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ORIGINAL ARTICLE

Visual representations as objects of analysis: the number line as an example

Anne Teppo · Marja van den Heuvel-Panhuizen

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Abstract Our paper examines the representational nature of number lines as they are used in instructional tasks. The examination is informed by a so-called mathedidactical analysis of the number line as a tool used in teaching students mathematics. This analysis led to the identification of a family of number line models, based on visual aspects of number lines each reflecting different forms and functions. In the article, number line tasks are unpacked to illustrate the visual representational components of particular number line models. We illuminate how these components of the models provide tools to locate whole numbers and integers, operate with them, and facilitate reasoning and understanding of underlying mathematical concepts.

Keywords Number line · Number line tasks · Representational aspects of number line · Subject-related analysis · Number line as tool · Number line as structural reference context

1 Introduction

> Number lines can be a taken-for-granted part of the mathematical landscape in many classrooms. They are taped to desks, posted on walls, printed in books, and shown on rulers. Because number line representations are ubiquitous from early elementary through high school, it is easy to assume that students and teachers have shared understandings of what these lines represent and of what sorts of mathematical thinking they make possible. (Strickland et al. 2005, p. 1)

The above quote describes a widespread view in mathematics education; that number lines are an everyday ingredient of mathematics learning and that they present a form of representation that is obvious in both its meaning and application. However, it cannot be assumed that students share the same perceptions as those of their teachers (Arcavi 2003; Cobb et al. 1992; Van den Heuvel-Panhuizen 2003a). Cobb et al. (1992) address this situation through a focus on the "experienced instructional transparency" of materials. They cite the classroom use of base-ten blocks to illustrate their perspective. In many cases, teachers incorrectly assume that their expert understanding of the blocks' "transparent instructional representation of place value notation" will also be evident to students, who have not as yet internalized this mathematical structure (ibid., p. 8). Further, the authors note that, "the problem of explaining how students make constructions compatible with those that the expert has in mind seems intractable as long as we fail to make our self-evident interpretations of external representations *an object of analysis*" (ibid., p. 9, emphasis added). This means that we, ourselves, as teachers or task designers or as researchers of mathematics education must take on the responsibility for this kind of analysis when discussing any didactical tool to be used in mathematics education.

This analysis is also necessary for the number line considered as a didactical tool, because research has found student performance on basic number line tasks to be

A. Teppo
Livingston, MT, USA
e-mail: arteppo@gmail.com

M. van den Heuvel-Panhuizen (✉)
Freudenthal Institute for Science and Mathematics Education, Faculty of Science & Department of Special Education for Cognitive and Motor Disabilities, Faculty of Social and Behavioural Science, Utrecht University, Princetonplein 5, 3584 CC Utrecht, The Netherlands
e-mail: m.vandenheuvel@fi.uu.nl

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problematic. For example, students may have difficulty correctly locating whole numbers when some of the regular tick marks are not shown on the number line (Diezman et al. 2010), using a number line incorrectly to solve simple addition word problems (Skoumpourdi 2010) or incorrectly interpreting whole number addition operations pre-drawn on a number line (Ernest 1985). As Ernest (1985, p. 422) notes, we should not assume that "because the number line model can be used to represent enactive procedures employed by young children its use is taken to be self-evident to children." Researchers have also cautioned that teachers, themselves, often do not have a deep understanding of the representational affordances and constraints present in the use of particular number lines within specific task contexts (Bobis and Bobis 2005; Gray and Doritou 2008; North Carolina Department of Instruction n.d.).

In this article, we carry out a theoretical analysis of the number line by unpacking the representational aspects of number lines used either as tools to order whole numbers and integers or to operate on them, or number lines used as objects to promote reasoning and reflection about underlying mathematical concepts. This analysis includes issues of visual apprehension, that is, understanding the visual features of a number line as carriers of mathematical meaning. In unpacking the number line, we take both a mathematical and a didactical perspective; which we refer to as a *mathedidactical analysis*.

2 Number lines considered as a representational system

A number line is more than just a figural drawing that is constructed by a set of arbitrary rules. It is part of a "semiotic representation" (Duval 1999). For any meaningful apprehension of a number line task to occur, it is necessary not only to notice a line's representational components, but also to understand how these components denote and convey information—how they "show the organization of relations between the representational units" (Duval 1999, p. 13).

A particular number line model is part of a representational *system* organized to highlight patterns of relations among numbers and operations. Such a system consists of a set of representational units and labeling conventions, and a collection of rules that describes the ways of working within this representation.

In any investigation of representation, an important question to address is, "What is representing what and in what ways?" (Kaput 1998, p. 266). Thus, it is necessary to unpack *how* a number line representation *visually* embodies particular aspects of the complex web of mathematical meanings ascribed to the notion of number, the structure of arithmetic operations on numbers, and density properties of the set of real numbers—comprising natural numbers (1, 2, 3, …), whole numbers (0, 1, 2, 3, …), integers (…, −2, −1, 0, 2, 3, …), rational numbers (such as ¼, or 0.25; 1/3 or 0.33333; 5/11 or 0.454545), and irrational numbers (such as $\sqrt{2}$ and π).

The mathematical meanings that can be ascribed to a number line vary considerably, according to an individual's selective focus on particular aspects of the line's representational units (Earnest 2007). Visually, these units consist of a *line segment*, which may be shown with *arrows* at each end and a set of *points* (tick marks) placed at equal *intervals* along the number line. Arcs or directed arrows drawn, between tick marks, are additional visual features that are used to model the operations of addition and subtraction.

The way that a given number line serves as a particular representation is also context-dependent and participant-dependent. Saxe's (2004) form-function framework provides a useful lens for unpacking representational aspects of number line use. Here, *form* refers to a number line's particular visual features such as the presence or absence of points, unit intervals, and zero. How one perceives and uses these forms determines their *function*. However, "confusion is very likely to happen if the nature of the number line model is not understood and the constituting aspects of its nature are not clearly recognized" (Van den Heuvel-Panhuizen 2008, pp. 25–26). Saxe (2004, p. 244) emphasizes the subjective nature of representational apprehension: "[I]n themselves, forms contain no intrinsic mathematical meaning. Rather, the meaning of a form emerges relative to the goals of individuals (and forms afford particular kinds of goals)."

It is important to note that we characterize the representational nature of a number line to be consistent with its use as a *formalized model*. A number line is a figural device, representing particular mathematical abstractions that make it possible to think about and operate with different types of number.

