M. Beishuizen, K.P.E. Gravemeijer
& E.C.D.M. van Lieshout (Eds.)

The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures

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Preface

Thanks to a grant of the Dutch Organization for Scientific Research (NWO) and additional funding from the research-schools NUOVO (Nijmegen) and ISED (Leiden) a so-called experts meeting could be organized at the University of Leiden in December 1996. The objective of this meeting was to discuss some central concepts in research projects in the USA, Germany, Belgium, and The Netherlands, that aim at the innovation of mathematics education in primary school. The immediate cause for organizing this meeting was in the effort of a number of Dutch researchers to get a research program on early arithmetic funded. The focus of this research program is on the optimalization of teaching and learning processes in early arithmetic. This optimalization concerns the relation between the amount of (explicit) guidance and support given by the teacher at one hand, and the cognitive capabilities of the student at the other hand. In this set up guidance does not only encompass the role of the teacher but also the choice of instructional activities. In relation to this, attention is given to the influence of context problems and models on strategies and procedures.

To strengthen their research-program proposal, the Dutch researchers wanted to profit from the expertise of their colleagues abroad. In preparation of the meeting, the Dutch researchers wrote five 'position papers', and they asked their foreign colleagues to use one of these position papers as a cathalyst for elaborating their own position in a reaction paper. These position papers and reaction papers and a summary of the discussion formed the basis of this book, that can be seen as the proceedings of this meeting. The meeting revolved around two central themes: theories on early arithmetic, and perspectives on how to support learning processes in early arithmetic. The international composition of the participants created an opportunity for an exchange of ideas developed in geographically separated research communities. Concerning views on early arithmetic, an example of such a difference could be that – as a crude generalization – mathematics-education researchers in the USA tend to put more emphasis on general number concepts, and place value than those in The Netherlands. While, in contrast, the Dutch give much more attention to number-specific strategies and procedures. Problem solving behaviors develop from informal strategies through 'progressive schematization', according to the realistic mathematics education (RME) view.

In The Netherlands, there is a tradition of emphasizing number relations. The objective is to help students develop a network of number relations in the same vein as what is recently proposed by Greeno with his environment metaphor. Research questions concern, among others, the role of context problems and models in facilitating favorable procedures and strategies. This research relates to the goal of developing sequences of instructional activities, that are seen as instrumental in teacher support. This can be contrasted again with the USA, where the emphasis on developing more sophisticated general number concepts seems to coincide with more emphasis on the role of the teacher in problematizing instructional tasks, then on the role of external experts designing, and sequencing instructional tasks. The German views – at least those advocating 'productive' early number teaching – come close to the Dutch RME-view but emphasize more investigations into (formal) mathematical number structures.

Another issue that is viewed differently in different research communities is the content and character of math problems. What is the role and importance of real-life problems? What is the role and importance of mathematical problems that aim at, what one might call 'number theory'? In relation to this the question of what preparation would be adequate for prospective teachers.

Apart from the differences in tradition between the represented countries in mathematics education – differences from which we can learn – the meeting also underscored the existence of truly international themes in the research of mathematics education. For example, the acknowledgement of the valuable role that informal problem solving can play, has lead to rethinking instructional methods in each of the countries. There seems to be also a common concern about the students' lack of realistic considerations in solving math problems.

Overall, we may say that, in line with the conference, this book has as an overarching perspective, that of the innovation of mathematics education. Within that perspective, the book deals in the broadest sense with what one might call educational development. Within this broader framework, the contributions vary from general expositions on mathematics education, instructional design, teacher training and teacher enhancement, to detailed analyses of task characteristics, and categories of strategies and procedures.

Contents

Participants Experts Meeting 14-15 December 1996, Leiden University[1]

Invited experts:
Prof. Dr. Tom Carpenter, University of Wisconsin-Madison, USA
Prof. Dr. Paul Cobb, Vanderbilt University, USA
Prof. Dr. Karen Fuson, Northwestern University, USA
Prof. Dr. Jens Lorenz, Universität Ludwigsburg, Germany
Prof. Dr. Christoph Selter, Universität Heidelberg, Germany
Prof. Dr. Elsbeth Stern, Universität Leipzig, Germany
Prof. Dr. Erik De Corte, University of Leuven, Belgium
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Organizing Dutch researchers:
Dr. Meindert Beishuizen, Leiden University, The Netherlands
Dr. Koeno Gravemeijer, Freudenthal Institute, Utrecht, The Netherlands
Dr. Hans van Luit, Utrecht University, The Netherlands
Dr. Ernest van Lieshout, Nijmegen University, The Netherlands
Dr. Gerard Seegers, Leiden University, The Netherlands

Invited audience:
Dr. Julia Anghileri, University of Cambridge, England
Prof. Dr. Monique Boekaerts, Leiden University, The Netherlands
Prof. Dr. Gaby Dutoit, University of Bloemfontein, South-Africa
Dr. Petra Scherer, Universität Dortmund, Germany
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Mrs. Jacqueline Besemer, Leiden University, The Netherlands
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Drs. Karel Groenewegen, Delfshaven School, Rotterdam, The Netherlands
Dr. Marja van den Heuvel, Freudenthal Institute, Utrecht, The Netherlands
Ms. Stephanie Juranek, Leiden University, The Netherlands
Drs. Ton Klein, Leiden University, The Netherlands

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[1] Apart from the invited foreign experts and organizing Dutch researchers a small audience participated in the meeting and the general discussions. Around the conference a visit was paid to the Freudenthal institute. Also a presentation was given by the authors' team of the new realistic textbook 'Wis & Reken', including some school visits.

Drs. Julie Menne, Freudenthal Institute, Utrecht, The Netherlands
Mr. Frans Moerlands, Publisher Bekadidact, Baarn, The Netherlands
Dr. Kees van Putten, Leiden University, The Netherlands
Dr. Bernadette van de Rijt, Utrecht University, The Netherlands
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Instructional design for reform in mathematics education

Koeno Gravemeijer
Freudenthal Institute, Utrecht University

1 Introduction

Today's reform in mathematics education has an international character. Although various countries differ in pace and manner of elaboration, common trends can be discerned. Globally, we can speak of a shift away from the 'transmission of knowledge' by teachers towards, 'investigation', 'construction', and 'discourse' by students. This reform is influenced by changes in ideas on goals, content, and character of mathematics education.

As far as the *goals* of mathematics education are concerned, there is a growing emphasis on the usefulness of mathematics. This trend is at least in part fostered by societal changes. The coming information society poses new demands to her citizens. These new demands are, for instance, illustrated by Paulos'(1988) notion of 'mathematical literacy'.

These societal changes in turn have their influence on the *content* of mathematics education. A more important factor, however, is the reconsideration of what it means to know and do mathematics. Where mathematics used to be seen as a ready-made system, as a product, the emphasis is now on the process of doing mathematics. While at the same time the notion of mastery of rules and procedures of mathematics is exchanged for the idea that students should have a deep understanding of their mathematics, and should be able to explain and justify it.

This shift is not detached of a change in theories about learning and epistemology, and this has its consequences for the *character* of mathematics education. Current theories on learning emphasize learning as an active process. At the same time, (radical) constructivists have undermined the belief in knowledge as an objective and transferable entity in favor of knowledge as individual, idiosyncratic constructions. This implies that nor teachers, neither students have a direct access to each other's understanding, which complicates matters seriously in education.

To some extent, the reform in mathematics education has taken the form of a reaction against the existing, now traditional, mathematics education. In this sense, the reform could be characterized as 'against teacher telling', 'against textbooks', 'against learning objectives', and so on. This has brought with it the genesis of taboo words, like, for instance, 'work sheets', and 'skills'.

Basically, the genesis of taboo words does not have to be harmful. It is only natural that an innovation develops its own language. And to some extent this is what happened. We now speak of 'activity sheets' instead of 'work sheets'. To be sure, these changes are well-grounded. The words 'work sheets', for instance, bring with them the image of students working on decontextualized and atomized subskills. Nevertheless, the phenomenon of taboo words brings with it the danger of creating blind spots.

In this respect, Cobb (1995) has pointed to such a danger in relation to the concept of 'basic skills'. While acknowledging that 'basic skills' are commonly associated with a reductionist view on instruction, he stressed that we should not be throwing the proverbial baby out with the bath water. A similar phenomenon seems to occur in relation to teachers' telling. Knowing that explanations by teachers may (and maybe, often do) effectuate in seeming results and misconceptions, appears to stimulate some scholars to be very reluctant towards teacher elucidation (this shows, for instance, in the work of Berdnarz, Dufour-Janvier, Potter, and Bacon, 1993; Lampert, 1989; Nemirovsky, 1994).

A similar reserve can be found in the way instructional tasks, or instructional activities, are dealt with (see for instance Hiebert e.a., 1996). Concerning the latter, I will argue in this chapter that current reform efforts in mathematics education will be seriously hampered, if not ample attention is given to the development of instructional sequences that fit the reform. In the second part of the chapter, I will elaborate upon the Dutch theory for realistic mathematics education (RME) as an example of an approach to instructional design that is in concordance with the gist of the reform. And I will try to elucidate how instructional sequences in line with this theory can help teachers shape the intended reform in practice.

Finally, I will address one specific type of instructional tasks, namely, instructional tasks that deal with real-life problems, and I will adduce arguments in support of this type of instructional tasks.

2 Reformed practice

The current reform movement is enacted on various levels. A large variety of mathematics educators is involved: teachers, teacher trainers, researchers, administrators, and so on. In the end, however, it will be the teachers who will have to make the innovation come trough. It will be the teachers who have to establish the reformed practice in their classrooms. So the question arises: How can teachers be helped in establishing such a reformed practice? What advice can be given? What support can be offered?

In this respect, research in mathematics education has already produced a number of valuable insights. Here we may think of research on informal solution procedures (Carpenter, T.P. and J.M. Moser, 1984), and on the development of mathematical concepts, like number (Steffe, Von Glasersfeld, Richards and Cobb, 1983), for instance. Further, research has shed new light on the situatedness of knowledge (Brown, Collins and Duguid, 1989), and last but not least, on the communication processes in the classroom (Cobb, 1987, Yackel and Cobb, 1995).

In this chapter I will take Cobb and Yackel's (1995) emergent / socio-constructivist perspective as a point of departure. This emergent perspective tries to enhance our understanding of mathematical learning by emphasizing that it is a process of both individual construction and of enculturation to the mathematical practices of particular communities. In this view, the class is seen as a community that develops it's own mathematics. The 'classroom community' develops it's own 'taken-as-shared meanings, interpretations and practices'. Together, one tries to constitute mathematical knowledge, while preserving everyone's individual responsibility. The goal is inter-subjective agreement, or as Cobb puts it: 'mathematical truths are interactively constituted' (Cobb, 1989). To put it differently, 'true' is that which is established as a truth by the classroom community. This, of course, asks for a certain manner of working. To some extent, the students are expected to behave like mathematicians. They have the obligation to explain and justify their own ideas and solutions, and they have the obligation to try to understand the ideas and solutions of others, and to ask for clarification and or to challenge them if necessary. In relation to this Cobb a.o. speak of 'social norms' (Cobb, 1989) in a similar fashion, Brousseau (1990) speaks of a 'didactical contract'. While analyzing this type of instruction, Cobb and Yackel (1995) developed the following interpretive framework (fig. 1).

Social Perspective	**Psychological Perspective**
classroom social norms	beliefs about own role, other's roles, and the general nature of mathematical activity in school
socio-mathematical norms	mathematical beliefs and values
classroom mathematical practices	mathematical conceptions

figure 1: interpretative framework for analyzing the classroom microculture (Cobb and Yackel, 1995)

This conceptual framework incorporates a social and a psychological perspective. The psychological perspective looks at the individual from a constructivist point of view, the social perspective looks at the classroom as a social community. According to Cobb and Yackel, these two perspectives complement each other. The corresponding components, they argue, are reflexively related. The 'classroom social norms', for instance, will influence the beliefs of the individual students. At the same time, the classroom norms can be characterized as shared knowledge. The social norms exist by the mercy of a good harmony between the individual beliefs and the classroom norms. Thus, in short, the total of the individual beliefs constitutes the classroom norms, but at the same time, the individual beliefs are shaped by the classroom norms.

The socio-math norms (Yackel and Cobb, 1993) show a similar reflexivity. Socio-math norms deal with issues as: what counts as a (different) solution, what counts as an insightful, or efficient solution, and what as an adequate explanation. Socio-math norms can only be established in a reflexive process. Students will use the reactions of the teacher to figure out what counts as an adequate explanation, or what counts as an efficient solution, but then there will have to be students that present explanations, or solutions that can be reacted upon by the teacher.

Classroom math practices too have this two-sidedness. At one hand, there is the mathematical practice that is accepted as a taken-as-shared practice at a certain moment in time. At the other hand, there are the individual insights, knowledge and abilities that lead to the constitution and acceptance of this practice. Again, the students actively contribute to the emerging practice, and at the same time the shared classroom math practices influence the individual insights, knowledge and dispositions.

3 Tasks

The importance of the classroom climate is especially stressed by Hiebert et al. (1996). Reform in mathematics education, they claim, can be condensed in one principle: 'students should be allowed to make the subject problematic' (Hiebert et al., 1996, 12). For, the practice of problematizing the subject will lead to the construction of understanding. The authors emphasize that this does not concern characteristics of the instructional tasks but characteristics of the classroom culture: 'Tasks are inherently neither problematic nor routine. Whether they become problematic depends on how teachers and students treat them.' (Hiebert et al., 1996, 16)

On the basis of an exemplary classroom episode, they show, how the character, meaning, and impact of a task is influenced by the classroom climate. A plain 'school task' – like finding the difference in height of two children who are 62 and 37 inches tall – they argue, can be problematized in such a manner, that it becomes a genuine problem that forms the basis for mathematical investigation that can lead to deep mathematical insight. In contrast, rich real-life problems may lose all their alleged qualities, if the students are not given the opportunity to problematize these real-life problems.

They argue that the question whether 'real-life' problems are better than 'school' problems is irrelevant. For them, the important questions are '(1) has the student made the problem his or her own, and (2) what kind of residue is likely to remain.' (Hiebert et al., 1996, 19). Furthermore, they claim that reflective inquiry and problematizing depend more on the student and the culture of the classroom than on the task.

This, in my view, is a false dichotomy. At one hand, there is no basis for reflective inquiry and problematizing if there is not something like a task, and, at the other hand, there is no task if the intended task is not constituted as a task in the classroom. In this regard, Cobb, Perlwitz, and Underwood (1992) differentiate between the instructional tasks as they are envisioned by their developers and the instructional activities as they are interactively constituted by teachers and their students in the classroom. Furthermore, how a task is perceived in a classroom, i.c. what instructional activities are realized in the classroom, does not solely depend on the classroom culture, or the social norms and the socio-math norms that are established. Whether a task will be perceived as routine, for instance, depends for a large extent on the experience that the students of that classroom have with that type of task at that moment in time. Mark that this is exactly the basis to discern 'math practices'. The criterion for delineating math practices is in what procedures one is not obliged to explain and justify anymore. As the math practices of the classroom community develop, (parts of) solution procedures that have to be explained and justified at a certain moment in time will have become routine and taken-as-shared later.

4 Hypothetical learning trajectories

Apart from the creation of the classroom climate, Hiebert et al. (1996) reserve a key role for the teacher in selecting and presenting tasks. What such a role involves has been the object of investigation of Simon (1995). His starting point is in the observation that the planning of instruction based on a constructivist view of learning faces an inherent tension. The teacher has to integrate his or her goals and direction for learning with the trajectory of students' mathematical thinking and learning. Simon analyses the teacher's role in terms of a process of decision making about content and task--as it emerged in a small classroom teaching experiment. To describe this role, he introduces the notion of a 'hypothetical learning trajectory' (HTL):

> 'The consideration of the learning goal, the learning activities, and the thinking and learning in which the students might engage make up the hypothetical learning trajectory (...).' (Simon, 1995, 133)

The key in this learning trajectory is in the thinking that the students might engage in as they participate in the instructional activities the teacher has in mind. Simon speaks of a hypothetical learning trajectory because the actual learning trajectory is not knowable in advance. Nevertheless, although individual learning trajectories may vary, learning often proceeds along similar paths. The teacher therefore can constitute a hypothetical learning trajectory based on expectations about such paths. The actual constitution of the instructional activities in the classroom and course of the teaching-learning process offer the teacher opportunities to find out to what extent the actual learning trajectories of the students correspond with the hypothesized ones. This will lead to new understandings of the students' conceptions. These new insights, and the experience with the instructional activities as such will form the basis for the constitution of a modified hypothetical learning trajectory for the subsequent lessons. Simon (1995) describes this process as a 'mathematics teaching cycle' (fig. 2).

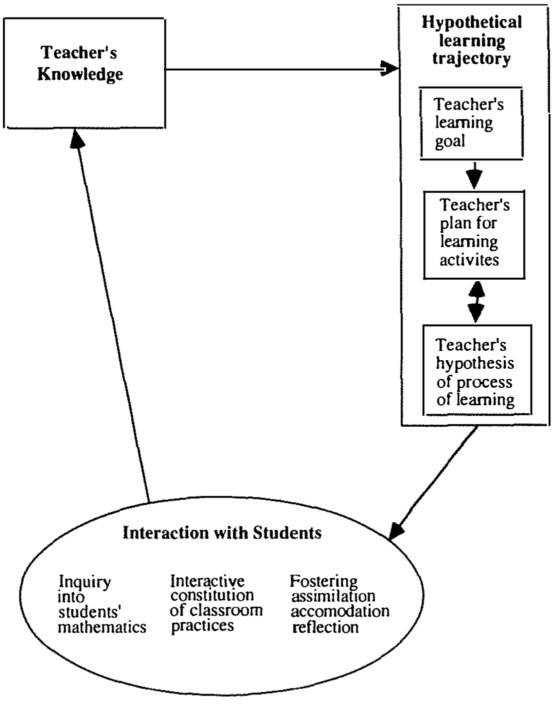
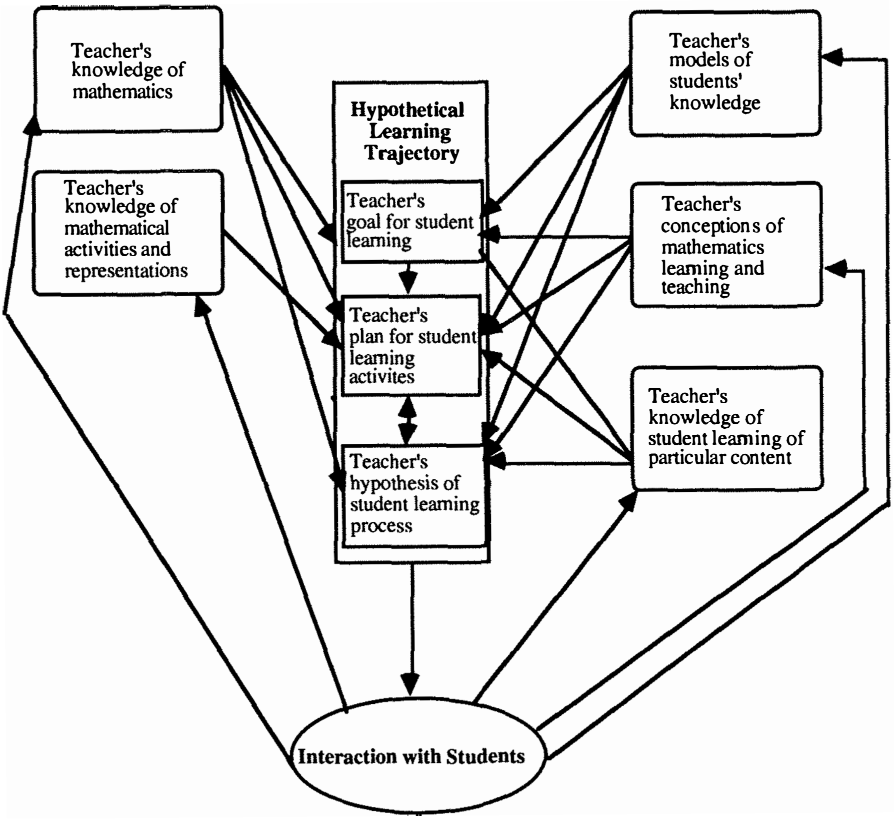


figure 2: mathematics teaching cycle (reprinted with permission from Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26,* 136)

For the generation of a hypothetical learning trajectory – in a process that Freudenthal would denote as a 'thought experiment' (Freudenthal, 1991) – the teacher relies on all sorts of knowledge. This, what we might call 'domain-specific' teacher knowledge, encompasses theories about mathematics teaching and learning, knowledge of learning with respect to a particular mathematical content, and knowledge of mathematical representations, materials, and activities (see fig. 3).



Note: Arrows indicate direction of influence. Connections and directions of influence are absent from the diagram to reduce complexity and to focus attention on relationships that are of particular interest in this discussion.

figure 3 (reprinted with permission from Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26,* 137)

Simon's analysis uncovers the complexity of selecting or creating adequate problems (or tasks). In doing so, he goes beyond the level of global requirements and criteria by focusing on the students' thinking and learning process. The HTL concerns, 'a prediction of how the students' thinking and understanding will evolve in the context of the learning activities.' (Simon, 1995, 136).

What Simon describes can be seen as short-term instructional design, with a few lessons as a unit. This 'micro-didactical' instructional design differs from traditional instructional design models (or 'first generation instructional design models' as Merrill, Li and Jones (1990) call them) in a fundamental manner. Traditional instructional design models concentrate on learning objectives. They focus on learning outcomes, while the process that leads to these learning outcomes is actually treated as a black box. For Simon, on the contrary, the teaching-learning process and especially the mental processes of the students are central.

