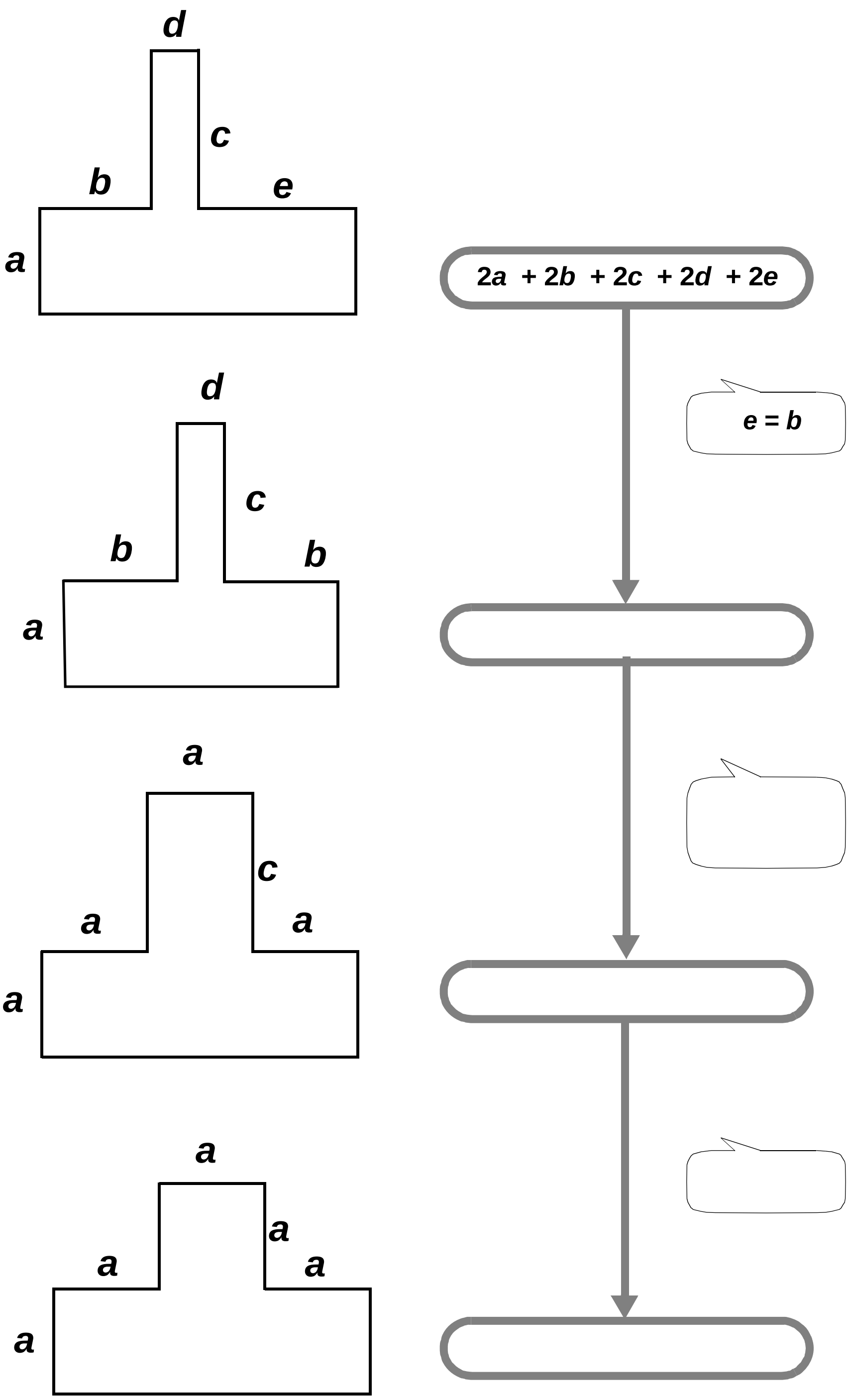
n	*H*	*R*	*V*
3			
4	9	16	9
5			
6			
7			

- Find direct formulas for ***V*** , ***E*** and ***F*** of a ***n***-sides tower.

V = ... ***E*** = ... ***F*** = ...

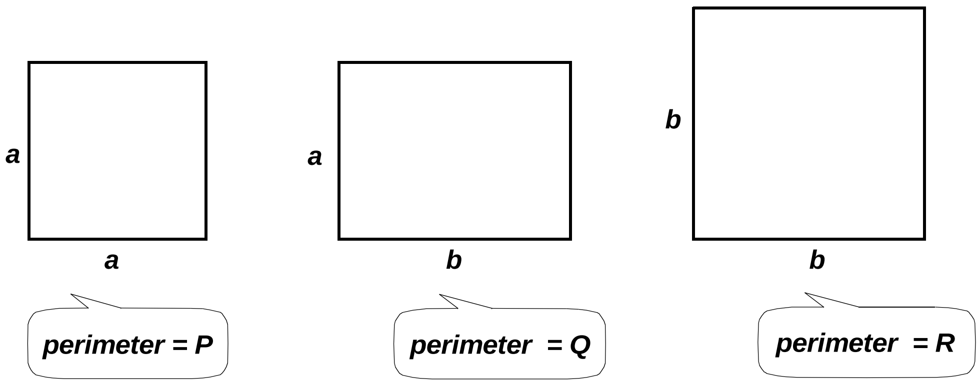
- For each tower valids: ***V*** + ***F*** = ***E*** + 2.
  How can you explain this?

Formulas for perimeters (I)



Formulas for perimeters (II)

Two squares and one rectangle in between:

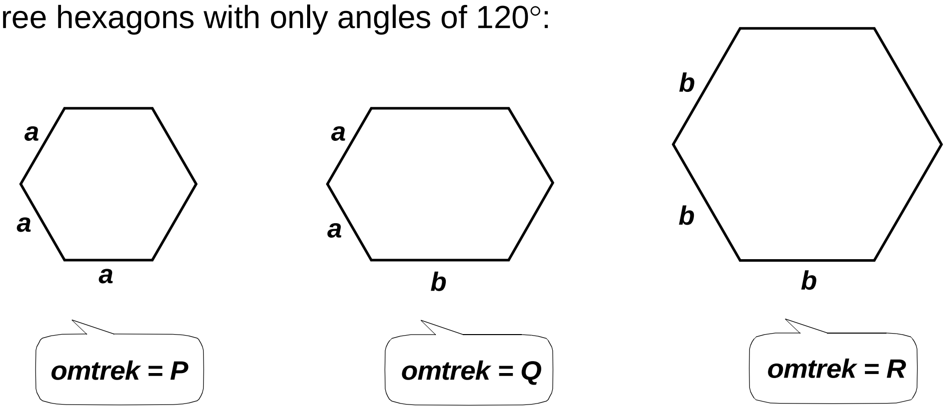


- Give formulas for the perimeters ***P***, ***Q*** and ***R*** related to ***a*** and/or ***b***:

  ***P* = ...** ***Q* = ...** ***R* = ...**

- Explain the relationship: $Q = \frac{1}{2}P + \frac{1}{2}R$

Three hexagons with only angles of 120°:



- Give formulas for ***P***, ***Q*** and ***R***:

  ***P* = ...** ***Q* = ...** ***R* = ...**

- Explain why $Q = \frac{2}{3}P + \frac{1}{3}R$ is valid.

- Design a hexagon with angles of 120° and perimeter ***S*** in such a way that the following relationship is valid: $S = \frac{1}{3}P + \frac{2}{3}R$

Polynomials and weights (I)

$$a + a + b + b + b + c + c + c + c = 2a + 3b + 4c$$

nine terms — *trinomial*

2, 3 and 4 are the **weights** of *a, b and c*

With the weights 2, 3 and 4 and the letter *a, b* and *c* you can make other trinomials like for example: $4a + 3b + 2c$.
There are totally six different trinomials in a, b, c with weights 2, 3, 4.

- Write down the other four trinomials.
- Add the six trinomials. Which trinomial is the result?

If it is known that $c = b$, you can make a binomial of $2a + 3b + 4c$.

$$2a + 3b + 4c, \quad c = b$$

↓

$$2a + 3b + 4b$$

↓

$$2a + 7b$$

You can do the same with the other five trinomials.

- How many different binomials do you get? Which ones?
- If you also know that $a = b$ you can simplify the binomials further. What is the result?

Polynomials and weights (II)

$w + w + x + y + y + y + y + y + z + z = 2w + x + 5y + 2z$

quadrinomial

***w* has a weight 2, *x* a weight 1, *y* a weight 5 and *z* a weight 2**

The weight 1 is often left out in polynomials but if you think it's more clear, you may write: $2w + 1x + 5y + 2z$

The sum of the weights in this polynomial is 10**.**

- Give five other quadrinomials in *w, x, y* and *z,* for which the sum of the weights is equal to 10.

- Add those five quadrinomials.
  What is the sum of the weights of the resulting quadrinomial?

If it is known that $w = x$ and $z = y$, then $2w + x + 5y + z$ can be simplified to a binomial:

$2w + x + 5y + 2z$
$w = x$ en $z = y$

↓

$2x + x + 5y + 2y$

↓

$3x + 7y$

Simplify your quadrinomials to binomials, when given: $w = x$ en $z = y$.

- Which binomials do you get?

- The sum of all these binomials is equal to a binomial.
  Which one?

Polynomials and weights (III)

In a college are held three tests for math in the last period before summer. Two tests are held in the 'normal' time (1 hour) and for one test the students have 2 hours).
To determinate the final score it's not fair to give the three results the same weight.
The teacher tells the students how he will calculate the final score:

$$\frac{A + B + 2C}{4}$$

In this formula *A* is the score for the first, *B* for the second one and *C* for the third (long) test.
Another formula which gaves the same result, is:

$$\frac{1}{4}A + \frac{1}{4}B + \frac{1}{2}C$$

Tanja has for the first test the score 4, for the second one the score 6 and for the last one the score 9.

- Calculate her final score, using both formulas.

- Calculate the final result for some other scores.

The teacher calculated a so called **weighted mean.**

Another weighted mean of *A*, *B* en *C* is for instance:

$$\frac{A + 2B + 3C}{6} \qquad \text{or:} \qquad \frac{1}{6}A + \frac{1}{3}B + \frac{1}{2}C$$

- Suppose that a teacher uses this formula, to determinate the final score of three tests. Can this formula be fair?

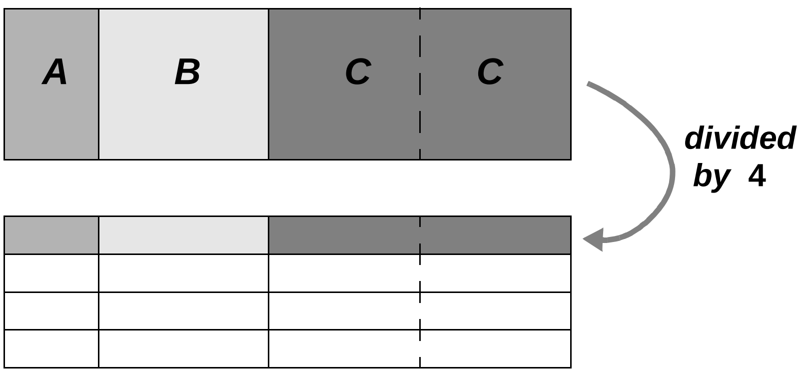
- Give two formulas for the 'normal mean' (each test has the same 'weight').

Equivalent (I)

$\frac{A + B + 2C}{4}$ and $\frac{1}{4}A + \frac{1}{4}B + \frac{1}{2}C$ are **equivalent**

That means:
whatever the numbers are that you substitute for *A, B* and *C*, the resulting values of both expressions will be the same.

This equivalency can be undersood looking at this diagram:



- Equivalent or not?

$\frac{A + 2B + 3C}{6}$ $\overset{?}{=}$ $\frac{1}{6}A + \frac{1}{3}B + \frac{1}{2}C$

$\frac{A + 2B + 3C}{6}$ $\overset{?}{=}$ $\frac{A}{6} + \frac{B}{3} + \frac{C}{2}$

$2 \times (5A + 3B + C)$ $\overset{?}{=}$ $10A + 3B + C$

$2 \times (5A + 3B + C)$ $\overset{?}{=}$ $2 \times (5A + 3B) + C$

$2 \times (5A + 3B) + C$ $\overset{?}{=}$ $10A + 6B + C$

Equivalent (II)

Here are ten expressions

A group of *equivalent* expressions do we call a *family*.

- Connect the members of the same family by a line.

$3X + 18Y + 63Z$

$3 \times (X + 6Y + 21Z)$

$3 \times (X + 7Z) + 6Y$

$3X + 3 \times (2Y + 7Z)$

$3X + 18Y + 21Z$

$3 \times (X + 6Y) + 21Z$

$3X + 6Y + 21Z$

$3 \times (X + 2Y + 7Z)$

$3 \times (X + Y) + 3 \times (Y + 7Z)$

Equivalent (III)

- Find as many expressions as you can which are equivalent with:

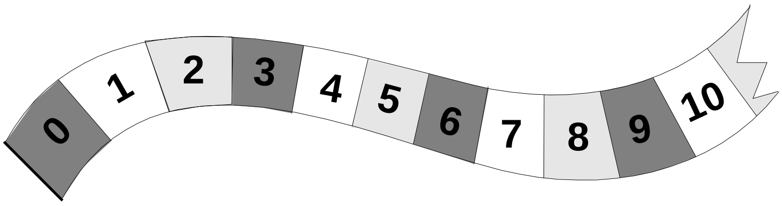
$$\mathbf{5}\boldsymbol{a} + \mathbf{10}\boldsymbol{b} + \mathbf{20}\boldsymbol{c}$$

Rule: each of the letters *a, b* and *c* may appear only once
in each expresion.

- The same for:

$$\frac{P + 2Q + 2R + 4S}{12}$$

Red-white-blue (I)



The number strip has a repeating pattern red-white-blue-red-white-blue, etc.

♦ What colour has the cell of number **100**? The same question for **1000**?

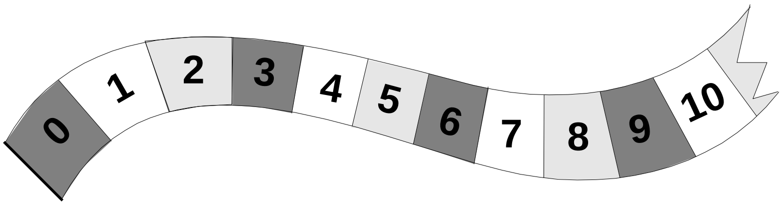
♦ Which of the numbers between **1000** and **1010** are in a red cell?

♦ Find a ’blue number’ of five digits.

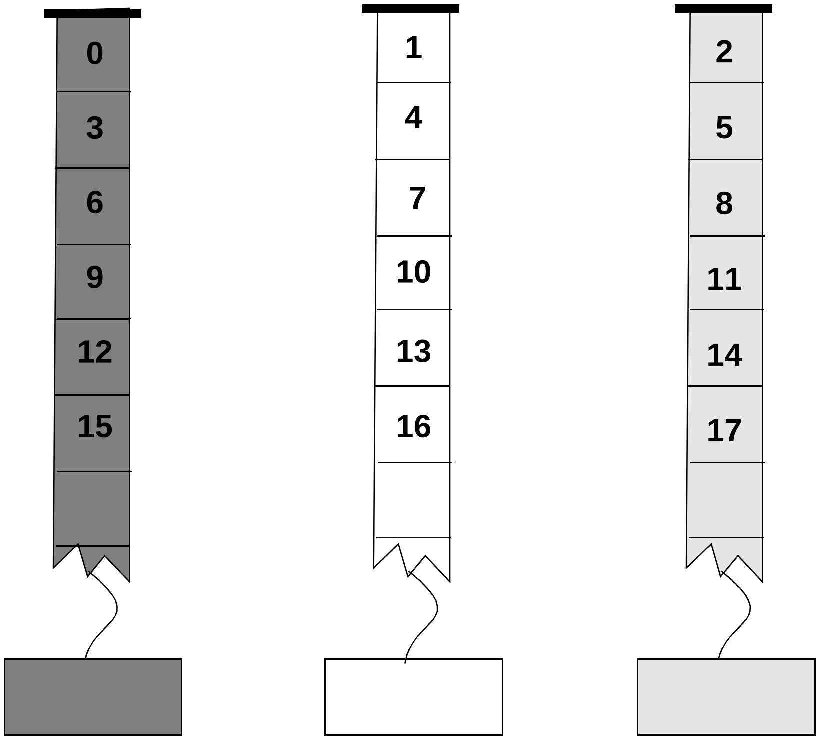
♦ If you add two ’red numbers’ , you always get a red number.
Do you think this true? Why??

♦ What can you say about the sum of two ’blue numbers’ ?

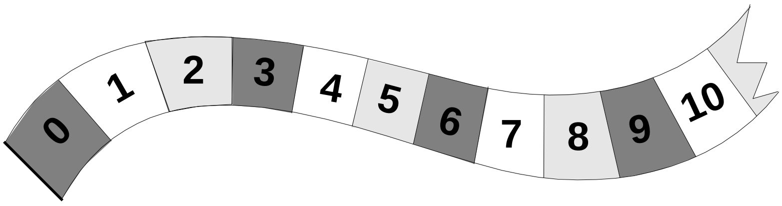
Red-white-blue (II)



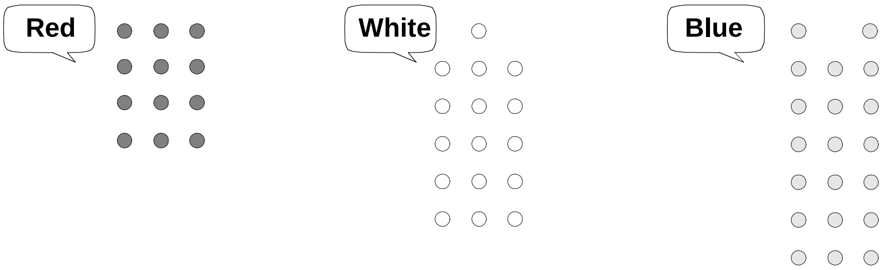
Below you see separate strips of the red, white and blue numbers .
♦ Give an expression to each number strip.



Red-white-blue (III)



The red, white and blue numbers can be represented by dot patterns.
For example:



From the dot patterns you can understand: **White + Blue = Red**
♦ Explain this.

♦Complete the chart for all combinations of colours.

+	**Red**	**White**	**Blue**
Red			
White			**Red**
Blue			

Different differences (I)

Isabelle did a job and earned e 100. She wants to buy a pair of sport shoes. They normally cost e 70 . So she expects e 30 will be left.
How lucky she is! Entering the shop she discovers that the pair she wants is reduced by e 8.

- How many Euro's does she have left?
- There are two ways to calculate this:

  a) **100 – (70 – 8)** and b) **(100 – 70) + 8**

  Which method a) or b) did you use?
  Explain, without looking at the result, that the alternative method is just as well.

Suppose that Isabelle already knew, that the price of the shoes was reduced, but she did not know how much ...
So she knew that more than e 30 should be left!

- Use the story to explain: $\mathbf{100 - (70 - \boldsymbol{a}) = 30 + \boldsymbol{a}}$

A more general equality: $\mathbf{100 - (\boldsymbol{p} - \boldsymbol{a}) = 100 - \boldsymbol{p} + \boldsymbol{a}}$

- Explain this equality.
  (You may suppose $p < 100$ and $a < p$).

A frequently occurring error: $100 - (p - a) = 100 - p - a$

- Invent a short story in which someone has $100 - p - a$ left from 100 Euros in stead of $100 - p$.

- Fill in the right expression: 100 – (..........) = $100 - p - a$

Different differences (II)

If you subtract **less**, there will be left **more**.

Example:

$$50 - 20 = 30$$
$$50 - (20 - x) = 30 + x$$

f you subtract **more**, there will be left **less**.

- Invent an example:

- Four conclusions. The first one is complete. Explain those.
- Complete the other three.

$y = 10 - x$
$z = 15 - y$
→ $z = 5 + x$

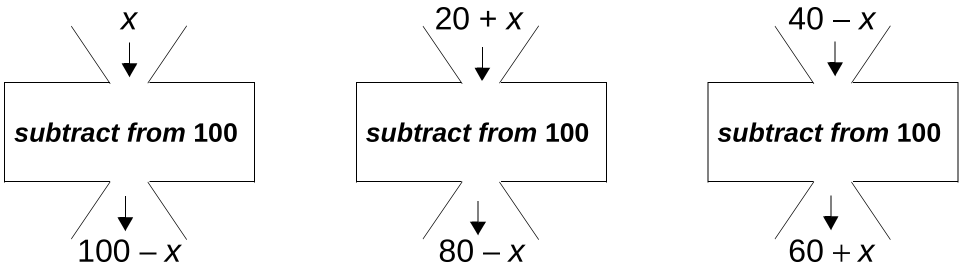
$y = 10 - x$
$z = 15 + y$
→ $z =$ ....

$y = 10 + x$
$z = 15 + y$
→ $z =$ ....

$y = 10 + x$
$z = 15 - y$
→ $z =$ ....

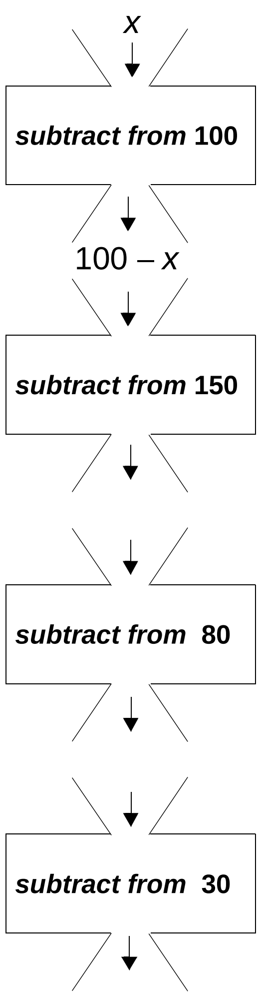
- Invent some other conclusions in the same style..
  Use other symbols than *x, y ,z.*

Different differences (III)



- The sum of INPUT and OUTPUT is 100 in each case. Check this!