Yet, we do not associate a number line with the word "geometric", because this line is not used as a shape or to represent a spatial situation. Here, we follow Duval's (1999) distinction between a geometric object and a drawing in which geometrical elements are used. Thus, while a number line may be constituted out of geometrical elements such as points and lines, the focus on labeling particular numbers and carrying out specific arithmetic operations places these objects outside a geometric register of representation. Therefore, we consider these number lines to be pictorial or figural, rather than geometric.

3 Mathedidactical analysis of the number line leading to a family of number line models

The *form* of a number line that is selected for a particular task should always be related to the *function* for which it is used as a model. There is not a "ubiquitous" number line that fits all number-related tasks. However, many "teachers are familiar with only the traditional number line with the whole numbers already in place" (North Carolina Department of Instruction n.d.), and students' difficulties with number line tasks have been observed in cases in which there is an overreliance on using this so-called "traditional" form (Bobis and Bobis 2005; Diezmann et al. 2010). It is important, therefore, to unpack the representational variation inherent in the different forms within the *context* of the function in which a number line is designed to serve as a model. With that goal in mind, we developed our classification of a family of five number line models on the basis of the *function* for which a particular model is used.

3.1 Our approach in analyzing the number line

In our mathedidactical analysis of the number line we examined, from a mathematical perspective, *what* mathematical content is represented by the number line and, from a didactical perspective, *how* this mathematical content is presented and students' understanding of the content is achieved through task design and prompted student activity. The mathedidactical analysis of the number line was based on our personal experiences as mathematics educators, a study of selected student texts for learning mathematics, and a review of relevant research literature. The goal of the analysis was to gain a better understanding of what this figural object is, both from a mathematical and a didactical perspective.

As we started to examine examples of number line tasks displayed in texts and reported in the literature, we began to identify differing accounts about the kinds of meaning that students, teachers, and researchers are assumed to assign to similar number line models (e.g., Diezmann et al. 2010). Such variation was also apparent in the ways that research tasks were framed and the resulting student productions were interpreted (c.f., Hannula 2003; Yanik et al. 2006). This range of interpretation led us to the realization that the "number line" was comprised of a set of representational "entities" that embodied different aspects of number and number relations and operations.

The variety of definitions presented in the literature for the number line representation provide further evidence of the breadth of mathematical and didactical meanings associated with this figural object. These definitions range from a focus on mathematical constructs, where the number line is considered as a linear representation of number, to functionally concrete descriptions of how to create the figural drawing.

Devlin (2008) describes this object as a representation of the "coherent and unified mathematical structure" of the real number system. Furthermore, according to Skemp (2002, p. 140), "[t]he number line is conceptual—it is a mental object, though we often use diagrams to help us think about it." Moreover, the number line is infinite. "[I]n our thoughts, we can think of a number line as going on and on to infinity" (ibid.). From another perspective, Lakoff and Núñez (2000, p. 279) describe the number line that "you learn in grammar school" as being a *metaphor* made up of "a conceptual blend […] of source and target domains […] in which entities are *simultaneously numbers and points.*"

Heefer (2011, p. 865) draws attention to the meanings that are to be assigned to the line's figural elements, stating that, "the number line is a *representation* of numbers on a straight line where points represent integers or real numbers and the distance between points matches the arithmetical difference between corresponding numbers." Freudenthal (1983) goes farther and describes the line's representational function from a didactical perspective, noting both a counting and a measurement interpretation.

> The device beyond praise that visualizes magnitudes and at the same time the natural numbers articulating them is the number line. (ibid., p. 101)
> How far is it from here to there on the number line? The little steps are counted. But you can also take the 'from here to there' between your thumb and forefinger, carry it back to 0 and read it off. Adding *n* to *m* can be performed by counting but it can also be done in one blow: the piece that is 'accomplished' at *n* is taken between the fingertips and carried over to *m*. (ibid., p. 102)

In spite of how a number line representation may be defined, researchers have recognized the individual interpretive act of engaging with a particular number line model in the context of either a classroom or research task. As Strickland et al. (2005, p. 1) point out, it is problematic to assume "that number lines have intrinsic meanings or singular interpretations."

Given the complexity of this representational system, we began our analysis by addressing the following questions.

- What does a number line represent and how does it do this?
- How is a number line used?
- Are there different number lines?
- What are the differences among the number lines and why are they different?
- What are the underlying conceptions of number supported by the different types of number lines?
- What are the strengths and limitations of the different number lines?

While synthesizing the answers to these questions, we identified a large variety of number lines. Contrasting and comparing various number line examples led to the development of a classification framework to characterize number lines in terms of (1) variations in their visual features, (2) the type of numbers involved, (3) how numbers and operations are represented by a number line, and (4) the didactical support that a number line provides. We used this framework to group the number lines according to similar characteristics into five different types of number lines that are commonly found in mathematics texts in primary and middle school and in teacher education programs for these school levels.

3.2 Family of number line models

Our characterization of a *family of number line models* is summarized in a form-function chart (see Appendix). In this chart the number line models are organized into five columns. The *function* of each model is to display information about numbers and relationships among numbers (as shown in the upper half of the chart); and to display operations on numbers (as shown in the lower half of the chart). The *form* of each number line is characterized by its key visual features that represent numbers and operations, and is illustrated with a visual image. In addition, for every model, information is included about the particular mathematical constructs and processes that are didactically supported by that number line. Finally, an example of an appropriate task illustrates how each number line can be used.

The *filled number lines*, in column 1, are characterized by equidistant points, or tick marks, that represent whole numbers. The upper image of the number line reflects the counting sequence and facilitates counting activities, positioning numbers, and exploring number order and relations. The lower image shows the carrying out of whole number calculations by counting-on or counting-back. Although in mathematics textbooks and in research literature these filled number lines are often called a “structured number line”, we decided not to use this name for these number lines, since all of the five types of number lines that we identified in our chart can be said to be “structured” in particular ways.

The *empty number lines*, in column 2 of the chart, focus attention solely on the order aspects of number. The upper image is characterized by ordered (but not necessarily equally spaced) points on a line segment. These points, which represent numbers, convey information about relationships related to number order. The lower image, which features an empty line segment, shows a calculation *strategy*. Starting from a freely positioned point on such a line, students can draw a sequence of jumps to visually represent the steps involved in carrying out a particular calculation.