In his article, Simon (1995) gives a detailed description of how he struggled with lessons on the area of a rectangle. There it shows how difficult it is to design a HTL. Simon comes to a good result by a process of conjecturing and checking. Knowing what Simon brings to this situation in knowledge and experience, it seems unfair to ask teachers to do the same – if we do not support them in one way or another. In fact teachers would have to do even more. For teachers do not only have to design instruction on a 'micro-didactical level', they also have to think of sequences of activities for various topics, and, they have to envision how to arrange all these activities in one school year. To give teachers a fair chance to succeed in this complex operation, they have to be given support by means of the design of sets of instructional tasks that are tailored to the current reform efforts. To be useful, these tasks have to be supplemented with a rationale or an instruction theory.

In the Netherlands, such instructional sequences and the accompanying theories are available, as the result of a long period of developmental research. Actually, what Simon does in his teaching experiment is what Dutch researchers have been doing over the last 25 years. Albeit, they have been developing 'local instruction theories', which are similar to Simon's hypothetical learning theories, but cover a longer period of time. These local instruction theories are the substrates of the instructional sequences that are developed within the framework of, what is now called 'realistic mathematics education' (RME). To give a good impression of this RME approach and its products, I will devote the rest of this chapter to an extensive description of realistic mathematics education as it emerged in The Netherlands.

5 Developing realistic mathematics education

At the end of the nineteen seventies, many countries introduced the so-called 'New

Math' to innovate their mathematics education. In The Netherlands, however, the New Math met the resistance of a group that organized itself under the name 'Wiskobas'. This group of primary-school mathematics educators joined force with secondary-school mathematics educators, and got the support of the famous mathematician Hans Freudenthal. Together they managed, (a) to keep the New Math textbooks out of the schools, and (b) to realize the erection of a National Institute for the Development of Mathematics Education, the IOWO. The IOWO got as its task to develop an alternative to the New-Math approach for mathematics education in primary and secondary school. As far as primary school was concerned, this also meant a departure from the traditional arithmetic, that had developed into a mechanistic training of rules and procedures. For secondary school it meant shift towards informal mathematics in applied situations, and an emphasis on 'mathematics for all' (see De Lange, 1987).

The point of departure for the IOWO was Freudenthal's (1971) philosophy of 'mathematics as a human activity'.

> It is an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter. This can be a matter from reality which has to be organized according to mathematical patterns if problems from reality have to be solved. It can also be a mathematical matter, new or old results, of your own or others, which have to be organized according to new ideas, to be better understood, in a broader context, or by an axiomatic approach (Freudenthal 1971, pp. 413-414).

It is important to note that this organizing activity – which is called 'mathematizing' in later publications – applies both to mathematical matter and to subject matter from reality. According to Freudenthal (1973), mathematics education for young children should start with mathematizing everyday-life reality. Besides the mathematization of problems which are real to students, there also has to be room for the mathematization of concepts, notations and problem-solving procedures. Treffers (1987) makes a distinction in this connection between horizontal and vertical forms of mathematization. The former involves converting a contextual problem into a mathematical problem, the latter involves taking mathematical matter onto a higher plane. Vertical mathematization can be induced by setting problems which admit of solutions on different mathematical levels.

Although Freudenthal espouses mathematics as an activity, he does not loose sight of mathematics as a product. In his view the process and the product should be connected; a combination of horizontal and vertical mathematizing should enable the students to reinvent mathematical insights, knowledge, and procedures. Freudenthal (1973, 1991) connects the idea of mathematizing with the principle of guided reinvention. According to the reinvention principle a learning route has to be mapped out along which the student could be able to find the result by him- or herself. The emphasis is on the character of the learning process rather than on inventing as such. The idea is to allow learners to come to regard the knowledge they acquire as their own, private knowledge; knowledge for which they themselves are responsible. On the teaching side, students should be given the opportunity to build their own mathematical knowledge store on the basis of such a learning process.

6 Theory-guided bricolage

Mark that the developmental work was guided by a philosophy of mathematics education, and not steered by some instruction theory. Although this philosophy proved to be a powerful heuristic, it did not give procedural guidelines of how to develop instructional sequences. Instead, the research question for the IOWO workers was to find out what instructional sequences, that would fit Freudenthal's philosophy, would be like. The aim of the research was to develop prototypical instructional sequences that could be used by textbook authors. This kind of research has an unmistakable element of tinkering in it. Like a handy man, the researcher can make use of all his or her domain specific knowledge concerning mathematics education: classroom experience, knowledge of text books, exemplary instructional activities, relevant research, and psychology. Note how this domain-specific knowledge of the developmental researcher corresponds with the teacher's knowledge Simon (1995) puts on the stage in his mathematics teaching cycle.

Like a handy man – or as the French say, a 'bricoleur' – the researcher will put the pieces of knowledge, and the suggestions for instructional activities together in such a way, that they fit his/her purposes. In this context the researcher's purpose is, to develop an instructional sequence that fits the adopted educational philosophy. That is to say, relevant ideas and examples will be selected and adapted, with the aforementioned philosophy of mathematics education in mind. Because of the overarching role of a theory or philosophy, we speak of 'theory-guided bricolage' (Gravemeijer, 1994). And, although the name theory-guided bricolage may suggest the contrary, we think that this manner of developing prototypes can take the shape of a research activity, denoted as 'developmental research'.

The core of the developmental research is in the cyclic alternation of 'thought experiments' and 'teaching experiments'. In a thought experiment, the researcher envisions how an instructional activity will work out in the classroom. Next the researcher will put the micro theory, that is embedded in the thought experiment, to the test in a teaching experiment.[1] In the teaching experiment the researcher goes in search of evidence that either confirms or refutes the theory of the thought experiment. Moreover, the researcher keeps an eye open for new possibilities. Next the outcomes of the teaching experiment are fed into the next thought experiment, which will be followed by another teaching experiment, and so on (see fig. 4).

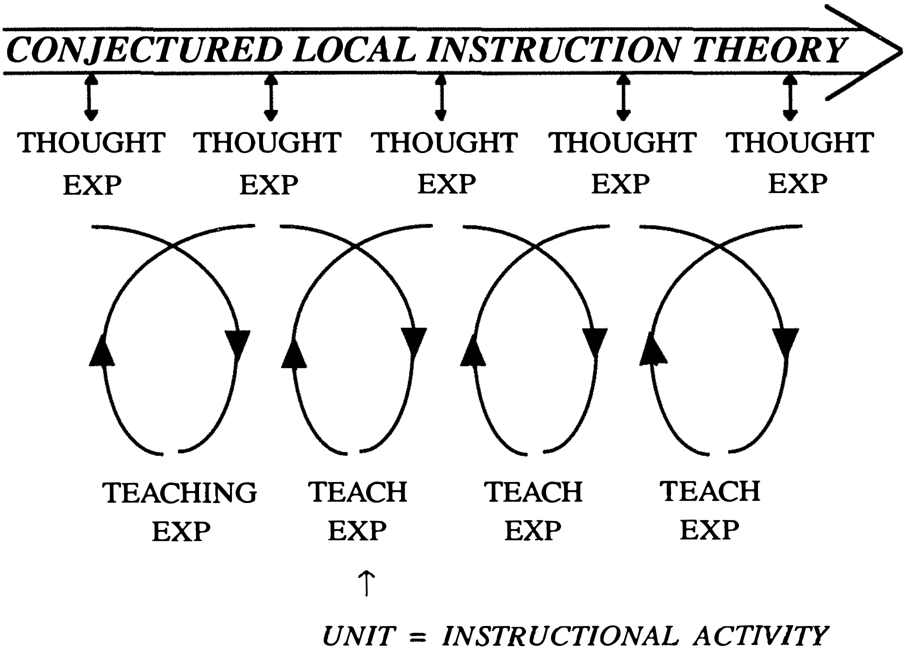


figure 4: developing a local instruction theory in a cumulative cyclic process

The process that governs this research process is very similar to that of the mathematical teaching cycles as described by Simon (1995). However, there are important differences. These concern the scope and the intent of these activities. Where a teacher may focus on a time span of one or two lessons, the intent of the developmental researcher is to develop instructional sequences and local instruction theories. This implies that the developmental researcher has to have a long-term learning process in mind. Moreover, the goal of the researcher is not to solve an immediate problem, but to foster an iterative, and cumulative, process of designing, experimenting, reflecting, and redesigning that results in a well-considered, and empirically grounded local instruction theory.

This implies that the mathematical teaching cycles serve the development of the instruction theory. In fact there is a reflexive relation between the thought, and teaching experiments at the micro level, and the local instruction theory that is being developed. At one hand, the preliminary local instruction theory guides the thought and teaching experiments, and at the other hand, the micro teaching experiments shape the final local instruction theory.

Note, that the cycles of thought and teaching experiments within the developmental process are not independent. Since the subsequent teaching experiments are carried out with the same students, each next teaching experiment starts with the residue of the preceding teaching experiments. Because of the cumulative interaction between the design of the instructional activities and the assembled empirical data, the intertwinement between the two has to be unraveled to pull out the proper instructional sequence. For it does not make sense to include activities that did not match their expectations, but the fact that these activities were in the sequence will have effected the students. Therefore adaptations will have to be made when the non-, or less-functional activities are to be left out.

Consequently, the instructional sequence will have to be put together as a reconstruction of instructional activities which are thought to constitute the effective elements of the sequence. Actually the theory that underlies the sequence can be seen as the result of the learning process, the researcher went through during the sequence of thought and teaching experiments. Thus, it is this learning process that has to justify the local instruction theory. Or as Freudenthal puts it:

> Developmental research means: 'experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience.' (Freudenthal, 1991, p. 161)

The process by which the theory is brought forth should justify the theory. In ethnographic research this methodological norm is labeled 'trackability'. The outsider should be able to retrace the learning process of the researcher. Trackability is essential for the teachers who want to use the instructional sequence developed in this process. If the norm of trackablility is truly fulfilled, the teachers can appropriate the experiences and considerations of the researcher, then they will have the disposal of a sound basis to make their own judgements, and to make their own adaptations.

That is to say, although the teacher may rely on a externally developed theory, and tasks, there is still room – and a need – for hypothetical learning trajectories. For, tasks will have to be trimmed to the specific situation of this teacher, with these students, at this moment in time. To make such decisions, the teacher has to construe hypothetical learning trajectories. Albeit, when construing these HTL's the local instruction theory can be used as a framework of reference. Let me elucidate this with the journey metaphor Simon uses.

Simon (1995, 136) uses this metaphor to explain his choice of the word 'trajectory'. The analogy between a learning trajectory and a journey is in the relation between the anticipated and the actual. For a journey, you will make a plan but when traveling you must constantly adjust because of the conditions you encounter. In a similar way, the learning trajectory the teacher anticipates for his students is a hypothetical learning trajectory that 'by definition' will differ from the actual learning trajectory.

This journey metaphor can be extended to elucidate the relation between local instruction theories and the hypothetical learning theories construed by the teachers. Like a journey, a long-term teaching-learning process can be planned in advance, and in a similar manner, the actual teaching-learning process has the be constituted in interaction with the conditions and developments one encounters. In this sense, an externally designed instructional sequence can function as a 'travel plan' for the teacher. Or better, the rationale, or the local instruction theory behind the instructional sequence forms the travel plan, and the availability of exemplary instructional activities enables the teacher to carry out this plan. Like a traveler, the teacher will have to adjust this plan continuously by construing HTL's that fit his/her interpretation of the actual situation.

Mark that, although the teacher and his or her students are unique, they will share similar experiences with the teachers and students who participated in the research experiments. Furthermore, the developers, who base their instructional sequences on extensive experimentation, will be well informed about the learning history of students – not of the students of 'classroom *x*', but instead of a sample of students. If teachers use the same set of instructional tasks as a basis for their instruction, the learning history of the students may be similar. Therefore, the learning route envisioned by the developer of the instructional sequence can be very relevant for the teacher who uses this sequence.

To give some insight in the general character of the local instruction theories developed in The Netherlands, I will discuss the over-arching 'domain-specific' instruction theory for realistic mathematics education.

7 Domain specific instruction theory

In the past two and a half decades, Freudenthal's philosophy, or global theory, is elaborated in many prototypes that represent local theories (e.g. local instruction theories on fractions, addition and subtraction, written algorithms, matrices, differentiating, and exponential functions).

In other words, global theory is concretized in local theories. Vice versa the more general theory can be reconstructed by analyzing local theories. In this manner, Treffers (1987) (re)constructed a domain specific theory for realistic mathematics education (RME. What he did was to try to make sense of twenty years of developmental work, carried out inside and outside IOWO and its successor OW&OC. In this way he was able to trace five characteristics of 'progressive mathematizing', as he denotes the actual elaboration of the reinvention principle. Progressive mathematizing in turn could be embedded in Van Hiele's level theory (Van Hiele, 1973) and Freudenthal's didactical phenomenology (Freudenthal, 1983).

Van Hiele distinguishes three levels of thought, which Treffers denotes as: an intuitive phenomenological level, a locally-descriptive level, and a level of subject-matter systematics (the level of mathematics as a formal system). These levels, which are subject-matter dependent, can be used for the global organization of an instructional course. The key for the distinction between the three levels is the notion of a (content specific) relational framework. At the lowest level the relational framework is as yet non-existent. Exploration of the subject matter area at this level may lead to the formation of fundamental relations, which may, in turn, be interconnected in such a way that a framework is created. As soon as the student has established such a framework the next level (which Treffers calls the descriptive level) has been reached. The highest level is unlocked when the relations themselves become object of investigation. In this way connections are made which allow for the construction of a logical and meaningful system.

This macro-structure that focuses on the mathematical content of an instructional sequence is completed with Freudenthal (1983)'s didactical phenomenology. Freudenthal puts an emphasis on a phenomenological exploration. Not only the mathematics, but also the phenomena in which the mathematics is embedded should be incorporated. Freudenthal proposes embeddedness as an alternative for embodiment in manipulatives. He argues for: 'starting from those phenomena that beg to be organized and from that starting point teaching the learner to manipulate these means of organizing' (Freudenthal, 1983, p. 32). From this starting point onwards, Freudenthal envisions a process of progressive mathematization, which passes through many minute levels. Since, 'the activity on one level is subjected to analysis on the next; the operational matter on one level becomes a subject matter on the next level' (Freudenthal, 1971, 417).

The third component of Treffers' sketch of a domain specific instruction theory for realistic mathematics education is expressed in characteristics of progressive mathematization. Treffers describes five characteristics.

The use of contextual problems: In realistic mathematics education, contextual problems do not just figure as applications at the end of a sequence. Contextual problems are also exploited as meaningful starting points from which the intended mathematics can emerge.

Bridging by vertical instruments: broad attention is given to models, model situations, and schemata, that rather than being offered right away, arise from problem-solving activities and subsequently can help to bridge the gap between the intuitive level and the level of subject-matter systematics.

Student contribution: the constructive element is visible in the large contribution to the course coming from the student's own constructions and productions.

Interactivity: explicit negotiation, intervention, discussion, cooperation, and evaluation are essential elements in a constructive learning process in which the student's informal methods are used as a lever to attain the formal ones.

Intertwining: the holistic approach, incorporates applications, implies that learning strands can not be dealt with as separate entities, instead an intertwining of learning strands is exploited in problem solving.

Notice how the relation between theory and development in realistic mathematics education differs from the traditional relation between theory and development. In the realistic approach, the theory applied in curriculum development is not a well-defined, fixed theory. The initial theory is global, to some extent vague, and open for adaptation. Application of an *a priori* theory is not under discussion, the theory functions as a guideline and it inspires developmental research. The more refined theory is an *a posteriori* theory: it is the reconstruction of a theory in action. To put it another way, the global basic theory is elaborated and refined in local theories. At the same time, the basic theory itself is developing. The central idea – mathematics as a human activity – remains the same, the relating theories however are adapted continuously.

8 Heuristics

One may note that Treffers' reconstruction of the theory underlying the prototypes developed in the spirit of the central idea, is rather descriptive in character. In the following I will try to recast the RME theory in a more prescriptive way, that is to say as heuristics for instructional development. I will distinguish three core principles: guided reinvention through progressive mathematization, didactical phenomenological analysis, and emergent models.

Guided reinvention through progressive mathematizing

According to the reinvention principle, the students should be given the opportunity to experience a process similar to the process by which the mathematics was invented. Thus a route has to be mapped out that allows the students to find the intended mathematics by themselves. To do so the developer starts with imagining a route by which he or she could have arrived at this outcome him- or herself. Here, knowledge of the history of mathematics can be used as a heuristic device. Knowing how certain knowledge developed may help the developer to lay out the intermediate steps, by which the intended mathematics could be reinvented.

The reinvention principle can also be inspired by informal solution procedures. Informal strategies of students can often be interpreted as anticipating more formal procedures. In this case, mathematizing similar solution procedures creates the opportunity for the reinvention process. In a general way one needs to find contextual problems that allow for a wide variety of solution procedures, preferably those which considered together already indicate a possible learning route through a process of progressive mathematization.

Mark that the reinvention process implies *long term learning processes*. Unlike learning sequences, where the learning path is chopped up in separate learning steps – which can be mastered independently – the reinvention process evolves as a process of gradual changes. Therefore intermediate stages always have to be viewed in a long term perspective, and not as goals in itself. In accordance with the notion of a level structure, an emphasis has to be given on guided exploration. One of the tasks of the developer is to construe this guidance by the design of a sequence of appropriate problems.

Didactical phenomenology

According to the didactical phenomenology (Freudenthal, 1983), situations where a given mathematical topic is applied are to be investigated for two reasons. Firstly, to reveal the kind of applications that have to be anticipated in instruction; secondly, to consider their suitability as points of impact for a process of progressive mathematization. If we see mathematics as historically evolved from solving practical problems, it is reasonable to expect to find the problems which gave rise to this process in present day applications. Next we can imagine that formal mathematics came into being in a process of generalizing and formalizing situation-specific problem solving procedures and concepts about a variety of situations. Thus it will be therefore the goal of our phenomenological investigation to find problem situations for which situation-specific approaches can be anticipated, and to find situations in which paradigmatic solution procedures may emerge that can foster vertical mathematization.

This elaboration of a didactical phenomenological analysis fits nicely with the idea of *free productions* (Streefland, 1990). After being introduced to a certain type of contextual problems students can be asked to generate similar problems. These free productions are beneficial for the students, since the making of these productions demands a reflection on the foregoing activity. This may make the students aware of, what they until then only knew in action. Free productions are beneficial for the developer too, since they may show informal strategies, notations, and insights that can be used in the sequel of the learning process.

Emergent models

The third heuristic is found in the role which emergent models play in bridging the gap between informal knowledge and formal mathematics. Whereas manipulatives are presented as preexisting models in product-oriented mathematics education, models emerge from the activities of the students themselves in realistic mathematics education. Crudely put, this means a model comes to the fore first, as a model that is a model *of* a situation that is familiar to the student. Next, by a process of generalizing and formalizing, the model gradually becomes an entity of its own. Only after this transition, it becomes possible to use this model as a model for mathematical reasoning (Streefland, 1985; Treffers, 1991; Gravemeijer, 1994). This transition from 'model-of' to 'model-for' implies a process of 'reification'. This notion of a process of reification fits with Sfard's (1991) account of mathematical development based on historical analyses. She shows that the history of mathematics is characterized by an ongoing process of reifications, in which procedures are reinterpreted as objects. In the model-of to model-for transition a similar shift takes place. That is to say, what is reified is *the process of acting with the model*, not the means of symbolization itself.

Note, that the model-of/model-for heuristic is the result of an effort to come to grips with an effective design practice. What a model is, is not sharp defined, but is rather constituted by examples. One of those examples is known as the 'double number line'.

This model originates in the process of modeling pacing as a form of linear measurement. The modeling can be done by marking intervals on a line. In this manner, a ruler is constituted as *a model of* pacing. When more refined measurement is concerned, smaller measurement units may be introduced and a double scale line can be constituted. With metric measurement, such a scale line may show, for instance, both meters and centimeters. Consequently, acting with this scale line may become *a model for* reasoning with units of a different rank, and more specific, for reasoning with decimal numbers.[2]

Other examples are in Van den Brink's (1989) passengerbus/arrow language, in the empty number line as model for addition and subtraction (Treffers; 1991, Gravemeijer, 1994), and in the use of repeated subtraction as a model for long division (Gravemeijer, 1997).

The examples show that these emergent models involve more than (the use of) some sorts of symbolization. Although symbolizing plays an important role. What is central, however, is a paradigmatic situation. It is the situatedness of the activity that is important. Here I do not mean situated in the context of the classroom – although one should not neglect this aspect. What I want to bring to the forefront here is the situatedness in connection with the suggested experientially real situation in which the problem is embedded. Students should be able to engage themselves in that situation, and think of the situation-specific strategies they could use to solve the problem in that situation. Although the students will not really act in the 'real' situation, in general. The objective is to solve the problem in an indirect manner; by imagining the situation, and by imagining acting in that situation. The latter can of course be facilitated by a form of symbolizing, or modeling.

The development of a model-of into a model-for can be illuminated by distinguishing the following levels (fig. 5):

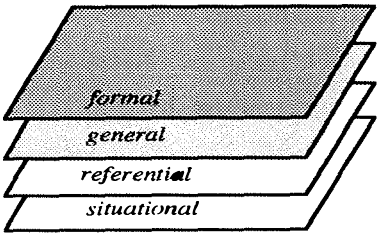


figure 5: levels

(1) the level of the situations, where domain specific, situational knowledge and strategies are used within the context of the situation (mainly out of school situations);
(2) referential level, where models and strategies refer to the situation which is sketched in the problem (posed in a school setting mostly);
(3) a general level, where a mathematical focus on strategies dominates the reference to the context;
(4) the level of formal arithmetic, where one works with conventional procedures and notations.