- Complete the chain :



The chain on the left corresponds to the following chain of differences:

$$\mathbf{30 - [80 - (150 - (100 - x))] = ...}$$

- This expression is equivalent with a very simple one. Which one?

- Make chains corresponding to:

$$\mathbf{10 - (9 - (8 - y))}$$

$$\mathbf{16 - [9 - (4 - (1 - a))]}$$

$$\mathbf{32 - [16 - [8 - (4 - (2 - k))]]}$$

- What are the three resulting binomials?

- Invent such a 'chain-exercise' by yourself.

Let it be true (I)

$X = 5$ → $20 + 5X = 45$

Check that this is true.

- Find a value for *X* that makes this true:

$X = \ldots$ → $20 + 5X = 35$

- Find in each of the following cases a value for X that makes the conclusion true.
  You may 'guess smartly' and check by calculating!

$X = \ldots$ → $25 + X = 125$

$X = \ldots$ → $25X = 125$

$X = \ldots$ → $\frac{1}{X+1} = \frac{1}{5}$

$X = \ldots$ → $\frac{1}{X-1} = \frac{1}{5}$

$X = \ldots$ → $X \times (X + 1) = 30$

$X = \ldots$ → $\frac{X}{X+1} = \frac{3}{4}$

Let it be true (II)

$$\frac{2}{A+1} = 1$$

$$\frac{B+1}{2} = 2\frac{1}{2}$$

$$\frac{C-1}{C+1} = \frac{3}{5}$$

$$E^2 + E = 110$$

$$(D-1)\times(D+1) = 99$$

$$5\times(F+10) = 80$$

$$2G = G + 17$$

$$\frac{1}{H} + \frac{7}{8} = 1$$

$$A + B + C + D + E + F + G + H = 60$$

Let it be true (III)

$$a = 4$$

$$a + 2b = 10$$

$$a + 2b + 3c = 28$$

$$a + 2b + 3c + 4d = 68$$

$$a + 2b + 3c + 4d + 5e = 93$$

$$a + 2b + 3c + 4d + 5e + 6f = 123$$

$$a + 2b + 3c + 4d + 5e + 6f + 7g = 200$$

- For which values of ***b, c, d, e, f, g*** all equalities in the triangle are valid?
- How do the answers change if ***a*** is not equal to 4, but to 8?

Generation problems

Today, Anja, her mother, her grandmother and her great-grandmother are together 200 years old.

Anja's mother was 30 years old when Anja was born.

Grandma was 25 years old when Anja's mother was born.

Great-grandma was 20 years old at the birth of Anja's grandma.

Suppose the age of Anja, her mother, her grandmother and her great-grandmoother respectively equal to ***a, b, c*** and ***d*** years.

- Write down everything you know about ***a, b, c*** and ***d*** in formulas.

- Calculate ***a, b, c*** and ***d***.

Peter, his father, his grandfather and his great-grandfather also are together 200 years old.

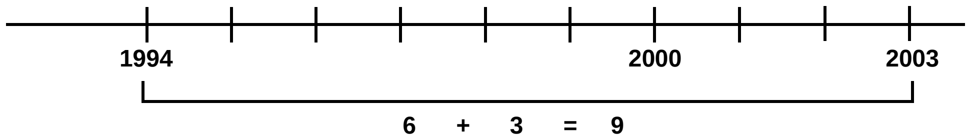
Father is 4 times as old as Peter, grandfather is $1\frac{1}{2}$ times as old as father, great-grandfather is $1\frac{1}{2}$ times as old as grandfather.

Suppose the age of Peter, his father, his grandfather and his great-grandfather are respectively equal to ***p, q, r*** and ***s*** years.

- Write down everything you know about ***p, q, r*** en ***s*** in formulas.

- Calculate ***p, q, r*** and ***s***.

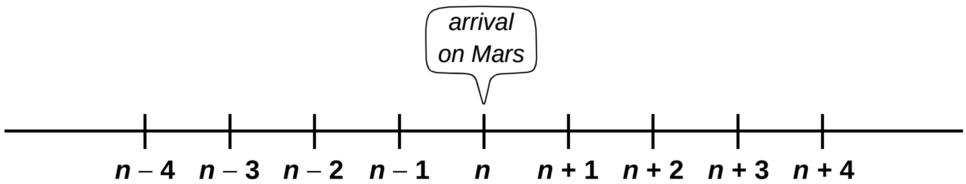
On the number line (I)



Between 1994 and 2003 are 9 years.

- How many years are there between 2011 and 1945?

In the year ***n*** astronauts from Earth land on Mars for the first time.
One year later they return to Earth. That will be in the year ***n* + 1**.
Again one year later the astronauts take an exhibition about their trip, around the world. That will be in the year ***n* + 2**.
The construction of the launching rocket began one year before the, landing on Mars, so this was in the year ***n* – 1**



Between ***n* – 1** and ***n* + 2** there are 3 years.
You may write:

$$(n + 2) - (n - 1) = 3$$

- How many years are there between ***n* – 4** and ***n* + 10**?
- Calculate:

$$(n + 8) - (n - 2) =$$

$$(n + 7) - (n - 3) =$$

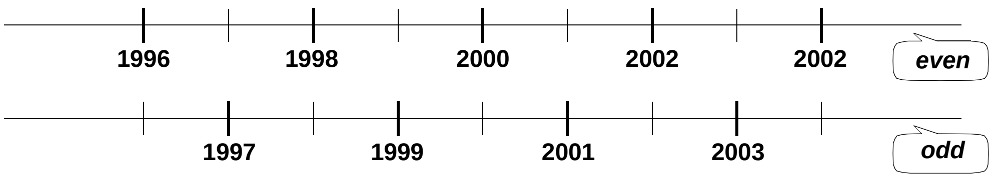
$$(n - 1) - (n - 4) =$$

$$(n + 3) - (n - 3) =$$

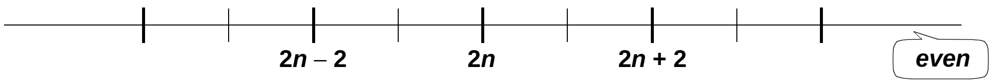
- How many years are there between ***n* – *k*** and ***n* + *k***?

On the number line (II)

Even and **odd** years

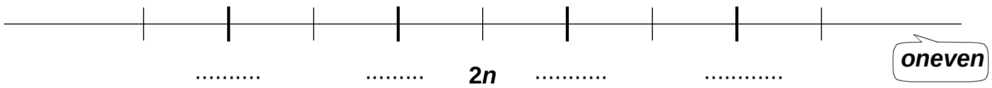


An **arbitrary** even year can be represented by **2*n***.
In two years it will be the year **2*n* + 2**, that is the even year that follows the even year **2*n***.
The even year that comes before **2*n*** is the year **2*n*** – **2**.



- What is the even year that follows the year **2*n* + 2**? And what is the even year that comes before the year **2*n*** – **2**?

The odd years are between the even years.



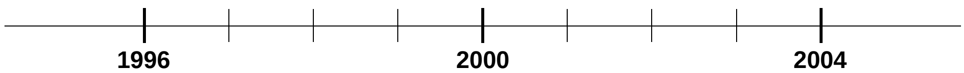
- Write expressions for the odd years on the number line’.
- Calculate:

(2*n* + 8) – **(2*n*** – **6) =**

(2*n* + 3) – **(2*n*** – **3) =**

(2*n* + 4) – **(2*n*** – **3) =**

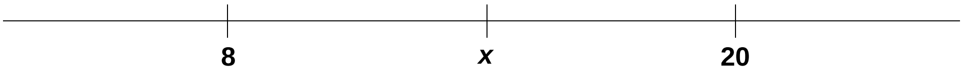
Olympic years are divisible by4.



- How can you represent an **arbitrary** Olympic year?
- Which Olympic year succeeds that year and which precedes it?

  Olympic wintergames are presently held in a year that exactly lies between two Olympic years.
- How can you represent an **arbitrary** year of wintergames?

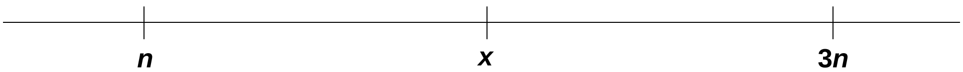
On the number line (III)



The number ***x*** is chosen in such a way that $x - 8 = 20 - x$ is valid.

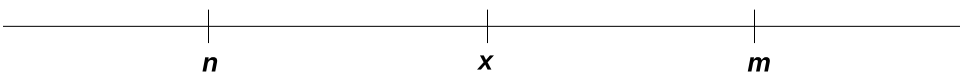
- Which value has ***x***?

- Which value has ***x*** if $x - 1971 = 2001 - x$?



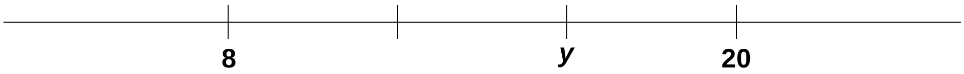
- $x - n = 3n - x$ → $x =$ .... (expression in ***n***)

- $x - 5n = 25n - x$ → $x =$ .... (expression in ***n***)



- $x - n = m - x$ → $x =$ .... (expression in ***n*** and ***m***)

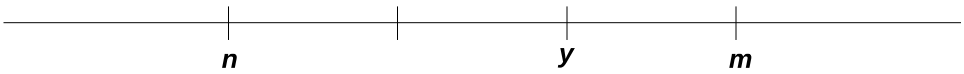
On the number line (IV)



The number $\boldsymbol{y}$ is chosen in such a way that: $\boldsymbol{y - 8 = 2 \times (20 - y)}$

- Which value has $\boldsymbol{y}$?

- Which value has $\boldsymbol{y}$ if: $\boldsymbol{y - 1971 = 2 \times (2001 - y)}$?

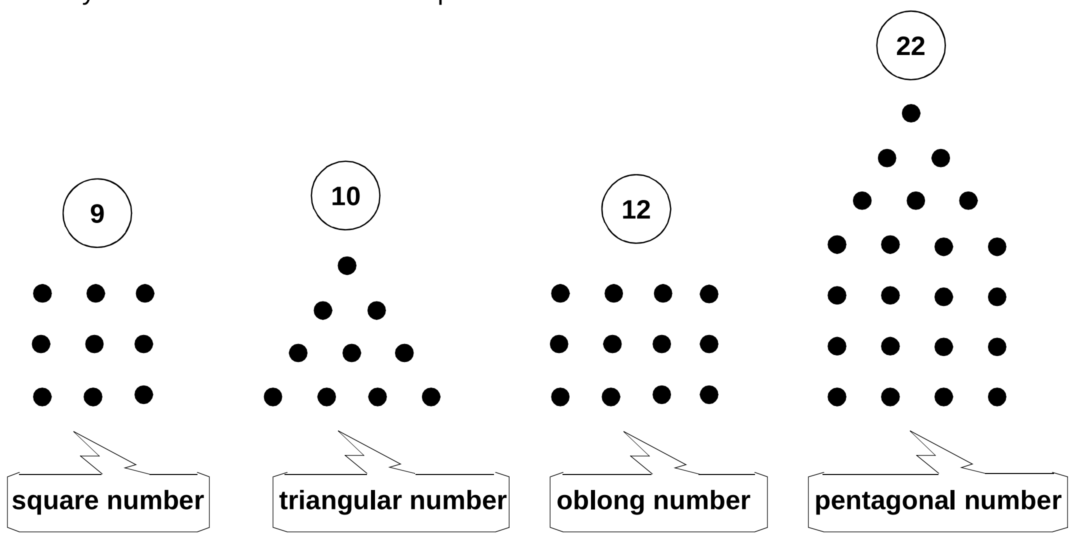


$\boldsymbol{y - n = 2 \times (m - y)}$

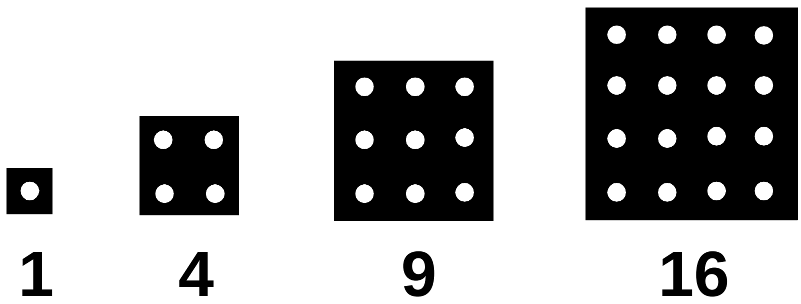
- Which expression (in $\boldsymbol{n}$ and $\boldsymbol{m}$) can you find for $\boldsymbol{y}$?

Dot patterns (I)

Nicomachos lived about the year 100 AC in Greece.
He wrote a book about the ’ admirable and divine properties of natural numbers.
Nicomachos sometimes used dot patterns to represent numbers.
Below you se the most famous examples:



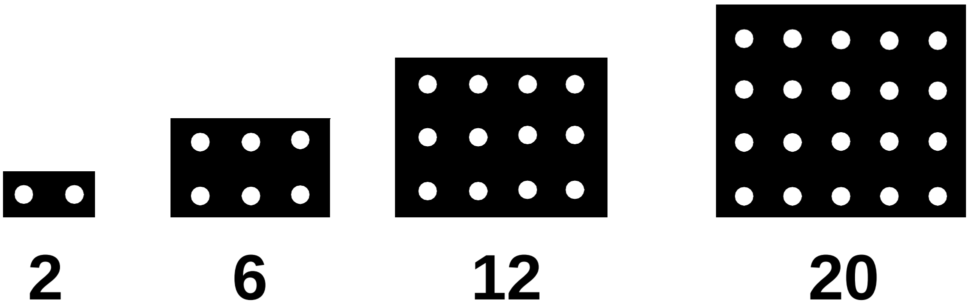
He gave every type a geometrical name. .
To begin with, consider the family of the *square numbers.*



- Write the next ten square numbers. You need not to draw the corresponding patterns (but you can ’see’ them in your mind).
- Consider the steps between consecutive square numbers.
  Do you see any rule? How can you see that rule in the dot patterns?
- 144 is a square number. How about 1444? And 14444?
  Use a calculator to investigate this.

Dot patterns (II)

The first four ***oblong numbers***



- Write the next ten oblong numbers. Continue the dot patterns in your mind.

- Consider the steps between succesive square numbers..
  Do you see any rule? How can you see that rule in the dot patterns?

- Is 9900 an oblong number? Explain your answer.

Take the mean of pairs of consecutive oblong numbers.
So:

the mean of 2 and 6 is 4
the mean of 6 and 12 is 9
the mean of 12 and 20 is 16

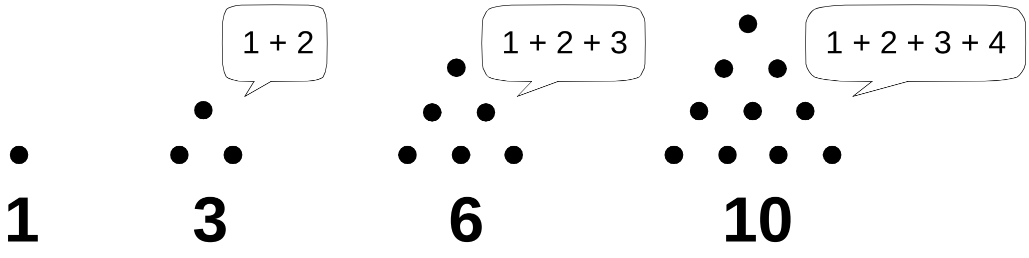
- Continue this at least five times.
  Which special numbers do you get as result?
  Try to explain your discovery.

Dot patterns (III)

Nicomachos was not the first person who used dot patterns for numbers. 600 jears before (about 500 BC) lived Pythagoras, a scholar who was the leader of a religious sect.

In the doctrine of Pythagoras ’natural numbers’ played the leading part. The device of him and his disciples was: ’*everything is number’.*

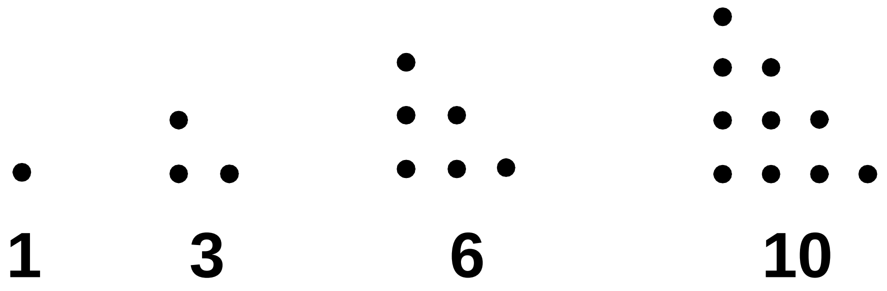
Their favorite number was 10, being the sum of 1, 2, 3 and 4.



10 is the fourth number in the sequence of the *triangular numbers.*

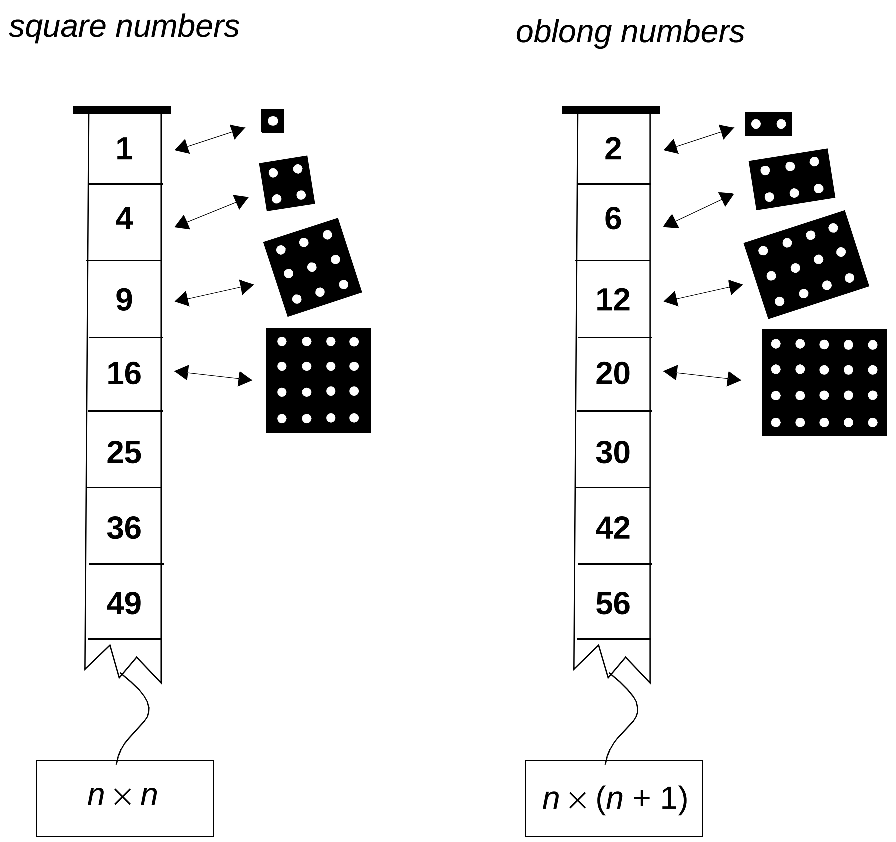
- Write the next ten triangular numbers.

The dot patterns of the triangular numbers can also be drawn in this way:



- Which special numbers do you get if each of the triangular numbers are doubled? How can you explain this using the dot patterns?
- Is 4950 a triangular number? Explain your answer.
- Calculate the sum of all natural numbers smaller than 100.

Strips and dots (I)



The expression for the square numbers is mostly written as: n^2
Pronounce: *n* square.

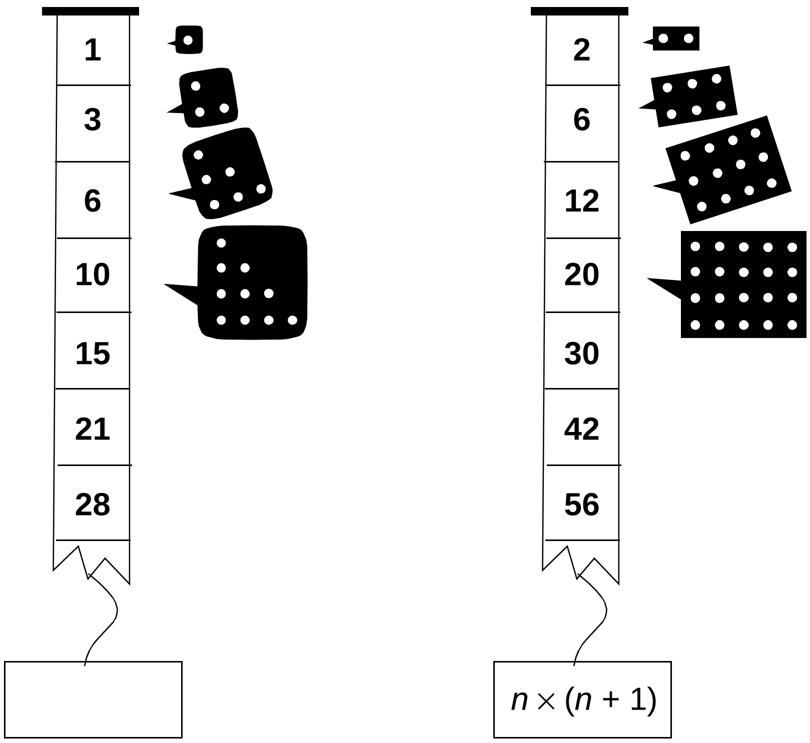
- Compare both strips. Which strip can you add to the left one to get the right one?

The expressions $n \times (n + 1)$ and $n^2 + n$ are equivalent.

- How can you explain this using dot patterns?

Strips and dots (II)

triangular numbers *oblong numbers*



Compare the triangular numbers with the oblong numbers.