The *directed-length number lines*, in column 3, utilize a measurement notion of number, where integers are represented by directed lengths—lines specified by both magnitude and direction. The upper image displays numbers as lengths measured from zero. In the lower image, addition and subtraction operations are represented by aligning and translating directed lengths. Directed-length lines support arithmetic operations on integers and reasoning about the structure of these operations.

On the *rational number lines*, top of column 4, the unit interval is divided up into equal sub-intervals, and rational numbers are represented as points or tick marks. These lines display rational number counting sequences and support the positioning of fractions and decimals. The *parallel rational number lines*, bottom of column 4, use sets of parallel lines with differently partitioned unit intervals to display and reason about equivalence relations. Numbers are represented as points or tick marks.

In the *proportional* and *double number line* models in column 5, numbers are represented by points or tick marks, which are proportionally positioned with respect to given boundary values. The *proportional number line* is used to display the approximate position of real numbers. The *double number line* consists of a single line with a double scale that displays paired points. Numbers above and below a single tick mark represent a ratio and are used to reason about a particular proportional relationship. It is important to note that both proportional number lines display *multiplicative* relationships among numbers, while the filled and directed-length number lines display *additive* relationships.

The “number representation” rows in the chart indicate fundamental differences in the underlying conceptions of number that are supported by different number line models, where numbers are associated either with points or with directed lengths. A point representation indicates a counting-based conception, where a number is derived from counting discrete objects. A directed length representation reflects a measurement-based conception, where a number is associated with the magnitude of a continuous length.

In the sections that follow we present our examination of number line tasks. Since a task that includes the use of a number line can serve a variety of instructional purposes, we focus on which number lines are incorporated in which tasks and how representational aspects of these number lines can elicit and foster particular mathematical understanding. Due to space limitations in this article, we include only examples that use the filled, empty or directed-length number lines with tasks involving whole numbers and integers.

4 Understanding the representational aspects of a number line

In this section we discuss representational aspects of counting-based and measurement-based number lines in relation to the ways these number lines appear in tasks. Although such number lines at first sight appear to be very simple models that represent, in an easy way, numbers and operations with numbers, it is a rather complex entity of which the constituting characteristics are not directly reflected in its visual features.

Within the two conceptions of number, we also examine in this section five functions of the number line's representational components. These functions, which are used to carry out number line tasks, include (1) representing the *location* of numbers; (2) representing *order relations* among numbers; (3) representing numbers as *operands* in calculations; (4) representing the *process* of carrying out operations; and (5) representing the *results* of operations. Our discussions of tasks will highlight how the visual features of different number line models are interpreted in terms of these features.

We unpack the various representational aspects of number lines by analyzing a filled number line task in Sect. 4.1 and several directed-length number line tasks in Sect. 4.2. Our discussions illustrate the importance of understanding these representational aspects in order to make meaningful connections between *what* is represented (the mathematical concepts) and *in what way* it is represented (the visual features).

4.1 Representational aspects of a filled number line

Children's initial exposure to number line activities utilize a counting-based conception of number, where the filled number line is typically employed as a visual aid to support the development of the whole number counting sequence (Everyday Mathematics n.d.; Skoumpourdi 2010). Activities may include extending a given sequence of numbers that are already labeled on the number line, demonstrating skip-counting by hops of two or three along a line, and determining the relative size of numbers by their left-to-right ordering. Subsequent activities include modeling whole number addition as a process of counting forward by steps of one unit (Everyday Mathematics n.d.). Here, while still using a filled number line, the intended instructional emphasis may reflect a measurement-based conception of number. However, this shift in representational meaning may not be apparent to students.

Research reported by Earnest (2007) illustrates the ambiguous nature of many children's understanding of whole number representation on a filled number line. The tasks used in a study with grade 5 children were designed to elicit information about how these students interpreted both tick marks and unit intervals. Follow-up interviews investigated whether a student located a point by noting the number of tick marks or the spacing between those tick marks. In his report, Earnest describes in detail an interview with one particular student.

Presented with the number line shown on the left in Fig. 1, the student quickly identified *Q* as 4. However, when the line shown on the right was drawn for her, she stated that the value of *Q* could be 3. Further questions suggested that she "used the tickmark with a counting function without regard to the distance between tickmarks" (Earnest 2007, p. 606).

Earnest's results highlight the importance of including *representational awareness* as part of the didactical intention of early activities on the number line. The filled number line, which is an instructional staple in many primary classrooms, may not be as self-evident a representation to students as teachers assume (Ernest 1985; Earnest 2007).

In his research, Earnest (2007, p. 604) asks, "for a student learning how to engage with the number line, what informs the relative salience of particular representational forms over others?" We take this question seriously in our analysis of particular number line tasks in terms of the representational awareness that is highlighted in suggested instructional contexts.

For example, consider how the task from the NCTM Illuminations (NCTM n.d.), shown in Fig. 2, illustrates the representational complexity of *points*, *spaces*, *intervals,* and "*hops*" when a number line is used to enact whole number addition. While the NCTM introduction to this task states that the number line model "highlights the measurement aspect of addition," we note below particular counting features of the teacher's narrative that accompanies the classroom implementation of the task. Given the emphasis on counting used in this example, we consider this task to employ a counting-based orientation, which requires a filled, rather than a directed-length number line.

In the classroom demonstration, addition is modeled by drawing arcs that span consecutive tick marks to represent how numbers are combined to determine a given sum through a process of counting-on. Here, we draw attention to the ways in which the different visual features of this particular model can be interpreted as representations of

Fig. 1 Interview tasks used by Earnest (2007, p. 604)

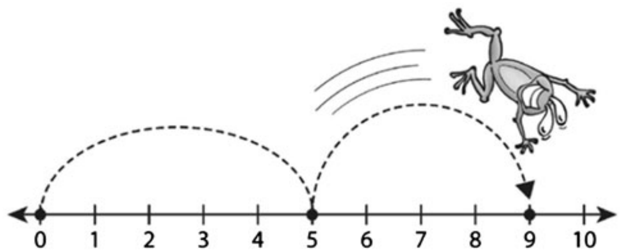


Fig. 2 Addition modeled as counting-on using a "hopping" metaphor

numbers, as arithmetic operands, as indicators of the process of addition, and as indicators of the results of calculations.

