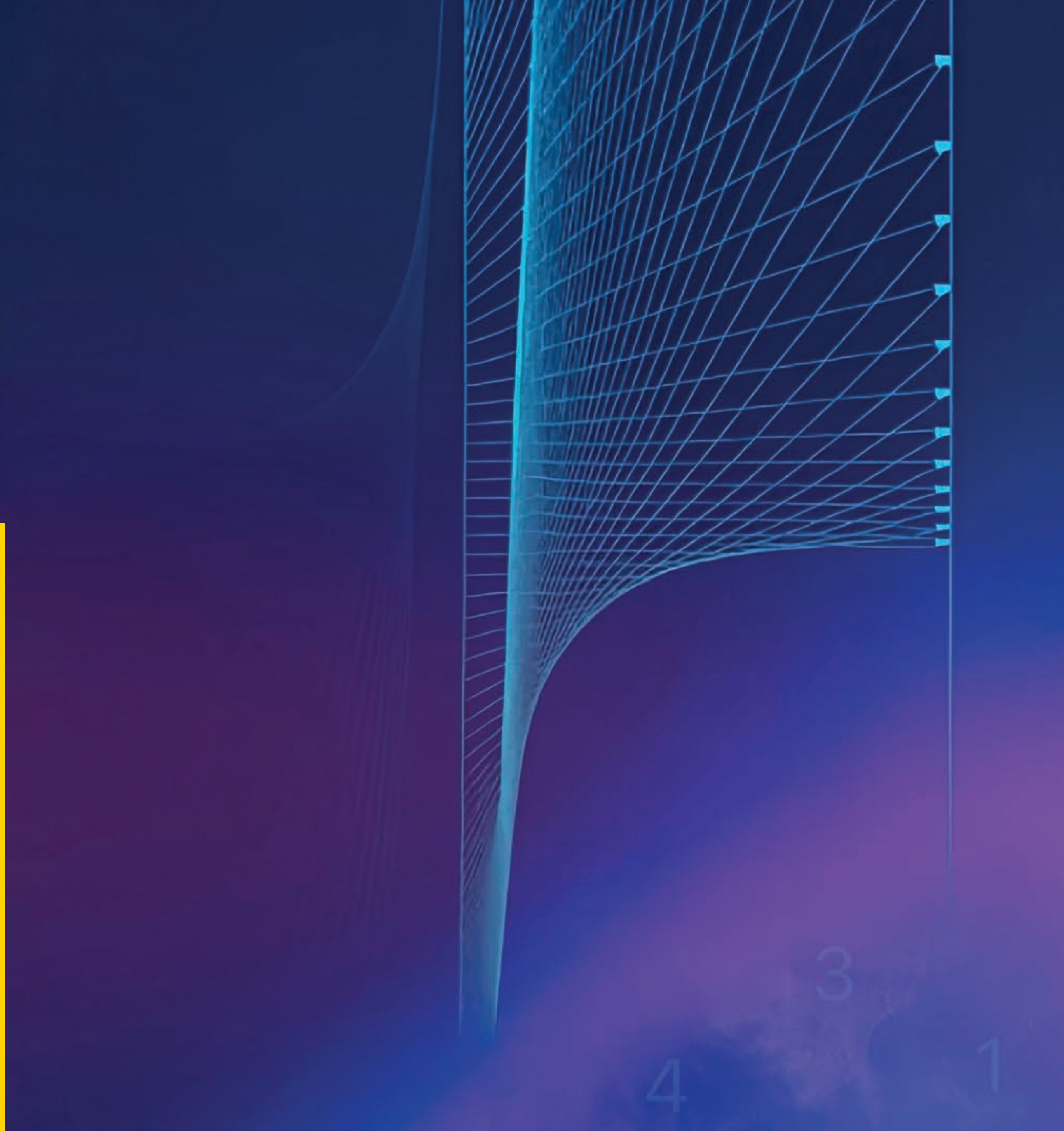

Hands-on: A digital-embodied path to functional thinking

HANG WEI

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Hands-on: A digital-embodied path to functional thinking

Hands-on: Een digitaal-belichaamde route naar functioneel denken

met een samenvatting in het Nederlands

Proefschrift

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Chapter 1 Introduction

Picture this: bustling markets, sacks of grains being measured and exchanged, or officers calculating how to divide harvests equitably. This was real-life mathematics for ancient traders and scholars, dealing with functional relationships (Li, 1987; Martzloff, 2007). A concrete example is captured in the 'Sumi problem' (Figure 1.1), which portrays a proportional conversion process. It details how to convert between different types of grain measured in consistent units. The functional relationship can nowadays be expressed mathematically as $\text{husked rice} = \frac{\text{millet} \times 3 \times 10}{5}$, where the equation ensures precision by accounting for measurement units (斗 and 升). By framing the problem as in the text, scholars could help traders and officers standardize conversions, ensuring consistency and accuracy across different scenarios. In practice, increasing the amount of millet by a factor k results in husked rice increasing by the same factor k, as expected from a direct proportionality. This functional approach also allowed for scalability and application of the same reasoning to different grain quantities. The underlying proportional reasoning shown here forms a foundation of what we now call functional thinking: identifying functional relationships between quantities and expressing them systematically.

1.1 Functional Thinking

Functional Thinking (FT) is a way of seeing the world. It manifests in specific aspects, including input-output thinking, covariation, correspondence, and mathematical object views. Input-output thinking helps students identify and apply functional rules in everyday contexts, such as calculating the total cost of cell phone bills with an input–output chain. Mathematical tools, e.g., the Algebra Arrows applet (Doorman et al., 2012) may support this. Covariation thinking supports understanding in physics, where concepts like velocity and acceleration depend on the simultaneous variation of time and position (Confrey & Smith, 1995). Correspondence thinking enables students to map functional relationships across multiple representations, such as graphs, tables, and formulas (Doorman et al., 2012; Pittalis et al., 2020). Mathematical object thinking views functions as objects with distinct representations and properties. These representations, e.g., nomograms, tables, graphs, formulas, or natural language, offer unique perspectives on a function's character. Recognizing functions as mathematical objects enables students to perform higher-order operations such as composition, transposition, and differentiation. Given the target education level of this thesis, the aspect of object formation is minimally emphasized.

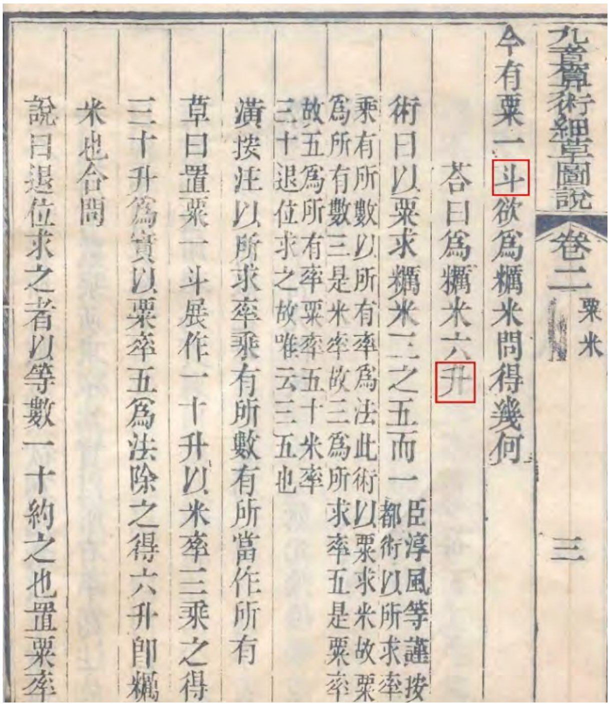
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Figure 1.1 Sumi problem: Grain distribution

In a world increasingly driven by data and mathematical models, the ability to think functionally is an important life skill. From understanding the slope of a line to modeling complex real-world phenomena, FT forms the foundation of many modern scientific and technological advances. For example, functional relationships underlie complex models for the spread of diseases. Such models rely on understanding how variables like infection rates, recovery rates, and population size interact dynamically over time (Brauer et al., 2009). Moreover, FT is also important for fostering other mathematical thinking skills, such as pattern recognition, dynamic reasoning, and the ability to generalize mathematical concepts. Pattern recognition helps students discern consistent relationships between variables (Cobb & Steffe, 1983). Dynamic reasoning builds on this by emphasizing how changes in one variable affect another (Thompson & Carlson, 2017), especially when time is involved as a parameter to reason both quantitatively and qualitatively (Keene, 2007). FT also acts as a prerequisite for generalization in some cases and lays the foundation for algebraic reasoning even in early grades (Blanton & Kaput, 2011). For instance, recognizing that doubling one quantity results in doubling another within proportional relationships (e.g., if $f(x) = 3x$, and x doubles from 2 to 4, y doubles from 6 to 12) supports the ability to generalize this pattern to all linear functions $f(x) = k \cdot x$. This early exposure to functional relationships fosters an intuitive grasp of algebraic structures. In this sense, FT provides the conceptual framework to analyze specific cases and abstract general rules.

Developing FT presents significant challenges. A common barrier is students' limited understanding of functions, often restricting them to proportional or linear relationships. This restricted view may prevent them from grasping more complex relationships such as exponential or quadratic models, which are fundamental to modeling real-world phenomena (Ellis et al., 2016). Furthermore, research shows that students often fail to connect different representations of functions, such as graphs, tables, and equations (Panasuk & Beyranevand, 2010). This difficulty limits their ability to flexibly interpret functional relationships and apply mathematical reasoning in different scenarios. Moreover, traditional teaching methods tend to focus on static representations, such as paper-based graphs or symbolic equations. While these representations are valuable, they lack the dynamic, interactive qualities that help students build robust, flexible understandings (Günster & Weigand, 2020). The rise of digital tools has made it possible to provide students with dynamic explorations of functional relationships. For instance, seeing how the graph of a function changes in real-time as one adjusts its parameters can illuminate the concept of trigonometric function in ways that static graphs cannot (Shvarts & van Helden, 2021). The role of digital technology in supporting FT will be discussed in detail in section 1.4.

1.2. Nomograms

There are many different ways to present functions, including natural language, equations, graphs, tables, and, slightly less common, nomograms. Nomograms, also referred to as arrow graphs or parallel axis representation, have been used as a visualization tool in mathematics to represent functions. A nomogram consists of two parallel number lines connected by arrows that represent a functional relationship. Specifically, these arrows map values on the input axis (x) to corresponding values on the output axis ($f(x)$), usually at regular intervals (Figure 1.2).

The history of this approach traces back to the mid-20th century, with significant contributions from various mathematicians and educators. Richmond (1963) used arrow graphs as a means to illustrate fundamental calculus concepts, such as continuity, derivatives, and composition of functions, using two parallel lines to represent the domain and range. He emphasized the pedagogical advantages of this representation, noting how it simplifies complex ideas like the chain rule and derivatives through visual intuition. Building on Richmond's work, Brieske (1978) expanded the use of mapping diagrams to demonstrate properties of continuous functions and derivatives. He highlighted their potential in helping students visualize the concepts that traditional Cartesian graphs struggle to convey, such as composition and continuity of composite functions. In 1996, Bridger refined mapping diagrams further, advocating for their use in fostering a dynamic perspective of functions. He introduced animated mapping diagrams, which allow users to interactively explore the relationship between input and output values, which innovatively introduced a process-oriented view of functions as associations rather than static graphs.

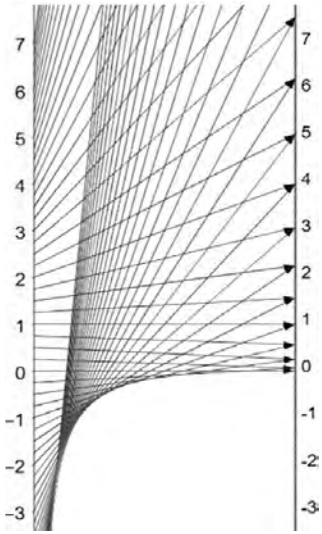

Figure 1.2 Nomogram representing the function $f: x \to x^2$

Note. This nomogram was created in GeoGebra by Rogier Bos (https://www.geogebra.org/m/snqnbnzz).

Closer to the field of secondary education, similar uses of nomograms like Parallel Axes Representations (Nachmias & Arcavi, 1990), and DynaGraphs (Goldenberg et al., 1992; Sinclair et al., 2009) have gained attention as fruitful methods for teaching and learning functions. These representations enable students to experience functions as processes rather than static entities. For example, students can manipulate one variable and immediately observe its effect on the output through DynaGraphs. Figure 1.3 shows how nomograms can be used to visualize relationships between variables. Figure 1.3a shows two different nomograms. The top one represents an increasing function with two convergent arrows, such as $f_1(x) = \frac{x}{2}$. Here, if $x_1 > x_2$, then $f_1(x_1) > f_1(x_2)$. The bottom nomogram represents a decreasing function with two intersecting arrows, such as $f_2(x) = -\frac{x}{2}$. If $x_3 > x_4$, $f_2(x_3) < f_2(x_4)$. Figure 1.3b compares two nomograms to their corresponding Cartesian graphs: $f_3(x) = |x|$, and $f_4(x) = -2x + 2$.

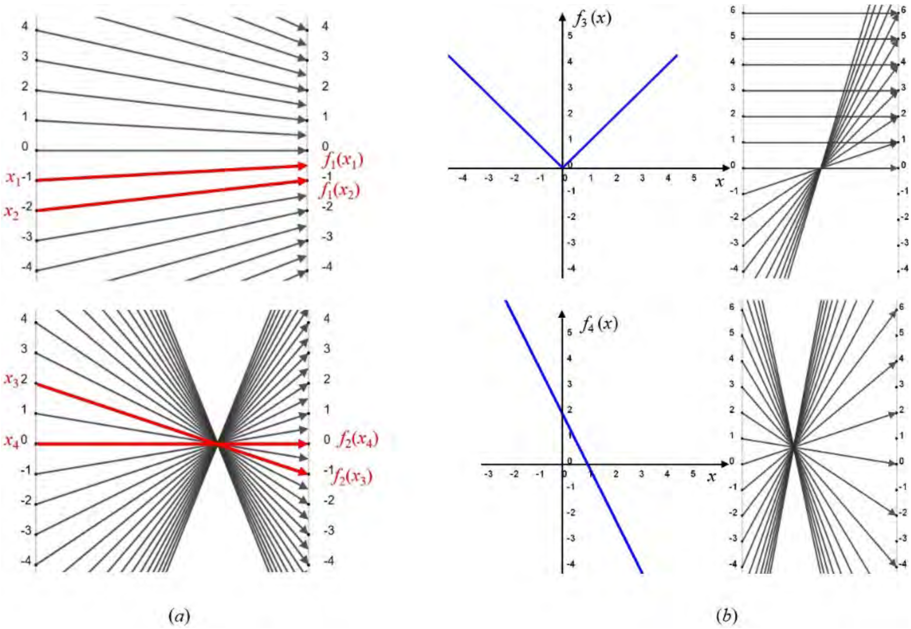

Figure 1.3 Nomograms representing different functions

In the context of this thesis, nomograms play a central role in exploring FT. This explicit visualization allows students to observe how input values are transformed into outputs through the function. As shown in Figure 1.4, a nomogram in a digital-embodied learning environment highlights this mapping process. Students can drag the two points vertically to adjust their values on the respective number lines and plot the target function. When the two points are placed correctly on the respective number lines according to the target function, the connecting arrow turns green, and its trace remains visible. However, if the placement of the two points does not accurately represent the target function, the arrow between them will turn red. In all, the use of nomograms provides students with a hands-on, interactive way to explore functions.

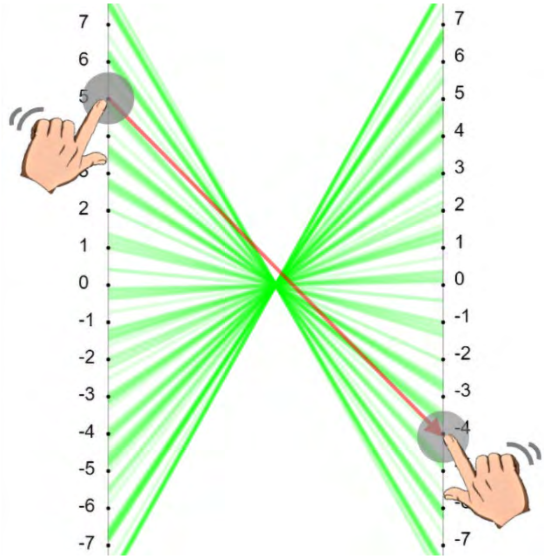

Figure 1.4 A digital-embodied nomogram representing $f: x \rightarrow -x$

1.3 Embodied learning

One of the challenges in mathematics education is helping students connect abstract concepts—often represented by symbolic notation—and the tangible experiences that give these concepts meaning. Embodied design provides a potential solution to these challenges by integrating physical actions with cognitive processes (Barsalou, 1999; Lakoff & Núñez, 2000; Abrahamson & Lindgren, 2014). According to this perspective, the body is not just a vessel for the brain; instead, cognition emerges through interactions between the mind, the body, and the artifact (Shvarts et al., 2021). Rather than treating perception, action, and reasoning as isolated processes, embodied cognition emphasizes the seamless integration of these elements. When students use their bodies to explore mathematical relationships (e.g., moving objects, gesturing, or physically enacting patterns), they create sensorimotor experiences that support their understanding of abstract concepts.

Building on these theoretical underpinnings, embodied design involves creating learning experiences that intentionally incorporate action and perception as central components of conceptual development (Abrahamson & Lindgren, 2014; Shvarts & Abrahamson, 2021). Embodied design involves creating learning experiences that intentionally incorporate action and perception as central components of conceptual development (Abrahamson & Lindgren, 2014). Instead of viewing abstract mathematical concepts as purely symbolic, embodied design situates learning in tasks that include students' bodily engagement, such as gesture, whole-body movement, or tangible manipulation. For example, in an augmented reality sandbox (Figure 1.5a), students adjust the steepness of planes while observing a rolling marble's trajectory, deepening their understanding of gradients and rate of change (Bos et al., 2022). Additionally, gesture-based learning, such as using hand movements to represent the sides of geometric shapes, facilitates comprehension of properties like symmetry and congruence. Similarly, whole-body movement activities, like walking along a number line taped on the floor (Vollmuller et al., 2023), help students grasp numerical relationships and operations through physical enactment. These embodied interactions create iterative loops between perception and action and further support the development of body potentialities for the designed tasks. For example, research suggests that two-hand movements can assist in understanding sine graphs. The movements clarify the connection between the arc length of a unit circle and the x-coordinate of the corresponding point on the emerging sine graph (see Figure 1.5b, Shvarts et al., 2021). While embodied tasks do not necessarily lead to full comprehension of these concepts, they provide students with sensorimotor experiences that can support their understanding and reasoning.

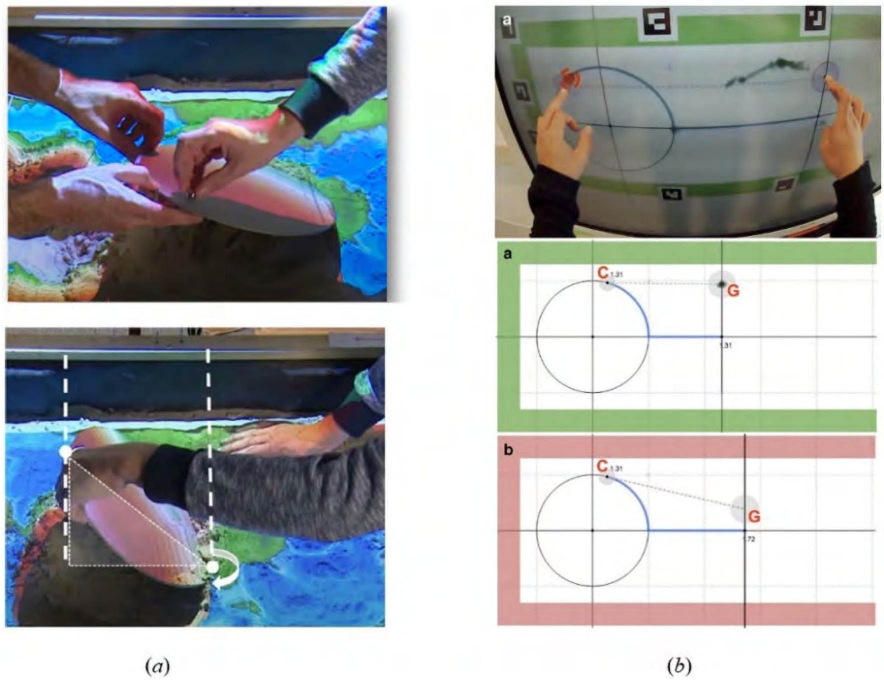

Figure 1.5 Embodied tasks (Bos et al., 2022; Shvarts et al., 2021)

1.4 Digital technology in the classroom

In recent years, the rapid integration of digital technologies in education has opened up new ways for exploring abstract mathematical ideas like functions. In today's classrooms, interactive, multi-touch screens and motion-sensing tools provide new opportunities for students to engage physically with mathematical concepts. These tools enable real-time feedback, foster collaborative learning, and help students visualize abstract ideas in a dynamic and intuitive way (Drijvers & Sinclair, 2024; Engelbrecht & Borba, 2024; Moreno-Armella et al., 2008). For example, students can use digital environments, such as GeoGebra or Numworx, to manipulate graphs dynamically, observing real-time changes as they adjust parameters or variables (Brown, 2015; Falcade et al., 2007; Lindenbauer, 2019; Rolfes et al., 2020; Shvarts & van Helden, 2023). This not only replicates but also amplifies the tangible experience of traditional physical instruments: students actively pull, drag, and transform mathematical objects on screen, embodying the relationships in a dynamic way.

Similarly, motion-sensing tools, like hand trackers or controllers integrated into AR and VR environments, allow students to map their physical movements to coordinate points on a graph (Malaspina & Malaspina, 2020). For instance, an AR application can generate a function graph based on the algebraic expressions, two-dimensional designs, or oral descriptions, allowing students to observe slopes at different points, rotate or flip the whole graph, or experiment with scaling (del Cerro Velázquez et al., 2021; Schutera et al., 2021). In an AR sandbox study (Bos et al., 2022), students were invited to roll a marble down a plane, while adjusting the plane's orientation and steepness. The AR sandbox can project real-time height lines onto the plane with the marble's trajectory perpendicular to these lines. Studies suggest that such embodied, spatial experiences can strengthen students' conceptual understanding by connecting abstract representations to real-world movement and observation (Bujak et al., 2013). By designing tasks that integrate digital and physical interactions, we can make the abstract nature of functions more accessible and intuitive. Students can therefore see, feel, and experiment with abstract concepts. Tools like the nomogram provided a tangible bridge between the physical and abstract when we embedded it in a digital-embodied learning environment.

Much has been achieved in using digital technology and embodied methods to make mathematical concepts more accessible, yet gaps remain—particularly around the process of "abstracting" the mathematical structures from these embodied experiences. Existing research emphasizes that although physical engagement can anchor conceptual understanding, students may struggle to identify which aspects of their bodily actions or spatial manipulations correspond to algebraic symbols and geometric features (Abrahamson et al., 2020; Bartolini & Martignone, 2020; Wittmann et al., 2012; Alibali & Nathan, 2013). In the context of covariational reasoning, for instance, embodied experiences can provide a concrete foundation for understanding how variables vary together. However, students may find it difficult to generalize these experiences into a broader understanding of functional relationships that can be expressed symbolically. This transition involves shifting from a procedural understanding—where concepts like functions or geometric figures are experienced through actions or measurements—to a more structural understanding, where these concepts can be represented and manipulated within formal symbolic systems.

Therefore, my PhD journey seeks to address this challenge by leveraging the potential of digital-embodied learning environments, which provides the rich intuitions generated by sensorimotor experience while systematically guiding students toward abstract mathematical reasoning. In doing so, I stand on the shoulders of giants, drawing connections between past innovations and future possibilities.

1.5 Overview of the thesis

This study addressed the research question: *How does hands-on work with nomogram tasks foster students' FT development in a digital-embodied learning environment?* By designing and implementing nomogram tasks in the digital-embodied learning environment, we aimed to uncover the mechanisms through which these interactive experiences support the growth of students' FT. To achieve this aim, four sub-studies form the body of the thesis (Figure 1.6).

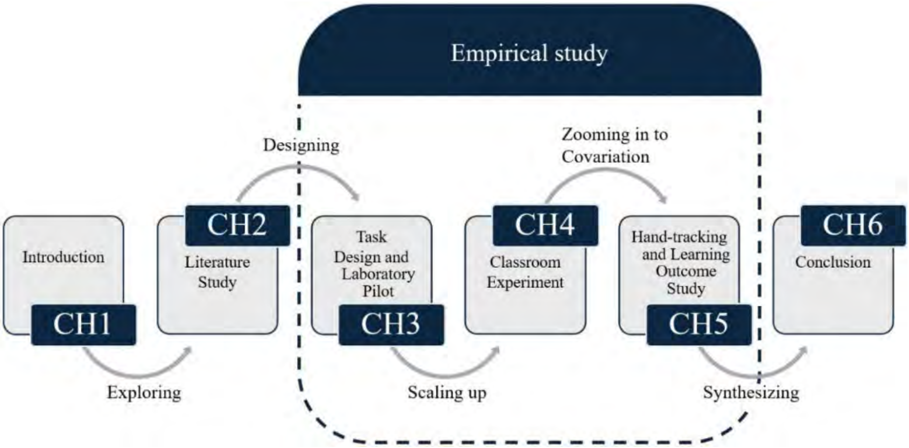

Figure 1.6 An overview of the chapters

Chapter 2: Systematic Literature Review

The first sub-study involved a systematic review of the existing literature on embodied approaches to fostering FT using digital technology. The review was conducted using databases such as Scopus, ERIC, and Web of Science, to answer the following research question and sub-questions:

- **RQ 1** How does research literature inform an embodied approach to FT using digital technology that invites abstraction?
- **RQ 1.1** Which role of technology is widely used in developing functional thinking?
- **RQ 1.2** What is known about different abstraction stages of functional thinking?
- **RQ 1.3** Which embodied approaches can be identified in the literature on developing functional thinking?

This study included a detailed analysis of 51 studies, categorizing them based on the types of embodied design (action-based, perception-based, and pseudo-embodiment) and their applications in fostering different aspects of FT (input-output, covariation and correspondence). The findings provide a comprehensive understanding of the current state of research and highlight gaps that this PhD project aims to address, such as the lack of empirical studies on the effectiveness of embodied designs in secondary school settings, limited exploration of how specific types of embodied tasks foster different aspects of functional thinking, and the need for frameworks that link embodied cognition to digital technology integration.

Chapter 3: Design of Learning Environments and Laboratory Pilot

The second sub-study focused on the iterative design and development of a digital-embodied learning environment. Drawing on theoretical frameworks such as embodied cognition and emergent modeling, tasks were designed to address three key aspects of FT: input-output, covariation, and correspondence. The study centered on the research question and sub-questions:

RQ 2	How can an embodied design using nomograms foster functional thinking?
RQ 2.1	How does a light ray context foster the students' meaning making of nomograms?
RQ 2.2	How do bimanual movement tasks foster covariational thinking?
RQ 2.3	How do different function representations and their conversions support a correspondence view on functions?

The design process followed a three-stage embodied design framework and involves creating hypothetical learning trajectories (HLTs), which outline the hypothetical learning processes and conceptualizations for each task. These HLTs are refined through iterative cycles of implementation, analysis, and redesign, using students' exploration, and interview data are analyzed to ensure alignment with the learning goals. Two main types of embodied tasks were developed: action-based tasks, which emphasize physical manipulation to ground mathematical concepts, and perception-based tasks, which leverage visual and sensory experiences to support learning. A laboratory-based experiment was conducted with two pairs of Grade 9 students to pilot the digital-embodied learning environment. The study involved a 90-minute intervention, followed by a 15-minute interview to gather students' reflections, which further informed subsequent redesigns.

Chapter 4: Classroom Implementation and Impact

The third sub-study investigated the implementation and impact of the digital-embodied learning environment in a classroom setting. The research question and sub-questions for this study were:

RQ 3	How can an embodied design using nomograms foster functional thinking in a classroom setting?
RQ 3.1	How does a digital-embodied design using nomograms affect the various aspects of functional thinking among students within a classroom setting?
RQ 3.2	How do the design features contribute to the development of functional thinking??

Conducted as a teaching experiment, this study included three 1-hour sessions with Grade 9 students from an international school in the Netherlands. Each session targeted specific aspects of FT: the first focused on input-output, the second on covariation, and the third on correspondence. Each session began with a brief introduction to the day's concept (5 minutes), followed by interactive tasks using tablets (45 minutes), and concluded with a whole-class discussion to consolidate learning (10 minutes). Pretests assessed baseline FT levels, while posttests measured learning gains after the final session. The classrooms, equipped with tablets, supported both individual and collaborative task exploration. This setup fostered active discussion and teamwork as students explored embodied tasks with real-time feedback. Data sources include pre and posttests, classroom observations, and mini interviews to assess improvements in students' FT levels.

Chapter 5: Hand-Tracking and Learning Outcome Study

The final sub-study zoomed in at the most noticeable aspect of FT, covariational reasoning (CR). This study investigated the following research question:

RQ 4	How do bimanual movements within a digital-embodied environment support students' covariational reasoning?

Two hypotheses guide the analysis:

Hypothesis 1	Higher levels of CR correlate with a shorter Time to fluency.
Hypothesis 2	Higher levels of learning gains in CR correlate with a longer Fluency Time Sum.

This study used hand-tracking technology to explore the relationship between students' hand movements and their CR. Both quantitative metrics (e.g., time to fluency, which measures how quickly students achieve smooth, coordinated hand movements, and fluency time sum, which reflects the total time spent in fluent coordination phases) and qualitative insights from hand-tracking trace analysis and post-task probes were analyzed. This mixed-method approach combined pre and posttests, hand-tracking data, and student reflections to comprehensively analyze how embodied interactions support CR development. The analysis framework of hand-tracking data was based on Thompson and Carlson's CR taxonomy, operationalized for the case of bimanual movement. It categorized students' CR levels from no coordination (L0) to smooth continuous covariation (L5). Students' bimanual movement patterns, reflected by trace continuity, coordination, and green feedback, were examined within this framework to reveal how perception-action loops and attentional anchors contribute to their improvements in CR.

Chapter 6: General conclusion and discussion

This final chapter synthesized the findings from the four sub-studies, providing a comprehensive overview of how the research contributes to the understanding and development of FT in mathematics education. By reconsidering the role of bodily movement through epistemological, ontological, and affective-cognitive lenses, the discussion highlighted both the affordances and challenges of embodied design for fostering FT. Additionally, the implications of these findings for pedagogy, digital learning artifacts, and future research directions were explored.

Chapter 2 An embodied approach to abstract functional thinking using digital technology: A systematic literature review

Abstract Embodied cognition has recently gained increasing attention in mathematics education research. However, little is known about ways to use an embodied approach to reach for mathematical abstraction. In this study, we investigate this topic, for the case of functional thinking (FT) using digital technology (DT), through a systematic literature study. We searched four databases, resulting in a corpus of 51 papers. As a result, we found that action-based and perception-based embodied approaches to contextual/situational abstraction are more prevalent in literature than pseudo-embodied approaches. In addition, the covariation and correspondence views on function are more frequently addressed compared to input-output and mathematical object views. We conclude with a discussion on the interplay of embodied approaches and abstraction in FT. For future research, we suggest investigating embodied approaches using digital technology for developing FT concerning different types of mathematical abstraction.

Keywords Functional thinking; Embodied cognition; Digital technology; Mathematical abstraction

This chapter is based on:

Wei, H., Bos, R., & Drijvers, P. (2023). An embodied approach to abstract functional thinking using digital technology: A systematic literature review. *The International Journal for Technology in Mathematics Education*, *20*(2), 75–92. https://doi.org/10.1564/tme_v30.2.2

2.1 Introduction

Since the beginning of the twentieth century, developing functional thinking (FT) skills is considered a central area of mathematics throughout primary, secondary, and tertiary education (Vollrath, 1986). Students need functional thinking—thinking in terms of relationships, interdependencies, and changes—for both later professional and daily life. FT includes understanding relations between real-world quantities, in the form of various mathematical functions. For example, public understanding of exponential growth turns out to be important to create support for measures during a pandemic, like the recent COVID-19 outbreak. In this regard, FT has received considerable attention from educators and researchers in past decades.

Technological advances have led researchers to explore novel ways to address FT in mathematics teaching and learning. A challenge in incorporating technology in mathematics classrooms is to identify and utilize various didactical functionalities of the digital technology (DT) (Drijvers et al., 2011), which will be detailed in the theoretical background section. Some more recent technologies, like touch screens, virtual/augmented reality and motion sensors, enable whole-body involvement in mathematics education. To meet the need for embodied design, Drijvers (2019) proposed an embodied instrumentation approach, offering design heuristics for embodied activities in a technology-rich environment. However, the extent to which embodiment has been explored for teaching functional thinking remains unclear. We are interested in studying the role of technology and embodiment in the teaching and learning of FT.

The development of FT progresses from concrete to more abstract notions of functional relationships. Researchers have theorized different layers and stages of abstraction in both general mathematics education and functional thinking (El Mouhayar & Jurdak, 2015; Ellis et al., 2016; Günster & Weigand, 2020; Tanişli, 2011). Treffers (1987) put forward horizontal mathematization and vertical mathematization, which can be viewed as stages of abstraction. We would like to study how students can be provided with an efficient and effective learning environment that fosters various abstraction stages of FT. In particular, we are interested to see how embodied learning environments facilitate the process of abstraction in the case of FT.

Overall, this systematic literature review addresses the following main research question: How does research literature inform an embodied approach to FT using DT that invites abstraction?

2.2 Theoretical background

This section presents four perspectives related to the literature review and ends with theoretical frameworks for this study.

Functional Thinking

The concept of FT can be traced back to the investigation of children's understanding of proportionality in functions (Inhelder & Piaget, 1958). The term "Functional thinking" was first used at the Meran Conference in 1905 (Vollrath, 1986). Researchers regard FT as a mathematical ability characterized by the following descriptions: (1) FT is a fundamental activity for working on functions (Vollrath, 1986); (2) FT is the ability to state, postulate, produce, and reproduce dependencies between variables (Freudenthal, 1983); and (3) FT is mathematical thinking on assumptions about dependency, that can be tested and, if necessary, revised (Burton, 1984).

With the development of mathematics education research, the definition of FT has diversified. For instance, an action/operational view stresses the operational and computational character of the function concept, considering a function as an input-output assignment. A dynamic view emphasizes the covariation of the dependent variable with the independent variable, or two variables depending on another one. A more static view, including the mapping view, leads to a more formal definition of function as a set of ordered pairs. Moreover, definitions of a function vary at different educational levels.

As described above, FT encompasses the process of describing, building, and reasoning about/with functions (Pittalis et al., 2020; Stephens et al., 2017). Specifically, three often-mentioned aspects of FT are (Confrey & Smith, 1995; Doorman et al., 2012; Vollrath, 1986):

1. Input-output thinking. Concerning input-output thinking, a function is regarded as an input-output assignment that helps to organize and to carry out a calculation process, in which pattern recognition related to pre-algebraic thinking is regarded as a first step.
2. Covariation and correspondence thinking. This aspect emphasizes that the dependent variable co-varies with the independent. The independent variable, while running through the domain set, causes the dependent variable to run through a range set, which includes the mapping view.

3. Mathematical object thinking. In this aspect, a function is a mathematical object which can be represented in different ways, such as arrow chains, tables, graphs, formulas, and phrases, each providing a different view of the same object.

Digital Technology

In this study, we focus on digital technologies that facilitate learning materials for FT or supports FT learning processes. Commonly used types of DT for FT include calculators or calculation software, dynamic geometry software (DGS), and computer algebra system (CAS).

Some of these technologies allow interaction of a more dynamic, interactive nature, such as dragging sliders and manipulating objects with a mouse. Studies point out that the action of dragging in DGS can potentially assist students in understanding dependencies in constructions through the recognition of mathematical invariances (Monaghan & Trouche, 2016). The most common draggable objects are sliders and (representations of) geometric objects (e.g., lines and points). When a slider is provided for a variable, the action of dragging can result in a continual reshaping of the figure according to the corresponding variable value (Lagrange & Psycharis, 2014).

Aside from dragging with a mouse, DT has evolved to be much more body-oriented: multi-touch screens, augmented and virtual reality platforms, motion sensors, and gesture-recognition systems offer students rich opportunities for embodied interactions (Shvarts et al., 2021). For example, motion detectors and object detection can be employed in the classroom to learn about the graphs of functions (Ferrara & Ferrari, 2020; Nemirovsky et al., 2013).

When utilizing DT for teaching FT, a challenge lies in combining the different mathematical functionalities and didactical functionalities of the DT (Drijvers et al., 2011). Drijvers (2018) identified five mathematical functionalities of DT: algebraic work, graphing tasks, statistical analyses, calculus procedures, and geometric jobs. Clearly, the mathematical functionality of a tool is intrinsically linked to mathematics itself. Some tools can serve multiple mathematical functionalities, such as GeoGebra, which can be used for algebraic work, graphing tasks and geometric tasks. In the context of FT, we restrict our attention in this study to four mathematical functionalities of DT: Number and Algebra, Graphing, Geometry and Calculus.

Drijvers et al. (2011) proposed three main didactical functionalities of DT in mathematics education: (1) doing mathematics, (2) practicing skills, and (3) developing conceptual understanding. These didactical functionalities are major factors influencing students' mathematics achievements (Gray et al., 2010). The three didactical functionalities of DT are not mutually exclusive but are intertwined. Based on the viewpoint of Drijvers et al., Young (2017) adapted these didactical functionalities and put forward three broad categories: (1) computation enhancement technologies; (2) instructional delivery enhancement technologies; and (3) presentation and modelling enhancement technologies. Concerning the domain of the function, Günster & Weigand (2020) provided a category system 'digital technologies (DT) usage'. This system encompasses six usages of DT in terms of function learning: (1) Variation within the learning arrangement, (2) Feedback through the learning arrangement, (3) Use of sliders, (4) Creating objects, and (5) Adjusting existing objects and (6) Zooming in and out.

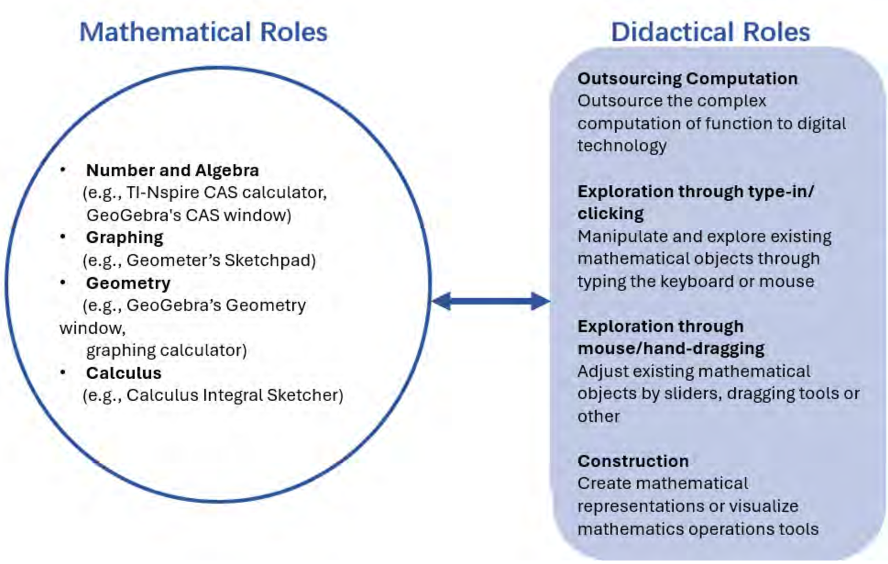

Figure 2.1 The mathematical and didactical roles of digital technology for FT

For the purpose of this study, we adapted both the mathematical functionalities and didactical functionalities to align with the particular types of digital technology. Instead of using the term "functionality", we use "mathematical role" and the "didactical role" of DT in the following text. The result of these adaptations can be found in Figure 2.1, which illustrates how this study combines the types of mathematical roles and didactical roles for our analysis.

An embodied approach to mathematics education

Several theories concern the role of the body in cognition and learning. Based on Conceptual Metaphor Theory in cognitive linguistics (Lakoff & Johnson, 1980), Lakoff and Núñez (2002) analysed the cognitive structure of mathematics, arguing that the kinds of everyday conceptual mechanisms, image schemas, aspectual schemas, conceptual metaphors, and conceptual blends are central to mathematics. Barsalou (1999) frames embodiment through grounding experiences. The role of sensorimotor experiences—perceptions, motor action, like gestures—in mathematics education has been stressed in many studies on embodied design (e.g., Abrahamson, 2016; Shvarts et al., 2021).

Abrahamson (2009) introduced a precise description of embodied design (a term first coined by Rompay and Hekkert (2001)) as a systematic and procedural design method. Initially, embodied design consisted of two types: perception-based design and action-based design (Abrahamson, 2009; Abrahamson & Lindgren, 2014). Action-based designs aim to ground mathematical concepts in students' natural capacity to adaptively solve sensorimotor problems. Perception-based designs aim to ground mathematical concepts in students' natural perceptual ability in their naive perceptions of a situation. Like the action-based genre, it is followed by a phase of reflection in which these views are developed. Additionally, Bos et al. (2022) propose another type of embodied design, incorporation-based design, in which a DT functionality is deliberately removed from the learning environment, inviting students to perform this functionality with their bodies.

For classification purposes, this paper suggests another type of embodied design—pseudo embodiment (Wei et al., 2022)—to capture embodied elements from existing studies, even if these studies do not explicitly use or mention embodied design (in Abrahamson's precise sense). In all, action-based and perception-based embodiments are relatively well established, while the incorporation-based embodiment is still at a rudimentary stage. In this study, we investigate how these four types of embodied approaches—action-based embodiment, perception-based embodiment, incorporation-based embodiment and pseudo embodiment—are involved in studies on the teaching of FT.

We focus on the following three aspects of embodied approaches: (1) Providing sensorimotor/gestural experiences, (2) Providing real-time or delayed feedback and (3) Mathematizing action-perception loops. With respect to aspect (2), we note that an embodied approach is often facilitated by feedback from DT. This often takes the form of continuous motion feedback, allowing students to discover and practice a new way of moving to provoke mathematical concepts (Alberto et al., 2021; Duijzer et al., 2019). Concerning aspect (3), we recall how Shvarts et al. (2021) theorize that mathematical knowledge emerges as part of a complex dynamic behavioral system that is constituted through multiple perception-action loops. Mathematical knowledge and meaning are then developed from those perception-action loops.

Mathematical abstraction

Piaget et al. (1966; 1977) distinguished two types of abstraction, which are empirical abstraction and reflective (or reflecting) abstraction. Empirical and reflective abstraction are contrasting but not completely exhaustive (Moessinger & Poulin-Dubois, 1981). In empirical abstraction, the processes are embedded in reality: shape, length, angle and so on. In comparison, reflective abstraction is drawn not from the object that is acted upon, but from the action itself.

To connect general abstraction with mathematical abstraction, we introduce horizontal mathematization and vertical mathematization (Treffers, 1987). Horizontal mathematization refers to the transformation from realistic problems to mathematical problems, which is the process of expressing contextual problems as mathematical problems. This process can be further divided into two types of abstraction: contextual/situational abstraction and referential abstraction (Gray & Tall, 2007; Mitchelmore & White, 2007; Gravemeijer & Stephan, 2002). Vertical mathematization refers to the mathematization after horizontal mathematization, which is the transformation process from specific mathematical problems to abstract concepts and methods. Vertical mathematization can be regarded as a reflective abstraction that requires the reconstruction of learners' mathematical concepts. This is completely different from empirical abstraction and can be further classified into two types of abstraction: particular abstraction and general abstraction (Blanton et al., 2005; Gray & Tall, 2007; Mitchelmore & White, 2007; Stephens et al., 2016).

Drawing from these theories, in this study, we set up a Function-Abstraction-Matrix (Table 2.1) consisting of two main dimensions: Abstraction stages and functional thinking. One dimension outlines the major abstraction stages that students may follow when abstracting in mathematics. The other dimension emerges from the three often-mentioned aspects of FT: Input-output, Covariation and correspondence, Mathematical object. We distinguish the following four stages of abstraction:

1. Contextual and situational abstraction: recognize properties of functions in real-life experience/contexts/situation. For example, recognize the linear relationship between the height of a plant and growth time in plant growth situations.
2. Referential abstraction: extract the properties of a function by abstracting from real-life situations, or their representations (e.g., ratio table), but not yet completely separated from the situation. Students refer to a situation by means of a given mathematical representation or using non-mathematical utterances. For example, students can use given function formulas/graphs to represent and model a distance-time relationship.
3. Particular abstraction: use graphs, symbols, and formulas to represent one particular decontextualized function/functional relationship. So, using mathematics as the new context rather than the real-life context, the mathematical representations are the "world in which we are" and as such, replace the initial context/situation. For example, identify one particular linear functional relationship using a formula, without a very general mathematical scope attached to it yet.
4. General abstraction: vertically mathematises general functions/functional relationships and constructs new structures. More on classes of mathematical objects and their properties and relations. For example, construct a composite function 'find $f(2x)$ if $f(x) = x^2 + 1$'.

Table 2.1 The function-abstraction-matrix

	Functional thinking		
Abstraction stages	Input-output	Covariation and correspondence	Mathematical object
Contextual and situational abstraction			
Referential abstraction			
Particular abstraction			
General abstraction			

The theoretical perspectives in this section enable us to present an integrated framework to study our overall research question. To thoroughly investigate this topic, we will focus on three research questions, each examining different aspects of the main question. First, we will determine the prevalent role of technology in fostering functional thinking:

RQ 1 Which role of technology is widely used in developing functional thinking? By understanding which mathematical and didactical roles are prevalent, we aim to uncover the potential relationships between these roles and their potential effective use within an embodied approach to functional thinking using digital technology.

RQ 2 What is known about different abstraction stages of functional thinking? Understanding the various abstraction stages associated with functional thinking enables us to recognize how an embodied approach might influence them. This understanding can help us tailor the embodied approach to effectively facilitate the progression through different abstraction stages, ultimately promoting a deeper and more nuanced grasp of FT.

RQ 3 Which embodied approaches can be identified in the literature on developing functional thinking? Exploring the embodied approaches found in the literature allows us to identify effective strategies and best practices for integrating embodiment and functional thinking in a DT-based learning environment. This knowledge can guide the design of an embodied approach that not only effectively supports functional thinking but also encourages abstraction, ensuring a comprehensive and engaging learning experience.

2.3 Method

To address the research questions, we carried out a systematic literature search, followed by a content analysis.

Systematic literature search

The literature search was conducted in four databases: ERIC, PsycINFO, Scopus, and Web of Science, and we searched for relevant studies published in peer-reviewed journals and written in English. Duijzer et al. (2019)

mentioned that some articles may not (yet) mention embodied cognition as the main or related theory but still apply its core characteristics. This reminded us that an arbitrary date limitation would reduce the available articles, so we did not restrict the publication dates of the articles. Also, qualitative studies, quantitative studies and mixed-method studies were collected simultaneously, and the qualitative data provided a main and supportive role by providing details of learning designs, and records of the analysis. In the course of our ongoing search attempts, we defined a query including the four key notions in our research question: Functional Thinking × (Embodiment OR Abstraction/Reification OR Digital Technology). See Appendix 1 for the full query. Our initial search, conducted on December 7, 2020, yielded 397 journal articles. After deduplication, 333 unique publications remained.

Screening for articles

The article screening phase was conducted within the FunThink Erasmus+ project and consisted of three rounds (Figure 2.2): The first round started with a quick scan of the detailed information, such as the title, abstract and keywords of the article, to judge each article's relevance to each of the four aspects: Functional Thinking (FT), Embodiment (EM), Digital Technology (DT) and Abstraction (AB). At the end of this round, 177 papers—empirical as well as theoretical papers—were initially collected with the help of ten coders from the FunThink Erasmus+ project. The second round was carried out in a more rigorous manner. Fifteen coders participated in the literature appraisal round, during which each coder read the full texts and finished a spreadsheet containing the core ideas and overall appraisal of each article. This resulted in the exclusion of eighty-four articles and the final selection of ninety-three articles for our review. Finally, we directly eliminated the articles coded 0 to 2 as they were deemed less relevant to our review study (*n*=42). As a result, fifty-one articles are included in the final corpus. To ensure rigor in the coding and appraisal process, each article was evaluated by at least three coders using the established criteria.