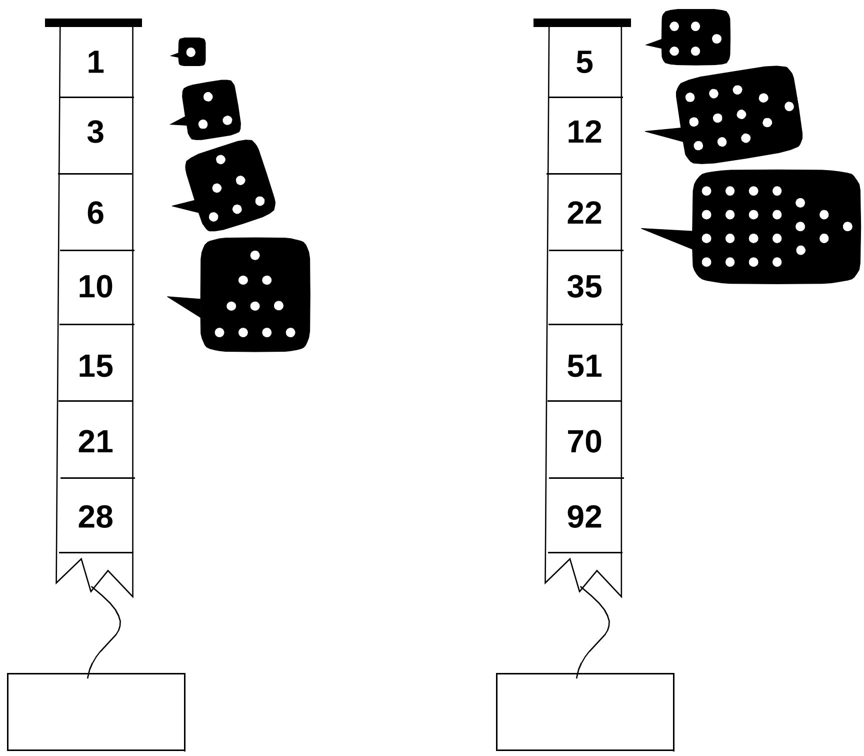
- Whic expression fits the strip of triangular numbers?
- Give one (or more) expressions which are equivalent with this.

Using the expression for triangular numbers you can calculate the sum of the first hundred natural numbers:

- 1 + 2 + 3 + 4 + 5 + ... + 98 + 99 + 100 = ____________

Strips and dots (III)

triangular numbers *pentagonal numbers*

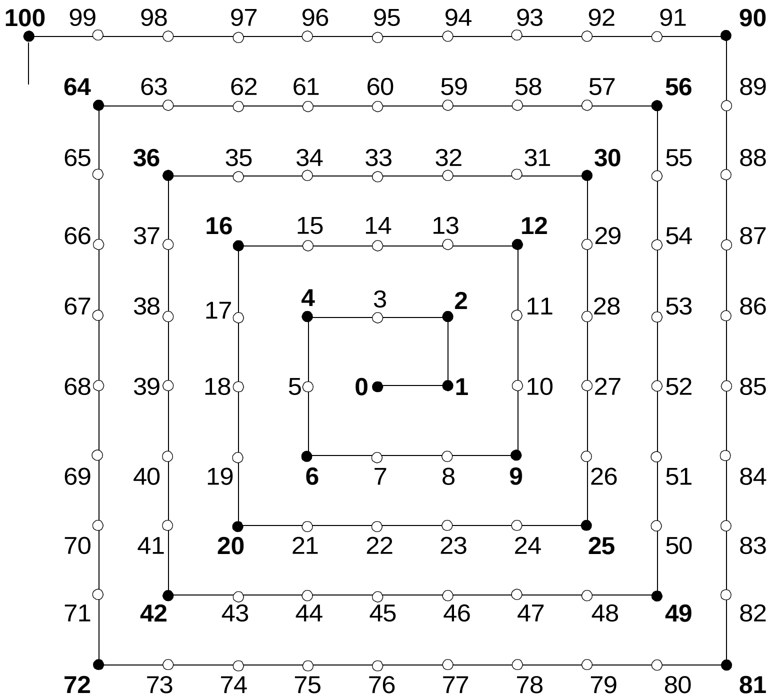


Compare the numbers of both strips.

- Which pentagonal number succeeds 92?
- Find an expression which represents the sequence of pentagonal numbers.

Number spiral (I)

You can make a number line in the shape of a spiral!



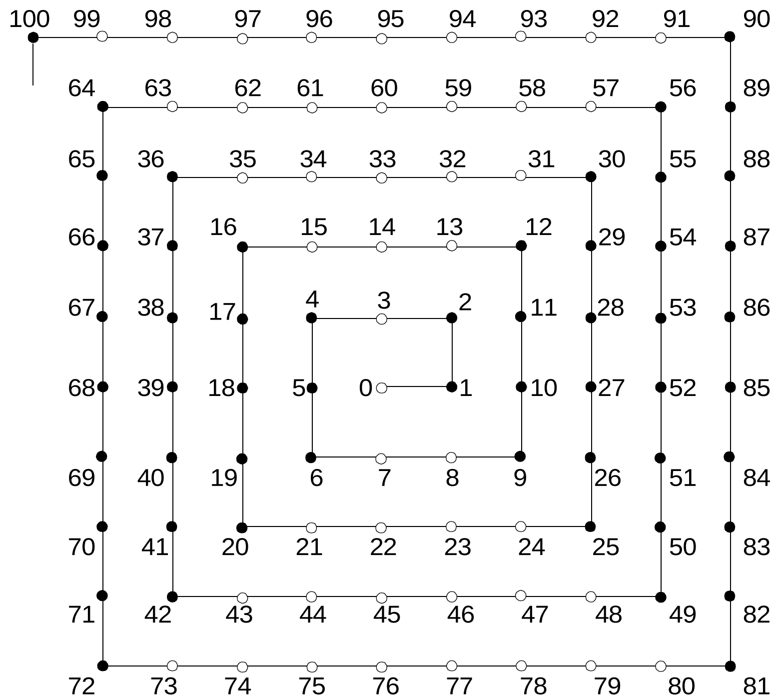
The black vertices of the spiral correspond with special families of numbers.

- Which families?

You can see in the diagram that every square number is lies exactly
in the middle between two oblong numbers.
For example: **49** lies in the middle between **42** and **56**.
Of course, for **49** = **7** × **7** and **42 = 6** × **7** and **56 = 8** × **7**

- The square number 144 lies in the middle between
  the oblong numbers .... and ....
- The square number 1444 lies in the middle between
  the oblong numbers .... and ....
- The square number n^2 lies in the middle between
  the oblong numbers .... and ....

Number spiral (II)



Now the dots on the vertical parts of the number spiral are black, the others white. If you add the black numbers form one vertical part, the result will be equal to the sum of the succeeding white numbers. You can check that:

1 + **2** = 3

4 + **5** + **6** = 7 + 8

9 + **10** + **11** + **12** = 13 + 14 + 15

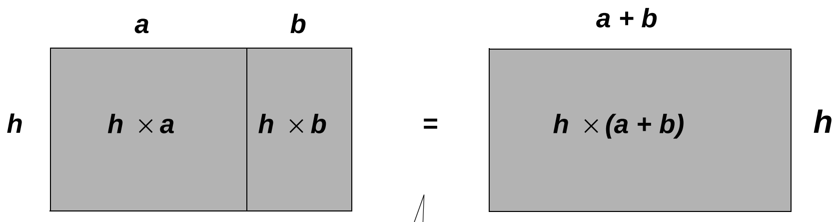
16 + **17** + **18** + **19** + **20** = 21 + 22 + 23 + 24

- What will be the next line of this scheme?
- You can check this line without calculating both sums.
  Hint: Mark the steps from ’black’ to white’.

Line *n* begins with the black number n^2 .

- Give an expression for the last black number on that line?
  How big are the he steps from ’black’ to white’.?
- Try to explain: ’black sum’ = ’white sum’

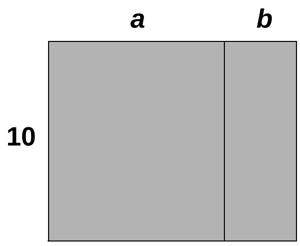
Geometric Algebra (I)



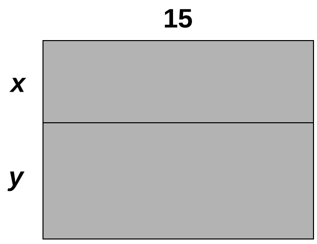
$h \times a + h \times b = h \times (a + b)$

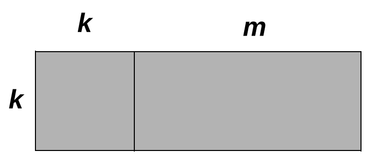
or

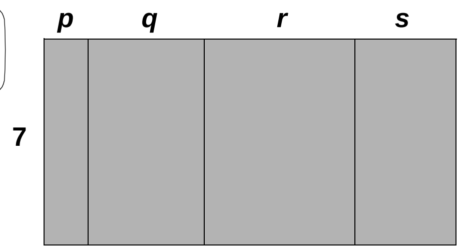
$ha + hb = h(a + b)$



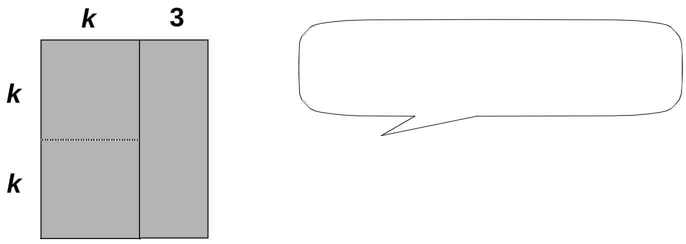
$10a + 10b =$ ....

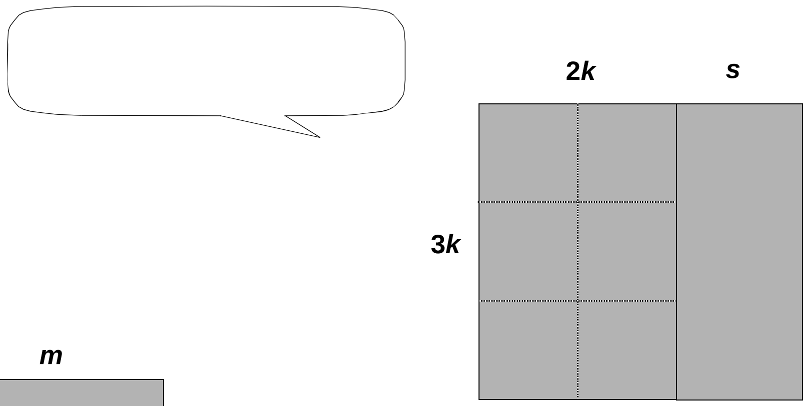


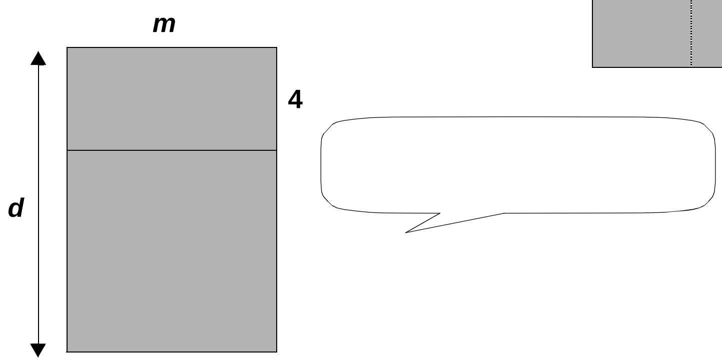


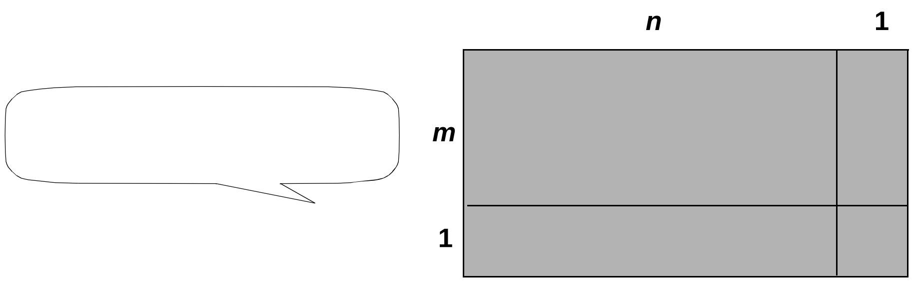


Geometric Algebra (II)

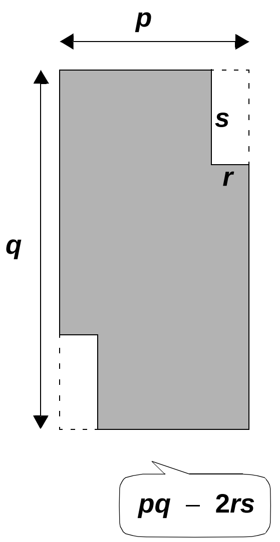


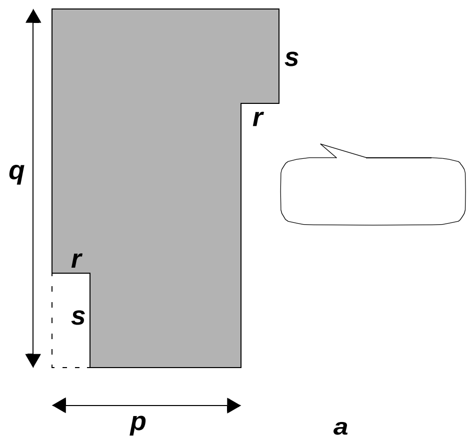


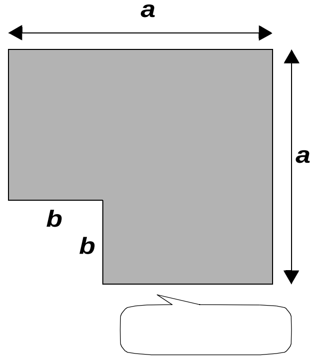


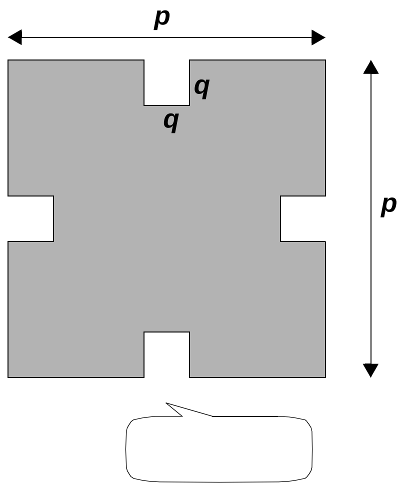


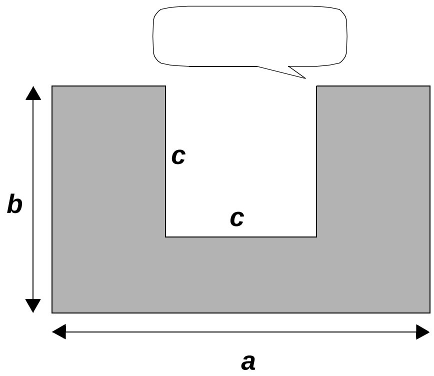
Geometric Algebra (III)

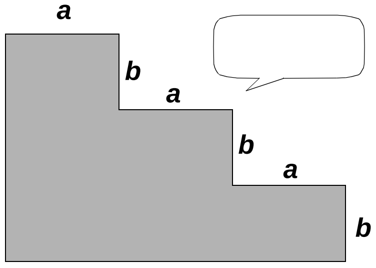






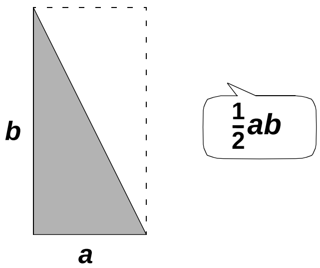


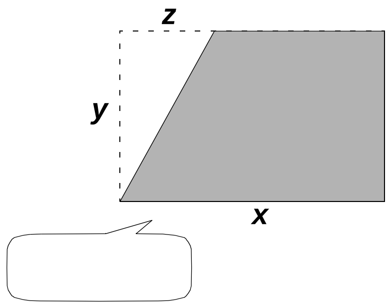


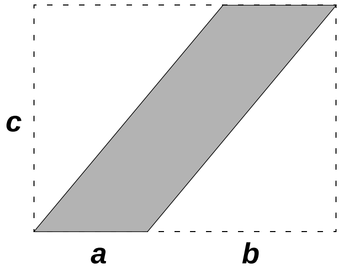


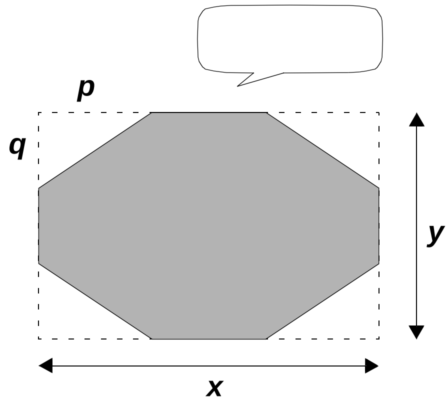
- Invent a figure with area $ab - 3c^2$
- Also one with area $p^2 + 4q^2$

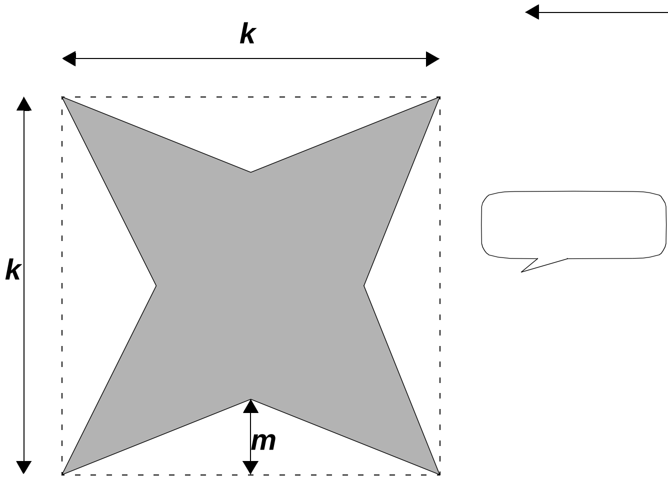
Geometric Algebra (IV)

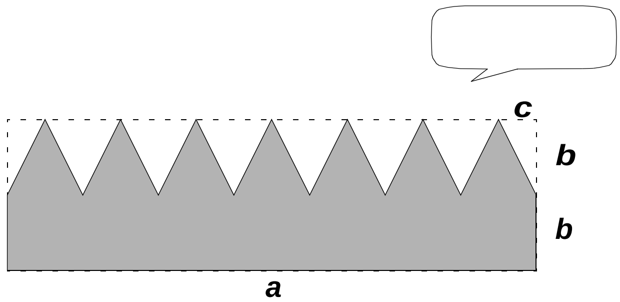












Isn't it remarkable? (I)

a and ***b*** are positive integers with the sum 10
S is the sum of the squares of ***a*** and ***b***, so: $S = a^2 + b^2$
T is the sum of the ***ab*** and ***ba***, so: $T = ab + ba$

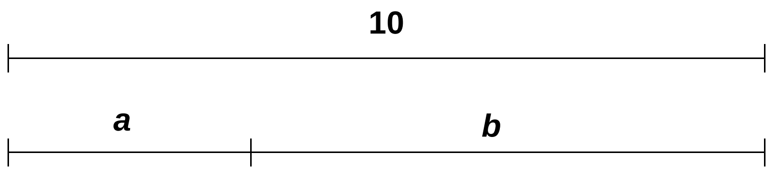
- Which values can ***S*** have? And ***T***? Fill in the table.

$a + b = 10$		S	T
a	b	$a^2 + b^2$	$ab + ba$

- Calculate the sum ***S + T*** in each case. What do you discover?

- Investigate what happens with ***S + T*** if ***a*** and ***b*** are decimal numbers with sum 10, for instance ***a*** = 3,8 and ***b*** = 6,2. Investigate some other examples. What did you find?

How remarkable? (II)



- Draw a square wit side ***a***. Also one with side ***b***.