In the discussion that follows, we focus our analysis solely on how the instructional context of the task emphasizes particular representational aspects of the use of a filled number line to model whole number addition. We do not comment on the didactical effectiveness of this task for promoting the development of computational understanding.

> The teacher instructions suggest:
> Tell the students that they will find sums using the number line model. Then display a large number line and a 5 + 4 domino, that is, a domino with 5 spots on the left side and 4 spots on the right. Then demonstrate with a counter how a hop of 5 is taken on the number line. <u>You may wish to encourage students to count aloud as the hop is made</u>. Then make a hop of 4, starting at the place the counter landed. You might choose to have them record what happened using the equation notation $5 + 4 = 9$, or to informally describe the moves this way: "If you take a hop of 5 spaces and then a hop of 4 spaces, you land on 9." You may wish to highlight the fact that in this model, spaces are counted, not points on the number line. (NCTM n.d.) (Underlining added by authors)

Notice the choice of particular words in the suggested classroom narration. The students are encouraged to *count* aloud as a "hop of 5" is made, and they are reminded that spaces are counted, not points. This particular instruction suggests that the drawing is not in agreement with the narration. Based on what is said here one might expect five hops of one (a counting-based orientation) instead of a continuous arc or "hop" from zero to five (a measurement-based interpretation). Moreover, there are other ambiguities in these instructions. If spaces are counted, then it would make sense to the students to perform a count as shown in Fig. 3, where the numbers in the squares are assigned to each space as the count is made.

The NCTM example illustrates subtle shifts in how meaning is assigned to the filled number line's visual features when the filled number line is used to model whole number arithmetic rather than to simply locate numbers. In this example, it is necessary to think of the visual representation in terms of a *coordination* between spaces and tick marks. Spaces are bounded by tick marks and the right most mark represents the ending value of a hop. In other words, a "hop of 5" is operationalized by moving a finger in an arc from one tick mark to the next, 5 times, on the number line. While the finger *covers* the spaces, the count is made by noting the tick mark at the *end* of each jump. This "hop arc" stands for both the numeral 5 (constituted from the count of five spaces) and the initial framing of the calculation as the location of the endpoint value representing the first addend.

When the diagram's visual features are regarded as representations of the addition *process*, further shifts in interpretation are required in the coordination of meaning between tick marks and spaces. In particular, notice that the beginning of the initial jump of "5" starts at *zero*, and thus includes *six* tick marks over the total jump of *five* spaces. Also, the jump of "4" moves across the tick marks representing the whole numbers 5 through 9. To understand why the total number of tick marks is ten and the number of spaces is nine, means being able to recognize the significance of spaces, endpoints of spaces, and tick marks, and how a *particular configuration of visual units* can be *counted* as a way to model addition on the filled number line. Notice also how the result of this calculation is "read off" the number line as the numerical value assigned to the tick mark representing the endpoint of the second addend. Here, the represented meaning shifts back from counting spaces to counting the number of tick marks drawn to the right of the zero location.

The equal spacing between tick marks can also be interpreted as measuring "units" from zero and used to locate whole numbers on the line in terms of measured

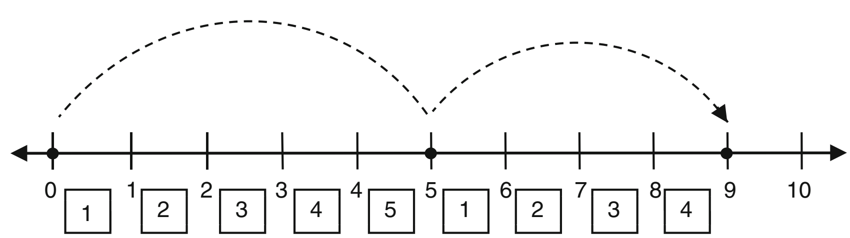


Fig. 3 Counting-on strategy modeled as counting spaces between tick marks

lengths (e.g., Diezmann et al. 2010). However, the emphasis in the above example treats the equal spaces more as "separators" between tick marks, where the equality of the spaces can be construed as representing the regularity of the counting sequence. In the next section we address the measurement interpretation of number in the context of directed-length number line tasks.

4.2 Representational aspects of a directed-length number line

The directed-length number line reflects a measurement-based conception that affects not only how numbers are apprehended but also how operations are represented. Within this conception, the numerals that label tick marks represent the values of the *endpoints* of *directed distances* measured from zero, and numbers are interpreted as specifically oriented *measured* line segments, or vectors.

On the directed-length number line, rules for modeling addition and subtraction of integers are a direct result of the way that positive and negative integers are positioned to the right and left of zero on the line, and the way in which numbers are represented as vectors that have both a magnitude (length) and a direction (designated by a head and a tail).

To illustrate the manner in which the visual features of a directed-length number line display the processes of integer addition and subtraction, we present examples taken from American preservice mathematics texts. Comparing the examples shown in Figs. 4 and 5 illustrates the complexity of these forms of visual representation.

A commonly used model in preservice texts employs the metaphor of "walking" on the number line to visually enact integer subtraction (Billstein et al. 2010; Çemen 1993; O'Daffer et al. 1998). The walker begins by facing to the right. Positive numbers are created by facing right and moving *forward* from zero the required number of units. Negative numbers are created by facing right and walking *backward* from zero the required number of units. Subtraction is accomplished by *turning around* and then walking either forward or backward, depending on the sign of the subtrahend. In the case of an addition the walker does not turn around but continues his walk either forward or backward, depending on the sign of the number to be added.

With respect to subtraction, the two directed-length number line diagrams in Fig. 4 make explicit the distinction between subtraction and the value of the subtrahend, as well as modeling the subtraction operation as the opposite of addition. In each diagram, the walker's movements are represented by arrows of appropriate lengths that are coordinated to the direction of the iconic "walker."