Content analysis

As a method to conduct literature reviews in a transparent, systematic, and rule-governed way, content analysis requires rigorously collecting, filtering, and classifying the existing research context (Mayring, 2004). During the analysis process, all fifty-one articles from the final corpus were included. These studies were categorized into three main classes based on the coding result: (1) digital technology and functionality; (2) embodied approach and cognitive contribution; and (3) abstraction stage and functional thinking. We classified the articles in each class based on different structural dimensions and related analytic categories provided in the theoretical background section. Triangulating the bibliometric findings with expert content analysis helped to reveal the role of different aspects in FT. The classification of studies into different aspects forms the basis and structure for the presentation of results in the following section.

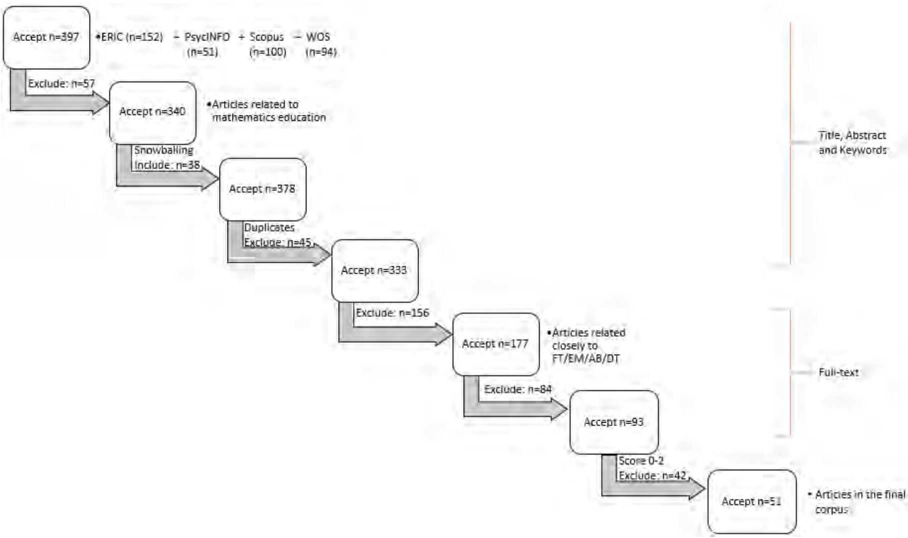

Figure 2.2 Flowchart of the literature selection

2.4 Results

This section starts with a descriptive analysis of the reviewed literature. To address our research questions, we first describe the results of the content analysis on functional thinking and the role of digital technology, its relation to abstraction and embodied approaches. Finally, we address the main topic, the interplay between embodiment and abstraction.

Descriptive analysis

The main goal of the descriptive analysis conducted in this study is to identify the occurrence distribution of each aspect (digital technology, mathematical abstraction, and embodied approach) as well as the current research trends. As evident from Figure 2.3, DT is the most frequent category, with embodiment being relatively new in its applications, and the category of abstraction the least, as not many studies explicitly mention both FT and abstraction.

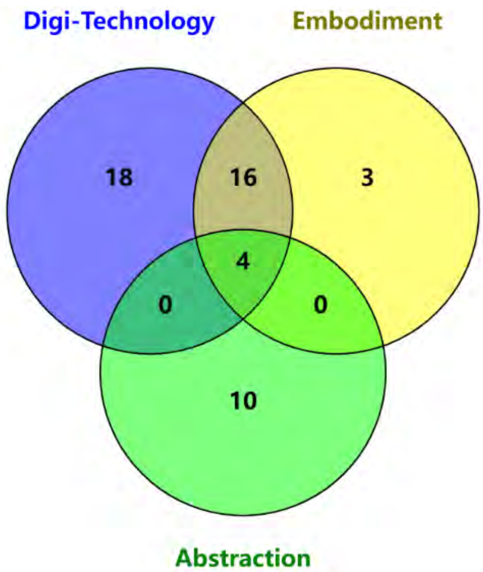

Figure 2.3 Venn diagram of the reviewed articles

The distribution of these fifty-one reviewed articles over publication year is shown in Figure 2.4, which shows that there has been an upward trend in the number of studies on the development of FT.

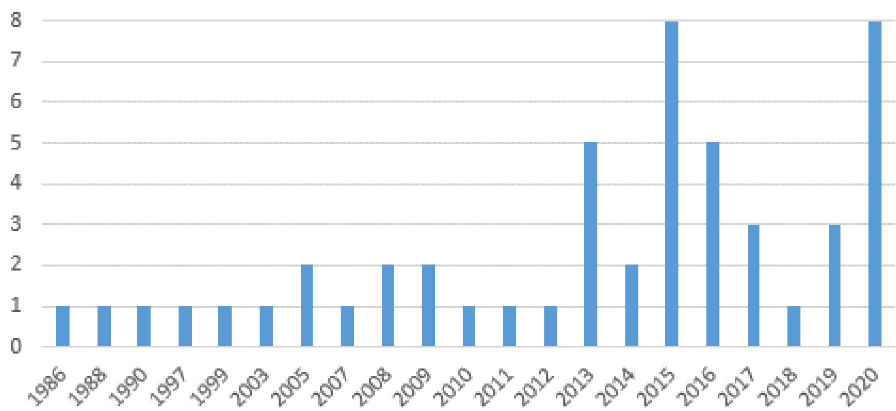

Figure 2.4 Distribution of publication year of the reviewed articles

With respect to the FT dimension, Figure 2.5 reveals that the reviewed studies tend to present only one aspect of functional thinking in one article. Only twelve (out of fifty-one) studies provide overlaps between different aspects of functional thinking. Among these overlaps, the covariation/correspondence thinking aspect and the mathematics object thinking aspect appears to be linked together, while the input-output thinking aspect seems more detached from the other two aspects.

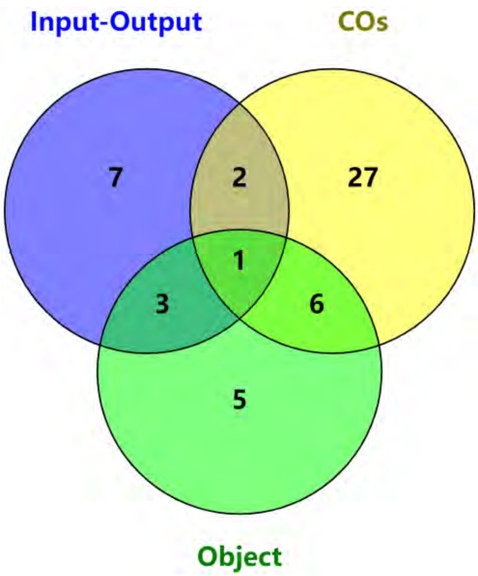

Figure 2.5 The relationship of functional thinking aspects

Functional Thinking and the role of digital technology

The mathematical and didactical roles of digital technology are analyzed in this section for the case of Functional Thinking. Figure 2.6 shows the heatmap of didactical and mathematical roles of DT. Among the four mathematical roles, the role of graphing has been widely used to develop FT. Algebra accounts for a substantial part of the mathematical role. As for the roles of geometry and calculus for FT, only a limited number of studies have mentioned and discussed this. Regarding the didactic role, the majority of studies encompass exploration and construction roles. The algebra role provides the opportunity to outsource computation but has not been thoroughly investigated in terms of the potentials offered by mouse or hand-dragging. This issue will be further explored in relation to the use of embodied approaches. Excluding the role of outsourcing computation, the graphing role appears in a relatively balanced distribution across the other didactic roles in the reviewed papers, suggesting a strong connection between FT development and graphing designs.

From the didactical role dimension, *Turtleworlds* allows *Exploration through mouse clicking and dragging*. From the mathematical role dimension, this software can provide the opportunity for *Geometry for FT*. In addition, Ferrara & Ferrari (2020) used *WiiGraph* software to engage pairs of students with functions through graphing motion, and one of their tasks, named *Line option for a+b.* The role of this technology can be classified as *Exploration* and *Construction* using *Graphing* for functional thinking. The graphing motion technology, which allows working with pairs of "positions over time"-graphs, provides students with the opportunity to observe in real-time the graph of the sum of two functions on the screen. So, in this case, in the task *WiiGraph* the roles of technology are *Graphing for FT*, *Exploration* and *Construction* of mathematical representations.

Altogether, the results indicate that the most common mathematical role of digital technology in the study of functional thinking is graphing, and the most frequent didactical role is allowing for exploration and construction.

Functional Thinking and abstraction

This section aims to provide an extensive overview of the status of research in the domain of functional thinking and abstraction stages. We classified major themes and issues in FT research. Besides the fourteen articles in the abstraction class, we also analyzed and positioned the other thirty-seven articles in the final corpus based on the Function-Abstraction-Matrix (Table 2.1).

As can be seen in the heatmap (Figure 2.7), the fact that cells near the diagonal of the matrix have a darker color suggests that there is a correlation between the abstraction stages and aspects of FT in the selected papers. The advanced FT aspects come with the advanced abstraction stages. Thirteen out of fifty papers only focus on one stage of abstraction/one aspect of FT. The others each cover more than one abstraction stage and/or more than one aspect of FT. For example, one teaching trajectory that recurs frequently is *Contextual/situational abstraction* to *Referential abstraction* to *Particular abstraction*.

Didactical Role of DT / Mathe-matical Role of DT	Outsourcing Computation	Exploration through type-in/clicking: manipulate and explore existing mathematical objects through typing the keyboard or mouse	Exploration through mouse/hand-dragging: adjust existing mathematical objects by sliders , dragging tools or other operations tools	Construction: Create mathematical representations or visualise mathematics
Algebra for functional thinking	Hong et al., 2015 (TI-Nspire CAS calculator); Brown, 2015 (graphing calculator with GridPic); Arzarello et al., 2005; Lagrange & Psycharis, 2014 (Casyopée)	Hong et al., 2015 (TI-Nspire CAS calculator); White, 2009 (Code Breaker); Getenet & Beswick, 2014 (Microsoft Mathematics); Lagrange & Psycharis, 2014 (Casyopée & Turtleworlds); Günster & Weigand, 2020; Heid et al., 2013; McCulloch et al., 2020 (mapping); Ogbonnaya, 2010 (Graphmatica); Jon, 2013 (TI-Nspire CX CAS handheld); Godwin & Beswetherick, 2003; Moore et al., 2019; Doorman et al., 2012 (AlgebraArrows); Paz & Leron, 2009 (DrScheme environment); Ayers et al., 1988 (UNIX shell scripts and pipes); Leinhardt et al., 1990	Lagrange & Psycharis, 2014 (Casyopée & Turtleworlds); Günster & Weigand, 2020; Heid et al., 2013; Doorman et al., 2012 (AlgebraArrows)	Hong et al., 2015 (TI-Nspire CAS calculator); White, 2009 (Code Breaker); Getenet & Beswick, 2014 (Microsoft Mathematics); Lagrange & Psycharis, 2014 (Casyopée); Günster & Weigand, 2020; Heid et al., 2013; Ogbonnaya, 2010 (Graphmatica); Lagrange & Psycharis, 2014 (Turtleworlds); Godwin & Beswetherick, 2003; Doorman et al., 2012 (AlgebraArrows); Paz & Leron, 2009 (DrScheme environment)
Graphing for functional thinking		Stylianou et al., 2005; Brown, 2015 (graphing calculator with GridPic); Ogbonnaya, 2010 (Graphmatica); Getenet & Beswick, 2014 (Microsoft Mathematics); Jon, 2013 (TI-Nspire CX CAS handheld); Lagrange & Psycharis, 2014 (Turtleworlds); Godwin & Beswetherick, 2003;	Nemirovsky et al., 2013; Ferrara & Ferrari, 2020; Nemirovsky et al., 1997; Stylianou et al., 2005; Lagrange & Psycharis, 2014 (Turtleworlds); Botzer & Yerushalmy, 2008 (Micro-computer-Based Laboratory); Hoffkamp, 2011; Lindenbauer,	Nemirovsky et al., 2013; Ferrara & Ferrari, 2020; Nemirovsky et al., 1997; Brown, 2015 (graphing calculator with GridPic); Ogbonnaya, 2010 (Graphmatica); Getenet & Beswick, 2014 (Microsoft Mathematics); Lagrange & Psycharis, 2014

		Doorman et al., 2012 (AlgebraArrows); Leinhardt et al., 1990	2019; Liang & Moore, 2020; Günster & Weigand, 2020; Heid et al., 2013; Rolfes et al., 2020; Falcade et al., 2007; Caglayan, 2015 (gestural expression); Moore et al., 2019; Doorman et al., 2012 (AlgebraArrows)	(Turtleworlds); Godwin & Beswetherick, 2003; Günster & Weigand, 2020; Falcade et al., 2007; Caglayan, 2015 (gestural expression); Moore et al., 2019; Eu, 2013; Roux et al., 2015; Doorman et al., 2012 (AlgebraArrows); Leinhardt et al., 1990
Geometry for functional thinking		Abrahamson et al., 2016; Lagrange & Psycharis, 2014 (Turtleworlds)	Lagrange & Psycharis, 2014 (Turtleworlds); Hoffkamp, 2011; Lindenbauer, 2019; Liang & Moore, 2020; Günster & Weigand, 2020; Rolfes et al., 2020; Falcade et al., 2007; van den Heuvel-Panhuizen et al., 2013 (Hit the target)	Stylianou et al., 2005; Lagrange & Psycharis, 2014 (Turtleworlds); Günster & Weigand, 2020; Falcade et al., 2007; Venter. 2020 (Image Functions Intervention)
Calculus for functional thinking			Miranda & Sánchez, 2019; Swidan et al., 2020 (Calculus Integral Sketcher)	Swidan et al., 2020 (Calculus Integral Sketcher)

Figure 2.6 Heatmap of the didactical roles and mathematical roles of DT

Functional thinking / Abstraction stages	Input-output	Covariation/ correspondence	Mathematical Object
Contextual/situational abstraction	Ellis et al., 2016; Lagrange & Psycharis, 2014 (Turtleworlds); Paz & Leron, 2009; Brown, 2015; Roux et al., 2015; McCulloch et al., 2020; Ferrara & Ferrari, 2020 (body movement); Arzarello et al., 2005; Doorman et al., 2012; van den Heuvel-Panhuizen et al., 2013	Ellis et al., 2016; Hitt & González-Martín, 2015; Nemirovsky et al., 2013 (hand movement); Nemirovsky et al., 1997 (move the contour analyzer by hand); Vollrath, 1986 (release a ball from slope); Abrahamson et al., 2016 (hand movement); Pinto & Canadas, 2018; Stephens et al., 2016&2017; Best & Bikner-Ahsbahs, 2017; Johnson et al., 2017; Pittalis et al., 2020; Leinhardt et al., 1990	McCulloch et al., 2020 (mapping); Botzer & Yerushalmy, 2008 (orthogonal components); Venter, 2020 (mapping)
Referential abstraction	Ellis et al., 2016; Roux et al., 2015; White, 2009; Ferrara & Ferrari, 2020; Arzarello et al., 2005; Doorman et al., 2012	**Ellis et al., 2016; Hitt & González-Martín, 2015; Nemirovsky et al., 2013; Nemirovsky et al., 1997; Ellis & Grinstead, 2008; Abrahamson et al., 2016; Liang & Moore, 2020; Moore et al., 2019; Blanton et al., 2015; Pinto & Canadas, 2018; Stephens et al., 2016&2017; Ellis et al., 2015; Best & Bikner-Ahsbahs, 2017; Johnson et al., 2017; Pittalis et al., 2020; Ferrari-Escoláet al., 2016; Leinhardt et al., 1990**	McCulloch et al., 2020 (mapping); Botzer & Yerushalmy, 2008 (orthogonal components); Venter, 2020 (mapping)
Particular abstraction	Ellis et al., 2016; Jon, 2013; White, 2009; Heid et al., 2013; Doorman et al., 2012	**Ellis et al., 2016; Swidan et al., 2020; Lagrange & Psycharis, 2014; Thompson et al., 2017; Ogbonnaya, 2010; Özgün-Koca, 2016; Getenet & Beswick, 2014; Miranda & Sánchez, 2019; Roux et al., 2015; Hong et al., 2015; Nemirovsky et al.,**	Swidan et al., 2020; Lagrange & Psycharis, 2014; Hong et al., 2015; Ayers et al., 1988; Botzer & Yerushalmy, 2008;

		2013; Ellis & Grinstead, 2008; Stylianou et al., 2005; Abrahamson et al., 2016; Günster & Weigand, 2020; Doorman et al., 2012; Rolfes et al., 2020; Hoffkamp, 2011; Falcade et al., 2007; Lindenbauer, 2019; Liang & Moore, 2020; Schwartz & Hershkowitz, 1999; Moore et al., 2019; Eu, 2013; Blanton et al., 2015; Pinto & Canadas, 2018; El Mouhayar & Jurdak, 2015; Best & Bikner-Ahsbahs, 2017; Ferrari-Escoláet al., 2016	Günster & Weigand, 2020; Godwin & Beswetherick, 2003; Schwartz & Hershkowitz, 1999; Doorman et al., 2012
General abstraction		Swidan et al., 2020; Lagrange & Psycharis, 2014; Doorman et al., 2012; Schwartz & Hershkowitz, 1999	**Swidan et al., 2020; Lagrange & Psycharis, 2014; Hong et al., 2015; Ayers et al., 1988; Caglayan, 2015; Günster & Weigand, 2020; Heid et al., 2013; Doorman et al., 2012; Falcade et al., 2007; Schwartz & Hershkowitz, 1999**

Figure 2.7 Heatmap of Functional Thinking aspects and abstraction stages

In the example of Abrahamson et al. (2016), the Mathematical Imagery Trainer helps students develop an understanding of proportional equivalence and get an initial insight into a covariation relation. Students are asked to keep the screen green by moving their hands in a fixed-interval gesture, from which the asked covariation relationship is presented. Results show that students discovered, enacted, and stated the relationship between two constant speeds (e.g., the left hand rises 1 unit per the right hand's 2-unit rise). Furthermore, students can use the rates of their hand motions to deduce that a fixed-interval rule is incorrect and that, instead, the distance between the hands must increase with height. This can lead to a further understanding of the continuous variation of co-variables. The understanding of the covariation aspect can derive from the sensorimotor experience without mathematical representations related to proportion, which also promotes contextual/situational abstraction. Therefore, this study is categorized in the cell *contextual/situational abstraction – covariation*.

Next, in Davis's study (2013), with the help of TI-Nspire™ CX CAS, students can develop an understanding of an input-output process with some input (independent variable), operating on it using some rules, and getting an output (dependent variable). For example, when exploring a property of quadratic functions, namely the x-intercept of the top, a parameter is input as a variable (a), another is input as a constant (b), and the x-intercept of the top (h) is the output (Figure 2.8). In this investigation, students make changes to a, which causes changes to h, the x-coordinate of the vertex. In this case, the task is based on a purely mathematical scenario without any real-life context, and the goal of this task is to investigate the property of only one type of function, quadratic. So, it belongs to the *particular abstraction – input-output* cell.

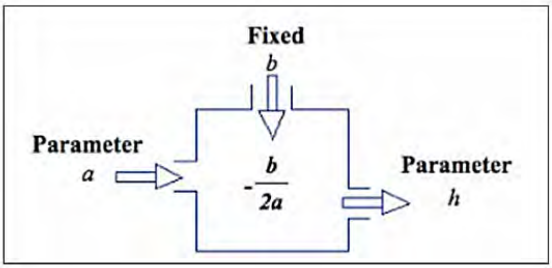

Figure 2.8 The parameter machine accepts a variable *a* and a fixed *b* and produces *h* (Davis, 2013, p. 6)

Functional Thinking and embodied approaches

In terms of embodied approaches used in reviewed studies, the pseudo-embodied approach shows considerable potential for providing students with FT-related sensorimotor experiences and helping to mathematise action-perception loops (Figure 2.9). Most reviewed studies adopted what we classify as an action-based approach to foster functional thinking. In contrast, studies in the genres of perception-based and incorporation-based approaches concerning FT so far are rare. As for the contributions of embodied approaches, providing sensorimotor/gestural experiences and mathematising action-perception loops seem to be more dominantly present.

We illustrate our classification using the example of *Drawing in Motion* (Nemirovsky et al., 2013). The prototype exhibit in this paper requires physical engagement and collaboration between two people who jointly produce a graph on a displayed Cartesian coordinate plane through a large LCD screen. Each participant controls one cartesian coordinate of a point with a handle. The two participants jointly draw on the screen by moving the handles. Every movement of the participants is shown on the screen by a movement of the tracing point, which can be regarded as real-time feedback to some extent since participants can sense their real-time location and shift their bodies accordingly. This instrument provides a new perspective of understanding function using the embodied approach, compared to the conventional ways of thinking about functions (e.g., dynamic/process and static/structural conceptions). It also provides the possibility for young learners to engage in the understanding of mathematical functions with the help of suitable mathematical instruments and embodied experiences.

In sum, this section shows that many studies adopting pseudo embodiment provide cognitive contributions to the development of FT, whereas not many studies have explored the feedback aspect of embodied approaches.

Type of contribution / Type of embodied approach	Provide sensorimotor/ gestural experiences	Provide real-time/delayed feedback	Mathematize action-perception loops
Pseudo embodiment	Doorman et al., 2012; Lagrange & Psycharis, 2014; Swidan et al., 2020; Hoffkamp, 2011; Lindenbauer, 2019; Liang & Moore, 2020; Günster & Weigand, 2020; Heid et al., 2013; van den Heuvel-Panhuizen et al., 2013; Rolfes et al., 2020; Falcade et al., 2007; Caglayan, 2015 (gestural expression); Godwin & Beswetherick, 2003	Lagrange & Psycharis, 2014; Günster & Weigand, 2020; Heid et al., 2013; van den Heuvel-Panhuizen et al., 2013; Falcade et al., 2007	Doorman et al., 2012; Lagrange & Psycharis, 2014; Swidan et al., 2020; Hoffkamp, 2011; Lindenbauer, 2019; Liang & Moore, 2020; Günster & Weigand, 2020; Heid et al., 2013; van den Heuvel-Panhuizen et al., 2013; Rolfes et al., 2020; Falcade et al., 2007； Caglayan, 2015 (gestural expression)
Action-based embodiment	Nemirovsky et al., 2013; Paz & Leron, 2009; Arzarello et al., 2005; Stylianou et al., 2005; Ferrara & Ferrari, 2020; Abrahamson & Bakker, 2016	Nemirovsky et al., 2013; Stylianou et al., 2005; Ferrara & Ferrari, 2020; Abrahamson & Bakker, 2016	Nemirovsky et al., 2013; Stylianou et al., 2005; Ferrara & Ferrari, 2020; Abrahamson & Bakker, 2016
Perception-based embodiment	Leinhardt et al., 1990; Vollrath, 1986; Ellis & Grinstead, 2008	Ferrara & Ferrari, 2020; Nemirovsky et al., 1997	Ferrara & Ferrari, 2020; Nemirovsky et al., 1997
Incorporation-based embodiment	Botzer & Yerushalmy, 2008		Botzer & Yerushalmy, 2008

Figure 2.9 Heatmap of embodied approaches and contributions

Embodied approaches to FT using DT inviting abstraction

Let us now examine the main research question on the ways in which embodied approaches may invite abstraction for the case of FT using DT. Figure 2.10 illustrates that different embodied approaches cover the four abstraction stages of FT. The transition from *Particular abstraction* to *General abstraction* occurs most frequently in the category of pseudo embodiment. There are only a limited number of designs in the pseudo embodiment category that contain more than two abstraction stages. Here, we show a concrete example that covers all four abstraction stages with the support of a multifunctional applet, *AlgebraArrows* (Doorman et al., 2012). Students are invited to develop an understanding of the input-output aspect of FT and experience the transition from *Contextual abstraction* to *Referential abstraction* by exploring real-life contextual tasks, such as a cell phone offer task. Next, they are provided with the opportunities to investigate the relationship between one function and its family of functions (e.g., $y = 2x - 3$ in the family of $y = ax + b$), which leads to the transition from *Particular abstraction* to *General abstraction*. Compared to the other three embodied approaches, few pseudo-embodied designs initiate from the *Contextual/situational abstraction* stage. In the case of van den Heuvel-Panhuizen et al. (2013), *Hit the target* provides students with sensorimotor experiences by letting them drag the arrows (using a mouse) to the bow and hit the target on the screen. As part of this task, students experience how the scores covary with their movement, and therefore develop covariational thinking from *Contextual/situational abstraction*.

It is noteworthy that none of the designs adopting action-based, perception-based and incorporation-based approaches cover the *General abstraction* stage. In these learning designs, students are invited to experience and explore given structures, but not to construct new mathematical structures while solving tasks. For instance, when using *WiiGraph*, a graphing motion technology, students work with pairs of positions over time graphs to explore how particular inputs lead to different outputs. Students' movements become the inputs, and function graphs are the outputs. During this activity, students can progress from a process of *Contextual/situational abstraction* to *Referential abstraction* and even on to *Particular abstraction.* They search for the functional rule and relationship between their movements and feedback from the software.

Abstraction stages / Type of embodied approach	Contextual/situational abstraction	Referential abstraction	Particular abstraction	General abstraction
Pseudo embodiment	Doorman et al., 2012; Lagrange & Psycharis, 2014; van den Heuvel-Panhuizen et al., 2013	Doorman et al., 2012; Liang & Moore, 2020	Doorman et al., 2012; Lagrange & Psycharis, 2014; Swidan et al., 2020; Hoffkamp, 2011; Lindenbauer, 2019; Liang & Moore, 2020; Günster & Weigand, 2020; Heid et al., 2013; Rolfes et al., 2020; Falcade et al., 2007; Godwin & Beswetherick, 2003	Doorman et al., 2012; Lagrange & Psycharis, 2014; Swidan et al., 2020; Günster & Weigand, 2020; Heid et al., 2013; Falcade et al., 2007; Caglayan, 2015
Action-based embodiment	Nemirovsky et al., 2013; Paz & Leron, 2009; Arzarello et al., 2005; Abrahamson & Bakker, 2016; Ferrara & Ferrari, 2020	Nemirovsky et al., 2013; Arzarello et al., 2005; Abrahamson & Bakker, 2016; Ferrara & Ferrari, 2020	Nemirovsky et al., 2013; Stylianou et al., 2005; Abrahamson & Bakker, 2016; Ferrara & Ferrari, 2020	
Perception-based embodiment	Leinhardt et al., 1990; Vollrath, 1986; Ferrara & Ferrari, 2020; Nemirovsky et al., 1997	Ferrara & Ferrari, 2020; Nemirovsky et al., 1997	Ellis & Grinstead, 2008; Ferrara & Ferrari, 2020;	
Incorporation-based embodiment	Botzer & Yerushalmy, 2008	Botzer & Yerushalmy, 2008		Botzer & Yerushalmy, 2008

Figure 2.10 The interplay of digital-embodied design and abstraction stages

In summary, the interplays between embodied approaches and abstraction processes suggest that there are differences between the pseudo embodiment category and the other three categories, such as the initial stage and end stage of abstraction. For the studies using pseudo embodiment, *Particular abstraction* is typically the initial stage, ending up with the *General abstraction* stage. As for action-based, perception-based and incorporation-based embodiments, *Contextual/situational abstraction* is the most common initial stage, but the *General abstraction* stage is not well-addressed in these three categories.

2.5 Discussion

Overview of the main findings

We carried out a systematic literature search, followed by a content analysis. In this section, we summarize the results based on three research questions and discuss the overall research topic.

> **RQ 1** Which role of the technology is widely used in developing functional thinking?

We distinguished two types of roles of digital technology, mathematical roles and didactical roles. For functional thinking, mathematical roles include Algebra for FT, Graphing for FT, Geometry for FT, and Calculus for FT. Didactical roles include Outsourcing Computation, Exploration (through type-in/clicking and mouse/hand-dragging), and Construction. From the literature, we conclude that Graphing and Algebra for FT are the most widely used mathematical roles, while there has been less attention on Geometry for FT. Concerning the didactical roles, a remarkable proportion of studies employ the *Exploration* and *Construction* roles for developing FT. The combination of *Algebra* role and *Exploration through mouse/hand-dragging* has not been investigated as much. Regarding the combination of different mathematical roles and didactical roles, most studies include more than one role.

The Mathematical roles of Graphing and Algebra commonly appear in GeoGebra, Geometer's Sketchpad and other software that support dynamic visualizations for functions. The tasks presented in this review study invite the development of students' functional thinking, mainly its covariation and correspondence aspects (Falcade et al., 2007; Lindenbauer, 2019). For example, some tasks support students in observing and exploring the influence of parameters on function graphs (Brown, 2015; Falcade et al., 2007; Davis, 2013; Lindenbauer, 2019). The designers of these tasks argue that dynamic visualizations are significantly more beneficial for learning than static ones. They mention other advantages of dynamic visualizations, such as the possibility to create more interesting learning environments, and giving real-time feedback to students (Lindenbauer, 2019; Rolfes et al., 2020; Roux et al., 2015).

A major trend in theory development and application pertains to teaching mathematics using DT. The evolution of existing theories highlights certain approaches, such as the Instrumental Approach, which is particularly relevant for fostering FT with DT. This approach examines the role of digital tools in shaping mathematical understanding and problem-solving processes (Sinclair et al., 2022). Digital technology offers students access to algebraic, graphical and numerical representations, and facilitates understanding of the relationship and transitions between different representations (Günster & Weigand, 2020). Several of our reviewed studies have highlighted the central role of DT in facilitating the transition between different representations, such as from the covariation aspect in the geometrical setting to the symbolic representation of functions (Brown, 2015; Heid et al., 2013; Lagrange & Psycharis, 2014; Ogbonnaya, 2010). In addition, DT provides students with intuitive access to graphical representations (Rolfes et al., 2020; Roux et al., 2015).

RQ 2 What is known about different abstraction stages of functional thinking?

We identified four main stages of abstraction in FT, including Contextual/situational abstraction, Referential abstraction, Particular abstraction, and General abstraction. We noticed that addressing advanced FT aspects comes with reaching for advanced abstraction stages. A substantial proportion of studies links up the Covariation/correspondence thinking aspect and the Mathematical object thinking aspect, while the input-output thinking aspect seems to be more independent of the other two functional thinking and has been investigated less.

As mentioned previously, there are three main aspects of FT: Input-output, Covariation and correspondence, and Mathematical object. This categorization is hierarchical in character, in the sense that we believe the three aspects also suggest a learning trajectory. Normally, students get their first introduction to FT from an input-output assignment that stresses the operational and computational characters, and then they start to engage in and recognize the dynamic process of covariational/correspondence reasoning. In the end, students view a function as a mathematical object with its own representations and properties (Doorman et al., 2012; Frey et al., 2022; Günster & Weigand, 2020; Hoffkamp, 2011). Accordingly, the internal hierarchy, combined with the evolutionary abstraction stages, suggests that higher FT levels and more advanced abstraction occur in relation to each other. Moreover, we note that some teaching trajectories, in accordance with different abstraction stages, reveal the possible levels of developing FT. Pittalis et al. (2020) describe students' functional-thinking modes, which consist of recursive patterning, covariational thinking, correspondence-particular, and correspondence-general factors. These levels or modes and our abstraction-function matrix all provide either detailed or concise descriptions for interpreting abstraction stages.

RQ 3 Which embodied approaches can be identified in the literature on developing functional thinking?

Within embodied design for FT, we distinguished the genres of Action-based embodiment, Perception-based embodiment, Incorporation-based embodiment, and Pseudo embodiment. We conclude that action-based embodiment is the most common approach, while perception-based and incorporation-based embodiment are rare. Moreover, although studies that use pseudo embodiment do not rigorously follow embodied design principles, they do provide perceptual or kinesthetic experiences and action-perception loops. Considering the cognitive contribution of embodied approaches, we found that providing sensorimotor/gestural experiences and mathematising action-perception loops are the most frequent contributions, while less consideration is given to the feedback aspect.

One notable tool used in the pseudo-embodiment category is the slider. There are two different settings of sliders: a) continuous slider (free movement on a bar without restriction), and b) discrete slider (static selection of particular values). A common use of a slider is to connect it to parameters controlling a family of functions. For example, students set up sliders for a, b, and c in the standard formula of a quadratic function, and manipulate sliders to keep a record of the coordinates of the vertices of the parabola (Davis, 2013).

The types of feedback from digital technology, real-time feedback and delayed feedback can influence embodied approaches. The core point of embodied design is whether there is real-time feedback (with mathematical meaning) on the movement. In this reviewed study, some tasks with real-time mathematical feedback support students' understanding of function concepts. In addition, some studies point out that feedback from representations on the screen might help students recognize their misconceptions and overcome them through additional interactions with the digital tool. Also, feedback may motivate students and evoke a curiosity that enables them to learn more effectively when receiving real-time feedback from the tool (Ogbonnaya, 2010; Özgün-Koca, 2016).

The main goal of this study is to investigate the interrelationship between digital technology, mathematical abstraction and embodied approaches. Sinclair et al. (2022) highlight that emerging trends in theory development challenge traditional binaries, such as the mind-body binary. In the context of teaching mathematics with DT, this shift has led to theoretical elaborations that attend to embodied ways of knowing, which can be essential for fostering FT through the use of digital tools.

The review results show some common configurations of task design in terms of abstraction stages for FT and digital-embodied approaches, such as the transition from Particular abstraction to General abstraction in the pseudo embodiment. These transitions in most action-based and perception-based embodiments cover Contextual/situational abstraction, Referential abstraction, and Particular abstraction. However, less attention has been paid to General abstraction in the categories of action-based, perception-based, and incorporation-based embodiments.

We believe that the transition between different abstraction stages can be addressed through embodied designs. Moreover, the distinction of different abstraction stages and different aspects of FT, as shown in the abstraction-function matrix, can inform the design of learning trajectories. Some studies report that students have difficulties representing the function independently from the mathematical context (e.g., a geometrical context) from which it arises (Hoffkamp, 2011; Miranda & Sánchez, 2019). In this case, real-life contexts providing *Contextual/situational abstraction* play a key role in inspiring students and sparking their imagination through embodied experiences or other real-life experiences.

As for *General abstraction* of FT, studies from the pseudo-embodiment category address it through the transition from *Particular abstraction* to *General abstraction*, such as the transition between different functions and transition within a family of functions (Günster & Weigand, 2020). The process of mathematics learning is intertwined with sensorimotor and perceptual aspects of using mathematical tools. Students are able to form abstract concepts through enacting bodily movements and to give meaning to mathematical representations (e.g., symbols, formulas, graphs) by invoking sensorimotor and perceptual patterns (Nemirovsky et al., 2013).

Limitations of the present study

Some limitations of our study are worth noting. Regarding the literature selection, only articles published in English were selected, and inclusion criteria in terms of FT may have led to the exclusion of some articles addressing primary education. Future studies may look at more detailed aspects of functions from the corpus at different educational levels. Additionally, the first round of coding was done in a large group, which could endanger coding uniformity, even if clear instructions were provided.

Implications and future directions

Our systematic literature review led to an Embodiment-Abstraction-Matrix, which outlines approaches and stages for fostering FT through DT. This matrix holds relevance for both teaching practices and future research.

In light of our engagement in embodied design, we believe that the combination of different representations should be conceptually related to FT. The primary consideration is aligning body movements with learning content. A major characteristic of embodied design involving DT is providing immersive interfaces to stimulate sensorimotor activity. When designing embodied tasks, it is essential to translate complex and abstract learning content into concrete body movements associated with input-output, covariation and correspondence, and mathematical object aspects of FT. Learning content should be presented in a visual, accessible and manipulable way, enabling students to perceive it through their bodies during the activity. Furthermore, encouraging students to explain and verbalize their action-perception experiences can promote understanding to their peers (Flood et al., 2020). Different learning content may also result in different roles for DT. DT offers opportunities for interaction and automated feedback, fostering the covariation and correspondence aspect of FT.

Notably, we posit that the sensorimotor/gestural experiences from embodied activities can supplement input received from other modalities (e.g., vision), allowing students to construct richer multimodal representations and facilitate more complex understanding (Drijvers, 2019). Studies involving action-based embodiment and perception-based categories provide students the opportunity to graph functions with their body movements (Ferrara & Ferrari, 2020; Nemirovsky et al., 2013). Graphing motion technology can lay the groundwork for mathematising action-perception loops by offering bodily foundations for mathematical concepts.

Drawing inspiration from the action-based task for proportion (Alberto et al., 2021), a nomogram can be used as an additional representation of FT (Friendly, 2008; Nachmias & Arcavi, 1990). Body movements employed while exploring nomograms enable students to experience the relationship between two variables, such as involving two hands or arms representing changes in two variables.

We also notice that feedback timing in embodied design is a subtle matter. Cognition is time-pressured and must be understood in terms of its functionality while interacting with the environment in real time (Wilson, 2002). We suggest that higher-level cognition can be developed with the aid of continuous real-time feedback from the learning environment. Further research is necessary for a more elaborated learning environment incorporating real-time feedback.

This study demonstrates the potential of embodied approaches, with or without DT, for developing FT in terms of mathematical abstractions. We hope our insights into the categories of embodied approaches and the main mathematical abstractions for FT will provide teachers, researchers, and curriculum designers with a spark of inspiration.

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Appendix 1: Query and Filters

Query and Filters

Functional thinking	("Function thinking" OR "Function reasoning" OR "Function relation" OR "math Function " OR "covariation reasoning" OR "Function approach" OR "thinking function") AND
Embodiment	(embod OR enactment OR sensorimotor OR kines OR perception OR action-perception OR "body motion" OR "physical experience" OR "physical participa") OR
Abstraction/Reification	(abstracti OR reification OR "math abstract" OR encapsulation OR "object formation" OR "concept imag" OR visualization) OR
Digital technology	("digital technolog" OR "digital tool" OR "physical tool" OR "ICT tool" OR ICT OR GeoGebra) AND
Domain	(math OR "math education" OR "math instruction" OR "physical science" OR science OR stem OR "teaching method" OR education OR learning) AND
Filter(s)	English language In SCOPUS and Web of Science, the limitations were set to journal articles and conference proceedings. In ERIC, the limitations were set to journal articles and peer-reviewed articles.

Supplementary Material

IO-A1:

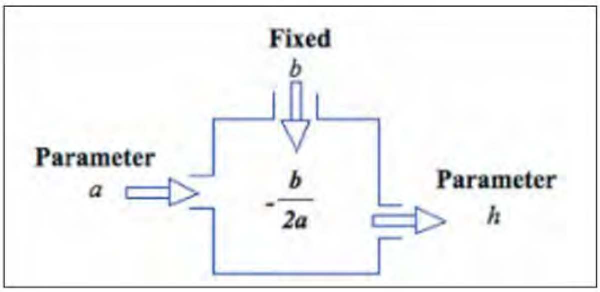
Ellis, 2016: Students' understanding of exponentiation can be developed from a repeated multiplication model in a scenario that Jactus grew by doubling its initial height every week. In this example, students can regard exponential function as an *input-output machine*, given a number and doing exponentiation with multiplication model. The plant growth context provides students with perceptual experience to facilitate *contextual/situational abstraction*.

IO-A2:

Ferrara & Ferrari, 2020: WiiGraph allows students to work with couples of positions over time graphs to explore how particular inputs lead to different outputs. Based on their observation of the '*a*+*b*' activity, two students moved in front of the sensor with a third line appearing in real-time on the screen, which is produced by adding the values of *a* and *b* over time. Students recognize that the result of '*a*+*b*' equals '*c*', that is, the graph of the third line is the sum of the movements of two people. Students' movements become the input and function graphs play as the output. The relationship' *a*+*b*=*c*' is the *input-output machine* in this case. And the process that students recognize and identify the relationship with given algebraic representation and given graphic representation using perceptual experience is *referential abstraction*.

IO-A3:

Jon, 2013: With the help of TI-Nspire™ CX CAS, students can get an understanding of the process involved at the beginning with some input (independent variable), operating on it through some rule, and producing an output (dependent variable). For example, when exploring the property of quadratic function (the x-intercept of the vertex), a parameter is input as a variable (*a*), another is input as a constant (*b*), and the x-intercept of the



vertex (h) is the output. In this investigation, students are making changes to a, which causes changes to h, the x-coordinate of the vertex. It is based on a pure mathematical scenario without any real-life context. And the goal of this task is to investigate the property of only one type of function, quadratic functions. So, it belongs to the *particular abstraction* stage.

(Davis, 2013, p.6)

COs-A1:

Abrahamson et al., 2016: The Mathematical Imagery Trainer helps students develop an understanding of proportional equivalence and initial insight of covariation relation. Students are able to make the screen green by putting their hands in a specific-interval gesture, which is the asked covariation. Students discovered, enacted, and stated the covariation of two constant rates (e.g., the left-hand rises 1 unit per the right hand's 2 unit rise). Furthermore, students can use the differing rates of their hand motions to deduce that a fixed-interval rule could be incorrect and that, instead, the distance between the hands must increase with height. That can lead to a further understanding of the continuous variation of co-variables. The understanding of the covariation aspect is based on the experience from action which also promote *contextual/situational abstraction*.

COs-A2:

Johnson et al., 2017: As a sequence of tasks for engendering covariational reasoning, Ferris wheels task and filling bottles task can promote transfer of covariational reasoning. The 'simpler' attributes of function, such as height and distance from Ferris wheels task, could prepare students more readily for further attributes, such as volume and height from a filling bottles task. Both tasks provide students with real-life contexts and the Ferris wheels task allows students to adjust the point (car) on the Ferris wheels and observe corresponding changes in the graph of function (width/height vs. distance). The given dynamic graphs and real-life situations are helpful for doing referential abstraction.

COs-A3:

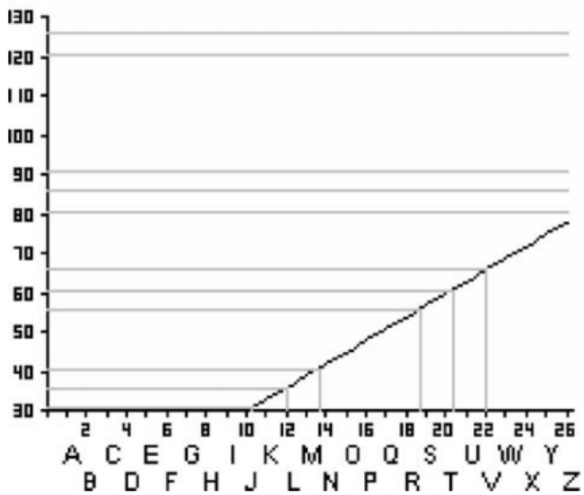
Günster & Weigand, 2020: A linearity task in GeoGebra environment was designed. The dynamic representations from learning environments help students to explore the relationship between side length and perimeter of polygons. When dragging the slider and adjusting the length of sides, students can observe how much the perimeter changes if the side length changes by 1 cm. This task uses a pure mathematical context asking students to investigate covariation relationship between variables, that is, linear relationship between side length and perimeter of polygons.

COs-A4:

Lagrange & Psycharis, 2014: The rectangle task in Casyopée lets students explore the position of a point M so that the area of the triangle is one third of the area of rectangle. The innovative functionalities in Casyopée, automatic modeling, allow students to understand key actions in the process of modeling a geometrical dependency into an algebraic function. Students can use function as a tool to solve geometrical question by constructing mathematical model. The process of building an algebraic model for describing the covariation between y_M (as an independent variable) and the area of the triangle BMC is general abstraction.

OB-A1:

McCulloch et al., 2020: The vending machine applet draws attention to the object view of function-each input (domain) should map to one output (range). It provides a context with which students are familiar. Students are asked to identify each vending machine as a function or non-function. They can get an understanding of the concept of function through the mapping process of using the vending machine applet. Therefore, it belongs to the contextual/situational abstraction.



OB-A2:

White, 2009: The Code Breaker applet provides both horizontal and vertical trace lines stretching from the y-axis x-axis, respectively, intersecting the candidate curve. These lines can serve as both a code-breaking resource and a scaffold to help students capture the object-like properties of function. For example, some students recognized that each plaintext letter should map to a single encoded output and creatively utilized this object-like property as problem-solving resources. The cryptographic context with given graphic representation develop the referential abstraction for conceptualizing functions.

(White, 2009, p.27)

OB-A3:
Swidan et al., 2020: The Calculus Integral Sketch (CIS) displays two Cartesian coordinate systems that are dynamically linked. These two coordinate systems show the function graph and indefinite integral of the function, respectively. Students can drag the function graph upward, which leads to an increase in the inclination of the antiderivative function graph. In this case, function is regarded as a mathematical object that can be submitted to higher-order processes, integral. Students investigate the function-derivative relationship in the mathematical context. The CIS provides a supportive environment for particular abstraction.

OB-A4:
Doorman et al., 2012 & Drijvers et al., 2013: The *AlgebraArrows* applet can support the construction of input-output chains of operations as a model of a dependency relationship. Students are provided with the opportunities to explore a family of functions (object view) which lead to the general abstraction stage. For example, students can investigate a family of functions representing braking distances for three different vehicles with the help of collapsed arrow chains. This learning environment supports the transition from a calculation understanding to an object understanding of functions by displaying different mathematical representations, such as arrow chains, tables, and graphs.

Chapter 3 Developing functional thinking: From concrete to abstract through an embodied design

Abstract In addressing the challenge of fostering functional thinking (FT) among secondary school students, our research centered on the question of how an embodied design can enhance FT's different aspects, including input-output, covariation and correspondence views. Drawing from embodied cognition theory and focusing on action- and perception-based task design that uses light ray contexts and different function representations, we developed a digital-embodied learning environment, using the nomogram as a central representation. Our pilot study, involving four eighth-grade students, provided insights into their physical interactions with these modules through a multi-touch digital interface. Analysis of video and audio recordings from the pilots, including students' hand gestures and verbal expressions, was guided by comparing hypothetical learning activities with the actual learning activities. The results show that (1) a concrete light ray context enables students to ground the abstract mathematical function concept, (2) the bimanual coordinating motion tasks, incorporating the covariation aspect of FT, allows students to connect their bodily experience with function properties, and (3) our embodied and dragging tasks support insight in the conversion between nomograms and graphs of functions, encouraging students' correspondence thinking by providing multiple perspectives to understand, reason about, and manipulate the function. In conclusion, our findings suggest the potential of digital-embodied tasks in fostering FT, evident in students' diverse strategies and reasoning.

Keywords Educational technology; Embodied design; Functional thinking; Mathematics education; Nomogram

This chapter is based on:

Wei, H., Bos, R., & Drijvers, P. (2024). Developing functional thinking: From concrete to abstract through an embodied design. *Digital Experiences in Mathematics Education*, *10*(2), 323–351. https://doi.org/10.1007/s40751-024-00142-z

3.1 Introduction

A mathematical function is an abstraction—it is an abstract concept representing a relationship between input and output, irrespective of the concrete meanings of those inputs and outputs. Still, concrete contexts can make abstract functions more meaningful to students, providing opportunities to attach meaning to the mathematical constructs the students develop (Van den Heuvel-Panhuizen & Drijvers, 2020). Students need FT—thinking in terms of relationships, interdependencies, and change—for both later professional and daily life (FunThink team, 2021). One of the main challenges teachers face is how to foster the different aspects of FT, especially the abstract ideas of variation and covariation that lay the epistemological foundation for students to develop robust conceptions of functions (Thompson & Carlson, 2017).

A specific representation called a nomogram may play an important role in the development of FT. Nomograms support FT by incorporating various representations and contexts, and have been emphasized by Thompson and Carlson (2017) for their importance in incorporating number lines and uniting two quantities' values in future study. The initial use of nomograms within a digital learning environment can be traced back to Nachmias and Arcavi (1990), who termed it the Parallel Axes Representation (PAR). Various forms of this function representation exist, such as the horizontally oriented DynaGraph (Sinclair et al., 2009). In our current research, and increasingly prevalent in recent studies, we use the term 'nomogram' to describe two number lines linked by a bundle of arrows, which shows how each input number on the left number line corresponds to an output number on the right (see Figure 3.1). It is a useful tool for developing FT due to its visual nature and ability to represent functional relationships using arrows.

The integration of embodied actions within abstract mathematical thinking enriches meaning-making processes by incorporating various sensory channels such as perceptual, auditory, tactile, and kinesthetic (Radford, 2009). Bimanual movement, referring to the coordinated use of both hands, has been incorporated into mathematical education as a means to foster an understanding of mathematical concepts, such as proportion (Abrahamson et al., 2016). The importance of the bimanual movement in mathematics education originates from Piaget's work, which proposed that children's understanding of their world and the concepts within it is deeply rooted in their physical interactions with the environment. More recent work in the field of embodied cognition expands upon Piaget's theories, arguing that our understanding of abstract mathematical concepts is grounded in the physical and sensorimotor experiences of our bodies (Abrahamson & Lindgren, 2014; Tall, 2004).