- Draw a rectangle with horizontal side ***a*** and vertical side ***b.***
  Also one with horizontal side ***b*** and vertical side ***a.***

- How can you explain that $(a^2 + b^2) + (ab + ba) = 100$**?**

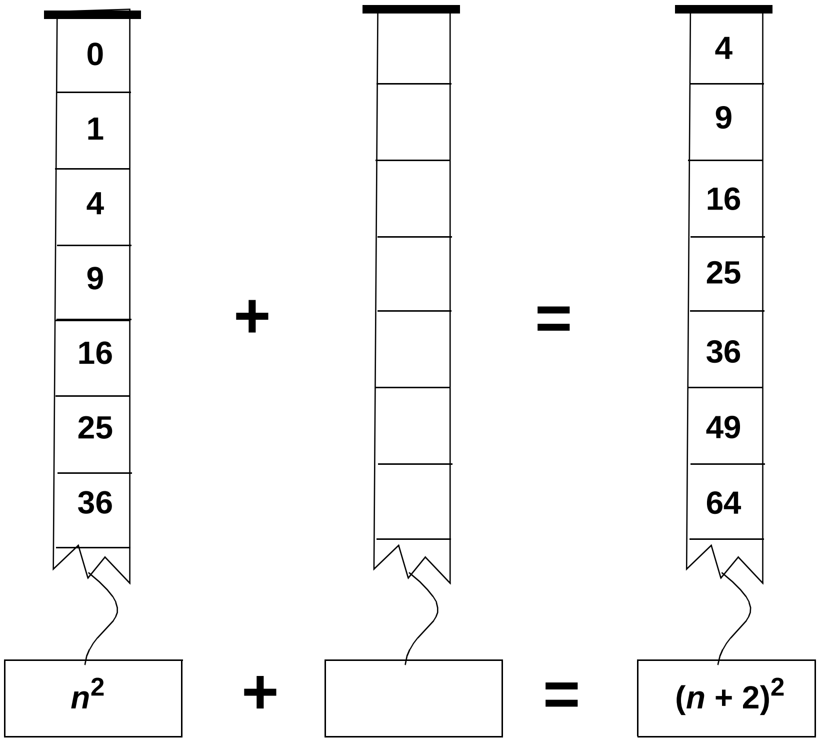
Strips and expressions (I)

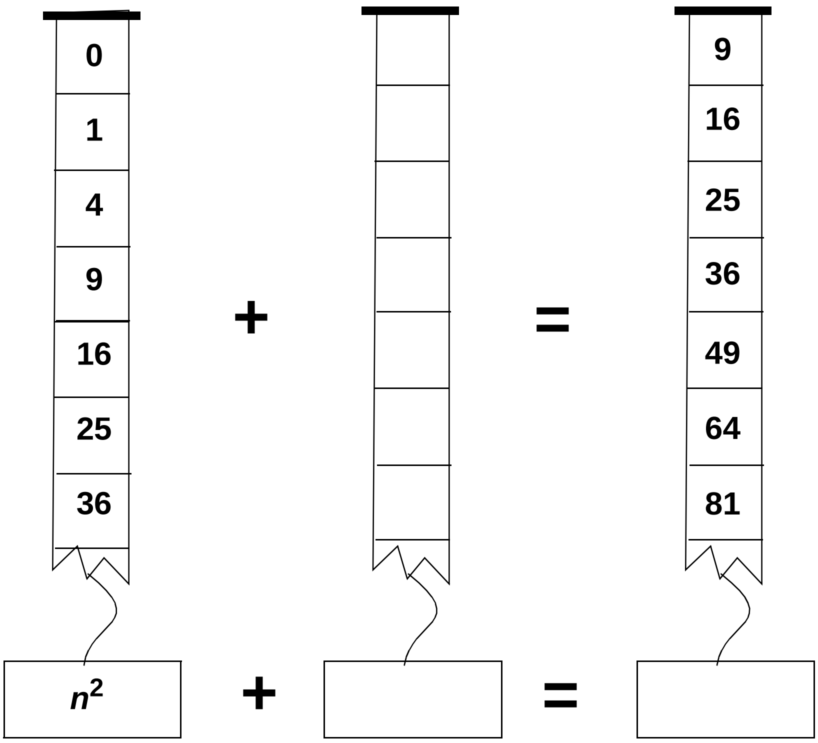
n^2	$(n + 1)^2$	$n^2 + 1$	$2n + 1$
0	1	1	1
1	4	2	3
4	9	5	5
9	16	10	7
16	25	17	9

- Fill in the missing numbers on the strips.
- Equivalent or not?

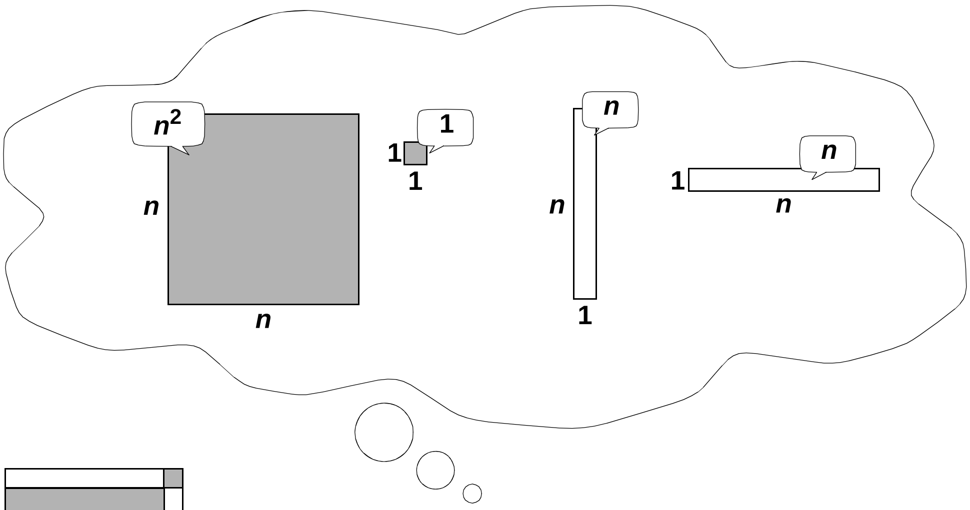
n^2	$\stackrel{?}{=}$	$n \times n$	$2n + 1$	$\stackrel{?}{=}$	$n \times n + 1$
$n^2 + 1$	$\stackrel{?}{=}$	$n^2 + 1^2$	$2(n + 1)$	$\stackrel{?}{=}$	$2n + 1$
$(n + 1)^2$	$\stackrel{?}{=}$	$n^2 + 1^2$	$(n + 1)^2$	$\stackrel{?}{=}$	$n^2 + 2n + 1$
$n^2 + 1$	$\stackrel{?}{=}$	$n \times n + 1 \times 1$	$(n + 1)^2$	$\stackrel{?}{=}$	$n^2 + 1 + 2n$

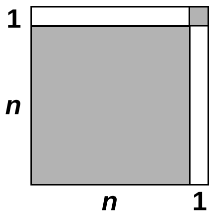
Strips and expressions (II)



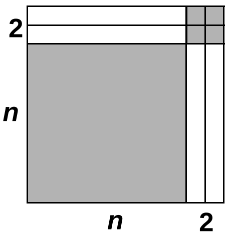


Formulas about squares

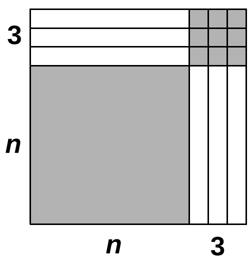




$$(n + 1)^2 = n^2 + 2n + 1$$



$$(n + 2)^2 = n^2 + 4n + 4$$



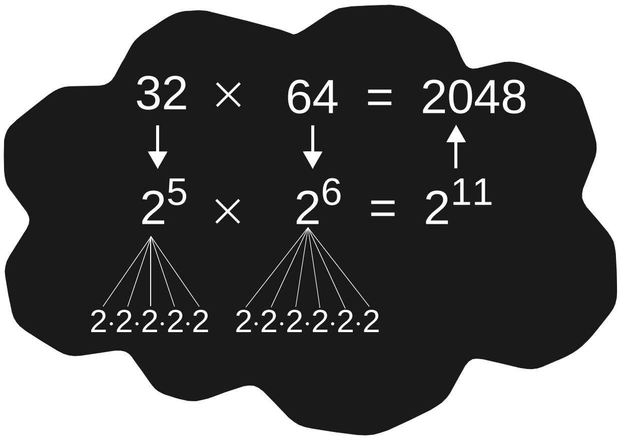
$$(n + 3)^2 = n^2 + 6n + 9$$

- How will this continue? Write the next three formulas..

Powerful tables (I)

Below you see a table with powers of 2:

n	2^n
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
11	2048
12	4096
13	8192
14	16384
15	32768
16	65536
17	131072
18	262144
19	524288
20	1048576



- Find the results of the following products using the table:

  16 × 8192 = ...

  8 × 16384 = ...

  512 × 512 = ...

  1024 × 1024 = ...

- Write all pairs of two positive integers with a product equal to

  1048576

Powerful tables (II)

fill in:

$243 \times 81 = ...$

$\downarrow \qquad \downarrow \qquad \uparrow$

$3^{\cdots} \times 3^{\cdots} = 3^{\cdots}$

Table with powers of 3

n	3^n
0	1
1	3
2	9
3	27
4	81
5	243
6	729
7	2187
8	6561
9	19683
10	59049
11	177147
12	531441
13	1594323
14	4782969
15	14348907
16	43046721
17	129140163
18	387420489
19	1162261467
20	3486784401

- Find the results of the following products using the table:

  $81 \times 19683 = ...$

  $2187 \times 59049 = ...$

  $6561 \times 6561 = ...$

  $729 \times 729 \times 729 = ...$

- Find the results of the following powers using the table:

  $81^3 = ...$

  $243^4 = ...$

  $27^5 = ...$

- Which number is smaller: 9^{10} or 10^9 ?

Powerful tables (III)

- Make a table with powers of 5 (until 5^{10})
  Design some problems which you can solve using this table.

n	5^n
0	1
1	5
2	
3	
4	
5	
6	
7	
8	
9	
10	

- The table with powers of 1 is very simple. Why?

- Do you know a table of powers with sharp rising results, which is very easy to write? Which one?

- If you put the tables with powers of 2 and 3 next to each other, and if you multiply the numbers on the same line, you get:

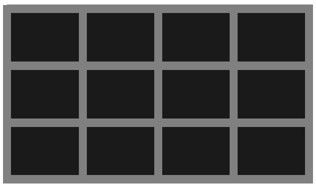
$$1 \times 1 = 1$$
$$2 \times 3 = 6$$
$$4 \times 9 = 36$$
$$8 \times 27 = 216$$

etc.

The results are just the powers of 6.
You can check this using your calculator.

- How can you explain without calculator: $2^{10} \times 3^{10} = 6^{10}$?

Divisors(I)



A slab of chocolat contains 12 pieces.

- For what numbers of persons fair sharing in whole pieces is possible?

If you gave a correct answer to the last question, then you found the divisors of 12. Maybe you forgot that it's possible to eat the slab alone, but also 1 is a divisor of 12.

- Complete the following divisor-table:

number	**divisors**	**number of divisors**
1	1	1
2	1 , 2	2
3	1 , 3	2
4	1 , 2 , 4	3
5	1 , 5	2
6		
7		
8		
9		
10		
11		
12	1 , 2 , 3 , 4 , 6 , 12	6
13		
14		
15		
16		
17		
18		
19		
20		
21		
22		
23		
24		
25		

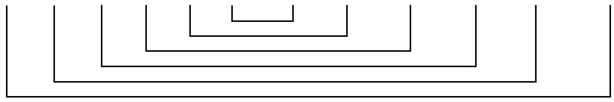
Divisors (II)

Consider the divisor-table of the numbers 1, 2, 3, ..., 25.
There are numbers with only two divisors
Such numbers are called **prime numbers**.

- What are the prime numbers below 25?
- What are the next three prime numbers?

May be you discovered alеready a smart way to find the divisors of a number. If you know one divisor, you can (mostly) find a second one immediately. So you can find pairs of divisors,.

Take for example the number 108.
The first pair of divisors is trivial: 1 and 108.
Because 2 is a divisor, you have immediately 54 (for $2 \times 54 = 108$).
The next divisor is 3, and you have also 36 (for $3 \times 36 = 108$).
Etcetera.
So we have: 1 , 2 , 3 , 4 , 6 , 9 , **12** , **18** , **27** , **36** , **54** , **108**



The bold numbers do you get as a present more or less.

- In the case of 108, you only have to 'try' the numbers 1 to 10. Explain this..
- Find in this way the divisors of: 88 ; 144 ; 210

The majority of numbers have an even number of divisors.

- For which type of numbers, the number of divisors is odd?

In the table with powers of 2 you can find the 8192.

- Which divisors has this number?
- How many divisors has 3^{10}?
- Find a big number by yourself, of which you quickly can give the number of divisors'.

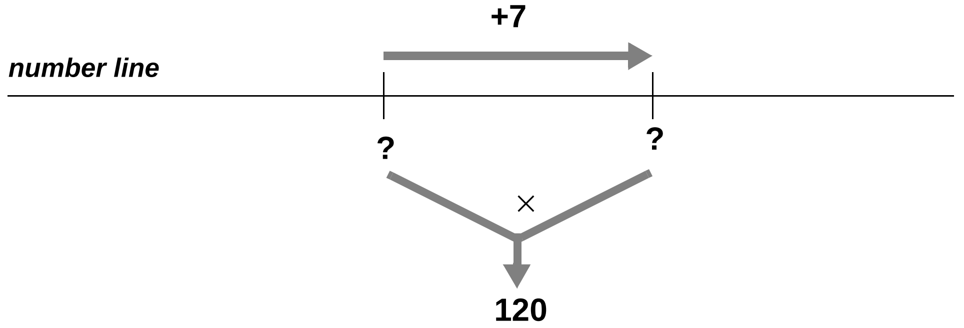
Divisors (III)

Of two natural numbers you know:

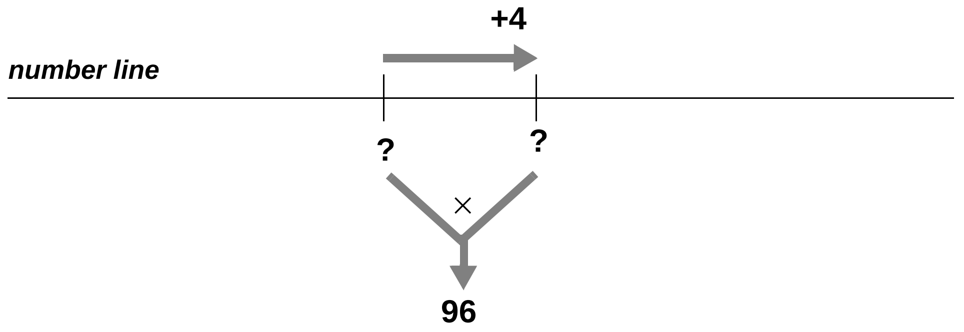
**** their product is* 80**
**** their sum is* 21**

- Which are the two numbers?

- Find the two natural numbers



- Find the two natural numbers



- Find a value for ***n*** in such a way that $n \times (n - 4) = 77$

- Find a value for ***p*** in such a way that $(p + 7) \times (p + 10) = 108$

Prime numbers

The Greek scholar Eratosthenes lived about 240 before Chr.
Eratosthenes was a universal man ; besides a mathematician he also was, historian, geographer, philologist and poet. He is famed for his meausurement of the earth.
Furthermore he found a smart way to find prime numbers ....

Look at the 'hundred-chart' belowr.

1 is no prime number, so the cell of 1 is shaded.
2 is a prime number , the cell of 2 remains white.
Then all the cells of multiples of 2 are shaded.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

3 is a prime number; the cell of 3 remains white.
***** Now shade the cells of all multiples of 3, which are not already shaded.

5 is a prime number; the cell of 5 remains white.
***** Now shade the cells of all multiples of 5, which are not already shaded.

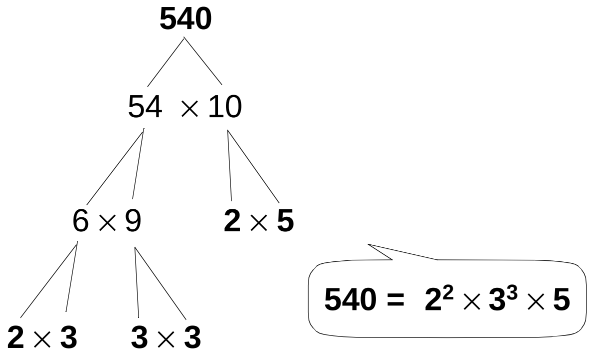
7 is a prime number; the cell of 5 remains white.
***** Now shade the cells of all multiples of 7, which are not already shaded

Now all white cells are containing a prime numberl!

* How can you explain this?
Hint: consider the prime number after 7, that is 11. What are the multiples of 11 smaller than 100? Why can you be sure without looking at the chart that the cells of these numbers are already shaded?

Prime factors

Prime numbers can be considered as the atoms for the natural numbers.
Every natural number is equal to a product of prime numbers.
For example: $540 = 2 \times 2 \times 3 \times 3 \times 3 \times 5$ or shortened: $540 = 2^2 \times 3^3 \times 5$
We say: 540 is *decomposed in factors* or *factorized*.
You can find such a decomposition in many ways. Try to split the number in two factors (bigger than 1 and smaller than the given number); if possible you do the same with both factors, and you repeat this process until you only have prime factors. For example with 540:



There are many other 'trees of decomposition'; the final result of all these trees has to be the same!

- Factorize in this way the following numbers:

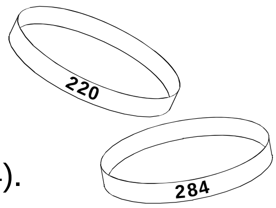
135	**704**	**2100**
135 = ...	**704 =**	**2100 =** ....

- Choose three other numbers by yourself and find the decomposition in prime factors.

Amicable numbers (I)

To clinch a close friendship between two persons, sometimes both friends wear an amulet or a ring with a short text inscribed on it.
In the age of Pythagoras (500 BC) they used sometimes the numbers **220** and **284** for such an inscription.
These two numbers are called ***amicable numbers***.

The mystery of the numbers 220 and 284 has something to do.
with their divisors.



- Find all divisors of 220.
  Add up all these numbers except 220..
- Do the same with the divisors of 284 (except 284).

If you did not make a mistake, you found a remarkable fact:

the sum of the divisors (except 220) of 220 = 284
the sum of the divisors (except 284) of 284 = 220

This property of a pair of numbers is very exceptional.
In later times famous mathematicians made a great effort to find other pairs of amicable numbers.

The French mathematician Pierre Fermat, who actually was a lawyer, discovered in 1636 a new pair: **17296** en **18416**.
Leonard Euler, a Swiss mathematician, whose portrait is on the Swiss banknotes, published in 1747 a list with thirty pairs of (big) amicable numbers; a few years later this list was extended to sixty pairs.
It's remarkable that all famous scholars didn't discover a rather small pair, namely **1184** and **1210.**
This pair was found by sixteen years old Italian boy, named Nicolo Paganini, in the year 1866.

- Check that **1184** and **1210** are amicable numbers.

Amicable numbers (II)

The Arabic mathematician Tabit ibn Qorra, who lived from 826 to 901, found a recipe to make pairs of amicable numbers.

His recipe was rather complicated as you can see below.

IF

$$p = 3 \times 2^{n} - 1$$

$$q = 3 \times 2^{n-1} - 1$$

$$r = 9 \times 2^{2n-1} - 1$$

and *IF* p, q, r *are three odd prime numbers*

THEN

$$A = 2^{n} \times p \times q$$

$$B = 2^{n} \times r$$

are AMICABLE numbers

- Take ***n* = 2**. Check if ***p, q, r*** indeed are prime numbers.
  Use the values of ***p, q*** and ***r*** to calculate ***A*** and ***B***.
  Now you see ...
- Take ***n* = 3**. Which of the numbers ***p, q, r*** is **not** a prime number.

For ***n* = 3** the recipe of Tabit ibn Qorra don't give amicable numbers!
For ***n* = 4** it does! If you don't have a long list of prime numbers, it is difficult to check if ***p, q, r*** are prime in this case. A computer has validated that this is true.

- Calculate ***A*** and ***B*** for ***n* = 4** and compare the result with the discovery of Pierre de Fermat (previous page).
- The numbers of Nicolo Paganini can not be made by the recipe above. Factorize these numbers and explain why not.

Obviously the formula of Tabit ibn Qorra don't give all pairs of amicable numbers !

Operating with powers (I)

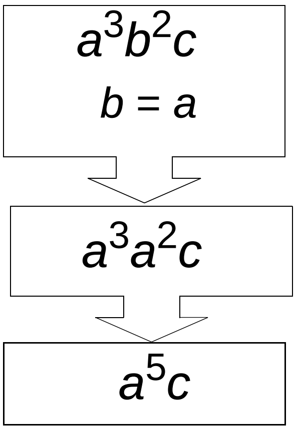
$$a \times a \times a \times b \times b \times c = a^3 \times b^2 \times c^1 = a^3b^2c$$

The **exponents** of *a, b,* and *c* are 3, 2 and 1.
(*note*: the exponent 1 is mostly not written)

With the exponents 3, 2 and 1 and the letters *a, b* and *c* also can be made other products, for example: ab^3c^2.
There are six different products that one can make using ***a, b, c*** and the exponents 1, 2, 3 .

- Write the other four products.
- Multiply the six products to each other.
  The result can be written in the form $a^{....}\ b^{....}\ c^{....}$
  Which exponents do you get?

If it is known that $b = a$, you can simplify a^3b^2c



You can do the same with the other five products.

- How many different products do you get? Which ones?
- If you also know that $c = a$ you can write each of these products as a **power** of *a*. Which one?

Operating with powers (II)

$$2m^3 \times 3m^2 = 2 \times 3 \times m^5 = 6m^5$$

$2 \times m \times m \times m$ **$3 \times m \times m$**

- Find other multiplications, as many as possible, with the same result:

.... $\times$ .... $= 6m^5$

.... $\times$ .... $= 6m^5$

....

....

....

....

....

....

....

....

....

....

Operating with powers (III)

$3a = a + a + a$ $\quad$ $a^3 = a \times a \times a$

$a = z^2$

$3z^2 = z^2 + z^2 + z^2$ $\quad$ $(z^2)^3 = z^2 \times z^2 \times z^2 = z^6$

$4z^2 + 3z^2 + 2z^2 - z^2 = 8z^2$

- On the places ____ you can fill in **+** , **–** or $\times$ to make the following equalities:

$4z^2$ ___ $3z^2$ ___ $2z^2$ ___ z^2 $= 10z^2$

$4z^2$ ___ $3z^2$ ___ $2z^2$ ___ z^2 $= 24z^8$

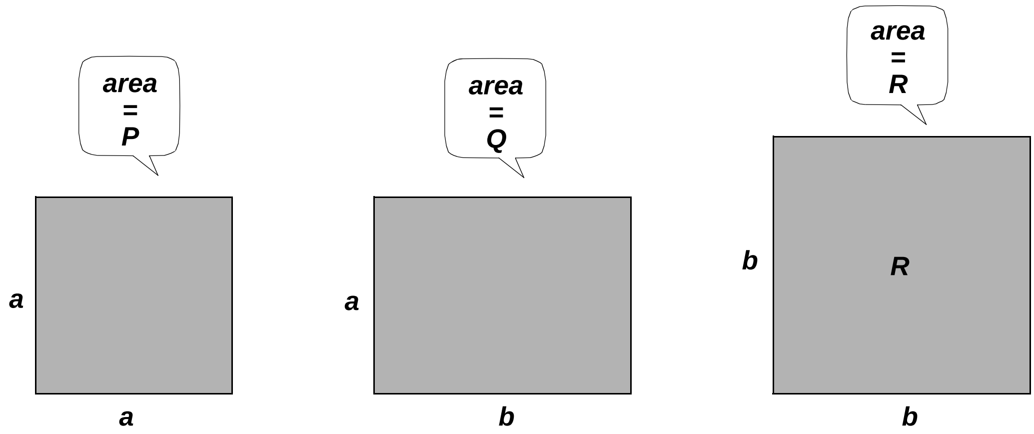
$4z^2$ ___ $3z^2$ ___ $2z^2$ ___ z^2 $= 14z^4$

$4z^2$ ___ $3z^2$ ___ $2z^2$ ___ z^2 $= 10z^4$

$4z^2$ ___ $3z^2$ ___ $2z^2$ ___ z^2 $= 2z^2$

Between two squares

A rectangle between two squares:



- Fill in expressions in ***a*** and ***b***: ***P*** = ... , ***Q*** = ... , ***R*** = ...