The next example of integer subtraction (see Fig. 5), also uses a directed-length number line, but here the iconic representation of the walker is changed into a more abstract representation that uses arrows (vectors). Both positive and negative numbers are represented by vectors, whose length

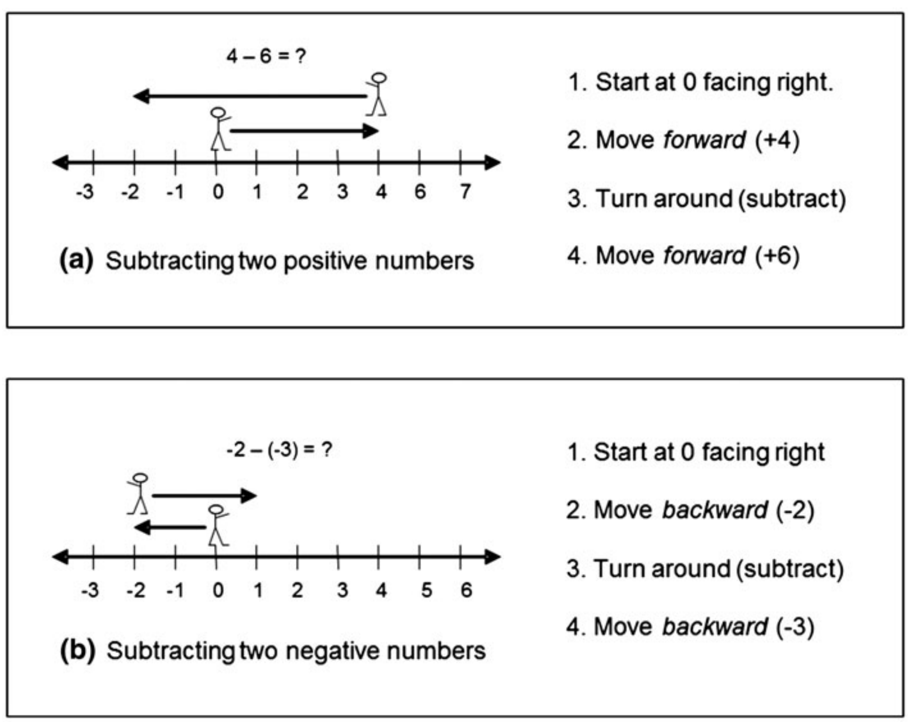


Fig. 4 Modeling integer subtraction using a "walking" metaphor

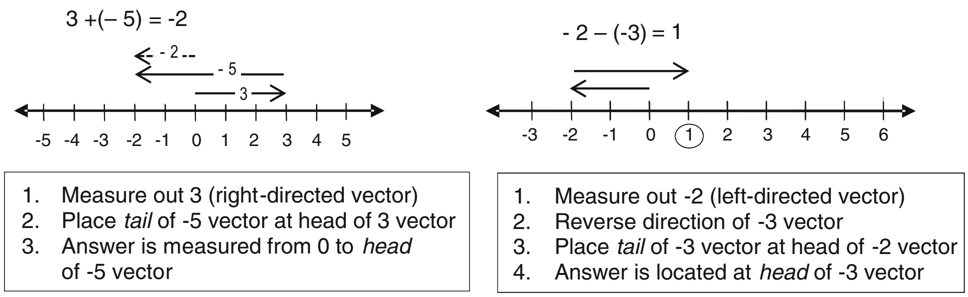


Fig. 5 Modeling integer subtraction using vectors

and direction are determined by placing the *tail* of the arrow at zero and the *head* of the arrow at the tick mark that represents the position of the given number on the number line. Arrows representing positive numbers face right and arrows representing negative numbers face left.

Once created, these number-vectors are moved back and forth along the number line to model the processes of integer addition and subtraction. For addition, the first addend is positioned with its tail at zero, and its head aligned with its respective number line tick mark. The arrow representing the second addend is moved along the number line until its tail is aligned with the head of the first addend, which is noted as a *tail-to-head alignment*. The result of the calculation can be read off on the number line by the location of the head of the second addend. This result can also be represented as an arrow with its tail at zero and its head at the "result" tick mark.

Subtraction is also modeled by translating and aligning an arrow. Subtraction begins with the arrow representing the minuend oriented in the appropriate direction with its tail located at zero. The subtrahend arrow is then moved to the head of the minuend and the direction of this arrow is *reversed* before its tail is aligned with the head of the minuend. As with addition, the result of the calculation can be read off the number line by the location of the head of the subtrahend.

The diagrams in Fig. 5 model integer operations using the tail-to-head alignment of directed lengths (vectors). The diagram on the left, taken from *Adding it up* (Kilpatrick et al. 2001, p. 92), models the addition of a positive and a negative integer. Notice that the result of the operation (dotted line vector) is represented here as a directed length measured from zero to the *head* of the second addend.

The diagram on the right, based on an example from a preservice text (Sonnabend 1993), models the subtraction of two negative integers. Notice that the arrow representing "−3" is shown facing to the right, which is the result of having its direction reversed under subtraction. Here, the result is represented by the value of the endpoint located at the head of the subtrahend.

In Fig. 4, numbers are created by "walking" a specified length, either moving forward for positive numbers or moving backward for negative numbers. In this *iconic* representation, the visual elements resemble an actual drawing of a person who creates a visible path by his or her movements. In contrast, in Fig. 5 numbers are represented by more abstract lengths, and the rule for representing an integer is *identical* for both positive and negative numbers, that is: create an arrow by locating the tail of the vector at zero and the head at the appropriate tick mark on the number line. As such, this form of representation is at an *indexical* level (Peirce 1998). At this level, unlike numbers represented by numerals (which are *symbolic*), the magnitude of each arrow still maintains a link to a real-world context as a countable set of units of length.

Commonalities between the two models for addition and subtraction of integers relate to a fundamental relationship between subtraction and addition—that is, subtracting a number produces the same result as adding its inverse. In the walking metaphor, this relationship is modeled by having the person *turn around* before completing the operation. In the same way, in the arrow model, the subtrahend is always *reversed* before being aligned with the head of the minuend.

The visual features of the walking metaphor create a set of rules tightly tied to a specific context. In contrast, in the directed-length arrow model the visible features are more abstract. This makes the latter number line model more generally applicable and, thus, a more powerful tool. An example of this power is also given in Fig. 10.

5 Using number lines as tools to operate and reason with numbers

The discussion in Sect. 4 focused on unpacking the *representational aspects* of the filled and directed-length number line models within the contexts of specific number line tasks. Section 5 shifts to an examination of how these representational aspects provide constraints or affordances to the ways in which such tasks can be *used as tools* to operate and reason with numbers.

5.1 Using the number line as a rule-governed or non-rule-governed tool for adding or subtracting whole numbers

In this section we examine how the use of particular number line models and the tasks in which they are involved promote either a fixed or a flexible way of adding or subtracting whole numbers. Tasks that steer towards an algorithmic solution tend to restrict the exploration and deepening of key concepts, while open-ended tasks encourage a growing understanding of underlying mathematical constructs and relations (Selter 1998).