The level-distinction illuminates, that the model initially comes to the fore as a context-specific model. At the referential level the model, refers to a concrete or paradigmatic situation that is experientially real for the students. The model derives its meaning from its relation with the situation it models. The model is used to support informal strategies that correspond with situation-specific solution strategies at the level of the situation. From then on, the role of the model begins to change. Strategies are more and more looked at from a mathematical point of view. The choice of a strategy is no longer dependent on its relation with the problem situation, but is governed by mathematical characteristics of the problem. The model gets a more general character, and a process of reification takes place, by which the model becomes an entity on its own. By then, the model becomes more important as a base for mathematical reasoning, than as a way to represent a contextual problem. As a consequence the model can become a referential base for the level of formal mathematics.[3]

The emergent character of the models in RME is visible in the way the levels are passed through. But, there is more. The label 'emergent' also refers to the way models are developed. In traditional, product-oriented mathematics education, models are derived from formal mathematics. This in essence is a top-down approach. In RME a bottom-up approach is striven for. The developers should not only present models that refer to experientially real situations, they should also look for models that fit the informal solution strategies of students. Thus, in a way the RME models emerge from the informal solutions of the students.

The emergent character also comes to the fore in the integrated development of model and mathematical meaning in a process of vertical mathematization and reification. Moreover, this dynamic model also changes in appearance over time, where traditional didactical models are fixed. Finally, the emergent approach should ensure that the reified model does not become detached from the originating contexts: the students will have to be able come up a contextual problem to 'concretize' a formal numerical task (see also Treffers, 1991).

9 Conclusion: Real-life problems

I like to conclude this chapter with a discussion on the indispensableness of real-life problems in mathematics education.

Real-life problems play a key-role in realistic mathematics education. Although one should be aware that 'realistic' does not stand for real-life problems. The use of the label 'realistic' refers to a foundation of mathematical knowledge in situations that are experientially real to the students. Context problems in RME do not necessarily have to deal with authentic everyday-life situations. What is central, is that the context in which a problem is situated is experientially real to students in that they can immediately act intelligently within this context. Of course the goal is that eventually mathematics itself can constitute experientially real contexts for the students.

Still, real-life problems are an important feature of realistic mathematics education, but that does not only hold for RME. Any reformed mathematics instruction should incorporate real-life problems; for two reasons: (1) since mathematics is rooted in real-life problems, and (2) for reasons of mathematical literacy.

The first argument for the use of real-life problems is that mathematics is rooted in everyday-life reality. If one takes a constructivist stance, one has to acknowledge that mathematics is socially and historically constituted. Or to put it differently:

mathematics is a human construction with a history. This history goes back to the genesis of mathematics in efforts to solve the kind of real-life problems we now denote as 'applications'. One cannot disconnect mathematics from these roots. In my view, mathematics cannot and should not be treated as an entity-in-and-of-itself without connections with a real-life reality. Once we have acknowledged that connections with everyday-life reality are essential, it is only natural to give real-life problems the same place as they had in history – as Freudenthal does.

The second argument for the use of real-life problems is in mathematical literacy as an educational goal. Mathematical literacy is not a goal that can be reached without a focused effort. Research shows that many students tend to neglect real-life meanings when they are solving problems in school (Verschaffel, in press, Greer, in press). When asked, for instance, how many planks of 1 meter one can make out of 4 planks of 2.5 meter, many the students answer 10 (Verschaffel, De Corte, Lasure, 1994).

It may be argued that this is a problem of 'school-math' socio-math norms that can be remedied easily by bringing this point to the attention of the students, and by changing the students' beliefs about what is expected from them in this regard. However, one should realize that these socio-math norms can not be developed in absence of real-life problems.

Further, mathematical literacy also asks for 'practical' mathematical knowledge. Mathematical literacy presupposes that one can judge the likelihood of the outcomes of mathematical calculations in reality – and likewise of the likelihood of what is presented as facts. To do so one has to have a framework of reference in terms of magnitudes in reality. One has to be able to round numbers sensibly, and to work with these rounded numbers in a flexible manner. Like Onno did (at the age of 12 years), when he read about a washing machine that claimed to be good for 5000 washings. 'That is a lot', he remarked, 'that one is good for more than ten years.' With the goal of mathematical literacy in mind, real-life problems have to be integrated in carefully designed instructional sequences that can be adapted refined and elaborated by autonomous teachers.

acknowledgement

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notes

1 Note that these are in fact micro-teaching experiments that concern the try out of one lesson.

2 Mark that 'the model' cannot be identified with one specific symbolization. As an heuristic, 'the model' in this example stand for the more general idea of a ruler. Thus, 'the mod-

el' encompasses all symbolizations that are used to represent this notion.

3 Note that the reification of the model coincides with the development of a mathematical framework of references that allows for its reification. Where the model first lends its meaning from the contexts it refers to, the model later takes its meaning from the connections within a mathematical framework. (In the example of one double number line, for instance, this framework would consist of concepts and insights concerning decimal numbers.)

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Models for reform of mathematics teaching

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1 Introduction

Constructing models of problem situations is a relatively intuitive strategy used by children throughout the world to solve problems in and out of school (Carpenter, 1985; Carpenter, Ansell, Franke, Fennema and Weisbeck, 1993; DeCorte, Verschaffel and Greer, 1996; Fuson, 1992; Gutstein and Romberg, 1996; Nunes, 1992; Verschaffel and DeCorte, 1993). It has been well established that without formal instruction children as young as five solve a variety of problems by modeling the action or relations in the problems. Initially the models maintain a great deal of fidelity to the problem situation as children use concrete materials or drawings to represent directly the problem situation, but over time the models become progressively more abstract such that children no longer need to actually directly model the problem situation with concrete materials. Thus, the development of children's early number concepts and operations can be portrayed as the progressive abstraction or progressive mathematization (Gravemeijer, this volume) of children's intuitive modeling strategies.

The picture of the development of children's mathematical thinking as the progressive abstraction of modeling strategies provides a coherent, principled basis for understanding the development of children's arithmetical thinking and problem solving abilities and for thinking about instruction that integrates problem solving and the development of arithmetic in the primary grades. Operations of addition, subtraction, multiplication, and division can be defined as formalizations of processes that children understand for solving certain types of problems. Rather than starting with the formal operations and developing manipulatives to give them meaning, instruction begins with problems that children can solve using intuitive modeling strategies. In this way formal mathematical operations are conceived as means of generalizing and symbolizing the informal models children already use and understand. In Freudenthal (1983) and Gravemeijer's (this volume) terms the focus is on the phenomena in which mathematical operations are embedded rather than on the embodiment of them in manipulatives.

Designing instruction to build on children's intuitive modeling strategies is relatively straightforward for developing basic understanding of arithmetic operations with relatively small numbers. Virtually all children model problems in predictable ways without explicit instruction, and about all that is necessary for a teacher is to provide children opportunity to solve problems, reflect on their solutions, and discuss their methods with other children. Certainly teachers may be more or less effective in engaging students in discussions of similarities and differences among alternative strategies and helping them to relate their intuitive strategies to formal mathematical operations and symbols (Fennema et al., 1996; Franke, Fennema and Carpenter, in press). But children develop progressively abstract strategies for solving basic joining, separating, comparing grouping, and partitioning problems without instruction, so the explicit way a teacher goes about helping children build on their intuitive strategies does not appear to be critical.

Furthermore, the intuitive strategies that children naturally use to solve problems are extremely robust and consistent. When teachers give students the opportunity to use their own strategies to solve basicproblems, the students are successful and their strategies are predictable. This makes it relatively attractive for teachers to let children use their own invented strategies to solve a variety of problems. Teachers immediately see that children can solve problems without formal instruction, they are surprised at the range and difficulty of problems that children can solve, and they can anticipate and understand the strategies that children use. All this serves to reinforce teachers' willingness to attempt to help students build on these informal strategies to develop understanding of basic number concepts and operations, and we generally have been successful in convincing teachers to adopt this perspective (Fennema et al., 1996).

The situation is not so straightforward when it comes to the question of developing base-ten number concepts and procedures for adding multidigit numbers. Whereas the modeling strategies that children use to solve problems and the abstractions of those strategies potentially represent relatively intuitive processes that could be constructed by children left to their own devices, base-ten number concepts and the standard algorithms for operating on multidigit numbers are socially constructed conventions that children will not learn independently. However, children bring all sorts of knowledge about base-ten numbers to instruction from recognition off repeating patterns in counting to knowledge of the number of pennies in a dime and the number of dimes in a dollar. Collectively a class of first grade children has quite a bit of informal knowledge of base-ten numbers that can serve as a basis for developing more formal notions of place value and inventing procedures for adding, subtracting, multiplying, and dividing multidigit numbers.

In the following section I portray one path for building students' multidigit concepts and procedures on their intuitive modeling processes by considering case studies of two first-grade classes as they develop increasingly abstract and sophisticated conceptions of and procedures for adding and subtracting multidigit numbers (Carpenter, Ansell, Levi, Franke and Fennema, 1996). Essentially the story of both cases parallel the story of the development of basic number concepts and operations, as the development of multidigit concepts is portrayed as the progressive abstraction of basic modeling strategies. Following that discussion, I provide a brief overview of the teacher development program from which these cases emerged. In the final section of the paper I relate this work to the principles of Realistic Mathematics Education characterized by Gravemeijer (this volume).

2 Extending the progressive mathematization of modeling strategies to multidigit numbers

Most of the students in both classes had been in kindergarten classes that provided extensive opportunity for problem solving (Carpenter et al., 1993), and at the beginning of the year the majority of students in both classes could solve a range of addition, subtraction, multiplication, and division word problems by modeling the action in the problems using individual counters. However, most students had limited knowledge of concepts of ten. About half the students in each class could count by 10s to count the number of sticks in a collection in which the sticks were bundled into groups of 10, but only one or two students in either class could use base-ten blocks to add 24 + 10.

Although most students had limited knowledge of place value and base-ten number concepts, instruction in neither class focused on activities whose purpose was to develop these concepts per se as is the case in most traditional mathematics instructional programs. Rather students solved a variety of problems with increasingly large numbers, and over time began to use increasingly sophisticated tools and procedures for solving them that embodied and made use of base-ten number concepts. Thus, we can trace the development of students' understanding of base-ten number concepts and multidigit procedures by examining the progression of strategies they used to solve addition and subtraction problems.

The development of students' multidigit concepts and procedures can be characterized as passing through three levels. Initially, almost all students used single counters to model problems, even when the numbers were in the 20s and 30s. At the next level the individual counters were replaced by base-ten materials in which individual counters were grouped in groups of ten so that students could construct and operate on numbers by counting collections of 10 rather than having to count individual counters. At first students were quite tenuous in the use of these materials. For example, in solving an addition problem, they often constructed each of the addends using the base-ten materials, but in counting the total, they counted each of the individual counters. Over time the students became more flexible and efficient in using the base-ten materials. At the final level, the operations on the base-ten materials were abstracted so that the students no longer actually used the materials, and solved problems using what we refer to as *invented algorithms*.

Invented algorithms are what we call the symbolic procedures that a number of participants in this conference have observed and written about that are constructed by students either individually or through interactions with other students. Like any algorithm invented algorithms reduce complex calculations to a series of simpler calculations. Unlike standard algorithms they are not necessarily designed for efficiency; instead they operate more directly on base-ten number concepts than with symbolic forms. Further, invented algorithms generally lack the repetitive, automatized quality of standard algorithms. Two of the most common invented strategies involve variants of strategies that we have labeled *combining units separately* and *incrementing,* which are illustrated by the following examples for adding 38 + 26 (For a more extensive discussion of invented algorithms, see Beishuizen, this volume; Carpenter et al., 1996; Fuson et al., 1997).

> Thirty and 20 is 50, and the 8 makes 58. Then 6 more is 64[1]
> Thirty and 20 is 50, and 8 and 6 is 14. The 10 form the 14 makes 60, so it's 64[2]

The levels are not rigid, and students who could use strategies at a given level often fell back on strategies of a previous level. In fact, for both the transition from using individual counters to using base-ten materials and the transition from using base-ten materials to using invented strategies, the numbers in a given problem had a significant influence on the strategy a student might use. Problems involving multiples of ten (30 + 50 or even 34 + 20), were more likely to elicit more advanced strategies than problems in which neither number was a multiple of ten. Similar patterns were observed for problems with numbers that did not require regrouping (34 + 25) and problems that did (34 + 28).

2.1 A synopsis of a year of instruction

Early in the year both teachers made base-ten materials available to the children to solve problems, but initially relatively little emphasis was placed on base-ten number concepts. Children were provided with tools that would afford solutions of problems using base-ten principles, but neither teacher demonstrated how to use the materials to represent base-ten numbers or to solve problems. Over the next several months, almost all the experiences that the classes engaged in with base-ten numbers involved solving problems involving addition, subtraction, multiplication, or division of multidigit numbers. Progress in the use of base-ten concepts and materials was slow and irregular as children began to use a variety of strategies to solve problems with larger numbers. During this time no single strategy prevailed at any point in time. Different children used quite different strategies, reflecting different conceptions of base ten-number concepts and of addition, subtraction, multiplication, and division. Each child also used different strategies to solve different problems, depending on the numbers in the problem, the operation, the materials available, and a variety of other factors.

Initially, several children in each class had limited notions about base-ten number concepts. They could count the tens as units, and with some encouragement they began to use the tens materials to solve problems. At first their use of the materials was somewhat limited, and during the first month of school, they were the only students to use base-ten materials to solve problems. As they discussed their solutions and shared them with the class, their solutions became more efficient and flexible; and other children began to begin to use the base-ten materials. Thus, although the teachers did not model the use of the base-ten materials themselves, these students provided the other students with models of how the materials could be used. The other students did not, however, immediately follow their lead. The evolution of use of base-ten materials was gradual, and there was a great deal of diversity within the class throughout the year. By the beginning of October only a handful of students in each class were using base-ten materials to model and solve problems, but by the end of November most of the students could use base-ten materials flexibly to model and solve problems.

Throughout the year students could use any appropriate strategy or materials, and it was not expected that students adopt a particular strategy once it had been introduced. But the teachers were not entirely passive in letting students choose strategies. They regularly probed individual students to determine whether they could use more advanced strategies, and they encouraged discussion of the more advanced strategies and the efficiency they provided as is illustrated by the following protocol as a student was describing her method of subtracting 27 – 4.

Karen: The answer is 23.

Ms. Keith: Oh, and how did, wait a minute. Are we going to let her just give the answer?

Class: No!

Ms. Keith: What are we going to say to her?

Students: How did you figure it out?

Karen: I figured it out with the [linking] cubes.

Peter: But how?

Karen: I'll show you. [Goes to get the cubes.]

Ms. Keith: Wait, wait, we want you to use words. It's very hard to put into words, some things, but it's a very good thing to do.

Karen: Okay. Well I had two sets of 10 and that made 20, and then I added a 7 to that.

Ms. Keith: Okay, you guys understand so far?

Students: Some say yes and some no.

Ms. Keith: Listen so you understand. She had two sets of 10, which would make her have [pause].

Students: Twenty.

Ms. Keith: Twenty, and then she had 7 more in a set.

Karen: And then I took away four on the 27.

Ms. Keith: On the 7 stack or on the 20 stack?

Karen: On the 7 stack.

Ms. Keith: Okay. Then what happened?

Karen: And then I got 23.
Ms. Keith: Did you go back and count how many?
Karen: Yes.
Ms. Keith: And when you counted, what did you do? How did you count?
Karen: One 2, 3.
Ms. Keith: You didn't use your tens?
Karen: Cause I had 20. Okay 10 and 10 make 20, and then 1, 2, 3.

figure 1

This exchange took place early in the year when base-ten number concepts were tenuous for most students. Ms. Keith helped Karen make explicit the relation between groups of ten and ten individual units and that units of ten can be counted. By having Karen explain what she did without the blocks, Ms. Keith encouraged Karen and the rest of the students to reflect on the numbers and the operations rather than just manipulate blocks.

As the rest of the class learned the concepts and procedures required to use the base ten materials to represent multidigit numbers and solve problems, the more advanced students began to abstract the strategies involving the base-ten materials and solve problems without them. These students started using invented algorithms in October and November but generally did not use them consistently until the beginning of December. Widespread use of invented algorithms emerged slowly, and it was not until February that a number of other students began to use them with any regularity.

By the end of the year, over two thirds of the students in each class could use invented algorithms to add two-digit numbers that required regrouping (38 + 26), and over half could use an invented algorithm to add three digit numbers (256 + 178). Almost all the rest of the students used base-ten blocks to solve a variety of addition, subtraction, multiplication, and division problems. Most students had more difficulty using invented algorithms to subtract, but about half of the students in each class used invented subtraction algorithms for at least some problems.

2.2 The transition to invented algorithms

We hypothesize that a major factor in the transition to using invented algorithms in these classes is reflection on and discussion of the blocks procedures. The invented algorithms that used are essentially abstractions of the blocks procedures, and in fact students verbal descriptions of blocks procedures sound very much like the invented algorithms that they come to use. Consider, for example, the following exchange that took place in the third-grade class Ms. Gehn was teaching the year following the case study (Carpenter, Levi, Ansell, Franke and Fennema, 1995). The student in the protocol was in transition from relying entirely on the physical manipulation of base-ten materials to using invented algorithms. The excerpt comes from a discussion of students' solution to a problem involving the sum 54 + 48. The children had used a wide variety of strategies to solve the problem, representing a range of understanding of base ten concepts. A number of children had solved the problem using tens blocks as one of their solutions. One child had used a hundreds chart, counting on by tens and then ones on the hundreds chart. Four children had solved the problem by first adding 50 and 40, then adding 4 and 8, and finally adding 12 to 90. The following exchange took place during the discussion of alternative solutions.

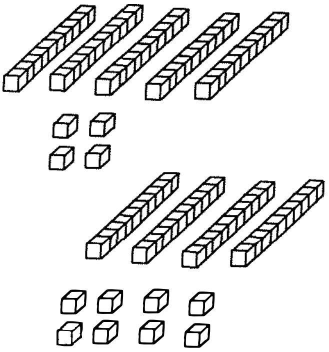

figure 2: Ellen's Base-Ten Block Solution for 54 + 48

Ms. Gehn: Now everyone go over to Ellen's desk.
Ellen: They don't need to go to my desk, I can tell them right here.
Ms. Gehn: I want them to go to your desk; I want them to see exactly what you showed me, and then you can tell me how you could do it without us having to go to your desk.

(The children move around Ellen's desk.)

Ellen: [Makes 54 and 48 with tens and ones blocks. See Figure 1] I knew this was 54, so I went 64, 74, 84, 94 [moves one ten block for each count] 95, 96... 102 [moves a one block for each count.]
Ms. Gehn: Now class what question am I going to ask her? Norman?
Norman: You didn't use the 54, did you have to make it?
Ms. Gehn: Good Norman, that is just what I was going to ask her. Ellen, did you need to make that 54?
Ellen: No.
Ms. Gehn: [Pulls the 54 away and covers it with her hand] Ok, now show me how you can solve the problem without the 54.
Ellen: 64, 74, ... [repeats the above strategy counting on without the 54].
Ms. Gehn: Ok, now you told me that you could do this without us moving to your desk. How would you have done that?
Ellen: Ok, I just put 54 in my head, and then I go 48 more. I go 54 [slight pause], 64, 74, 84, 94, [She puts up a finger with each count to keep track of the tens. At this point she has 4 fingers up. She puts down her fingers and puts them up again with each count as she continues counting by ones.] 95, 96, 97... 102.

figure 3

In this one exchange, we see three related but quite distinct strategies: directly modeling the problem using tens bars, abstracting the first quantity and counting the second quantity, and counting on by tens using fingers. The three strategies represent successive levels of abstraction. In the first strategy, the objects in the problem were represented directly with the blocks. In the second strategy, the quantity representing the first set was abstracted, and Ellen counted on starting with the number in the initial set, counting the blocks representing the set that was joined to the initial set. In the final strategy, the counting words no longer were linked to physical materials. The counting words themselves were counted by keeping track of the counts on fingers. The fingers did not act like the blocks did in the first two strategies; they were not surrogates for the blocks. They played a very different role. As Ellen counted on by ten from 54, she was not counting imaginary collections of ten. She was using her fingers to keep track of how many counts of ten she had made. The counting sequence itself had become an object of reflection, and as such it could be counted (Steffe, Von Glasersfeld, Richards and Cobb, 1983).

Although the strategies represent quite different conceptions of the problem, there is a clear relation between them. Each strategy represents an abstraction of the one that precedes it. Furthermore, the verbal descriptions of the strategies are remarkably similar. In each case Ellen went through the same counting sequences; it is the referents for the counts that changed. Ellen described three different strategies. It is not clear whether she could have generated the third strategy if she had not first actually modeled the problem using ten blocks. She seems to be in a transition from modeling to using more abstract strategies. She even referred to her abstraction as a description of what she did with the blocks when she said that she could tell the class what she did without going to her desk.

2.3 Discussion of the cases

Although there were fundamental differences in the organization and operation of the two classes, they had much in common. In both classes instruction was orchestrated to help students build on their existing knowledge. All good instruction must take into account what students know and the goals of instruction. It is a question of emphasis. With traditional instruction the focus tends to be on what students are expected to learn. In the two cases described above, the focus was decidedly on what students knew.

The focus on building on each students' thinking was reflected in the multiplicity of strategies that different students used at any given point throughout the year. In traditional instruction, individual lessons have particular objectives. Students are expected to learn a specific concept or skill, and the next lesson moves on to a new objective for which the concept or skill learned in the preceding lesson may be a prerequisite. In contrast, the lessons in the two cases were not organized around a hierarchy of objectives, with each lesson dedicated to meeting a particular objective. Students often solved the same problem, but the different strategies they used represented very different points in the evolution of their understanding of multidigit concepts and operations. Thus, the concepts that some students were developing in October other students were learning in December or February. Some students progressed further than other students. The goal was not that every student reach a certain point but that each student extend their knowledge as far as they could. What was critical was each extension represented a deepening understanding, in other words that each new strategy could be related in a meaningful way to concepts and strategies that the student already understood.