Below you see six formulas with ***P, Q*** and ***R***.

- Investigate which formulas are correct.

$Q^2 = PR$

$Q = \frac{P + R}{2}$

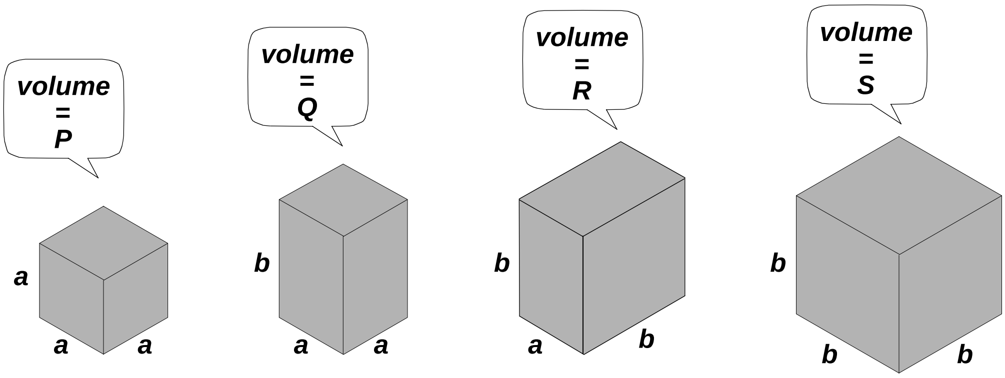
$Q = \sqrt{PR}$

$2Q = P + R$

$Q - P = R - Q$

$\frac{Q}{P} = \frac{R}{Q}$

Between two cubes



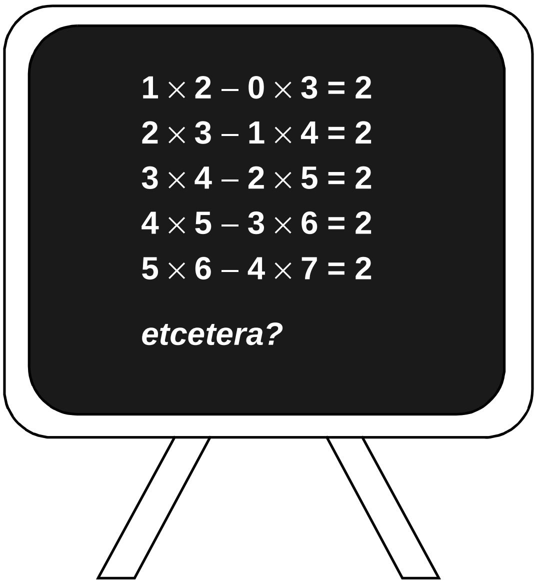
- Fill in expressions in ***a*** and ***b***: ***P*** = ... , ***Q*** = ... , ***R*** = ... , ***S*** = ...
- Check the following formulas:

$Q^2 = PR$ $\qquad$ $Q = \sqrt{PR}$ $\qquad$ $\dfrac{Q}{P} = \dfrac{R}{Q}$

- Invent at least three formulas with only ***Q***, ***R*** and ***S***.

- Also invent some formulas with ***P***, ***Q***, ***R*** and ***S***.

You can count on it (I)

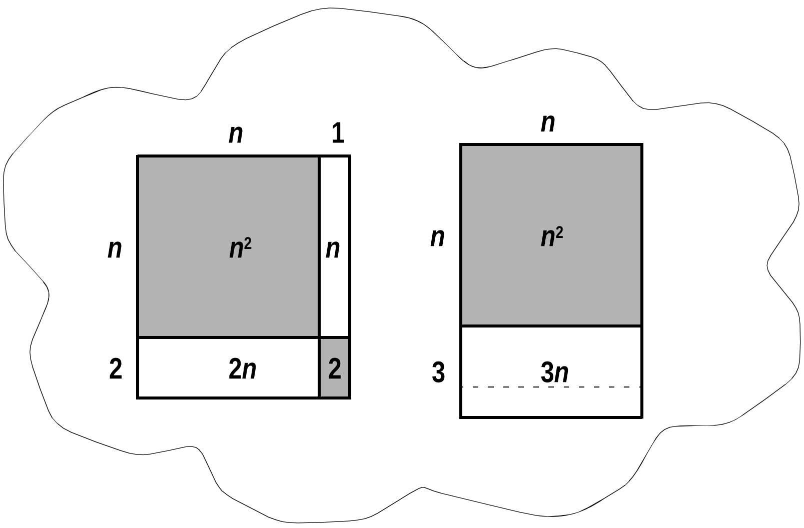


- Check the calculations on the blackboard. Continue the sequence with some more lines. What do you think?
- Give some other calculations, fitting in the sequence, with numbers between 100 and 1000. Check if the result will be 2?

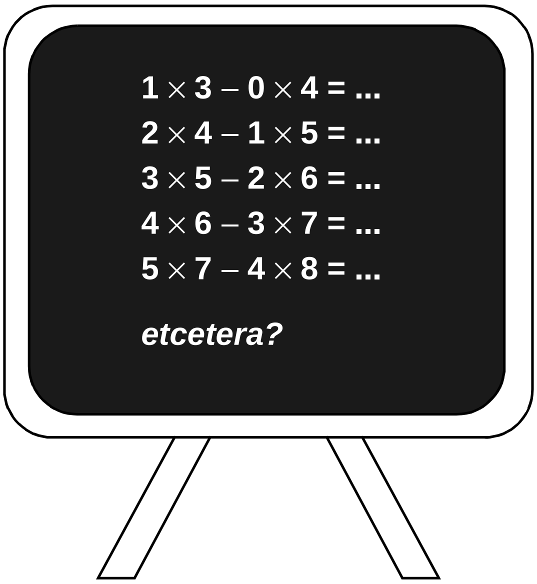
This is the general rule:

$$(n + 1) \times (n + 2) - n \times (n + 3) = 2$$

- How can you explain the validity, using the pictures in the cloud?



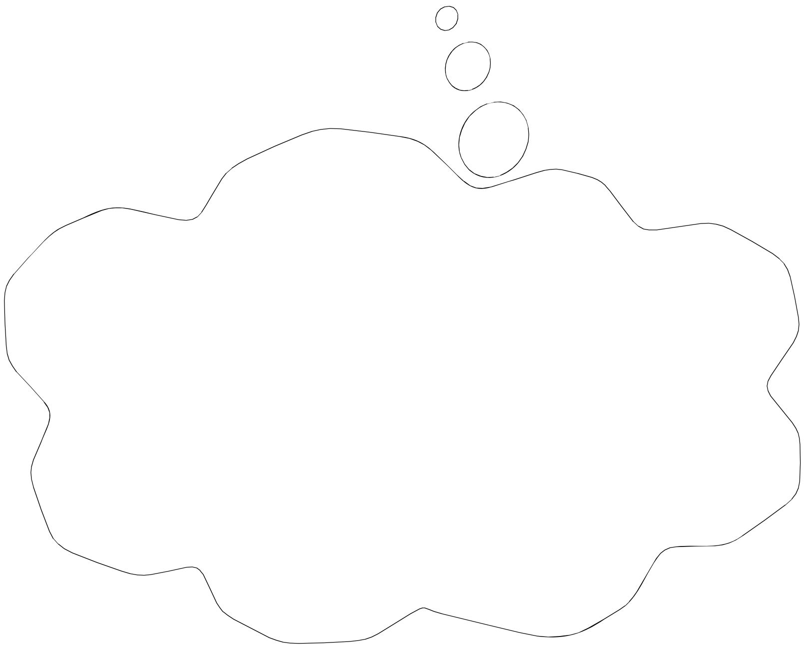
You can count on it (II)



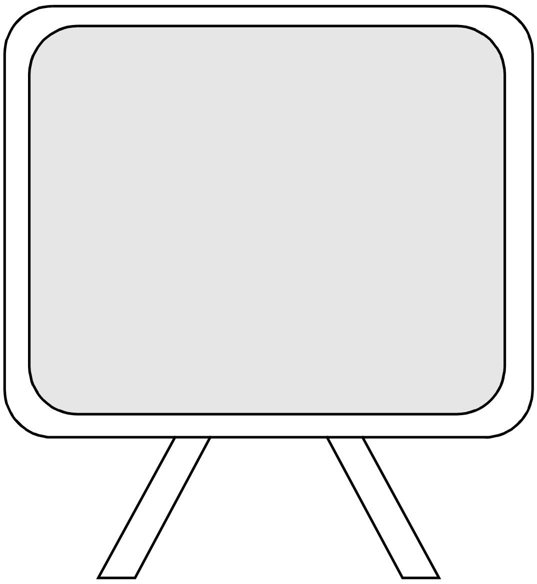
- What is the regularity in this sequence of calculations?

- Which formula corresponds with this sequence?

- Draw a picture that explains the formula:



You can count on it (III)

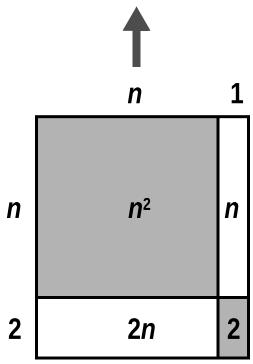


- Design a similar sequence of calculations (with the same result on each line).

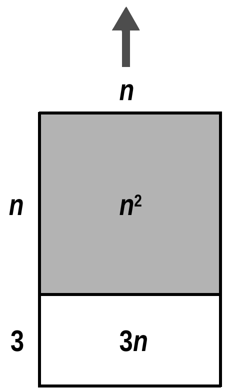
- Give a formula that fits the sequence.

You can count on it (IV)

$(n + 1) \times (n + 2) = n^2 + 3n + 2$



$n \times (n + 3) = n^2 + 3n$



$(n + 2) \times (n + 3) =$ ....

$n \times (n + 5) =$ ....

– ....

$(n + 2) \times (n +$ ....$) =$ ....

$n \times (n +$ ....$) =$ ....

– 10

....

....

– ....

Dictation

in words	in symbols
the sum of ***n*** and ***m*** is reduced with **5**	$n + m - 5$
the sum of ***n*** and **8** is multiplied by ***n***	$(n + 8) \times n$
the product of ***n*** and ***m*** is reduced with ***k***	
the sum of the squares of ***n*** and ***m***	
the square of the sum of ***n*** and ***m***	
the sum of 2 times ***n*** and 3 times ***k***	
the sum of ***n*** and **6** is multiplied by the difference of ***n*** and **6**	
the product of the third powers of ***n*** and ***m***	
the square of 5 times ***n***	
5 times the square of ***n***	

The price of algebra (I)

Algebra takes time, and time is money.
Below you se a detailed 'price' list

Price list:	
operations +, –, ×, :, /	1 point each time
taking a square	2 points each time
taking the 3rd power	3 points each time
taking the 4th power	4 points each time
etc.	etc.
using variables	1 point each time
parenthesis and numbers	free

Example 1: what is the price of $\mathbf{3}\boldsymbol{n} + \boldsymbol{m}$?

3	**number**	**free**
n	**using variable**	**1 point**
$3 \times n$	**multiplication**	**1 point**
m	**variable**	**1 point**
$(3\times n) + (m)$	**addition**	**1 point**
	total price	**4 points**

Example 2: what is the price of $(3n + m)^2$?

$3n + m$	just calculated	**4 points**
$(3n + m)^2$	**square**	**2 points**
	total price	**6 points**

- Find the price of:

  $n^2 + 3n$

  $n \times (n + 3)$

  $(n + 1) \times (n + 3)$

  $n^2 + 4n + 3$

The price of algebra (II)

$n^2 + 3n$ and $n \times (n + 3)$ are equivalent expressions.
They have not the same prices yet! (See preceding page).

- Here are pairs of equivalent expressions. Make sure of it.
  Find out for each pair which of the two is the cheapest..

$n + n + n + n$ and $4 \times n$

$n \times n \times n \times n$ and n^4

$n + n + n + n$ and $2 \times n + 2 \times n$

$(m + 1)^2$ and $(m + 1) \times (m + 1)$

$(m + 1)^2$ and $m^2 + 2m + 1$

$a^2 \times a^3$ and a^5

$(a^3)^2$ and a^6

The price of algebra (III)

- Compare the prices of $\boldsymbol{a^4b^4}$ and $\boldsymbol{(a^2b^2)^2}$

Both expressions are equivalent, but the second one is 2 points cheaper.

- Try to find an expression as cheap as possible, which is equivalent with $\boldsymbol{a^4b^4}$

You can write $\boldsymbol{n^{15}}$ in various ways (it's to say: replacing by an equivalent expression).
Here are some possibilities:

$$n \times n \times n \times n \times n \times n \times n \times n \times n \times n \times n \times n \times n \times n \times n$$

$$(n \times n \times n \times n \times n)^3$$

$$n^{10} \times n^5$$

- Which of them is cheaper than $\boldsymbol{n^{15}}$?
  Try to find an equivalent expression with the lowest price.

- Find the cheapest expression equivalent with $\boldsymbol{x^{24}}$

Splitting fractions (I)

Unit fractions

4000 years ago the mathematicians in Egypt worked with unit fractions, like:

$$\frac{1}{2}\ ,\ \frac{1}{3}\ ,\ \frac{1}{4}\ ,\ \frac{1}{5}\quad \text{etc.}$$

Thus witrh fractions with a **numerator** equal to **1**.

A fraction with another numerator than **1,** always can be split up in unit fractions. This one is easy:

$$\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

It is less insipid if it is demanded to use as few unit fractions as. possible, in this case:

$$\frac{3}{4} = \frac{2+1}{4} = \frac{1}{2} + \frac{1}{4}$$

Another example:

$$\frac{19}{24} = \frac{12+6+1}{24} = \frac{1}{2} + \frac{1}{4} + \frac{1}{24}$$

- Try to split into unit fractions, as few as possible:

$$\frac{7}{8} = \frac{\dots + \dots + \dots}{8} = \frac{1}{\dots} + \frac{1}{\dots} + \frac{1}{\dots}$$

$$\frac{5}{6} = \dots\dots\dots\dots = \dots\dots\dots\dots$$

$$\frac{13}{16} = \dots\dots\dots\dots = \dots\dots\dots\dots$$

$$\frac{13}{18} = \dots\dots\dots\dots = \dots\dots\dots\dots$$

$$\frac{99}{100} = \dots\dots\dots\dots = \dots\dots\dots\dots$$

Splitting fractions (II)

Numerator and denominator of a fraction can be divided (or multiplied) by the same number; the value of the fraction does not change then.

Examples: $\frac{5}{15} = \frac{1}{3}$ (numerator and denominator are divided by 3)

$\frac{b}{5b} = \frac{1}{5}$ (numerator and denominator are divided by *b*)

This rule is applied with splitting into unit fractions.

Examples: $\frac{n+1}{3n} = \frac{n}{3n} + \frac{1}{3n} = \frac{1}{3} + \frac{1}{3n}$

$\frac{a+b}{ab} = \frac{a}{ab} + \frac{b}{ab} = \frac{1}{b} + \frac{1}{a}$

- Split into unit fractions as few as possible :

$\frac{x+y}{xy} =$

$\frac{k+m+n}{kmn} =$

$\frac{p+1}{pq} =$

$\frac{p+1}{p^2} =$

- Fill in the correct expressions:

$$\frac{\ldots + \ldots + \ldots + \ldots}{abcd} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

Fractions on strips (I)

Fill the empty cells:

1	−	$\frac{1}{2}$	=	$\frac{1}{2}$
$\frac{1}{2}$	−	$\frac{1}{3}$	=	$\frac{1}{6}$
$\frac{1}{3}$	−	$\frac{1}{4}$	=	$\frac{1}{12}$
$\frac{1}{4}$	−	$\frac{1}{5}$	=	$\frac{1}{20}$
$\frac{1}{5}$	−	$\frac{1}{6}$	=	$\frac{1}{30}$
	−		=	
	−		=	

$$\frac{1}{n} - \frac{1}{n+1} = \square$$

1	+	$\frac{1}{2}$	=	$\frac{3}{2}$
$\frac{1}{2}$	+	$\frac{1}{3}$	=	$\frac{5}{6}$
$\frac{1}{3}$	+	$\frac{1}{4}$	=	$\frac{7}{12}$
$\frac{1}{4}$	+	$\frac{1}{5}$	=	$\frac{9}{20}$
$\frac{1}{5}$	+	$\frac{1}{6}$	=	$\frac{11}{30}$
	+		=	
	+		=	

$$\frac{1}{n} + \frac{1}{n+1} = \square$$

Fractions on strips (II)

Fill the empty cells:

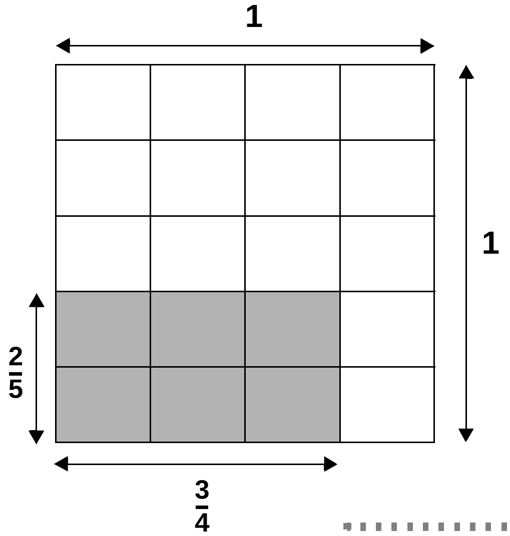
1		$\frac{1}{2}$		$\frac{1}{2}$
$\frac{1}{2}$		$\frac{1}{3}$		$\frac{1}{6}$
$\frac{1}{3}$		$\frac{1}{4}$		$\frac{1}{12}$
$\frac{1}{4}$	$\times$	$\frac{1}{5}$	$=$	$\frac{1}{20}$
$\frac{1}{5}$		$\frac{1}{6}$		$\frac{1}{30}$

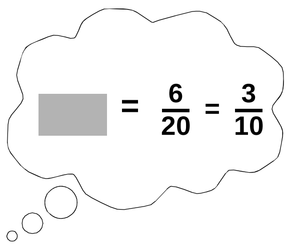
$\frac{1}{n}$	$\times$	$\frac{1}{n+1}$	$=$	

1		$\frac{1}{2}$		2
$\frac{1}{2}$		$\frac{1}{3}$		$\frac{3}{2}$
$\frac{1}{3}$		$\frac{1}{4}$		$\frac{4}{3}$
$\frac{1}{4}$	$:$	$\frac{1}{5}$	$=$	$\frac{5}{4}$
$\frac{1}{5}$		$\frac{1}{6}$		$\frac{6}{5}$

$\frac{1}{n}$	$:$	$\frac{1}{n+1}$	$=$	

Multiplying fractions (I)





$$\frac{3}{4}\times\frac{2}{5} = \frac{3\times 2}{4\times 5} = \frac{6}{20} = \frac{3}{10}$$

- Draw a picture to show: $\frac{4}{5}\times\frac{2}{3} = \frac{4\times 2}{5\times 3} = \frac{8}{15}$

- Calculate: $\frac{3}{5}\times\frac{1}{4} =$ $\qquad$ $\frac{3}{5}\times\frac{4}{5} =$

  $\frac{3}{4}\times\frac{1}{5} =$ $\qquad$ $\frac{1}{5}\times\frac{1}{4} =$

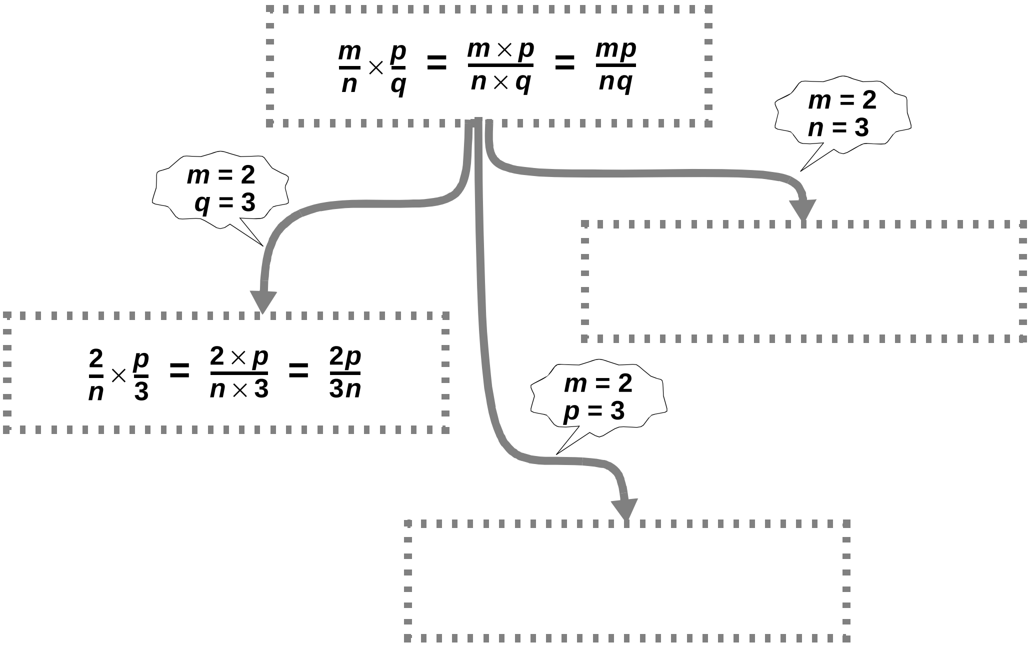
- Calculate: $\boldsymbol{p^2 + q^2 + 2pq}$ for $\boldsymbol{p} = \frac{1}{3}$ and $\boldsymbol{q} = \frac{2}{3}$