The particular set of tasks shown in Fig. 6 is excerpted from a student activity page from an American grade 1 text (Maletsky and Roby 2004, p. 318). The complete activity consists of eighteen vertically aligned calculations that require students to subtract with crossing the ten, followed by a task in which the students have to write the number sentence "11 − 4 = 7" that belongs to the number line drawing on the right.

On this activity page, the filled number line is used as a *rule-governed tool* to model whole number subtraction. To enact the process, the minuend is located by its numerically labeled tick mark. Subtraction is performed by counting back from this point, until the count matches the value of the subtrahend. The answer is then identified as the number associated with the tick mark reached on the last count. Gray and Doritou (2008) note a similar emphasis on a counting forward or back type of procedural use of the filled number line in British primary instruction.

This kind of use of the filled number line provides few opportunities for students' conceptual development. In tasks like the above there is little left for the students to do but follow the directions. Students are not invited to think about the operation of subtraction itself or to explore how number facts can be used in "smart" ways to promote informal, but meaningful alternatives to rule-bound calculation processes (Van den Heuvel-Panhuizen 2008). In such a fixed task, the diagram becomes a kind of *figural algorithm* or counting tool. Because of its explicit, procedural approach, the activity tends to promote what Cobb et al. (1992, p. 5) describe as an "algorithmatization of mathematics and the disappearance of conceptual meaning."

Such closed tasks are typical of a traditional approach to education. It is assumed that students transfer learning from the results of carrying out a mathematical procedure that has been pre-structured into the material or task interface. Freudenthal (1991) uses the term *anti-didactical inversion* to refer to this approach to education. Here, the world of mathematics is imposed on the students in a ready-made way. They are not allowed to create this world for themselves by mathematically organizing and structuring rich contexts.

The number line can also be used as a *non-rule-governed tool*, as shown by the set of empty number line diagrams in Fig. 7, which are used to display solution strategies for the problem "14 − 6 = ?" Here the didactical situation draws its power from the number line's open representation. Instead of consisting of pre-planned materials, the open-ended task encourages students to mathematize the activity from within their existing levels of knowledge (Selter 1998; Van den Heuvel-Panhuizen 2008).

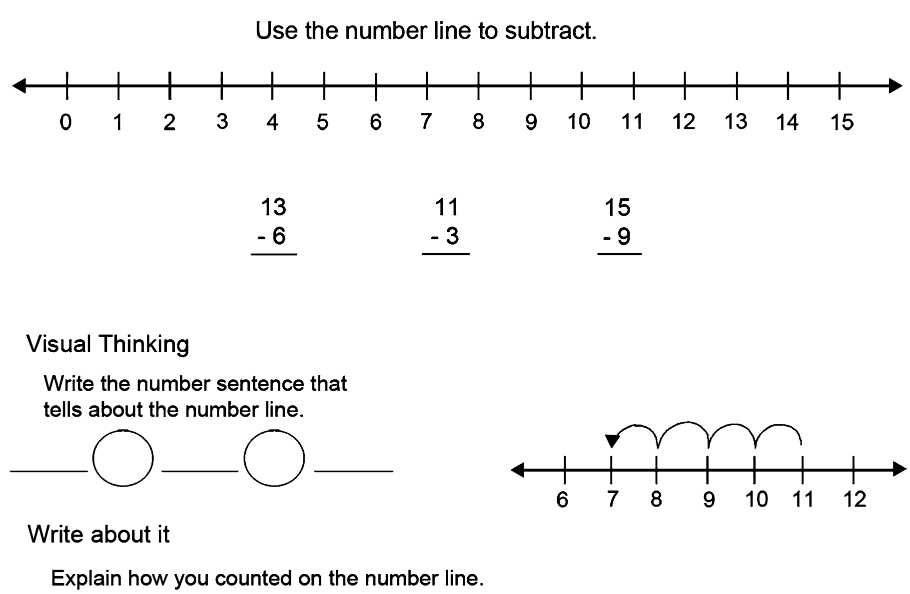


Fig. 6 Student activity page displaying a counting-back subtraction structure

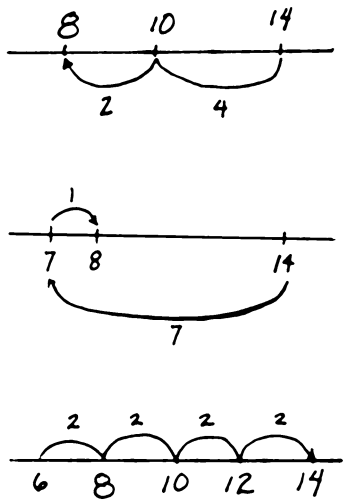


Fig. 7 Using the empty number line to solve "14 − 6" in different ways

The number line provides a *representational potential* that supports thinking flexibly about numerical and operational relationships. In the first diagram, the problem "14 − 6 = ?" is solved by first jumping back to 10 and then jumping back the remaining 2. In the second diagram the knowledge of number doubles prompts the strategy in which a backwards jump of 7 is followed by a forward jump of 1 that compensates for the over-jump. In the third diagram the problem is solved by skip counting by 2 s from 6 to reach 14, producing a result of 8. In each case, the empty number line is used to record a sequence of operations that reflects a student's personal way of decomposing and combining numbers to find the result.

Both the closed task (Fig. 6) and the open-ended task (Fig. 7) elicit counting or jumping actions on a number line to model subtraction with crossing the ten. However, from a didactical perspective, the tasks are very different in terms of developing students' calculational fluency. The task in Fig. 6 encourages a fixed, rule-governed approach to arithmetic in which procedural skills and number sense are not integrated. Here, the answer to a given calculation can be read directly off the number line. In contrast, the empty-number line subtraction in Fig. 7 encourages students to operate more flexibly with numbers, and consider how they can be combined in a variety of ways to achieve the desired result.

By facilitating an array of solution strategies, the empty number line invites students to reflect on and further develop mathematical concepts and connections (Selter 1998). Selter's study evidenced that number line models can direct student activity. Similarly, in another study (Peltenburg et al. 2010) it was found that tools in general can make a difference. When in a digital environment children were provided with manipulatives, they showed a clear preference for direct subtraction procedures (taking away), whereas when they were provided a number line tool, indirect addition or subtraction procedures (e.g. adding on) were frequently used as well.