Neither of the teachers demonstrated or modeled target problem-solving strategies that they expect students to learn. Rather they immediately started with problems that the students could solve using informal modeling and counting strategies. Throughout the year students solved problems that were similar to the problems that they solved at the beginning of the year; the numbers changed somewhat as the year progressed, but the problems were essentially the same and could be solved with the same modeling strategies that students used at the beginning of the year. What characterized learning in these classes was the evolution of the strategies that students used to solve the problems. The evolution was characterized by the adoption of progressively more efficient strategies that drew on increasingly sophisticated multidigit number concepts. The transition from using single counters to using base-ten materials represented not only a more efficient way to represent large numbers, it was based on at least an implicit understanding that units of ten could be counted and the relation of collections units of tens and ones to the number names used to designate them. The development of invented algorithms meant that students were able to reflect on the operations on the blocks to the point that they could abstract them. These transitions were made throughout the year for different students.

2.4 Developing understanding: forging connections

A benchmark for assessing whether instruction is promoting the development of understanding is to be able to characterize how it provides for the construction of critical connections that give meaning to the newly learned ideas. The strategies that students developed represented reasonably natural extensions of existing strategies and were perceived as such. Using base-ten materials to solve problems was another form of modeling using objects collected into groups of 10; invented strategies were abstractions of the strategies using base-ten materials. Because of the range of problems that students solved, they frequently moved back and forth from modeling with tens to using invented strategies for different problems, and they also were encouraged to solve problems in more than one way. Thus, when students adopted more advanced strategies, they did not forget about the more basic strategies that the new strategies were related to. Furthermore, because different students in each class used different strategies, strategies were continuously juxtaposed and compared as the students shared their strategies.

2.5 Reflection and articulation

Sharing strategies was a prominent feature of both classes that appeared to play a critical role in students developing more advanced strategies and connecting them to existing strategies. The emphasis was always on the strategies that students used, and for almost every problem they solved they shared their strategies. These sharing sessions served a number of purposes. Because students were expected to share their strategies, they needed to use a strategy that they understood well enough that they could explain it. This seemed to mitigate against students imitating strategies that they saw other students use unless they understood them.

Another consequence of sharing was that the more advanced students modeled strategies for other students. Too often demonstrations of procedures by teachers or students focus on the external behaviors involved in carrying out a procedure and mask the underlying cognitive processes and decisions involved in the solution. This was not the case in the sharing sessions in these classes. Students not only shared their answers; they made the thinking involved in solving the problem visible.

There appeared to be a subtle but important difference, however, in having students rather than the teacher model the strategies. When a teacher models a particular procedure, an implicit message may be conveyed that this is the preferred method and the students should strive to adopt it. The same status was not attributed to strategies modeled by students. Although the teachers did engage students in discussions of the relative merits of different strategies, both teachers went out of their way to not convey that everyone should try a given method, and the students appeared to adopt this same perspective. When a particularly elegant or efficient strategy was shared, the other students did not immediately attempt to use it, and we observed little evidence of students attempting to imitate a strategy that was beyond them.

Another important aspect of sharing strategies was that students not only needed to be able to solve a problem; they needed to be able to explain their solution. The necessity of articulating their solution processes appeared to encourage students to reflect on their solutions. In fact the articulation of strategies often became a form of public reflection as is illustrated in Ms. Keith's interaction with Karen. This reflection appeared to play a critical role in the transition to using invented algorithms and in maintaining connections between invented algorithms and the operations on base-ten materials on which they were based.

The sharing of strategies as a regular part of instruction also represented the primary means for the two teachers to assess their students' understanding. Because they listened to how students solved problems during instruction, it was not necessary to conduct separate assessments.

Finally, the sharing of strategies communicated to the students that their thinking was valued and that they had something important to contribute. Almost universally the students were anxious to share how they had solved a problem, and the students expected any visitor to the classes to want to listen to them described their thinking. These perceptions contributed to the students' feeling of confidence in their ability to learn and understand mathematics, that they had responsibility for their own learning, and that it was under their control.

2.6 Notation

Although the classes shared a number of critical features, there were interesting differences as well. One potentially important difference was in the use of notation. Although students in Ms. Gehn's class regularly wrote number sentences for problems that they solved, the number sentences generally did not play an important role in the students' solutions of the problem. For the most part, the students solved the problem with blocks or in their heads, and did not record their methods of solution. In contrast Ms. Keith provided students with notations they could use to record their solution processes. In the second week in November, Ms. Keith introduced a notation to represent children's solutions when they used an invented algorithm. Up to that time, the few children who used invented algorithms, did them in their heads, and there was no record of what they had done. As one child, Jason, explained his solution to a problem involving the sum 52 + 28, Ms. Keith wrote the following: 50 + 20 -> 70 + 8 -> 78 + 1 -> 79 + 1 = 80[3]. She did not tell Jason that he should use this notation, she simply used it to record what he had done. In the ensuing weeks she continued to use the notation when students presented invented algorithms.

Later in the year Ms. Keith introduced another notation, which the students called the *pull down method.* Whereas the arrow notation was effective in representing solutions in which calculations in which a running total was kept, the pull down method allowed numbers to be combined in a variety of ways. In the example in Figure 4, tens and ones are first combined separately and then the two sums are combined.

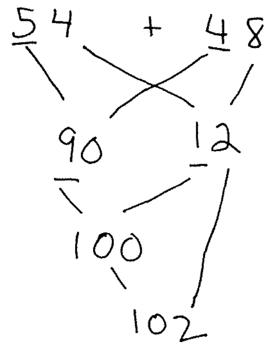

figure 4: Pull-Down Notation for 54 + 48

These notations provided a frame of reference so that everyone could see how numbers had been combined. This facilitated discussion and comparison of alternative strategies. Over time the students began to appreciate that the notation could provide them a record of what they had done so that they could share it more easily with the class. It also provided a record of the numbers they had combined so that they could check their work. Over the next few months students began to adopt the notations to record their work.

The consequences of introducing such notation as early as the first grade raises interesting issues. The potential benefits are illustrated by the experiences of students in Ms. Keith's class. On the other hand, notations tend to restrict the strategies that students are likely to use. For example, the arrow notation supports the use of sequential procedures but does not easily afford a combining units separately strategy (Beishuizen's 1010 strategy). Beishuizen (1993), Gravemeijer (1994), and Klein, Beishuizen and Treffers (in press) make a strong case that the open number line notation, which is similar in many respects to Ms. Keith's arrow notation, supports the development of students strategies. However, they are working from a somewhat different set of assumptions than Ms. Gehn and Ms. Keith. Their explicit goal is to constrain students' strategies, as they consider sequential strategies to be more effective in the long run, particularly for subtraction problems. We too have found that relatively few students successfully use a combining-units-separately strategy to subtract. We did find some students that essentially treated the difference in the units as generating a negative number that was subsequently combined with the tens. Some students actually used negative number notation; others simply subtracted the difference in the units (e.g. 83 – 27 -> 80 – 20 = 60, 7 – 3 = 4, 60 – 4 = 56). These particular strategies were not used by many students, but very few students used buggy versions of the combining-units-separately strategy. Most students either shifted from a combining-units-separately strategy to a sequential strategy or modeled the problem with base-ten materials (Carpenter et al., 1996).

Thus, even if certain notations do support the acquisition of invented algorithms, I would propose that when they should be introduced would be an open question. At what point do we want to limit children's constructions? Having both combining-units-separately and sequential strategies in play at the same time affords children the opportunity to make connections between sequential- and collected-multiunit concepts. Students in both classes were learning multidigit concepts and operations with understanding, and I think it is an open question when and how notation should be introduced for the greatest long term benefit. Moreover, it is a question for which there may not be a simple answer.

3 Cognitively Guided Instruction

In this chapter I have focused primarily on the parts of our research that deal with the development of children's mathematical thinking and the instructional contexts that foster that development, but it is impossible to separate entirely those aspects of our research from the rest, and it is necessary to have some perspective of the entire program in order to understand the context for my comments about the relation of our research to the characterization of Realistic Mathematics Education portrayed by Gravemeijer (this volume). For the last 12 years Megan Franke, Elizabeth Fennema, and I have been engaged in an integrated program of research studying the development of students' mathematical thinking; instruction that influences that development; teachers' knowledge and beliefs that influence their instructional practices; and how teachers' knowledge, beliefs, and practices are influenced by their understanding of students' mathematical thinking. Like the developmental research characterized by Gravemeijer, our research has been cyclic. We started with explicit knowledge about the development of children's mathematical thinking (Carpenter, 1985) which we used as a context to study teachers' knowledge of students' mathematical thinking (Carpenter, Fennema, Peterson and Carey, 1988) and how teachers might use knowledge of students' thinking in making instructional decisions (Carpenter, Fennema, Peterson, Chiang and Loef, 1989). We found that although teachers had a great deal of intuitive knowledge about children's mathematical thinking, that knowledge was fragmented and as a consequence generally did not play an important role in most teachers' decision making (Carpenter et al., 1988). If teachers were to be expected to plan instruction based on their knowledge of students' thinking, they needed some coherent basis for making instructional decisions. To address this problem, we designed a teacher development program called *Cognitively Guided Instruction (CGI)* the goal of which was to help teachers construct conceptual maps of the development of children's mathematical thinking in specific domains from which they could deduce hypothetical learning trajectories for their students (Carpenter, Fennema and Franke, 1996).

In the workshops and in subsequent interactions with teachers, we focused entirely on children's mathematical thinking. Our goal was to study how teachers would use knowledge of students' thinking, and as a consequence we provided no curriculum materials nor did we specify explicit principles for instruction. However, the teachers were given opportunity to discuss implications of their emerging understanding of children's thinking for their instructional practice. Thus, teachers' plans for instruction were not constructed in isolation but as part of a community of practice, and our conception of the development of teachers emerging knowledge, beliefs, and practice are consistent with the emergent/socio-constructivist frame described by Cobb and Yackel (in press) and Gravemeijer (this volume) (Franke, Fennema and Carpenter, in progress).

In a series of studies we found that learning to understand the development of children's mathematical thinking could lead to fundamental changes in teachers' beliefs and practice and that these changes were reflected in students' learning (Carpenter et al., 1989; Fennema, et al, 1993; Fennema et al., 1996). The studies provided sites for studying the development of children's mathematical thinking in situations in which their intuitive strategies for solving problems were a focus for reflection and discussion. These studies provided new perspectives on the development of children's mathematical thinking and the instructional contexts that support that development (Carpenter et al., 1993; in press), which has lead to revisions in our approach to teacher development.

4 CGI and RME

Although Gravemeijer's description of RME is cast as heuristics for instructional development and CGI is principally concerned with teacher development, there is remarkable correspondence between the principles he articulates and the principles underlying CGI. The principles apply to us as researchers engaged in teacher development and to CGI teachers engaged in developing their own instructional theory and practice. The similarities start with the basic theoretical perspectives. Gravemeijer characterizes the Dutch researchers as engaged in developing 'local instructional theories' from which more general theory can be reconstructed (p. 16, 18). This is precisely the approach we have taken in CGI. We have helped teachers develop domain specific knowledge about children's thinking, which not only provides a basis for teachers to construct local instructional theories about the domain, it also serves as a context for them to construct more general theory about instruction and the development of children's thinking (Carpenter et al., 1996). As do RME researchers CGI teachers develop, test, revise, and retest theories about the development of children's thinking and instruction that supports that development, a process we have labeled *practical inquiry* (Franke et al., in press).

4.1 Heuristics for instructional development

Gravemeijer outlines three core principles of heuristics for instructional development: (1) guided reinvention through progressive mathematization, (2) didactical phenomenological analysis, and (3) emergent models.

4.2 Guided reinvention through progressive mathematization

According to the reinvention principle a learning route should be mapped out along which students can be expected to construct the desired results themselves. This is precisely what we saw in Ms. Gehn' and Ms. Keith's classes. The direct modeling strategies used by the students in those classes and the abstraction of those strategies correspond to horizontal and vertical mathematization processes described by Treffers (1991) and Gravemeijer (1994, this volume). Treffers and Gravemeijer describe five characteristics of progressive mathematization, all of which characterize instruction in the two classes described above.

1 *The use of contextualized problems.* Virtually all problems were set in contexts that the students understood and could deal with intelligently. They were the starting points upon which formal operations and symbols were based.
2 *Bridging by vertical instruments.* Models and strategies emerged over time as a result of solving problems, and the emerging models and strategies represented a line of progression from intuitive informal strategies to more abstract procedures.
3 *Student contribution.* Students constructions represented the primary source of new strategies.
4 *Interactivity.* Students informal methods were used as a basis to develop formal procedures through a constructive learning process that involved constant discussion of and reflection on alternative strategies. In fact discussion and reflection represented primary mechanisms contributing to students adopting more progressively more sophisticated strategies.
5 *Intertwining.* Although the discussion of the two cases focused on the development of multidigit concepts and operations, the teachers did integrate these ideas throughout other learning strands. For example, in Ms. Gehn's class graphing and measurement activities played key roles as children were learning to solve problems involving comparisons, and a unit on money provided a familiar context in which many children made significant progress in using invented algorithms.

4.3 Didactical phenomenology

The goal of phenomenological investigation is 'to find problem situations for which specific approaches can be generalized, and to find situations that can evoke paradigmatic solution procedures that can be taken as the basis for vertical mathematization.' (Gravemeijer, this volume, p. 24). The taxonomy of problem types that underlies our analysis of the development of children's conceptions of whole number operations certainly fits this bill. In the words of Freudenthal (1983) the taxonomy represents 'phenomena that beg to be organized (p. 32).' Because the teachers develop a clearly articulated understanding of the different classes of problems that give meaning to operations of addition, subtraction, multiplication, and division, they provide opportunity for students to develop integrated, multifaceted conceptions of these operations. Subtraction, for example, is not just conceived as taking away, but also in relation to comparing, part-whole relations, and missing addend situations. Because the problems can be solved with a variety of strategies representing different levels of abstraction and understanding, they provide a basis for vertical mathematization.

4.4 Emergent models

For both RME and CGI, emergent models bridge the gap between informal knowledge and formal abstract strategies. In both cases models emerge from the activities of the students themselves. There is a transition from modeling situations to models as a basis for mathematical reasoning. Gravemeijer distinguishes four levels of model development starting with situational, and moving to referential, then general, and finally formal. Students in the CGI classes described in the case studies fell in the middle two levels. By the beginning of first grade they were able to operate at the referential level, constructing concrete models to represent situations. The case studies traced the development of these direct modeling strategies to a general level in which the focus was on the strategies used to solve the problem rather than on modeling the context of the problem. At each level the activities of the prior level become an object of reflection. At the referential level, children are able to reflect on and represent a situation. They do not have to actually act within the context of the situation. They can use counters to represent the qualities described in a problem, but they still construct physical models that are consistent with the context of a problem. They solve partitive division problems by partitioning objects into a given number of sets, and they solve measurement division problems by making sets of a given size. At the next level students use derived fact strategies (Carpenter, 1985) or invented algorithms that focus more on the general mathematical structure of the operation than the specific context of the problem. I am just beginning to study how the strategies that children use at the general level become objects of reflection such that children can form generalizations about the procedures themselves and formalize their ways of notating those generalizations.

Throughout the transitions to more abstract models, the new models do not become detached from the original model. In both RME and CGI, models are constructed through a bottom up approach. Problems are selected that support the informal modeling strategies of students and provide a basis for extending them. Classroom norms that encourage reflection on and discussion of alternative strategies insure that strategies at each of the levels remain connected.

4.5 Problem selection

Instruction in both RME and CGI classes is based on problem situations that are meaningful for the students. 'Problems in RME [and CGI] do not necessarily have to deal with authentic everyday-life situations. What is central is that the context in which a problem is situated is experientially real to students in that they can immediately act intelligently within this context (Gravemeijer, this volume, p. 27).' This is essentially the position that I along with Hiebert, Fuson, and others proposed in a recent article. We argued that what was critical was that tasks become problematic for students. 'Tasks are inherently neither problematic nor routine. Whether they become problematic depends on how teachers and students treat them (Hiebert et al., 1996, p. 16).' The two CGI teachers in the case studies constructed most of their own problems based on the problem taxonomy they had learned in CGI workshops. Although they took advantage of problem situations that arose naturally in their classes, many of the problems they used were simple word problems. What was critical was that the tasks were dealt with as genuine problems by the students and that they could act intelligently with the context of the problems.

Although we share the same fundamental perspective on problems, I believe that there are some differences between RME and CGI in how problems are conceived. The analysis upon which CGI teachers' selection of problems is based on the semantic structure of problem types. We focus on distinctions among problems that characterize critical differences in the ways that students think about and model them rather than on the situational context in which problems are embedded. Thus, we are concerned with distinctions between measurement and partitive division, because those differences are real for students. On the other hand teachers must instantiate the analysis by setting problems in specific situational contexts. I may be mistaken, but I sense that for RME problem structure is more in the background and context is more in the foreground than is the case with CGI.

A second potential difference resides the conception of *guided exploration* articulated by Gravemeijer (this volume). In the two cases described above neither teacher construed that a major part of her role was to 'design a sequence of appropriate problems (p. 24)' to guide progressive mathematization. Although Ms. Gehn did vary the numbers in problems somewhat as the year progressed, neither teacher constructed a systematic progression of problems to encourage vertical mathematization. The basic analysis of the development of children's mathematical thinking does provide a framework that could serve as the basis for constructing such a sequence of instructional activities, but that was not the route that the teachers pursued. Throughout the year different students operated at different levels in solving the same problems, and student progress was characterized by the changes in strategies that students used rather than in the types of problems they solved.

5 Conclusion

Although CGI has focused explicitly on the development of teachers' knowledge and we have not been engaged in developing curriculum materials, we are not proposing that there is a dichotomy between developing teachers understanding of children's thinking and curriculum development. We have never proposed to teachers in our program that they not use curriculum materials developed by others, and many teachers do follow a standard curriculum, particularly when they are starting to work out how to base instruction on student thinking. In practice, however, many CGI teachers do not rely on a specific curricular program and develop many of their own problems, but that is due in large measure to their dissatisfaction with available materials. Many of the teachers believe that the materials available to them do not reflect what they know about the development of children's thinking or do not afford them flexibility to build on their students' thinking. By the same token, we recognize that curriculum may be designed to support the development of teachers' understanding of students thinking (Cobb, Wood, Yackel, 1990; Gravemeijer, 1994). In fact I would propose, that this is an essential feature of any curriculum that has as a goal to bring about fundamental reform in mathematics instruction.

The stance we have taken is driven to a large extent by a concern for maintaining coherence and focus in our research not by fundamental opposition to curriculum development. Rather than representing points of conflict, I believe that the multiple perspectives represented at this meeting offer a basis for triangulation to help us figure out what is essential and what is not in the different paths we have followed for bringing about the fundamental vision of reform of mathematics instruction that I sense is shared by most of the participants in this conference.

acknowledgement

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notes

1 Beishuizen (this volume) labels this particular procedure 10s and categorizes it as a decomposition procedure, because the 38 is decomposed. He distinguishes it from what he labels as N10, in which 20 is added directly to 38. We classified both 10s and N10 as sequential strategies, because the addition is done sequentially. The case can be made, as Beishuizen, Van Putten, and Van Mulken (1997) do, that the 10s strategy represents an intermediate strategy between 1010 and N10. We have found, however, that with American students in our classes it is difficult to distinguish between N10 and 10s strategies, and students frequently use them interchangeably. When asked to elaborate on an N10 strategy, students often give a 10s explanation. This may reflect the fact that students invented algorithms are based on collected multiunits rather than models supporting se-

quential multiunits (Fuson, 1992).

2 Beishuizen (this volume) calls this procedure 1010.

3 There are direct parallels between this notation and the empty number line discussed by Beishuizen (1993), Gravemeijer (1994), Klein, Beishuizen and Treffers (in press).

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Instructional design for teacher education

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1 Introduction

All over the world calls for reforming school mathematics can be heard. These calls are urgent, but certainly not new (Becker and Selter, 1997). In the past there have been several official reform documents, like the general circular of the Danish Ministry of Education from the year 1900 (see Niss, 1997); reform projects, like for example the Nuffield project (Chambers and Murray, 1967); reform movements, like the Association of Teachers of Mathematics (1967); or reform writers, like for example Gattegno (1971) or Colburn (1832/1970).

Nowadays, we see an even broader consensus as expressed in many official documents such as the U.S. *Curriculum and Evaluation Standards* (NCTM, 1989), the Dutch *Proeve van een Nationaal Programma* (Treffers, De Moor and Feijs, 1989), the English *Mathematics Counts* (Cockcroft, 1982) or the Australian *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990).

2 Instructional design – the neglect of a scientific task

However, the desired reform cannot happen by solely setting the scene. Practising teachers need to know how to deal with the subject matter in a way differing from the so-called traditional one. The teachers' knowledge base certainly has to consist of (domain-specific as well as topic-specific) *background* knowledge, but also of specific knowledge about instructional activities and material. It is beyond doubt that several teachers have made important contributions in this field.

Nevertheless, most of them simply do not have the time, the motivation and the necessary background to *professionally* develop and evaluate learning environments. What makes the task of designing really difficult is the fact that it cannot be restricted to isolated nice examples. Instructional design always has to consider the necessity to integrate all its products into coherent conceptions' – local and global instruction theories (Gravemeijer, this volume).

Despite the difficulty of transforming general guidelines into meaningful activities that meet the demands of the reform, it seems to me, as if this is often not regarded as a task of scientific value. If you, for example, have a look at the international top-level journals of our discipline – like *Educational Studies in Mathematics* or the *Journal for Research in Mathematics Education* – you can get the impression that working on the *realisation* of the reform is not regarded as to be something really scientific. You can read, for example, several – surely inspiring – articles about semantic structures of multiplication word problems or informal strategies for solving problems with a multiplicative structure, but you rarely find any sound advice about how to deal with multiplication table *in the classroom.* And this topic surely is not an exception.

In this context Gravemeijer (Gravemeijer, this volume) elaborates that mathematics education has a fundamental and clear-cut responsibility to develop practical suggestions within the capacity of an average teacher (Griffiths, 1983, 360). These suggestions should of course not be applied blindly, but adapted and refined by *autonomous* teachers (Gravemeijer, this volume).