In the context of our digital learning environment that features nomograms, the multi-touch and real-time feedback capabilities of digital technology (DT) enable students to investigate and construct nomograms using bodily movements, specifically a bimanual dragging motion (Figure 3.1). Correctly positioning the two points on respective number lines causes these arrows to change to green, with their trajectory remaining visible, allowing for real-time feedback akin to the principles seen in the Mathematical Imagery Trainer for Proportion (MIT-p) (Abrahamson & Trninic, 2011). By traversing the entire nomogram, the students can visualize the complete function, which may include intersecting arrows, parallel arrows, and other distinctive features. However, if the placement of the two points does not accurately represent an input-output pair, the arrow between them will turn red. Inspired by the MIT-p, our evolved digital-embodied nomograms, potentially named MIT-$f(x)$, extends to various functional relationships, making it a more versatile tool for developing FT. It achieves a range of instructional strategies, from using number lines and fostering corresponding quantities' relationship, to reinforcing the facilitation of smooth continuous covariational and correspondence reasoning by providing continuous movements on the two number lines. In this regard, the main research question guiding our study is: *How can an embodied design using nomograms foster functional thinking?* In order to explore the multifaceted nature of FT, we have formulated the following specific research questions through the lens of three aspects of FT: Input-output, Covariation, and Correspondence:

RQ 1 How does a light ray context foster the students' meaning-making of nomograms?

RQ 2 How do bimanual movement tasks foster covariational thinking?

RQ 3 How do different function representations and their conversions support a correspondence view on functions?

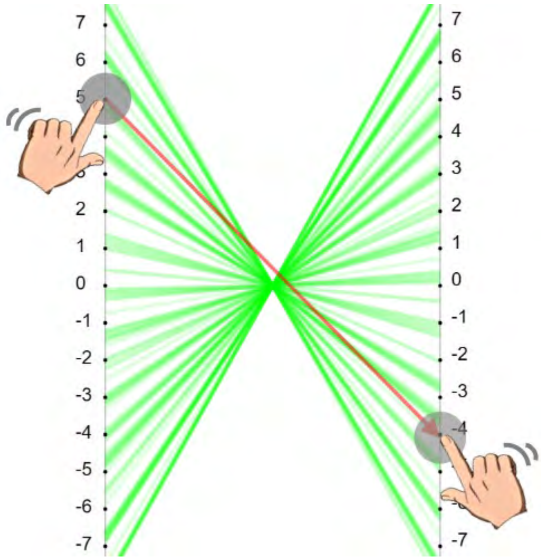


Figure 3.1 Movement on a nomogram

3.2 Theoretical framework

Two theoretical lenses—FT and Embodied Learning—serve as the foundations for our investigation in this study. By intertwining these theoretical perspectives, we aim to examine how the interactive, dynamic, and physically engaging nature of embodied learning can bolster FT.

Functional Thinking

FT, as a process of describing, building, and reasoning about/with functions (Pittalis et al., 2020; Stephens et al., 2017; Thompson & Carlson, 2017), consists of four main aspects: Input-output thinking, Covariation thinking, Correspondence thinking, and Mathematical object thinking (Confrey & Smith, 1995; Doorman et al., 2012; Wei et al., 2023; Vollrath, 1986). These four aspects indicate how to understand the concept of mathematical function through different characteristics of functions:

1. Input-output thinking: A function is regarded as an input-output assignment that helps organize and carry out a calculation process (Doorman et al., 2012). It is considered the initial stage of understanding function, especially with the help of a special representation, an arrow chain (Freudenthal, 1983). Moreover, recognizing patterns and structures are linked to this aspect. For example, the recursive pattern is seen as how to get a number in a sequence when the previous number or numbers are given (Frey et al., 2022; Stephens et al., 2017).

2. Covariation thinking: This aspect emphasizes the relationship between two variables, primarily focusing on how changes in the independent variable cause corresponding shifts in the dependent variable. The emphasis is on the simultaneous change or movement of both variables (Confrey & Smith, 1995; Doorman et al., 2012; Thompson & Carlson, 2017).
3. Correspondence thinking: It is more about the pairing relationship between the two variables and being able to represent them with multiple representations, such as arrow chains, tables, graphs, formulas, and phrases (Doorman et al., 2012). Correspondence thinking highlights that each value of the independent variable aligns with a unique value of the dependent variable. Instead of emphasizing the simultaneous change, as in covariation, correspondence thinking underscores the direct association or pairing of values between the two variables (Pittalis et al., 2020; Smith, 2008). This aspect incorporates the mapping view, facilitating a holistic understanding of functional relationships (FunThink team, 2021).
4. Mathematical object thinking: A function, in this aspect, is seen as a mathematical object with its own representations and properties. Within this aspect, a function is recognized as part of a family of functions (Sfard, 1994), subject to higher-order operations such as composition, transposition, and differentiation. Concerning the scope of our study, this aspect receives minimal emphasis.

In this paper, our focus is specifically on the first three aspects of FT. Various aspects of FT are embedded in our task based on function representations, like tables, graphs, and formulas. Some studies argue that dynamic visualizations could be significantly more beneficial for learning functions than static representations (Brown, 2015; Falcade et al., 2007; Lindenbauer, 2019, Ten Voorde et al., 2023). For example, some tasks with interactive dynamic visualizations support students in observing and exploring the influence of parameters on function graphs. In addition, there are some other advantages of dynamic visualizations, such as providing the possibility to create more interesting learning environments, and facilitating understanding of the relationship and transitions between different representations (Günster & Weigand, 2020; Lindenbauer, 2019; Rolfes et al., 2020; Roux et al., 2015). Therefore, the aforementioned interactive, dynamic digital-embodied nomogram serves as a central feature.

In addition, we considered the design heuristic of the emergent model (Gravemeijer, 1999). The emergent model includes four levels of activity: task setting, referential, general, and formal. To support the design hierarchy of our tasks, we work with emergent modelling activity in three levels: situational, referential, and general, following our previous literature study (Wei et al., 2023). The use of these three levels will be further discussed in the *Design* section with a combination of hypothetical learning trajectories.

Embodied learning

Embodied learning is an educational approach that integrates bodily movements and physical experiences into the learning process. It operates on the premise that cognition is not only confined to the mind but involves the entire body (Barsalou, 1999; Lakoff & Nunez, 2000). The theory of embodied cognition is foundational to understanding the mechanisms and efficacy of embodied learning, as it provides the theoretical underpinning for how bodily engagement can enhance cognitive processes. According to this perspective, cognitive activities such as problem solving, memory, and learning are not just abstract mental tasks but are connected to sensory-motor systems. This connection implies that physical actions and sensory-motor experiences can shape and facilitate cognitive processes (Barsalou, 2008; Glenberg, 1997). Highlighting the role of the body in cognition and learning, embodied learning has gained quite some attention in mathematics education research (Bos et al., 2021; Drijvers, 2019; Lakoff & Nunez, 2000; Shvarts et al., 2021). The theoretical foundations concerning embodied learning in our study include the following two aspects: Embodied design and embodied instrumentation. Embodied Design leverages embodied cognition to create learning environments and materials that prompt students to engage physically and perceptually with mathematical concepts. Similarly, Embodied Instrumentation combines embodied cognition with an instrumental approach to learning, emphasizing the coupling between the learner, the physical tools or artifacts, and the tasks at hand (Shvarts et al., 2021).

Embodied design

Regarding the design and use of embodied cognition in the mathematics classroom, Abrahamson (2009) introduced a well-defined notion of embodied design (a term first coined by Van Rompay and Hekkert (2001)) as a systematic and procedural design method. It consists of two types: action-based design and perception-based design (Abrahamson, 2009; Abrahamson & Lindgren, 2014).

Action-based designs aim to ground mathematical concepts in students' natural capacity to adaptively solve sensorimotor problems. In action-based design, the sense of meaning comes from being able to achieve a target outcome using both a naïve and an instrumented strategy with a technological system. For example, to teach the concept of a parabola, learners can be encouraged to manually plot a series of green isosceles triangles, which collectively form a U-shaped trace (Palatnik et al., 2023; Shvarts & Abrahamson, 2019). In this case, the sensorimotor coordination pattern manifesting the parabola concept necessitated preserving the triangle by equating the distances from a point to the directrix (CB) and focus (CA) (see Figure 3.2). This task requires students to engage in a physical exploration of the parabola's geometric properties, specifically its reflective symmetry and the definition involving distances to the focus and directrix. This method allows them to intuitively grasp the parabola shape by acting on, combining naïve (manually tracing) and instrumented (keeping isosceles triangles equal to learn about the parabola) strategies within a technological system (the graph and its representation of geometric figures).

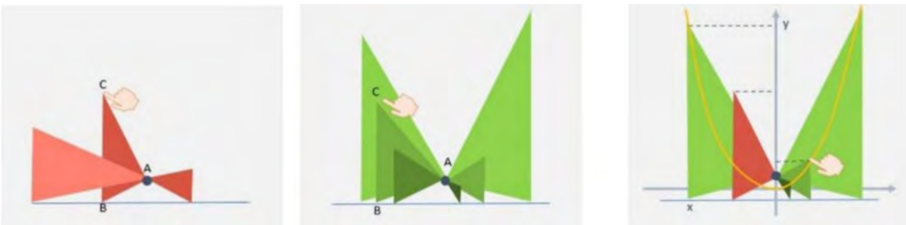


Figure 3.2 Sensorimotor coordination patterns of a parabola $y = x^2$ (Palatnik et al., 2023, p.170)

Perception-based designs aim to ground mathematical concepts in students' natural perceptual ability in their naive perceptions of a situation. Like the action-based genre, it is followed by a phase of reflection in which these views are developed. This approach involves the manipulation of students' perceptual fields or having them engage in activities where they discern patterns, identify relationships, or perceive variations. For instance, in a study on teaching the gradient using an augmented reality sandbox (Bos et al., 2022), students were invited to roll a marble down a plane, adjusting its direction and steepness (Figure 3.3). The sandbox projected real-time height lines onto the plane, with the marble's trajectory perpendicular to these lines, indicating the steepest direction. In perception-based design, the sense of meaning arises when someone can make the same inferences from both direct and indirect observations of a given phenomenon. From the rolling marble experience students are invited to make inferences about the fact that the height lines and the gradient are perpendicular. Action-based and perception-based designs help create a richer, multisensory learning environment where learners can make sense of abstract concepts by enacting and perceiving them physically.

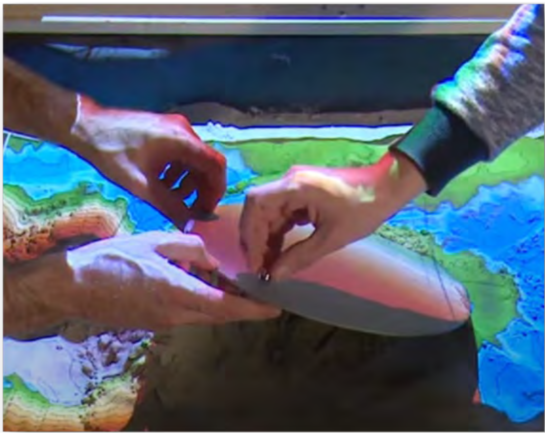


Figure 3.3 Rolling a marble down a plane in the augmented reality sandbox (Bos et al., 2022)

Building upon the existing literature on embodied design, our study also draws on the embodied-design procedure to inform our task design. Three fundamental steps are emphasized by Abrahamson (2014), namely *Phenomenalization*, *Concretization*, and *Dialog*. *Phenomenalization* involves creating an intuitive situation related to the topic being learned. It starts by identifying a generic schema or pattern underlying the topic, and then developing a scenario where applying this schema provides a solution. *Concretization* involves creating a visual model of the situation. The goal here is to decide on a formal disciplinary model related to the problem, devise a visual version of it, identify symbols that can represent the student's solution strategy, and create incentives for the learner to use these symbols to understand the problem. In the *Dialog* stage, the learner is guided through the process of using informal actions to solve the problem situation, constructing a formal visual solution, and reflecting on the relationship between their intuitive understanding and the visualization of the situation. The application of Embodied Design in mathematics education can profoundly reshape teaching and learning experiences. By weaving these stages into our task design (see details in the *Design* section), we aim to offer a more accessible and engaging learning experience, promoting a deeper comprehension of FT.

Embodied Instrumentation

In line with the design views of this study, we considered the Embodied Instrumentation theory (Drijvers, 2019) for the elaboration of task designs. As a combination of embodied cognition and an instrumental approach, Embodied Instrumentation underscores the amalgam between the body, artifact, and the cognitive scheme involved when DT is used in mathematics education. The term 'instrumentation' here refers to the process through which an artifact (a tool, technology, etc.) becomes a part of the student's conceptual scheme. A scheme is an invariant organization of activity for a certain type of situation (Vergnaud, 2009). Expanding on this idea, Shvarts et al. (2021) emphasize the nuanced and complex nature of action regulation, which occurs through dynamic functional systems involving both the body and the artifact in perception-action loops (Figure 3.4). Significantly, perception-action loops are the lynchpin of this body-artifact functional system. These loops are central to a complex dynamic system of behavior, with perception and action existing as intertwined processes within the interaction and coupling with the learning environment. For example, initial perception emanates from the interaction with and/or observation of the artifact, guiding the students' actions at the same time. Concurrently, the actions reciprocally offer feedback and verification, thereby generating a new perception or preserving the existing one. Unlike conventional mental schemes, these functional systems are decentralized and can be expanded through the inclusion of artifacts. In this context, an artifact is not only an external tool but becomes a part of the system, contributing to the way learners interact with and comprehend mathematical concepts.

The aforementioned theories offer a framework to develop FT, particularly in the context of using digital technologies. The Embodied Design theory, with its bifocal approach of action-based and perception-based designs, promotes the grounding of mathematical concepts within students' sensory-motor coordination. The Embodied Instrumentation theory stresses the relationship between the body, artifact, and cognitive scheme when employing DT in mathematics education. These two theoretical perspectives both centered on the principle of embodied cognition. They emphasize the need to consider students' physical interactions and perceptual experiences, and underscore the role that artifacts play in shaping these experiences. The fusion of these theories directs the way of designing a natural and engaging learning environment for the development of FT.

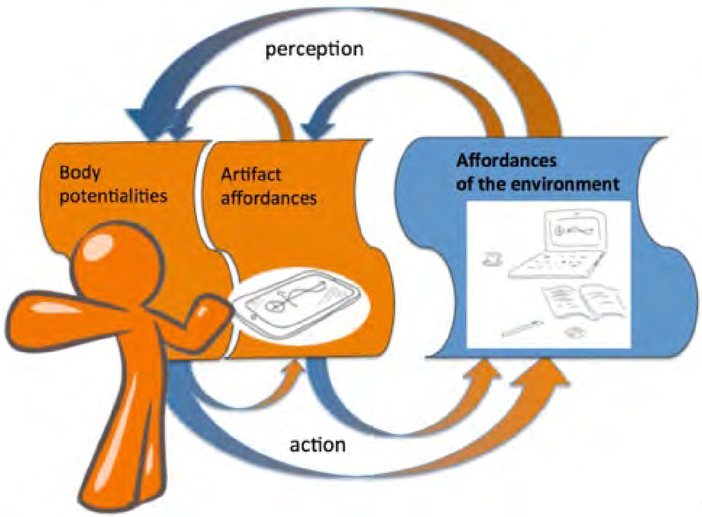


Figure 3.4 Body-artefact functional system in interaction with the environment (Shvarts et al., 2021, p.451)

3.3 Methods

To address the research questions, we conducted a design-based study. This study was structured into two main phases: the initial design phase, where we developed and refined the digital-embodied learning materials, and the subsequent case study phase. In the case study phase, we observed and analyzed the experiences of two pairs of 14-year-old students.

Design

Our aim is to design a digital-embodied learning environment for fostering FT. As a first step, we drew on the principles of the Embodied Design framework (Abrahamson, 2014) to construct an overarching architecture for FT. This architecture, which is grounded in our theoretical foundations, includes three critical stages: *Phenomenalization* - light ray context; *Concretisation* - embodied nomogram model; *Dialog* - a series of questions/tasks. The core concept is FT, and the implementation of a light ray context in our embodied design builds upon the historical use of nomograms, PAR, and DynaGraphs in mathematics education (Nachmias & Arcavi, 1990; Sinclair et al., 2009).

This series of designs introduces the concept of function through a real-life context that is familiar to students, namely light rays. The innate connection between the light ray context and the nomogram model serves as the cornerstone of our design. Imagine a scenario with a sun or a light bulb illuminating an object, casting shadows in varying positions (Figure 3.5)[1]. We hypothesize that the students' existing familiarity with light rays and their effects, such as shadows, will help them understand the linear patterns that form the basis of nomograms. Specifically, light travels in straight lines and objects can obstruct light, creating shadows. Therefore, students' perceptual experience that their shadow follows their bodily movement affects predictable object-shadow positions, and can help them grasp the concept of predictable relationships between variables in nomograms. Students mathematize their intuitive perception of the situation using nomograms. The illustrative diagrams, including Figure 3.5, are intended to support this foundational experience rather than to provide a comprehensive exploration of light and shadow geometry (Gravemeijer & Doorman, 1999).

We use the light ray context as the model for *Phenomenalization*. In particular, two aspects of the context implicitly convey fundamental properties of the nomogram. First, the directedness of the light, from hand to shadow, carries over to the nomogram where arrows go from input element to image element. Second, in a nomogram one only draws a finite number of arrows, even though arrows are virtually sprouting from every point on the input axis. The situation is the same for light rays: we draw only a finite number, even though we know there are light rays through every point. As described in the introduction section, digital-embodied nomograms can offer a tangible, hands-on experience that enables learners to comprehend the relationship between two variables. This understanding is facilitated through both visual (perceptual) and tangible (action) modes. And these nomograms can effectively communicate a learner's solution strategy using specific visual cues, such as the trace of an arrow and algebraic symbols. Hence, digital-embodied nomograms serve as an ideal model for the *Concretization* phase of the learning process. The *Dialog* stage is where the mathematical concepts are consolidated and where the learners can see the relationships between their intuitive actions or strategies and the formal mathematical concepts. It encourages learners to navigate through problem scenarios using their informal approaches, subsequently guiding them to

[1] The geometric representations in Figures 3.5 and 3.6 are intentionally simplified for clarity, despite deviations from shadow shapes in reality. Empirical evidence from our later study indicates that these simplifications did not detract from student understanding or engagement, supporting their use for educational purposes.

craft formal mathematical solutions. Our learning environment is structured to enhance this experience: students are presented with questions, urging them to consolidate and articulate their insights. Additionally, the collaborative design of some tasks stimulates interactive dialogue between peers. And there is another opportunity for students to seek clarifications from tutors, which ensures that misconceptions are addressed, thereby safeguarding a holistic, dialogic learning journey.

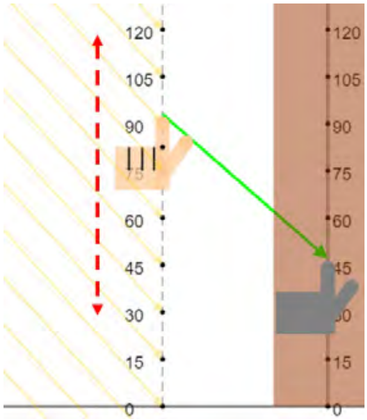


Figure 3.5 A hand and its shadow under the sun

Beyond the general design principles discussed above, the specific design ideas for each task are outlined in an HLT (Bakker, 2018; Simon & Tzur, 2012). We elaborated the HLT based on the embodied instrumentation theory (Drijvers, 2019) and emergent modeling (Gravemeijer, 1999). A full description of the sequence for each learning module, referred to as the HLT, can be accessed at the provided link: https://bit.ly/FTnomogram. The table describing the HLT comprises multiple horizontally-arranged components, which include task numbers, task descriptions, mathematical objectives, students' activities (incorporating practices/techniques for utilizing artifacts, and levels of the adapted emergent mode), and the conceptualization of various aspects of FT. The arrangement of the three HLTs is sequentially organized in a vertical manner, reflecting both the aspects of FT and the levels of the emergent model.

In Module 1, tasks begin with situational activities that incorporate input-output thinking within a light ray context. Following several varied light ray tasks, the real-life context faded, and the required movement shifts from unimanual to bimanual. This gradual shift introduces covariation thinking through referential activities. To illustrate, Figure 3.6a presents a task built on the context of a cardboard tree and its shadow on a screen. Here, students have the opportunity to delve into the geometric meaning of both additive and multiplicative terms of the linear relation $y = 3x - 2$ by manipulating the input, either moving the tree's apex/base or the entire tree. When students modify the tree's apex or base position, a commensurate change in the shadow's magnitude is observable, governed by a specific multiplicative factor, in this case "3", which is geometrically an enlargement factor. The additive factor, in this case "–2", is the image of the zero on the input axis.

In Module 2, the tasks initiate with referential activities that adopt semi-nomograms (nomograms without numbers), eventually leading to general activities. This module emphasizes mathematical contexts, highlighting covariation thinking and initial correspondence thinking. The presentation of various representations, such as nomograms and formulas, lays the foundation for introducing correspondence thinking. An example can be discerned in Figure 3.1, where a task is based on the function $y = -x$. In this activity, students are tasked with acting in a specific bimanual motion—moving both hands in opposite directions at the same speed. This motion mirrors the mathematical relationship encapsulated in the function $y = -x$: a positive increment in x induces an equal decrement in y, and vice versa. Such a coordinated movement not only embodies the interrelation between the two variables but also holds the promise of assisting students in transitioning from concrete actions to an abstract mathematical conceptualization of functions. The act of moving both hands in opposite directions at the same speed embodies covariation thinking. It enables students to directly perceive that during the act of moving, every position of the left hand on the input number line corresponds to a right-hand position on the output line, as a result of an inverse movement. This kinesthetic experience is expected to reinforce the understanding of functions as covariational relationships between variables where the change of one variable directly influences the change of another in a specific manner.

Module 3 comprises general activities centered on covariational thinking, integrating various representations of functions. The core idea of this learning module is the conversion between these representations, with special emphasis on the transition between nomograms and function graphs. Using both unimanual and bimanual motions, students delve into the correspondence aspect of FT. Through an action-based design, depicted in Figure 3.6b, students can adjust arrows on the nomogram, observing the resulting point on the corresponding function graph: $y = x^2$. The semi-coordinate system offers a visible and tangible representation of these shifts: the left hand's movement directly influences the point's horizontal position, while the right hand dictates its vertical movement. Notably, students are exposed to the subtleties of their hand movements' acceleration, which mirrors the change in the derivative of the function $y = x^2$. As they approach $x = 1$, the left hand's movement needs to decelerate relative to the right hand. In contrast, distancing from $x = 1$ necessitates the left hand to progressively outpace the right. This hands-on experience emphasizes the covariational relationship between the two variables, demanding both speed and coordination to align the point accurately on the function graph. Consequently, students are expected to get insights into the conversions between nomogram and function graph, all while grounding their understanding in an intuitive, embodied experience.

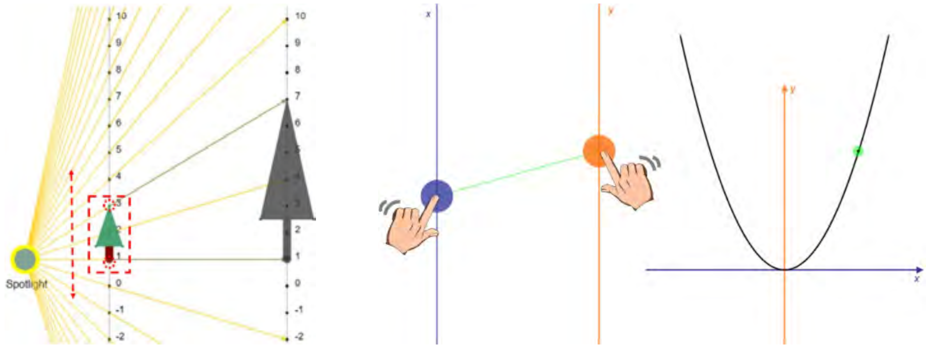


Figure 3.6 (a) Cardboard tree and shadow task; (b) Maintaining a point on the function graph

Case study participants

For our case study, we selected two pairs of students who were in the pre-university stream of secondary education in the Netherlands, each pair comprising students aged 14 years, to participate in the intervention. These students were chosen by their teacher for their collaborative and communicative abilities, and their willingness to take part.

The starting points and preliminary knowledge of the students were: experience with number lines; basic algebra to describe relations between quantities/variables; using algebra for modeling situations; basic skill with graphs in coordinate systems. This foundational knowledge is essential to grasp the concepts introduced in the intervention and effectively engage in the learning modules.Intervention

The intervention was carried out with the two pairs of students, first in November 2022 and then in January 2023. Each intervention session, covering three learning modules, took 90 minutes, followed by a 15-minute interview to gather the students' reflections and insights. Students were given digital-embodied nomogram tasks within a learning environment that featured a multi-touch screen (Figure 3.7). This setup provided an interactive platform for the students to explore and engage with the tasks. Throughout the intervention, the students were encouraged to collaborate and ask questions from the teacher during their work.

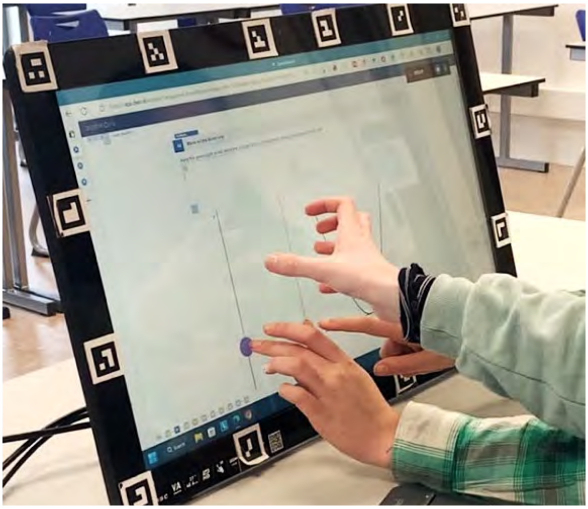


Figure 3.7 Students worked on a multi-touch screen

Data collection and analysis

The two primary data collection methods were video and audio recordings. The video recordings served a dual purpose: first, they tracked the physical activities of the students, including their hand motions and gestures, which are of prime interest in our study. Second, they captured the on-screen activities and students' solutions in the answer boxes for each task. This gave a clear picture of how students were engaging with the digital-embodied nomogram tasks, and how they manipulated the tools provided within the digital learning environment. In addition, audio recordings of the post-intervention interviews were collected. This dataset was then fully transcribed to facilitate the subsequent analysis.

Central to our data analysis was understanding students' usage of digital-embodied nomogram affordances in their conceptualization of FT. We selected key segments where these affordances were distinctly used for further exploration. We qualitatively analyzed participants' actions, explanations, and discussions, comparing our anticipated outcomes

(Hypothetical Learning Activities, HLA) with the actual events (Actual Learning Activities, ALA). This comparative lens provided insights into several key areas: practices and techniques with artifacts, conceptualizations of FT, action-perception loops in action-based tasks, and attentional anchors for action-based tasks. This comparative analysis is important for addressing the research questions. For instance, comparing conjectured and actual practices of and techniques for using artifacts helped us to assess the degree to which the light ray context promotes students' interpretative capabilities with respect to nomograms. Likewise, we could determine how the bimanual tasks have been a catalyst for fostering covariational thinking by contrasting hypothetical and actual action-perception loops. Through this comparison process, enriched with participant quotes, we not only identified patterns in the students' interactions but also gained insights in our design's efficacy and suggested areas for improvement.

3.4 Results

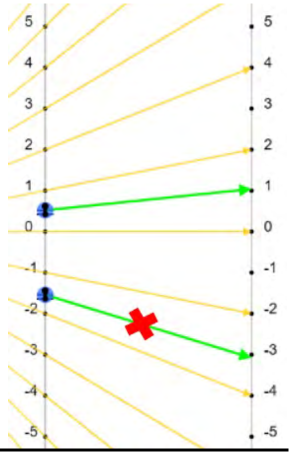
The result section provides empirical data on how FT can be fostered through the three digital-embodied learning modules, each corresponding to one aspect of FT. For each learning module, we present one or two exemplary tasks and describe participants' activity. We compare the HLA with the ALA, highlighting the similarities and discrepancies between them, accompanied by examples and quotes from participants. The redesign ideas are also presented.

Learning Module 1: Light ray context and nomogram

Figure 3.8 Students' hand gestures while exploring the bulb-mosquito task

Example 1: In the bulb and mosquitos' shadows task (see Figure 3.8), students can move the positions of two mosquitos and observe how the positions of their shadows change correspondingly. The relationship between the position of the mosquito and its shadow is: $\text{height\_shadow} = \text{height\_mosquito} \cdot 2$

HLA	Hypothetical practices / techniques for using artifacts: By moving one input (mosquito) once on the number line, students can recognize and distinguish various patterns of relationship (light rays). The relationship established between the mosquito and its corresponding shadow in this instance is a proportional relationship. Hypothetical conceptualizations of FT: During the exploration of the bulb-mosquito context, students are able to manipulate the mosquito’s position and subsequently observe the resulting shadow location, thus facilitating an understanding of the input (mosquito) and output (mosquito’s shadow) relationship (coordination of input values and output values), and covariation thinking (output covaries when students change the input).
ALA	Actual practices / techniques for using artifacts: Student Pair 1: The students adjusted the position of each mosquito individually and noted the gradient of the light rays (arrows). After the tutor explained the question, their attention shifted toward the input-output (mosquito-shadow) relationship. Student Pair 2: The students manipulated the positions of the two mosquitos simultaneously, assessing the inter-mosquito distance and the distance between their shadows. Rather than a point-to-point relationship, they focused more on the interval-to-interval relationship (Chunky continuous covariation). They prioritized the distance between the two mosquitos in relation to the corresponding distance between their shadows, instead of observing how the output depends on the input. One of the students said: “The distance between all the lines is getting bigger and bigger... Well, I don’t know if I’m saying it right, But the distance between that line is getting bigger.” Actual conceptualizations of FT: Student Pair 1*:* The students phrased their findings, saying, "The shadow is at double the height of the mosquito's height." This description shows their comprehension of the relationship between input (height of the mosquito) and output (height of the shadow). Student Pair 2: The students gave a more descriptive explanation, focusing on the distance between the two mosquitoes. They described the rule as "the distance between the light rays doubles, as the light rays are further away from the light source". They showed an understanding of chunky covariation (interval-to-

	interval), an advanced level of FT, compared to the input-output (point-to-point) level.
Comparison	While the actual learning trajectory of Pair 1 closely followed the hypothetical trajectory, Pair 2 showed an unexpected but advanced level of FT. In other words, concerning the potential given by the task, students could prioritize the inter-mosquito distance and the distance between their shadows, instead of observing how the output (shadow) depends on the input (mosquito position). The students may have found the interval-to-interval relationship more intuitive or engaging to explore (because there are two mosquitos available), which diverted their attention from the point-to-point relationship.
Ideas for Redesign	To prevent the unnecessary distractions on the screen, we plan to reduce the complexity by removing one of the mosquitos. Notably, these students are relatively high achievers, and pair 2 invested six times as much time to solve the task. 

Learning Module 2: Bimanual nomogram tasks

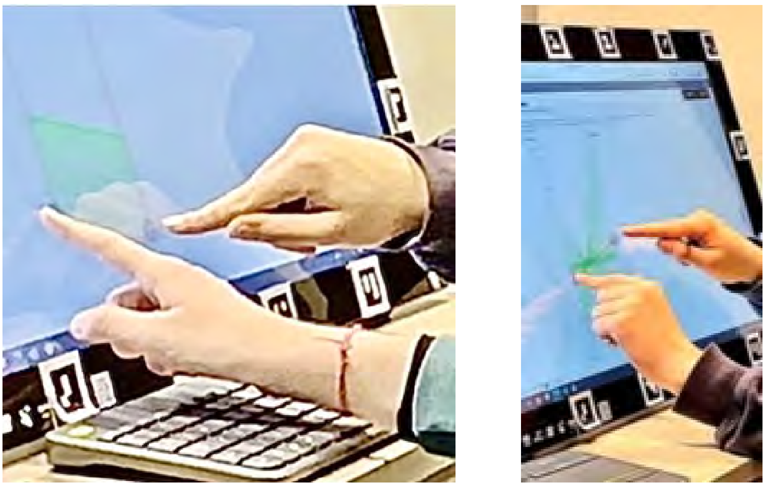
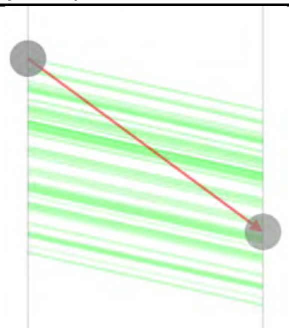


Figure 3.9 Students explored nomograms (a) collaboratively, and (b) individually

Example 2.1: In the ‘Keep the arrow green with your neighbor’ task (see Figure 3.9a), students can construct the arrows of the semi-nomogram (without numbers) with their peers by controlling one point per person. The relationship between the input and output is: $\text{output} = \text{input} - 2$

HLA	Hypothetical action-perception loops: During the exploration, students adjust the arrow’s endpoints; one student moves one end, while the other student adjusts the opposite end. They aim to maintain the arrow’s green color, indicating a correct relationship between the heights of the two ends. If the points misrepresent the function’s nomogram, the arrow turns red, prompting students to correct their positioning to maintain the green indication. Initially, students employ subtle or slight movements in segmented (“chunk”) motion strategies, which help them get familiar with this new movement. As students traverse the input axis, they visualize the function’s complete pattern, observing intersecting and parallel arrows among other features. Hypothetical practices / techniques for using artifacts: Students move two points on two lines vertically with height differences on both sides and adjust the moving speed (same speed for both sides) and direction (same direction on both sides) based on the feedback of the arrow (green = positive, red = negative). Hypothetical conceptualizations of FT: Students’ simultaneous and smooth manipulation of the two points can foster an understanding of covariation between two variables based on bimanual movements. This can be achieved by drawing an analogy between their physical experiences (heights of hands) and mathematical meaning (dependent and independent values). In addition, the whole set of arrows, as shown by their trace, can contribute to the development of correspondence thinking.
ALA	Actual action-perception loops: Student Pair 1: The students were observed to adjust the position of one end of the arrow (student 1) and then move the other end (student 2) until the arrow turned green. After finding several green arrows, students started to move the two points together and keep the arrow green all the time. In the end, they adjusted the arrow smoothly and perceived the relation between the heights of the two ends of the arrow to meet the positive feedback, which is a green arrow. Student Pair 2: The students adopted a strategy similar to pair 1.

	Actual practices / techniques for using artifacts: Student Pair 1: The students became adept at moving vertically along two straight lines with both hands, which laid the foundation for the later tasks. Student Pair 2: The students followed a practice similar to pair 1. In addition, it was observed that the students focused on the speed of their movements and came to the insight that the speed of the two points' movements must be consistent in order to maintain parallelism between the arrows in the nomogram of $\text{output} = \text{input} - 2$. Actual conceptualizations of FT: Student Pair 1: The students connected their observations to the previous learning module. They referred to the light-shadow context, which was not actually present "The sun's rays come from one side again, so the green arrows run in one direction." This task saw the emergence of situational reasoning since they used the light-shadow context to explain what they observed from this new task. But limited covariational thinking was observed. Student Pair 2: The students made a connection between their physical movements and the geometric attributes of the nomogram, stating, "By trying to both go down/up at the same speed…make sure you both move your fingers down at the same speed so that the lines stay parallel all the time". This enabled them to comprehend how one variable changed in relation to another based on their bodily experience.
Comparison	The HLA and ALA are primarily aligned, as students followed the anticipated process of adjusting the arrow and maintaining the green color; moving two points on two lines vertically and adjusting their movement based on the arrow's feedback. Their reflection suggests a strong bond between their bodily experience and the mathematical meaning. The HLA was an accurate description of the learning process, and the instructional strategies and materials were successful in guiding students along the desired learning trajectory.
Ideas for Redesign	The alignment between the HLA and ALA indicates that this series of tasks was successful in facilitating the desired learning outcomes. There will be no adaptation. 

Example 2.2: In the task 'Describe the rule with x and y' (see Figure 3.9b), students need to complete the nomogram (with numbers) by moving first and then describe the rule with x and y. The relationship between the input and output is: $y = -2 \cdot x$.

HLA	Hypothetical action-perception loops: During the exploration, students are expected to move the two points together while keeping the arrow green to find the moving patterns based on the numbers on the two lines. Hypothetical practices / techniques for using artifacts: Students are expected to move two points on the two number lines simultaneously and maintain the moving speed (different speeds for both sides) and direction (opposite direction on both sides) to hold positive feedback. Hypothetical attentional anchor: An intersection point of the traces of the arrow. Hypothetical conceptualizations of FT: With the appearance of numbers on both lines, students are expected to strengthen their understanding of input-output pairs and covariational quantities with referential reasoning. Students can develop a connection between a rule that determines their movement to a more abstract rule/relationship between two variables, which entails a form of correspondence thinking.
ALA	Actual action-perception loops: Student Pair 1: The students were found to first move one unit on the left number line, then adjust the right point until they observed the arrow getting green. After noticing a pattern in the traces of the arrows, they adjusted their movement based on the trace. Student Pair 2: These students quickly found some green arrows and subsequently shifted their attention to the point where the existing arrows intersected. They then moved their hands and completed the nomogram by aligning the arrows with the intersection point. Actual practices / techniques for using artifacts: Student Pair 1: Students practiced a strategy of moving one unit by one unit, conjecturing the possible moving pattern, and then following the traces of the arrow to move two points smoothly. Student Pair 2: The students noticed that a shortcut to creating green arrows easily on the nomogram involved 'rotating' the arrow around the intersection point. Actual attentional anchor: An intersection point of the traces of the arrow.

	Actual conceptualizations of FT: Student Pair 1: The students adopted a top-down strategy, first identifying several input-output pairs, such as 0 to 0, 2 to -4, -1 to 2, and then performing calculations to give the rule with x and y. They primarily focused on the mathematical aspects rather than the motion aspects of this task, which led them to covariation thinking. Student Pair 2: The students connected their movement pattern and the geometric attributes of the nomogram. Their statement, "we have to make sure the line stays on this center point (intersection point) … and from there, one point (moves) down and the other (point moves) up." They quickly recognized that the opposite motion of their hands corresponds to a '-' symbol in the formula. Subsequently, by analyzing number pairs on the nomogram, they deduced the additive factor. This indicates their realization of the initial covariational relationship between the coordinated hand movements.
Comparison	In the action-perception loops, the discrepancy lies in the students' (pair 1) initial approach to moving one unit on the left number line and adjusting the right point until the arrow turned green, instead of moving the two points together while keeping the arrow green. In the attentional anchor, the students focused on the intersection point of the traces of the arrow, rather than the green arrow itself. This shows that this task might not have been perceived as action-based. The students paid more attention to the mathematical aspects, viewing the nomogram as a representation rather than a tool that provides physical experiences to facilitate mathematization.
Ideas for Redesign	The step-by-step approach could become an alternative pathway to understanding the moving patterns. As for students' focus on the intersection point to complete the nomogram, future study will consider if it is necessary to redirect students' focus to the green arrow, for example, make the traces invisible while students move the two points and reveal the traces when the movement covers most of the target traces.

Learning module 3: Transition tasks of multiple representations for functions

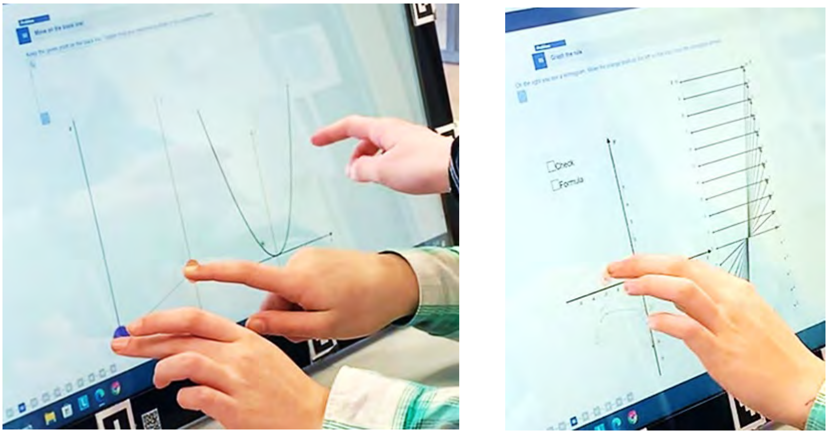


Figure 3.10 Students explored the transition between (a) a nomogram and corresponding function graph; (b) a function graph and corresponding nomogram

Example 3.1: In the task 'Move on the line' (see Figure 3.10a), students can adjust the arrow in the nomogram and try to keep the corresponding point moving on the function graph. The relationship between x and y is: $y = x^2$

HLA	Hypothetical action-perception loops: Students are expected to explore the relationship between their hand movements and the corresponding point in the coordinate system, observing how the left and right-hand movements affect the point's horizontal and vertical positions, respectively. Hypothetical practices / techniques for using artifacts: Students should become aware of the relationship between their hand movements and the corresponding point's movements to accurately adjust their actions; left-hand movement controls the horizontal movement of the point (left/right), while right-hand movement controls the vertical movement (up/down). Hypothetical conceptualizations of FT: Speed and coordination are crucial to keep the point on the function graph. Students are expected to comprehend how their hand movements simulate the covariational relationship between the two variables. They can develop an understanding of the conversion between representations, such as from an arrow in a nomogram to a point on the function graph, based on their physical experience. Additionally, students are expected to recognize that the vertical axes in both representations remain consistent. Another crucial observation is the orientation of the

	horizontal axes; they are rotated at a right angle, a geometric detail can be mirrored in the student's movement.
ALA	Actual action-perception loops: Student Pair 1: Initially, students moved the right point on the nomogram, observing its impact on the vertical movement of the green point in the Cartesian coordinate system. They then moved the left point and recognized its influence on the point's horizontal movement. After figuring out the effect of each hand's movement on the point's position in the Cartesian coordinate system, they started to move the two hands simultaneously on the nomogram to ensure the point moved along the corresponding function graph. Student Pair 2: These students adopted a reverse strategy. They were observed initially moving both hands together to see the point's movement in the Cartesian coordinate system. They then experimented with moving one point at a time to understand the influence of the left number line on the point's horizontal movement and the right number line on the vertical movement. Actual practices / techniques for using artifacts: Student Pair 1: The students demonstrated the same understanding as hypothesized, using their left hand to control the point's horizontal movement (left/right) and their right hand to control the vertical movement (up/down). Student Pair 2: Similar to Pair 1. Actual conceptualizations of FT: Student Pair 1: The students built the full connection between the two representations, nomograms and function graphs. They described the findings of the function $y = x^2$ as "you have to move the x circle up slowly and the y circle you have to move up faster", indicating that when x is greater than 1, y changes at a faster rate than x. This shows their understanding of the correspondence relationship of x and y for the function $y = x^2$. They incorporated their bodily experiences to explain the conversion between different function representations. Student Pair 2: The students gave a similar statement on the conversion between nomograms and function graphs.
Comparison	Both HLA and ALA demonstrated similar practices/techniques for using artifacts and shared similar action-perception loops, indicating that the hypotheses were effective and accurate in guiding students' engagement and focus. However, the difference lies in the initial strategy used by the students in pair 1 and pair 2. Pair 1 started by moving the right point and then the left point, while pair 2 initially moved both hands together. Concerning the conceptualization of FT, both pairs were able to build the full

	connection between the two representations and used their physical experiences to explain the conversion between different function representations. These findings suggest that different students may adopt different strategies through exploration, but either way can lead them to the final learning goal.
Ideas for Redesign	The HLA and ALA have no conflicts considering the learning goal of this task, which is 'Conversion between different representations of functions, nomogram and function graph'. There will be no adaptation for this task.

Example 3.2: In the task 'Graph the rule' (see Figure 3.10b), students can use a digital pen on the screen to plot the function graph according to a given nomogram. There is a colorful arrow in the nomogram showing the position of the pencil in real time. The functional rule in this task is: $y = |x|$

HLA	Hypothetical action-perception loops: When completing the function graph, students are expected to move the digital pen (orange point) in the Cartesian coordinate system and observe the corresponding color-changing arrow on the nomogram. Students can use a top-down approach, first having an overview impression of the nomogram and guessing the function, and then moving the digital pen to draw the graph, with the color-change arrow signifying the location of the point in the Cartesian coordinate system. When the point aligns with the target function graph, the corresponding arrow in the nomogram turns green, and the trace of the pen remains visible. Or students can adopt a trial-and-error method, working in small steps to keep the green arrow on the given nomogram by loops. Hypothetical practices / techniques for using artifacts: Students are expected to first move the pen horizontally or vertically to determine the effect of the movement, and then focus on plotting several separate points that fit within the nomogram. Eventually, they should be able to plot the function graph. It is plausible they may need to erase and restart the entire canvas several times before they can smoothly plot the function graph. Hypothetical conceptualizations of FT: It is a reverse task of conversion between nomogram and function graph. By 'matching' the color-changing arrows on the nomogram and the pen's position in the Cartesian coordinate system, students' comprehension of input-output pairs and their corresponding locations in the Cartesian coordinate system is reinforced. The integration of nomogram, function graph and

	formula in this task allows students to experience how different representations work together to represent one rule/function, which leads to correspondence thinking.
ALA	Actual action-perception loops: Student Pair 1: The students first found a point (1,1) that turned the arrow green. They then began to freely move the pen to observe its effect on the color-changing arrow on the nomogram. After discerning the connection between the pen and the color-changing arrow, they tried to plot the left part (second quadrant) of the function graph in small steps by loops. Student Pair 2: The students initially moved the pen in an unstructured manner, attempting to find number pairs and to turn the color-changing arrow on the nomogram green. They found that even with free pen movement, when it reaches a certain area, the trace of the pen could remain visible. After being redirected by the tutor, they refocused on the task goal, which was to plot the function graph based on the given nomogram, rather than to move the pen freely. Actual practices / techniques for using artifacts: Student Pair 1: Initially, students plotted a few points in the coordinate system while observing the corresponding nomogram and then lined up these points to complete the function graph. Student Pair 2: The students first moved the pen randomly and got some pen traces, which formed part of the accurate function graph. Then they used a strategy similar to pair 1 to plot the graph. Actual conceptualizations of FT: Student Pair 1: The students identified the connection between the movement of the pen and the color-changing arrow on the nomogram, and then further got an understanding of the conversion between the function graph and the nomogram. Their use of the number pairs indicates a deep development of the input-output aspect of FT. When giving the formula of the function, they described it as “x times -1 (second quadrant)…Always the same number (first quadrant)”. They have not learned absolute function, so provided this kind of stepwise formula, when $x < 0, y = -x$, and when $x > 0, y = x$. This shows a strong ability to transfer between different function representations, which exemplifies the correspondence aspect of FT. Student Pair 2: The students also grasped the conversion between the function graph and the nomogram. They figured out the patterns of the number pairs from the given nomogram. When explaining the findings, they described it as “(in the first

	quadrant) the graph should be a 45-degree straight line,…, it is also a 45-degree straight line but towards the opposite direction (in the second quadrant)". Their explanation suggests that they have also got an understanding of correspondence thinking.
Comparison	The difference in the action-perception loops is the initial strategy used by both pairs of students. In terms of the use of artifacts, both pairs demonstrated a grasp of the connection between pen movement and the color-changing arrows on the nomogram, and applied similar strategies to plot the function graph. In conceptualizing FT, both pairs of students recognized the conversion between the function graph and the nomogram, and identified patterns from the given nomogram, indicating developed correspondence thinking.
Ideas for Redesign	The confusion about the task goal implies a need for clearer task instructions. To avoid this confusion, the task goal of graphing the function in the coordinate system will be explicitly stated at the beginning to guide student focus, for example, highlighting the goal in the instruction and the title of the tasks. In addition, the pen's traces could potentially cause a misinterpretation of the task. We will modify this aspect to preserve all traces rather than exclusively maintaining the correct one.

The results section elucidates how FT can be fostered through three digital-embodied learning modules, each targeting a different aspect of FT. Through detailed examples and participant activities, we compared the differences in HLA and ALA. Moreover, the comparison and redesign ideas presented suggest ways for refining the design to better align with students' perceptual and physical experiences and learning needs. These results set the stage for a broader research of the implications of digital-embodied learning.