5.2 Reasoning about the nature of the results obtained by adding or subtracting whole numbers

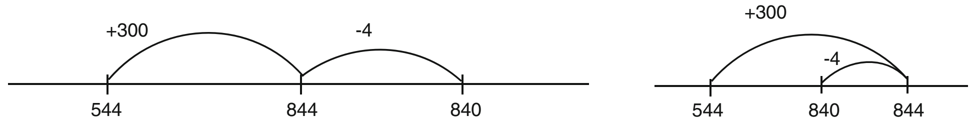
A number line may also be used as a *structural reference context* (Steinbring 1997) to construct connections between the visual features of a particular number line model and underlying mathematical relations and structures. Here, the representational function of the number line shifts from a model *of* a particular task to a model *for* reasoning about the more abstract mathematical concepts implicit in the problem situation (Streefland 1993; Van den Heuvel-Panhuizen 2003b).

The pair of diagrams shown in Fig. 8 illustrates how the empty number line can serve as both a model for a particular problem and as a structural reference context for reflecting on the nature of the results that are represented on the number line. These diagrams were produced by students in a fourth-grade German class following a sequence of lessons designed to introduce the empty number line (Dettmer-Kratzin 1997, p. 50). In an earlier class discussion, the students established that it was necessary to preserve numerical order when positioning numbers on an empty number line, but that equivalent spatial distances need *not* be maintained.

In spite of this earlier discussion, when the students were asked to solve the problem "544 + 296", one student drew the diagram shown on the left in Fig. 8 to represent his solution strategy of "544 + 300 − 4". It was not until the teacher questioned his organization that other students examined the diagram more critically. Upon further directed discussion, it was agreed that his drawing violated the "main rule" of the empty number line; preserving numerical order. The diagram on the right was then drawn to model the calculation steps within the *representational constraints* of the empty number line's form.

A further examination of the students' diagrams illustrates how reasoning about more general properties of the results of whole number addition and subtraction can be promoted. In particular, the number line requirement of left-to-right order makes visible, in the right-hand diagram, important arithmetic characteristics, namely that addition of whole numbers results in a sum greater than either addend; and subtraction of a number produces a difference less than the minuend. These characteristics are only implicit in the symbolic representation "544 + 300 − 4 = 844". In fact, the linear left-to-right syntax of this

Fig. 8 Students' alternative empty number line models for finding "544 + 296"



number sentence provides no structural reference context from which students can begin to construct meaning for the relations that exist among the values of the operands and the result for any arithmetic operation.

5.3 Conceptualizing subtraction

The number line can also serve as a structural reference context for thinking about the concept of subtraction. Tasks can be designed to illustrate that learning to calculate is more than simply learning a particular calculation procedure. It also requires an understanding of number relationships and properties of operations (see also Van den Heuvel-Panhuizen and Treffers 2009).

Consider our constructed examples of directed-length number lines shown in Fig. 9. The question below each number sentence frames the different ways in which the operation "$3 - (-2) = ?$" may be conceptualized.

In Fig. 9a, subtraction is expressed symbolically in a left-to-right order. The problem in Fig. 9b asks for the missing addend. In Fig. 9c the directed distance between the two operands is requested.

In Fig. 9a, the equation expresses subtraction as "taking away", yet applying the rule of "change the direction of the subtrahend arrow" produces a diagram that seems to represent addition. The visual representation reminds us that subtracting a negative number is equivalent to adding a positive number, a property of subtraction not evident in the symbolic equation. In Fig. 9b, the diagram represents an "adding on" solution. This approach is similar to that in Fig. 9c, where finding the difference is interpreted on the number line as counting the units between the heads of the arrows representing the subtrahend and minuend.

The structure of subtraction can be expressed even more abstractly, as shown by the diagram in Fig. 10 (Dr. Math 2001). Here, numbers are represented by tick marks on an empty number line, where neither a zero location nor a unit interval is specified. On this empty number line, A and B represent any real number and only order aspects of numbers are important: A is represented as less than B.

Subtraction is defined in terms of the missing addend as the distance "traveled" to move from the subtrahend to the minuend. For the equation $B - A = ?$, the operation is diagramed as $A + ? = B$ (shown as the lower arrow in Fig. 10; here, the missing addend is a positive number). For the equation $A - B = ?$ (the lesser number minus the greater number), the operation is diagrammed as $B + ? = A$ (shown as the upper arrow in Fig. 10; here, the missing addend is a negative number). Since the values of A and B are unspecified, it is impossible to locate them by

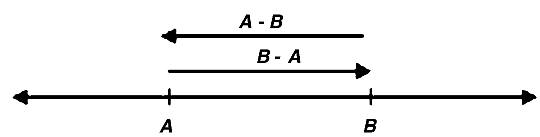
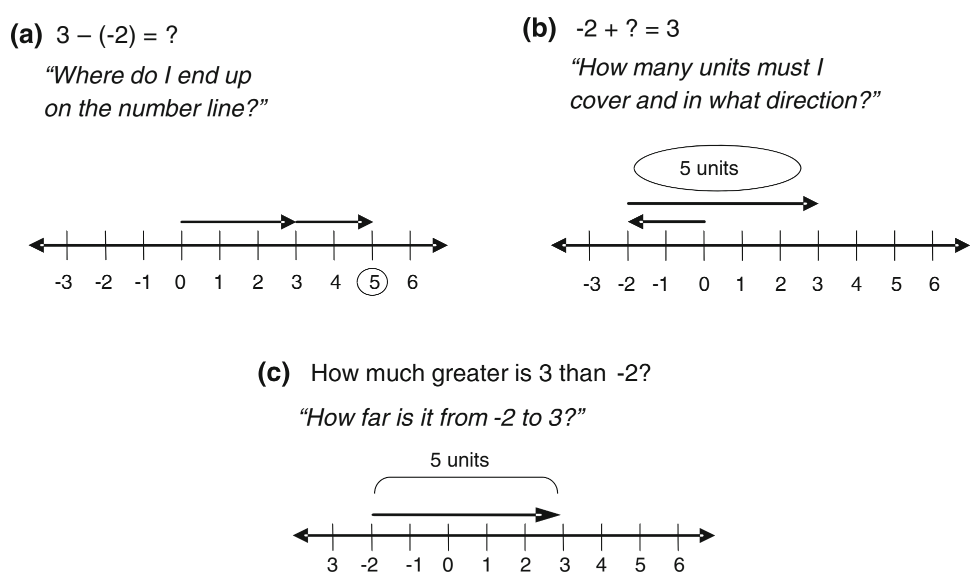


Fig. 10 Representation of the concept of subtraction as a directed difference

Fig. 9 Alternative ways to represent and conceptualize a given subtraction operation



directed distances measured from zero. Instead, they are indicated by tick marks and the missing addend is represented as a vector, or a directed distance.