It is encouraging that the Dutch ideas of developmental research, showing the right degree of *practical proximity* and *theoretical distance*, are making strides towards acceptance as an important field of research (Streefland, 1991; Gravemeijer, 1994; Freudenthal 1991, 147-180). Developmental researchers try to overcome the RDD-model of innovation, in which research, development and dissemination were more or less strictly separated and linearly ordered.

On the contrary, Freudenthal (1991, 159) argues that 'practice, at least in education, requires a cyclic alternation of research and development.' Consequently, scientific knowledge is not only the *input* for developmental activity, but also the *output.* The developer's visions about how the teaching/learning process proceeds (*thought* experiments) are put into practice (*practical* experiments). What happens in the classroom is consequently analyzed and the results are used to continue the developmental work. Continual observation and recording of individual learning processes should be at the heart of such research (Streefland, 1990). 'This process of deliberating and testing leads to a product that is theoretically and empirically founded – well-considered and well-tried' (Gravemeijer, 1994, 113).

This view implies that development does not receive lower esteem than testing: In harmony with this notion, Wittmann's (1995) plea for *mathematics education as a design science* calls attention to the importance of creative design for change – design that is understood as 'the execution of a thought experiment of teaching and learning both' (Streefland, 1993, 116).

It is my well-considered opinion that the discipline of mathematics education has neglected developmental research in the past. In making this remark I am fully aware that it is so easy to be misunderstood. Thus, I clearly want to point out that developmental research should, of course, *not* monopolize the landscape of mathematics education. There are many topics for research that can hardly be solved exclusively by developing and testing learning environments.

3 The crucial role of teacher education

If reform is the aim, alternative curricula, alternative textbooks and teaching materials, and alternative forms of assessment (Van den Heuvel-Panhuizen, 1996; Clarke, 1992) are essential. However, these are all mediated through the teacher, specifically through the teacher's beliefs about how to organize and facilitate children's learning of mathematics and about his/her own relation to mathematics or to the nature of learning (Fennema and Franke, 1992; Lerman, 1993). That the teacher in return is influenced by the textbook clearly indicates the importance of developmental research.

Nevertheless, the teacher is one, and maybe *the key element* in change (Brown, Coone and Jones, 1990; Thompson, 1992). In this context, teacher education – preservice as well as inservice – plays a crucial role. As Cooney (1994, 109) has shown, our discipline seems to have some backlog in this respect: Progress in teacher education is much less apparent than in teaching practice at school.

But reform efforts will seriously be hampered, if reform in teacher education is not regarded as being as urgent as reform at schools. This implies that there is a clear-cut need *for instructional design for teacher education*!

Thus, I want to devote the rest of my paper to this issue. In doing so, I am aware that I do not seem to be very close to the conference-topic: *The role of contexts and models in the development of mathematical strategies and procedures.* However, I want to relate my contribution to Gravemeijer's chapter (this volume). His main point, as I understand it, was to discuss the problem how teachers can be supported in order to make our common reform efforts successful. As I am mainly engaged in preservice, I will not explicitly deal with inservice teacher education. However, I have the feeling that my remarks are somehow also relevant for the latter.

Before I come to examples (sections 4 and 5), I briefly want to sketch my background 'philosophy' of preservice teacher education in order to place the following in a broader context. As Bromme's (1994, 81) meta-analysis has shown, offering teacher students the necessary background knowledge surely is a precondition for their professionality as teachers. However, teachers actually become professionals *while* they are teaching and reflecting on their teaching. The main goal of teacher education should consequently be that the teacher students prepare themselves for their forthcoming professional self-development.

Thus, teacher education should not *primarily* be an apprenticeship training that provides recipes and methods that are directly applicable in the classroom, but it should *first and foremost* assist prospective teachers in developing their autonomy. This implies to support them in increasing their degree of awareness – about mathematics, about children's mathematical learning, about the quality of teaching material and so forth (Selter, 1995; see Dewey, 1904; Walther, 1984, 71).

In the following I will sketch two introductory courses specifically designed for primary teachers – one in mathematics and one in didactics – aiming at fulfilling this goal. Both elements – mathematical and didactical education – are *equally important*, as good teaching must always take into account both: the subject matter on the one hand and children's thinking and ways of encouraging its development on the other (Dewey, 1976).

4 An introductory course in mathematics

We know that the various components of professional teachers' knowledge are intertwined and that mathematical knowledge – naturally – is one of them. There surely is no simple relation like, the more mathematical knowledge, the better the teaching'. But the evidence available suggests that teacher knowledge of the content can – to a great extent – positively influence teaching practice (Fennema and Franke, 1992; Bromme, 1992, 105; Brophy, 1991, 352).

In addition to the mathematical knowledge, the mathematical attitude is of importance: One's conception of what mathematics is affects one's conception of how it should be dealt with at school (see Hersh, 1986, 13), or as Thom (1973) has put it: 'All mathematical pedagogy ... rests on a philosophy of mathematics.' We know that these belief systems are not static, but 'dynamic, permeable mental structures, susceptible to change in light of experience' (Thompson, 1992, 140).

In this context, the teacher students' first encounter with mathematics at the teacher education institution plays a crucial role. Probably more than follow-up courses it offers the opportunity to encourage the teacher students to develop a lively relation to (the activity of doing) mathematics. In the following, I want to describe an introductory course in mathematics that was designed by my colleagues Gerhard Müller, Günter Krauthausen and myself.

4.1 General overview

Several years ago, I could not imagine that a calculus course could be something appropriate for prospective primary teachers. In the meantime, it seems to be possible to me that it could be integrated into the curriculum, *if* it was treated in close relation to phenomena and in critical distance to exaggerated symbolism.

However, why not take topics that are more closely related to what is required in primary schools? Freudenthal (1991, 147) once quoted the mathematician Landau who said 'Number theory is good; thanks to it one can get a Ph. D.' In my opinion, number theory is also good for primary teacher education, because several of its problem contexts are so rich that they can also be used in primary school, provided they are made accessible for children. In addition, the relations to the history of mathematics are not just apparent for us (Euclides, Fermat, Gauss), but also available for teacher students.

The course that I want to sketch now offers almost no 'real-world' mathematics and definitely has to be complemented by other courses that deal with real-life phenomena. Nevertheless, as I hope to show in the following, the primary school-related contexts I will be presenting can be regarded as being realistic – in the sense Realistic Mathematics Education understands this term: meaningful and being full of relations for the learners (Geravemeijer, this volume).

Altogether 550 first-year students attended the course. The course was composed of 14 units. Each unit consisted of a 90 minutes lecture and a small group work with 25 teacher students and one student tutor. Due to these circumstances it does not make much sense to think about other forms of organization – the 550 students were our reality!

The general 'philosophy' behind the course read as follows:

> 'According to the famous mathematician and mathematics educator Hans Freudenthal, mathematics should not primarily be seen as a finished product, but as a human activity. Thus, this course aims not only at 'introducing' you into mathematical contexts, but especially into the activity of doing mathematics. It is essential that you actively and critically go over the lecture parts and that you engage in preparing your homework thoroughly as well as in discussing your findings during the small group sessions.'

Each of the following topics was dealt with during one lecture and the corresponding small group work session:

1 Substantial learning environments for pupils and for teacher students
2 From the history of arithmetic (origin of numbers, Egypt, Babylon, ...)
3 Counting – a variety of strategies
4 Combinatorical problems
5 Numbers and counting in different place value systems
6 Arithmetic in different place value systems
7 Arithmetical sequences and progressions (Sylvester's theorem, Fibonacci-numbers, ...)
8 Primes (infinity of primes, formulas to find primes, distance between primes, ...)
9 Geometric numbers (triangular numbers, square numbers, ...)
10 Square numbers (difference between square numbers, last digits, ...)
11 Divisors (number of divisors, sum of divisors, perfect numbers)
12 Divisibility (rules of divisibility, divisibility in different systems)
13 Magic squares (methods of construction, different kinds of magic squares)
14 More substantial learning environments for pupils and for teacher students.

However, it was not just the topic that was important, but also the way in which it was treated. Without going into it in depth, I should mention that it was our aim to run the course with as few formulas as possible and to keep in touch with the phenomena, as premature symbolism is the enemy of understanding.

4.2 Examples from unit 1

As a representative example I would like to show the students' homework from the first unit. The activities centered around so-called substantial learning environments suited for primary school children as well as for primary school teacher students. The students had to prepare them at home and discuss them in a two hour session at university. In the corresponding lecture the mathematical background of two similar learning environments was described: number chains and arithmetic triangles (see Becker and Selter, 1997).

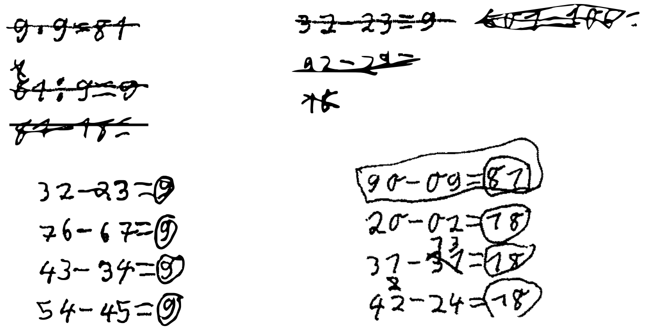
The first unit was meant to be a paradigmatic example for what we had in mind for the entire course: Taking substantial problem contexts that are related to primary school arithmetic and reflecting on them on an advanced level. Due to space constraints, I cannot go into detail here. I thus have to leave it to the reader to work on the problems.

4.2.1 Subtracting reverse numbers

Take a two digit number and its reverse number. You get the reverse number, if you interchange the two digits. Subtract the smaller from the bigger number (Examples: 52 – 25 = 27 or 90 – 09 = 81).

a Which results are possible? Why exactly these? How many different ways of arriving at each result are there?

b Assume that second graders are working on the reverse numbers-problem. Benni joins you during the lesson and says: 'Look here!' How would you react?



~~9·9=81~~
~~81:9=9~~
~~32-23=9~~

32-23=9
76-67=9
43-34=9
54-45=9

90-09=81
20-02=18
31-13=18
42-24=18

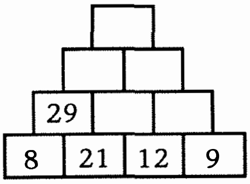
Benni

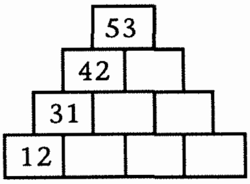
c Transfer the rule to three digit numbers and work on the questions given in a!

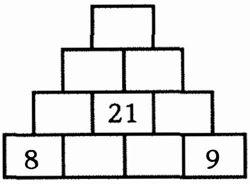
4.2.2 Number walls

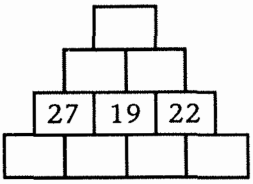
The rule for the number walls activity is as follows: Except for the bottom row the number to be entered in each stone is the sum of the numbers of the two stones directly underneath (29 = 8 + 21). The bottom row consists of the so-called bottom stones; the stone on the top is called top stone.

a Enter the missing numbers!

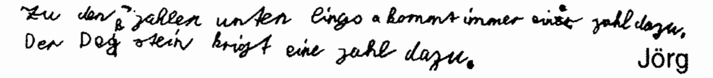








b How does the top stone change, if you add 1 (2, 3, 4, …, *n*) to the first (the second, the third, the fourth) bottom stone? Compare your findings with those that second graders made.



Es kommt 13 dann 14, 15, 16. die Linke Seite verendert sich immer.	Heinz	At first it is 13, then 14, 15, 16. The left side always changes.
Erst waren es 13 dann 14. Erst waren es 33 dann 34. Erst waren es 62 dann 63.	Bernd	At first it was 13, then 14. At first it was 33, then 34. At first it was 62, then 63
Zu den zahlen unten lings komt immer eine zahl dazu, Der Deg stein krigt eine zahl dazu.	Jörg	The numbers in the left bottom stone always get one bigger. The top stone is getting one bigger.
Ich gehe ~~imer~~ immer einen mehr, bei der Zahl lings unten bei der Zahl in der mitte lings und nach bei der Zahl lings oben.	Helga	I always take one more in the left bottom stone, in the left stone in the middle and also in the top stone.

c Transfer this investigation to number walls with 2, 3, 5, 6, … bottom stones!

4.2.3 Differences of square numbers

Try to represent integers as differences of two square numbers, like for example: 8 = 9 – 1; 25 = 36 – 9 or 95 = 144 – 49.

Building on children's mathematical thinking

8 *Standard algorithms for addition and subtraction*
Zone of proximal development

9 *Standard algorithms for multiplication and division*
Progressive mathematisation, part 2: vertical component

10 *Geometry in grades 1 and 2*
Fundamental ideas as guidelines, part 2: geometry

11 *Geometry in grades 3 and 4*
The idea of a spiral curriculum

12 *Problems with real-world mathematics*

13 *Real-world mathematics in grades 1 and 2*
Operative principle revisited

14 *Real-world mathematics in grades 3 and 4*
Learning as a constructive and social process revisited
Natural differentiation (instead of prestructured individualisation)

5.2 Examples from unit 5

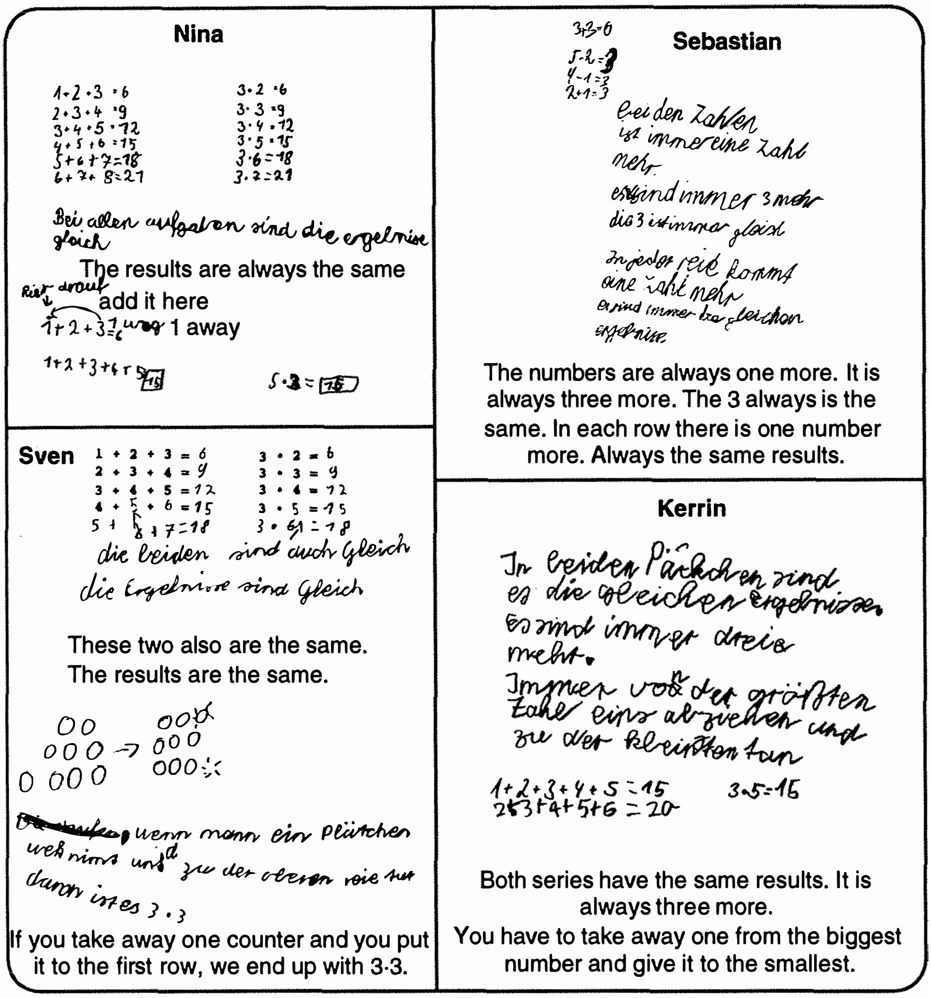
In the following I will present three activities from unit 5 that the students had to prepare at home. This unit dealt with multiplication and division in the domain of 1 to 100 as well as with the operative principle which was introduced in the corresponding lecture: It holds that to understand a piece of mathematics, the pupil needs to understand the *effects* that certain *operations* have on certain *objects*. Also a possible teaching-learning route for multiplication table was described (Selter, 1994).

5.2.1 Number stairs

Second graders worked on the so-called number stairs-problem. They had to solve the following two series of addition respectively multiplication problems: $1 + 2 + 3 =$, $2 + 3 + 4 =$, $3 + 4 + 5 =$, ...; $3 \cdot 2 =$, $3 \cdot 3 =$, $3 \cdot 4 =$, ...

a Put down the next tasks in each series and work them out. What do you notice? Explain your findings in different ways, among others in a way that second graders in your opinion would use!

b Now analyse the following documents of second graders. To what extent do they meet your expectations, to what extent do they differ?



Nina

1+2+3 =6
2+3+4 =9
3+4+5 =12
4+5+6 =15
5+6+7=18
6+7+ 8=21

3·2 =6
3·3 =9
3·4 =12
3·5 =15
3·6=18
3·7=21

Bei allen aufgaben sind die ergebnise gleich

The results are always the same

Rief drauf

add it here

1+2+3=6 weg 1 away

1+2+3+4+5 15

5·3=15

Sebastian

3+3=6
5-2=3
4-1=3
2+1=3

bei den Zahlen ist immer eine Zahl mehr.
es sind immer 3 mehr
die 3 ist immer gleich
In jeder reie kommt eine zahl mehr
es sind immer die gleichen ergebnisse

The numbers are always one more. It is always three more. The 3 always is the same. In each row there is one number more. Always the same results.

Sven

1 + 2 + 3 = 6
2 + 3 + 4 = 9
3 + 4 + 5 = 12
4 + 5 + 6 = 15
5 + 6 + 7 = 18

3 · 2 = 6
3 · 3 = 9
3 · 4 = 12
3 · 5 = 15
3 · 6 = 18

die beiden sind auch Gleich
die Ergebnisse sind Gleich

These two also are the same.
The results are the same.

[illegible] wenn mann ein Plätchen weg nimt und zu der oberen roie tut dann ist es 3·3

If you take away one counter and you put it to the first row, we end up with 3·3.

Kerrin

In beiden Päckchen sind es die gleichen ergebnisse.
Es sind immer dreie mehr.
Immer von der größten Zahl eins abziehen und zu der kleinsten tun

1+2+3+4+5 =15
2+3+4+5+6 =20

3·5=15

Both series have the same results. It is always three more.
You have to take away one from the biggest number and give it to the smallest.

c Read the paper 'Objekte – Operationen – Wirkungen' by Wittmann (1985) that describes the operative principle. Illustrate its main ideas by means of the number stairs-problem!

d Put down further activities that are situated within the number stairs-problem context. For example, the difference between two addends does not have to be 1 ...

5.2.2 How many is 60 : 4?

In several interviews third graders were given context problems as well as 'context-free' problems that had not been dealt with at school beforehand. Apart from other tasks, Lina was given a card with 60 : 4 =. The interviewer also asked: 'What is 60 divided by 4?' In second grade Lina had spent considerable time on multiplication and division at the basic level, but taking bigger numbers was something new.

We are joining the interview in progress. Previously Lina has guessed 10, 20, 18, and 21. In checking by multiplication, she found that none of these were correct. The interviewer asked her to explain her thinking process. As we join them, she has just started thinking about 16.

a Before you start to read the transcript you should anticipate different ways of how third graders would work out 60 : 4!

b Subsequently study the transcript and describe Lina's and the interviewer's ways of thinking!

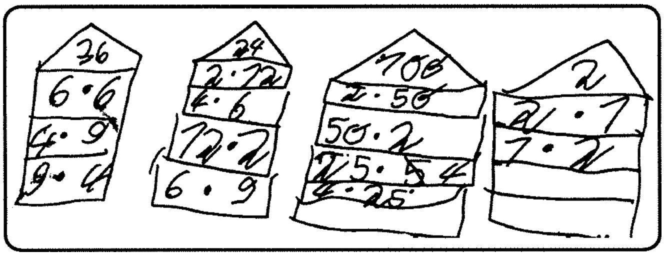
c What are, in your opinion, possible reasons for the misunderstandings that seem to occur?

d Now, formulate two context problems fitting to 60 : 4, one for quotative and one for partitive division! Hypothesise about children's solution strategies regarding these problems!

L: Well, 16 times - oh, 16 times 4 is, ... 4 tens are 40, then 46 and plus 4, 50, 52 plus 6 is 58. That won't work.
I: Why did you say plus 6?
L: Sorry?
I: You just said plus 6. 52 plus 6 is 58.
L: Yes.
I: Why did you say 6?
L: Because I was trying to find 16 times 4. I had to use a 6, because first I did the 4 tens and then the 6s.
I: But if you're finding 16 times 4, you wouldn't use four 6s, you would use six 4s. But you already know that 10 times 4 is 40. You just said it.
L: Yes.
I: What times 4 is 20? (L thinks and looks confused) Does this help?
L: What times 4 tens makes ... or ?
I: 10 times 4 is 40.
L: Yes.
I: How many do I need to get to 60?
L:20
I: And what times 4 is 20?
L: Sorry? What times 4 is 20? (L whispers) 8, 12, 16, 20 (says aloud) Oh I forgot to count. Let me count. (L counts by fours on her fingers) 5.
I: Hmm, so now you know that 10 times 4 is 40 and 5 times 4 is 20...? (I raises tone of voice into a question)
L: (after 24 seconds, L hesitates) 5? Wrong or right?
I: (after 25 seconds) The 4 goes into 40 ten times and into 20 five times. 10 times into 40 and 5 times into 20. And 40 plus 20 is 60. How many times does it go into 60?
L: The 4?
I: If we have the 10 from the 40 and the 5 from...
L: 15.
I: 15, right?
L: Hm.