3.5 Conclusion and Discussion

The overarching question addressed in this paper is how an embodied design can foster abstract FT. To provide more comprehensive insights, we divided it into three sub-questions, each focusing on a specific aspect of FT. In the following, we reflect on how the results obtained from each learning module contribute to answering these sub-questions.

RQ 1 How does a light ray context foster the students' meaning-making of nomograms?

The light ray context, an integral part of our embodied design, served as an important instrument in facilitating students' understanding of nomograms and the input-output aspect of FT. As advocated by Abrahamson & Lindgren

(2014), students need guidance to take action and move their bodies in specific ways, simulating key mechanisms and spatial relations. This approach helps them understand and apply functional analogies in the targeted knowledge domain. In our scenario, we have embedded numerous elements within the light ray context that render the nomogram as a function representation. Firstly, the orientation of the light rays (arrows) underscores the principle that, with a function, it is the inputs that consistently map onto the outputs, and not the reverse. Secondly, a single light ray can only correspond to a unique point in the shadow, similar to the function rule where input can only determine one specific output. Thirdly, in nomograms, only a finite number of arrows are drawn, despite every point on the input axis theoretically having an arrow, similar to how we depict only a limited number of light rays even though they pass through every point. Lastly, the visualization of two hands in the hand-shadow tasks offers an analogy for the subsequent dual-hand motion task on a nomogram. The object placed in front of the light source symbolizes the input, while the resulting shadow represents the output. The direction of the arrows offers students clear guidance regarding the mapping from input to output values.

Though still partial and vague in this learning module, students' grasping of the mathematical meaning of the nomogram—as a function representation—was accomplished through and manifested in their bodily actions, gestures, artifacts (the learning environment), and mathematical symbols (Radford, 2009). As shown in our observations, students were able to draw connections between their actions – such as changing the light source's type or position – and the resulting changes in the light ray patterns on the nomogram. While experimenting with two light sources, the sun or a bulb, the learning environment supports the students in the meaning making of nomograms for two types of functions. The parallel nomograms are interpreted as the result of sunlight, which represents adding to or subtracting from the input values. The divergent nomograms, with a focal point left of the input lines, are ascribed to a bulb or spotlight, which enlarges the input values to some extent (see Example 1). In addition, the perceptual experience provided by the light ray contexts allows students to construct the mathematical meaning of using nomograms. For instance, the geometric patterns resulting from different light ray contexts left a deep impression on the students, enabling them to refer back to these contexts even in subsequent learning modules (as seen in Example 2.1).

In conclusion, the light ray context has been a productive situational tool in our embodied design, fostering a deeper understanding of the function representation of nomograms through a tactile and sensory-engaging approach.

RQ 2 How do bimanual movement tasks foster covariational thinking?

In our embodied nomogram tasks, the bimanual movements offer an opportunity for students to physically explore the relationship between two variables in a function, with the left hand representing x, and the right hand y, respectively. This enables students to physically experience the covariation between the two variables. As prompted by Alberto et al. (2022), our embodied nomogram tasks also cover two learning phases, a qualitative stage (nomograms without numbers) and a quantitative stage (nomograms with numbers). In the qualitative stage, by moving both hands to maintain the green status of the arrow, students understood that these variables were related and their hands' movements should be coordinated (see example 2.1). In the quantitative stage, students adopted a more systematic, quantitative approach to their movement patterns. For example, they used a strategy of moving one unit at a time on the left number line, and then adjusting the right point accordingly (see example 2.2). This shows they made connections between their movement and mathematical reasoning of the discrete numerical values associated with each point.

Concerning the bimanual movement in the nomogram-function graph tasks, students used both hands to manipulate the points that represented the value of the variables in the nomogram and observe the corresponding changes of the variables in the Cartesian coordinate system. A clear example of how this facilitated understanding of the covariational relationship can be seen in a task that involved keeping a point moving along a function graph (see example 3.1). To keep a point moving along the function graph $y = x^2$, students quickly realized they had to move their right hand (controlling y) faster than their left hand (controlling x) when x was bigger than 1. This active engagement provided a tactile foundation for their comprehension of the different function representations. This grasp was evident in their explanations such as, "the (y) point goes faster than the (x) point" or "when I move my left hand, the point goes left or right (on the nomogram), and when I move my right hand the point runs vertically (in the Cartesian coordinate system)".

Moreover, in line with continuous feedback used in previous studies in mathematics education, a notable feature embedded in our tasks was the continuous real-time feedback provided by the color-changing arrow on nomograms (Abrahamson, 2014; Alberto et al., 2019; Shvarts et al., 2021).

Continuous feedback has proven to be a promising tool, promoting new sensorimotor coordination through students' exploration and interaction with the learning environment. This was apparent in our tasks as students adapted their hand movements in real time to maintain the 'green' of the color-changing arrow. Students could immediately see the impact of their hand movements on the function and adjust their strategies accordingly. This loop of action, observation, and reaction has the potential to reinforce understanding continuous covariation in a dynamic, iterative way. The continuous feedback provided in these tasks was not only real time but also visually intuitive, using color changes (i.e., the arrow turning green or red) as an indication of correctness (Alberto et al., 2022). Such feedback, coupled with the simultaneous manipulation of the variables, enabled the students to experience the complex interrelationship between the two variables, thereby constituting a body-artifact functional system for covariational reasoning using nomogram.

In this manner, the integration of bimanual movement into the learning process not only aligns with the properties of functions itself but also leverages recent advancements in DT to provide a novel, hands-on approach to the foster of covariation thinking.

RQ 3 How do different function representations and their conversions support a correspondence view on functions?

Various function representations, such as arrow chains, tables, graphs, formulas, and nomograms, allow for different types of functional reasoning, fostering a holistic understanding of functions, which is the core of correspondence thinking. In our tasks, the conversions between different representations – nomogram, formula, and function graph – were intentionally designed to help students transfer them smoothly based on concrete experience. The formula and function graph, a more conventional representation, helped students further consolidate the functional relationship in a symbolic and graphical way.

According to previous research (e.g., Ainsworth, 1999; Duval, 2006), using multiple representations, especially the transitions between them, can deepen students' conceptual understanding and encourage more flexible thinking. When students are trained to transition smoothly between different function representations, they are better positioned to anticipate and operate on the function. This anticipatory ability allows students to deduce implications in one representation based on insights obtained from another. For instance, during the plotting function graph task (see example 3.2), students were given the opportunity to conduct the unimanual movement on one representation (function graph) while observing the corresponding changes on another (nomogram). The changes induced by the unimanual movement were related to the overall shape of the nomogram or slope of the function graph, encouraging students to anticipate outcomes before initiating the plotting process.

In summary of the findings on RQ3, the use of different function representations and conversions between them in our study encouraged correspondence thinking by providing multiple perspectives to observe, understand, reason about, and manipulate the function. These tasks boosted an understanding of the function as a correspondence relationship, promoting a general understanding towards an object view of function.

In addressing the limitations of this case study, several elements deserve further consideration. First, the subjects of our study are students from the pre-university stream, suggesting that these students may have a solid foundational knowledge, potentially enabling them to better grasp abstract concepts than their peers. This skews the findings, as the approaches used might have different effects on students from various learning backgrounds. Other uncontrolled factors could have influenced the outcomes as well, like learning perceptual preferences and familiarity with digital tools. Second, although the students' performance was closely observed and analyzed, their strategies and thought processes were inferred from their behaviors and verbal expressions, possibly introducing some degree of bias. For instance, although some students employed a top-down strategy yielding correct responses, their subsequent explanations connected these answers to our questions, including embodied elements we expected. This could lead to the misinterpretation that tasks were addressed using a mere embodied approach. Thirdly, the design intricacies, despite their novelty, might have been overly complicated for some participants. The confusion evident in initial tasks emphasizes the need for more explicit instructions in future designs. Adjustments, such as explicitly stating the task goal and refining the movement traces to maintain clarity, could help avoid misinterpretations. Last, the embodied nature of our tasks heavily relies on the nomograms. We could question if students are truly grasping the mathematical concepts, or if they are mastering the manipulation of this specific tool.

Adding to our conclusions, we want to emphasize the importance of design considerations for effective embodied tasks. First, the importance of providing students with a concrete experience, as exemplified by the light ray context in our design. This concrete, situational context served to anchor abstract mathematical concepts, allowing students to build on their intuitive understanding of the physical world—such as noticing how the presence of light sources affects the environment around them—how sunlight creates shadows that change during movement, or how a flashlight casts a shadow when its beam is obstructed. We acknowledge that while students might not have a detailed understanding of the geometric properties of light diffusion, their general awareness of how light and shadows interact due to movement is sufficient for the learning goals. And students frequently referred to this context in subsequent tasks and modules, illustrating the lasting value of such metaphors in function learning with nomograms. Another crucial design consideration is the integration of an interactive, dynamic learning environment that offers real-time feedback (Abrahamson, 2014; Alberto et al., 2022; Shvarts et al., 2021). The students responded positively to the interactive nature of our tasks, with both pairs indicating that they enjoyed engaging with the tasks and observing the changes in a real-time manner. Notable comments included the rewarding experience of seeing the arrows turn green, the ease of understanding how the graphs work through the dynamic lines, and the preference for this interactive learning environment over traditional textbooks. This immediate, sensory feedback provided by the tasks fosters a more engaging, intuitive, and satisfying learning experience, highlighting the potential benefits of integrating such elements into mathematics learning. At the heart of our embodied design is the intentional and tight coupling of learning goals with target tasks. We aimed to impart an understanding of functions as relationships between two variables. To this end, the two-hand coordinating motion served as an ideal task: it is an accessible action that affords both stability and dynamism. The stability of this action—consistent action type across all tasks, regardless of function type — could facilitate the emergence of body-artifact functional systems (Shvarts et al., 2021), while the dynamics allows for a wide range of movement patterns, mirroring the various properties of functions. This close alignment between task and learning goal was crucial in ensuring students develop new body potentialities and create new affordances under the embodied learning environment.

In conclusion, this study highlights the advantages of integrating digital-embodied nomogram tasks to foster FT. Such an approach appears to deepen students' grasp of abstract mathematical concepts by providing concrete experience. Furthermore, these findings offer a robust foundation for future research in FT, embodied learning, and the role of DT in mathematics education.

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Chapter 4 A digital-embodied design for functional thinking in the classroom

Abstract To address the challenge of teaching functional thinking (FT), this study proposed to ground mathematical reasoning in tactile experiences and investigated whether a digital-embodied design using nomograms enhances FT in a classroom setting. A teaching experiment was conducted with 39 9th- grade students across three 1-hour sessions, each dedicated to one aspect of FT: Input-Output, Covariation, and Correspondence. In Module 1, real-life contexts and application of function rules invite Input-Output thinking. In Module 2, bimanual coordination tasks with nomograms target Covariation. In Module 3, the transitions between different function representations focus on Correspondence. Data from pretests, posttests, classroom observations, and mini interviews demonstrated significant improvements on all aspects of FT, especially Covariation. Key design features—real-life context, bimanual coordination movements, real-time feedback, and various function representations—helped students bodily engage with functions, supporting smooth transitions from sensorimotor experiences to mathematical reasoning. In conclusion, integrating digital-embodied tools into classroom may support FT development.

Keywords Functional thinking; Embodied design; Digital technology; Mathematics education; Classroom implementation; Nomogram; Parallel axes representation

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4.1 Introduction

Embodied cognition represents a paradigm shift in the understanding of how cognitive processes are rooted in the body's interactions with its environment. According to this perspective, cognition is not only a product of abstract mental functions but is deeply rooted in the physical experiences, perceptions and actions of the body (Barsalou, 1999; Lakoff & Núñez, 2000; Varela et al., 1991). In the field of mathematics education, embodied learning has shown promise in reshaping our perspectives on how students understand complex and abstract mathematical ideas. Research suggests that mathematical cognition is tied to sensorimotor experience, as students' sensemaking of abstract mathematical concepts includes gestures, spatial reasoning, and bodily movements (Abrahamson & Lindgren, 2014; Nemirovsky et al., 2013). The integration of digital technologies into this framework has led to the creation of digital-embodied learning environments, which allow students to engage with abstract concepts through interactive digital tools (Georgiou et al., 2021; Pittalis et al., 2024). These environments, through tools such as virtual manipulatives or motion-based interfaces, provide tangible and concrete bodily experiences that align physical actions with mathematical concepts, making these concepts more accessible (Pittalis & Drijvers, 2023; Shvarts et al., 2021; Wilson, 2002). The embodied approach using digital technology leads to immersive and interactive learning experiences. This approach not only enriches the learning process but also aligns constructivist theories of active knowledge construction, with the view that knowledge is constructed through abstract reasoning and tactile interaction with the world.

Since the beginning of the twentieth century, functional thinking (FT) has emerged as a crucial topic in mathematics education (Thompson, 2008). FT is essential for modeling real-world problems and engaging in complex problem solving. Despite the recognized importance of FT in developing mathematical literacy, students often struggle with the abstract nature of functions and their representations (Ellis et al., 2016; Tanışlı, 2011; Stephens et al., 2017; Thompson & Carlson, 2017). For example, grasping the dynamic nature of covariation can be challenging—understanding how two quantities change in relation to each other—especially when students are required to visualize or mentally manipulate these relationships (Castillo-Garsow et al., 2013; Thompson & Carlson, 2017). Recent research suggests that digital-embodied learning environments offer a promising solution direction. By bodily interacting with digital representations of functions, students could better understand covariation and other abstract concepts by grounding their learning in sensorimotor activities (Duijzer et al., 2019; Shvarts & Abrahamson, 2019). Thus, the incorporation of digital-embodied learning environments in teaching FT presents considerable potential.

Transitioning embodied learning from controlled laboratory research to a real classroom is challenging. The practical issues, including teacher involvement, varying student abilities, and technical limitations, make the adoption of embodied learning difficult on a larger scale. Previous studies also highlight the nuanced relation between technology integration and learning outcomes, especially on how digital tools like GeoGebra, interactive whiteboards and tablets may enhance conventional teaching methods (De Vita et al., 2018; Duijzer et al., 2019; Günster & Weigand, 2020). The transition from laboratory settings to classroom environments amplifies these complexities, revealing variations in implementation effectiveness due to factors such as scaling for larger groups, accommodating diverse student abilities, and local contextual dynamics (Alberto et al., 2022; Cai et al., 2020; Kosmas & Zaphiris, 2023). Empirical evidence suggests that the success of technology-enhanced learning environments depends on how well these variables are managed. The effectiveness of digital tools in improving learning outcomes often varies based on how they are integrated into the classroom, with factors such as class size, teacher preparations, and student diversity playing key roles (Alberto et al., 2022; Drijvers, 2019). Given these challenges, this study aims to further investigate the potential of an embodied learning approach in classroom settings.

4.2 Theoretical Framework

The integration of digital-embodied learning environments into mathematics education represents a shift toward more interactive and tangible methods of teaching abstract concepts, such as FT (Abrahamson et al., 2021; Drijvers, 2019). This study is anchored in several key theoretical frameworks including Embodied Design, Functional Thinking, and the use of Digital Technology. It specifically explores the role of a digital-embodied design using nomograms in fostering FT within a classroom setting.

Delving deeper into embodied cognition within a mathematics classroom, Abrahamson (2009) refined the concept of Embodied Design as a design methodology. This innovative method involves creating learning environments and resources that require students to use their bodies in learning activities, thereby grounding abstract mathematical ideas in physical experience. The core premise is that cognitive processes are influenced by bodily interactions, which suggest that physical engagement can enhance conceptual understanding (Abrahamson & Lindgren, 2014; Barsalou, 1999). Embodied design in mathematics education has been shown to improve students' grasp of complex subjects, such as FT, by making the abstract nature of mathematical concepts more accessible and intuitive (Nathan & Walkington, 2017). For instance, the use of manipulatives, gestures, and motion-capture technology allow students to internalize mathematical ideas through physical actions and visual representations (Duijzer et al., 2019; Pittalis & Drijvers, 2023; Shvarts & Abrahamson, 2019). Embodied design can be categorized into different strands: action-based, perception-based, and incorporation-based design (Abrahamson & Lindgren, 2014; Bos et al., 2022; Wei et al., 2023). Action-based designs ground mathematical concepts in students' natural capacity to adaptively solve sensorimotor problems (Palatnik et al., 2023; Shvarts & Abrahamson, 2019). For example, students can manipulate a triangle's vertex to discover its equidistant properties, tracing a parabola (Shvarts & Abrahamson, 2019), while in a histogram example, students can move balls and bars to represent data, reinventing the histogram through actions (Boels & Shvarts, 2023). These designs promote conceptual understanding by enabling students to discover dynamic relationships with their sensorimotor experiences. Perception-based designs emphasize students' perceptual capacity, supporting the understanding of mathematical structures through perceptual sensitivity of phenomena (e.g., ratios, balance; Abrahamson, 2012; Tancredi at al., 2021). Incorporation-based designs intentionally remove a digital artifact's functionality, prompting students to internalize and perform this functionality through bodily experience (Bos et al., 2022; Botzer & Yerushalmy, 2008). Across these embodied design strands, the learning process unfolds through iterative perception-action loops (Shvarts et al., 2021), where actions generate perceptual feedback that guides subsequent action.

Functional Thinking is a key component of school mathematics, emphasizing the understanding of functions as objects, their representations, and the relationships between those representations (Vollrath, 1986). The development of FT is crucial for students' ability to model real-world situations mathematically and to solve complex problems (Kaput, 1998). In this study, we focus on three aspects of FT: Input-output (IO) thinking, which focuses on calculation processes through input-output assignments and pattern recognition (Doorman et al., 2012; Frey et al., 2022; Stephens et al., 2017); Covariation (COV) thinking, emphasizing the dynamic relationship between two variables and their covaried relationships (Carlson et al., 2002; Confrey & Smith, 1995; Doorman et al., 2012; Thompson & Carlson, 2017);

and Correspondence (COR) thinking, which delves into the unique pairing of these variables across multiple representational forms, such as arrow chains, tables, graphs, formulas, and phrases (Doorman et al., 2012; Pittalis et al., 2020; Smith, 2008). Another frequently discussed aspect of FT, namely mathematical object thinking, was excluded from this study due to the educational level being considered. Within the mathematical object aspect, a function is recognized as part of a family of functions (Sfard, 1991), including higher-order operations such as composition, transposition, and differentiation (Wei et al., 2023). These aspects enable the development of a holistic understanding of functional relationships, from recognizing sequences and patterns to interpreting and mapping variable interactions (Wei et al., 2024).

Additionally, we explore an innovative approach to function representation in teaching and learning FT, using a graphical calculation tool known as a nomogram (see Figure 4.1). The idea of using nomograms in digital learning settings first emerged with the work by Nachmias and Arcavi (1990), who introduced the Parallel Axes Representation. Nomograms, including various formats like the horizontal version DynaGraph identified by Sinclair et al. (2009), are helpful in visualizing functional relationships. A nomogram comprises two parallel number lines, which represent values of the input and output variables respectively. Points on the axes are connected by arrows or lines that illustrate the functional relationship mapping from input to output. Within our digital-embodied learning environment, each axis features a movable point, allowing students to simultaneously adjust the x and y values by moving both their hands. This type of bimanual movement—coordinated actions involving both hands—can enhance students' understanding of mathematical concepts by providing a concrete, physical experience of abstract ideas (Abrahamson et al., 2014; Jaber et al., 2024). As students manipulate the points, the interface offers real-time feedback: when the points correctly represent an input-output pair, the arrow between them changes from red to green, and the green arrows remain visible. If the points do not correspond to a valid pair, the arrow turns red and leaves no trace. This real-time feedback is similar to the features in the Mathematical Imagery Trainer for Proportion (Abrahamson & Trninic, 2011), where color change provides immediate, intuitive guidance to students. This dynamic, hands-on interaction aligns with the principles of embodied cognition and constructivist learning theories. In this way, the digital-embodied nomograms make the often-challenging abstract concepts of FT more tangible, potentially fostering a deeper understanding of functions through interactive, embodied experiences.

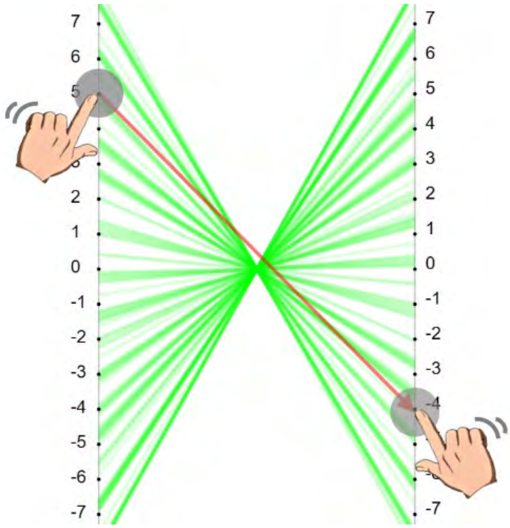

Figure 4.1 A Digital-embodied nomogram for the function $x \rightarrow -x$

Advancements in digital technology have transformed educational practices, allowing for the development of interactive and engaging learning environments. Research in mathematics education has increasingly focused on the potential of digital tools to support conceptual understanding and problem-solving skills (Doorman et al., 2012; Roschelle et al., 2010). Digital-embodied learning environments, which combine physical interaction with digital representations, offer a unique platform for students to explore mathematical concepts in a dynamic and intuitive manner. For example, to teach the concept of a parabola, learners can manually plot a series of green isosceles triangles that collectively form a U-shaped trace (Palatnik et al., 2023; Shvarts & Abrahamson, 2019). This task asks students to physically explore the parabola's geometric properties by preserving the equal distances from a point to the directrix and focus. Through this hands-on interaction, students intuitively grasp the reflective symmetry and structure of the parabola by combining both manual tracing and digital feedback. Additionally, constructivist learning theory emphasizes the importance of learners constructing their own knowledge through active engagement with the environment. This theory is also consonant with the use of digital-embodied learning environments, as it advocates for instructional designs that allow students to explore, experiment, and to make sense of mathematical concepts through direct manipulation and interaction. The use of digital-embodied nomograms in teaching FT aligns with this theory by enabling students to visualize and manipulate functional relationships, thereby developing conceptual understanding that integrates these tools into their mathematical reasoning.

In summary, the theoretical background of this study integrates embodied design, the importance of FT in mathematics education, and the affordances of digital technology. This study aims to address the overarching research question:

Research Question How can an embodied design using nomograms foster functional thinking in a classroom setting?

This question is split up into two sub-questions to explore the roles of the digital-embodied learning environment's impact on students' mathematical reasoning:

Sub-RQ1 How does a digital-embodied design using nomograms affect the various aspects of functional thinking among students within a classroom setting?

Sub-RQ2 How do the design features contribute to the development of functional thinking?

The key design features considered in this study include real-life contexts, bimanual coordination movement, real-time feedback, and multiple representations. By exploring these questions, the study aims to contribute to effective mathematics teaching and learning while offering insights into how digital-embodied learning environments can be designed to enhance students' understanding and application of FT in a classroom setting.

4.3 Method

This study was structured as a teaching experiment conducted in two Grade 9 classes, aimed at exploring the impact of digital-embodied designs on students' functional thinking within a classroom setting.

Module Design

The design comprised three learning modules designed around the concept of digital-embodied design, specifically using nomograms to enhance FT. As outlined in our previous study (Wei et al., 2024), the design process began with the development of Hypothetical Learning Trajectories (HLT), including a detailed hypothetical learning progression (see Appendix 2 for an example). Each learning module, as shown in Figure 4.2, was structured to progressively build on students' understanding and application of FT concepts in varied contexts, with both real-life and pure mathematical scenarios. Different types of functions, such as linear functions, quadratic functions and the absolute value function, are addressed. For those interested in exploring these modules further, access is provided through the following link: https://embodieddesign.sites.uu.nl/activity/functional-thinking/.

In Module 1, the foundational stage, the emphasis on light ray contexts introduces students to IO thinking through engaging real-life scenarios. The light rays from the object to its shadow in the nomogram are reflected by arrows going from the input to the output element, with only a limited number of arrows drawn, even though they could originate from every point on the input axis. This module transitions from unimanual to bimanual movements, also laying the groundwork for covariational reasoning (Figure 4.2a). In this 'Bulb and mosquito's shadow' task, students can move the position of the mosquito and observe how the position of its shadow changes correspondingly. The relationship between the position of the mosquito and its shadow is: $\mathrm{height\_shadow} = \mathrm{height\_mosquito} \times 1.5$.

Module 2 builds upon this by shifting from light ray contexts to purely mathematical ones, replacing them with semi-nomograms—nomograms without numbers—and then formal nomograms (Figure 4.2b). This transition guides students toward a richer perceptual and kinesthetic experience in mathematical reasoning. Tasks involving bimanual movement reflect COV, where the coordinated movement of both hands mirrors the covariation between two variables. In this 'Keep the arrow green' task, students aim to maintain the arrow's green color, indicating a correct relationship between the heights of the two ends. The relationship between the input and output in this task is: $\mathrm{output} = -2 \times \mathrm{input} - 4$.

Module 3 focuses on COR thinking, inviting students to explore the transitions between nomograms, function graphs, and formulas through both unimanual and bimanual activities (Figure 4.2c). This module is pivotal in enabling students to perceive and act on the dynamic visualization of various functional relationships, fostering a deep, intuitive understanding of the connections between action-perception loops and function representations. In this 'Find the domain and range' task, students can first adjust the arrow in the nomogram and try to keep the corresponding point moving on the function graph. After that, they can predict the domain and range by observing the range of the arrow's traces on the nomogram. The relationship between x and y is: $y = \sqrt{x+2}$.

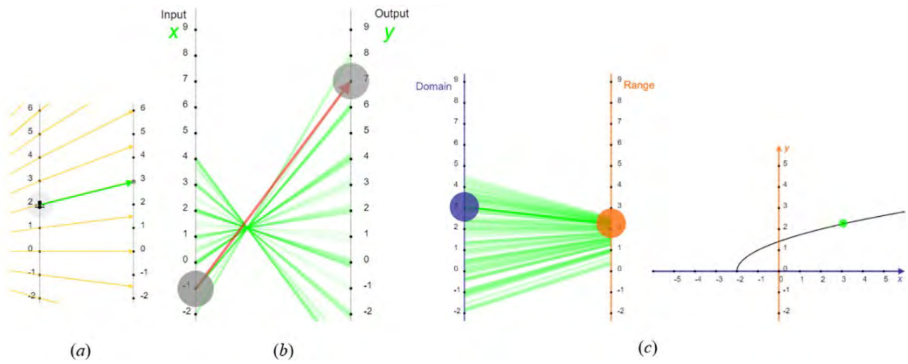

Figure 4.2 Examples of tasks from each module: (a) Bulb and mosquito's shadow; (b) Keep the arrow green; (c) Find the domain and range.

Instruments

To comprehensively assess students' FT development and learning processes, this paper employed three data collection instruments: Pretest and posttest, Answer boxes in the digital-embodied learning environment, and Mini interviews.

Pretest and Posttest

The pretest and posttest were administered in a paper-pencil format, designed in line with the three facets of FT: input-output, covariation, and correspondence. Each test consisted of 15 items, scored on a 5-point scale to capture levels of student understanding, resulting in a maximum score of 75 points per test. These tests comprised two types of items: those closely related to the FT learning module content ("close assessment", nine items) and those relevant to the broader curriculum but varying in specific contexts ("proximal assessment", six items). This was done to measure both direct learning outcomes and generalized skill application (Ruiz-Primo et al., 2002). The posttest was designed to maintain structural and conceptual equivalence with the pretest. It featured variations in numbers or contextual settings to prevent rote memorization while keeping the cognitive demand of the tasks. Of these fifteen items, nine were categorized as close assessments, incorporating contexts similar to those encountered in the learning modules, like light and shadow tasks, and comparisons between nomograms and function graphs. The remaining six items were proximal assessments, introducing contexts not explicitly covered in the learning modules, such as scenarios involving a moving walkway and function tables.

The validity of the assessment tool was established through a multifaceted approach, including expert reviews from both experienced mathematics teachers and researchers in mathematics education research. These experts ensured the assessment items were aligned with the core aspects of FT, thereby affirming the content validity of the tool. The pilot of the pretest to 51 students allowed for a classical test analysis, including calculations of p-value, variance, item reliability index (Rir), and item-total correlation (Rit), culminating in a Cronbach's alpha (α) of 0.71. This result indicates satisfactory internal consistency and thereby affirms the reliability of the assessment tool. Based on the insights garnered from this pilot study, several adjustments were made to enhance the assessments' clarity and simplicity. These adjustments included the removal of an item deemed overly simple and modifications to the test instructions to ensure greater clarity. Such refinements were crucial in enhancing both the validity and reliability of the assessments. These steps, from pilot testing to statistical analysis and subsequent adjustments, underscore the thorough process undertaken to ensure that the assessments accurately and reliably measure students' FT (detailed test example available in Appendix 1).

To ensure the reliability of scoring the pretest and posttest, two independent coders evaluated the work. The second coder reviewed 50% of the total data. Inter-rater reliability was assessed by calculating Cohen's Kappa, which yielded a coefficient of 0.96, indicating an excellent level of agreement between the coders' independent assessments. After their independent assessments, a consensus discussion was conducted to reconcile any differences. In addition to the reliability checks, the normality of the distribution of the difference scores between the pre- and posttests was assessed using the Shapiro-Wilk test, yielding the following statistics: W_{diff} (38) = .98, p = .70. These results support the assumption that the distribution of difference scores can be considered normal. Therefore, a paired t-test was used to compare the pretest and posttest scores of students, addressing Sub-RQ1.

Answer Boxes in the Digital-embodied Learning Environment

Students provided written responses in answer boxes within the learning modules after completing embodied tasks. These responses captured reasoning processes and FT understanding. A four-level grading system was built: Integrated reasoning, Basic reasoning, Simple observation, and No reasoning, where "No reasoning" includes both incorrect reasoning and blank responses (detailed grading rubric available in Appendix 4). For example, in task 2.1, where students were required to describe how they keep an arrow green through bimanual movement on the nomogram $y = x + 1$, the reasoning was classified as follows:

- Integrated Reasoning: Demonstrates comprehensive reasoning by integrating movement speed, direction, and the maintenance of geometric relationships (e.g., fixed angles or height differences) in their explanations.
- Basic Reasoning: Shows reasoning that focuses on one aspect (either speed or direction) but lacks consideration of geometric relationships or the simultaneous integration of multiple factors, such as the angle/height difference. An exemplar response is, 'Make sure that both your hands are going the same direction at the same time/speed.'
- Simple Observation: Makes simple observations without engaging with the concept of bimanual coordination or the deeper FT required to interpret or manipulate the nomogram. An exemplar response is, 'It turns green at a certain angle.'
- No reasoning: Simple statements without focusing on the bimanual movement. For instance, an example from a student, 'I move very slowly one finger at a time.'

To confirm the reliability of grading students' responses in the answer boxes, two independent coders reviewed the submissions. The second coder analyzed 20% of the responses. Following their independent evaluations, a discussion was held to reconcile any discrepancies. The inter-coder reliability for these assessments was quantified using Krippendorff's alpha, which produced high coefficients of 0.80 for the answer box gradings. It signifies a robust level of agreement between the coders post-discussion and reinforces the reliability of the grading process.

Mini interviews

The development of the mini interview protocol was derived from the core tasks identified within the HLT (Wei et al., 2024). This alignment is essential for gathering detailed information on students' progression along these predefined learning paths. The primary objective of the mini interviews is to encourage students to reflect upon and elaborate on their written responses. This is achieved through a carefully structured series of open-ended questions and follow-up queries based on their initial reactions. This method is designed as an interactive dialogue that prompts deeper student engagement with their learning processes, thereby yielding richer, more detailed insights into their learning progression along the HLT (Drijvers, 2003). The mini interviews were carried out by the research team during the teaching experiment. Following the questions from the interview protocol, the researchers conducted interviews with students, either individually or in pairs. Most interviews lasted about one to two minutes, were audio-recorded, and then transcribed.

To guide the qualitative analysis of data collected from mini interviews, a detailed codebook was developed. This codebook outlines specific categories and descriptions for coding students' interactions with the digital-embodied learning environment, focusing on aspects of FT (Wei et al., 2023) and embodied learning (Abrahamson & Lindgren, 2014; Shvarts et al., 2021). It includes codes for input-output thinking, covariation thinking, correspondence thinking, action interaction, perception interaction, and the perception-action loop. Additionally, it addresses students' difficulties, strategies, and improvements in FT, along with emerging themes that link embodied experiences to mathematical understanding. To ensure the reliability of coding the interview quotations, two independent coders reviewed the transcriptions. The second coder coded 15% of the transcription. After independently completing the coding, a discussion was conducted to resolve any discrepancies. The inter-coder reliability was assessed using Cohen's Kappa, resulting in a coefficient of 0.73, indicating a substantial level of agreement between the coders.

Participants

A total of 39 grade-nine students, aged 14 to 15 years, were enrolled in the study from an international school located in the Netherlands. These students, all from two classes, were inclusively recruited for the research, with no selection criteria applied. All students and their parents provided informed consent, and participation was voluntary. The students were average to high achievers and had a foundational understanding of functions, including experience with number lines, basic algebra to describe relationships between quantities or variables, using algebra for modeling real-world situations, and basic graphing skills in coordinate systems. They had not yet studied the formal definition of a function. To ensure uniformity in the research conditions, all participants received the same instruction and engaged in identical tasks within the digital-embodied learning environment.

Intervention

The experiment spanned over three sessions for each class, with each session lasting one hour. These sessions were scheduled to ensure a consistent and immersive learning experience for the participants. To facilitate this, every participant was equipped with a tablet (Figure 4.3). Each session was planned to include a 5-minute introduction or review phase, a 45-minute exploration period, and a 10-minute recap session (detailed teaching manual example available in Appendix 3). The introductory phase aimed at setting the stage for the day's activities, revisiting key concepts from previous sessions or introducing new ones relevant to the day's tasks. During the exploration period, students engaged with the digital-embodied learning environment in a self-directed manner. They were encouraged to interact with the tasks, discuss their findings, and collaborate. There are a few tasks, such as the one where two students plot a nomogram together, that required collaborative learning, with each student manipulating a different point. This hands-on period was crucial for students to discover and apply concepts of FT within the digital-embodied learning environment. To conclude each session, a 10-minute recap was conducted to facilitate a whole-class discussion. It was carried out by the researcher and served to highlight the learning goals of the session, to address common challenges encountered by students during their exploration, and to reinforce key concepts.

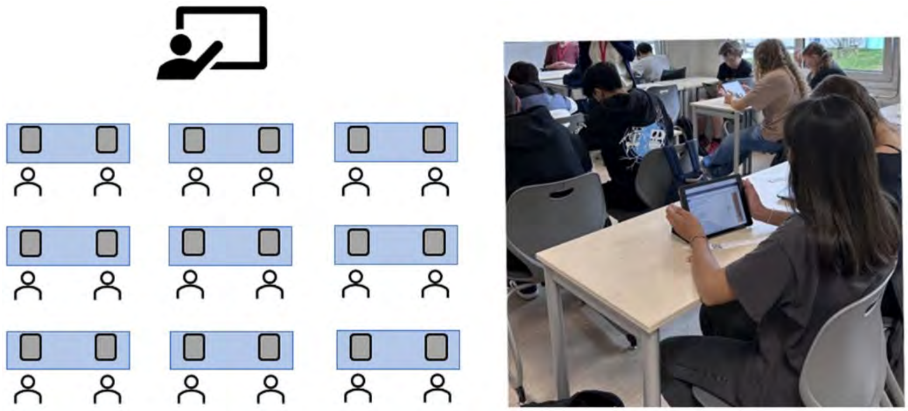

Figure 4.3 Classroom setup featuring digital-embodied nomogram tasks for FT

Throughout the experiment, researchers acted as tutors, providing guidance and support to students as they navigated through the learning modules. Classroom teachers assisted with managing classroom dynamics and ensuring that the sessions progressed smoothly. This collaboration ensured that the educational environment was conducive to both exploration and learning, allowing for an effective investigation into the role of digital-embodied designs in promoting FT within a classroom setting.

Data Collection and Analysis

Pretest and Posttest

The pretest and posttest were administered before and after the three-session teaching experiment. These assessments aimed to measure the levels of FT among students at two points in time, providing a basis for evaluating the impact of the teaching intervention. The analysis of these scores employed a paired t-test to assess the improvement in students' FT levels before and after the teaching intervention, addressing Sub-RQ1.

Answer Box in the Digital-Embodied Learning Environment

The data derived from students' written responses in the answer boxes were from the digital-embodied learning environment. All student responses in the answer boxes were saved under each student's individual account. After the experiment, the research team downloaded these responses and evaluated them using the four-level grading system described in the Instruments section. Beyond assigning scores, the analysis also involved identifying representative examples of student responses to illustrate typical reasoning patterns. These examples provide additional insight into the students' learning processes, contributing to answering Sub-RQ2.

Mini interview

Data gathered from transcripts of mini interviews conducted with students during their interaction with the learning modules provided rich insights. Researchers used the same codebook (outlined in the Instruments section) to categorize students' actions, difficulties, and conceptual understandings of FT. The analysis highlighted typical or noteworthy patterns of student explanations, showing how the design features influenced their thinking. The analysis also included selecting representative cases of students' articulations, which are presented in the *Results and Interpretations* section. This method allowed us to collect comprehensive data in addition to task completion and to gain insights into students' problem-solving processes and mathematical reasoning (Lobato et al., 2012), thus addressing Sub-RQ2.

By systematically integrating the three instruments, this study triangulates quantitative and qualitative data to provide a comprehensive picture of students' FT development. Table 4.1 summarizes how each instrument contributes to addressing the research questions.

Table 4.1 Summary of instruments and their contributions to the research questions

	Pretest and Posttest	Answer Box Responses	Mini interview
Sub-RQ1: How does a digital-embodied design using nomograms affect the various aspects of FT among students within a classroom setting?	Quantitatively measures students' FT levels before and after the intervention: - Addresses gains in input-output, covariation, and correspondence aspects of FT. - Paired t-test determines overall improvement in FT.	Enables analysis of specific reasoning patterns connected to FT aspects.	Clarifies subtle changes or misunderstandings not visible in written responses.
Sub-RQ2: How do the design features contribute to the development of functional thinking?		Demonstrates how design features (real-life context, bimanual movement, real-time feedback, multiple representations) shape students' written reasoning and responses.	Probes how students experience each design feature in real time to uncover their strategies, highlighting the role of each feature in fostering FT.

4.4 Results and Interpretations

This section presents the results derived from three sources of analysis: (1) pretest and posttest data, including both scores and examples of students' answers, (2) students' responses in the answer boxes of the digital-embodied learning environment, and (3) insights from mini interviews.

Pretest and Posttest Results

Quantitative data analysis indicated a significant improvement in students' functional thinking levels (Sub-RQ1). A paired *t*-test was conducted to compare the pretest and posttest scores of students. Overall, there was a statistically significant increase in the posttest scores (M = 50.05, SD = 8.65) compared to the pretest scores (M = 33.09, SD = 10.76, $t(38)$ = 10.18, p < .001). The effect size, Cohen's d = 1.74, indicated a substantial improvement in students' FT levels, as evidenced by an average increase of 16.96 points between the pretest and posttest scores (Maximum score is 75). Figure 4.4 illustrates the variability in improvement across different pretest scores, reflecting students' initial FT levels. The results indicate that students, regardless of their starting FT levels, showed significant gains following the intervention. Notably, those with lower initial FT scores tended to demonstrate greater improvement than students with higher pretest scores.

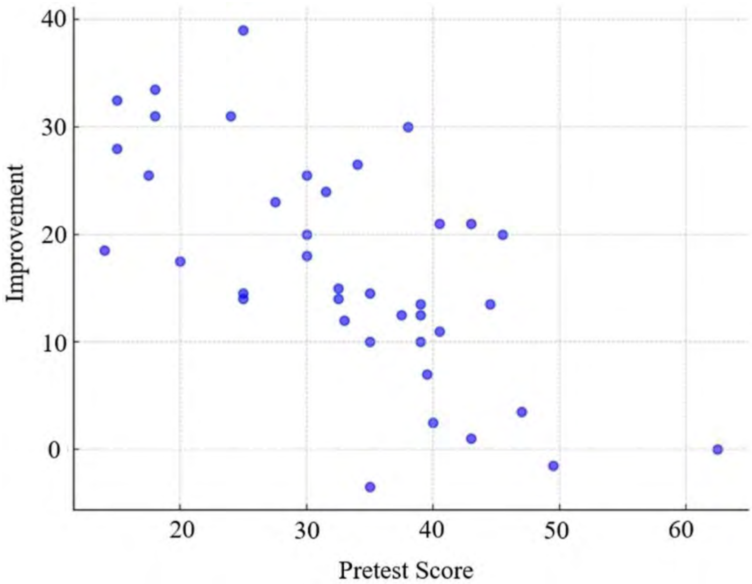

Figure 4.4 Relationship between students' pretest scores and their improvement

When diving into the results on functional thinking aspects—Input-Output, Covariation and Correspondence—the results show improvements across these facets (as detailed in Table 2). The assessment framework for these FT aspects comprised a differentiated number of items: three items for the IO aspect, five items for the COV aspect, and seven items for the COR aspect, with each item scored on a 5-point scale.

A closer examination of Table 4.2 indicates an improvement in students' FT levels. In the COV aspect, the mean score increased from 1.7 to 3.5, reflecting a stronger grasp of the relationship between variables. The COR aspect's mean score experienced a rise from 2.0 to 2.8, highlighting a deepened understanding of pairing relationships and the use of multiple representations. The IO score also improved, from 3.5 to 4.5, showing the teaching intervention's effectiveness, albeit with a smaller relative gain due to the ceiling effect. These improvements affirm the positive impact of the lesson on all three facets of functional thinking within the classroom setting.

Table 4.2 Pretest vs Posttest for each aspect of FT (Maximum score is 5)

	IO	COV	COR
Pretest	3.5	1.7	2.0
Posttest	4.5	3.5	2.8
Improvement	1.0	1.8	0.8

The posttest responses revealed students' enhanced understanding of the COR aspect of FT—using different representations, such as nomograms, to depict functional relationships. For instance, in the pretest (Figure 4.5a), a student adopted simple geometric shapes to illustrate the context provided by the test item. In contrast, the posttest showed a marked evolution in the student's approach: the term 'nomogram' was explicitly used, and a correctly plotted nomogram was employed to accurately address the question (Figure 4.5b). While this adaptation is expected, its significance lies in how students transitioned from informal and context-based representations to a structured mathematical tool. This progression was not isolated but observed across different items, signifying a broader adaptation among students to use this new representation as a mathematical tool. Furthermore, this adaptation served as evidence of the development of COR thinking, as students were able to represent functional relationships through multiple representations, including nomogram and formula in this case.

Results from Students' Responses in the Answer Boxes

To assess the extent to which students achieved the learning goals outlined in our HLT, we coded their responses in the answer boxes using the above-mentioned four-level grading system: No reasoning, Simple Observation, Basic reasoning, and Integrated reasoning. All tasks were categorized into eight groups based on their goals and forms. This allowed us to systematically measure how different aspects of FT were developed, as relevant for Sub-RQ1. This section first provides a brief overview of the task groups' information and how design features were embedded in each group of tasks. It then presents the coding results, showing the distribution of students' reasoning levels across different task groups. Finally, two representative examples are analyzed to further demonstrate how students' responses reflect their reasoning levels and how their thinking may have been influenced by the design features (as questioned in Sub-RQ2).

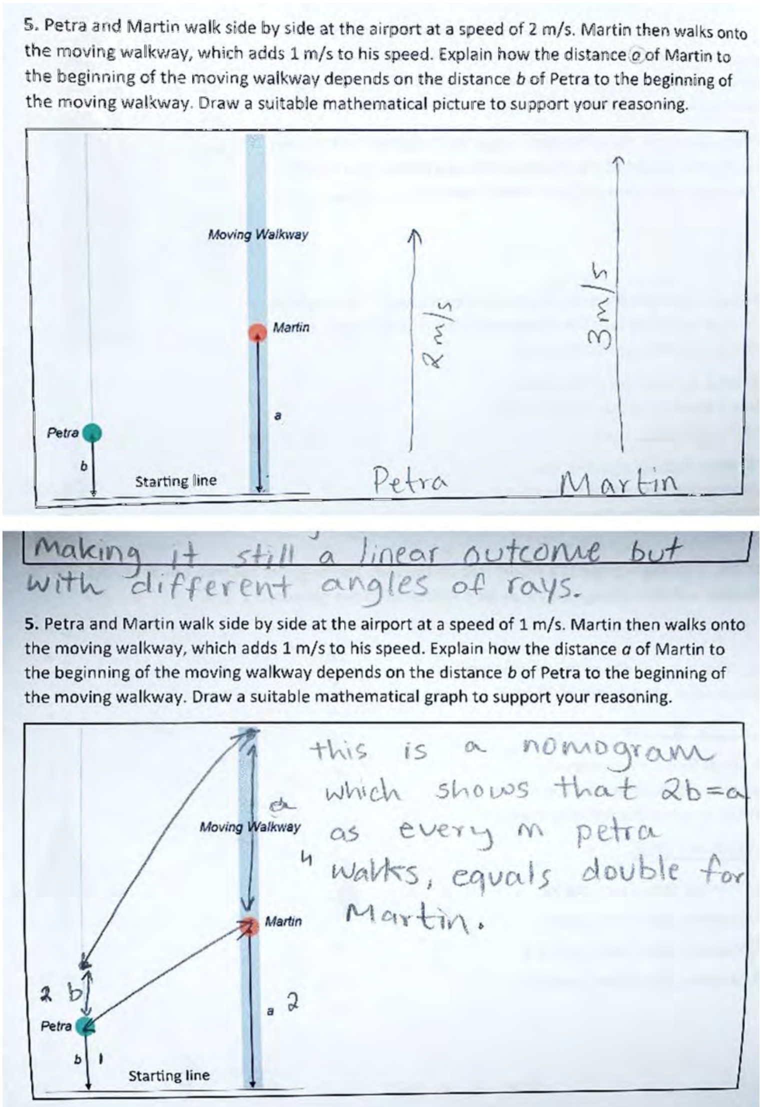

Figure 4.5 **(a)** The same student’s response in the pretest and **(b)** in the posttest

Figure 4.6 shows the percentage distribution of students’ reasoning levels across eight groups of tasks, offering a quantitative lens through which to view their learning progression in alignment with the HLT. Tasks 1.1-1.5 introduced the IO aspect through unimanual tasks embedded in real-life contexts involving foundational functions like $y = x + b$ and $y = k \cdot x$. In these tasks, students manipulated an object along the input axis, receiving real-time feedback as they observed corresponding changes on the output axis. In Tasks 1.6-1.8, a different real-life scenario was introduced with more complex functional relationships, such as $y = k \cdot x + b$. Students interacted with these tasks either by moving an object or adjusting its height along the input axis to observe changes on the output axis. Task 1.9 served as a transition point, introducing the nomogram as a mathematical tool to represent functions. With a shift from real-life contexts to abstract mathematical representations, Tasks 2.1-2.4 focused on the COV aspect by requiring students to coordinate bimanual movements on semi-nomograms. These tasks allow them to synchronize their two-hand movements, reinforcing the covariation between input and output values. As students adjusted both hands simultaneously, they also received real-time feedback by color-changing cues. Then in Tasks 2.5-2.9, nomograms were introduced with formulas, connecting multiple function representations. The integration of the Cartesian coordinate system in Tasks 3.1-3.3 requires students to explore the relationship between nomograms and function graphs. Students are supposed to convert between these different function representations. Tasks 3.4-3.7 reversed the process, and asked students to plot function graphs based on given nomograms. Finally, Tasks 3.8-3.9 formally introduced the concept of functions, focusing on defining input values, output values, domain, and range. Students plotted nomograms and used them to determine the domain and range of functions (detailed group information is available in Appendix 2).

Since tasks within each group share similar formats but vary only in functional relationships, percentages for each reasoning level were calculated as the average across all tasks within a group. Each task group corresponds to a different stage of the HLT. We excluded the final task of each module from the analysis since too many students did not complete these tasks, due to lack of time. These omitted tasks were designed only as extra challenge exercises intended for students who managed to complete the first nine tasks within the allotted time.