The level of abstraction in the diagram supports reasoning about the *structure* of subtraction, rather than focusing simply on finding *results*. Note that, whatever the sign of the two numbers, subtracting a lesser from a greater number always produces a positive value, represented by a right-facing arrow, and subtracting a greater from a lesser number yields a left-facing arrow, or a negative value.

6 Summary

The visual aspects of a number line diagram are relatively straightforward, consisting of a set of tick marks displayed along a line segment. It was only when we began to carefully examine these visual features as *objects of analysis* from a *representational* perspective that we became aware of the range and complexity of mathematical meanings that can be supported by this common didactical tool.

One aspect of this complexity relates to how the visual features of the filled and the directed-length number lines are interpreted within either a counting-based or a measurement-based conception of number. Understanding the distinctions between these two conceptions helps to clarify the conventions for representing and the rules for operating with whole numbers and integers on either the filled or the directed-length number line.

Our analysis of number line tasks also examined how representational aspects of different number line models can direct student activity. Comparisons of the use of either the filled or the empty number line illuminate how the presence or absence of equally-spaced tick marks promotes more fixed or flexible engagement with addition and subtraction tasks.

The representational aspects of the number line models can also be exploited to promote reflective engagement with mathematical constructs. We presented examples of tasks in which the number line served as a structural reference context that shifted the number line from being a model *of* a particular context to that of a model *for* reasoning about underlying mathematical relations and structure. In these examples, the representational constraints of each type of number line were used to highlight important abstract concepts.

We suggest that an explicit representational analysis of the visual features of different number line models can inform the manner in which these didactical tools can be used in classrooms and in research to support and investigate children's reasoning about numbers and operations. While the filled number line frequently appears in early grade instruction as a way to introduce students to whole number sequences, the emphasis in this model on discrete counting may impede further conceptual understanding. The empty number line has been suggested as an effective model to move students forward (North Carolina Department of Instruction n.d.).

The classroom use of various number line tasks is anything but straightforward. Teachers must become aware of the representational nuances and types of meaning that are both explicit and implicit within the contexts of specific number line tasks, and care must be taken to understand the limitations and affordances of each type of model within particular learning trajectories (Bobis and Bobis 2005; Van den Heuvel-Panhuizen 2008). Additionally, the results of research investigating students' problematic use of particular number line models (e.g., Diezmann et al. 2010; Earnest 2007) highlights the importance of explicitly attending to students' development of appropriate representational apprehension.

As a final comment, we note that this article addresses only a small corner of the wide variety of number line models employed throughout the k-12 curriculum. Moving into middle school and beyond, as students encounter rational numbers, investigate proportional relationships, and later study algebra, concerns of representational awareness must be continually addressed to enable these students to confidently employ different number line models as they are contextually embedded within a variety of classroom tasks.

Appendix

	FAMILY OF NUMBER LINE MODELS					
	FUNCTION	Display of numbers				
		1	2	3	4	5
	Name	FILLED NUMBER LINE	EMPTY NUMBER LINE	DIRECTED-LENGTH NUMBER LINE	RATIONAL NUMBER LINE	PROPORTIONAL NUMBER LINE
	Type of Number	Whole numbers	Whole numbers	Integers	Rational numbers	Real numbers
FORM	Physical features	Infinite line with equidistant points and zero point	Line segment with ordered points	Infinite line with unit intervals and zero point	Infinite line with equally subdivided unit intervals and zero point	Line segment with boundary points
FORM	Visual image	0 1 2 3 4 5 6	17 29 87	3 -2 -1 0 1 2 3 4	0 $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ 1 $1\frac{1}{4}$	0 100
FORM	Number representation	Equidistant points	Ordered points	Directed lengths	Points	Points
	Didactical support	Exploring the number sequence, number order and relations	Ordered positioning of numbers	Measurement conception of numbers	Rational number sequences, density property of numbers	Relative position of numbers
	Example of tasks	Continue the number sequence What number comes before 3 and what comes after 3?	Put 87, 17, 29 in order on the line	Show 3 on the line Show -2 on the line	Continue the fraction sequence	Find the approximate position of 89
	FUNCTION	Display additive operations			Display equivalences relationships	Display multiplicative relationships
	Name	FILLED NUMBER LINE	EMPTY NUMBER LINE	DIRECTED-LENGTH NUMBER LINE	PARALLEL RATIONAL NUMBER LINES	PROPORTIONAL DOUBLE NUMBER LINE
FORM	Physical features	Infinite line with equidistant points and zero point	Line segment with ordered points	Infinite line with unit intervals and zero point	Parallel infinite lines with equally subdivided unit intervals and zero points	Line segment, double set of scales, paired points
FORM	Visual image	0 1 2 3 4 5 6	+20 16 35 -1 36	4 3 -2 -1 0 1 2 3 4	0 $\frac{1}{2}$ 1 0 $\frac{1}{4}$ $\frac{2}{4}$ 1 0 0.1 0.2 0.3 0.4 0.5 1	0 ? 108 0 9 36
FORM	Number representation	Equidistant points	Ordered points	Directed lengths	Points	Points
FORM	Operational representation	Using jumps	Using jumps	Aligning and orientating directed lengths		
	Role of model	Carrying out operations	Tracking operations	Demonstrating operations	Modelling equivalent rational numbers	Selecting calculation strategies
	Didactical support	Calculating by counting-on or counting-off	Promoting flexible calculation strategies	Operating with integers	Reasoning about equivalence relationships	Reasoning with proportions, fractions and percents
	Example of tasks	Find the answer to 2+3 Find the answer to 2x3	Find the answer to 16+19	Find the answer to 3 - 4	What is the decimal equivalent of $\frac{1}{4}$?	Find ? if: ? is to 9 as 36 is to 108

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