5.2.3 Multiplication houses

Second grade children were dealing with the so-called multiplication houses. The rule was as follows: A number not larger than 100 was entered into the 'roof' of the house. The children were to find all the tasks from the multiplication table (1·1 up to 10·10) having this roof number as their result. For each new product a new floor could be built. Task and swap task were regarded as to be different.



In the beginning of the lesson the teacher had selected certain roof numbers. Subsequently, the children were free to choose numbers themselves. While working some children had the idea to build the 'highest' and the 'flattest' house (in the domain of 1 to 100). They forgot the initial limitation to use just tasks from the multiplication table. After some time Simone joined the teacher and told him proudly: 'Look here, I've got it. 100 is the highest and 2 the flattest house!'

a Are these really the right solutions?

b How do you react? Give reasons!

c What further activities can you think of within the context of multiplication houses?

d Imagine that one teacher claims that the multiplication houses activity is boring and to difficult for second graders, whereas a colleague of his insists on exactly the opposite. Give arguments for both positions. What is your own opinion?

5.3 The role of children's own productions

As I have shown in section 4, taking children's own productions (Streefland, 1990, Selter, 1994) as reference points for the *mathematical* training of primary teachers appears to make sense: They are suited to activate (or to keep up) the teacher students' motivation, as they can demonstrate that their mathematical training is relevant for their future job. The role of these own productions can be manifold: They can be taken as the starting point of the mathematical reflection and encourage the teacher students to work on understanding the background of given problem contexts. But they can also be investigated later in order to compare the own mathematical activity with that of the children. But these are just two of the ways possible.

Via the examples of this section I hope to have shown that children's documents can also be fruitful reference points for *didactical* discussion. Videos, transcripts or children's written work can, among several aspects, fulfill the important purpose of sensitizing for their perspectives (Selter and Spiegel, in press). The prospective teachers can learn that children's thinking often should not be regarded as a *deficit*, but as a *different* way to approach the same thing. Using children's documents can help teacher students to understand that they often think different ...

- ... from what we think,
- ... from what we assume they think,
- ... from other children and
- ... from what they thought a couple of moments ago dealing with basically the same thing.

6 Three heuristics revisited

After describing some aspects of my design work as a teacher educator, I come back to Gravemeijer's chapter now. In my opinion, the general 'philosophy' about teaching and learning that seems to be the basis of his paper as well as of the domain-specific instruction theory of Realistic Mathematics Education reads – simplified – as follows: 'Encourage the learner to and assist him in develop(ing) his own thinking instead of telling him what and how to think. Consequently take a bottom-up instead of a top-down approach of 'teaching'.'

In order to let these ideas come true on the classroom level, Gravemeijer (this volume) mentions three heuristics for instructional design: guided reinvention through progressive mathematizing, didactical phenomenology and emergent models.

As his chapter does *not explicitly* deals with instructional design for *teacher education*, I would like to take up his three heuristics and try to apply them to my previous remarks. For me they are rather *descriptive* in character, as I did not use them as guidelines for my own design work. However, as we will see in the following, some aspects related to them were indeed (intuitively) present from the very beginning.

6.1 Guided reinvention through progressive mathematizing

The first heuristic reads as follows:

> 'According to the reinvention principle, the students should be given the opportunity to experience a process similar to the process by which the mathematics was invented.

> Thus a route has to be mapped out that allows the students to find the intended mathematics by themselves' (Gravemeijer, this volume).

Thus, the developed courses should be devoted to the 'philosophy' of viewing mathematics as an activity: Learning mathematics should primarily be understood as *doing* mathematics (see Gravemeijer, 1994, 91). In Freudenthal's (1991, 49) terms,

> 'the learner should reinvent mathematizing rather than mathematics; abstracting rather than abstractions; schematizing rather than schemes; formalizing rather than formulae; algorithmizing rather than algorithms; and verbalizing rather than language.'

Not the application of a mathematical standard procedure, of ready-made mathematics should be the actual aim, but solving the problem itself on a more or less formal level:

> 'Rather than fitting the problem into a predesigned system, one describes it in a way that allows us to come to grips with it' (Gravemeijer, 1994, 92).

In this context, I see clear parallels between teaching practice and teacher education. Like teaching should aim at building on children's knowledge in order to encourage its further development, teacher education should build on the teacher students' abilities.

Aiming at such a similarity is also important, as the teacher students' own *learning histories* have a strong influence on their *teaching philosophies* (see Bauersfeld, 1993, 1). It was often mentioned that teachers teach as they were taught themselves (Shuard, 1984; Cooney, 1994, 107). Thus, teachers' courses need to be organized in a way that provides them with experience in learning that they will want *their students* to experience (Wittmann, 1989; Becker, 1992, 254), without making the mistake not to challenge them intellectually.

The bottom line of these remarks reads as follows: Learning mathematical subject matter in teacher education courses (via a bottom-up approach) is important, but developing an attitude of *mathematical* inquiry is even more important. Thus, it is not primarily the *quantity* of *subject matter*, but first and foremost the *quality* of *activities* the teacher students are engaged in that counts!

As I see a clear parallel between the activities of mathematizing and of didacticizing, these remarks appear to be also applicable for the didactical parts of the studies: Learning subject-matter related didactics (via a bottom-up approach) is important, developing an attitude of *didactical* inquiry even more.

In this context the five tenets of Realistic Mathematics Education (Treffers, 1987, 247-250; Gravemeijer, this volume) seem to be relevant for the design of courses in teacher education – mathematical as well as didactical – as I want to show by commenting on our experiences in designing the two courses described (see sections 4 and 5):

- *Use of contextual problems*: Contextual problems did not figure just as applications, but also as starting points for mathematical (didactical) activity and reflection.

We tried to use 'realistic' problem contexts which, in our opinion, carried the potential to be meaningful for our teacher students. They could serve as the starting point for the prospective teachers' mathematical (didactical) activity and reflection. I will further comment on this issue in section 6.2.

- *Bridging by vertical instruments*: Paradigmatic models, schemes, diagrams, ... were the vehicles of progressive mathematisation (didactization).

We were looking for models, partly in form of representative examples, that had a so-called amphibian-like status. These should be *concrete* enough to be relatable to the teacher students' experience, but they should also be *general* enough to represent a whole class of situations in order to encourage further learning processes. I will come back to this issue in section 6.3.

- *Student contribution*: Teacher students were encouraged to decisively influence the teaching-learning process via their own constructions and own productions.

We tried to design problems that were accessible to all of our students, so that they all – regardless of their level of knowledge – were encouraged to contribute. However, as Gravemeijer (this volume) mentions, good problems alone do not guarantee the success of learning processes, as well as a good learning atmosphere on its own does not. Whether guided reinvention takes place or not was thus not only dependent on our selection of appropriate problems, but also on our abilities to keep up or to create a good 'classroom culture'. In return our success in doing so was dependent on our teacher students' (prior) experiences and perceptions. Surely these remarks about reflexivity are also true for the next point.

- *Interactivity*: Explicit negotiation, intervention, discussion, cooperation and evaluation were key elements that carried the potential to influence each teacher students' learning path as well as the whole teaching-learning process.

Our goal was to design meaningful activities that could not only be dealt with on different levels, but that also had a common core in order to be made a topic of negotiation and discussion. For that purpose the problems finally selected should be solvable in different ways and should (at least partly) have more than one solution. It was our ambition that the chosen activities provoked discussion that was rooted in different perspectives on a *mathematical* or a *didactical problem*.

- *Intertwining*: Learning strands were intertwined and the long-term learning process was taken into account.

We were aiming at identifying problems that were substantial on the one hand and that fitted into the coherent *conception* of the course as well as into our general conception of teacher education on the other. Following the idea of a spiral curriculum several problem contexts were used repeatedly during one course as well as in related courses. What in my opinion is true for teaching practice also appears to be relevant for teacher education: Less may be more – a smaller number of substantial learning environments to be revisited and investigated on different levels and with different goals in mind seem to be more promising than an excessive variation of (isolated) activities.

6.2 Didactical phenomenology

In writing about *desired* heuristics of instructional design for teacher education, we need to remind ourselves that existing models are often rooted in traditions appearing to be extremely powerful: one of them is that of mathematical and didactical formalism.

If you, for example, take the *mathematical* courses designed on the basis of formalism it is quite common, that prospective primary teachers are confronted with courses in logic and set theory and that they have do deal with relations, functions and algebraic structures in a formalistic way. Like first grade children they first of all have to learn to speak the formalistic language of mathematics – sometimes for things that are obvious to them – instead of being engaged in meaningful mathematical *activity*.

The well-meant attempts to guarantee mathematical understanding through conceptual and formal accuracy, often result in exactly the opposite (see Wittmann, 1996, 315). Certainly, the mathematical form can be concrete and meaningful. But, as a rule, this is not the case, if it is *identified* with the *foundations* of understanding. The effort to take the mathematical form – the final product of mathematical activity – as the starting point for learning, was sharply criticized by Freudenthal (1973, 103) as *anti-didactic inversion*. He claimed that mathematical form should not be the *input*, but (at best) the *output* of mathematical activity.

In contrast to the anti-didactic inversion, to turning the learning process upside down, Freudenthal (1983) advocated the *didactical phenomenology*. Didactical phenomenology is understood as starting from phenomena that are meaningful for the learner, that beg to be organized and that stimulate learning processes (see also Thom, 1973). 'We can imagine that formal mathematics came into being in a process of generalizing and formalizing situation-specific problem solving procedures and concepts about a variety of situations' (Gravemeijer, this volume). Thus, the goal of a phenomenological investigation is to find problem situations for which situation specific approaches can be generalized, and to find situations that can evoke paradigmatic solution procedures that can be taken as the basis for vertical mathematization (Gravemeijer, this volume).

As a consequence for our mathematical parts of teacher training, we were striving for an informal, a problem- and process-orientated MATHEMATICS – in capital letters, following Wittmann (1995) – instead of a formalistic, a prestructured, a finished (closed) mathematics – the 'small' mathematics. Striving for a non-linear, holistic understanding was a central goal for the didactical courses as well. For example, we introduced the didactical principles not as absolute truth, but as condensed experience of others that was offered to our teacher students. We always tried to take care that they were aware of the fact that there are no general solutions for didactical problems in forms of recipes.

In designing the courses that I have sketched in the previous sections, we always tried to figure out realistic problems that encourage our teacher students to engage in mathematical respectively didactical activity. In this context, I would like to quote Gravemeijer (Gravemeijer, this volume) on the meaning of the word realistic:

> 'The use of the label 'realistic' refers to a foundation of mathematical knowledge in situations that are experientially real to the students. Context problems in RME do not necessarily have to deal with authentic everyday-life situations. What is central, is that the context in which a problem is situated is experientially real to students in that they can immediately act intelligently within this context. Of course the goal is that eventually mathematics itself can constitute experientially real contexts for the students.'

In this sense 'realistic' might be better expressed by 'common sense' (Gravemeijer, 1994, 94).

In looking for realistic phenomena it appeared to be one (surely not the only) promising way to take substantial learning environments – instead of isolated activities that are hardly related to teaching practice – as the integrating core of teacher education (Wittmann, 1984). We used them as *reference points* for *mathematical* as well as for *didactical reflection* (Walther, 1984). As we tried to integrate documents of teaching-learning situations, especially children's own productions, in form of videos, transcripts or written work, we were striving for increasing the degree of proximity to the teachers' reality.

6.3 Emergent models

The third heuristic focuses on the role that emergent models play in bridging the gap between informal and formal knowledge. Whereas I could easily relate the first two heuristics to my design work for teacher education – seeing them as having intuitively been present and perceiving them as a helpful orientation for the forthcoming –, I experienced some difficulties in recognizing a similar degree of relevance for the third one. In order to reduce the probability that fundamental misunderstandings occur, I initially want to sketch my understanding of it.

Gravemeijer distinguishes between manipulatives and models. In this context, he clearly argues in favor of a bottom-up instead of a top-down approach: 'Whereas manipulatives are presented as preexisting models in product-oriented mathematics education, models emerge from the activities of the students themselves in realistic mathematics education' (Gravemeijer, this volume).

I totally agree – regardless whether children or teacher students are concerned – that models should be given such an *epistemological* status instead of serving as a means of transmitting knowledge (Wittmann, 1994). The primary aim of the use of models should not be regarded as to illustrate mathematics (didactics) from an expert point of view; rather, they should support (teacher) students in constructing mathematics (didactics) starting from their own perspective (Gravemeijer, this volume).

As they should encourage development by being semi-*concrete* and semi-*abstract*, they have an amphibious-like property. Semi-concrete means that it is possible to apply them to solve problems in special contexts (that can be real-life as well as 'pure' mathematical); semi-abstract implies that they can be used for a variety of problems. Thus, models can be applied to all levels of understanding, from very preliminary, context-bound ones to very advanced, abstract ones (Van den Heuvel-Panhuizen, 1995, 3).

In this context, Gravemeijer (this volume) identifies four levels of understanding: the level of the *situations*, a *referential* level, a *general* level and the level of *formal arithmetic*. Models are placed at an intermediary level between the situational and the formal knowledge (Gravemeijer, 1994, 102). As such the switch from the model-of to the model-for level is important: 'A model comes to the fore first, as a model of a situation that is familiar to the student. Next, by a process of generalizing and formalizing, the model gradually becomes an entity on its own. Only after this transition, it becomes possible to use this model as a model for mathematical reasoning' (Gravemeijer, this volume).

Thinking in terms of teacher education I certainly see the relevance of this bottom-up approach: Not prescribing teacher students how to solve a problem and how to put down its solution, but encouraging them to use and to express their own mathematical power. The goals of our courses were a little more complex: They did not only have to support our students to bridge the gap between their 'informal' and the so-called formal *mathematics*, but also between 'informal' and 'formal' *didactics*. In addition, it was not just the subject matter, but also the mathematical (didactical) attitude that we regarded as to be important.

While designing the courses described (sections 4 and 5) we were looking for problems that potentially had the status of being concrete and of being abstract at the same time. I think that the selected learning environments represent, at least to some extent, this property. They are concrete as you can immediately work on them, but they are also general, as they can be used to develop and to concretize abstract mathematical or didactical concepts. The number stairs problem (section 5.2.1), for example, is concrete on the one hand, but also a paradigmatic example for the operative principle. The subtracting reverse numbers problem (section 4.2.1) is concrete, but it is also connected to a famous theorem from number theory, Kapreka's theorem. As children and teacher students are engaged in solving the problem, they are in principle doing the same, as the famous mathematician did.

However, these representative examples themselves cannot directly be *identified* with models. As far as I understand it, models are to emerge in the process of dealing with them. One could argue that the heuristic of emergent models is something special for the learning of mathematics at school, or even more specialized, as Gravemeijer (this volume) put it, for the learning of *arithmetic*. All examples given (decimals, addition and subtraction with small numbers, long division) are examples from the arithmetic learning strand.

To sum up my difficulties with the third heuristic: Is the idea of emergent models a general heuristic for instructional design for all mathematical learning strands and all age groups or does it have to be restricted? Could it, in principle, also be helpful with respect to the *didactical* learning strand? How could this heuristic serve as a reference point for my further design work? Do the problems presented in sections 4 and 5 have to be replaced, as they do not carry the potential to let models emerge? The more I think about these questions the more I get the impression that I need to learn more about it, first and foremost via discussing further examples.

7 Conclusion

Coming back to the courses that I briefly sketched in sections 4 and 5, I finally want to make one point. Gerhard Müller and myself simply did not have the time and the money to conduct developmental research at its best. What we did was 'just' to start from a global 'philosophy' of mathematics teacher education and from a local theory of how our topics (arithmetic respectively introduction into didactics) could be organized and to design activities that tally with this general orientation and form a coherent conception.

The results of the try-out of these prototype courses will be used for a revision of the specific course as well as they have already shaped our general ideas. However, I would have to go too much into detail, if I described that in a thorough way. Thus, what I presented should be understood as to be well-considered on the one hand, but as to be preliminary on the other.

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What can research on word and context problems tell about effective strategies to solve subtraction problems?

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Abstract

Starting from the observation that solving subtraction problems remains difficult for many poorly performing elementary school students, research was discussed that showed which procedures children effectively use to solve subtraction problems. Study 1 showed that word problems are able to trigger off qualitatively very different but effective solution strategies. One of them, the indirect addition strategy, was discussed as an interesting alternative to direct subtraction. Study 2 showed (1) the relation between the operation (adding or subtracting) and the mental arithmetic strategy (jump or decomposition strategy) used to solve two-digit addition and subtraction problems and (2) the superiority of a jump strategy to a decomposition strategy for subtraction problems. The implications for teaching subtraction were discussed and proposals for future research were made.

1 Introduction

In the field of addition and subtraction of two-digit numbers many children have especially difficulties in mastering subtraction problems with trading (for an overview see Fuson, 1992 and also Beishuizen, 1993). Many children who perform poorly in mathematics or who have learning difficulties never even attain this level in the course of their elementary school career. Then how can these skills be improved?

In this chapter I will address the relation between mathematical problem solving and word problems, contexts, models and the location and magnitude of the given numbers. The main focus is on difficulties that children encounter while trying to solve subtraction problems and ways to improve teaching. However, much research has to be done before a clear picture can be drawn. Therefore this chapter is inevitably incomplete and kaleidoscopic. I shall discuss this topic aided by the research results of others and by more detailed research reports of our group.

Traditionally, teaching mathematics has focused on decontextualized number sentences. However, an approach based on the informal solution procedures, which children bring along into school, has gained growing interest due to publications of e.g. Carpenter, Hiebert, and Moser (1983), Carpenter and Moser (1984), Carraher, Carraher, and Schliemann, (1987), De Corte and Verschaffel (1987), and De Corte, Verschaffel, and De Win (1985). In the Netherlands the reforming of mathematics instruction has led to so-called realistic methods in which very much attention is paid to presenting students with real problems, mostly in the form of pictures with text (see e.g. Gravemeijer, 1994). These problems (or so-called contexts) are constructed with the aim to foster 'self-invented' or informal solutions. When I hereafter use the term context I shall both mean the problem format as used in the realistic instruction methods, and the purely textual word problem format.

In this chapter I shall discuss whether and how contexts can possibly play a part in improving teaching of subtraction problems from 20 up to 100. First I shall review a few relevant findings from research on simple arithmetic word problems. These findings concern the influence of the semantic and syntactic structure on the representation of set relations and the resulting solution procedure. Then I shall shift the focus of attention to the field of adding and subtracting with numbers from 20 up to 100. I will deal with the influence of the type of operation (adding or subtracting), and of contexts on the calculation procedures with these larger numbers. It is here were the importance of the empty number line as model for mental arithmetic enters. Finally I shall make a few suggestions for research aimed at improving mental arithmetic in the field of 20 up to 100.

2 Study 1: direct subtraction and indirect addition

The influence of the semantic structure and number characteristics of simple word problems has been studied extensively (Carpenter, Hiebert, and Moser, 1983; Carpenter and Moser, 1984; De Corte and Verschaffel, 1987; De Corte, Verschaffel, and De Win, 1985; Jaspers and Van Lieshout, 1994a; Van Lieshout, Jaspers, and Landewé, 1994). In most studies simple addition and subtraction problems below 20 have been used. For the present discussion the problems of which an example is given in Table 1 are relevant. The Change (Ch) 1 through 4 problems were presented to children in many studies. This also holds for the Combine (Cb) 1, 2, and 3 problems, although the Cb-3 in the literature is known as Cb-2. So the confusing fact is that there are different Cb-2 problems, possibly because in their often cited study Riley, Greeno and Heller (1983) distinguished only two types of combine problems, viz. the Cb-1 and the Cb-2 (the Cb-3 in Table 1). Their argument, not to distinguish six types of combine problems as they did for the change and compare word problems, was that the combine situation consisted of only a whole and two equivalent parts.

Therefore only two semantically different problems were possible: one in which the whole is unknown and one in which either of the two parts is unknown. However, the Cb-2 and Cb-3 problems are often solved differently, which will be shown hereafter.

Type	Example
Change (Ch)	
Ch1	Mary had 6 apples. Mary got 2 apples more. How many apples does Mary have now?
Ch2	Mary had 6 apples. Mary gave 2 apples away. How many apples does Mary have left?
Ch3	Mary had 2 apples. Mary got some apples more. Now Mary has 6 apples. How many apples did Mary get?
Ch4	Mary had 6 apples. Mary gave some apples away. Now Mary has 2 apples. How many apples did Mary give away?
Combine (Cb)	
Cb1	Mary has 2 apples. Peter has 6 apples. How many apples do Mary and Peter have together?
Cb2	Together Mary and Peter have 6 apples. Mary has 2 apples. How many apples does Peter have?
Cb3	Mary has 2 apples. Peter also has some apples. Together Mary and Peter have 6 apples. How many apples does Peter have?
Compare (Cp)	
Cp1	Peter has 2 apples. Ann has 6 apples. How many more apples than Peter does Ann have?
Cp2	Ann has 6 apples. Peter has 2 apples. How many apples less than Ann does Peter have?

table 1: examples of word problems

The change 1 and combine 1 problems are, depending on the level of the child's skill and the availability of manipulatives, solved by a direct modeling counting-all strategy, a counting-up-from-one-of-the-given-numbers strategy or by applying a known additive fact (Carpenter, Hiebert, and Moser, 1981; De Corte and Verschaffel, 1987). The other problem types of Table 1 are subtraction problems.

To illustrate the findings regarding subtraction problems from several studies, some yet unpublished results of our own work (Van Lieshout et al., 1994) will be presented, which will also show the resemblance of the Ch3 to the Cb3. The first aim was to show that subtraction word problems, which are the same in the magnitude of the given numbers (below 10), but differ in the semantic structure and in the order of small and large numbers, have a different impact on the problem solving strategies in terms of direct subtraction and indirect addition. The set of problems for which clear predictions in terms of these two strategies can be made are the Ch3, Cb3, Cp1, Cp2, and Ch2 (see Table 1). Knowing which word problems trigger off indirect addition can be of help in devising a mathematics curriculum in which indirect addition is taught. The second aim was to show that the indirect addition strategy is at least an equivalent alternative to direct subtraction considering the expected success of the strategy in finding the correct solution.