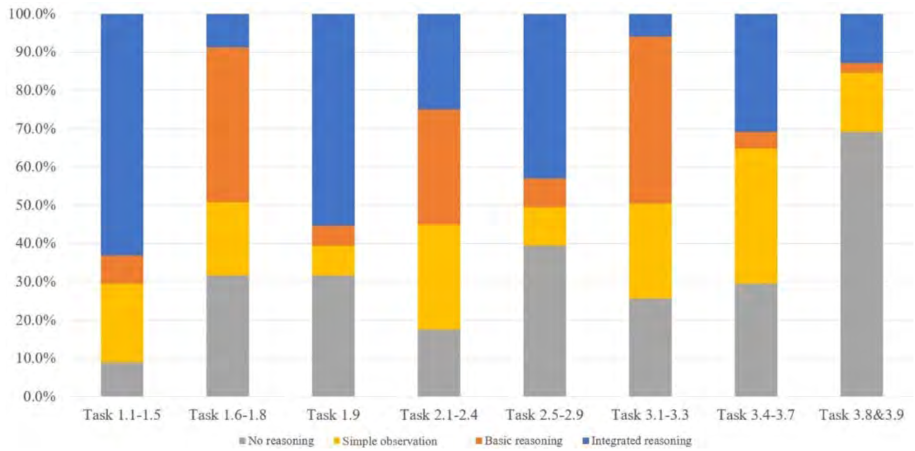

Figure 4.6 Student reasoning levels by task group from the three learning modules

In Module 1, foundational concepts of FT are introduced through the tangible and context-rich scenarios of light ray contexts, engaging students in IO thinking. Tasks 1.1-1.5 exhibit a high degree of integrated reasoning (63.16%), indicative of students successfully navigating initial situational activities that center on IO thinking within concrete contexts, such as manipulating an object and its shadow under light rays. This suggests that the tangible, context-rich tasks effectively introduced students to the foundational concepts of FT. As the tasks evolve to require higher abstract thinking with the shift to complex context (1.6-1.8), a noticeable decrease in integrated reasoning to 8.77% and an increase in no reasoning, imply a gap between task complexity and students' prior knowledge or skill sets. The resurgence in integrated reasoning (55.26%) in task 1.9, despite being a conclusive task introducing the nomogram, could be attributed to students synthesizing earlier concepts with the geometric interpretation of the linear relation introduced in this task, aiding in the conceptual leap required.

Module 2 advances the trajectory by incorporating semi-nomograms and formal nomograms, furthering students' perceptual and kinesthetic experiences of FT. The balanced distribution of reasoning levels in tasks 2.1-2.4, with a notable peak in basic reasoning (30.00%), indicates an appropriate level of challenge that is accessible yet progressively abstract. Nonetheless, the contrasting integrated reasoning (43.00%) and no reasoning (39.50%) in tasks 2.5-2.9 suggest a divergence in students' abilities to apply their understanding of bimanual coordination to mathematical reasoning, highlighting the need for nuanced instructional support. For example, students with integrated reasoning can provide the correct formulas for the nomograms while students with no reasoning, although some of them tried to plot the nomograms, cannot provide the required formulas for the functions.

In the concluding Module 3, the emphasis is on correspondence thinking, where students deal with the complexities of transitioning between multiple representations of functions, including nomograms, function graphs, and formulas. The majority of basic reasoning (43.59%) in tasks 3.1-3.3 implies that the students are struggling with the module's content, possibly due to the abstract nature of transitioning between nomograms and function graphs. The variability in comprehension becomes more pronounced in the latter tasks (3.4-3.7), where simple observation (35.26%) is prominent, and the most challenging tasks (3.8 and 3.9) result in a majority displaying no reasoning (69.23%). This could indicate that the tasks may be too advanced for some students, or that the connection between bimanual movements and the graphical representations of functions, particularly the correspondence aspect of FT in the action-based design of the tasks, requires more explicit instruction or redesign.

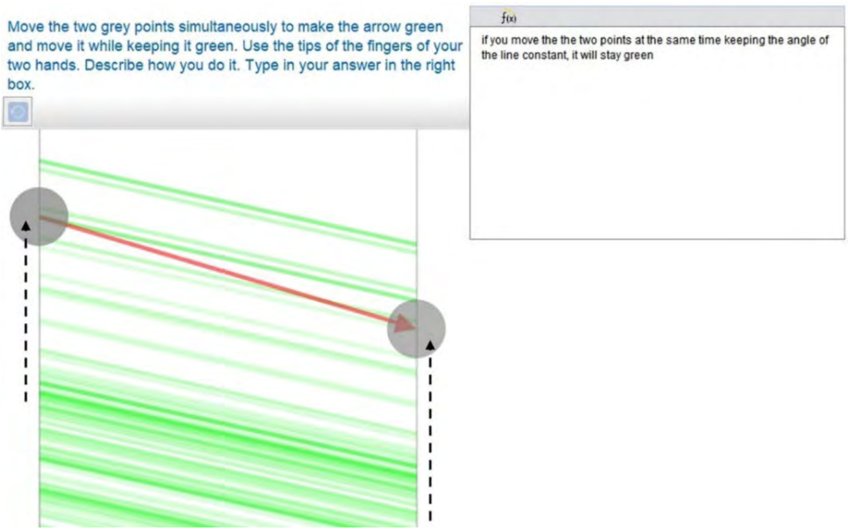

Figure 4.7 An integrated reasoning example for task 2.1

An example of Integrated reasoning is demonstrated in Figure 4.7, where a student articulates the required bimanual movement, stating "move the two points at the same time" and "maintain a constant angle." The color-changing feedback, described by the student as "it will stay green", provided real-time perceptual cues. This kind of mechanism allows students to confirm and keep the correct angle of the target nomogram. By continuously adjusting their bimanual movement based on these visual feedback cues, the student was able to stabilize their coordination and consolidate the target action of the nomogram. This interaction with the digital-embodied learning environment facilitates perception-action loops, that reinforce the connection between movement and mathematical representation.

Figure 4.8 illustrates an example of Basic reasoning, where a student wrote "just move your fingers around a lot", implying simultaneous circular motion of both fingers. This response, while recognizing the need for vertical movement in two directions, falls short in detailing the nuances of speed or height adjustments.

Overall, the distribution of reasoning levels across the tasks suggests that while the early stages of the HLT align with students' competencies, leading to high levels of strong reasoning, as tasks progress in complexity and abstraction, there is a clear need for additional support. This indicates the importance of designing teaching interventions that account for the diverse learning paces and comprehension levels of students, ensuring that the transition from concrete to abstract mathematical reasoning is accessible and effectively facilitated.

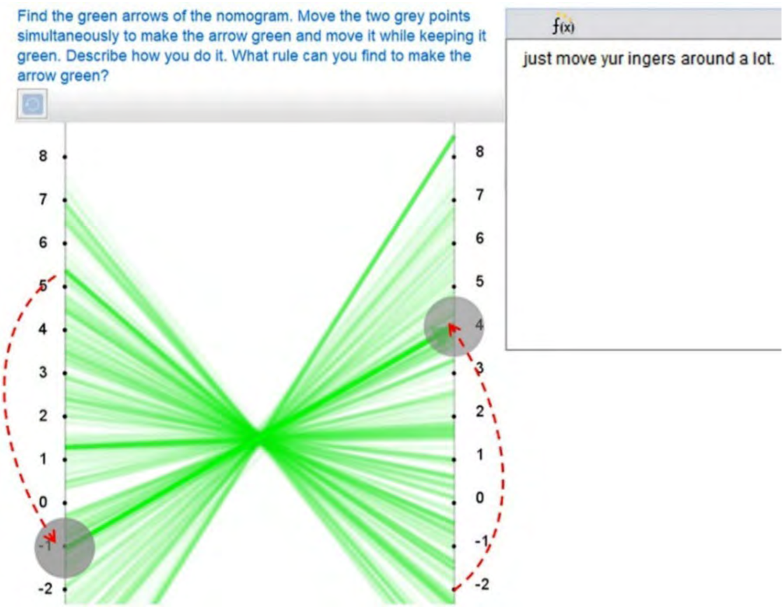

Figure 4.8 A basic reasoning example for task 2.5

Results from Mini interviews

The qualitative data from coding reports of the mini interviews provided depth to these findings. Patterns of understanding evolved from initially superficial to more conceptually grounded as students progressed through the learning modules. This section presents students' expressions in different types of digital-embodied tasks, focusing on the following design features: real-life contexts, bimanual coordination movement, real-time feedback, and multiple function representations (Sub-RQ2). To ensure a comprehensive interpretation, we distinguish between typical responses, which represent common reasoning patterns, and best practice examples, which show more advanced reasoning processes.

The use of Real-life Contexts

This section focuses on the integration of real-life contexts within the learning modules, specifically, on how real-life contexts enrich students' learning experiences and FT. Through the lens of real-life contexts, students encounter mathematical phenomena in settings that are both familiar and meaningful to them. The insights offered by these students emphasize a conceptual shift: from concrete observations [perception] to the abstraction of functional relationships. The following two quotes are typical responses that were observed in similar forms in multiple mini interviews.

> *Students 231006 in Task 1.5*: So I saw that the bulb on [left of] the mosquito. It was on different positions. And these different positions meant different shadows. And I saw that if the mosquito was on the 1, and then its shadow would have been at 1.5 (See Figure 4.2a; $\mathrm{height_{shadow}} = \mathrm{height_{mosquito}} \times 1.5$) … [when asked to compare scenarios] because the bulb, it's like into one direction, but the sun is circular. So it's has rays everywhere. And then also the rays of the sun could be stronger than the bulb. (See Figure 4.9a; $\mathrm{height_{shadow}} = \mathrm{height_{hand}} - 15$)

> *Student 231016 in Task 1.8:* I found that the size of the shadow was always two times the size of the tree, no matter how big or small you made the tree, the size of the shadow is always twice the size of the tree … if you put the bottom of the tree at 1, the bottom of the shadow also at 1 … for example, when I make it 4 units long

from 1 to 5, you can see that the shadow goes from 1 to 9, which is 8 units long. (See Figure 4.9*b*; $\text{height}_{\text{shadow}} = \text{height}_{\text{tree}} \times 2$).

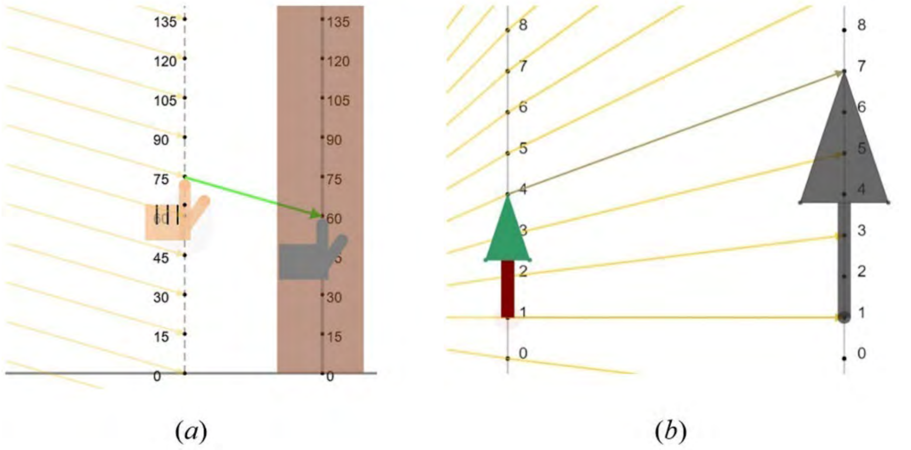

Figure 4.9 Tasks illustrating **(a)** A hand and its shadow under parallel light rays; **(b)** Adjustment of a cardboard tree and its shadow

Analysis: Initially, students grapple with the task of directly mapping inputs to outputs—a foundational mathematical skill in the field of FT. As they interact with these real-life contexts, they begin to identify underlying patterns and relationships. Notably, the emergence of a multiplicative understanding signifies a key development in their mathematical reasoning. Students move beyond mere observation, applying their insights to generalize about proportional relationships. This transition marks a critical step towards mathematical abstraction and reasoning, where specific instances serve as a springboard for the derivation of general principles. This progression indicates their deeper comprehension of linear relationships. Importantly, this journey is facilitated by their engagement with real-life contexts using both observation [perception] and movement [action], which provide a tangible framework for exploring and internalizing abstract mathematical concepts. However, we also observed examples where students applied their everyday experiences, leading to misunderstandings in some cases. For example, when Student 231006 was asked to compare sunlight and bulb light, he focused on the strength of the light rays rather than how the shadow's position is influenced by different light sources. This highlights the need for clear guidance when shifting from familiar real-life contexts to more abstract mathematical reasoning.

Bimanual Coordination Movement

This section explores the reflections of students engaged in tasks requiring the simultaneous manipulation of two variables, showing how embodied learning through bimanual coordination movement enhances understanding of covariational relationships. The following examples show that bimanual tasks enable students to physically engage with and thus internalize function concepts by embodying the relationships between variables. The direct manipulation of points to reflect linear relationships provide a concrete context through which abstract mathematical principles become tangible. This physical interaction is crucial in making abstract concepts accessible through sensorimotor experiences. Student 231010's response is a typical response similar to many other students, while Student 231108's response can be seen as the best practice response.

> *Student 231010 in Task 2.4*: So the relationship that we found between the two different points and how to keep them, how to keep the line green is that you have to move the gray points at the same speed. So at a constant speed but in the opposite directions. (See Figure 4.1; $\text{output} = -\text{input}$)
>
> *Student 231108 in Task 2.7:* You have to multiply it with like [there is] a minus sign. Because when it goes higher it [the other point] turns into negative. The positive number on the x turns into negative on the y. So that means there has to be like a negative number somewhere because it has to bring it down. (See Figure 4.2b; $\text{output} = -2 \times \text{input} + 4$)

Analysis: Student 231010's experience illustrates an understanding of inverse relationships, as the student articulates the necessity of moving two points at constant but opposite speeds [action] to maintain a specific visual indicator (a green line), embodying the functional relationship that output equals the negative of the input. Student 231108's reflection further delves into the concept of linear relationship, where the manipulation of one variable (input) directly influences the other (output) through a specific linear equation [action], in this case, highlighted by a negative multiplication factor. This student's observation about the transition of a positive number on the input-axis to a negative on the output-axis [perception], facilitated by a multiplication with a negative number. Moreover, we observed that most of the students began with relatively static bimanual movements, first adjusting one point and then the other until the arrow turned green. After several attempts, they began to recognize a rough pattern in the relationship between the points and gradually coordinated both hands, resulting in smoother, more fluid movements. Although some of them did not grasp the exact pattern immediately, this iterative process of refining their bimanual movements helped them move closer to understanding the underlying functional relationship. However, as tasks became more complex—asking students to write down the functional rule as a formula—some students changed their movement habits. They adopted a more static approach, using discrete bimanual movements to identify integer pairs, which made deducing the formula easier.

Real-time Feedback

This section delves into the impact of real-time feedback on student learning, particularly through the mechanism of color-changing cues that signal correct or incorrect actions. These cues provide immediate feedback, allowing students to observe the effects of their actions immediately, adjust their strategies, and understand the dynamics of covariational relationships. The immediacy of the feedback ensures that students can quickly correct misconceptions and refine their embodied misconceptions, leading to a more engaged and effective learning experience. The following two quotes are typical responses that were observed in multiple mini interviews.

> *Student 231101 in Task 2.4:* You have to maintain a constant speed. And we have to maintain it to get the angle right. Otherwise, if you go, if one goes too fast and one goes too slow and there's no coordination in it, then we're not going to get the right angle and then there's not going to be a green color.
> *Student 231114 in Task 2.7*: I first moved the lines until I could find out. I could see which ones are green. Then I kept moving. Then I checked how much I got, like the relationship between the left and the right.

Analysis: Students' experience from the above quotations exemplifies the iterative process of learning facilitated by real-time feedback. By experimenting with different positions and movements until the desired feedback (green color) is achieved [perception], students engage in a process of hypothesis testing in action, adjustment, and re-evaluation [action]. This process not only aids in discovering the underlying functional relationships but also in automatizing and describing these concepts through repeated, feedback-informed practice. In addition, real-time feedback acts as a bridge between action and perception. This kind of action-perception loops supports the development of a more intuitive grasp of mathematical concepts, as students learn to anticipate the outcomes of their actions based on previous feedback.

Multiple Function Representations

The feature of multiple function representations within learning modules provides pedagogical value in exposing students to diverse ways of visualizing and understanding functions. This analysis focuses on how the digital-embodied tasks involving the manipulation of sliders to adjust a nomogram and its transformation into a Cartesian coordinate system play a role in developing students' FT. Manipulating an object in one place [action], while simultaneously observing another object moves or changes accordingly in a different place [perception] fosters the connection between the objects and invites the student to make sense of the functional relationships. Student 231002's response is a typical response similar to many other students, while Student 231104's response can be seen as the best practice response.

> *Student 231002 in Task 3.7*: So like the left side is the x-axis. And then the right side is the y-axis. Because when I'm like moving with the slider, when the slider goes to the left, the left side turns into the x-axis as it like becomes flat [horizontal].
>
> *Student 231104 in Task 3.9*: I think the domain represents the x-axis on the graph. And the range presents the y-axis as well.

Analysis: The act of moving a slider [action] and observing the corresponding transformation of a nomogram into a Cartesian coordinate system [perception] enables students to concretely understand the transitions between different function representations (student 231002). This physical manipulation, coupled with the visual changes observed, supports action-perception loops, and helps concrete the transition from the nomogram to the function graph. Specifically, this involves rotating the input axis and projecting the endpoints of arrows onto the two perpendicular axes in the coordinate system. And Student 231104's engagement with the task demonstrates an attempt to link the graphical characteristics of functions with their mapping view, specifically noting the relevance of domain and range within the function graph. Students begin to experience the intricate ways in which these various representations—graphical, numerical, and algebraic—intersect and complement each other in depicting the same functional relationships by directly connecting them visually [perception].

The qualitative insights gathered from the coding reports have enriched our understanding of students' FT development. While the mini interviews reveal diverse experiences, they suggest that the digital-embodied learning environment can facilitate students' journey from recognizing simple patterns to abstracting and applying functional rules. For example, students moved from concrete observations, such as mapping inputs (objects) to outputs (their shadows) using light rays, to exploring more complex contexts, like controlling a slider to convert a nomogram into the corresponding function graph in the coordinate system. However, it is more accurate to describe these instances as specific cases where particular design features—such as real-life contexts, bimanual coordination, real-time feedback, and multiple function representations—supported students in recognizing patterns and linking them to abstract functional rules. This points to the importance of these features in providing tangible experiences that help with the transition from concrete to abstract understanding in FT.

4.5 Conclusion and Discussion

In this section, we will first synthesize the results from the previous section into answers to the research sub-questions. These answers lead to a conclusion on the main research question. After considering the limitations of the study, we will reflect on these conclusions from a theoretical and practical perspective in the discussion section.

Conclusion

This study embarked on an exploration of the impact of a digital-embodied design using nomograms on fostering FT in a classroom setting. From the results of both quantitative and qualitative analyses, we conclude that such a design can enhance students' understanding of function concepts. The findings uncovered the dynamics of learning processes, strategies, and the components within the digital-embodied learning environment that correlate with improvements in FT.

Building on this main conclusion, we now explore the sub-research questions in detail. The first one concerned how a digital-embodied design using nomograms affects the various aspects of FT among students within a classroom setting. From the quantitative analysis of pretest and posttest scores, we conclude that students showed a statistically significant improvement across all three FT aspects. Students with lower initial FT levels demonstrated greater improvement than higher achievers. The detailed assessment of COR, COV, and IO scores highlights the specific areas that benefited from the use of digital-embodied nomograms. The teaching intervention particularly improved students' COV thinking, with a relatively smaller gain in the IO aspect due to the ceiling effect. Compared to the gains observed in other FT aspects, the advancement and average score increase in the COR aspect are less pronounced, suggesting specific challenges or limitations in fully grasping or applying correspondence thinking. These findings suggest that while the intervention was broadly effective, certain areas of FT received more noted benefits. From the qualitative analysis of students' responses in the digital-embodied learning environment, we conclude that students' reasoning levels varied based on the nature of the tasks and their initial FT levels. Specifically, Module 1's concrete, tangible scenarios supported students' comprehension of the IO aspect, leading to a majority demonstrating strong reasoning abilities. However, as tasks became more abstract in later modules, disparities in reasoning levels became apparent. The subsequent introduction of semi-nomograms and formal nomograms in Module 2 aimed to deepen students' perceptual and kinesthetic engagement with FT. Yet, this shift also marked a point where the theoretical underpinnings and practical applications began to diverge for some students. Module 3 further explored this challenge by focusing on correspondence thinking and transition between various function representations, such as nomograms, function graphs, and formulas. The prevalence of average reasoning and weak reasoning indicates students' struggles with the abstract nature of these transitions. These findings suggest that while the digital-embodied design positively impacts FT development, students' progress is influenced by their initial FT levels and the nature of the tasks.

Turning to the second sub-research question on how specific design features contribute to FT development, we conclude that features such as real-life contexts, bimanual coordination, real-time feedback, and multiple representations play an important role in facilitating students' transition from concrete sensorimotor experiences to abstract mathematical reasoning. Real-life contexts, as introduced in Module 1, helped students connect familiar experiences to mathematical concepts, supporting the development of the IO aspect. Students' vertical movement along the number lines serve as a simulation of key mechanisms, specifically through the manipulation of an object (input) and its shadow (output). This simulation guides students to take action and move their fingers in mathematically relevant ways on digital-embodied nomograms. As tasks became more abstract in later modules, the use of bimanual coordination allowed students to physically explore covariational relationships, supporting a concrete understanding of variable interdependence. This hands-on approach allowed students to internalize the concept of covariation through direct manipulation. The integration of real-time feedback further contributed to students' learning by providing immediate cues, such as color changes, to signal correct or incorrect bimanual movements. This feedback mechanism enabled students to adjust their strategies instantly and supported the development of action-perception loops, the core of deeper engagement within a digital-embodied learning environment. Additionally, engaging with functions in multiple representations, such as nomograms, function graphs, and formulas, allowed students to express ideas in different forms. The successful retrieval of one representation could activate others, which helps students to integrate these distinct pieces into a cohesive understanding of functional relationships and promote the development of correspondence thinking.

Overall, we conclude that the use of digital-embodied nomograms leads to marked improvements in students' FT, particularly for those with lower initial proficiency. By integrating real-life contexts, bimanual coordination movements, real-time feedback mechanisms, and multiple function representations, this comprehensive approach fosters a deeper understanding of function concepts in the classroom setting.

Limitations

This study's insights into the use of digital-embodied nomograms for fostering FT are subject to several limitations that could affect the findings' generalizability and depth. The specificity of the sample may restrict the generalization of results toward various educational settings. The short-term nature of the assessment overlooks long-term retention of FT developments, and the specific use of digital tools may not be feasible in all classrooms due to technological constraints. Additionally, the potential ceiling effect observed in the IO aspect suggests that initial student proficiency could mask the intervention's impact, and the study did not thoroughly explore the variability in instructional support and its effectiveness. The lack of a control group further limits claims about learning gains and comparisons with traditional methods. While factors such as classroom dynamics, student variability in mathematical proficiency, and the role of the teacher may influence how the intervention translates from a controlled setting to a real classroom, this study focused on determining whether the learning modules have a similar positive impact in a classroom environment as predicted in the HLT. Future research could further investigate these aspects by incorporating control groups and systematical analyses of student engagement and learning outcomes. Finally, qualitative insights for Sub-RQ2 are based on selected examples rather than a systematic trend. Future studies could adopt a more structured approach to analyzing qualitative data. Addressing these limitations in future research will be essential for developing a more comprehensive understanding of digital-embodied learning environments' role in the development of FT.

Discussion

How do these findings feedback shed light on our initial theoretical framework, consisting of notions on embodied design, functional thinking and digital technology? Grounded in this theoretical framework, our research emphasizes the efficacy of employing digital-embodied nomograms to enhance students' FT in a classroom setting. The results align with the theoretical underpinnings proposed in studies such as Abrahamson and Lindgren (2014) and Drijvers (2019), highlighting the transformative potential of integrating physical engagement and digital representations in learning abstract mathematical concepts. These findings resonate with the core principles of embodied cognition, suggesting that cognitive processes are deeply rooted in by bodily interactions. The perceptual and sensorimotor experiences students gained within the digital-embodied learning environment can substantially enhance conceptual understanding (Bos et al., 2022; Duijzer et al., 2019; Pittalis & Drijvers, 2023; Shvarts & Abrahamson, 2019).

Specifically, the exploration of the design features is noteworthy. Several studies highlight the importance of using real-life contexts in mathematics education (Freudenthal, 1971; Gravemeijer & Doorman, 1999; Laurens et al., 2017; Sembiring et al., 2008). Integrating real-world contexts into mathematics instruction can enhance students' engagement and understanding by making abstract concepts more concrete and relatable. The light shadow context used in the first learning module exemplifies this principle by allowing students to discover functional rules through experimentation and observation in a setting that mirrors their everyday experiences. Moreover, research emphasizes the importance of connecting mathematical concepts to student's potential educational experiences and intuitive understandings (Freudenthal, 1971; Nemirovsky et al., 1998). By engaging with the physical world, such as through the light shadow context, students can connect their informal knowledge with formal mathematical concepts.

The exploration of covariation through bimanual coordination tasks serves as an effective method for introducing students to the concept of variables' interdependent changes. Fostering covariational thinking has always been a challenge in mathematics education, especially the abstract nature of continuous variation and covariation (Carlson et al., 2002; Thompson & Carlson, 2017). We explored bimanual coordination, referring to the coordinated use of both hands to interact with learning environment, as a potential method. It has been incorporated into mathematical education as a way to foster an understanding of abstract mathematical concepts, such as proportion for primary school students (Abrahamson et al., 2016). In this study, as students adjust one variable with one hand and another variable with the other hand, they can observe in real-time how these variables covary. The sensorimotor experience with the mathematical content not only makes the learning experience more engaging but also embeds a concrete, experiential understanding of covariation. In addition, the kinesthetic experience provided by bimanual interaction enhances memory retention and conceptual understanding (Black et al., 2012). As revealed through the mini interviews, by physically engaging with mathematical concepts, students can form more concrete understandings of covariation. Through the bimanual interactions, students often describe covariational relationships in terms of one variable moving upward (increasing) while another moves downward (decreasing). It shows how students integrate digital-embodied nomograms into mathematical reasoning. Moreover, real-time feedback environment allows students to experiment with inputs and directly observe and adjust the corresponding outputs. The real-time visual feedback provided by digital-embodied nomogram tasks complements textual mathematical explanations, thus catering to a broader range of learning preferences. By incorporating these design features, the tasks adeptly present the dynamic relationships between variables, embodying the principles of covariational thinking and helping with overcoming the difficulties in covariational reasoning (Carlson et al., 2002; Thompson, 2008; Thompson & Carlson, 2017). However, as noted in the results section, there is a noticeable shift in students' bimanual movement from continuous to relatively discrete movements. Insights drawn from mini interviews suggest this shift may be due to a commonly used strategy, where students focus on identifying specific integer pairs during discrete movements. When asked to provide a formula, students frequently rely on these integer pairs to identify patterns. The behavioral transition also signals an important direction for future research. It calls for further investigation into how bimanual movements, especially different types of bimanual movement—continuous versus discrete—influence the development of FT. Future studies could explore their impact across the various aspects of FT to uncover understandings of how physical interactions with mathematical content can support or hinder the learning progression.

Compared to our previous study, which was conducted in a controlled laboratory setting with only two student pairs (Wei et al., 2024), this study explores the implementations of employing a digital-embodied design using nomograms in a classroom environment. The freedom afforded to students in this setting allows them to independently explore mathematical concepts and communicate with peers, while features such as collaborative tasks and whole-class discussions enhance peer to peer interactions and collaborative learning processes. By synthesizing these insights, this study shows the considerable potential of digital-embodied learning environments with their rich design features to foster FT. It adds valuable perspectives to the discourse on instructional design strategies within the realm of mathematics education.

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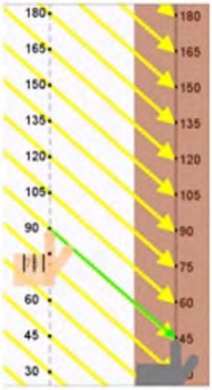
Appendix 1: Pretest and posttest examples

IO **1.** Tom is playing a hand-shadow game in front of a wall. We use arrows to connect Tom's hand and its shadow. For instance, the green arrow in the picture below indicates that when the height of Tom's index finger is 90 cm, the shadow of his index finger is 45 cm.

If Tom places the top of his index finger at a height of 140 cm, can you find the height of the shadow of the top of his index finger?
The height of shadow of Tom's index finger is at __________cm.



Below, a spotlight projects a model tree on the wall. The height of the tree can be varied. The shadow will then change height as well. Please finish the questions 2, 3, 4.

COV **2.** What is the height of the model tree if the shadow has a height of 24 cm? __________ cm

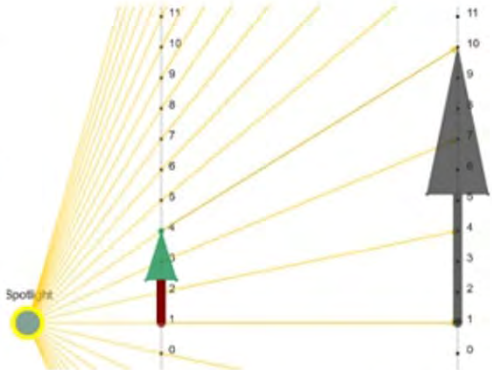
COV **3.** What formula describes the relationship between the **top** of the model tree and the **top** of its shadow?

A. Shadow_top =Tree_ top ×2.5

B. Shadow_ top =Tree_ top ×3

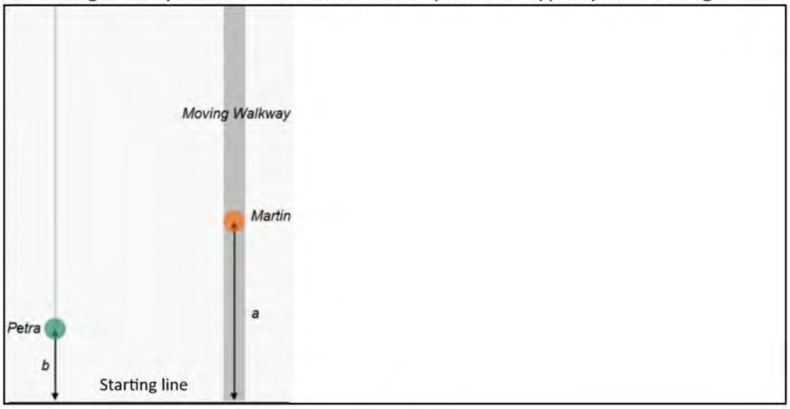
C. Shadow_ top =Tree_ top ×2+2

D. Shadow_ top =Tree_ top ×3-2



COR **4.** In the hand - shadow game, you can use a spotlight, or a bulb, as a light source. What difference does this make in the way you compute the height of the shadow from the height of the hand, compared to the case where the light source is the sun?

COR **5.** Petra and Martin walk side by side at the airport at a speed of 1 m/s. Martin then walks onto the moving walkway, which adds 1 m/s to his speed. Explain how the distance *a* of Martin to the beginning of the moving walkway depends on the distance *b* of Petra to the beginning of the moving walkway. Draw a suitable mathematical picture to support your reasoning.



2

IO **6.** The table shows some inputs and outputs of a given rule. Complete the empty cells.

INPUT	OUTPUT
0	-1
5	14
7	20
10	29
12	
	44

COV **7.** Based on the table in task 6, which statement is correct? ____________

A. When the input value increases by 20, the output value increases by 50.
B. When the input value increases by 25, the output value increases by 75.
C. When the input value increases by 50, the output value decreases by 115.
D. When the input value increases by 100, the output value decreases by 300.

Appendix 2: HLT of module 2

Task	Description of task	Mathematical goals	Student (embodied) activities • action-perception loops • Emergent modeling (situational, referential, general) Practices / techniques for using artefacts (stable action/perception constitution)	Assumptions about how instructional activities support abstract functional thinking that leads to mathematical goal: Conceptualizations of different aspects of function, e.g., expressed in verbalizations
2.1-2.4	Based on the experiences from previous tasks, let students explore the relationship between the two hands' heights and get familiar with the movement on a nomogram  1. Individual task 2. Collaborative task	• Identify a relationship between the positions of two hands • Describe covariation and correspondence relationships between two patterns using non-mathematical language	• As try out the tasks with two hands, students recall the hand-shadow tasks and copy the movements • Referential: When two hands move simultaneously, students experience the relationship between their hands' positions • By practicing the two hands' movements on a nomogram, students get action proficiency so that they are able to explore the following tasks with a digital nomogram • The movement pattern from individual tasks could lead to first covariation reasoning, and then, correspondence reasoning. The movement pattern from collaborative tasks could lead to covariation reasoning more • Action-based design: attentional anchor: the green arrow between two points	• Conceptualizations of covariation aspect: when moving the two points simultaneously and smoothly, students develop the understanding of how the covariation happens between two variables based on their two hands' movements which draws an analogy between physical experience (heights of hands) and mathematical meaning (dependent and independent values)
			• Action-perception loop: students adjust the position of the arrow and perceive the relation between the heights of two ends of the arrow to meet the positive feedback which is a green arrow • Practices for using artefacts: move two points on two lines vertically and adjust the moving speed/direction based on the color of the arrow (green=positive, red=negative)	
2.5-2.9	Nomogram with numbers  1. Numerical expression 2. Algebraic expression Overview of nomogram  1. Match task	• Identify a relationship between the heights of two hands • Describe covariation and correspondence relationships between two number lines using mathematical language • Identify a functional rule from a nomogram • Represent functional rules using formula and nomogram	• Referential: When students try out the tasks with numbers on two lines, they focus on not only the moving direction but how far/tall their hands go to make the arrow green • Action-based design: attentional anchor: the green arrow between two numbers • Action-perception loop: students move the two points and observe the relation between their positions and adjust the two points based on the color of the arrow, so that feedback is positive=relation between input values and output values is satisfied • Practices for using artefacts: move two points on the two number lines and keep the moving speed and moving direction so as to hold the positive feedback • General: Students get an overview of nomograms and corresponding function formulas. Students reason the relationship between input and output values by observing the nomogram and matching the value pairs with the formula	• Conceptualizations of input-output aspect: With the numbers showing up on the two lines, students reinforce the understanding of input-output pairs • Conceptualizations of covariation aspect: When finding the rules of making arrow green, students think more mathematically with the help of numbers on the two lines. They realize that the movements of two hands have covariation and/or correspondence relationships. In addition, by observing the nomograms for different relations, students get deeper understanding about how a covariational relation can be represented by a nomogram • Conceptualizations of object aspect: By switching between different representations, nomogram and formula, to represent functional rules, students realize that a relation/function can be represented in different ways with the same essence
	2. Complete algebraic expressions for functions			

Appendix 3: A teaching instruction of module 2

Lesson 2: Introduction to nomogram

- **5-minute review:** Recap the previous lesson and introduce Module 2.

  Example:

  In our previous lesson, we explored the input-output relationship from nomograms. And specifically, two different light sources. You all worked on creating and manipulating the nomograms and saw how changing the input affected the output.

  Now that we have a solid understanding of the input-output relationship, we're ready to build on that knowledge in today's lesson. Today, we'll be exploring another relationship, again using nomogram tasks.

  During today's lesson, we'll engage in hands-on activities that will help you visualize and explore the covariational relationship. You'll be working to plot nomograms and analyze how the variables covary.

- **45-minute work in pairs**: Students work on Module 2 tasks using the DME platform. Conducting two rounds of mini-interview, task 2.5 and 2.7

- **10-minute recap:** Hold a whole-class discussion on the salient tasks during the exploration and recap the session. (plot nomogram y=2x-1)

  Example:

  Let's go over the tasks quick. First half of the tasks are two-hands movements on different nomograms. And then rest ask for more mathematical reasoning, like using formula to describe the rule you found from the nomograms.

  Question1: how do you see from the nomogram that the relation is linear?

  Question 2: For example, if we have y=ax+b, how do you find a and b from the nomogram?

  Question 3: Does anyone want to share their insights about the covariational relationship of a rule/relation? Can someone explain how to write a formula to represent a relation or rule given a corresponding nomogram?

  (back up) Question 4: we also worked on a task for non-linear relations, if now I give you this relation y=x^2. Can you plot its nomogram? anyone would like to give it a try?

Appendix 4: Answer box grading rubric

Module 1:

Score	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6	Task 7	Task 8	Task 9
Integrated Reasoning	Correctly concludes: Height of the shadow=height of the hand -45	Correctly concludes: Height of the shadow=height of the hand -15	Accurately concludes: The light source is behind 90. It is not parallel like previous tasks. S=1.5h-45	Correctly concludes the shadow's size is double that of the mosquito. S=2*mosquito	Correctly concludes the shadow's size is 1.5 times bigger than the mosquito. S=1.5*mosquito	Find out two extension lines of two different arrows and line them up to find the position of the bulb.	The size of the shadow doesn't change and s=3*tree (height) or s=3*tree-2 (position)	Correctly relates the shadow's size to the tree's height with accurate calculations. S=2*tree (height) or s=2*tree-1 (position)	Accurately concludes the relationship between input and output: Output=input+2
Basic Reasoning	Height of the shadow=height of the hand +45 or only states the height difference is 45	Height of the shadow=height of the hand +15 or mention the height difference is 15	Mention the position of 90 or the shadow dramatically increases/decreases when its away from 90	Mention the movement of the shadow is depends on the movement of the mosquito with wrong	Mention the movement of the shadow is depends on the movement of the mosquito with wrong	Find the position and mention the 0/horizontal line	The size of the shadow is three times bigger or s=3*tree (height) or s=3*tree-2 (position)	The size of the shadow is two times bigger or s=2*tree (height) or s=2*tree-1 (position)	

Score	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6	Task 7	Task 8	Task 9
				formula/number	formula/number				
Simple Observation	Only mention linear relationship without formula/number	Only mention linear relationship without formula/number	Explain the differences based on the angle of lights explain the shadow/outcome/output	Mention the movement of the shadow is depends on the movement of the mosquito without formula/number	Mention the movement of the shadow is depends on the movement of the mosquito without formula/number	Only find the position without detailed explanation	Only mention the size of the shadow doesn't change or bigger than the tree	Only mention the size of the shadow doesn't change or bigger than the tree	Only graph the nomogram
No reasoning	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer

Module 2:

Score	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6	Task 7	Task 8	Task 9
Integrated Reasoning	Two hands/fingers move towards the same direction with the same speed, or have the fixed height difference/ with the same angle/with left hand lower than right hand	Left hand moves slower than right hand towards the same direction	Two hands/fingers move towards the same direction with the same speed, but has the fixed height difference/with the same angle/with left hand higher than right hand	Two hands/fingers move towards the different/opposite direction with the same speed	y=-x+3 two hands/fingers move towards the different/opposite direction with the same speed	y=-2x	y=-2x+4	Checked by DWO	y=x+5 y=2x+1
Basic Reasoning	Two hands/fingers move towards the same direction with the same speed	Two hands/fingers move towards the same direction with different speed or the	Two hands/fingers move towards the same direction with the same speed or two hands'	Two hands/fingers move towards the different/opposite direction or the line between two hands	y=-x+3 or two hands/fingers move towards the different/opposite direction with the same speed				

Score	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6	Task 7	Task 8	Task 9
		angle changes when move further	movement has the fixed height difference/ the same angle	intersect on the one point in the middle					
Simple Observation	Two hands/fingers move together with the same speed or two hands/fingers move towards the same direction	Two hands/fingers move together with different speed	Two hands/fingers move together with the same speed or two hands/fingers move towards the same direction	Two hands/fingers move together with the same speed or two hands/fingers move towards different direction	y=-x or y=x+3 or two hands/fingers move towards the different/opposite direction with the same speed			Only one match is correct	Only one formula is correct
No reasoning	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer

Module 3:

Score	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6	Task 7	Task 8	Task 9
Integrated Reasoning	Two hands/fingers move towards the same direction with the same speed horizontally	When x>0, left hand first moves slower than right hand, and then move faster than right hand when x<0, left hand moves towards different direction and fisrt slow and then fast	y=3, x doesn't matter keep the right point at 3, and move the left point anywhere	y=|x|	y=x-3	y=-2x-2	Rotate the x-axis 90 degree	Domain [-2,5/∞] range [0,2.5/∞] or other numbers they found from the graphs	Domain [-5/∞,5/∞] range [-1,1] or other numbers they found from the graphs
Basic Reasoning	Two hands/fingers move towards the same direction with the same speed	Only mention the direction or only mention the speed change (first slow and then fast)	Only write down y=3 or only mention the movement correctly				Mention the connection between x- and y-axis in nomogram and function graph mention the x-axis become	Only find domain or range correctly	Only find domain or range correctly

Score	**Task 1**	**Task 2**	**Task 3**	**Task 4**	**Task 5**	**Task 6**	**Task 7**	**Task 8**	**Task 9**
							perpendicular to y-axis		
Simple Observation	Two hands/fingers move together with the same speed or two hands/fingers move towards the same direction	Two hands/fingers move together with different speed		Only graph the function	Only graph the function	Only graph the function	Mention the input-output values on each graph	The domain is x-axis range is y-axis	The domain is x-axis range is y-axis
No reasoning	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer	Provides incorrect reasoning or no answer

Chapter 5 An embodied approach to covariational reasoning: A hand-tracking study

Abstract Covariational reasoning is critical for understanding functional relationships in mathematics. Yet, many students struggle to understand varying quantities and to conceptualize covariation. This study explores the research question of how bimanual movements within a digital-embodied learning environment can support students' covariational reasoning, appropriating Thompson and Carlson's (2017) covariational reasoning taxonomy. The intervention consisted of three lessons involving seventy-six Grade 9 students. Data included hand tracking data, post-task probes, and pretest and posttest results. We examined the relationship between students' bimanual coordination and their covariational reasoning levels, addressing two hypotheses: (H1) Higher levels of covariational reasoning correlate with a shorter time to reach fluency in the bimanual coordination, and (H2) higher levels of learning gains in covariational reasoning correlate with longer time spent on fluently performing the bimanual coordination. As results, the pretest and posttest scores demonstrated significant improvements in students' covariational reasoning after the experiment. Quantitative analyses of hand-tracking data indicated that students with higher initial covariational reasoning levels achieved fluent bimanual movement more quickly than others (supporting H1), while those with greater learning gains spent more time consolidating their understanding in fluency phases (supporting H2). Qualitative findings showed how the interplay of perception-action loops, attentional anchors, and real-time feedback facilitated the internalization of covariational relationships. This study highlights the potential of an embodied approach to fostering covariational reasoning and introduces a framework for analyzing embodied learning through the integration of hand tracking, probes, and assessments.

Keywords Covariational reasoning, Embodied design, Hand tracking, Digital-embodied learning, Mathematics education

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Wei, H., Bos, R., & Drijvers, P. (Under review). An embodied approach to covariational reasoning: A hand tracking study. *Educational Studies in Mathematics.*

5.1 Introduction

Covariational reasoning is important for understanding relationships between variables in dynamic contexts. Covariational reasoning is defined as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002; p. 354). Insight in covariation is a prerequisite for the interpretation of functional relationships and the modeling of real-world phenomena. Research highlights the need for teaching approaches that support students' transition from asynchronous interpretations to simultaneous reasoning within covariational reasoning (Carlson et al., 2002; Johnson, 2012; Paoletti & Moore, 2017). Such supportive teaching approaches may involve tasks that explicitly link physical actions to symbolic abstract outcomes. For example, tasks might include dynamic feedback (e.g., color changes or animations) that visually represent covariational relationships (Abrahamson & Trninic, 2011). These features serve as resources for students to manifest their covariational reasoning through gesture, coordination, and symbolic expression.

Digital-embodied learning environments have emerged as powerful platforms for fostering conceptualization of mathematical concepts through embodied interaction (Abrahamson et al., 2011; Abrahamson et al., 2023; Flood et al., 2020; Georgiou & Ioannou, 2019; Jaber et al., 2024; Pittalis et al., 2024; Pittalis & Drijvers, 2023). These environments allow students to explore functional relationships by directly manipulating variables or graphical representations with real-time feedback. Such approaches align with theories of embodied cognition, which suggest that learning is deeply rooted in the body's interactions with the environment (Lakoff & Núñez, 2000; Varela et al., 1991). In particular, Abrahamson et al. (2014) demonstrated how primary school students' understanding of proportions emerged and improved through bimanual tasks that required them to manipulate objects with two hands simultaneously. Students need to discover a correct movement strategy guided by perceptual feedback, that is screen turning green. This intuitive strategy prompted students to qualitatively articulate their strategies and later quantitatively describe them. This approach initiated broader research on mathematics embodied design: that bodily experience can support formal conceptualization when tasks are specifically designed to connect sensorimotor coordination with symbolic outcomes (Abrahamson et al., 2011; Alibali & Nathan, 2012; Turgut, 2022).

Despite the potential of these environments, gaps remain in our understanding of how specific tools or tasks contribute to the development of covariational reasoning. While digital and embodied tools such as sliders, dynamic graphs, and gesture-based controls are designed to support such reasoning, more research is needed to test hypotheses about how these tools foster covariational reasoning. First gap lies in understanding the mechanisms by which embodied learning tools facilitate the transition from physical interaction to mathematical reasoning, a process where initial sensorimotor discoveries enable the articulation and formalization of mathematical concepts. As Abrahamson et al. (2020) note: “Participants in embodied-design activities discover and develop concept-grounding enactive processes, even when they are not aware that or what they are learning” (p. 18). A second gap involves understanding how different design features, such as real-time feedback or the level of immersion, impact students’ mathematical reasoning processes (Christopoulos et al., 2024; Hulse et al., 2019). To address these gaps, some studies point out the importance of analyzing learning activities and mathematical reasoning from different lenses, for example, through the lens of embodied learning processes, which ground reasoning within the symbolic semiotic register (Abrahamson & Sánchez-García, 2016; Nathan & Alibali, 2021). Advances in hand-tracking technology, such as the Geometry Touch app (Sepp et al., 2023), and tools leveraging virtual reality, gesture-based interaction modes, and delayed feedback mechanisms (Chatain et al., 2022), offer new opportunities to explore embodied learning processes. Moreover, these tools together with data analysis methodologies like recurrence quantification analysis (Abdu et al., 2025; Tancredi et al., 2021) provide quantitative evidence that mathematical learning is a process of sensorimotor reorganization. The third gap lies in explicitly connecting these dynamic, process-oriented findings to the established cognitive frameworks (e.g., Thompson & Carlson’s covariational reasoning taxonomy) and outcome-based measures of mathematics education research.

In this study, we explore how digital-embodied learning environments support the development of covariational reasoning. We apply mixed methods, including the analysis of hand-tracking data, qualitative insights from post-task probes, and pretest and posttest performance evaluations. Our analysis builds on the covariational reasoning (CR) framework developed by Thompson and Carlson (2017), which we adapt to our research lens of embodied interaction and its use of hand-tracking data. Detailed information on these methods will be provided in the subsequent sections. Through this integration of methods, this study not only evaluates students’ progress in CR but also contributes to the broader understanding of how digital-embodied learning environments support mathematical reasoning.

5.2 Theoretical Background

In this section, we address covariational reasoning, then embodied learning, and draw on these perspectives to articulate how sensorimotor coordination can ground covariational reasoning, and then present research questions and corresponding hypotheses.

Covariational reasoning

Thompson and Carlson's (2017) taxonomy describes a five-level progression of covariational reasoning. Lower levels involve recognizing that changes in one quantity correspond to changes in another (L1: Pre-coordination of values; L2: Gross coordination of values levels). Higher levels require coordinating specific values (L3), interpreting simultaneous changes over fixed intervals (Chunky continuous covariation; L4), and perceiving continuous and smooth covariation (L5).

Many students face challenges in developing covariational reasoning. These include difficulties in coordinating simultaneous changes in variables (Carlson et al., 2002), focusing on discrete values over continuous relationships (Bagossi, 2024; Thompson & Carlson, 2017; Wilkie, 2020), and connecting physical experiences to abstract representations (Abrahamson et al., 2014; Shvarts & Abrahamson, 2019). The first two challenges are often linked. For instance, a student may adjust one variable at a time, seeing the relationship as static or sequential. This tendency to focus on discrete values becomes evident in tasks involving graphs, where students often interpret individual points without understanding the continuous nature of the curve (Carlson et al., 2002; Johnson, 2012).

Embodied learning

To foster covariational reasoning, researchers have turned to embodied learning, an approach suggesting that our thinking and understanding are connected to our bodily experiences and how we interact with the world (Barsalou, 2008; Lakoff & Nunez, 2000). A core principle is sensorimotor coordination, with which students actively manipulate objects or digital representations to develop mathematical concepts (Abrahamson, 2021; Abrahamson & Bakker, 2016; Shvarts et al., 2021).