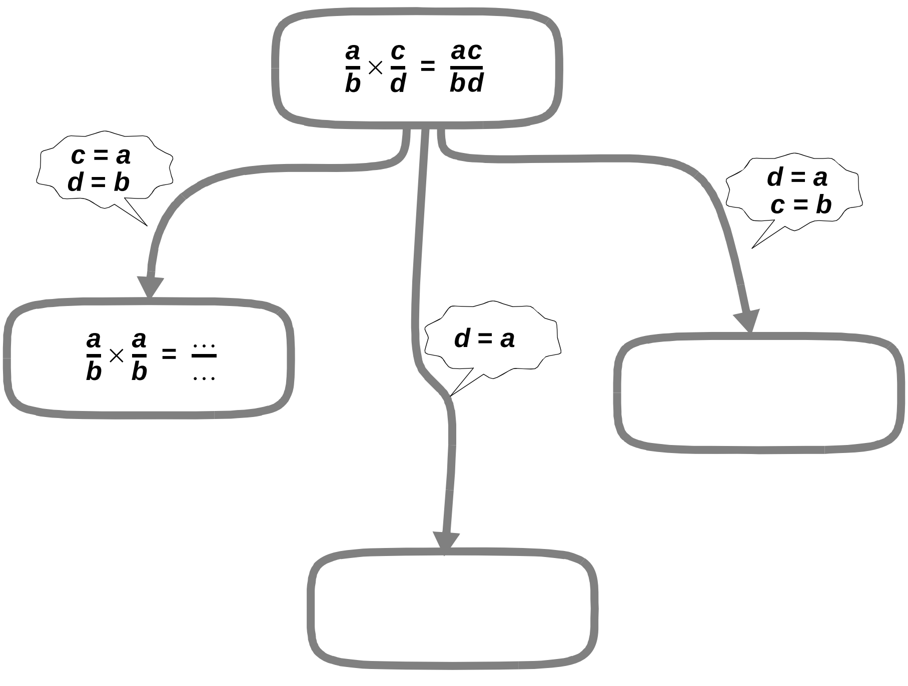
  Also for: $\boldsymbol{p} = \frac{2}{5}$ en $\boldsymbol{q} = \frac{3}{5}$

- Calculate: $\boldsymbol{p^2} - \boldsymbol{q^2}$ for $\boldsymbol{p} = \frac{3}{4}$ and $\boldsymbol{q} = \frac{1}{4}$

  Also for: $\boldsymbol{p} = \frac{5}{8}$ and $\boldsymbol{q} = \frac{3}{8}$

Multiplying fractions (II)





Mediants (I)

You have seen that:

$$\frac{m}{n} \times \frac{p}{q} = \frac{m \times p}{n \times q} \quad \text{......... (A)}$$

Maybe you should think that:

$$\frac{m}{n} + \frac{p}{q} \overset{?}{=} \frac{m + p}{n + q} \quad \text{......... (B)}$$

- Calculate in this way the 'sum' of $\frac{3}{5}$ and $\frac{1}{4}$

  Check that this result is lying *between* $\frac{3}{5}$ and $\frac{1}{4}$
  so it con not bethe real sum of both fractions!

- How can you calculate the right sum of $\frac{3}{5}$ and $\frac{1}{4}$?

Formula (B) is not a good recipe to add fractions,
but it is recipe to find intermediate fractions, so-called *mediants*.

- Check for some examples, that the values of $\frac{m+p}{n+q}$ always lies between the values of $\frac{m}{n}$ and $\frac{p}{q}$

- Design some examples, in which the value of $\frac{m+p}{n+q}$ lies in the middle between te values of $\frac{m}{n}$ and $\frac{p}{q}$

In a group of 31 students are much more girls than boys
(19 to 12).

In a parallel group (29 students) it is just the reverse (12 to 17)

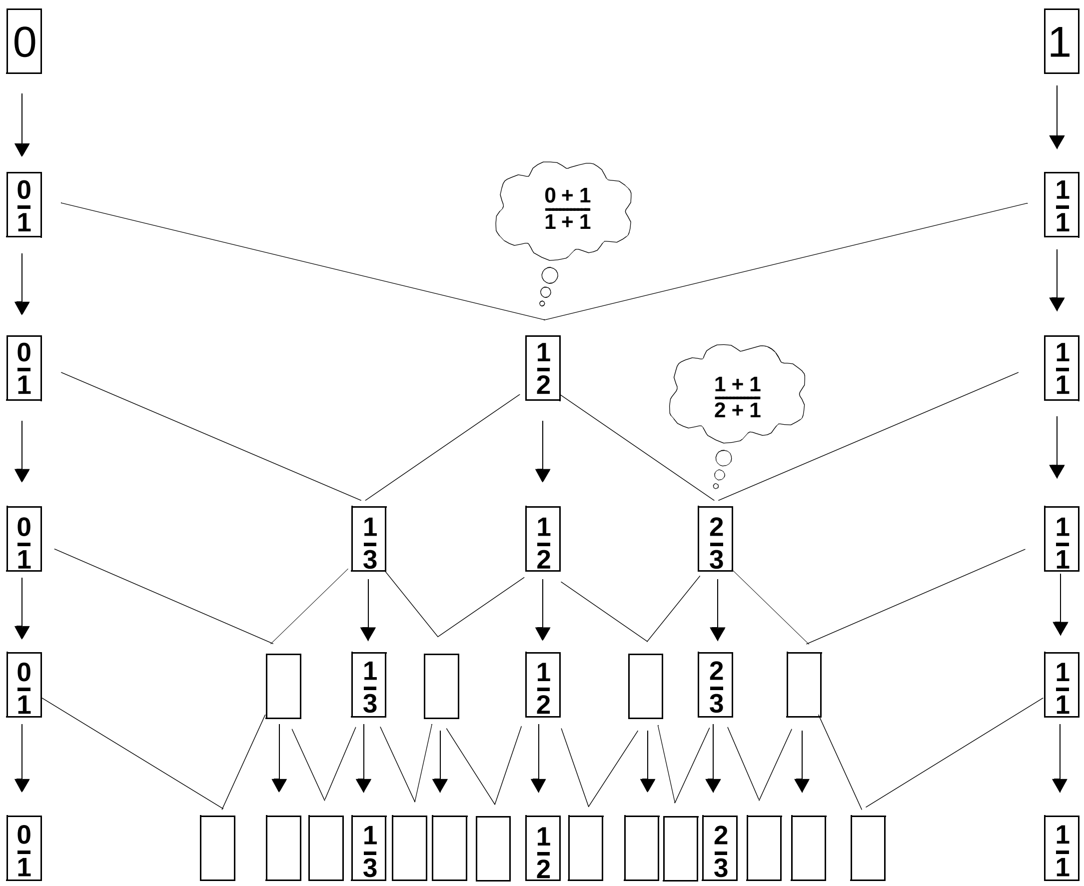
The two groups are joined during the lessons of P.E.
and then the girls and boys are separated.

- Check that this junction results in a more balanced distribution of numbers of girls and boys.
- What has this example to do with the concept of mediant?

Mediants (II)

With the formula for mediants, can be made step by step new fractions.

- Fill the empty cells in the 'tree' below.



Adding fractions

The correct formula for adding fractions is (unfortunately) much more complicated than the mediant formula. Here it is:

$$\frac{m}{n} + \frac{p}{q} = \frac{m \times q + n \times p}{n \times q}$$

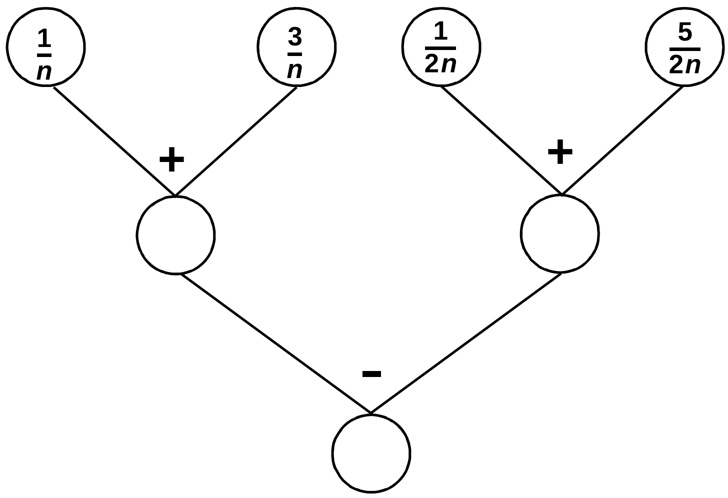
- Check this formula for the fractions $\frac{3}{5}$ and $\frac{1}{4}$
- Design some examples by yourself to check the addition formula.
- Which (simpler) formula do you get if $m = 1$ and $p = 1$?
- Which (insipid) formula do you get if $n = 1$ en $q = 1$?

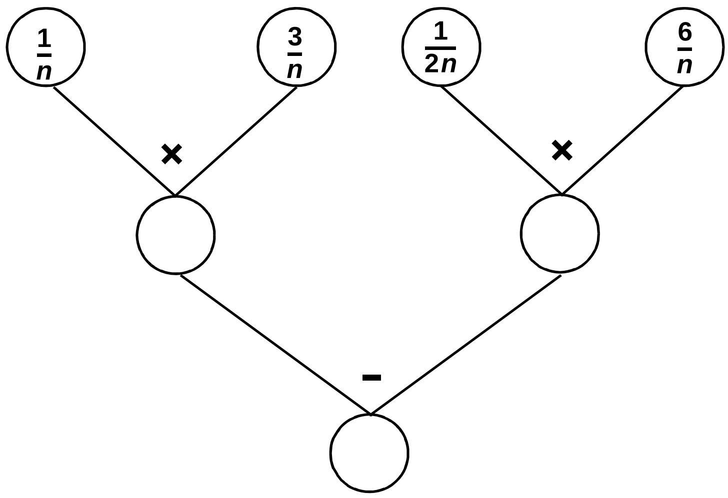
The addition formula is in many cases more complicated than necessary, but it's always correct.
Here is an example, in which you can add fractions without any formula:

$$\frac{3}{n} + \frac{2}{n} = \frac{5}{n}$$

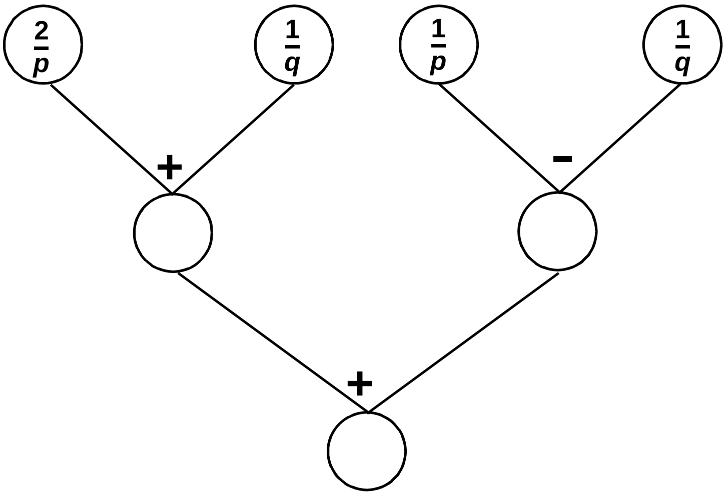
- If you should use the addition formula (with $m = 3$, $p = 2$ en $q = n$) you should have $$\frac{3}{n} + \frac{2}{n} = \frac{3 \times n + n \times 2}{n \times n}$$ Show that the result is equal to $\frac{5}{n}$
- Find one fraction equal to: $\frac{m}{3} + \frac{m}{2}$
- Also for: $\frac{m}{3} + \frac{p}{2}$

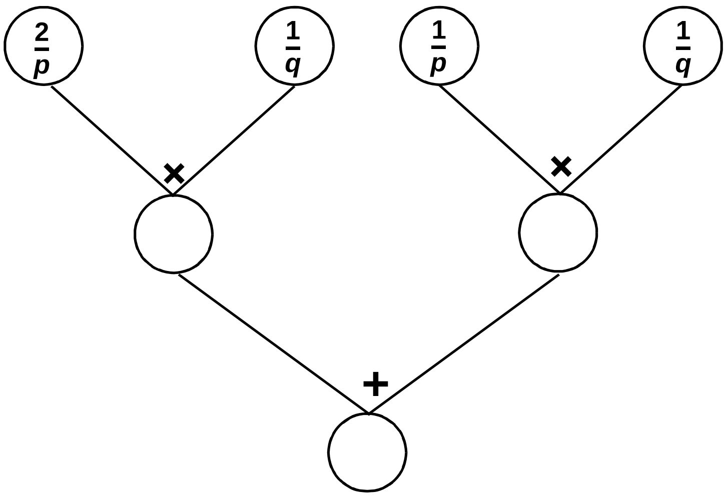
Trees with fractions (I)





Trees with fractions (II)





More fractions

$$\frac{a}{2} + \frac{a}{3} + \frac{a}{6} = \ldots$$

$$\frac{2}{a} + \frac{3}{a} + \frac{6}{a} = \ldots$$

$$\frac{b}{2} + \frac{b}{4} + \frac{b}{6} + \frac{b}{12} = \ldots$$

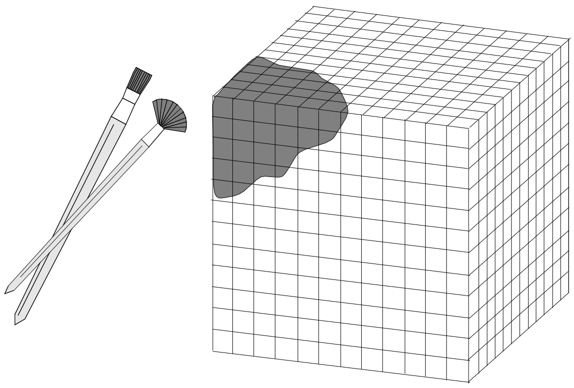
$$\frac{2}{b} + \frac{4}{b} + \frac{6}{b} + \frac{12}{b} = \ldots$$

$$\frac{c}{2} + \frac{c}{6} + \frac{c}{10} + \frac{c}{12} + \frac{c}{15} + \frac{c}{60} = \ldots$$

$$\frac{2}{c} + \frac{6}{c} + \frac{10}{c} + \frac{12}{c} + \frac{15}{c} + \frac{60}{c} = \ldots$$

- Design one pair of additions in the same style.

Painting a cube (I)



One cube ($12 \times 12 \times 12$ cm) is glued from little wooden cubes, of 1 cm^3.

- How many small cubes are used?

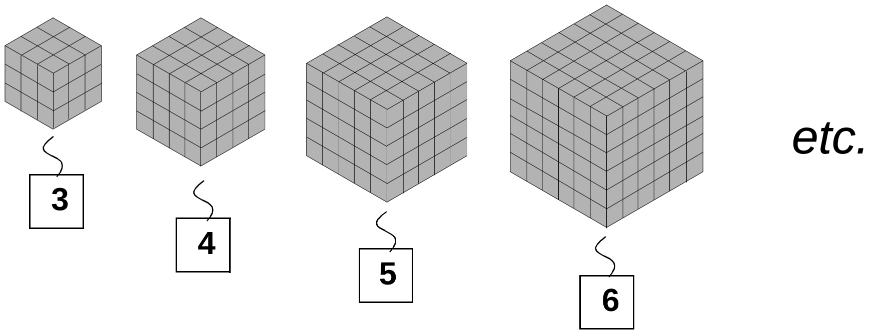
The faces of the big cube are painted red.
There are small cubes that get 3 red faces.

- How many?
- How many small cubes get only 2 red faces?
- How many get only 1 red face?
- How many small cubes are not painted at all?

After an idea of Pierre van Hiele.

Painting a cube (II)

A sequence of painted cubes:

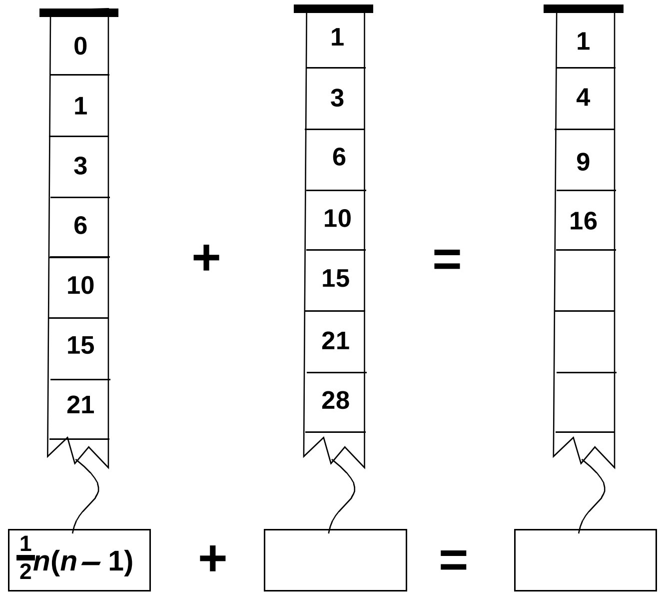


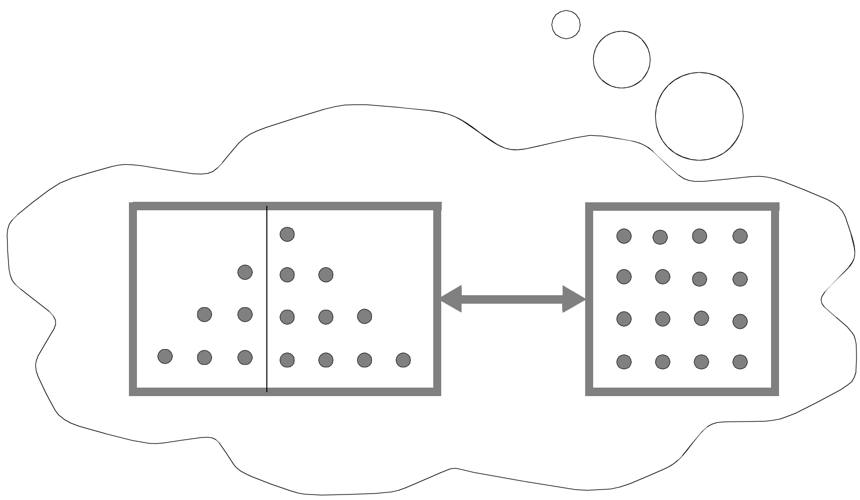
The first cube is 3 × 3 × 3 cm , the second one 4 × 4 × 4 cm, etc.
Look at the table below; $\boldsymbol{r}$ represents the length of an *edge* in cm.
You can read the value of $\boldsymbol{r}$ on the labels in the drawing.
The number of small cubes with 3 painted faces, is called $\boldsymbol{A_3}$
$\boldsymbol{A_2}$, $\boldsymbol{A_1}$ and $\boldsymbol{A_0}$ represent the number of small cubes with 2, 1 and 0 painted faces. $\boldsymbol{A_{tot}}$ represents the total number of small cubes.

r	A_3	A_2	A_1	A_o	A_{tot}
3	**8**	**12**	**6**	**1**	**27**
4					
5					
6					
7					
8					

- Check the first row in the table and complete the table.
- Which regularities do you notice?
- Explain the formula: $\boldsymbol{A_2 = 12(r - 2)}$
- Try to find expressions in $\boldsymbol{r}$ for $\boldsymbol{A_1}$, $\boldsymbol{A_o}$ and $\boldsymbol{A_{tot}}$

Triangular numbers and squares



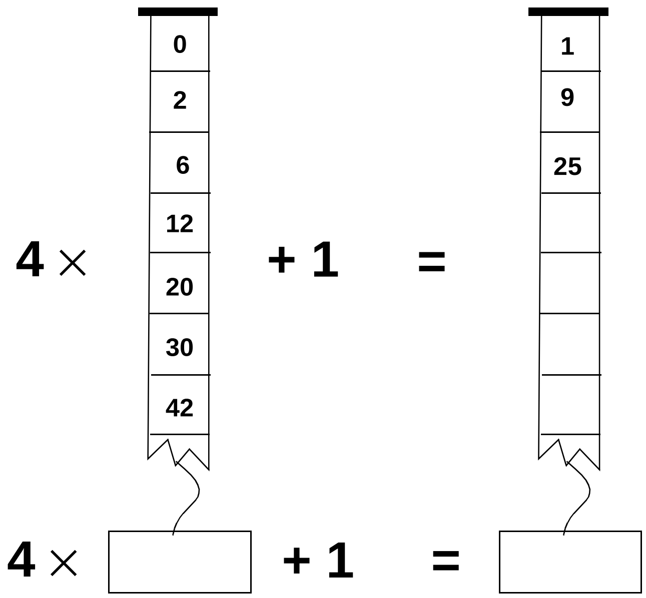


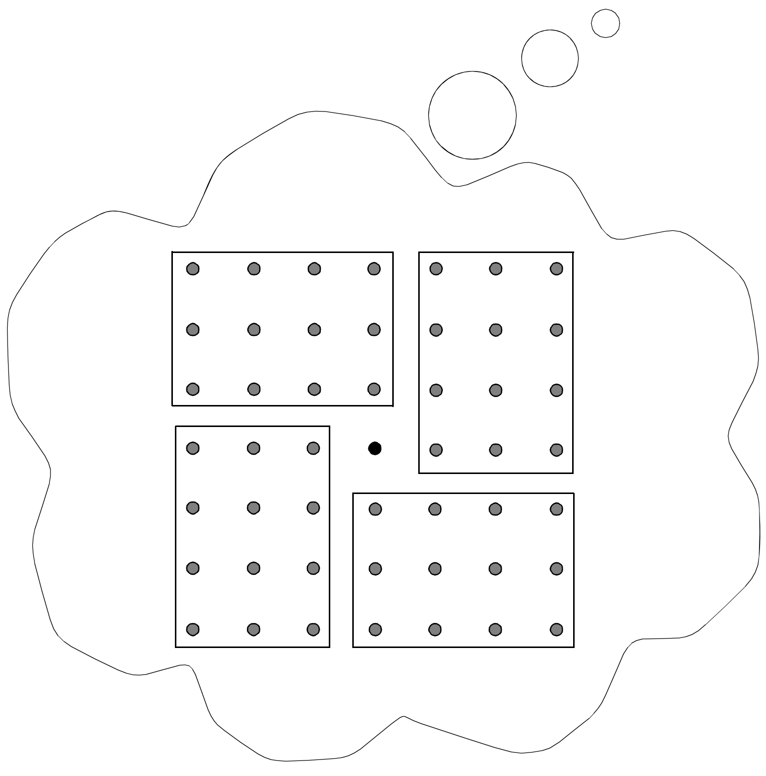
$$\tfrac{1}{2}n(n-1) + \tfrac{1}{2}n(n+1) = n^2$$

- Explain :

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 100

Oblong numbers and squares





$$4n(n + 1) + 1 = (2n + 1)^2$$

A remarkable identity (I)

A rectangular field has a length of 31 m and a width of 29 m.