With regard to the first aim, which was demonstrating the influence of the semantic structure and the order of the numbers on the choice of a strategy, a closer look at the different subtraction word problem types is necessary. The Ch3 problems describe chronologically the unknown change from a start set to an end set. Therefore it was expected that in accordance with many other studies indirect addition starting from the start set is the favourite procedure. The fact that the first number mentioned in the problem text is the smaller one of the two could also be the reason to start with the first number. The Cb3 problems do not contain a description of a changing situation. However, as in the Ch3 problems, the first number is the smaller one of the two, whereas, like the Ch3 problems, the unknown one is situated between the first and second number. The structure of both problem types could be adequately described with a number sentence with the format: small + unknown = large.

Contrary to the Ch3 problems, the Ch2 problems describe chronologically how a start set is decreased with a known change. Therefore it was hypothesized that in agreement with many other previous studies, direct subtraction, starting from the start set, is the favourite procedure. The order of numbers, first the larger one, then the smaller one and finally the unknown one, could also be the reason for the application of a direct subtraction procedure. The same holds for the Cp2 problems. First the larger number is given, next the smaller one, and finally in the last sentence the unknown quantity is described. The corresponding number sentence that describes the structure of both problems best is: large – small = unknown. The choice of strategies in case of the Cp1 problems is more difficult to predict than in case of the Cp2 problems. To be sure, the position of the smaller and larger numbers are the same as in the Ch3 and Cb3 problems, however, the position of the unknown one is different: not between the two numbers but after the two numbers such as Ch2 and Cp2 problems.

Of course these are not the only possible subtraction problem types. For other problem types it is more difficult to predict the relative use of both strategies. They are also much less investigated and besides, they are more difficult due to linguistic complexities and therefore less important for the purpose of strategy triggers in a teaching situation. Perhaps there is one exception: the Change 4 problem type (see Table 1). This is not a difficult problem, but because of its structure it will be more a problem that due to its semantic and syntactic form (large - unknown = small) triggers off an indirect subtraction strategy starting from the start set.

In sum, in the set of subtraction problems discussed, both semantic and syntactic factors are assumed to be operative in influencing the choice of strategy. The semantic factor bears on the chronological description of a change in quantity whereas the syntactical factor pertains to the order of the smaller and larger number and the unknown one. The hypothesis was that when both factors are in agreement with a small + unknown = large format the children will choose an indirect addition strategy, and when both factors correspond to a large - small = unknown form the children will opt for a direct subtraction strategy.

The second aim was to show that the indirect addition strategy is an effective method in finding the correct solution to different types of subtraction problems. Fuson (1992) gave several reasons why counting up to (indirect addition) would be easier than counting down (direct subtraction). The original study was not devised to compare the effectiveness of the counting up to and counting down procedures. However, one could wonder whether the indirect addition and direct subtraction procedures in general, which in fact denote several procedures varying from direct modelling with materials up to number facts usage, would also differ in effectiveness.

2.1 Method

2.1.1 Participants

One hundred and seven participants of two regular primary schools and two special schools for mildly mentally retarded (MMR) children in an urban area participated in the study. They were selected by asking schools for children with sufficient reading and computational skills. A reading level of at least level four of the AVI (a Dutch technical reading test) was required, which is approximately the reading level of children halfway grade two and which means that fluently reading the used word problems aloud was possible. Furthermore they had to be able to add numbers with a sum of 10 or smaller and to subtract from a number not larger than 10, irrespective whether the answer was produced by rote or by counting. In their classes the children received a paper-and-pencil test consisting of 24 simple addition and subtraction word problems and the Standard Progressive Matrices (Raven, 1960), a non-verbal test for intellectual development.

Based on their individual score on the word problem test as much as possible pairs of participants were formed with a difference of one at most in the total score. One of the pair was a MMR child the other one a nonretarded (NR) child and both were of the same sex. In order not to loose some low scoring participants in two pairs of the lowest scoring children a difference in score of 4 was allowed. As a result of this matching procedure 23 MMR and 23 NR children (28 boys and 18 girls) remained for more extensive testing and observation. Some children were not always available for testing, therefore the actual number, depending on the data that were used, was somewhat lower.

The mean proportion correctly solved word problems in the matched groups was. 57 (*SD* = .28) for the MMR and .55 (*SD* = .27) for the NR children. The group means did not differ: t (44) = 0.18, $p > .05$. Also the mean raw score on the Raven Standard Progressive Matrices of the children of the matched groups did also not differ: t (39) = 0.05, $p > .05$, $M = 29.8$ (*SD* = 8.1) for the MMR and $M = 30.1$ (*SD* = 8.6) for the NR children. These mean scores are comparable with approximately the 70th percentile for the NR and the 20th percentile for the MMR children (according to the 1979 British norms, see Raven and Summers, 1986). The Raven scores of one NR child and three MMR children were not available. Matching MMR and nonretarded children resulted in a significant age difference of 4.4 years: t (43) = 30.23, $p < .0001$. The MMR children had a mean age of 12.1 years whereas the mean age of the other children was 7.7 years.

The two regular schools used realistic math methods (*Operatoir Rekenen* and *Wereld in Getallen*), the two special education schools used more traditional methods (*Zo Reken Ik Ook* and *Niveaucursus Rekenen*).

2.1.2 Material and procedure

Fourteen of the 24 problem types in the paper-and-pencil test, used for the selection of the subjects, were the same 14 types of word problem types as described by De Corte and Verschaffel (1986) and Riley et al. (1983). So the 14 problem types consisted of the three semantic main categories Change (six problem types), Combine (two problem types), and Compare (six problem types). These problem formulations differed in two aspects from the original problem texts of De Corte and Verschaffel (1986) and Riley et al. (1983). Whereas, originally, two persons' names had been used in the change types, we only used one name. An example of the original problem is:'Mary had 6 apples. She gave 2 apples to John. How many apples does Mary have left?'. First we changed this type into:'Mary had 6 apples. *Mary gave 2 apples away.* How many apples does Mary have left?'. (The modified sentence is italicized.) Secondly, in our word problems, all pronouns were replaced by the names to which they referred.

The remaining ten problem types were the first ten of the same standard types as described before, however, they contained distracting, irrelevant information. Since these problems are not relevant for the present discussion, no further details will be given. A description of these problems and the results of it can be found in Van Lieshout et al. (1994). As mentioned before for the present presentation only the Ch3, Cb3, Cp1, Cp2, and Ch2 are relevant. Also some results regarding the strategies used to solve the Ch4 problem type are presented as a contrast to the five main problem types.

Immediately after the first paper-and-pencil session, children of both school types were selected and matched for individual testing and observation. A second and third test session respectively occurred approximately two and four months after the first session. The first test took place in December and January. Each individual test session consisted of administering the 14 standard word problem types on separate cards at random. However, the problems were presented with and without material. Material consisted of either cubes or a 'number line' without numbers but with marks. The child could mark positions on this line such as the number from which he or she intended to start counting. In the individual test session the use of modelling, counting and the use of number facts was recorded. A word problem was never immediately followed by the same problem type in an other material condition.

The numbers used in all word problems met strict criteria: All numbers and the correct answer were larger than 0 and smaller than 10, each of the two or three given numbers were different, and the correct answer or an incorrect answer computed by adding or subtracting any combination of the given numbers was not allowed to be the same as one of the given numbers. Within each test version persons' names, object names and numbers were unique for each word problem. Across the test versions, persons' names, object names and numbers were randomly varied.

The strategies were scored into the same categories as used by De Corte and Verschaffel (1987). They made a distinction between material, verbal and mental strategies. Within these main categories several strategies for the addition and subtraction problems were distinguished separately. The material subtraction strategies scored were: (1) Separating from the larger set of objects until the smaller number of objects is removed and counting the remaining set, (2) separating to the smaller number of objects by removing objects from the larger set of objects and counting the number of objects removed, (3) adding on objects to the smaller set of objects until the larger number is attained and counting the added objects, and (4) matching the objects of the smaller set to objects of the larger set and counting the number of unmatched objects in the larger set. The verbal strategies scored were: (1) Counting down from the larger number with the smaller number as the number of counting words and stating the last counting word as the answer, (2) Counting down to the smaller number, starting with the larger number and counting the number of counting words, and (3) counting up to the larger number, starting from the smaller number and counting the number of counting words. The categories of the mental

$\eta^2 = .21$. The Problem type × Strategy interaction was analysed with a contrast between the polynomial trend in the problem type effect and the contrast between indirect addition and direct subtraction strategy (contrasts: 1, -1, and 0 for respectively indirect addition, direct subtraction, and indirect subtraction). The polynomials were fitted in a sequence of problem types that was the same as the order of problem types in Figure 1. Corroborating the relations depicted in Figure 1 and in agreement with the hypothesis this analysis revealed that the difference in proportion of problems solved by either the indirect addition strategy or the direct subtraction strategy differed significantly on the linear trend in the problem types, $F(1, 42) = 267.16$, $p = .000$, $MSE = 0.10$, $\eta^2 = .86$. (Some higher order polynomials also interacted with the contrast between addition and indirect subtraction.)

It is clear that Ch3 problems are predominantly solved by an indirect additive strategy and Ch2 problems are mainly solved by a direct subtraction strategy with the other problem types somewhere in between. Note that the results of the Cb3 problems resemble those of the Ch3 problems. To give a clearer picture of the rather specific relation between problem type and preferred strategy, Figure 1 also shows the proportion of strategies used to solve the Ch4 problem type. It shows that the indirect subtraction strategy, which is hardly used with the other problem types, is often applied in case of a Ch4 problem.

The significant Group × Strategy interaction was analysed while using the previously mentioned contrast between indirect addition and direct subtraction strategy. The difference between the proportion of the use of indirect addition and direct subtraction strategy differed significantly in both groups, $F(1, 42) = 11.08$, $p = .002$, $MSE = 0.11$, $\eta^2 = .21$. The MMR group was less inclined to use the indirect addition strategy than the direct subtraction strategy (see Figure 1), whereas the reverse was true for the NR group.

2.2.3 Strategies and performance

In order to establish the influence of type of strategy on the success of finding the correct answer, multiple regression analyses were performed with the proportion correctly solved word problems per type as criterion variable. Due to the low frequency of occurrence of the indirect subtraction strategy in the present problem types this strategy was left out of the regression analyses. In each analysis controlling variables as age, the raw score on Raven's non-verbal intelligence test, and the condition of learning difficulties were always entered first. The direct subtraction strategy was entered next and finally the contribution of indirect addition was added. Tables 2 and 3 show the results of the analyses. From Table 2 it can be concluded that the frequency of using the indirect addition strategy explained a significant part of the variance in the number of correct solutions. This even applies if the problem type mainly triggers off direct subtraction (Ch 2, Cp 2, see Figure 1), although in those cases the contribution to the total variance was small (see ΔR^2 in Table 2). Table 3 shows that after performing the last step in the regression analyses both strategies, direct subtraction and indirect addition, contributed significantly to variance in the number of correct answers, although the size of the βs depended on the match of the nature of the proble and the strategy used.

Step	R^2	ΔR^2	ΔF	p
Change 2				
1	.17	.17	2.40	.08
2	.72	.56	70.04	.000
3	.79	.06	9.97	.003
Change 3				
1	.07	.07	< 1	.46
2	.07	.00	< 1	.94
3	.89	.71	111.73	.000
Combine 3				
1	.01	.01	< 1	.94
2	.03	.02	< 1	.39
3	.73	.70	87.27	.000
Compare 1				
1	.02	.02	.25	.86
2	.22	.20	8.68	.006
3	.86	.65	157.86	.000
Compare 2				
1	.14	.14	1.89	.15
2	.38	.24	13.67	.001
3	.47	.09	5.50	.025
Note. Step 1: Age, Raven, and Group (possession of mental retardation). Step 2: Use of direct subtraction. Step 3: Use of indirect addition.				

table 2: summary of hierarchical regression analyses for variables predicting correct answers for each word problem type ($N = 40$) I: explained variances

Variable	β	*SE B*	β
Change 2			
Age	0.00	0.00	.14
Raven	0.00	0.00	.03
Group	0.03	0.15	.08
Direct subtraction	0.89	0.10	.94****
Indirect addition	0.69	0.22	.29**
Change 3			
Age	0.00	0.00	.12
Raven	0.00	0.00	.02
Group	0.06	0.16	.13
Direct subtraction	1.08	0.17	.69****
Indirect addition	0.91	0.09	1.00****
Combine 3			
Age	0.00	0.00	-.42
Raven	0.00	0.00	.04
Group	-0.16	0.20	-.35
Direct subtraction	0.94	0.14	.78****
Indirect addition	0.91	0.10	1.00****
Compare 1			
Age	0.00	0.00	-.44
Raven	-5.86	0.00	-.02
Group	-0.25	0.16	-.49
Direct subtraction	0.94	0.08	.84****
Indirect addition	0.96	0.08	.91****
Compare 2			
Age	0.00	0.00	-.47
Raven	0.00	0.00	.08
Group	-0.19	0.24	-.48
Direct subtraction	0.52	0.12	.76****
Indirect addition	0.85	0.27	.35*

*$p < .05$; **$p < 0.1$; ****$p < .0001$.

table 3: summary of hierarchical regression analyses for variables predicting correct answers per word problem type ($N = 40$) II: parameters for each predictor in the last step

When the regression analyses for each material condition (without material, cubes and number line) were separately done the results were essentially the same as with the combined analyses.

2.3 Discussion

Subtraction problems with an additive structure viewed from a semantic and syntactic perspective, the Change 3 and Combine 3, were mostly solved with an indirect addition strategy. Subtraction problems with a subtractive structure, the Change 2 and the Compare 2, were mostly solved with a direct subtraction strategy. As shown before in many other studies children spontaneously applied an additive strategy to certain subtractive problems instead of a direct subtraction procedure, which has been taught in many mathematics curricula for many years. Obviously, the semantic and syntactic structure of the word problem is very compelling in the child's choice for a solution procedure. A striking fact is the resemblance between the Combine 3 problem and the Change 3 problem. Both of them strongly triggered the indirect addition strategy. De Corte and Verschaffel (1987) also showed that the Combine 3 was mainly solved by an indirect addition strategy. Since the semantic structure of the Combine 3 problems contains no clue in such direction, it must have been the location of the smaller, unknown, and larger number in the problem text (the syntactical structure) that caused this choice. The resemblance between the Change 2 and Compare 2 problems in triggering a direct subtraction strategy also seems to be a result of the position of the known and unknown numbers in the text and less the result of a correspondence in semantic structure.

The Change 3 and Combine 3 problems are not the only problem types that trigger an indirect addition strategy. Also some equalize problem types possess this quality. For example, Carpenter and Moser (1984) already showed that what at the time they called a Join Missing Addend problem was mainly solved by adding-on and counting-up-from given strategies. The original problem read: 'Kathy has 3 pencils. How many more pencils does she have to put with them so she has 15 pencils altogether?'

Interestingly, in the present study the problem type strategy effect was different for the NR children than for the MMR children. The latter group of children used more direct subtraction and less indirect addition strategies than the former group. It is difficult to understand how this can be a result of having or not having learning difficulties. Another explanation is based on the fact that the children with MMR were much older than the other children. As a result the children with MMR surpassed the other children in the number of years they received mathematics instruction. Perhaps many years of (traditional) mathematics instruction changes this strategy effect. (The effect cannot be attributed to higher competence acquired by the longer period of mathematics instruction the children with MMR got. For the two groups were matched on their word-problem solving performance.) This length-of-schooling explanation seems to be confirmed by yet unpublished results of the Beishuizen group (De Joode, 1996). Students of grade 3 used the indirect addition strategy less frequently and the direct subtraction strategy more frequently to solve Change 3 and 2 problems than the students of grade 4. (This effect could be the result of more flexible solution strategies as students get more mathematics education and less automatically choose the strategy that corresponds with the structure of the word or context problem, but that explanation is contradicted by the results of De Joode.)

Probably the teaching habits or curriculum contents for the students in both studies (Study 2 and De Joode) were biased towards direct subtraction.

Generally, the analysis of word-problem solving performance in the present study showed that the problem types did not differ from each other in this respect. So the problems that were mainly solved by indirect addition were not more often solved correctly than the problems that were mainly solved by a direct subtraction strategy. However, in this study no instruction was given to improve indirect addition or direct subtraction procedures. If instruction had been given, application of the indirect addition procedure could have resulted in more correctly solved subtraction problems. Fuson (1986; Fuson and Fuson, 1992) showed that teaching children a counting-up-from given strategy for subtraction problems raised their performance to the level of addition problems, although earlier the school had put considerable effort into improving counting down.

Nevertheless, in the present study the degree to which indirect addition was used appeared to be a good predictor of subsequent success in finding the correct answer, even in the case of subtraction problems that mainly trigger an direct subtraction strategy. This result shows the importance of indirect addition as a valuable alternative to direct subtraction in the domain of small numbers. However, it should be noted that the use of a direct subtraction strategy also appeared to be a good predictor of success.

This does not imply that the Change 3 or Combine 3 are always easier to solve than a bare number sentence in a canonical format. (A number sentence in a noncanonical format such as 3 + ? = 8 cannot be solved until a child is taught the meaning of that number sentence.) Because of linguistic obstacles the child can have problems with building a correct representation of the problem situation. Cummins, Kintsch, Reusser, and Weimer (1988) showed that the performance on verbal problems was considerable lower than on numerical problems. The form of the number sentences matched the structure of the word problems. However, children can be helped to overcome difficulties in understanding word problems (see e.g. Jaspers and Van Lieshout, 1994b; 1994c).

After having stressed the importance of the indirect addition strategy (adding on, counting up to and retrieving an indirect additive fact) I now will turn to the domain of mental arithmetic with two-digit numbers.

3 Study 2: mental arithmetic procedures to solve problems with numbers from 20 up to 100

Whereas in the area of word problem solving the solutions procedures can be described in terms of modeling, counting, and number facts, in the area of mental addition and subtraction from 10 up to 100 it seems that it is the calculation method that matters. The method of calculation appears to have a strong influence on the correctness of the answers, especially with subtraction problems (Beishuizen, 1993), which are generally more difficult than addition problems. In the study (Linssen, 1996) to be reported here we wanted, first, to substantiate Beishuizen's claim that some calculation methods are more successful than others and, secondly, to devise and test context problems that would trigger off the more successful strategy.

Beishuizen (1993; Wolters, Beishuizen, Broers, and Knoppert, 1990) studied extensively jump and decomposition procedures in mental arithmetic with addition and subtraction problems between 20 and 100. He called the decomposition procedures 1010 procedures, because the tens are added or subtracted separately from the ones. Take for example '52 – 28 ='. Both 52 and 28 are decomposed in tens and ones and the tens and ones are processed separately. The solution steps could then proceed as follows: First 20 is subtracted from 50 (50 – 20 = 30), next, after deciding that 8 cannot be subtracted from 2, 8 is subtracted from 12 (12 – 8 = 4) and finally the two results are added while compensating for the borrowing of one ten (20 + 4 = 24). Beishuizen (1993) coined the term N10 for the jump procedure, because the problem solver leaves the number that is the starting point of the calculation unsplit. The solution of the example problem '52 – 28 =' could proceed as follows: First 20 is subtracted from 52 (52 – 20 = 32) and finally 8 is subtracted from 32 (32 – 8 = 24) possibly with the decomposition of 8 (32 – 2 – 6 = 24). In line with the Beishuizen results we expected in this study 2, conducted by Linssen (1996), to find two main strategies: the 1010 and the N10 strategy.

Beishuizen (1993) showed that the N10 procedure is less prone to errors than the 1010 procedure. Especially subtraction problems with borrowing (such as the example given before) are less often solved correctly with the 1010 than with the N10 procedure, mainly due to the well-known 'smaller-from-larger' error (2 – 8 = 6, so 30 + 6 = 36 and '36' becomes the false answer). For addition problems it is less a matter of concern whether children use the N10 or 1010 procedure, in fact they use the latter procedure more often with addition than with subtraction problems (Beishuizen, 1993). In the present study, like Beishuizen we expected to find the 1010 procedure used with subtraction problems to be more risky, that is to say less successful, than the N10 procedure. For addition problems, we expected no difference in effectiveness of both procedures.

Beishuizen (1993) also showed that the N10 procedure is more often used in combination with subtraction problems than in combination with addition problems, whereas most arithmetic methods in the Dutch schools try to teach the application of the N10 both to addition and subtraction problems. Probably many children by themselves discover that in case of subtraction problems the N10 procedure is more safe than the 1010 and that in case of addition problems the 1010 procedure is just as appropriate as the N10 procedure. Therefore, in the present study it was expected that the N10 procedure would be applied more frequently to subtraction than to addition problems and that the opposite would hold for the 1010 strategy.

In order to explore the possibility of eliciting jumping strategies rather than decomposition strategies, contexts were developed that were either neutral or were expected to encourage jumping. This jumping-triggering quality was created in two different ways: the geometric and the number prestructuring method. The first method, the geometric method, was developed with taking into account the findings of Klein, Beishuizen, and Treffers (1995) concerning the effect of using the empty number line (Treffers, 1991). They taught primary school children to use the empty number line as a tool for solving addition and subtraction problems. This empty number line is considered to be a model for representing the numbers and the operations on numbers. The number line represents the numbers as counting numbers and therefore should fit in with early informal problem solving strategies as counting up and counting down. After counting ones, steps of tens could be the natural continuation of these strategies into the field of addition and subtraction of numbers from 10 up to 100. For example, '42 + 36 =' could be solved as '42 plus 10 is 52, plus 10 is 62, plus 10 is 72, plus 6 is 78'. Eventually, this ordinal orientation should support the establishment of the N10 jump strategy (Fig. 2). The 1010 decomposition procedure is incompatible with the number line because in this procedure the answer is not attained by a sequence of steps along the number line but instead requires parallel tracks for the ones and tens. The line should be 'empty' to prevent the children from reverting to primitive counting strategies and reading off the answer which occurs if the line is marked with numbers. For an extensive discussion of the empty number line, see Gravemeijer (1994).