Action-based embodied design, for example, creates learning activities that challenge students to solve motor problems by coordinating physical actions in specific ways (Abrahamson & Bakker, 2016; Abrahamson &

Lindgren, 2014). Abrahamson and Trninic (2011) introduced a task where primary school students learned about proportion by moving two handheld controllers simultaneously. To make the screen green (indicating correctness) with two controllers, students had to discover the specific coordinated way to position their hands at different relative heights. This feedback-guided exploration enabled students to first grasp the proportional relationship intuitively and articulate it qualitatively, before they later formalized this understanding using grids and numbers. While performing the proportion task, stable perception-action loops emerge as students act perceive feedback, and adjust their movements (Abrahamson & Mechsner, 2022). As these actions stabilize, attentional anchors may emerge—specific perceptual or spatial configurations (e.g., a diagonal line between hands in the above proportion task) that facilitate perceptuomotor action (Abrahamson & Sánchez-García, 2016; Shvarts et al., 2021). Stabilization is identified when a student's action becomes fluent, efficient, and consistently aligned with the emerging attentional anchor. Once stabilized, these anchors not only improve motor performance but also prepare students for mathematization and support the transition from enacted movement to symbolic reasoning. In digital-embodied learning environments, the interactive interface becomes an integrated part of these loops, creating body-artifact functional systems where (Shvarts et al., 2021). Students' actions, digital-feedback, and emerging attentional anchors collectively develop the understanding of mathematical concepts (Pittalis et al., 2024; Pittalis & Drijvers, 2023; Shvarts et al., 2021).

In sum, this perspective informs our study by shaping both the design of the digital-embodied learning environment and the interpretation of students' bimanual movements as evidence of emerging covariational reasoning.

An embodied view on covariational reasoning

This study builds on the perspective that covariational reasoning can be supported—and made observable—through embodied interaction. We propose that coordinated hand movements within a carefully designed digital-embodied environment can reflect a student's coordination of the underlying mathematical quantities (Flood et al., 2020; Pittalis et al., 2024; Pittalis & Drijvers, 2023). Specifically, we use action-based embodied design to engage students in bimanual coordination tasks aimed at fostering and revealing their covariational reasoning. Our learning environment uses nomograms (Figure 5.1; Nachmias & Arcavi, 1990; Sinclair et al., 2009)—

parallel number lines connected by arrows representing functional relationships—to engage students in tasks requiring bimanual coordination.

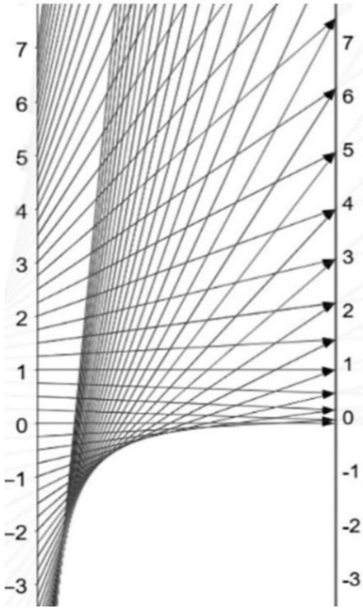


Figure 5.1 Nomogram of the function $f: x \to x^2$

To systematically analyze students' sensorimotor activity, we developed an embodied operationalization of Thompson and Carlson's CR framework to intertwine covariational reasoning and embodiment (Table 5.1). The original framework is based on students' reasoning as inferred primarily from verbal utterances. Our adaptation, made a priori, maps each level of CR onto patterns of bimanual coordination observable through hand-tracking data. For instance, where the original CR taxonomy refers to anticipating how changes in one variable affect another, our operationalization identifies corresponding motor behaviors, such as how one hand's movement affects the other in a coordinated way. Table 5.1 thus serves as our primary analytical tool connecting students' embodied actions to their level of CR.

Table 5.1 Original vs operational taxonomy of CR levels

Original CR framework (Thompson & Carlson, 2017)	CR framework in this study on bimanual movement
L0: At the *no coordination level*, the person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.	**L0**: At the *no coordination level*, the student does not envision the variables varying together. Bimanual movements focus on adjusting one hand or the other independently, with no coordination between the two movements.
L1: At the *pre-coordination of values level*, the person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.	**L1**: At the *pre-coordination of values level*, the student envisions changes in the positions of both hands but asynchronously—one hand moves first, followed by the other, and so on. Movements are sequential rather than simultaneous, and the student does not anticipate creating pairs of values through coordinated actions.
L2: At the *gross coordination of values level*, the person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.	**L2**: At the *gross coordination of values level*, the student forms a loose image of how movements of one hand correspond to movements of the other, such as "one hand moves up while the other moves down." However, the coordination is imprecise, and the relationship is viewed as a general pattern rather than a connection between specific pairs of values.
L3: At the *coordination of values level*, the person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).	**L3**: At the *coordination of values level*, the student coordinates specific positions of one hand with specific positions of the other, intentionally creating discrete pairs of values (x, y). Movements are more deliberate, reflecting an anticipation of forming clear pairs, such as distances or positions along a scale, though still limited to discrete points.

L4: At the *chunky continuous covariation level*, the person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.	**L4**: At the *chunky continuous covariation level*, the student envisions simultaneous changes in the movements of both hands, representing the two variables varying together targeting discrete intervals. Continuous movements occur in intervals, called "chunks," and have not yet been integrated into an uninterrupted movement.
L5: At the *smooth continuous covariation level*, the person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.	**L5**: At the *smooth continuous covariation level*, the student achieves smooth simultaneous bimanual movements, demonstrating an advanced understanding of how the variables vary together in a continuous manner.

The nomogram environment (Figure 5.2) enables students to experience functions as processes rather than static entities (Wei et al., 2024). Students can simultaneously adjust two movable points on two number lines, starting with an otherwise blank canvas. This interaction offers real-time feedback: an arrow connecting the two points turns green and leaves a trace for correct input-output pairs or turns red and disappears for incorrect pairs. This real-time feedback invites students to actively participate in perception-action loops. For example, color changes steer students' attention toward achieving coordination goals. The development of stable perceptual patterns for organizing movement occurs through the emergence of attentional anchors. As the anchor stabilizes, it can function as a self-generated, immaterial artifact (Abrahamson & Bakker, 2016). It is a new cognitive structure that students begin to notice, reflect upon, and express using symbolic forms such as number lines or algebraic formulas. In this way, attentional anchors support reification of covariation.

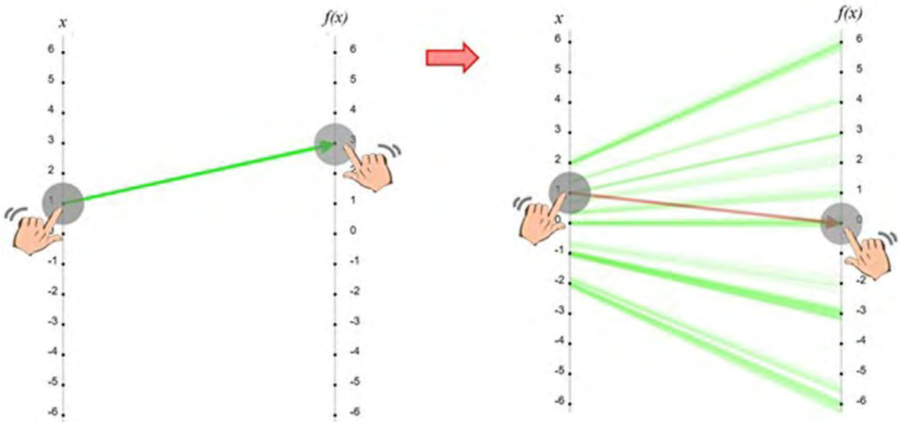


Figure 5.2 The accumulated results of exploring the function $f : x \rightarrow 3x$ on the nomogram

Hand tracking has been increasingly adopted in educational research as a tool for analyzing fine-grained motor actions and their role in learning processes. Lindgren and Johnson-Glenberg (2013) emphasized that hand tracking can capture embodied cognitive processes by revealing how students' real-time motor adjustments reflect their coordination of perception, action, and emerging conceptual development. This research tradition has modeled the nonlinear dynamics of discovering and stabilizing movement strategies in relation to task feedback. For example, Tancredi et al. (2021) analyzed the discovery and stabilization of bimanual coordination patterns in an embodied mathematics task. A subsequent study has integrated eye-tracking data to further detail the perceptual learning involved (Abdu et al., 2025). Building on this foundation, we integrated post-task probes with quantitative analysis of hand tracking and pretest and posttests data. We illustrate how the micro-level process of stabilizing bimanual coordination corresponds to macro-level learning gains in covariation reasoning.

The research question guiding the study, with its corresponding hypotheses, is as follows:

> *How do bimanual movements within a digital-embodied environment support students' covariational reasoning?*

We hypothesize that:

H1: Higher levels of CR correlate with a shorter Time To Fluency.

Students with higher initial CR levels are expected to coordinate their sensorimotor activity with the nomogram task faster. Previous research suggests that higher CR involves anticipating how changes in one variable affect another, reducing cognitive load (Moore & Carlson, 2012). While the nomogram introduces a new embodied format, students with higher initial CR are likely to interpret its affordances more efficiently because they already practice in anticipating and constructing continuous covariation. This aligns coordination dynamics research, which emphasize that students' performance is shaped by their intrinsic coordination patterns (Kostrubiec et al., 2012). Thus, Time To Fluency is defined as the time it takes students to first achieve smooth continuous movement. Students with higher CR levels are expected to reach fluency with fewer trial-and-error attempts.

H2: Higher levels of learning gains in CR correlate with a longer Fluency Time Sum.

This hypothesis assumes that extended time in the fluency phase supports learning, especially for students with initially weaker CR. For these students, fluency becomes a space for reasoning through action. As they repeatedly adjust their movements in response to feedback, they refine their sensorimotor patterns that may evolve into coordinated dynamical Gestalts (Alberto et al., 2022), which in turn serve as attentional anchors for symbolic reasoning and articulation. Thus, students with lower initial CR levels may benefit more from extended fluency phases, while students with a higher initial CR level are expected to spend less time in the fluency phase to achieve similar proficiency. Total fluency time is expected to correlate positively with learning gains.

5.3 Methods

This study was structured as a 3-hour teaching experiment conducted in four Grade 9 classes (N = 76) across two countries.

Design of the Digital-Embodied Learning Environment

The design of the three learning modules focuses on the three aspects of functional thinking: input-output, covariation, and correspondence (Doorman et al., 2012; Wei et al., 2024). Design features include using real-life context, bimanual coordination movement, real-time feedback, and multiple representations (Wei at al., 2025). The modules are embedded in the Numworx platform (https://www.numworx.nl/), equipped with multi-touch and real-time feedback capabilities.

In Module 1, students engage with input-output thinking through tasks involving light ray contexts, such as finding relationships between an object and its shadow across different configurations. The module gradually transitions from unimanual to bimanual interactions, providing an introduction to covariation. Module 2, the focus of this study, addresses covariation by progressing to abstract nomograms. Students are asked to manipulate two points simultaneously, each representing a variable, to coordinate covarying input and output values along labeled axes and reason about formulas. The bimanual movements with real-time feedback foster the development of perception-action loops. In Module 3, students interact with dynamic transformations between nomograms, function graphs, and formulas, to foster correspondence thinking by encouraging students to coordinate physical actions with symbolic and graphical representations. The modules can be found at:

https://embodieddesign.sites.uu.nl/activity/functional-thinking/.

Study Design

We used a fully mixed, concurrent, quantitative-dominant mixed-methods approach (Leech & Onwuegbuzie, 2009). Our analysis centers on covariation, as this aspect showed the most significant improvement (Wei et al., 2025). Quantitative measures, including hand-tracking data and pretest and posttest scores, were used to assess CR development and bimanual coordination. Post-task probes provided real-time qualitative insights into students' covariational reasoning and embodied experiences, aligned with Robinson's (2023) descriptive and explanatory categories.

The teaching experiment spanned three one-hour sessions. Each session included a 5-minute introduction, a 45-minute exploration period, and a 10-minute recap, with each student using a tablet. The introduction outlined the learning goals and procedures, encouraging students to discover functional relationships through their bodily experience and connecting to school mathematics while maintaining an exploratory feeling. During the exploration period, students interacted with the digital-embodied tasks independently or collaboratively. The 10-minute recap led by the researcher facilitated discussion, reinforced key concepts, and addressed common challenges (Wei et al., 2025). Prior to the first session, all students completed a pretest, and one week after the final session, they completed a posttest.

Instruments

The pretests and posttests, administered in paper-pencil format, assessed three aspects of FT: input-output (IO, 3 items), covariation (COV, 5 items), and correspondence (COR, 7 items). The tests also included items targeting input–output and correspondence reasoning—aspects addressed in other intervention sessions—to provide a comprehensive measure of students' FT. By analyzing both the full FT scores and the COV scores separately, we were able to evaluate general learning outcomes as well as specific gains in covariational reasoning. The COV scores were used in both the descriptive and inferential statistical analyses reported in the Results section.

Each test included 15 items, comprising nine "close assessments" mirroring the learning modules' contexts (e.g., nomogram plotting) and six "proximal assessments" testing broader curriculum applications (e.g., moving walkway scenarios) to measure generalized skills (Ruiz-Primo et al., 2002). Items were a mix of open-ended and multiple-choice formats, scored on a 6-point scale (0-5 points for each item; maximum 75 points). The validity of the pretests and posttests includes expert reviews from both experienced mathematics teachers and researchers in mathematics education research. For open-ended items, the scoring rubric assigned 0 points for an incorrect or irrelevant response, 3 points for a partially correct response that demonstrated correct reasoning but was incomplete or imprecise, and 5 points for a fully correct response that included clear reasoning and appropriate mathematical representation. For multiple-choice items, the rubric assigned 0 points for an incorrect response and 5 points for a correct response. Figure 5.3 shows some sample items from the pretest. The complete assessment is available at the following link: https://bit.ly/FTitem

A pilot study involving 51 students, 33 from China and 18 from the Netherlands (the same two countries where the teaching experiments were later conducted), was carried out under standard classroom conditions. This pilot supported a classical test analysis, including calculations of p-value, variance, item reliability index (Rir), and item-total correlation (Rit). Based on the results and expert feedback, one overly simple item, for which over 95% students got full score, was removed. The refined instrument demonstrated satisfactory internal consistency (Cronbach's $\alpha = 0.71$). To ensure scoring reliability, two independent coders evaluated the work (the second coder scored a randomly selected 50% of the total dataset). Initial agreement was excellent (Cohen's $\kappa = 0.91$), and all subsequent discrepancies were resolved through a consensus discussion.

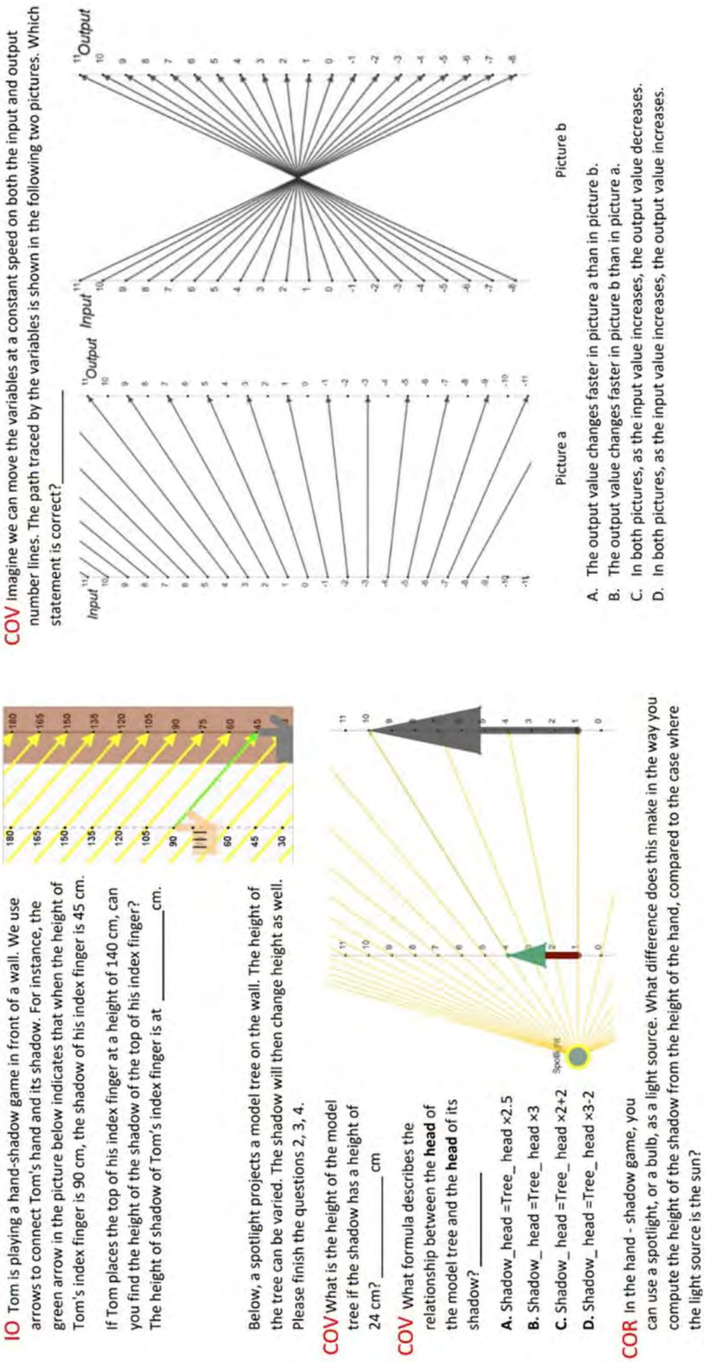


Figure 5.3 Examples of IO, COV and COR items

A protocol for the post-task probes was developed based on the Hypothetical Learning Trajectories, including a detailed hypothetical learning progression (Wei et al., 2024). The aim of the post-task probes was to encourage students to reflect on and expand upon their written responses through open-ended questions and follow-up prompts (Drijvers, 2003; Robinson, 2023). The probes were designed to provide a window into how students were beginning to make sense of their sensorimotor experiences (e.g., “Can you explain how you made it green?”). This approach allows researchers to better trace the emerging conceptual structures that students construct as they attempt to “language” their embodied experiences.

Participants

Seventy-six Grade 9 students, aged 14 to 15 years, were enrolled in the study from two international schools in the Netherlands (two classes, 39) and China (two classes, 37). We employed a convenience sampling strategy, selecting schools that had existing collaborations with the research team and access to the necessary technological infrastructure for digital-embodied tasks. Recruitment was inclusive with no selection criteria, and informed consent was obtained from all students and parents. The students, who were average to high achievers, had foundational knowledge of functions (e.g., number lines, basic algebra, graphing) but had not formally studied functions.

Data Collection

Hand-tracking Technology

Hand-tracking data were collected from Numworx log files that captured the locations of two draggable points (representing each hand) on the screen every millisecond. These data allowed us to classify students’ bimanual movements into two phases: Exploration and Fluency (Tancredi et al., 2021). During the Exploration phase, students initially explored the movement rule and made trial attempts to understand it. In the Fluency phase, students’ movements became more consistent and smoother, which could indicate that they had developed a better practice of the bimanual coordination and understanding of covariational relationships.

To distinguish these phases, we calculated the moving average of the differences between the actual and target positions of the right hand, assuming that the left hand’s position determined the target. The variable Ly represents the left hand’s movement, while Ry represents the right hand’s movement. By plotting these hand positions over time on a continuum, we were able to identify detailed trajectories of the student’s bimanual movement process, specifically differentiating between the Exploration and

Fluency phases, as discussed in the Result section. These trajectories were further analyzed to plot each student's time versus hand position data, providing insights into their bimanual CR levels based on the operational taxonomy in Table 5.1.

The target position varied by task, with each task reflecting different functional relationships. For each task, we set a specific threshold to distinguish the phases based on task difficulty. Movements with distances between the actual and the targeted position below this threshold were classified as Fluency, indicating accuracy in coordinating the two variables. For example, in a task like $f\colon x \to x - 2$, which requires parallel hand movements with a fixed height difference, a threshold of 0.2 units on the number line was set; when students deviated beyond this threshold, the system prompted adjustment by turning the arrow red. If the deviation exceeded three seconds, it was classified as part of the Exploration phase.

Post-task probes

To capture students' learning progression along the hypothetical learning trajectories (Wei et al., 2024), we conducted brief, one-to-two-minute post-task probes during the teaching experiments. Conducted by the research team, these probes were held individually or in pairs. All probes were audio-recorded and later transcribed for analysis. While not every student was asked, there were no predetermined selection criteria. As students completed tasks, researchers approached them for a short conversation.

Data Analysis

To explore the potential relationship between bimanual task performance indicators—specifically, time to fluency (TTF) and fluency time sum (FTS)—and improvement in covariational reasoning (as measured by the COV scores in pretest and posttest), we conducted a Spearman's Rank Correlation test. This non-parametric test was chosen because the bimanual data are not normally distributed, and the COV item scores in the pretest and posttest are ordinal.

The module includes ten tasks in total, seven of which are bimanual tasks relevant to this analysis. First, trials in which a student did not reach the Fluency phase were excluded, as TTF could not be measured (188 samples removed). Second, to prevent mislabeling brief moments of fluency, any fluency period lasting less than three seconds was reassigned to the Exploration phase. This three-second threshold was informed by prior research in embodied mathematics education and movement science, which emphasize the importance of sustained motor patterns for meaningful interpretation. (Kelso, 1995; Tancredi et al., 2021). Based on these considerations and pilot observations of typical task durations (Wei et al., 2024), trials with TTF values above 150 seconds (indicating prolonged difficulty in achieving initial fluency) and FTS values shorter than 3 seconds or longer than 400 seconds (indicating unusually extended practice) were excluded (92 samples removed). After all exclusions, approximately 250 valid samples (258 samples in TTF, 252 samples in FTS) remained for the final correlation test.

For the qualitative analysis of hand-tracking data, we used the operationalization of CR for bimanual movements to systematically observe and categorize how these movements reflect different levels of CR.

We developed a detailed codebook to analyze the transcriptions of the post-task probes, focusing on functional thinking aspects (Wei et al., 2023) and embodied learning (Abrahamson & Lindgren, 2014; Shvarts et al., 2021). Codes included input-output, covariation, correspondence thinking, and embodied learning elements like perception-action loops, as well as students' challenges, strategies, and progress. The second coder coded 15% of the transcription, leading to an inter-coder reliability of κ = 0.73, which indicates moderate agreement.

5.4 Results

This section first presents quantitative findings and correlation analyses conducted to test our hypotheses. These analyses focus on bimanual performance indicators—TTF and FTS—and their relationship to covariational reasoning. We then provide qualitative insights from the hand-tracking data and post-task probes.

Quantitative Results

Overall learning gains

The analysis of pretest and posttest scores revealed significant improvements in students' FT, including CR, after the intervention. The overall FT score (out of 75 points) increased from 38.13 (SD = 12.82) to 50.49 (SD = 9.59). A paired t-test confirmed that this improvement was statistically significant, t (75) = -8.66, $p < .001$, with a large effect size (Cohen's $d = 1.09$). Table 5.2 shows pretest and posttest scores across all three FT aspects.

To examine the specific COV aspect, we analyzed the subset of five COV items. The mean score for the COV items increased from 2.08 to 3.59 (out of 5), and a paired t-test again revealed a significant improvement, t (75) = -9.67, $p < .001$, with a large effect size (Cohen's $d = 1.11$).

Table 5.2 Mean and standard deviation of pretest and posttest scores (*N*=76)

	Total score M (SD)	IO	COV	COR
Pretest	38.13 (12.82)	3.65	2.08	2.41
Posttest	50.49 (9.59)	4.28	3.59	2.81

Note. Maximum total score = 75. Maximum sub-score (IO, COV, COR) = 5. Sub-score standard deviations were not reported.

Hand-tracking data

To test the two hypotheses mentioned above, a correlation analysis was conducted to examine the relationship between students' CR level (five items from pre- and posttest, respectively) and two key performance indicators from hand-tracking data: TTF and FTS.

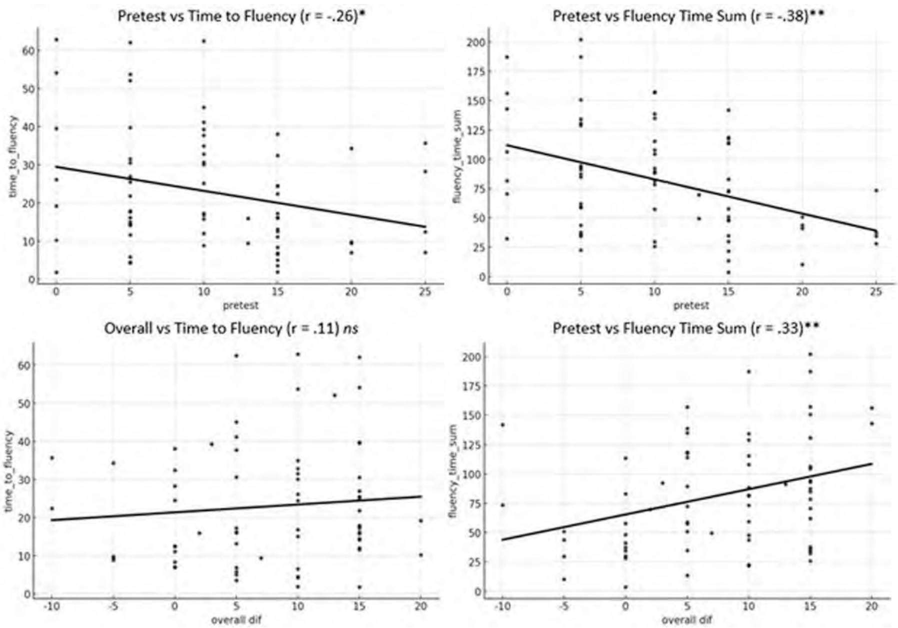


Figure 5.4 Correlation between pre/post-test scores and bimanual performance metrics

Note. $p < .05$*, $p < .01$**, $p < .001$***; *ns* = not significant.

Based on the above figure, we discuss each correlation test result as follows:

- Pretest scores vs. TTF: A weak negative correlation indicates that higher pretest scores are associated with slightly shorter TTF. This supports H1 and suggests that higher initial levels of covariational thinking predict faster achievement of the fluency phase. Stu-dents who start with higher CR levels transition to fluency phase faster.

- Pretest scores vs. FTS: A moderate negative correlation suggests that higher pretest scores are associated with a shorter FTS, supporting H2. This implies that students with stronger CR before the intervention show more efficient fluency practices overall.
- Overall improvement vs. TTF: A nonsignificant positive correlation suggests no meaning-ful relationship between improvement in CR and TTF. TTF is more reflective of initial CR levels than of learning gains.
- Overall improvement vs. FTS: A moderate positive correlation indicates that greater im-provement in CR is significantly associated with longer FTS. This supports H2, as it suggests that students who demonstrated greater learning gains in CR spent more time in the fluency phase.

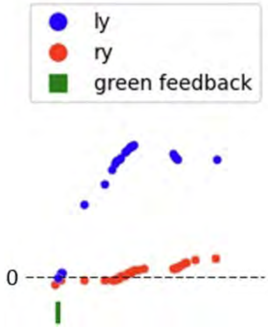
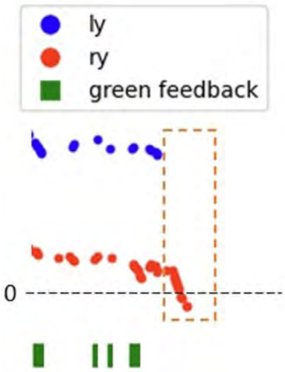
Qualitative insights from hand-tracking data

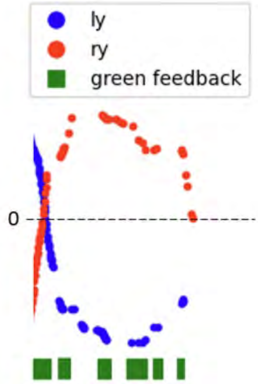
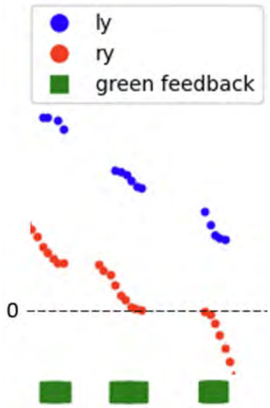
In this section, we first provide exemplary hand traces for each level of the adapted CR taxonomy, demonstrating how bimanual movement patterns reflect CR levels. We then present two cases to clarify the interpretation of the exploration and fluency phases through two students' entire learning process.

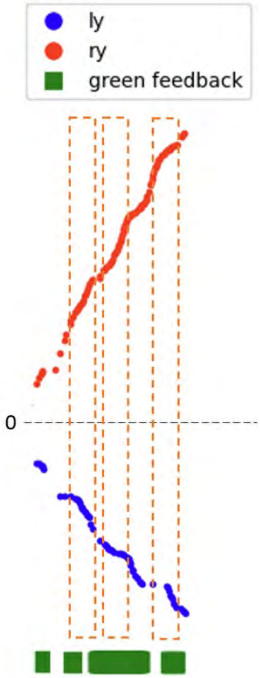
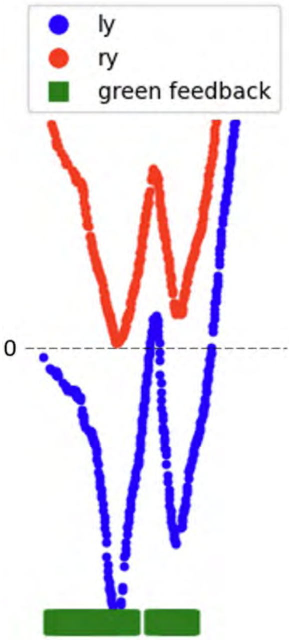
Exemplary hand traces across CR levels

Table 5.3 presents the application of our bimanual CR taxonomy, outlining the progression of students' understanding from no coordination (L0) to smooth continuous covariation (L5). Each level is illustrated with representative hand-tracking data, highlighting key features in students' hand traces, green feedback patterns they received, and their corresponding CR. By analyzing specific patterns in these hand traces, along with the corresponding green feedback when values aligned, we highlight the temporal and embodied aspects of students' reasoning. These examples were chosen for their ability to key features, transitions, and variations in bimanual coordination observed across our broader dataset. By focusing on these targeted samples, we demonstrate how the CR taxonomy is operationalized in students' bimanual movements and how learners develop increasingly sophisticated understandings of covariation.

Table 5.3 Exemplary hand-tracking observations and insights across CR levels

CR Level	Hand-tracking Example and Observations
L0: No coordination	**Task:** $f: x \rightarrow 3x$ **Observation:** One hand, left, represented in blue, moves while the other, right, represented in red, remains stationary. Student then moves another hand to test correct positions for green feedback. We can only see intermittent traces/lines instead of a continuous trace/line without green feedback. No bimanual coordination movement.  **Insight:** The student does not yet show awareness of a covariational relationship; movements have frequent pauses or sudden changes, suggesting independent actions between the two hands.
L1: Pre-coordination of values	**Task:** $f: x \rightarrow x - 2$ **Observation:** Hands move sequentially in very small movements around the same place; one hand adjusts, followed by the other, creating a “stop-and-go” pattern. There are some intermittent and short-lived periods of green feedback during sequential movements, but these are lost when the other hand adjusts with a larger movement (the orange box moment).  **Insight:** Shows initial awareness of covariational relationships, but no simultaneous coordination or pairing of values.

CR Level	Hand-tracking Example and Observations
L2: Gross coordination of values	**Task:** $f: x \to -x$ **Observation:** Hands move in opposite directions with inconsistent coordination. A mirrored pattern (one hand moves up while the other moves down) arises, but with imprecise alignment. Green feedback is more sustained than in earlier level (longer duration time). **Insight:** Indicates an emerging understanding of general patterns between variables but lacks precision in forming specific pairs. 
L3: Coordination of values	**Task:** $f: x \to x - 2$ **Observation:** The left hand moves in three separate small intervals around three separate values and lifts in between. For each value, the right hand is moved down until green feedback occurs. Then both hands move down together for a while. **Insight:** Demonstrates an understanding of how the variables relate at specific small intervals around discrete values, rather than performing a smooth transition across the entire domain. 

CR Level	Hand-tracking Example and Observations
L4: Chunky continuous covariation	**Task:** $f: x \to -2x$ **Observation:** The traces show both hands stopping at specific positions to form pairs, and smoother movements in between (the orange boxes) compared to the previous level. These pauses suggest that the student is focusing on aligning hand movements to create accurate (x,y) pairs. There is more sustained green feedback, which shows that the student successfully coordinated the bimanual movements within these chunks. **Insight:** Indicates deliberate, accurate bimanual movements within specific intervals, or “smooth chunks.” The absence of smooth transitions between these intervals shows that the coordination is not yet globally continuous. 
L5: Smooth continuous covariation	**Task:** $f: x \to x + 1$ **Observation:** The hand traces exhibit synchronized, continuous coordination between the left and right hands, with the general flow of movement aligning with the target covariational relationship (the green feedback is well-sustained). **Insight:** Illustrates the student’s ability to achieve smooth, continuous coordination between variables, signifying an advanced level of covariational thinking. The sustained green feedback highlights the ability to maintain accuracy over the duration of the task, suggesting that the student can apply the bimanual movement fluently in real-time. 

Cases of two students' learning processes

The following two cases show how CR levels are related to the exploration and fluency phases throughout students' entire learning processes.

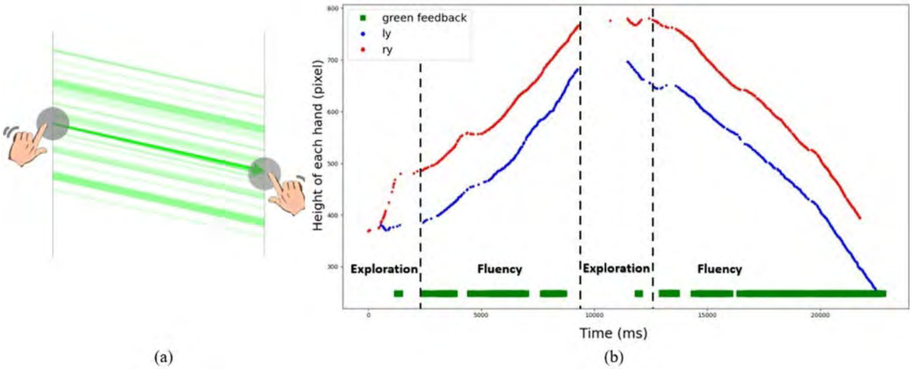


Figure 5.5 (a) Bimanual movement for $f: x \to x - 3$; (b) A student's hand movements across phases

Case 1: Figure 5.5 presents a student's hands movement trace for the function $f: x \to x - 3$. This student moved the right point upward significantly without getting green feedback, causing a decrease in movement speed and a slight repositioning of the right point (L0-L1). Subsequently, the student realized the necessity of moving both points in parallel, which led to achievement of the fluency phase (L4). As the student approached the top of the screen, he paused and began exploring different directions, entering a second exploration phase (L1-L2). Finally, both points were moved downward smoothly with green feedback (L5).

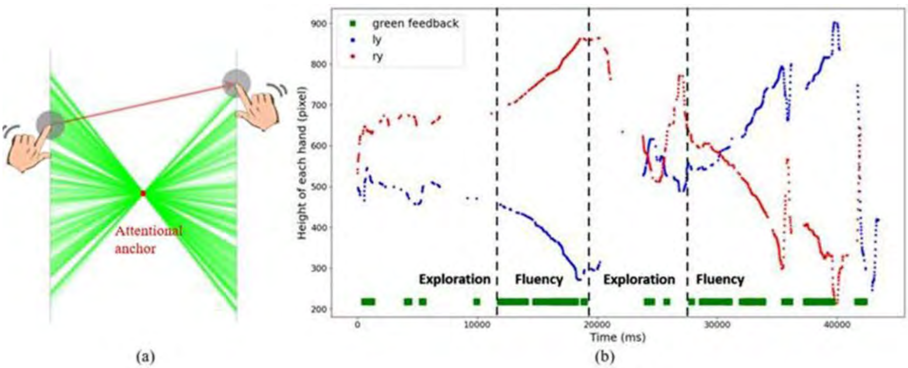


Figure 5.6 (a) Bimanual movement of $f: x \to -x$; (b) A student's hand movements across phases

Case 2: For the task in Figure 5.6, students were assigned to complete the nomogram of the function $f: x \rightarrow -x$. This student also went through multiple learning phases. After the initial exploration phase (L1-L2), his hand movements displayed a pattern: his right hand moved upwards while his left hand moved downward simultaneously (L5). In the second Exploration phase, he adopted an inverse approach, moving his right hand downward while the left hand moved upward (L1-L2). As movement smoothness improved again, the student achieved fluency, showing a geometrically symmetrical movement akin to the properties of the target nomogram (L4). After navigating the entire screen, he described his findings in the answer box: "To get the green line, I moved the two dots in opposite ways which created this pattern." Many students also noted the intersection point as an attentional anchor to adjust their movements. As one student remarked, "we can get an intersection point through several green lines," implying how this visual feedback supported their coordination strategy.

Qualitative insights from post-task probes

The following quotes from post-task probes illustrate students' reasoning process. The analysis reflects individual learning moments, including shifts in recognizing covariational relationships and the emergence of perception-action loops.

In the task $f: x \rightarrow x - 2$, Student A stated, "*Yes, this line [the arrow connected by the two grey points]. I first make it parallel, and after finding it turns green, it can go down at a constant speed. This ensures that each line is parallel to it and can ascend constantly*." Student A observes whether the line is parallel and whether it turns green [perception]. Based on that, the student adjusts hand movements to ensure parallelism and a constant rate of descent [action]. The green color reinforces the correctness of the coordinated movements. In contrast, Student B noted: "*For the third one [task], they [y and x] increase and decrease together... the angle is constant. So, the values of y and x should be relatively easy to calculate*." Student B focuses more on specific pairs of *x* and *y* values. The student first notes change in *x* and *y* values [perception] and then identifies the need for synchronized two-hand movements to maintain a constant angle/slope [action]. By associating the consistent downward movement of the arrow/line with a fixed difference [−2], students begin to perceive and enact the functional relationship between the input and output variables. This embodied experience supports the emergence of an attentional anchor that stabilizes the students' coordination movement.

In the task $f: x \to -x$, students focus more on the dynamics of speed and direction. Student C described, "*The movement direction of your left and right hands? Opposite. You need to keep this point here [intersection point] ... The speed is the same. It can maintain its green color*." This student describes the directionality (opposite) and synchronization (constant speed) of the hand movements [action], and perceives the intersection point as an attentional anchor for correctness using color feedback for verification [perception]. Student D elaborated on this in more detail: "*You have to maintain a constant speed…if one goes too fast and one goes too slow and there's no coordination in it, then we're not going to get the right angle and then there's not going to be a green color*." Student D emphasizes the importance of maintaining speed and coordination between the hands [action] to get constant green feedback. This task is designed to build an understanding of the inverse relationship, where an increase in one variable results in a corresponding decrease in the other. The necessity of matching speeds to maintain the angle reinforces the concept of proportional inverse changes.

For the more complex linear function $f: x \to -2x + 4$, students began to connect their movements directly to symbolic mathematical properties. Student E mentioned, "*The one number is going down. The other one going up. Like one [input] is starting from positive to negative. The other one [output] from negative to positive … From below zero the input is decreasing by one and the output is increasing by two. And above zero the input is increasing by one and the output is decreasing by two*." Student E aligns the movements with the slope's meaning: for every unit increase in *x*, *y* decreases by two units, which shows the understanding of the function's directional change and the ability to articulate the relationship using symbolic language. These utterances reflect a shift from sensorimotor exploration to expressing covariational structure verbally, suggesting that the student is beginning to re-describe their sensorimotor pattern in terms of formal mathematical reasoning.

In summary, through bimanual tasks, students ground their prospective understanding of covariation concepts in perception-action loops. These embodied experiences enable students to transform their interactions with the digital-embodied environment into stabilized sensorimotor patterns, which they articulate symbolically or verbally as function concepts, such as slope and rate of change.

5.5 Discussion

To answer the main research question on how bimanual movements within a digital-embodied environment support students' covariational thinking, the results have shown that bimanual coordination tasks prompt covariational reasoning through evoking perception-action loops. These loops intertwine kinesthetic dynamics with mathematical reasoning, corroborating the action-based genre of embodied design. The two cases, along with the data in Table 5.3, show how students actively use real-time feedback, in the form of color-change cues, to adjust their hand movements, thereby transforming sensorimotor experiences into articulated covariation concepts. Evidence from students' pretest and posttest performances, as well as their reasoning during the post-task probes, demonstrates an improved understanding of functional relationships, and COV aspects in particular.

The results support hypothesis H1, higher levels of CR correlate with a shorter time to fluency, suggesting that higher initial CR levels are associated with shorter TTF. The weak negative correlation between pretest CR levels and TTF suggests that students with higher initial CR levels achieved fluency more quickly in bimanual tasks than their peers with lower starting levels. Interestingly, the correlation between overall improvement in CR and TTF was not significant. This indicates that the speed of transition to fluency is more closely tied to students' initial CR levels than to the extent of their improvement over time. These results suggest that TTF reflects how efficiently students can translate their initial understanding into action but does not necessarily drive the further development of CR, which could potentially be explained by a ceiling effect.

The findings support H2, higher levels of learning gains in CR correlate with a longer fluency time sum. The positive correlation between overall improvement in CR and FTS highlights that student who showed learning gains engaged more extensively in fluency phases. The finding suggests that FTS is both an indicator and, to some extent, a potential contributing factor to covariational reasoning development. The role of FTS appears more complex than initially hypothesized. A moderate negative and significant correlation between pretest scores and FTS suggest that students with stronger initial CR tend to spend less time in the fluency phase. This finding can support for the claim that higher initial proficiency allows students to develop efficient coordination strategies and reduce the need for extended fluency practice. Conversely, longer duration in fluency phases may provide students with lower initial CR levels opportunities to refine and solidify their understanding.

From a theoretical perspective, the analysis of hand tracking data expanded on Thompson and Carlson's (2017) taxonomy by operationalizing the covariational reasoning levels through the embodied bimanual tasks. The six levels, ranging from no coordination (L0) to continuous covariation (L5), provided a basis for interpreting embodied learning through observable hand tracking and qualitative data. While the results show that hand tracking data can present differences in CR levels, such as trace continuity, coordination, and green feedback, the interpretation of these patterns needs careful consideration (see Table 5.3). Lower levels (L0-L2) are characterized by uncoordinated or loosely coordinated movements (discrete trace; no or intermittent green feedback), while higher levels (L4-L5) exhibit smoother, simultaneous bimanual coordination. The sustained green feedback and smooth trace patterns observed in L5 reflect advanced covariational reasoning. The patterns of hand tracking traces (intermittent or smooth) and feedback (continues green bars or intermittent ones) thus reflect students' embodied learning processes. Importantly, this study extends the original CR framework by operationalizing its levels through observable movement patterns in the context of bimanual tasks. This adaptation supports the broader application of the CR framework in embodied learning research. Linking specific hand movements to covariational reasoning offers valuable insights for both researchers and educators. While this alignment shows potential, it should be noted that using hand-tracking data as a direct means of assessing CR would require further exploration.

The bimanual data in this study can also be interpreted from a reification perspective as elaborated by Shvarts et al. (2024). Together with (Abrahamson, 2021), these scholars identify two steps in reification processes: (1) developing sensorimotor coordination that brings forth a new perceptual structure and (2) crystallizing this perceptual structure into a mathematical artifact. Our findings reflect both stages. As seen in tasks like $f(x)=-x$, students in our study developed stable bimanual coordination strategies, such as maintaining mirror symmetry movement across the intersection point (Table 5.3_Level 2) guided by real-time feedback. This aligns with step 1, in which new coordination gives rise to an attentional anchor, which is the intersection point formed by green lines. Our data also reflects aspects of the second step described by Shvarts et al. (2024), by which students began to articulate the functional relationships discovered through movement, translating their embodied experience into verbal and symbolic representations. Our data illustrates how mathematical artifacts—like the idea of a subtraction sign or directionality—can emerge through reflection on sensorimotor synergies that have stabilized through actions. This interpretation is further enriched by Bos's (2022) perspective on reification, which describes it as a shift from a series of actions on objects to a cohesive single process. In Bos's view, the achievement of smooth bimanual movements (characteristic of higher CR levels, see Table 5.3, Level 4 and 5) can itself be interpreted as an embodied manifestation of reification. Unlike Sfard's commognitive theory (2008), where reification shows through the introduction of a noun or pronoun about this process, Bos's and our approach interpret smooth bimanual movement as a non-verbal sign of reification. Reaching bimanual fluency in our task signifies more than motor skill acquisition; it embodies a grasp of the function as a coherent covariational artifact. This view also aligns with Kaput et al. (2008), who highlight the importance of enabling students to create meaning through their interactions with mathematical representations. By scaffolding students' sensorimotor actions and encouraging reflection, digital tools act as reified artifacts that embody mathematical meaning. For example, the integration of real-time feedback allowed students to directly perceive the consequences of their movements, make immediate adjustments, and refine their coordination strategies. This finding resonates with prior research (Drijvers, 2015; Tall, 2004; Turgut, 2022; Weigand et al., 2024), that argued that digital tools facilitate the connection between action and thought, supporting students in transitioning from exploration to reasoning and symbolic representation.

Methodologically, our study integrates two research traditions: the dynamic systems analysis of embodied interaction and the cognitive analysis of conceptual development in covariational reasoning. We complement a growing research program that uses dynamic systems theory and fine-grained interaction data, such as hand-tracking, to model learning as a nonlinear process of stabilizing sensorimotor coordination (Abdu et al., 2025; Tancredi et al., 2021). Our study makes a contribution by investigating what these stabilized coordinations signify in terms of students' conceptual reasoning. We achieved this by integrating two additional data sources: post-task probes, which enriched the analysis by capturing students' verbal reflections on their strategies and reasoning processes (Shvarts et al., 2021), and pretest and post-test scores, which provided a measure of macro-level conceptual change. This kind of integration also aligns our work with a trend in the field to use mixed methods to create a more robust picture of mathematics learning (Johnson et al., 2024). Our approach allows us to demonstrate how the micro-level emergence of sensorimotor fluency corresponds to macro-level gains in students' mathematical understanding. In doing so, we respond to the call to study the "microgenesis of multimodal conceptual development" (Abdu et al., 2025) by linking specific embodied actions to measurable learning outcomes.

Limitations and Future Directions

While the findings offer promising insights, we acknowledge two limitations. First, this study used a convenience sample from two international schools in the Netherlands and China. This choice was guided by pragmatic factors, including access to tablets and established partnerships with the research team. Our goal at this stage was not generalizability, but to explore whether identifiable patterns of embodied covariational reasoning could emerge in diverse but comparable contexts. Second, our analysis centered on specific performance indicators from the hand-tracking data (Time to Fluency and Fluency Time Sum). While effective, these metrics touch the surface of the complex sensorimotor processes involved. Other potentially informative dimensions of movement (e.g., recurrence rate, determinism, trapping time; Abdu et al., 2025) were not explored.

This study has implications for future research on covariational reasoning in digital-embodied environments. First, there is a need to develop and apply advanced analytical methods to better capture the micro-dynamics of embodied learning. In particular, techniques such as Recurrence Quantification Analysis (Tancredi et al., 2021) with multimodal data (Abdu et al., 2025) can be used to explore the transition between different CR levels corresponds to shifts in the stability of the eye-hand systems. Second, a controlled experimental design is needed to systematically examine the causal relationships between embodied interaction and conceptual development. Randomized control trials with larger and more diverse student populations would help validate the observed patterns and enhance the generalizability of the findings.

In line with other studies that highlight the potential of sensorimotor engagement for mathematical learning (Abrahamson & Sánchez-García, 2016), this study contributes to the growing evidence emphasizing how sensorimotor experiences can improve conceptual understanding. Sensorimotor learning provides an entry point for students with different prior academic achievements to engage deeply with mathematical concepts. This study corroborates that claim, demonstrating that embodied tasks enable students to access complex ideas through intuitive, action-based exploration.

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Chapter 6 General conclusion and discussion

In this closing chapter, we first provide an overview of the study as a whole. Next, we discuss its limitations (Section 6.2) and summarize its scientific and methodological contributions (Sections 6.3 and 6.4). In section 6.5, we zoom out and reflect in retrospect on the role of the body in mathematics learning, the study's central theoretical paradigm. The chapter concludes with implications for future research and educational practice.