- Give a fast estimation (in m^2) of the area.
- How many m^2 does this esitmation deviate from the exact area?

- The same two questions for a field of 41×39 meters

- 51×49 → estimation: 50×50 = ... ; exact result = ... ; deviation = ...

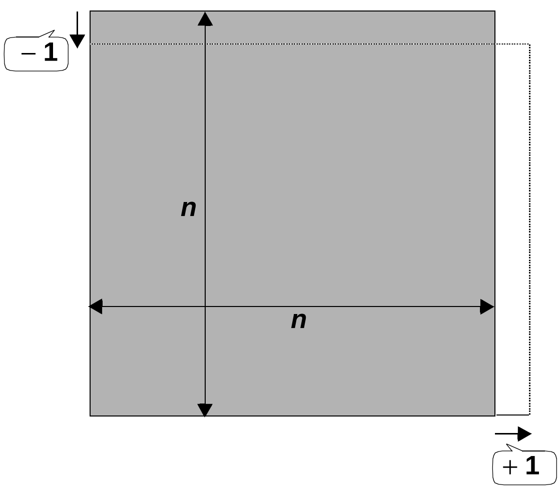
- Make a similar scheme for 61×59.

- Explain from the picture:

The difference between $\boldsymbol{n \times n}$ *and* $\boldsymbol{(n + 1) \times (n - 1)}$ *is equal to* **1**

or as a formula:

$$(n + 1) \times (n - 1) = n^2 - 1$$



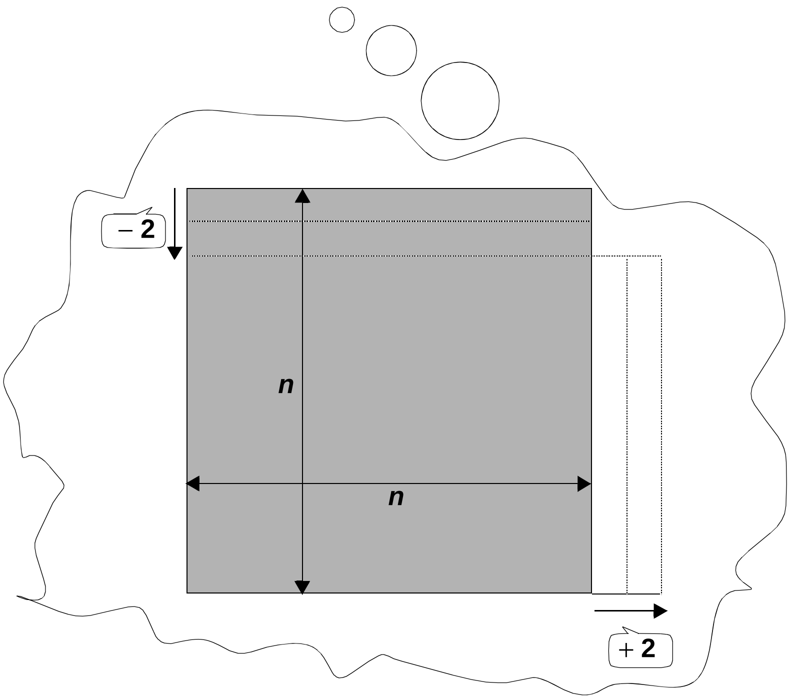
A remarkable identity (II)

- 32×28 → estimation: $30 \times 30 =$ .... ; exact result = ... ; deviation = ...

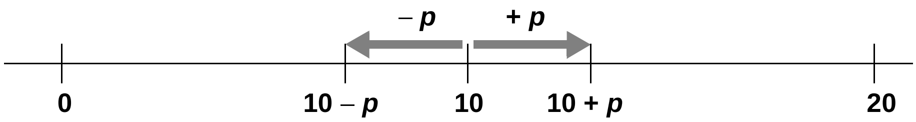
- Make a similar scheme for 42×38.

- Also for 52×48.

- Find a general rule.



A remarkable identity (III)



- Look at the number line and fill:

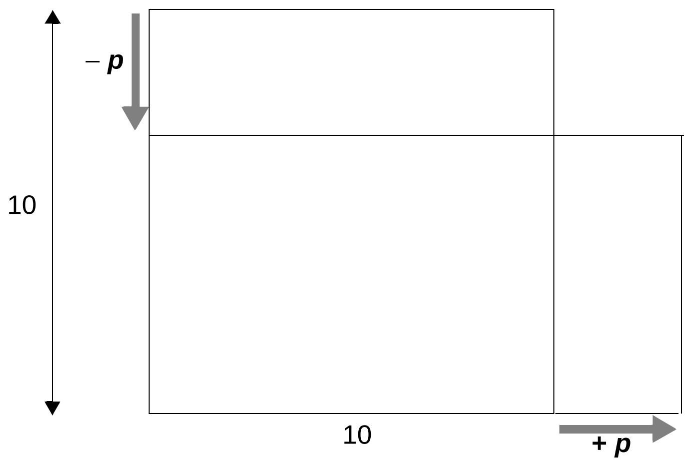
  (10 + *p*) + (10 – *p*) = ...

  (10 + *p*) – (10 – *p*) = ...

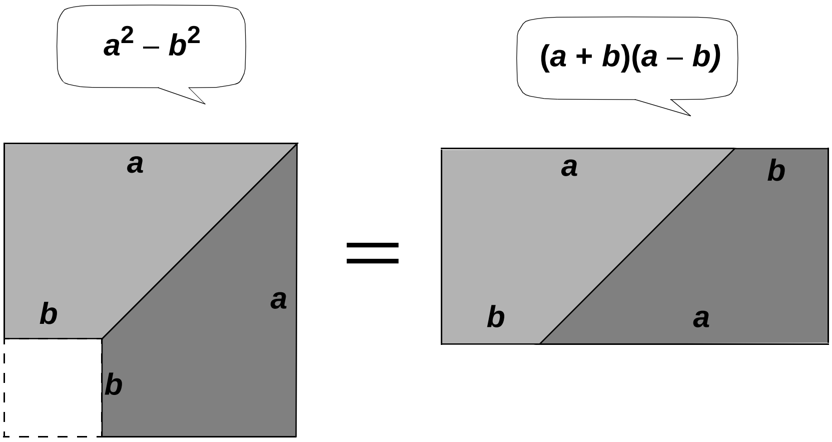
August thinks that '**10 + *p*** times **10 – *p*'** is equal to 100.

His reasoning: **10** timesl **10** is **100**
10 – ***p*** is ***p*** less than **10,**
but **10 + *p*** is ***p*** more than **10,**
so they compensate each other**.**

Is August right?



A remarkable identity (IV)



a^2 minus b^2 = $(a + b)$ times $(a - b)$

example:

$54^2 - 46^2 = (54 + 46) \times (54 - 46) = 100 \times 8 = 800$

- Calculate in this manner, without calculator:

  $52^2 - 48^2$ = ... = ... = ...

  $67^2 - 33^2$ = ... = ... = ...

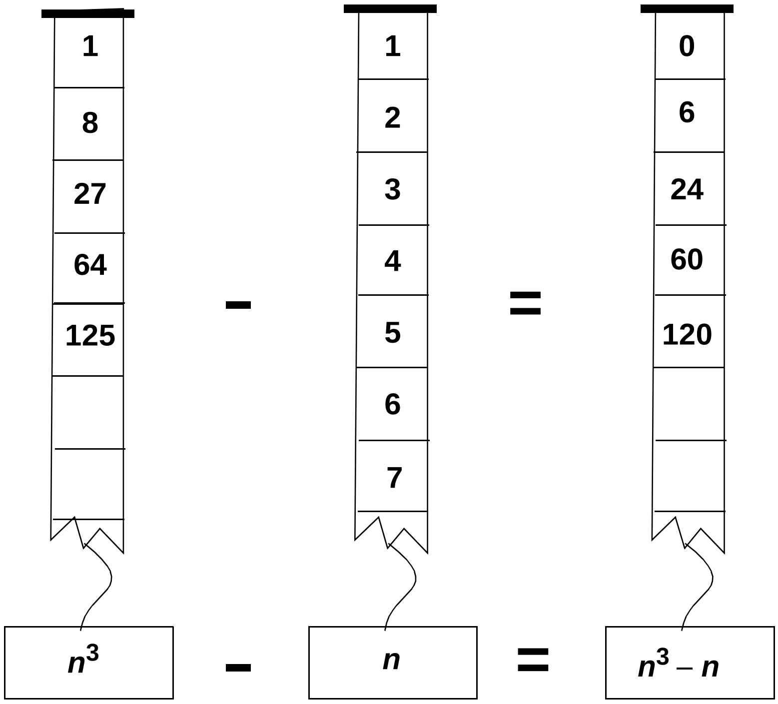
  $501^2 - 499^2$ = ... = ... = ...

- Design some exercises in the same style.

- **100** times **100** is more than **100 + *p*** times **100 – *p***
  How many more?

- n^2 is more than ***n* + 10** times ***n* – 10**
  How many more?

Divisible by 6



- Fill in the empty cells.

Look at the strip on the right side. It seems to be that all numbers in that strip are divisible by 6.

- Take some other values for n and investigate of this is also true in those cases.

In the strip with label $n^3 - n$ one can find a nice pattern.

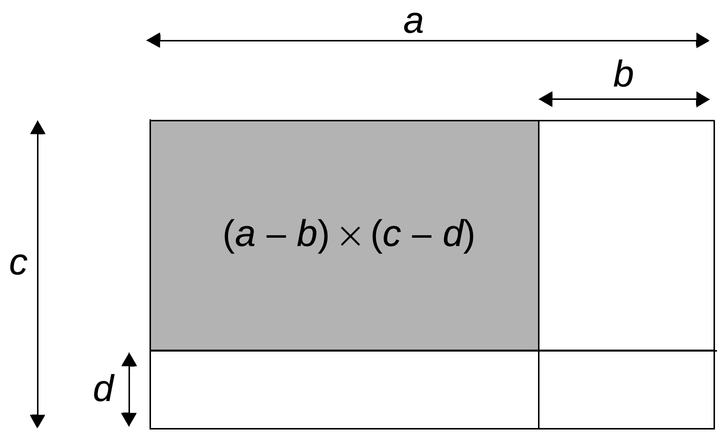
To begin with 6: $\mathbf{6} = \mathbf{1} \times \mathbf{2} \times \mathbf{3}$

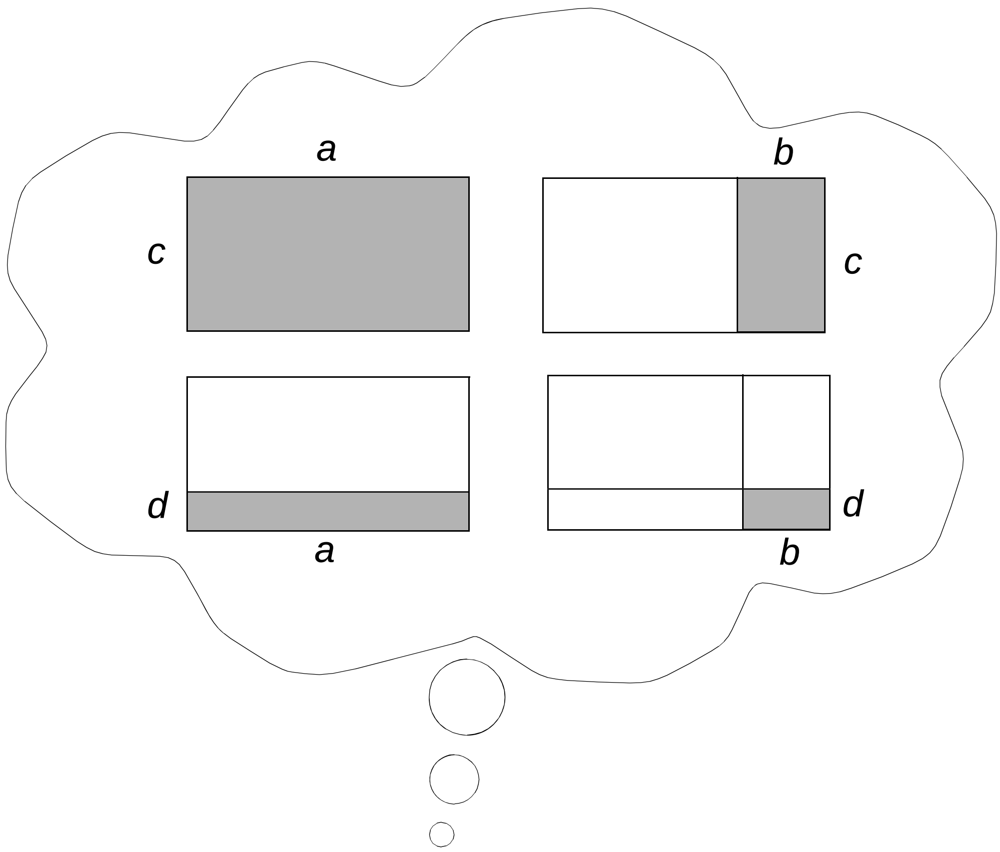
And then: $\mathbf{24} = \mathbf{2} \times \mathbf{3} \times \mathbf{4}$

$\mathbf{60} = \mathbf{3} \times \mathbf{4} \times \mathbf{5}$

- Will this be continued? Try some cases.
- Complete (and checkr!) : $n^3 - n$ = **...........** × **...........** × **...........**
- Try to explain why $n^3 - n$ is divisible by **6**, for each value of n.

Multiplying differences (I)

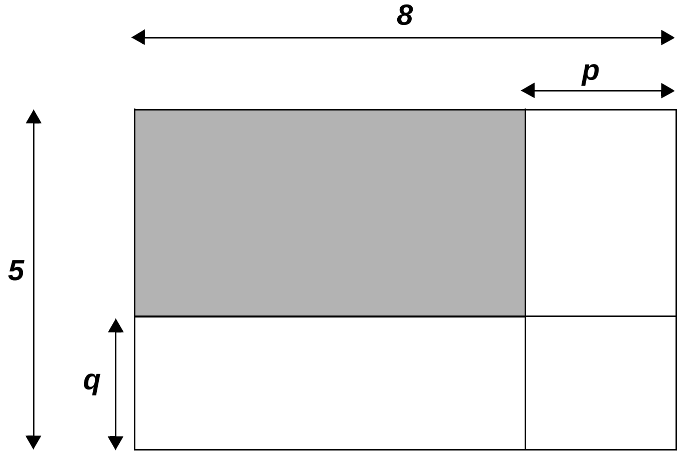




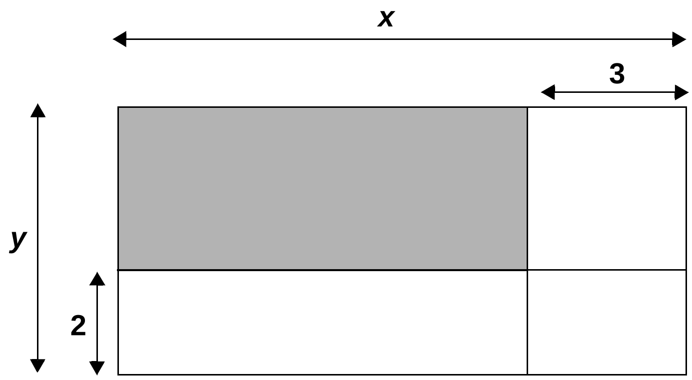
$$(a - b) \times (c - d) = a \times c - a \times d - b \times c + b \times d$$

- Explain this formulafrom the pictures
- Is the formula true if $a = b$? Explain your answer.

Multiplying differences (II)



$(8 - p) \times (5 - q) =$ ...40 – ....



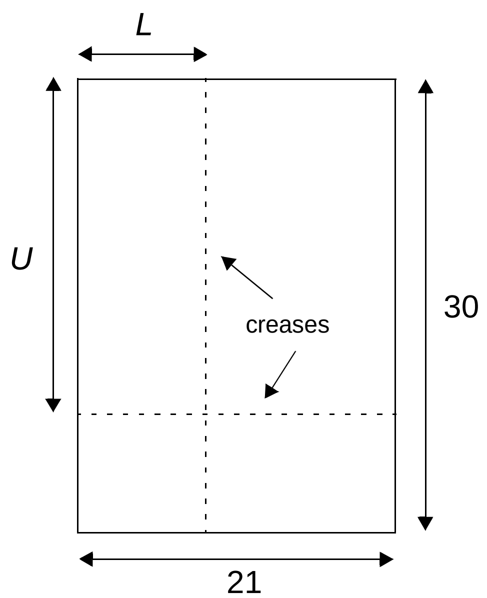
$(x - 3) \times (y - 2) =$ ....

- Design your own product of differences with corresponding picture.

Multiplying differences (III)

A sheet of paper, size A4, has dimensions 210 and 297 mm.
Rounded to centimeters: 21 and 30 cm.

- Take a sheet A4 and fold it twice: one crease parallel with the shortest edge and one parallel with the other edge.
  The distances from the edges are chosen abritary.



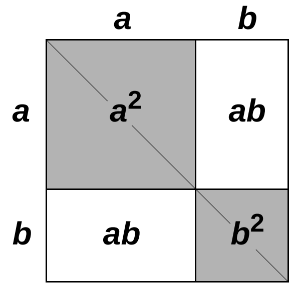
Call the distance from the 'vertical' crease to de left side *L*
and the distance from the 'horizontal' crease to de upper side *U.*

- Unfold the paper; now you se four rectangles.

  Write in each of those rectangles an expression in *L* and *U* for the area of that rectangle.

- What will be the result if you add the four expressions?
  Check this.

Squares of sums (I)



$$(a + b)^2 = a^2 + b^2 + 2ab$$

- Explain the equivalence of $(a + b)^2$ and $a^2 + b^2 + 2ab$ from the picture.

- Mental arithmetic:
  calculate $a^2 + b^2 + 2ab$ in the case that $a = 37$ and $b = 63$

- Suppose $b = a$. The equivalence of both expressions must stay valid..
  Verify without using a picture that $(a + a)^2$ and $a^2 + a^2 + 2aa$ are equivalent indeed.

- Suppose $b = 2a$ and check the equivalence.

- Substitute $a = 2p$ and $b = 3p$ in $(a + b)^2$ en $a^2 + b^2 + 2ab$
  Verify the equivalence.

Squares of sums (II)

The next poem (translated from Dutch) is from the book "*A lark above a pasture*" of the author K. Schippers.
The title of this poem is a complicated algebraic formula.
The poem finishes with another formula which looks nicer.
In this way the poet will express the beauty of his lover.

> $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$
>
> I say your beauty minus your eyes is *a*
> the spirit that darts in you is *b*
> your eyes
> *c*
> added and at least given a square:
> $(a + b + c)^2$

- The last nice expressionis equivalent with the complicated first one.
  Explain this from the picture.

	a	***b***	***c***
a	a^2	***ab***	***ac***
b	***ab***	b^2	***bc***
c	***ac***	***bc***	c^2

- Substitute ***c*** = 0 in both expressions.
  Which formula do you get?

- Suppose ***a = b = c***. Both expression can be written using only one letter.
  Check by calculating that they are equivalent..

- The square of 111 has a funny result: 12321.
  You can check this by a calculator, but you can also use the two expressions from the poem ....

Squares of sums (III)

Next step: the square of: $\boldsymbol{a + b + c + d}$

You will believe that the rule begins as follows:

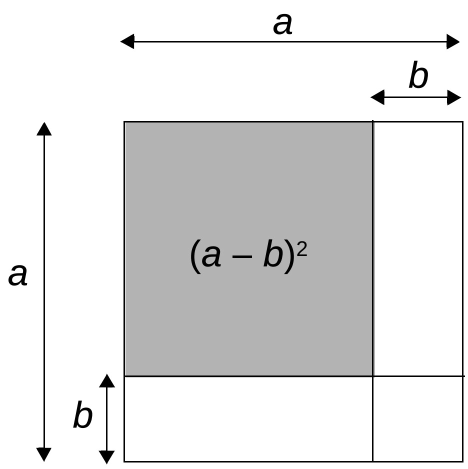
$\boldsymbol{(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab +}$ **...............................**

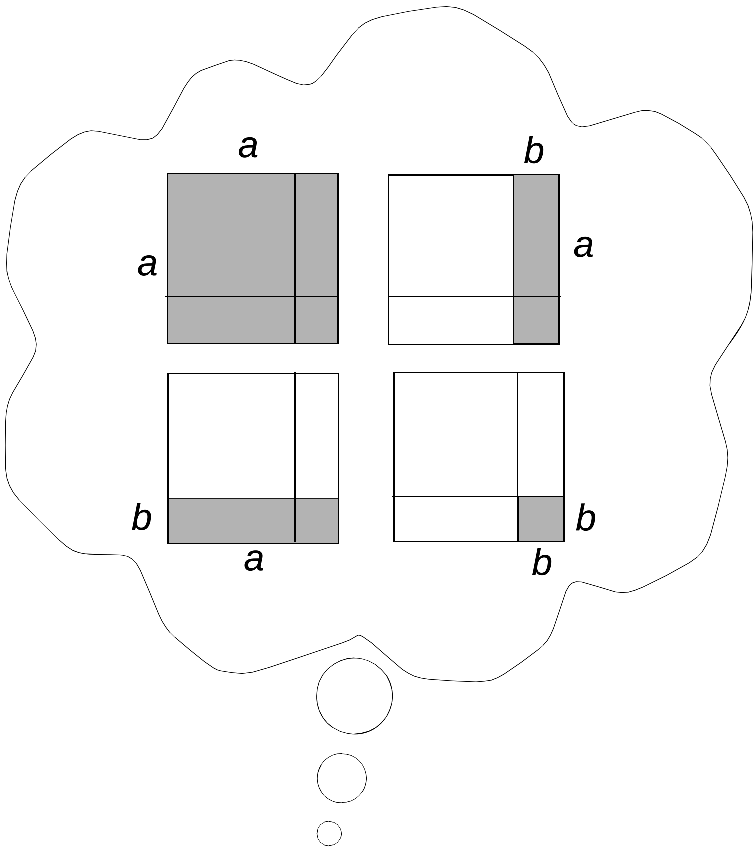
- Complete this rule and draw a picture to convince yourself (or other people) that the formula is correct.

- Use the rule to explain: $1111^2 = 1234321$

- Suppose $\boldsymbol{a = b = c = d}$. Write the two expressions with one letter and check by calculation that they are equivalent.

Squares of differences (I)

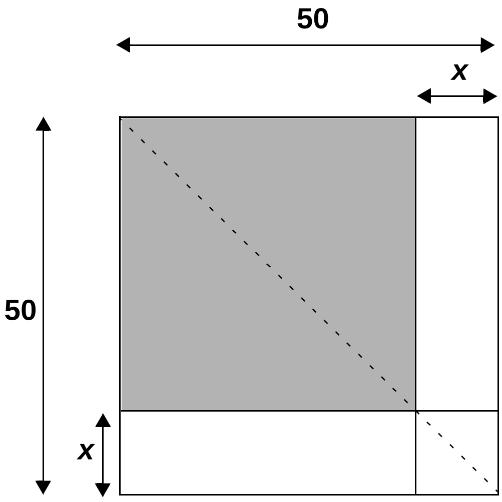




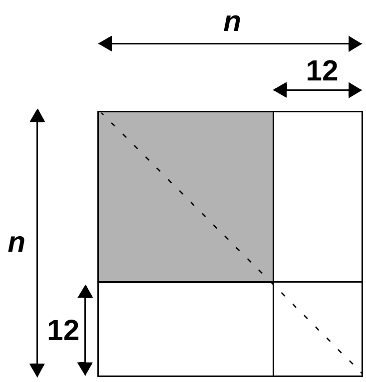
$$(a - b)^2 = a^2 - 2ab + b^2$$

- Explain this formula from the pictures.
- Is the formula correct for $a = b$? Explain your answer.

Squares of differences (II)



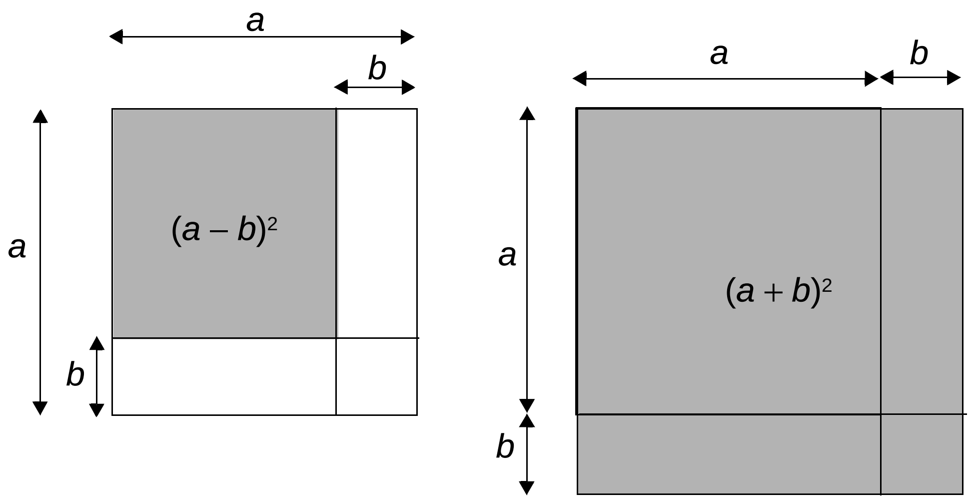
$(50 - x)^2 = 2500$ ....



$(n - 12)^2 =$ ....

- Design your own square of a difference and draw the corresponding picture.

More about squares



$(a - b)^2 + (a + b)^2 = 2a^2 + 2b^2$

- Explain this formula.
- Mental arithmetic: $99^2 + 101^2$
- Also: $49^2 + 50^2 + 51^2$

$S = (n - 2)^2 + (n - 1)^2 + n^2 + (n + 1)^2 + (n + 2)^2$

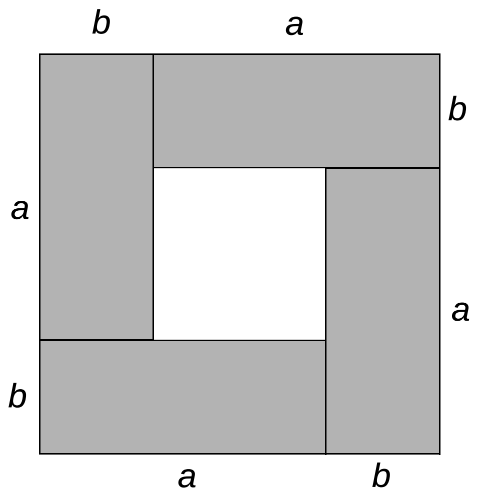
Assertion: *S* is divisible by 5, for each integer value of *n*

- Investigate of this assertion is true. If it's true, try to give an explanation.

$T = (n - 3)^2 + (n - 2)^2 + (n - 1)^2 + n^2 + (n + 1)^2 + (n + 2)^2 + (n + 3)^2$

- Which number will divide *T*, regardless the value of *n*? Give an explanation.

Sum, difference, product



$(a + b)^2 - (a - b)^2 = 4ab$

- Explain this formula from the picture.
- Check the formula in the case: *a* = *b.*

$S^2 - D^2 = (S + D)(S - D)$

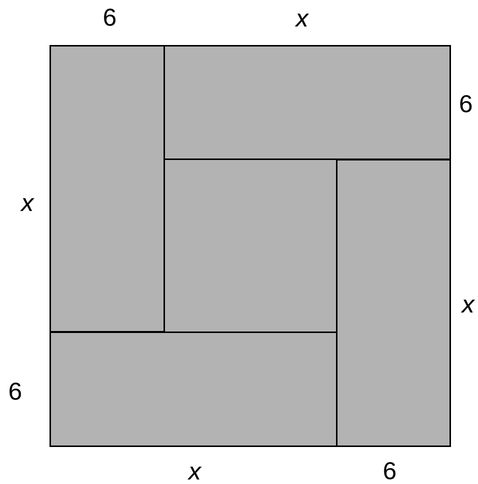
- Suppose ***S*** = *a* + *b* and ***D*** = *a* – *b*
  Substitute this in the preceding formula.
  This leads to the formua in the black rectangle.

S represents the sum, ***D*** the difference and ***P*** the product of two numbers.

- Complete this table:

S	***D***	***P***
13	3	
22		96
	19	120
	0	576

Equations with squares (I)

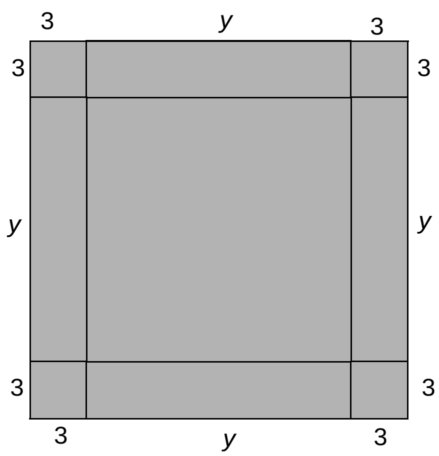


area = 441

or:

$(x + 6)^2 = 441$

- Calculate the value of *x*.

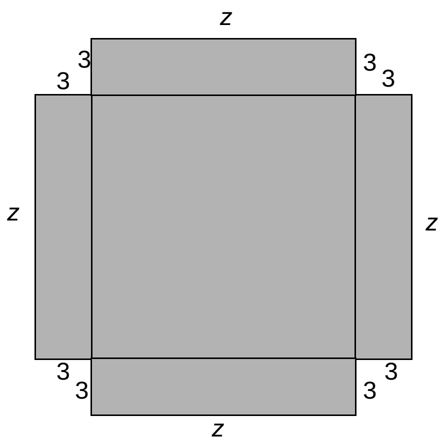


area = 400

or:

$(..........)^2 = 400$

- Calculate the value of *y*.



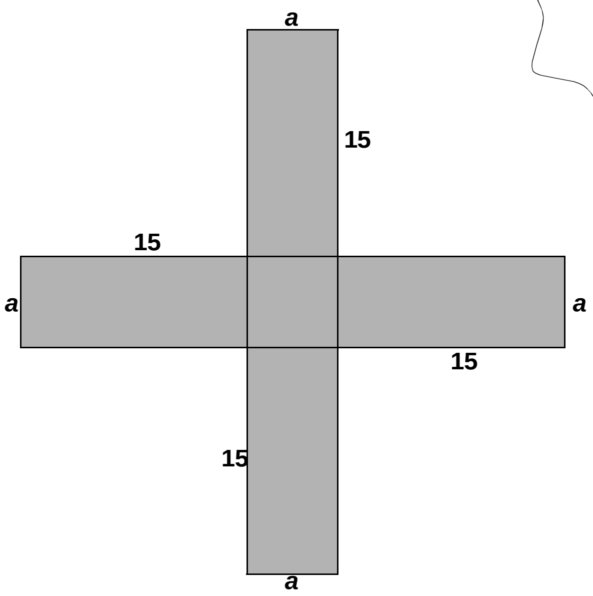
area = 364

or:

$z^2 + 4 \times 3z = 364$

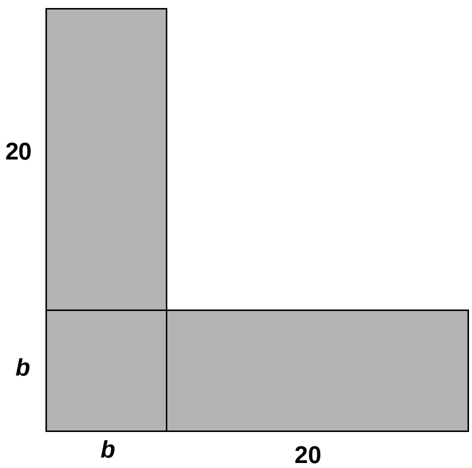
- Calculate the value of *z*.

Equations with squares (II)



area = 396
or:
.... = 396

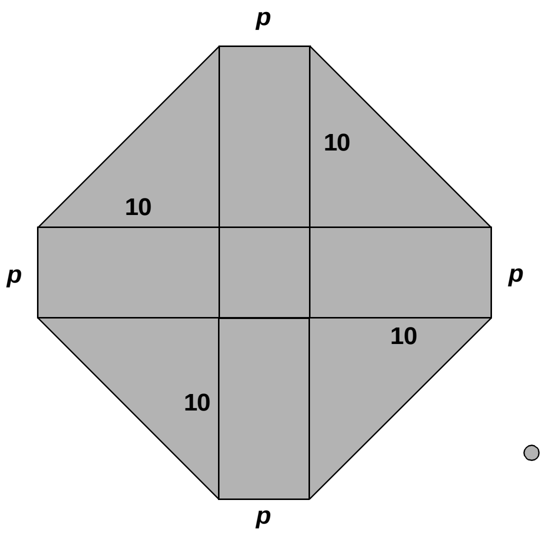
- Calculate the value of *a*.



area = 384
or:
.... = 384

- Calculate the value of *b*.

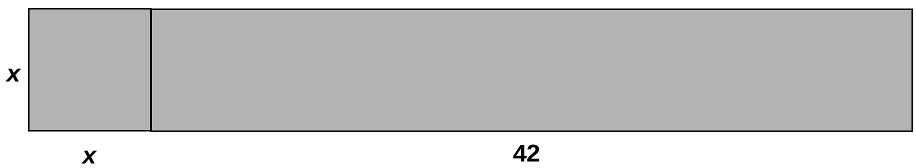
Equations with squares (III)



area = 425
or
p^2 + ... + ... = 425

- Calculate the value of *p*.

area = 343
or:
.... + ... = 343



- Calculate the value of *x*.

Antique equation (I)

The square of the with 3 reduced fifth part of a group of monkies was hidden in a cave.
Only one monkey, who climbed in a tree, was visible.

How many monkies were there totally?

Bhaskara, India (1114 - 1185)

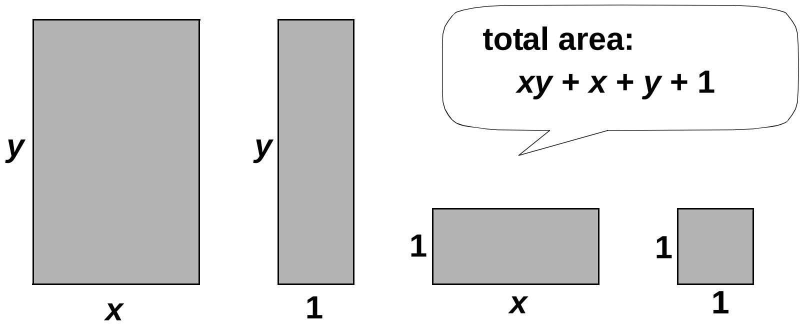
Antique equation (II)

The son of Pritha, rancorous by the fight,
used a quiver full of arrows
to kill Karna:
- with half of his arrows he repeled
the arrows of his enemy;
- with 4 times the square root of the total
number of arrows he killed his horse;
- wih 6 arrows he brought down Salya;
- with 3 he destroyed his sunshade and bow
- with 1 arrow he hitted the head of the fool.

How many arrows were shoot by Arjuna,
the son of Pritha?

Bhaskara, from Vya Ganita (= 'calculus of square roots')

Factorizing (I)



- How can you make one rectangle with the four pieces?
- Explain: $xy + x + y + 1 = (x + 1)(y + 1)$
- Complete: $xy + 2x + 2y + 4 = (x + ...)(y + ...)$

  How can you explain the equivalence by a picture?
- Fill in the missing numbers:

  $ab + 2a + 3b + 6 = (a + ...)(b + ...)$

  $pq + 5p + 6q + 30 = (p + ...)(q + ...)$

  $xy + 11x + 9y + ... = (x + 9)(y + ...)$

  $mn + ...\, m + ...\, n + 200 = (m + 20)(n + ...)$

- Suppose $b = a$, $q = p$, $y = x$ and $n = m$, then you get (fill in):

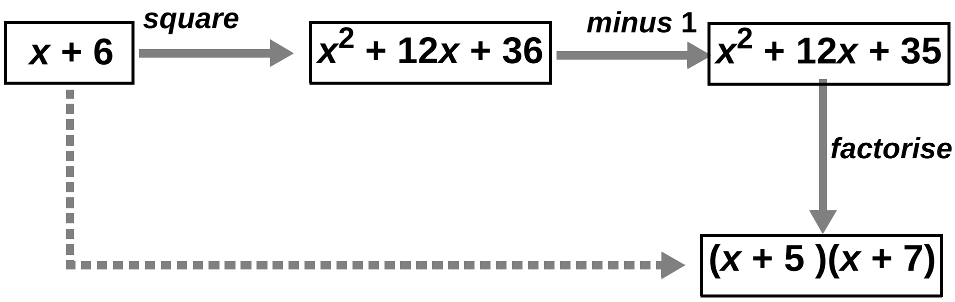
  $a^2 + 5a + 6 = (a + ...)(a + ...)$

  $p^2 + 11p + 30 = (p + ...)(p + ...)$

  $x^2 + 20x + ... = (x + 9)(x + ...)$

  $m^2 + ...\, m + 200 = (m + 20)(m + ...)$

Factorizing (II)



- Check the calculation above.

- Make a similar calculation, starting with respectively:

  $x + 4$, $y + 10$, $z + 11$, $p + 1$

- Create some other examples yourself.
- Try to give a general rule?

Formula with cubes(I)

In an old algebra book one can find this formula:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Without a real explanation, you cancarry out some tests
to investigate if the formula could be correct.

- Exchangel ***a*** and ***b***.

$(a + b)^3$ ↓ $(b + a)^3$ — *equivalent!*

$a^3 + 3a^2b + 3ab^2 + b^3$ ↓ .... — *equivalent?*

- Suppose $b = 0$

  Then: $(a + b)^3 =$ .... and $a^3 + 3a^2b + 3ab^2 + b^3 =$ ....

- Suppose $b = a$

  Then : $(a + b)^3 =$ . ... and $a^3 + 3a^2b + 3ab^2 + b^3 =$ ....

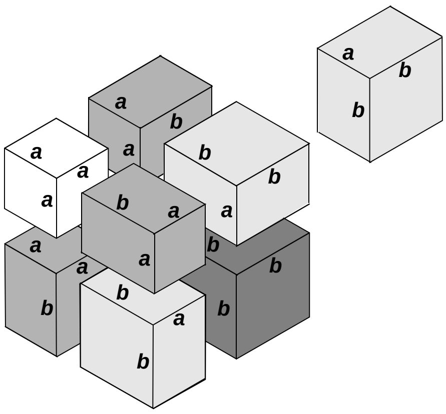
- Suppose $b = 10a$

  Then: $(a + b)^3 =$ . ... and $a^3 + 3a^2b + 3ab^2 + b^3 =$ ....

All these tests don’t give any garantee that the formula is correct,
they only give some trust.

Look at the eight blocks.

- How can they used to explain the formula at the top of this page?

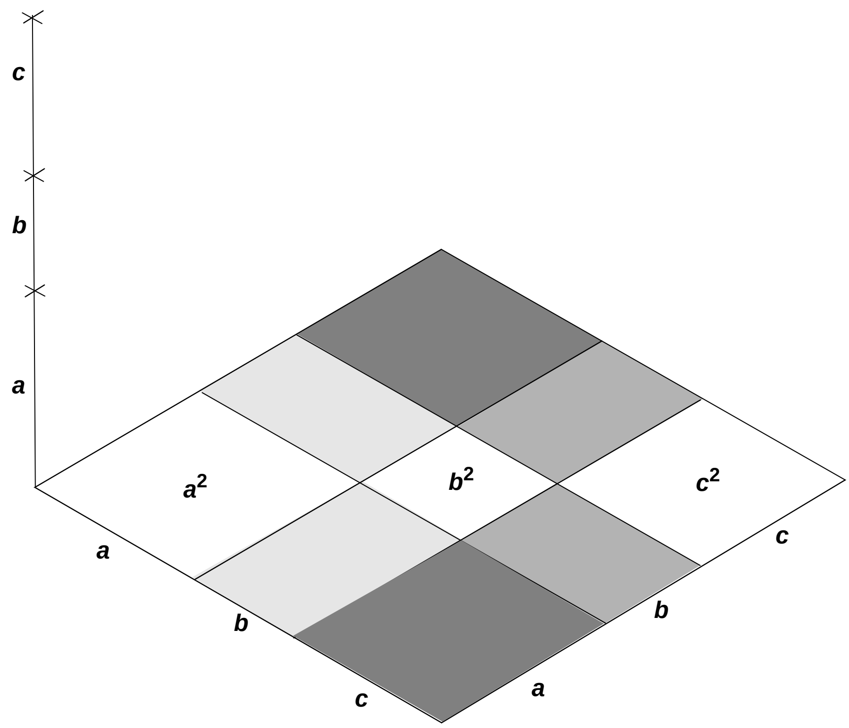


Formula with cubes (II)

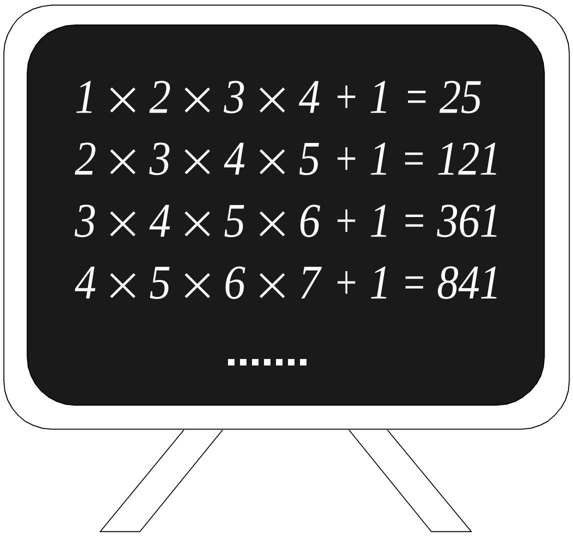
$(a + b + c)^3 = a^3 + 3a^2b +$ .... $+ c^3$

- Complete this formula.

Hint: look at the picture and imagine that it will be extended to a cube building with three floors, respectively with height ***a***, ***b*** and ***c***. On each floor are nine 'rooms'. You can find an expression for the volume of each room......



A remarkable pattern



$$1 \times 2 \times 3 \times 4 + 1 = 25$$
$$2 \times 3 \times 4 \times 5 + 1 = 121$$
$$3 \times 4 \times 5 \times 6 + 1 = 361$$
$$4 \times 5 \times 6 \times 7 + 1 = 841$$
....

- Check that the the four results are squares.

- How should you continue the list? Give two lines more.
  Check that the results are squares again.

The product of any four consecutive whole numbers added to 1 is a square!

- Try to prove this rule by using algebra.

  Here are two hints:
  * compare the product of the two inner factors with the product of the two outer ones;
  * look for a relationship between these products and the base of the square.