6.1 Research overview

The overarching aim of the thesis was to explore how to foster students' functional thinking (FT) within a digital-embodied learning environment. To achieve this aim, we focused on key aspects of FT, including input-output, covariation, and correspondence, and identified how specific design features (e.g., real-life contexts, bimanual movement, and multiple representations) can support students' co-emergence of physical action and mathematical reasoning. Chapter 2 lays the foundation by mapping existing research on embodied approaches and digital tools in developing FT, foregrounding gaps, particularly regarding how students move from bodily engagement to abstract conceptions of function. Chapter 3 concerns designing and piloting a digital-embodied learning environment. Chapter 4 extends this design to a classroom context, evaluating its effectiveness with a larger group of students and exploring how design principles translate into real instructional practice. Finally, Chapter 5 zooms in on the micro-processes of covariational thinking using hand-tracking data, through a more detailed analysis of the sensorimotor patterns that support learning gains. Together, these four sub-studies form a coherent trajectory from conceptual groundwork to design, classroom implementation, and in-depth analysis of embodied mathematical learning.

In **Chapter 2**, we surveyed the research landscape where FT and embodied cognition intersect with digital technology. This systematic review of 51 papers clarified the roles that technology plays from an either didactical or mathematical angle, the stages of mathematical abstraction embedded in FT, and the diversity of embodied strategies. Results highlight that most research uses graphing and algebraic roles of DT, often through dynamic software like GeoGebra, to promote covariation and correspondence, while input-output perspectives and geometry-based tasks remain relatively under-explored. Four main abstraction stages (contextual/situational, referential, particular, and general) emerge in these studies, suggesting that higher-level abstractions typically require dynamic representations and multiple function views. Although action-based and perception-based embodied tasks are common, the potential of continuous real-time feedback remains underutilized. At this point, we recognized that the MIT proportion task could be generalized to any function and that its mathematized version aligns with a known mathematical concept: nomogram in a parallel axes system. By revealing these gaps and opportunities, the systematic review motivates the design, implementation, and analysis of digital-embodied tasks in the subsequent sub-studies and guides the thesis toward nomogram-based interventions that can foster deeper FT.

Building on the insights from the systematic review, we investigated how an embodied design, centered on nomograms, can help students develop FT in **Chapter 3**. Grounded in a light ray context, the designed learning environment uses input-output mappings as a metaphor: rays (or arrows) map from an object (input) to its shadow (output), representing different rules described by functions. By manipulating parameters for contexts, such as sunlight vs. spotlight (representing additive or multiplicative relationships), students were observed to interpret nomograms as function representations. In doing so, they linked everyday intuition (e.g., shadow patterns) with mathematical structures like parallel or divergent rays. Central to this design are bimanual movement tasks, which encourage students to physically coordinate two variables along the nomogram's input and output axes. Real-time color feedback (green/red) cues them to adjust their hands until the correct relationship is maintained. This tactile process fostered a deeper grasp of functional relationships. For example, students actively experienced how one variable must speed up or slow down relative to the other. In the meantime, the embodied tasks intentionally integrate different function representations, prompting students to convert between nomograms, formulas, and function graphs. This conversion practice invites a correspondence view of functions, helping students anticipate how changes in one representation affect another. While the approach proved engaging and conceptually rich for the small group of 14-year-old participants (from the pre-university stream), the findings also reveal design complexities, such as the risk of tool-driven rather than concept-driven learning. Overall, by iteratively comparing Hypothetical Learning Activities to Actual Learning Activities, Chapter 3 revealed how specific design features—such as light ray contexts, bimanual coordination, and real-time feedback—can nurture function concepts when tightly coupled to the targeted mathematical content. These outcomes set the stage for broader classroom applications and refinements in subsequent sub-studies.

Now that we have identified the potential of digital-embodied tasks, we transported the digital-embodied nomogram designs into authentic classroom conditions to examine its feasibility on a broader scale of Grade 9 students (N=39) in **Chapter 4**. In a series of three learning modules, students interacted with real-life contexts (e.g., light and shadow), performed bimanual coordination tasks, received real-time feedback (green/red arrows), and navigated multiple function representations. Quantitative findings (pretest-posttest gains) demonstrated significant improvements across the three main aspects of functional thinking: input-output, covariation, and correspondence. Students initially weak in FT showed notable gains, while high achievers appeared to reach a performance ceiling in input-output aspect. However, the progression to more advanced correspondence thinking is relatively modest, suggesting deeper challenges in mastering representation conversion tasks (e.g., transitioning between nomograms, formulas, and function graphs). Qualitative data (answer boxes within the environment, mini interviews) clarified how key design features scaffold the transition from concrete sensorimotor experiences to abstract mathematical reasoning. Real-life contexts anchor students' early understanding; bimanual movements strengthen covariation awareness; continuous color feedback fosters immediate strategy adjustments; and multiple representations broaden students' grasp of functions as correspondences between variables. Chapter 4 affirmed the feasibility and educational potential of digital-embodied tasks in a classroom setting: embodied experiences and digital representations, when deliberately aligned, can drive significant learning gains in FT. These outcomes pave the way for a deeper, micro-level exploration of how covariational thinking evolves in Chapter 5.

In **Chapter 5**, we delved deeper into the micro-processes of FT by investigating covariational reasoning (CR) in a digital-embodied environment. Specifically, it examined bimanual hand movements through a hand-tracking data, aiming to link sensorimotor fluency with conceptual development in CR. Quantitative analyses showed that students with higher initial CR levels tended to reach "smooth, coordinated" movements more quickly (shorter Time to fluency), reflecting how existing conceptual understanding supports efficient motor coordination. Students who made greater learning gains in CR typically spent more time in fluent coordination phases (longer Fluency time sum), which suggests that prolonged interaction in embodied exploration can foster deeper reasoning development. Additionally, hand-tracking data provided a microlens into perception-action loops, showing how students relied on real-time color cues to iteratively refine their movements and conceptualize continuous covariation. Supporting evidence from pre/post assessments and mini interviews corroborated that bimanual tasks heighten understanding of relationships between variables. We operationalized Thompson and Carlson's (2017) CR taxonomy to an embodied, bimanual task context, providing a tool to better analyze learning activities and mathematical reasoning through the lens of embodied learning processes. This allowed us to analyze patterns in students' hand-tracking data and classify their CR levels using the bimanual CR framework, ranging from no coordination (L0) to smooth continuous covariation (L5). Methodologically, Chapter 5 showed how combining hand-tracking metrics (time to fluency, fluency time sum) with qualitative insights can capture both the "how" and the "why" of students' reasoning development. These findings reinforced that digital-embodied environments—especially those featuring coordinated hand movements and real-time feedback—can foster sensorimotor interaction with functional relationships in combination with CR.

By tracing this path, the thesis integrated conceptual, design-based, and empirical angles. It showed how embodied interactions, digital feedback, and mathematical representations together shape students' understanding of functions. The four sub-studies in the chapters thereby converged on the central conclusion that meaningful bodily engagement, if aligned with carefully structured nomogram tasks and activities, can serve as a potent way for students' FT development.

6.2 Limitations

The generalizability of the findings and the extent to which they capture the full complexity of students' learning processes remain open questions. These limitations are synthesized below, categorized into contextual, methodological, and theoretical concerns.

Contextual and design limitations

Firstly, the digital-embodied tasks required access to tablets, multi-touch screens, and specialized software (e.g., Numworx). Such tools may not be feasible in resource-limited classrooms, restricting broader implementation. Technical issues during the interventions, such as software glitches or hardware limitations, could have affected students' learning experiences. Investigating how similar embodied learning principles could be adapted to low-tech or non-digital environments would enhance scalability.

Secondly, some tasks may be too complex or not optimally designed to guide students toward the learning goals. For example, in Module 3 (e.g., finding domains and ranges for trigonometric functions through nomograms, Figure 3.6 (Chapter 3), some tasks were perceived as overly abstract or technically challenging, leading to student disengagement or reliance on trial-and-error strategies. These issues highlight the need for further refinement of task sequences and possible extra instructional support to ensure that tasks effectively guide students toward the expected learning outcomes.

Methodological limitations

Firstly, the empirical studies were conducted with relatively small and specific samples (e.g., the pilot study with four eighth-grade students; classroom studies with 76 ninth-grade students). The lack of diversity restricts the applicability of the findings to other contexts, such as public schools or different cultural environments and limit generalizability of the results. Further research with more diverse populations would help evaluate the broader applicability of these interventions.

Secondly, data collection and analyses mostly concentrated on immediate outcomes, often measured right after interventions, whether students would apply their newly formed FT to novel problems remains unknown. This raises questions about the durability of the learning gains and whether the embodied approach leads to sustained improvements in students' FT over time. Studies investigating how embodied learning might impact broader reasoning or problem-solving skills over semesters or years would be informative.

Thirdly, the assessment of students' learning primarily relied on pre-/post-tests and qualitative analysis of hand-tracking data. Ceiling effects were observed on some input-output test items (Chapters 4 & 5). Interpreting hand movement patterns as direct indicators of cognitive processes is complex. For instance, distinguishing between different levels of CR based on hand traces requires careful calibration and relied on researcher interpretation. The complexity of analyzing hand tracking data and ensuring its accuracy in reflecting students' reasoning limits the robustness of conclusions drawn from these measurements.

Finally, key embodied metrics like movement fluency and coordination operationalized using task-specific thresholds (Chapter 5). The lack of standardized embodied metrics makes it difficult to compare findings across studies. This risks misclassification of exploration vs. fluency phases through hand-tracking data. Establishing a standardized framework for embodied learning metrics, validated across different digital tools and instructional contexts, would enhance the reliability and generalizability of future research in this area.

6.3 Scientific contributions

Empirical evidence on how bodily movement shapes functional thinking

A scientific contribution of this thesis is the empirical evidence linking physical, hand-based coordination to the development of FT.

Findings on hand movement patterns with qualitative data (Chapters 3 & 5)

Hand trace continuity (the consistency of hand movement paths) and feedback mechanism (the real-time color-changing cues) offer qualitative perspectives on how movement underpins mathematical cognition. These data, which record the continuity, speed, and path consistency of hand movements, reflect students' iterative adjustments as they coordinate two variables to achieve fluency. For instance, hand trace continuity—measured as the smoothness of movement paths—showed students progressing from irregular exploratory patterns to consistent trajectories. The hand-tracking data do not merely record movement; they embody the learning process itself, making visible the micro-adjustments through which students developed precise CR. Such close connections between bodily motion and conceptual thinking echo other studies on gesture and learning, which highlight embodied interactions support students to express mathematical relationships as grounded cognitive actions, rather than embedding external artifacts directly into their cognition. (Alibali & Nathan, 2012; Abrahamson & Sánchez-García, 2016; Goldin-Meadow, 2009).

These findings also align with perception-action loop theories (Abrahamson & Sánchez-García, 2016; Shvarts et al., 2021), suggesting that repeated "micro-adjustments" of the body form more stabilized sensorimotor patterns, which they articulate symbolically or verbally as function concepts, such as slope, intercept, and directionality. As students "keep the arrow green," they continuously reconcile what they see, how they move, and what they understand about the function's behavior, gradually solidifying covariational reasoning. For instance, a student in Task 2.4 noted that "if one hand goes too fast or too slow, there's no coordination…and we don't get the green color." This iterative cycle of noticing an error, adjusting movement, and checking the arrow fosters micro-level perception-action loops (Shvarts et al., 2021).

Insights from quantitative data (Chapter 5)

By analyzing TTF, the interval students need to achieve smooth, synchronized hand movements, and FTS, the total duration in that fluent movement state, we uncovered correlations between embodied performance and CR development. Higher initial CR levels tended to correlate with shorter TTF: Students with stronger initial covariational thinking reached fluent bimanual coordination more quickly. This suggests that pre-existing conceptual grounding accelerates motor adaptation. Greater CR gains often accompanied longer FTS: Students who started with lower initial CR spent more time exploring in a fluent phase and gain more robust improvements. This extended “hands-on” coordination appears to enable iterative refinement of functional relationships, which reflects the positive role bimanual tasks play. Although correlation tests cannot prove direct causation, the findings show that coordinated bodily interaction and real-time digital feedback could be levers for CR development.

Overall, researchers have argued that embodied approaches can accelerate the path from action to symbolic representation by anchoring abstract concepts in embodied experiences (Lakoff & Núñez, 2000; Nemirovsky, 2011; Duijzer et al., 2019). This thesis provides with that argument to the specific domain of bimanual movements with empirical evidence.

Contribution to embodied design approaches

A second contribution of this thesis lies in its integration of embodied design principles with digital technology to foster FT. Building on the theories of embodied design (Abrahamson & Lindgren, 2014), this thesis uses Hypothetical Learning Trajectories (HLTs, Bakker, 2018; Simon & Tzur, 2004) to guide the iterative development of tasks that connect sensorimotor experience with FT.

Chapter 2 revealed limited empirical work on how secondary students move from embodied actions toward higher-level abstractions of functions. It also emphasized a need for embodied tasks that better integrate real-time feedback and dynamic visualizations of functions. Our learning environment, grounded in the Embodied Design framework (Abrahamson, 2014), was deliberately developed to address these gaps. Below, each module is explained in terms of how it addresses a research gap, what the design is like, and describes an illustrative example.

Module 1: Transition from light ray contexts to bimanual tasks

Targeted gap: Calls for studies to explore how context-based activities could move students beyond a simple "input-output" view toward deeper functional relationships in real-time, dynamic settings.

Design and example: Students begin by investigating how shadows' position and size change as an object moves next to different light sources (Figure 4.9, Chapter 4). Students are placed in a meaningful scenario that highlights basic function rules (e.g., $y = 3x - 2$). Following these situational tasks, students gradually shift from unimanual to bimanual activities, which is designed to strengthen covariational reasoning. Through coordinating two hands—one controlling the tree (input) and the other tracking the shadow (output)—students experience how changes in one variable dynamically affect the other.

Illustrative outcome: Students who initially described tasks in everyday language (e.g., "the shadow gets bigger when I move the tree to the right") begin to articulate them in function-like terms (e.g., "When I raise the tree by 1 unit, the top of the shadow goes up by about 3 units"). In doing so, Module 1 responds to the systematic review's call for more robust evidence of how contextual, embodied actions can ground abstract concepts.

Module 2: Semi-nomograms, covariation, and early correspondence
Targeted gap: Chapter 2 noted the need for using dynamic visualizations to promote FT, especially the covariation aspect.

Design and example: Module 2 begins with semi-nomograms, nomograms without labeled numbers, to help students first focus on motion rather than numeric precision. As students gain fluency, labeled axes are introduced to connect embodied actions with formal function notation. Throughout, bimanual coordination and real-time color feedback guide students from trial-and-error movements to a systematic, smooth coordination. For instance, maintaining a green arrow in a $y = -x$ task requires moving both hands at an equal but opposite speed, which quickly reveals how input increments correspond to output decrements.

Illustrative outcome: Students begin to articulate patterns such as "when x increases by 1, y decreases by 1" based on physical coordination, rather than symbolic manipulation. This supports early covariational reasoning and addresses the systematic review's gap regarding how to help students unify bodily experience with dynamic representations.

Module 3: Conversions between nomograms, graphs, and formulas
Targeted gap: Chapter 2 pointed out the need for tasks that integrate

multiple function representations and support advanced function aspects like correspondence or mathematical object.

Design and example: The core tasks involve dynamic conversion practice conversion practice where students transfer between adjusting the nomogram arrow and predicting its corresponding point on the Cartesian graph and vice versa. When students move their hands along input-output axes, they can observe the domain/range constraints: beyond certain input values, the arrow can no longer stay green nor the point align with the function graph. Although formulas are not always explicitly given, students begin to hypothesize how a formula might look when "mapped" onto their bimanual movements. They infer that valid (x, y) pairs follow a consistent pattern, sometimes articulating early algebraic ideas based on their bimanual movements and the observed visual mapping (e.g., "You have to multiply it with like [there is] a minus sign. Because when it goes higher it [the other point] turns into negative. The positive number on the x turns into negative on the y.")

Illustrative outcome: Through these dynamic, real-time tasks and accompanying dialogues (mini interviews, discussions), students develop a grasp of a function as a consistent mapping from inputs to outputs, including analyzing domain and range. This address both aspects highlighted by Chapter 2: a need for dynamic, real-time tasks and an emphasis on advanced function aspects.

Overall, the embodied design structure, spanning from a contextual anchor to formal mathematical dialog, advances the literature by demonstrating how embodied design principles guide task designs that suitable for secondary mathematics classrooms.

Insights into technology-integrated mathematics education

A third contribution lies in the insights gained on how digital tools, specifically multi-touch tablets, digital-embodied nomograms, and hand-tracking software, can deepen students' conceptual understanding. These tools also give researchers a richer window into students' embodied learning processes.

By using multi-touch interfaces, students could physically manipulate multiple variables and experience functions as live, dynamic processes. This aligns with findings from Chapter 2 (RQ1.1) and Chapter 4 (RQ3.2), both of which stressed the importance of receiving real-time feedback in developing a more flexible, coordinated understanding of function concepts. Several students mention how they continuously moved or slide objects in order to immediately see the changes of functional relationships. For example, one of the students said: “I first moved the lines until I could find out. I could see which ones are green. Then I kept moving. Then I checked how much I got, like the relationship between the left and the right.” In a non-digital or mouse-only setup, the interface typically forces a one-variable-at-a-time approach and repeatedly testing multiple positions might be cumbersome. The student’s references to “move,” “see green,” and “check” suggest that multi-touch gestures and real-time color feedback on the device lowered the barrier to experimentation.

Real-time feedback (arrow turning green/red) emerged as an effective mechanism for prompting immediate adjustments in students’ movements and thinking. The systematic incorporation of color, motion, and number lines created multimodal cues that continually invited students to reconcile their physical actions with symbolic or graphical outcomes. Some students mention “moving points” at a “constant speed” or in “opposite directions,” referencing how the system turns red or green if their hands move incorrectly or correctly. These gestures make covariation physically intuitive, as each hand directly represents an axis of change. Because of the color-changing feedback, students can correct or refine their speeds/directions in time, which help to build a solid embodied sense of function relationships. The real-time feedback illustrates how digital technologies (with instantaneous color-coded feedback) can amplify the role of sensorimotor interaction (Drijvers, 2019). This aligns with calls for deeper research into how dynamic visualizations and bodily gestures together influence understanding (Wilkie, 2020).

In some mini interviews across Modules 2 and 3, students express curiosity or satisfaction in “turning the arrow green.” For example, one student commented, “I like it. It’s like an interactive task…you have to find out…to keep thing green". This kind of spontaneous expression of positivity is direct evidence that the touchscreen-based nomogram tasks feel more engaging. Taken together, these comments from the mini interviews point to the positive impact of tablets and digital-embodied nomograms, underscoring how a multi-touch screen, real-time feedback environment motivates students.

6.4 Methodological Contributions

Integrative use of systematic review, design-based research, and mixed methods

The thesis is methodologically improved in that it combines a systematic literature review, design-based research, and a mixed-methods approach: Chapter 2 systematically mapped the state-of-the-art in embodied cognition and FT to ensure the design efforts were informed by recognized gaps: e.g., minimal empirical work on real-time feedback and bimanual tasks for secondary school students. Chapter 3 applied DBR principles (Bakker, 2018; Cobb & Steffe, 2010) through iterative design cycles, guided by embodied design principles and emergent modeling (Abrahamson & Lindgren, 2014; Gravemeijer, 1999). Each cycle refined the tasks to better align learning goals with task features (nomograms, real-time feedback, bimanual movement), and to adapt interventions for real classroom settings (Alberto et al., 2022; Boels & Shvarts, 2023).

A mixed method further strengthened the study. Quantitative instruments (pre/posttests, bimanual movement measures) provided objective measures of learning gains, while qualitative data (video observations, answer boxes, mini interviews, hand tracking data) offered rich insights into students' strategies, misunderstandings, interactions with digital tools, and embodied learning process. Triangulating hand movement data (millisecond-level traces of bimanual motion), dialogue (mini interviews), and written artifacts (pre/posttests and answer boxes) improved the trustworthiness of conclusions, revealing why certain embodied design choices succeeded and how students progressed along hypothetical learning trajectories. Together, these integrative methods provide a robust framework for conceptualizing, enacting, and analyzing digital-embodied learning interventions in mathematics education.

Novel tools and analyses for embodied interaction

A second methodological contribution lies in the analytic approaches for capturing and interpreting students' embodied interactions:

Hand-tracking technology in the Numworx software: Numworx records the x-y coordinates of each "draggable point" (representing the student's hands) every millisecond. As learners move their left and right fingers, the system logs detailed information on locations of the draggable points. We then calculated direction, velocity, and distance from the target functional relationship. Because Numworx also supports programming the color-changing feedback (green arrow for correct alignment, red arrow for misalignment), the hand tracking data can be aligned with students' attempts and on-screen feedback. This makes it possible to analyze how quickly and how smoothly students achieve the "green" fluency phase.

Operationalizing Thompson and Carlson's (2017) taxonomy of CR: Thompson and Carlson propose a hierarchy of covariational reasoning, ranging from no coordination of variables (L0) to smooth continuous covariation (L5). We adapted these descriptors to the case of bimanual movement (e.g., identifying where students show discontinuous, stepwise adjustments vs. continuous, synchronized hand movements). Based on that, we examined trace continuity (how smoothly or discontinuously students moved) and color changing feedback patterns. For example, L0-L2 levels might exhibit repeated red-arrow interruptions (no green feedback) and intermittent hand traces, while L4-L5 levels showed continuous, fluent traces that always triggered green feedback.

This thesis offers a structured framework to analyze students' movement patterns in an embodied environment. It can be adopted or adapted by future researchers interested in sensorimotor learning. For example, it can be used to observe how specific changes in movement traces coincide with improvements in test scores or interview data, or to distinguish when and how many times students shift from discrete, trial-and-error hand movements (lower CR levels) to smooth, integrated bimanual control (higher CR levels) within a task in the domain of function.

6.5 Rethinking the role of the body in mathematics learning

Now that we have summarized the study's results, its limitations and its scientific and methodological contributions, we turn to a broader perspective and rethink the role of the body in mathematics learning, a key paradigm when we started out this thesis' trajectory.

One of the key contributions of this thesis is providing empirical evidence that bodily movement can play a role in fostering FT. The findings suggest that gestures and coordinated bimanual movements help students make sense of input-output relationships, covariation, and correspondence. However, movement alone is not sufficient to invite conceptual development. Mathematical meaning emerges from a complicated interplay between the body, artifacts, social interactions, and environmental affordances.

Any theory of learning or teaching is grounded in a particular stance on how we know the world around us (epistemology) and on the nature of reality (ontology). In mathematics education, these philosophical foundations frame not only our understanding of how students come to know mathematics but also how we design instructional environments, interpret students' actions, and justify methodological decisions.

In the following, we revisit our initial ideas on the role of the body in mathematics learning from an epistemological, an ontological, and an affective and cognitive perspective.

Epistemological perspectives: How do action and perception shape mathematical knowledge?

Traditionally, many have assumed a Platonic or Cartesian view of mathematics: knowledge exists "out there" in an abstract, infallible realm, separate from human sense-making. However, both constructivist and enactivist theories have called this assumption into question. They proposed that knowing is a dynamic process emerging from the student's engagement with the environment (Piaget, 1955; Varela et al., 1991; von Glasersfeld, 1989).

In contrast to these traditional views, the embodied cognition lens – central in this thesis – addresses the epistemological question—How do we come to know mathematics?—in a different way (Barsalou, 2008; Glenberg & Robertson, 2000; Lakoff & Núñez, 2000; Varela et al., 1991). Rather than viewing knowledge as discovered in a purely abstract or symbolic form, embodiment theory challenges the Cartesian dualism of mind and body, positing that mathematical understanding arises from sensorimotor engagement with the environment, social interactions, and cultural artifacts. This perspective emphasizes the role of action and perception, which are recognized as the driving forces of cognition. Students do not just store and retrieve static representations of mathematical concepts; instead, they continuously enact and reconstruct these concepts through repeated cycles of bodily movement and sensory feedback (Abrahamson & Lindgren, 2014; Shvarts et al., 2021). For example:

- Action involves physical engagement, such as moving arms with sensors that control the heights of dots on a screen (Abrahamson & Trninic, 2011), dragging a series of isosceles triangles to plot a parabola (Palatnik et al., 2023; Shvarts & Abrahamson, 2019), or keeping balance on a board to represent changing quantities (Tancredi at al., 2021).
- Perception includes both visual and tactile feedback, including observing height difference between height lines (Bos et al., 2022),

noticing color cues in a situation (Abrahamson, 2012), or feeling the relative speeds between hand movements (as in bimanual tasks).

The literature claims that epistemologically, then, mathematical ideas (e.g., proportionality, covariation, function) emerge through embodied interaction. This resonates with Piaget's (1955) constructivist principle that learners build knowledge structures through active adaptation to, and organization of, their experiences.

While action and perception facilitate embodied learning, an epistemological challenge is the gap between experiential knowledge and formal symbolic representations. Many students successfully perform bodily or artifact-based tasks but struggle to articulate these strategies in algebraic or conventional mathematical language. This tension shows the epistemic leap from tacit, sensorimotor schemas to explicit, universally recognized symbolic systems.

In this thesis, we build upon these ideas from embodied cognition and demonstrate how specific design choices can facilitate this emergence and help connect the experiential-formal gap. We elaborate on this in two main ways.

First, we illustrate how functional thinking can be actively constructed through iterative perception-action loops within our embodied nomogram tasks. For example, in Chapter 4, students explored tasks that involved manipulating inputs and outputs on a nomogram and observing the correspondent changes on the cartesian function graph. Interacting with this transition between different function representations prompted students to connect their actions (e.g., sliding points) with the perception of dynamic visual feedback (like points projections). Over time, these repeated cycles with nomogram tasks enabled students to reify processes of covarying two variables (e.g., adjusting two variables in tandem) into a more concrete concept of a functional relationship. Epistemologically, the knowledge of "function" did not preexist in an abstract domain, the students constructed this notion through interactive experiences. Additionally, hand-tracking data from Chapter 5 provided micro-level evidence of this constructive process. Students adjusted their bimanual movements based on real-time feedback (e.g., color cues indicating correctness). This suggests they were developing an internal, predictive sense of the functional rule governing the task, which helps to anticipate the consequences of their movements, continually refining this internal forward model through their actions (Shvarts & Abrahamson, 2023). The real-time feedback allowed students to detect discrepancies between their anticipated outcome and the actual sensory feedback received. Such perception-action loops, driven by this forward model, reflect and confirm what an embodied epistemology anticipates: knowledge emerges through repeated loops of acting and perceiving.

As a second means to use and adapt findings from literature, we aim to addresses the experiential-formal gap through using congruent representations (Aziz-Zadeh et al., 2006; Segal, 2011). A congruent representation is one where the structure of the physical action or perceived phenomenon shares a structural analogy with the formal mathematical concept. Epistemologically, gesture-based insights can serve as a potential foundation. For example, in our nomogram tasks, the bimanual action for a function like $y = -x$ involves moving hands in opposite directions at similar speeds (As x increases, y decreases at the same rate). This physical experience of "oppositional, synchronized movement" is congruent with the mathematical idea of an inverse relationship where an increase in x corresponds to a proportional decrease in y. The design of the bimanual nomogram task, therefore, aims to make the physical experience a direct, felt analogy of the mathematical structure, and thereby grounding the developing abstract concept in a concrete, sensorimotor way.

Ontological perspectives: What is the nature of mathematical concepts in a digital-embodied environment?

Ontologically, this thesis aligns with perspectives considering mathematics as a human practice, in which concepts are not pre-existing entities but emerge from active, situated interactions within sociocultural and material contexts (Brown et al., 1989; Lave & Wenger, 1991; De Freitas & Sinclair, 2014). More specifically, we draw on an ecological onto-epistemology where mathematical objects arise through a body–artifacts functional dynamic system (FDS) (Shvarts et al., 2021; Shvarts et al., 2024). These FDSs form when students' bodily potentialities (e.g., gestures, perceptual structures) coordinate with the affordances of physical or digital artifacts to fulfill a functional goal. Therefore, rather than anchoring concepts in mental representations (internalist approaches) or in external symbolic notations (externalist approaches), this perspective sees mathematical concepts as entities that construct by action and perception.

In this thesis we built on these notions in two ways. First, in a digital-embodied environment, where physical actions (gestures, movements, and manipulations) intertwine with responsive interfaces, mathematical concepts take on a dynamic, process-like character. The central concept in our case, function, is not an abstract set of ordered pairs. Instead, it arises as relationships-in-action, taking shape through embodied activities such as dragging points on a coordinate plane. This ongoing coupling of bodily movements and visual feedback demonstrates that a concept is not merely represented but co-enacted in real time. Students' repetitive actions gradually stabilize into FDS. As these synergies become fluent, students develop new ways of perceiving the environment (e.g., "seeing" how a slope changes). Importantly, these newly stabilized actions are not only left to bodily memory. They become "crystallized" in artifacts with cultural meanings, such as an intersection point on the graphs or formulas (Shvarts et al., 2021).

Our second ontological perspective concerns the bimanual movements. An enactive ontology highlights that mathematical concepts become meaningful through doing, especially the iterative, mediated interplay of actions, reflections, and symbolic artifacts (Varela et al., 1991; Nemirovsky, 2003). In the thesis, bimanual movements, supported by the multitouch devices, could serve as a dynamic, pre-symbolic patterns of action (Abrahamson & Trninic, 2011). These patterns embody aspects of a mathematical concept (e.g., a specific rate of change) before it is formally articulated. Over time, these enacted patterns can be reified into more durable cultural artifacts, e.g., algebraic notations or function graphs. The digital-embodied tasks do not just reveal students' preexisting understanding, they help generate that understanding. This illustrates a view that the concept emerges from body-artifact synergy (Shvarts et al., 2021).

Reflecting in retrospect on this thesis, a question emerges: how stable and transferrable are these concept enactments outside the specific environment? If the concept "lives" in the synergy of action and digital feedback, does it remain an effective tool for understanding once the digital support is removed? Drawing on reflective abstraction (Boonstra et al., 2023), we can see that the process of naming, re-describing and internalizing newly crystallized artifacts can help students transfer their understandings to different contexts. In other words, the internalization of action-based strategies enables students to reactivate these embodied understandings outside the original environment. In our designed learning environment, this reflective abstraction is further supported through purposeful reflection (via an "answer box" in the digital interface), discussion (mini interviews), and re-notation (conversions among multiple function representations). While the immediate feedback from multitouch devices sparks realizations, the stability and portability of these realizations depend on structured opportunities for students to formalize, verbalize, and symbolically encode their enacted experiences. This reflective layer ensures that what begins as situated action can evolve into a more transportable form of mathematical concept. Through action, perception, and cultural artifact creation, knowledge, initially embedded in action, can be transformed into a more generalized understanding.

Affective and cognitive perspectives: How do emotion, engagement, and cognition interact in embodied mathematics learning?

Mathematics learning is a multifaceted experience shaped by not only cognitive, but also emotional, motivational, and embodied dimensions. The situatedness of embodied learning activities underscores the importance of understanding how student engagement, emotional responses, and cognitive processes interrelate (Eynde et al., 2006; Hannula, 2012). In this section, we reflect on how insights from theories on affect, motivation, and cognition, have been addressed in this thesis, examining both individual and social perspectives to illuminate this interaction.

Research suggests that physiological and affective states, such as interest or frustration, can guide students' attention, affect how they approach tasks, and shape their sense-making of mathematical ideas (Hannula, 2012; Power & Dalgleish, 2015). A key driver of exploration in this process is curiosity, which sparks divergent inquiry and encourages students to try different strategies or persist in the face of challenge. Thereby, curiosity can potentially activate more varied and sustained cognitive engagement (Goldin, 2000; Hannula et al., 2016). When students are curious, they are more likely to experiment with new methods or perspectives and even a broader range of problem-solving skills, prior knowledge, or reflective thinking than they would if simply following a prescribed procedure.

Beyond individual experiences, socio-constructivist theories highlight that emotions and motivation are often shaped and coregulated within socially constructed, such as influenced by classroom interactions and norms (Eynde et al., 2006). In digital-embodied learning settings, the interplay of physical interaction, emotional arousal, and cognitive reflection drives mathematical sense-making (Cross, 2009; Hannula et al., 2016).

Even if in the research process leading to this thesis the affective perspective has played a minor role, we can in retrospect identify two aspects: appeal to curiosity, and engagement in social interaction. As for the first, the color-changing feedback mechanism in this thesis aims at raising students' curiosity: the arrow turning green led to sustained engagement as the students sought to maintain the desired state. Observations revealed emotional peaks (e.g., cheering upon success) and sharing of strategies. Students share these “wins” with peers, sparking discussions about their strategies. As they strive to maintain the green color despite dynamic changes tied to their movements, their curiosity persists, motivating further exploration. Interview data reinforce this: one student described the nomogram as “Nomogram is amazing, it’s very magical.” Notably, the immediacy of the color-changing feedback amplified emotional responses and prompted them to reflect on their current strategy and to consider if their approach was working and why. As one student reflected, “I found 3 or 4 points, and I tried…thinking if it might be squares, or addition…then realized only a multiplicative relationship fit all those points.”, a statement showing how iterative experimentation, fueled by inquisitiveness, led to conceptual breakthroughs. Such moments highlight the bidirectional relationship between affect and cognition: while curiosity drives exploration, successful outcomes reinforce self-efficacy, fostering further cognitive risk-taking (Bandura & Wessels, 1997; Hannula et al., 2016).

Second, engagement and curiosity have a social dimension. Chapters 4 and 5 of this thesis investigate this in a classroom context, where students had the chance to collaborate within a digital-embodied environment to solve tasks. Students faced initial challenges in coordinating their movements but overcame them through joint effort. One student noted, “It was a little difficult at first because we couldn’t move it properly. Yeah. But after some time, it started moving like really smoothly...Once you have the coordination, it’s easy.” Another described synchronized actions: “My hand would be on the output line and I would go up; when I went up, her hand was on the input line going down—the same angle and speed… Once you get the fixed speed, it’s easy to make all lines come green.” These experiences underscore how collaboration fosters a shared understanding of the task’s functional relationships, as students articulate strategies and co-construct knowledge. This kind of collaborative process also improves emotional safety, which refers to an environment where students feel secure enough to take intellectual risks, make mistakes, and express uncertainty without fear of negative judgment from peers or the teacher. Emotional safety can often be improved by positive collective efficacy, which is the group’s shared belief in its capability to succeed (Karau & Williams, 2014; Klassen & Krawchuk, 2009). In such an environment, students interpret confusion or mistakes as opportunities for discovery rather than personal shortcomings (Eynde et al., 2006). Over time, repeated emotional “wins” and productive struggle can refine students' attitudes and identities, even helping them view themselves as capable problem solvers (Hannula et al., 2016).

6.6 Implications for future research

This thesis opens up avenues for deeper and broader investigations on embodied learning. Future research may pursue the following directions to advance both the theoretical and practical applications.

Rethinking Assessment

Traditional assessments may not fully capture the learning outcomes of an embodied approach to functional thinking. Much of students' emerging understanding described in this thesis unfolds through coordinated hand movements, quick adjustments when an arrow turns red, and spontaneous discussions of strategies. Therefore, written tests alone would offer an incomplete picture of the students' progress. Chapter 5 demonstrates how hand-tracking technology can complement these traditional measures, revealing students' movement patterns, use of feedback, and engagement over time. Although this approach enriched the research team's post-hoc understanding of students' learning processes, future work could prioritize dynamic, real-time assessment tools that empower teachers to act during lessons, for example in the following ways:

- Real-time analytics: Develop AI-driven platforms that interpret hand-tracking/gesture logs or gaze patterns data during tasks to provide immediate feedback to teachers (Darvishi et al., 2024; Šola et al., 2024). These platforms would employ machine learning algorithms to process sensor data (e.g., from tablets or motion-capture systems) and detect patterns indicative of learning states (Mitra & Acharya, 2007; Pellas et al., 2020). For example, for hand-tracking data, algorithms could identify hesitation (e.g., prolonged pauses between movements), miscoordination (e.g., inconsistent bimanual synchronization), or fluency (e.g., smooth, continuous adjustments) during tasks like nomogram manipulation (Pellas et al., 2020). As for gaze patterns, eye-tracking might reveal whether students focus on critical features (e.g., intersection points in a nomogram) (Scheiter et al., 2019; Shvarts & Abrahamson, 2019; Strohmaier et al., 2020). Such data could prompt timely intervention or targeted scaffolding, ensuring that students receive immediate support when their learning trajectory begins to falter. By transforming these embodied metrics into actionable classroom intelligence, teachers can respond more precisely to students' needs.

- Post-hoc analysis: In addition, the stored log files would enable longitudinal analysis to refine teaching and design strategies. The log data can be used to identify whether students' movement preferences, such as discrete adjustments over continuous motion. By aggregating all students' data together, it might reveal systemic challenges (e.g., difficulty interpreting negative slopes) so as to prompt iterative redesign of modules to include scaffolding steps (e.g., pre-task warm-ups for learning goals). Machine learning could also correlate movement patterns with learning outcomes to recommend individualized task sequences (e.g., students needing more fluency practice receive additional bimanual exercises).
- Reflective tasks: In parallel, structured reflection can deepen students' metacognitive awareness of how physical actions map onto abstract mathematical concepts. Asking students to keep a brief journal or digital log, describing specific bimanual movements, unexpected adjustments, and links to mathematical ideas. It is important to encourage them to articulate the embodied processes behind their understanding. This practice not only helps students internalize concepts but also provides teachers with evidence of learning that may otherwise remain hidden in physical action.
- Performance-based assessments: Performance-based tasks can offer richer opportunities to demonstrate embodied understanding. For instance, students might be asked to recreate or extend a function within the digital-embodied environment, explaining their choices and describing their movement patterns. Such tasks shift the emphasis from static symbolic understanding to dynamic mathematical structure building, which reflects the core of embodied mathematics. By observing students' evolving strategies and justifications, teachers and researchers gain a multidimensional view of students' conceptual growth that traditional assessments might overlook.

In summary, broadening assessment methods through including real-time analytics, post-hoc analytics, reflective tasks, and performance-based assessments might better align with the embodied nature of students' learning. These methods not only illuminate students' developing proficiency but also guide teachers in delivering timely, personalized support.

Nomograms for advanced function types

Much of this thesis focused on linear, quadratic, or absolute value functions to build foundational FT. A fruitful extension would be to explore more complex function families (e.g., exponential, trigonometric, and piecewisedefined functions) and advanced calculus content (e.g., function composition or transformations, derivative and gradient). This extension aims to investigate whether embodied learning through nomograms continues to improve students' abstract understanding across diverse mathematical content and educational levels.

- Function composition: Brieske (1978) notes that nomograms excel at visualizing function composition, where one function's output becomes another's input. The visualization of function composition is less intuitive in a cartesian coordinate system. Through nomograms, students could trace arrows from an input through multiple mappings, concretizing the abstract layering of functions and the chain rule (e.g., see a GeoGebra example: https://www.geogebra.org/m/fxhvnnhp).
- Derivatives and instantaneous rate of change: Nomograms can represent the derivative, sometimes conceptualized as a "local multiplier," where the scale factor between input and output intervals approximates this rate of change. While Richmond (1963) was one of the first to highlight nomograms in mathematical contexts, recent work by Bos and Brinks (2024) extends this visual interpretation. They use nomograms to represent instantaneous rate of change as an enlargement factor relative to a local focus, which visualizes the derivative as a scaling effect at a point (e.g., see https://www.geogebra.org/m/hap8j44e). This approach mirrors how the tangent line approximates the graph in traditional calculus but more engaging (Bos et al., 2019).
- Gradients in multivariable calculus: Brieske (1978) highlights nomograms' role in multivariable settings, such as $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ functions, where they can represent linear transformations like stretches along axes. Similarly, Inselberg's (2009) parallel coordinates (nomograms) provide a visual solution related to gradients under certain conditions. For example, if a scalar function f in $\mathbb{R}^N$ is such that its partial derivatives (the components of its gradient ∇f) are linear functions of the variables (e.g., if f is a quadratic function), then setting ∇f equal to a constant vector b yields a system of N linear equations. The solution to this system could then be found and visualized as the intersection of these N hyperplanes within the parallel coordinate system. This provides a specific geometric interpretation for finding points with particular gradient values.

By pursuing these directions, researchers can uncover nomograms' full potential as an embodied representation across different levels of mathematics education. Future research should aim to empirically validate their efficacy and refine their implementation.

Scalability and accessibility considerations

While the thesis shows success in specific contexts, broader adoption of digital-embodied design requires addressing scalability and accessibility, particularly in different educational settings, large classrooms, and resource-limited environments.

- Scalability in large classrooms: Investigate how digital-embodied learning can be effectively implemented in large classrooms without compromising student engagement and individualized feedback. This may involve exploring group-based interactions, teacher-guided demonstrations, or hybrid digital-paper approaches.
- Low-tech options: To extend the reach of embodied learning, research could explore low-cost, non-digital alternatives, such as paper-based nomograms, manipulation with physical artifacts, or classroom-scale embodied activities. These alternatives could serve as effective options for digital environments while maintaining embodied learning features.

By addressing these considerations, future research can help bridge the gap between innovative embodied learning environments and their practical large-scale implementation. Scaling up the digital-embodied learning environment requires thoughtful adaptation that ensures that all students have the opportunity to have a meaningful role in the lesson.

6.7 Implications for educational practice

The journey of this thesis–from conceptualizing embodied learning environments to assessing students' evolving functional thinking–offers a roadmap for reimagining mathematics education. At its heart lies a simple yet profound idea: mathematics is not just abstract symbols on a page, but a dynamic structure of movement, emotion, and discovery. Here, we translate these insights into actionable strategies for educators, curriculum designers, and policymakers, weaving together the threads of nomograms, embodied pedagogy, and classroom innovation.

A new character in the function curriculum: The nomogram

One of the stars of this thesis is the nomogram. Nomograms did not start in classrooms, they were born out of necessity in the 19th century, before calculators or computers existed. Engineers, scientists, and mathematicians needed a way to perform complex calculations quickly and accurately. Their brilliance lies in the ability to transform symbolic equations into a visual, manipulable form. This historical context explains why nomograms appeared: they were practical tools for a pre-digital world, making complex math accessible through geometry. Today, this same accessibility makes them a useful educational tool, turning abstract functions into something students can see and touch.

In today's classrooms, functions are typically taught through the lenses of tables, algebraic formulas, Cartesian graphs, and mapping diagrams. Each has its strengths, but they often leave students viewing functions as static objects rather than dynamic relationships. Nomograms step in to fill this gap, offering a hands-on, kinetic complement that ties these representations together. Take an algebraic representation like $y = 2x + 1$. It is precise and concise, but it is also abstract, it does not show how y changes with x. Students might memorize the rule without truly grasping the relationship. A nomogram, however, lets them explore it physically. In the digital-embodied nomogram of $y = 2x + 1$, they can move one hand along an x-axis and another along a y -axis. Through this, they feel the pattern, their y -hand speed is always doubling the x-hand speed. It is a similar situation for plotting $y = 2x + 1$ on a coordinate plane. It is a straight line with a slope of 2 and a y -intercept of 1. Yet, this standard depiction can feel static. Students see the line but may struggle to sense the motion of two variables covarying, since a Cartesian graph merges changes in both x and y into one geometric object. By contrast, a nomogram enables them to literally move x and y and observe how they covary along the scales. This "enacted representation" provides direct experiences of how a change in one variable affects the other, especially for those who find traditional graphs or formulas abstract.

So, why not include nomograms into the mathematics curriculum? At the secondary level, where functions often trip up students, nomograms could join the cast of representations alongside graphs and formulas. Teachers might weave them into lessons, offering ready-made digital tools and activities. For students who struggle, nomograms could be the gentle guide that leads them into the world of FT, turning abstract ideas into something they can get their hands on. For example, coordinating bimanual movements to maintain a green arrow taught students to intuit inverse relationships ($y = -x$) or proportionality ($y = 2x$) before they wrote formulas. The nomogram, with its roots in real-world contexts like light and shadow, offer a missing link in curricula dominated by static graphs and equations.

Bringing nomograms into schools is not without challenges. They require multi-touch devices (like tablets or interactive whiteboards) that support bimanual input and real-time feedback. For teachers, the key is sequencing (see Appendix 3 in Chapter 4 for exemplary lesson plans):

1. Real-Life Context: Start with a relatable scenario, like how shadows change with light position. It mirrors how one variable depends on another.
2. Nomogram Exploration: Introduce digital-embodied nomograms where students manipulate variables and observe outcomes.
3. Linking Representations: Guide students to translate their nomogram actions into graphs and formulas, highlighting the connections among these different representations of the same relationship.
4. Reflection: Use whole-class discussions or short written exercises so students can solidify their new insights and connect the embodied experience to formal symbolic language.

Embodied Classrooms

The thesis demonstrates the effectiveness of digital-embodied tasks in authentic classroom settings (Chapter 4), showing significant learning gains in FT. These tasks can be implemented using accessible technologies like multi-touch tablets or interactive whiteboards. This section explores how teachers can adopt embodied learning to create interactive, student-centered mathematics lessons.

Embodied learning aligns with contemporary trends in mathematics education that emphasize active, student-centered approaches. Research shows that students develop deeper understanding when they actively construct knowledge rather than passively receive it (Freeman et al., 2014; Hiebert & Grouws, 2007). Traditional teaching often focuses on rote procedures, but embodied learning invites students to engage physically with concepts. In this thesis, students were invited to move their actively interact with the tasks. This process also mirrors the pedagogical principle of multiple representations, where students explore mathematical ideas through different formats (e.g., visual, symbolic, kinesthetic) to build flexibility and insight (Goldin & Shteingold, 2001). By incorporating movement, embodied tasks add a kinesthetic dimension that makes abstract notions like covariation palpable.

Effective classroom activities often follow a progression from inquiry-based exploration (students freely manipulate variables) to guided reflection (students discuss patterns or write brief explanations) to formalization (students connect their movements to symbolic notations or standard function graphs) (Artigue & Blomhøj, 2013; Bakker, 2018; Cobb et al., 2003). It is recommended to start with a context that is meaningful to the students, for example through being close to real-life:

1. Embodied tasks are effective when grounded in familiar contexts or phenomena (e.g., shadows and light sources), which students then link to more abstract representations (Nathan & Walkington, 2017).
2. After physically enacting relationships, teachers can gradually prompt students to express these relationships in formal mathematics language (e.g., algebraic formulas and Cartesian graphs), developing a dynamic-to-formal trajectory of understanding (Drijvers, 2019).
3. Teachers can also use these embodied tasks to facilitate mathematically rich discussions. For example, after a group completes a correct "green arrow" diagram, the teacher might have them explain their reasoning. Another group might discuss why their arrow fluctuates between green and red to reveal partial or incorrect understandings. These moments are critical for bridging sensorimotor experiences with formal mathematics language.

Overall, this thesis argues that nomograms should be recognized not as a replacement but as an enhancement in function education. Their embodied, interactive nature complements static graphs, tables, and algebraic formulas. Digital-embodied nomograms provide an additional lens through which students can experience and internalize the dynamic nature of functions. Structured lesson plans that connect embodied experiences with formal mathematics notation through inquiry, guided exploration, and reflective discussion are key to ensuring that nomograms serve as a cohesive representation in function teaching and learning.

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Summary

Functional thinking (FT)—understanding relationships between variables through aspects such as input-output, covariation, and correspondence—is an important skill in mathematics and in everyday life. However, students often struggle with its abstract nature. Traditional teaching, often relying on static pictures, does not effectively build dynamic reasoning about how variables change together. This difficulty is aggravated by the abstract character of functions, which are frequently taught as fixed rules rather than as dynamic relationships between changing quantities.

This thesis addresses these challenges through integrating digital technologies using the lens of embodied cognition, which claims that the body, environment and artifact work together in shaping learning. The main innovation is the use of nomograms. A nomogram is a visual tool that maps functional relationships through parallel axes and arrows (Figure 1). In this thesis, we reimagine digital nomograms as dynamic, interactive tools that connect students' sensorimotor experiences with formal mathematics. The overarching question guiding the thesis is: How do nomogram tasks foster students' FT development in a digital-embodied learning environment?

To answer this question, the thesis comprises six chapters: An Introduction (Chapter 1), four core sub-studies (Chapters 2-5), and a General Conclusion & Discussion (Chapter 6). These sub studies progress from identifying research gaps to designing, piloting and refining digital-embodied nomogram learning environments, implementing them in regular classrooms, and finally, to analyzing bimanual movements at a micro-level. Together, these sub studies provide a holistic view on how nomogram tasks in a digital-embodied setting develop students' FT, including theoretical grounding and practical application.

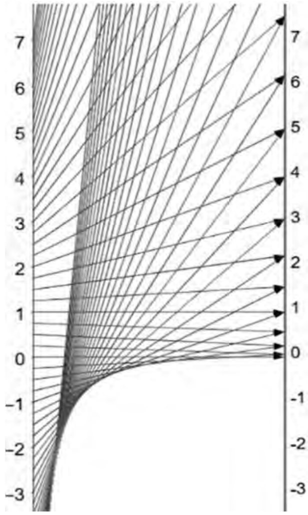


Figure 1 Nomogram of the function $f: x \rightarrow x^2$

Chapter 1 contextualizes FT within broader mathematical practice. It argues that a digital-embodied learning environment using nomograms could address challenges in FT teaching and learning. The chapter introduces the main research question and explains the choice of a design-based approach, in which iterative cycles of design, piloting, analysis and redesign offer a systematic method to study students' understanding of FT. Figure 2 provides an overview of the sub studies in this thesis.

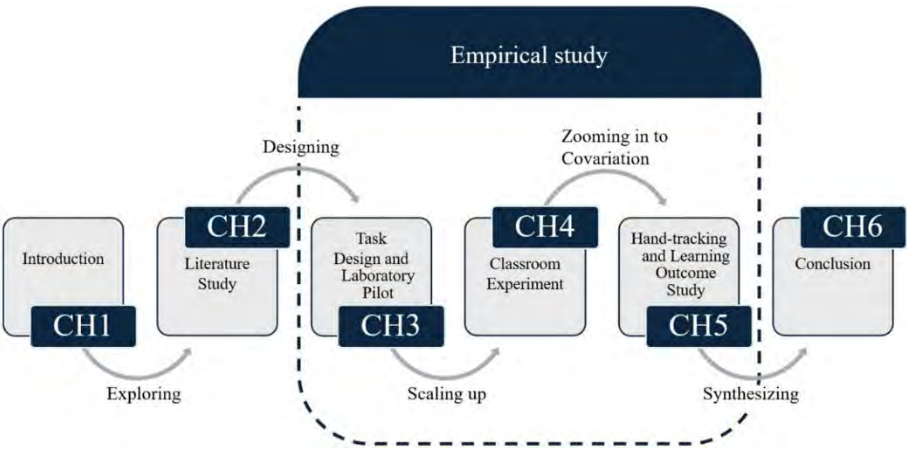


Figure 2 An overview of the studies and the chapters

In **Chapter 2**, we surveyed existing research on FT, embodied cognition, and digital technology to answer these research questions:

RQ 1 How does research literature inform an embodied approach to FT using digital technology that invites abstraction?

RQ 1.1 Which role of technology is widely used in developing functional thinking?

RQ 1.2 What is known about different abstraction stages of functional thinking?

RQ 1.3 Which embodied approaches can be identified in the literature on developing functional thinking?

This systematic review of 51 papers clarified the roles that digital technology plays from an either didactical or mathematical angle, the stages of mathematical abstraction embedded in FT, and the diversity of embodied strategies. Results show that most studies use graphing and algebraic roles of digital tools, often through dynamic software like GeoGebra, to promote covariation and correspondence. The input-output aspect of FT and geometry-based approaches remain relatively under-explored. Four main abstraction stages (contextual/situational, referential, particular, and general) emerge in these studies, suggesting that higher-level abstractions typically require dynamic representations and multiple function representations. Although action-based and perception-based embodied tasks are common, the potential of continuous real-time feedback remains underutilized. At this stage, we recognized that the MIT proportion task (Abrahamson & Trninic, 2011) could be generalized to any function and that its mathematized version aligns with an existing mathematical representation: the nomogram in a parallel axes system. By identifying these gaps and opportunities, the systematic review motivated the design, implementation, and analysis of digital-embodied modules in the subsequent sub-studies.

Building on the insights from the systematic review, we investigated in **Chapter 3** how an embodied design, centered on nomograms, can help students develop FT. It addresses the following research questions:

RQ 2 How can an embodied design using nomograms foster functional thinking?

RQ 2.1 How does a light ray context foster the students' meaning making of nomograms?

RQ 2.2 How do bimanual movement tasks foster covariational thinking?

RQ 2.3 How do different function representations and their conversions support a correspondence view on functions?

Grounded in a light ray context, the learning environment uses input-output mappings as a metaphor: rays (or arrows) map from an object (input) to its shadow (output), representing different rules described by functions. By manipulating parameters for contexts, such as sunlight vs. spotlight (representing additive or multiplicative relationships), students were observed to interpret nomograms as function representations. In doing so, they linked everyday intuition (e.g., shadow patterns) with mathematical structures like parallel or divergent rays. Key to this design are bimanual movement tasks, which encourage students to physically coordinate two variables along the nomogram's input and output axes. Real-time color feedback (green/red) signals them to adjust their hand positions until the correct relationship is maintained. This tactile process did indeed foster a sensorimotor experience of functional relationships. For example, students actively experienced how one variable must speed up or slow down relative to the other. In the meantime, the embodied tasks intentionally integrate different function representations, prompting students to convert between nomograms, formulas, and Cartesian function graphs. This conversion practice invited a correspondence view of functions, helping students see how changes in one representation affect another. While the approach proved engaging and conceptually rich for the small group of 14-year-old participants (from the pre-university stream), findings also showed design issues, like the risk of tool-driven rather than concept-driven learning. Overall, by iteratively comparing Hypothetical Learning Activities to Actual Learning Activities, Chapter 3 revealed how specific design features—such as light ray contexts, bimanual coordination, and real-time feedback—can nurture function concepts when tightly coupled to the targeted mathematical content. These results set the stage for design refinements and broader classroom application in the subsequent sub-studies.

In **Chapter 4**, we moved the digital-embodied nomogram intervention to authentic classroom conditions to examine its feasibility on a broader scale of Grade 9 students (N=39). The research questions for this chapter are:

RQ 3 How can an embodied design using nomograms foster functional thinking in a classroom setting?

RQ 3.1 How does a digital-embodied design using nomograms affect the various aspects of functional thinking among students within a classroom setting?

RQ 3.2 How do the design features contribute to the development of functional thinking?

In three digital-embodied learning modules, students interacted with real-life contexts (e.g., light and shadow), performed bimanual coordination tasks, received real-time feedback (green/red arrows), and navigated multiple function representations. Quantitative findings (pretest-posttest gains) demonstrated significant improvements across all three aspects of functional thinking: input-output, covariation, and correspondence. Students initially weak in FT showed especially notable gains, while high achievers appeared to reach a performance ceiling in the input-output aspect. Overall, progress in more advanced correspondence thinking was relatively modest, suggesting challenges in mastering representation conversion tasks (e.g., transitioning between nomograms, formulas, and function graphs). Qualitative data (answer boxes within the digital learning environment, mini interviews) clarified how key design features scaffolded the transition from concrete sensorimotor experiences to abstract mathematical reasoning; a process in which initial sensorimotor experiences enable the articulation and formalization of mathematical concepts. Real-life contexts anchored students' early understanding; bimanual movements strengthened covariation awareness; continuous color feedback fostered immediate strategy adjustments; and multiple representations broadened their grasp of functions in various forms. As such, Chapter 4 confirmed the feasibility and educational value of digital-embodied modules in a classroom setting, reinforcing that embodied experiences and digital tools can drive significant learning gains in FT. These outcomes pave the way for a deeper, micro-level exploration of how covariational thinking evolves in Chapter 5.

In **Chapter 5**, we delved deeper into the micro-processes of FT by investigating covariational reasoning in a digital-embodied environment. The following research question was investigated:

RQ 4 How do bimanual movements within a digital-embodied environment support students' covariational thinking?

Specifically, this chapter examined bimanual movements through hand-tracking data, aiming to link sensorimotor fluency with conceptual development in CR. We operationalized Thompson and Carlson's (2017) CR taxonomy to an embodied, bimanual task context, providing a tool to better analyze learning activities and mathematical reasoning through the lens of embodied learning processes. This allowed us to analyze patterns in students' hand-tracking data and classify their CR levels using the bimanual

CR framework, ranging from no coordination (L0) to smooth continuous covariation (L5). Quantitative analyses showed that students with higher initial CR levels tended to reach smooth, coordinated movements more quickly (shorter Time to fluency), reflecting how existing conceptual understanding supports efficient motor coordination. Students who made greater learning gains in CR typically spent more time in fluent coordination phases (longer Fluency time sum), which suggests that prolonged interaction in embodied exploration can foster deeper reasoning development. Additionally, hand-tracking data provided a microlens into perception-action loops, showing how students relied on real-time color cues to iteratively refine their movements and conceptualize continuous covariation. Supporting evidence from pre-post assessments and mini interviews corroborated that bimanual tasks heighten understanding of relationships between variables. Methodologically, Chapter 5 showed how combining hand-tracking metrics (Time to fluency, Fluency time sum) with qualitative insights can capture both the "how" and the "why" of students' reasoning develops. These findings reinforced that digital-embodied environments—especially those featuring coordinated hand movements and real-time feedback—can foster sensorimotor interaction with functional relationships in combination with CR.

Chapter 6 synthesizes the findings from the four sub-studies, revealing how they collectively address the overarching question. This synthesis leads to the following conclusions.

- Nomograms can be effective tools to foster input-output thinking, covariational reasoning, and representation conversion within correspondence thinking, especially when they are augmented with real-time feedback and bimanual tasks.
- Embodied design features (particularly coordinated bimanual movements) create attentional anchors for abstract functional relationships, helping students "feel" how changes in one variable correspond to changes in another.
- The design is practically feasible in regular classroom settings, with empirical evidence of learning gains and positive engagement.

Theoretical implications include a deeper understanding of embodied cognition in mathematics education: students' bimanual movement fluency develops concurrently with mathematical thinking. The findings indicate that body-artifact functional dynamic systems – involving bimanual coordination, real-time feedback, and interactive digital representations – facilitate the mathematization of functional relationships. Methodologically, this thesis offers a replicable framework for future design-based research in digital-embodied learning. Specifically, it combines systematic review, iterative environment design, classroom testing, and fine-grained sensor data (hand-tracking) to analyze students' learning processes.

Limitations of the study include reliance on specific digital tools (e.g., multi-touch tablets), the relatively homogeneous student samples, and the short-term nature of the interventions. Future research is invited to (a) explore low-tech or mixed reality adaptations of nomograms, (b) conduct longitudinal studies for sustained improvements in students' FT, (c) integrate machine learning or AI-driven analytics for real-time scaffolding, and (d) extend nomogram-based approaches to more advanced functions across different educational levels. While limitations suggest caution in generalizing findings, this thesis offers a robust framework for both future research and classroom innovation. Specifically, this innovation contributes to mathematics education by demonstrating how to effectively introduce the nomogram into the function curriculum as a dynamic, interactive representation; and by illustrating the principles and practicalities of establishing embodied classrooms where students actively engage with mathematical ideas through bodily movement and interaction, fostering a more active and student-centered learning experience.

摘要

函数思维（Functional Thinking）作为数学教育中的一项重要能力，不仅涉及变量关系的多维度理解（如输入-输出、协变关系及对应的表征方式），更是在连接数学抽象与现实应用中发挥着桥梁作用。尽管函数概念在数学课程中占据基础地位，学生却因其高度抽象性而难以理解。传统教学模式往往侧重于静态图表与公式推导，难以有效支持学生发展动态的协变推理能力，导致函数规则沦为机械记忆和计算的对象。这一教学困境亟待创新性的教学方法予以突破。

本研究以具身认知理论（Embodied Cognition）为理论基础，结合数字技术的发展前沿，提出一种基于诺模图（Nomogram，亦称“列线图”）的数字化具身学习环境（Digital-embodied learning environment），旨在通过身体动作与数学概念的深度融合，重构函数思维的教学路径。诺模图作为一种历史悠久的函数可视化工具，通过平行坐标轴与几何映射直观呈现变量间关系（如图1所示）。本研究将其重新设计为动态交互界面，整合实时反馈与双手协调操作的机制，引导学生在感知运动的过程中内化函数的多重表征（如代数式、图像与情境化模型）。本文的核心研究问题是：在数字化具身学习环境中，基于诺模图的任务如何促进学生函数思维的发展？

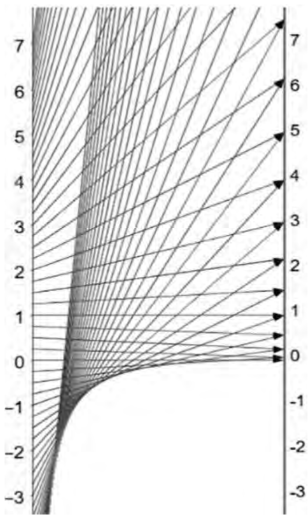


图 1 函数 $f\colon x \to x^2$ 的列线图

为系统探究这一问题，研究通过以下四个阶段的子研究递进展开：

摘要

1. 理论建构与发现研究空白（第二章）：系统性文献综述，梳理函数思维与具身学习的研究现状，明确理论基础与研究缺口；
2. 任务设计与试点（第三章）：基于诺模图开发具身化任务，开展小规模教学实验来初步验证其教学潜力；
3. 课堂规模化应用（第四章）：在真实课堂中实施优化后的方案，检验其可行性与普适性；
4. 微观机制分析（第五章）：利用手部运动追踪技术（Hand-tracking），量化分析身体动作与数学思维之间的关联机制。

这些子研究提供了一个整体视角，揭示了数字-具身环境中的诺模图任务如何通过理论基础、实践应用和微观分析促进学生的函数思维发展。最终，第六章整合研究发现，提出具身化函数教学的理论模型与实践框架，为未来相关研究与教学实践提供依据与参考。

第一章：引言

第一章将函数思维置于更广泛的数学实践背景下，提出基于诺模图的数字-具身学习环境作为应对函数思维教学与学习挑战的创新路径。本章介绍了主要研究问题，并阐述了基于设计的研究方法的依据，即通过任务设计、试点与优化的迭代过程，为培养和研究学生函数思维的发展提供系统化的方法支持。图2展示了本论文中各子研究的整体概览。

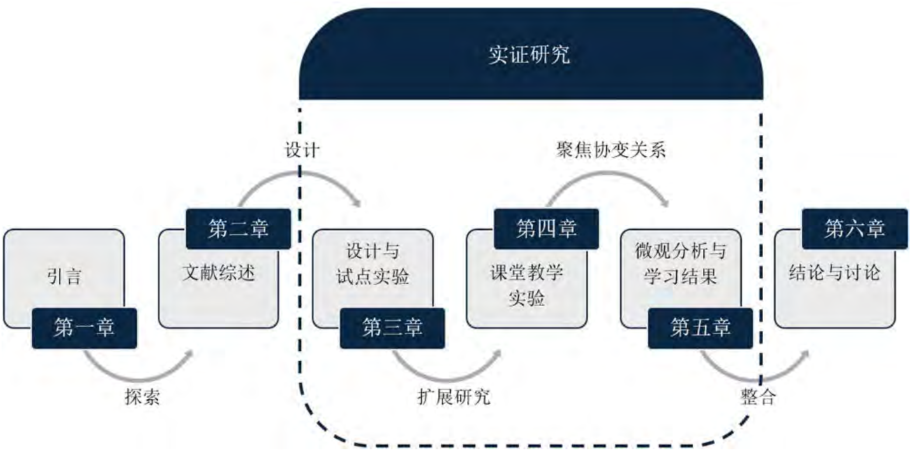


图2 研究概览

第二章：文献综述

第二章通过系统文献综述（Systematic literature review）奠定研究基础，探讨当前运用数字技术支持函数思维具身化发展的研究现状，并提出以下研究问题：

研究问题1：现有文献如何为运用数字技术的具身方法提供支持，以培养学生对抽象函数思维的理解？

子研究问题1.1：数字技术在函数思维发展的过程中广泛扮演了哪些角色？

子研究问题1.2：关于函数思维的不同抽象阶段，现有研究揭示了哪些内容？

子研究问题1.3：在发展函数思维的文献中，已被识别的具身方法有哪些？

基于对51篇相关文献的系统梳理，本章总结了函数思维中的数学抽象阶段、具身策略的多样性，以及数字技术在教学或数学视角中的角色等。结果表明，大多数现有研究主要通过GeoGebra等动态软件的绘图和代数功能，促进协变与对应的理解，而输入-输出视角和几何类任务的探索仍较为有限。研究归纳出函数思维发展的四个主要抽象阶段：情境（Contextual）、指代（Referential）、具体（Particular）和一般（General），并指出高阶抽象通常依赖动态表征和多种函数视角的支持。

此外，尽管动作与感知导向的具身任务在现有研究中占主导地位，持续实时反馈这一关键机制——作为错误纠正和深入参与的关键驱动力——仍较少被充分利用。基于上述发现，本研究提出诺模图具备潜力，通过协调性手势等身体动作与函数概念建立更紧密的联系：其平行坐标轴结构有助于输入-输出映射，而动态交互则可实现“操作-反馈-修正”的学习闭环。

本章节通过文献综述揭示了研究空白与机会，为后续子研究中数字-具身任务的设计、实施和分析提供理论支持，并明确论文基于诺模图开展干预研究的方向。

第三章：数字-具身学习环境的设计与试点

第三章基于综述的研究见解，围绕“光线-影子”这一隐喻情境，开发了一套数字化诺模图学习环境，并在一所中学中选取4名14岁资优生开展试点研究。研究围绕以下核心问题展开：

研究问题2：使用诺模图的具身设计如何促进学生的函数思维？

摘要

子研究问题2.1：“光线-影子”情境如何支持学生对诺模图的意义建构？

子研究问题2.2：双手运动任务如何促进协变思维？

子研究问题2.3：多种函数表征及其转换如何支持学生发展函数的对应观点？

本研究以“光线-影子”情境为依托，设计的具身学习环境以输入-输出映射为隐喻：光线（或箭头）从物体（输入）映射到影子（输出），代表不同的函数规则。通过设置阳光与聚光灯等情境参数（影响加法或乘法关系），学生逐步将诺模图视为函数的表征形式，实现日常直觉（如影子模式）与数学结构（如平行或发散光线）的连接。核心任务为诺模图中的双手运动任务：诺模图界面设有左右两数轴，分别对应输入变量（x）与输出变量（y）。学生需协调双手同时操作控制两轴上的输入值和输出值，使连接输入和输出的“映射箭头”保持绿色（表示当前操作符合函数规则）。例如，在线性函数关系$y = x + 1$中，左手匀速上移时，右手需以相同速率向上移动，且与始终高出左手一个单位长度的距离；而在非线性函数关系如$y = x^2$中，右手则需先减速后加速以匹配相应的函数规律。当操作偏离函数规律时，箭头转为红色，促使学生反思变量间的协变关系。这种动觉互动过程加深了学生对函数关系的感知与理解，例如，学生能主动感知一个变量相对于另一个变量的加速或减速变化。

同时，具身任务整合了多种函数表征形式，引导学生在诺模图、公式和函数图像（平面直角坐标系）之间转换，从而强化函数的对应观点。试点结果表明该方法具有较高的吸引力和概念丰富性，但也暴露出设计复杂性带来的挑战，如存在工具驱动而非概念驱动的风险。

通过比较假设学习轨迹（Hypothetical learning trajectory）与学生的实际学习轨迹，本章进一步揭示了“光线-影子”情境、双手协调操作与实时反馈机制等特征如何与目标数学内容紧密结合，并为后续课堂层面的评估与实施奠定基础。

第四章：课堂规模化实施与效果验证

本章节将数字-具身诺模图设计引入真实课堂，检验其在九年级数学课堂中的可行性与效果（N=39），并围绕以下研究问题展开：

研究问题3：在课堂环境中，基于诺模图的具身设计如何促进学生的函数思维？

子研究问题3.1：数字-具身设计如何影响学生函数思维的不同维度？

子研究问题3.2：设计特征如何具体支持函数思维的发展？

优化后的教学方案在两个九年级班级中进行教学实验，设计包含针对函数思维水平的前后测和三个60分钟的学习模块。前后测结果显示，学生在输入-输出、协变与对应思维三个函数思维维度上均有显著提升，特别是协变维度。最初函数思维能力较弱的学生的进步尤为明显，而成绩优秀的学生在输入-输出方面的表现似乎达到了上限。总体而言，对应思维方面的进步相对较小，这表明学生在掌握表征转换任务（如在直观图、公式和函数图之间转换）方面面临挑战。

定性数据（答案框记录与快速访谈）揭示了具体设计特征如何支持学生从感觉运动体验到抽象推理的过渡：现实情境锚定早期理解，双手运动增强协变意识，颜色反馈促使即时调整，多重表示拓宽对应理解。特别是实时反馈与多表征联动显著降低了认知负荷，对基础薄弱学生尤为有效；然而过度具身化可能导致“动手不动脑”的问题，需平衡操作活动与反思讨论之间合理分配时间。本章验证了基于诺模图的数字-具身任务在课堂教学中的可行性和教育价值，强调当具身体验与数字表征形成协调互动时，能够有效促进学生函数思维的发展。

第五章：微观分析

第五章深入研究数字具身环境中协变推理的微观过程，利用手部追踪技术采集学生在诺模图任务中的运动轨迹，并围绕以下研究问题展开：

研究问题4：双手运动如何支持学生的协变思维？

通过对手部运动数据的分析，本研究尝试连接感觉运动的流畅性与协变推理的概念发展，提取了两项关键指标：进入流畅运动的时间（Time to Fluency）和流畅运动总时长（Fluency Time Sum）。定量分析结果显示，协变推理初始水平较高的学生更快进入流畅运动状态，反映出概念理解有助于身体动作的协调控制；而学习进步较大的学生在流畅运动阶段停留时间更长，表明具身探索中的持续互动促进思维发展。这些模式可以与Thompson和Carlson的协变推理分类法相对应，研究将运动模式与概念理解水平（从L0无协调到L5平滑协变）进行匹配。手部追踪数据揭示出“感知-行动循环（perception-action loop）”：学生依赖实时的颜色反馈来优化动作并概念化连续协变。

结合前后测结果与访谈资料，本研究进一步验证了双手任务在增强变量关系理解方面的有效性。方法上，本研究整合定量的运动数据

与定性的访谈资料，捕捉学生概念转变的“方式”和“原因”，从而拓展了连接感觉运动与函数思维发展的理论框架，强化了数字-具身环境促深度互动的结论。

第六章：结论与讨论

第六章综合四个子研究的主要发现，回答本研究的总体问题，并确认以下结论：

- 诺模图在输入-输出关系、协变推理和对应思维等维度上，是一种有效的表示转换工具，尤其在实时反馈和双手操作任务的主持下表现突出；
- 具身设计特征（如情境设定，双手运动和实时反馈）为抽象的函数关系提供了具体的物理锚点，有助于学生“感知”变量变化的动态过程。
- 教学实验证明该设计在真实课堂中的实施具有可行性，学生在函数思维方面取得显著学习收获，并表现出积极参与的态度。

理论上，本研究深化了数学教育中对具身认知的理解：双手运动的流畅性与数学思维的同步发展过程表明，身体-工具动态系统（Body-artifact functional dynamic system）能够有效支持函数关系的内化与数学化。

在方法上，研究提供了一个可复制的设计研究框架，结合系统性文献综述、迭代任务设计、课堂测试和手部追踪数据技术，深入分析学生的学习轨迹与实际学习过程。

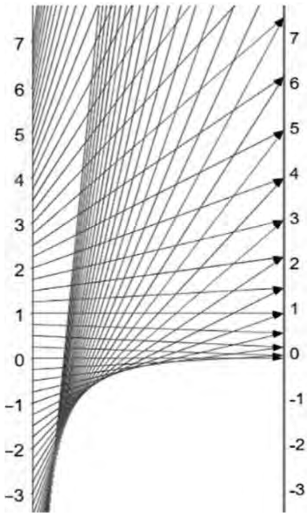
本研究的局限性包括对特定工具（如多点触控平板）的依赖、样本的同质性以及干预时间的限制。未来的研究方向建议包括：(1) 探索诺模图的低技术或混合现实改造方案；(2) 进行追踪研究，以实现学生函数思维的持续改进；(3) 集成机器学习或人工智能驱动的分析工具以提供实时反馈支持；(4) 将基于列线图的方法扩展到不同教育阶段的函数学习中。

综上所述，本研究通过理论创新与实证探索，为数字时代的函数思维教学提供了一条具身化的发展路径。诺模图不仅是一种教学技术工具，更是一座连接身体经验与数学抽象的认知桥梁。当学生的手指在屏幕上舞动时，他们不仅在操作数据，更在重构对函数本质的理解——这一过程，正是具身认知理论在数学教育中的生动而有力的写照。

Samenvatting

Functioneel denken (FD) – het begrijpen van relaties tussen variabelen via aspecten als input-output, covariatie en correspondentie – is een belangrijke vaardigheid, zowel in de wiskunde als in het dagelijks leven. Leerlingen worstelen echter vaak met de abstracte aard ervan. In het traditionele onderwijs, dat veelal steunt op statische afbeeldingen, leren leerlingen slechts in beperkte mate om dynamisch te redeneren over hoe variabelen samen veranderen. Dit probleem wordt versterkt doordat functies, die vaak worden aangeleerd als vaste regels in plaats van dynamische relaties tussen veranderende hoeveelheden, zelf ook abstract zijn.

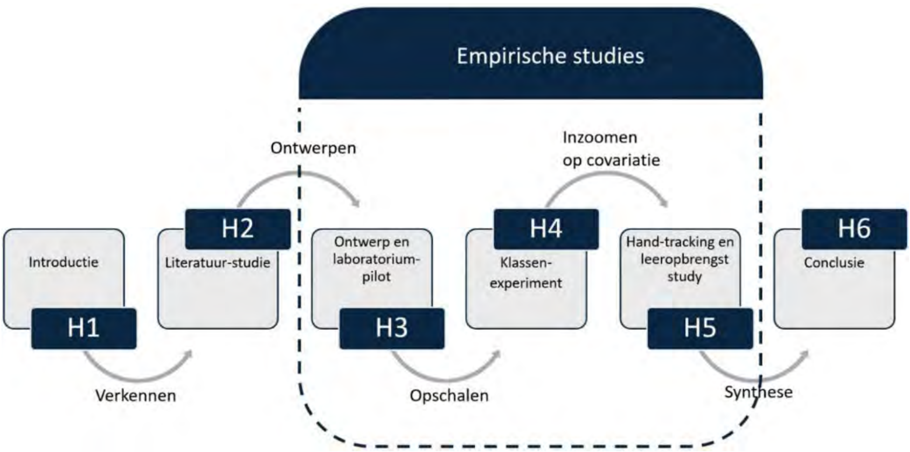
Dit proefschrift pakt dit probleem aan door digitale technologieën in het onderwijs in te zetten vanuit het perspectief van *embodied cognition* (belichaamde cognitie), een theorie die stelt dat lichaam, omgeving en hulpmiddelen samenwerken bij het leren. De belangrijkste innovatie is het gebruik van nomogrammen. Een nomogram is een visueel hulpmiddel dat functionele relaties weergeeft met parallelle assen en pijlen. In dit proefschrift gebruiken we digitale nomogrammen als dynamische, interactieve tools die de sensomotorische ervaringen van leerlingen verbinden met formele wiskunde. De overkoepelende onderzoeksvraag van dit proefschrift is: Hoe bevorderen opdrachten met nomogrammen de ontwikkeling van FD bij leerlingen in een digitale, belichaamde leeromgeving?



Figuur 1 Nomogram van de functie $f: x \to x^2$

Deze vraag wordt beantwoord in zes hoofdstukken: een inleiding (Hoofdstuk 1), vier centrale deelstudies (Hoofdstukken 2–5), en een conclusie en discussie (Hoofdstuk 6). De vier deelstudies richten zich op het identificeren van hiaten in onderzoek, het ontwerpen, testen en verfijnen van digitale, belichaamde modules rond nomogrammen, de implementatie ervan in klaslokalen, en tot slot de analyse van bimanuele (tweehandige) bewegingen op microniveau. Samen bieden deze deelstudies een holistisch beeld van hoe nomogram-opdrachten in een digitale, belichaamde setting het FD van leerlingen bevorderen, inclusief een theoretische onderbouwing en een toepassing in de onderwijspraktijk.

Hoofdstuk 1 plaatst FD in de context van de bredere wiskundepraktijk. Er wordt beargumenteerd dat een digitale, belichaamde leeromgeving met nomogrammen een manier is om uitdagingen in het onderwijzen en leren van FD aan te pakken. Het hoofdstuk introduceert de centrale onderzoeksvraag en licht de keuze toe voor een ontwerpgericht onderzoek (design-based research), waarbij iteratieve cycli van ontwerpen, testen, analyseren en herontwerpen een systematische methode bieden om het begrip van FD bij leerlingen te bestuderen. Figuur 2 geeft een overzicht van de deelstudies in dit proefschrift.



Figuur 2 Overzicht van deelstudies en hoofdstukken

In **Hoofdstuk 2** hebben we bestaand onderzoek naar FD, embodied cognition en digitale technologie geanalyseerd om de volgende onderzoeks(deel)vragen te beantwoorden:

1 Hoe kan de onderzoeksliteratuur bijdragen aan een belichaamde benadering van FD met digitale technologie die abstractie uitlokt?

1.1 Welke rol van technologie wordt veel gebruikt bij de ontwikkeling van functioneel denken?

1.2 Wat is er bekend over verschillende abstractiefasen van functioneel denken?

1.3 Welke belichaamde benaderingen zijn te vinden in de literatuur over de ontwikkeling van functioneel denken?

Deze systematische review van 51 artikelen verhelderde de rollen die digitale technologie speelt vanuit een didactisch of wiskundig oogpunt, de stadia van wiskundige abstractie binnen FD, en de diversiteit aan belichaamde strategieën. De resultaten tonen aan dat de meeste studies digitale tools inzetten voor grafische en algebraïsche toepassingen, vaak via dynamische software zoals GeoGebra, om covariatie en correspondentie te bevorderen. Het input-output aspect van FD en geometrische benaderingen blijven relatief onderbelicht. Vier hoofdfasen van abstractie (contextueel/situationeel, referentieel, particulier en algemeen) komen naar voren in deze studies. Dit suggereert dat abstracties op een hoger niveau doorgaans dynamische representaties en meerdere representaties van functies vereisen. Hoewel actie- en perceptiegerichte belichaamde opdrachten gebruikelijk zijn, wordt het potentieel van continue, directe feedback nog te weinig benut. In dit stadium realiseerden we ons dat de MIT-verhoudingstaken (Abrahamson & Trninic, 2011) gegeneraliseerd kunnen worden naar elke functie en dat de gemathematiseerde versie hiervan overeenkomt met een bestaande wiskundige representatie: het nomogram in een parallel assenstelsel. Door deze hiaten en mogelijkheden te identificeren, motiveerde de systematische review het ontwerp, de implementatie en de analyse van de digitale, belichaamde modules in de volgende deelstudies.

Voortbouwend op de inzichten uit de systematische review onderzochten we in **hoofdstuk 3** hoe een belichaamd ontwerp, gericht op nomogrammen, leerlingen kan helpen FD te ontwikkelen. Het behandelt de volgende onderzoeks(deel)vragen:

2 Hoe kan een belichaamd ontwerp met nomogrammen functioneel denken bevorderen?

2.1 Hoe bevordert een context met lichtstralen de betekenisgeving van nomogrammen door leerlingen?

2.2 Hoe bevorderen bimanuele (tweehandige) bewegingstaken covariationeel denken?

2.3 Hoe ondersteunen verschillende representaties van functies en de omzetting daartussen een correspondentieperspectief op functies?

De leeromgeving, gebaseerd op een context met lichtstralen, gebruikt input-outputrelaties als metafoor: stralen (of pijlen) lopen van een object (input) naar zijn schaduw (output) en representeren zo verschillende regels die door functies worden beschreven. Door parameters voor contexten te manipuleren, zoals zonlicht versus spotlight (die additieve of multiplicatieve relaties vertegenwoordigen), bleken leerlingen nomogrammen te interpreteren als representaties van functies. Hierbij legden ze verbanden tussen alledaagse intuïtie (bijv. schaduwpatronen) en wiskundige structuren zoals parallelle of divergerende stralen. Centraal in dit ontwerp staan bimanuele bewegingstaken, die leerlingen aanmoedigen om fysiek twee variabelen te coördineren langs de input- en outputassen van het nomogram. Directe kleurfeedback (groen/rood) geeft aan dat ze hun handposities moeten aanpassen om de juiste relatie te behouden. Dit tactiele proces bevorderde inderdaad een sensomotorische ervaring van functionele relaties. Leerlingen ervoeren bijvoorbeeld actief hoe de ene variabele moet versnellen of vertragen ten opzichte van de andere. Tegelijkertijd integreren de belichaamde opdrachten bewust verschillende representaties van functies, waardoor leerlingen worden aangezet om te schakelen tussen nomogrammen, formules en Cartesische functie grafieken. Deze oefening in omzetting stimuleerde een correspondentieperspectief op functies en hielp leerlingen inzien hoe veranderingen in de ene representatie een andere beïnvloeden. Hoewel de aanpak boeiend en conceptueel rijk bleek voor de kleine groep 14-jarige deelnemers (vwo-leerlingen), toonden de bevindingen ook ontwerpissues, zoals het risico dat het leren meer door de tool dan door het concept werd gestuurd. Al met al laat hoofdstuk 3 door het iteratief vergelijken van hypothetische leeractiviteiten met werkelijke leeractiviteiten zien hoe specifieke ontwerpkenmerken – zoals contexten met lichtstralen, bimanuele coördinatie en directe feedback – functieconcepten kunnen voeden wanneer ze nauw gekoppeld zijn aan de beoogde wiskundige inhoud. Deze resultaten legden de basis voor verfijning van het ontwerp en bredere klassikale toepassing in de volgende deelstudies.

In **hoofdstuk 4** pasten we de interventie met digitale, belichaamde nomogrammen toe in authentieke klassituaties om de haalbaarheid ervan op grotere schaal te onderzoeken bij 39 derdeklassers (klas 3 vwo). De onderzoeksvragen voor dit hoofdstuk zijn:

3 Hoe kan een belichaamd ontwerp met nomogrammen het functioneel denken bevorderen in een klassituatie?

3.1 Hoe beïnvloedt een digitaal, belichaamd ontwerp met nomogrammen de verschillende aspecten van functioneel denken bij leerlingen in een klassituatie?

3.2 Hoe dragen de ontwerpkenmerken bij aan de ontwikkeling van functioneel denken?

In drie digitale, belichaamde leermodules werkten leerlingen met levensechte contexten (bijv. licht en schaduw), voerden ze bimanuele coördinatietaken uit, kregen ze directe feedback (groene/rode pijlen) en navigeerden ze tussen meerdere representaties van functies. Kwantitatieve bevindingen (vooruitgang tussen voor- en natoets) toonden significante verbeteringen aan op alle drie de aspecten van functioneel denken: input-output, covariatie en correspondentie. Leerlingen die aanvankelijk zwak scoorden op FD lieten bijzonder opmerkelijke vooruitgang zien, terwijl hoogpresterende leerlingen een prestatieplafond leken te bereiken op het input-output aspect. Over het algemeen was de vooruitgang in het meer geavanceerde correspondentiedenken relatief bescheiden, wat wijst op uitdagingen bij het beheersen van taken die het omzetten van representaties vereisen (bijv. de overgang tussen nomogrammen, formules en grafieken van functies). Kwalitatieve data (antwoordvakken in de digitale leeromgeving, mini-interviews) verduidelijkten hoe belangrijke ontwerpkenmerken de overgang van concrete sensomotorische ervaringen naar abstract wiskundig redeneren ondersteunden; een proces waarbij initiële sensomotorische ervaringen de articulatie en formalisering van wiskundige concepten mogelijk maken. Levensechte contexten verankerden het aanvankelijke begrip van leerlingen; bimanuele bewegingen versterkten het bewustzijn van covariatie; continue kleurfeedback stimuleerde onmiddellijke aanpassingen van strategieën; en meerdere representaties verbreedden hun begrip van functies in verschillende vormen. Hoofdstuk 4 bevestigde de haalbaarheid en educatieve waarde van de digitale, belichaamde modules in een klassituatie, en onderstreepte dat belichaamde ervaringen en digitale tools aanzienlijke leerwinsten in FD kunnen opleveren. Deze resultaten baanden de weg voor een diepgaander onderzoek op microniveau naar hoe covariationeel denken zich ontwikkelt, zoals beschreven in hoofdstuk 5.

In **hoofdstuk 5** doken we dieper in de microprocessen van FD door covariationeel redeneren in een digitale, belichaamde omgeving te onderzoeken. De volgende onderzoeksvraag stond centraal:

4 Hoe ondersteunen bimanuele bewegingen binnen een digitale, belichaamde omgeving het covariationeel denken van leerlingen?

Specifiek onderzocht dit hoofdstuk bimanuele bewegingen via hand-trackingdata, met als doel sensomotorische vloeiendheid te koppelen aan conceptuele ontwikkeling in covariationeel redeneren (CR). We hebben de CR-taxonomie van Thompson en Carlson (2017) geoperationaliseerd voor een belichaamde, bimanuele taakcontext. Dit leverde een instrument op om leeractiviteiten en wiskundig redeneren beter te analyseren vanuit het perspectief van belichaamde leerprocessen. Dit stelde ons in staat patronen in de hand-trackingdata van leerlingen te analyseren en hun CR-niveaus te classificeren met behulp van het bimanuele CR-raamwerk, variërend van geen coördinatie (L0) tot soepele continue covariatie (L5). Kwantitatieve analyses toonden aan dat leerlingen met een hoger initieel CR-niveau doorgaans sneller soepele, gecoördineerde bewegingen bereikten (kortere 'Time to fluency' – tijd tot vloeiendheid), wat weerspiegelt hoe bestaand conceptueel begrip efficiënte motorische coördinatie ondersteunt. Leerlingen die grotere leervorderingen maakten in CR, brachten doorgaans meer tijd door in vloeiende coördinatiefasen (langere 'Fluency time sum' – totale vloeiendheidstijd). Dit suggereert dat langdurige interactie bij belichaamde exploratie de ontwikkeling van dieper redeneren kan bevorderen. Daarnaast boden hand-trackingdata een gedetailleerd inzicht in perceptie-actiecycli, waaruit bleek hoe leerlingen vertrouwden op directe kleurfeedback om hun bewegingen iteratief te verfijnen en continue covariatie te conceptualiseren. Ondersteunend bewijs uit voor- en natoetsen en mini-interviews bevestigde dat bimanuele taken het begrip van relaties tussen variabelen vergroten. Methodologisch toonde hoofdstuk 5 hoe het combineren van hand-tracking maten (Time to fluency en Fluency time sum) met kwalitatieve inzichten zowel het 'hoe' als het 'waarom' van de ontwikkeling van het redeneervermogen van leerlingen kan vastleggen. Deze bevindingen versterkten het idee dat digitale, belichaamde omgevingen – vooral die met gecoördineerde handbewegingen en directe feedback – sensomotorische interactie met functionele relaties kunnen bevorderen in combinatie met CR.

Hoofdstuk 6 synthetiseert de bevindingen van de vier deelstudies en laat zien hoe deze gezamenlijk de overkoepelende onderzoeksvraag beantwoorden. Deze synthese leidt tot de volgende conclusies.

- Nomogrammen kunnen effectieve hulpmiddelen zijn om input-outputdenken, covariationeel redeneren en het omzetten van representaties binnen correspondentiedenken te bevorderen, vooral wanneer ze worden uitgebreid met directe feedback en bimanuele taken.

- Belichaamde ontwerpkenmerken (vooral gecoördineerde bimanuele bewegingen) creëren aandachtsankers voor abstracte functionele relaties, waardoor leerlingen kunnen 'voelen' hoe veranderingen in de ene variabele overeenkomen met veranderingen in de andere.
- De praktische haalbaarheid van dit ontwerp werd aangetoond in reguliere klassituaties, met empirisch bewijs van leerwinst en positieve betrokkenheid.

Theoretische implicaties omvatten een dieper begrip van embodied cognition in wiskundeonderwijs: de vloeiendheid van bimanuele bewegingen van leerlingen ontwikkelt zich gelijktijdig met hun wiskundig denken. De bevindingen wijzen erop dat functioneel-dynamische systemen van lichaam en hulpmiddel – met bimanuele coördinatie, directe feedback en interactieve digitale representaties – de mathematisering van functionele relaties vergemakkelijken. Methodologisch biedt dit proefschrift een repliceerbaar kader voor toekomstig ontwerpgericht onderzoek naar digitaal, belichaamd leren. Specifiek combineert het een systematische review, iteratief ontwerp van de leeromgeving, klassikale tests en fijnmazige hand-tracking data analyse om de leerprocessen van leerlingen te analyseren.

Beperkingen van het onderzoek betreffen onder meer de afhankelijkheid van specifieke digitale hulpmiddelen (bijv. multi-touch tablets), de relatief homogene groepen leerlingen en de korte duur van de interventies. Toekomstig onderzoek zou zich kunnen richten op (a) het verkennen van low-tech of mixed reality-aanpassingen van nomogrammen, (b) het uitvoeren van longitudinale studies naar duurzame verbeteringen in het FD van leerlingen, (c) het integreren van machine learning of AI-gestuurde analyses voor directe ondersteuning (scaffolding), en (d) het uitbreiden van nomogramgebaseerde benaderingen naar meer geavanceerde functies op verschillende onderwijsniveaus. Hoewel de beperkingen tot voorzichtigheid manen bij het generaliseren van de bevindingen, biedt dit proefschrift een robuust kader voor zowel toekomstig onderzoek als innovatie van de lespraktijk in de klas. Concreet draagt deze innovatie bij aan het wiskundeonderwijs door te laten zien hoe het nomogram effectief kan worden geïntroduceerd in het curriculum voor functies als een dynamische, interactieve representatie. Bovendien illustreert het de principes en praktische aspecten van het creëren van 'belichaamde klaslokalen' waar leerlingen actief met wiskundige ideeën aan de slag gaan via lichamelijke beweging en interactie, wat een actievere en leerlinggerichte leerervaring bevordert.

Publications related to this thesis

Wei, H., Bos, R., & Drijvers, P. (2022). Embodied approaches to functional thinking using digital technology: A bibliometrics-guided review. In U.T. Hodgen, J., Geraniou, E., Bolondi,G. & Ferretti, F. (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)*. Free University of Bozen-Bolzano and ERME. https://hal.science/hal-03748999

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Wei, H., Bos, R., & Drijvers, P. (2022). *Embodied approaches to functional thinking using digital technology: A bibliometrics-guided review.* Paper presented at the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12), Online, February 2–6.

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Curriculum Vitae

Hang Wei was born in August 1995 in Xinjiang, China. After completing her secondary education at Kuitun First Senior High School, she enrolled at Beijing Normal University, where she obtained her bachelor's degree in Mathematics and Applied Mathematics in 2017. Driven by a strong interest in mathematics education, she continued her academic journey with a master's degree in Education (Mathematics) at Beijing Normal University, which she completed in 2020. During her studies, she undertook teaching internships at Urumqi No. 23 High School and Beijing No. 8 High School, which deepened her interest in classroom-based educational research.

In June 2021, Hang started her PhD at the Freudenthal Institute, Utrecht University, under the supervision of Prof. Paul Drijvers and Dr. Rogier Bos, with funding from the China Scholarship Council (CSC). Her research explores how digital-embodied designs can support the development of functional thinking in secondary mathematics education. As part of her doctoral work, she actively contributed to the Erasmus+ FunThink project, where she conducted literature reviews, co-developed digital learning modules, and coordinated international collaboration to implement and refine these modules in classroom settings.

Alongside her academic work, Hang enjoys traveling, doing sports, photography, and spending time with friends and family.

Acknowledgement

As fate would have it, I have just received the final round of revisions for the last chapter. With that done, I begin to write these acknowledgements, a look back and summary of my four-year doctoral journey.

It all began in the autumn of 2019 in Beijing, with my first meeting with Paul, arranged by Professor Cao. This is a story I have told many times to many people over the past four years, but each time I tell it, and even now as I write it down, my heart is filled with gratitude and excitement. I call it the most important turning point in my life. I thank Paul for embracing me with his immense generosity, giving me the determination and courage to continue on the academic path. Due to the pandemic, it was not until mid-2021 that I set foot in the Netherlands. Through Paul's introduction, I had the great honor of meeting my co-supervisor, Rogier, and becoming his first PhD student. At his warm invitation, I spent my first summer in the Netherlands with his two cats, Poesjkin and Ziggy. Through his own experiences, Rogier made me realize the diversity of life, and I began to rethink the meaning of life, of existence, and of pursuing a PhD. There are countless stories between the three of us and endless thanks I wish I could give. Above all, I want to say: your academic wisdom and life philosophies have become a part of me, it is a treasure I will carry forward and share in my future!

At the Freudenthal Institute (FI), there are so many people full of passion for life and for academia. The first colleague I met, and my deskmate, was Lonneke. She cared for me like a mother, proofreading my first article. That was the first time I felt such sincere help from a colleague, and it taught me how to interact and get along with my peers. Unfortunately, because we graduated at different times, we had few opportunities to connect after my first year. But when we met again at a conference years later, you said you were happy to see how much I had grown and progressed. I am sincerely grateful and admire your personal charisma.

Anna is, in my heart, an undisputed female role model! Your passion, rigor, and dedication to academia have profoundly influenced me. I truly love the EMIG meetings you organize and always look forward to the articles you recommend and the discussions with everyone. What I admire even more is your understanding and support for foreign minorities like myself. In a bilingual environment like the Netherlands, it is often inevitable that we are excluded from conversations, but you could always sense my timid retreats

and encouraged me to speak up bravely. I hope that one day I can become an outstanding scholar, just like you!

To Michiel, who invited Anna and me to go ice skating often! You are always so approachable and capture people’s attention with unexpected knowledge. Your understanding and tolerance of different cultures are deeply impressive; I never thought a colleague would wish me a happy Chinese National Day. Your care and support for students have also inspired me, so meticulous that you even think of us when choosing flight seats. I believe the FI will flourish under your leadership!

Next, I also had a period of close collaboration with Gerben on the FunThink project. As a high school teacher pursuing a PhD at the same time, your dedication to your work and passion for teaching are admirable. You are one of the few colleagues I’ve met who replies to emails outside of work hours. Your reliability and efficiency give a great sense of security, you are truly dependable!

To my paranymph, Jacoliene! You are my lifesaver! My first impression of you was, "This person is so cool!" You have contradictory and charming charisma. I don’t know if you still remember our first close interaction when I went to your school for data collection. I had been stood up by another school at the last minute and felt like my world was collapsing. Thankfully, you appeared like an angel to lead me out of the dilemma, and that very afternoon I successfully collected data at your school. Later, when you showed me around Sydney after a conference, our conversation about life made me see how truly incredible you are. I still remember the moving tears in your eyes when you spoke of your cat passing away and the beautiful bracelet you had made from its memory. You are a genuine, free-spirited person, with a life full of love, family, and adventure. I truly admire you, and I am so honored to have you by my side.

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Looking back on these four years, I once believed that doing a PhD would give me all the answers. But now I realize that life doesn’t always have to be about finding the answers. Sometimes, it’s about learning to walk the path, to live the questions. The answers can arrive in their own time.

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Understanding how things change together is an important skill, both in mathematics and everyday life, from personal finance to public health. Learning about these relationships, in mathematics usually modeled as functions, can be challenging for students because the ideas may feel abstract and disconnected from their experience. Regular teaching with static formulas and graphs often fails to help students make sense of how the values of variables co-vary dynamically. This thesis explores an innovative way to make the learning of functional thinking, and of covariation in particular, more concrete and interactive. To do so, we designed a digital learning environment that uses a visual representation called a nomogram. Our key innovation was to get students to use both hands to explore these mathematical relationships. By moving points on two parallel lines on a screen, students experience directly how changes in one variable affect another variable, based on the function rule. The learning environment provides real-time feedback: a green arrow appears for a correct pairing, and a red one for an incorrect match. This feedback helps students adjust their movement and develop an understanding of covariation. Through a series of studies, from initial design pilots to trials in regular classrooms, we found this hands-on approach helped students grasp complex mathematical ideas in an embodied way. The action of moving the two hands and the perception of the feedback allowed students to develop a strong feeling for how functions work. By connecting physical action with abstract thinking through the use of digital technology, this research demonstrates that sometimes the best way to understand an abstract concept is to get your hands on it.