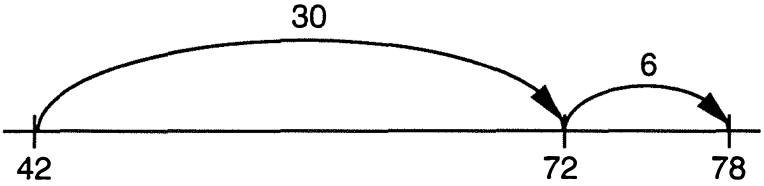


figure 2: solution of the problem 42 + 36 = with the jump procedure on the empty number line

According to the idea of the supporting role of a number line the first N10-triggering method consisted of presenting the children with problems depicting the situation with a scale-like representation. This representation was meant to support the solution as if a number line was present. And because a number line solution can be considered as incompatible with a 1010 procedure, in this condition more N10 strategies were expected. The second method, the number prestructuring method, consisted of prestructuring the problem in terms of the decomposition of the second number in tens and ones, whereas the first number was only presented as a whole number. If the problem was e.g. '52 – 28=', the context problem contained the numbers 52, 28, 20 and 8.

3.1 Method

3.1.1 Participants

Seventy-one students of grade 3 (Dutch groep 5) of two urban primary schools participated in the study. The number of participants in one of the schools was 29. In the other school the participants belonged to two different groups, one consisting of 14 the other of 28 students. Both schools used the Dutch *realistic* math method *Rekenen en Wiskunde.*

3.1.2 Procedure

The three groups participated in two test sessions. During the first session the children were presented with 18 number sentences (bare problems). During the second session they were presented with 18 context problems. During the first session the students were allowed to use 45 minutes to complete the test. For the second session one hour was available. When a child finished before the end of the session, he or she could do another task selected in agreement with the teacher in advance. During the instruction preceding the sessions the experimenter asked the children to write down their solution steps. To make sure that the children would write down the steps a practise session preceded the two test sessions. In this practise session the experimenter invited one of the children to write down his or her solution of an addition problem with carrying on the blackboard. If the steps were correctly written down, the experimenter asked whether there was another child that had solved the problem with other steps. If there was such a child, he or she was asked to write his or her steps on the blackboard. If there was not such a child the experimenter herself demonstrated the procedure. At least one N10 and one 1010 procedure were demonstrated by the students or experimenter. Next all children were allowed to practise with some problems the procedure of writing down the solution steps. In the meantime the experimenter checked the actions of each child. If she found a child who did not write down the solution steps, the experimenter asked the child to write down its solution on the blackboard and asked the other children to help this child.

In the two test sessions the experimenter referred to the practise session and again demonstrated the write-down procedure in the same way as in the practise session.

3.1.3 Materials

In both test sessions the same 18 problems were used. Half of the problems were addition problems with the correct answer below 100. The other half were subtraction problems with the minuends below 100. None of the units of the numbers were equal to 8 or 9 in order to prevent 'rounding' strategies. No numbers with a round ten value (e.g. '20' or '50') were used. All problems required carrying out. The problems were divided in six blocks of three problems each. On each page of the test booklet one block was presented. Three of the blocks contained one addition and two subtraction problems. The other three blocks consisted of one subtraction and two addition problems.

For expository reasons the context test for the second session is explained first. The context problems of session 2 consisted of an introductory sentence that gave a sketch of the situation and sometimes contained one of the given numbers. Between this sentence and the last sentence a picture was located that not only depicted the situation but also contained one or more essential numbers which were not contained in the text. There were three types of contexts: neutral, prestructured, and geometric. Each block contained one neutral, one prestructured, and one geometric context. Figure 3 shows examples of the three context types. The order of the three types of problems was counterbalanced by using all six possible sequences resulting in 18 problems. Half of the blocks consisted of addition problems whereas the other half consisted of subtraction problems. Table 4 shows the problems used (without the context). Another five versions of this test were constructed. These versions resulted from systematically moving the first two blocks to the back of the list of problems until the order of the first version was arrived at the top again. This action produced two new test versions. The remaining three versions were created by reversing the order of the first three sequences. The different versions were created to prevent the children from cheating and to control for order effects.

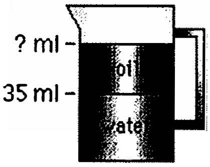
(a) Neutral
John has 52 guilders.

He buys this football.
How many guilders does John have left?
... guilders

(b) Prestructured
In the morning mother bought a box of chocolates.

In the evening 27 chocolates are missing in the box.
Rob has eaten 20 of them and Peter has eaten 7 of them.
How many chocolates are left in the box?
... chocolates

(c) Geometric
A measure contains water and oil.



27 ml oil floats on the water.
How many ml liquid the measure contains in total?
... ml

figure 3: examples of a neutral, prestructuring, and geometric problem

Neutral		Prestructured		Geometric	
36 + 25 = 54 + 37 = 37 + 15 =	52 – 34 = 45 – 16 = 72 – 45 =	47 + 16 = 56 + 35 = 57 + 25 =	45 – 26 = 55 – 27 = 32 – 17 =	35 + 27 = 57 + 26 = 47 + 34 =	92 – 35 = 34 – 16 = 85 – 26 =

table 4: addition and subtraction problems used in the three N10-triggering conditions

The six versions of problem orders were used twice: one for the bare problems of session 1 and the other one for the context problems of session 2. Thus the bare problem test of session 1 was constructed in the same way. Each child was presented with only one version of the list with bare problems and with only one version of the list with context problems. For the bare problems the distinction between neutral, prestructured, and geometric was meaningless save for the fact that each bare problem served as a control for the same problem in context format.

3.1.4 Scoring

The strategies were scored by the experimenter as '1010' (e.g.: 63 – 27 = ?; 60 – 20 = 40; 3 – 7 = ?; 13 – 3 = 10; 10 – 4 = 6; 40 – 10 = 30; 30 + 6 = 36), 'N10' (e.g.: 63 – 27 = ?; 63 – 20 = 43; 43 – 3 = 40; 40 – 4 = 36), including several variants, '10s' (starting as 1010 but switching to a N10-like procedure, e.g.: 63 – 27 = ?; 60 – 20 = 40; 40 + 3 = 43; 43 – 3 = 40; 40 – 4 = 36, and 'Other' (including the N10-like A10 procedure, e.g.: 63 – 27 = ?; 63 – 3 = 60; 60 – 20 = 40; 40 – 4 = 36). For a review of these procedures and their variants, see Beishuizen (1993), Beishuizen, Van Putten, and Van Mulken (1997) and Klein and Beishuizen (1994). The strategies of ten randomly chosen participants were scored again by a second observer in order to determine the degree of agreement between the two observers. Kappa attained the high value of .91.

3.2 Results

Table 5 shows the distribution of strategies. The proportion of problems solved with a 10s strategy was relatively low and was not used for the analyses. As can be inferred from Table 5 the proportion of problems solved with another strategy than N10, 1010, and 10s was close to zero. So the A10 procedure, which was scored into this 'Other' category, seldom occurred.

	Addition		Subtraction	
Strategy	*M*	*SD*	*M*	*SD*
N10	.52	0.45	.59	0.44
1010	.32	0.43	.20	0.35
10s	.11	0.28	.16	0.33

table 5 : proportion of strategies by operation

An overall alpha level of .05 was used for all statistical tests. All reported statistics of repeated measures ANOVAs were based on the multivariate method of analyzing.

3.2.1 Operation and strategy

A 2 (operation) × 2 (strategy) ANOVA yielded significant main effects, $F(1, 70) = 4.85$, $p = .03$, $MSE = 0.01$, $\eta^2 = .07$ respectively $F(1, 70) = 11.93$, $p = .001$, $MSE = 0.52$, $\eta^2 = .16$, and a significant interaction effect, $F(1, 70) = 8.81$, $p = .004$, $MSE = 0.08$, $\eta^2 = .11$. Simple effect analyses showed that the mean proportion of N10 strategies differed between the addition and subtraction problems, $F(1, 70) = 123.65$, $p = .000$, $MSE = 0.358$, $\eta^2 = .64$. The same was true for the mean proportion of 1010 strategies, $F(1, 70) = 37.07$, $p = .000$, $MSE = 0.26$, $\eta^2 = .35$, although the direction of the difference was opposite to the difference in the N10 strategies (see Table 5). So, in accordance with the expectation, the N10 procedure was applied more frequently to subtraction than to addition problems whereas the opposite was true for the 1010 strategy.

3.2.2 Strategy and performance 1

Based on the number of N10 strategies per student, the total group of students was devided into two nearly equally large groups: a N10 group ($n = 35$) and a non-N10 group ($n = 36$). In the N10 group always-respond-with-a-N10 strategy was the modal response category and occurred in 54% of all cases. The mean proportion of N10 strategies in this group was .94 (*SD* = 0.09). In the non-N10 group never-respond-with-a-N10 strategy was the modal response category and this strategy occurred in 44% of all cases. In this group the mean number of N10 strategies was .18 (*SD* = 0.23). The proportion of correctly solved problems was subjected to a 2 (strategy) × 2 (operation) ANOVA with Strategy as between subjects factor and operation as within subjects factor. Both main effects turned out to be significant, $F(1, 69) = 7.80$, $p = .007$, $MSE = 0.05$, $\eta^2 = .10$, respectively $F(1, 69) = 20.11$, $p = .000$, $MSE = 0.03$, $\eta^2 = .23$. The interaction between Strategy and Operation attained also statistical significance, $F(1, 69) = 14.14$, $p = .000$, $MSE = 0.03$, $\eta^2 = .17$. The interaction pattern is shown in Figure 4. Analyses of simple effects yielded a significant operation effect in the non-N10 group, $F(1, 70) = 34.84$, $p = .000$, $MSE = 0.03$, $\eta^2 = .33$, and a nonsignificant operation effect in the N10 group, $F(1, 70) = 0.17$, $p = .68$, $MSE =$

0.05, $\eta^2 = .00$. Clearly, as expected, subtraction problems were most difficult for the students who seldom or never used the N10 strategy, whereas their addition performance did not differ from the students who did use the N10 strategy.

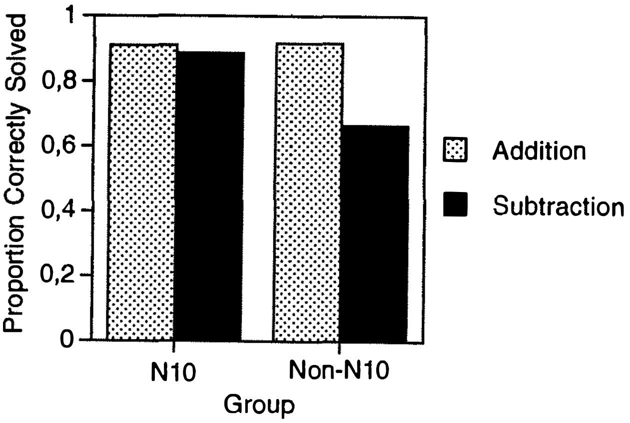


figure 4: proportion correctly solved problems by group (N10 users and Non-N10 users) and by operation (addition and subtraction).

3.2.3 Strategy and performance 2

The relation between strategy and performance can be studied from a slightly other viewpoint by analyzing this relation with the help of multiple regression analysis. This more easily enabled us not only to analyses the influence of the N10, but also the role of the 1010 strategy. The mean proportion correctly solved addition and subtraction problems were analyses by two separate regression analyses. The first analysis concerned the addition problems. It tested the influence of strategies on success in solving these problems. A hierarchical test procedure was applied. The proportion of N10 strategies was entered first in order to test the 1010 effect. As expected the 1010 strategy explained a significant part of the variance in the proportion of correctly solved addition problems (Table 6). The N10 strategy did not add a unique contribution. However, this analysis is limited in its value because of a near-ceiling performance with addition problems (M = .91, *SD* = 0.12, compare with subtraction problems: M = .77, *SD* = 0.29).

Variable	β	*SE* β	β
Step 1			
N10 strategy	-0.02	0.03	-.08
Step 2			
N10 strategy	0.05	0.05	.19
1010 strategy	0.10	0.05	.36*
Note. R^2 = .01 for Step 1(p = .53); ΔR^2 = .06 for Step 2 (p = .046). *p = .046.			

table 6: summary of hierarchical regression analysis for strategies predicting correct answers to addition problems (N = 71)

The second analysis (Table 7) tested the influence of N10 strategies on success in solving subtraction problems. A hierarchical test procedure was applied in which the proportion of 1010 strategies was entered first. As Table 7 shows the proportion of N10 strategies explains a highly significant part of the variance in the proportion correct answers after controlling for the influence of the 1010 strategy. The proportion 1010 strategies itself lost its significant negative influence in the first step after the proportion of N10 strategies was controlled for. A plausible explanation of the negative beta in step 1 is that successful problem solving occurred mainly after applying a N10 strategy. Applying a N10 strategy precluded the occurrence of a 1010 strategy. So, a negative relation between using 1010 and success was forced, which was removed by controlling for the use of N10 strategies.

Variable	β	*SE* β	β
Step 1			
1010 strategy	-0.21	0.10	-.26*
Step 2			
1010 strategy	0.04	0.12	.05
N10 strategy	0.32	0.09	.48**
Note. R^2 = .07 for Step 1(p = .03); ΔR^2 = .14 for Step 2 (p = .001). *p = .03; **p = .001.			

table 7: summary of hierarchical regression analysis for strategies predicting correct answers to subtraction problems (N = 71)

In sum, the results of both regression analyses are largely in agreement with the hypothesis: For addition the 1010 strategy is successful, whereas for subtraction the

N10 strategies proves to be the best strategy. The failing effectiveness of the N10 strategy for addition problems was not expected.

3.2.4 Contexts and N10

The number of N10 strategies was subjected to a 2 × 2 × 3 (Operation × Context × Problem type) repeated measures ANOVA. 'Context' refers to the nature of the problem: context or bare. 'Problem type' refers to the three problems that either belonged to the geometric, prestructured or neutral category. Only the operation effect was significant, $F(1, 70) = 4.40, p = .04, MSE = 2.02, \eta^2 = .06$. However, the experimenter noticed that in one of the classes after reading and viewing the context the majority of the children first wrote down a number sentence before they started to write down the solution steps. Perhaps this was induced by an intervening remark of the teacher who asked the children to write down a number sentence before starting to solve the problem. Therefore it was decided to enter the distinction between children who immediately wrote down the number sentence (the 'decontextualisers') and those who started to write down the solution steps (the 'contextualisers') as a between subjects design factor in the analyses. Those children that in 90% of all context problems immediately wrote down a number sentence formed the decontextualisers group. Nearly all the children from the class in which the 'decontextualising' was observed became member of this group. The mean proportion of decontextualised context problems in this group ($n = 29$) was .98 ($SD = 0.03$). The mean proportion of decontextualised context problems in the remaining group ($n = 42$) was .03 ($SD = .11$).

The ANOVA was repeated with 2 × 2 × 2 × 3 (Group × Operation × Context × Problem type) design with repeated measures on the latter three factors. Again, the operation effect was significant, $F(1, 69) = 4.90, p = .03, MSE = 2.03, \eta^2 = .07$. The proportion N10 was higher in the solutions of the subtraction problems than in the solutions of the addition problems ($M = .87$, respectively $M = .76$). The a priori contrast between both N10 trigger conditions (geometric and prestructured) and the neutral condition interacted significantly with Group, $F(1, 69) = 4.05, p = .048, MSE = 0.12, \eta^2 = .06$. No other contrasts between the trigger conditions interacted significantly with the group effect. Figure 5 shows the interaction effect. In this figure the proportion of N10 solutions with bare problems has been subtracted from the proportion N10 solutions with context problems. Furthermore, the data of the two N10 triggering conditions have been combined into one category. The analyses of simple effects within both groups did not reveal any further significant contrast effects, Decontextualisers: $F(1, 70) = 1.17, p = .28, MSE = 0.12, \eta^2 = .02$, Contextualisers: $F(1, 70) = 2.01, p = .16, MSE = 0.12, \eta^2 = .03$. Seemingly, together, the nonsignificant hypothesised effect (in the nondecontextualisers group) and the nonsignificant unexpected reversed effect in the decontextualisers group, are responsible for the significant interaction.

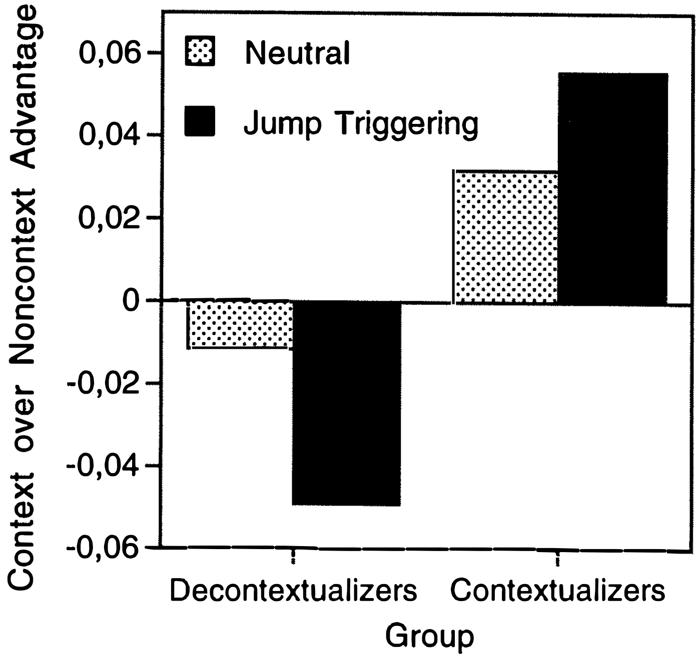


figure 5: advantage (expressed as difference in proportion of N10 strategies) in jump-triggering quality of neutral and jump-triggering context problems over bare numerical problems (noncontext problems) for the decontextualisers and the contextualisers group.

3.3 Discussion

The present study showed that children applied mainly two strategies: the N10 or jump strategy and the 1010 or decomposition strategy. As predicted, the N10 procedure was applied more frequently to subtraction than to addition problems whereas the opposite was true for the 1010 strategy. Children who were the most frequent N10 users performed better in subtraction than children who were the least frequent N10 users. Furthermore, the more the children applied a N10 strategy to subtraction problems the more problems they solved correctly. The 1010 strategy appeared not to be a predictor of success in finding the correct answer. The N10 strategy indeed seems to be a more safe strategy to solve the most difficult problems in the area of adding and subtracting up to 100: viz. subtraction problems with borrowing.

It also seemed that the reverse is true for addition problems. The more the children used a 1010 with addition problems the more successful they were in solving these problems, whereas the N10 did not have any effect. This finding seems to contradict the general usefulness of the N10 strategy as claimed e.g. by Beishuizen

(1993). If a person applies a 1010 strategy to addition problems and a N10 strategy to subtraction problems this perhaps testifies flexibility and insight into the efficiency of procedures and the difficulties of certain problems. So this could be a warning to oversimplify instruction and to withdraw any instructional support to the use of the 1010 strategy. On the other hand it has been remarked that perhaps a ceiling effect of the number of correctly solved addition problems has distorted the real relations between the two types of strategies and the correctness of the solutions. Future research has to address this question by including as participants children who are less able in solving addition problems.

Except for the latter case all foregoing conclusions support the work of Beishuizen (1993). The results of the present analysis are virtually identical to his results. This gives support to the idea to stimulate the application of the N10 strategy to subtraction problems, as Klein, Beishuizen, and Treffers (1995) have already done. However, the causal role of the N10 and 1010 strategies in attaining success or failure would be more firmly established by a well controlled (teaching) experiment than by the present correlative approach with existing strategy groups.

The results of the attempt to trigger off N10 strategies by means of especially designed contexts were disappointing. There was indeed a context effect but it was only present after splitting up the group in students who seemed to first decontextualise the context into a number sentence and children who did not. However, it was not clear whether this effect was caused by the expected effect in the latter group or the unexpected inverse effect in the former group. Besides, the relevant effect sizes were very small. A possible explanation for this disappointing result is that the children were already very used to their own strategic choices and not sensitive to contextual influences. Once again this would be a plea for having participating less experienced children in the study. Another explanation is that the context was not powerful enough and can not be made more powerful because the children have to learn to use the number line before they can profit from it. Gravemeijer (1994) pointed to several problems that children encounter when using the empty number line for the first time. So, perhaps one cannot expect immediate effects of our types of contexts.

4 General discussion

In study 2, which was focussed on mental arithmetic up to 100, no subtraction contexts that could trigger off an indirect addition strategy were used. However, Beishuizen et al. (Beishuizen, et al., 1994; Klein and Beishuizen, 1994) did present children with several context problems e.g. with the Change 3 structure. Problems like these appeared to trigger off much more indirect addition than direct subtraction strategies. De Corte and Verschaffel (1987, p. 376, Table 7) already showed results from which can be deduced that the predominant mental strategy to solve Cb-3 problems was the retrieval of an indirect additive derived fact, e.g. 5 + ? = 12: '5 plus 5 equals 10 and 10 plus 2 equals 12, so the answer is 2 plus 5, which equals 7'. The findings of Beishuizen et al. are in agreement with the many studies (such as Study 1) in which this phenomenon of indirect addition in the field of arithmetic up to 10 or 20 manifested itself. In Beishuizen's results the context problems also proved to trigger off more indirect addition than the bare problems.

Another important finding of Beishuizen et al. (Beishuizen, Torn, and Klein, 1994; Beishuizen, Van Putten, and Van Mulken, 1997; Klein and Beishuizen, 1994) was that in the solutions of the context problems a N10-like variant emerged. This variant, the A10 strategy 3, starts like the N10 strategy proper with the smaller number. However, the first step consists of 'filling up' the number to the nearest ten. Now it is time to look at the solution steps in the N10, 1010 and A10 procedures used to solve subtraction problems in a direct subtraction and an indirect addition way in more detail. Table 8 shows a few examples. In this table the procedures described are based on the aforementioned work of Beishuizen et al. and can be considered as a start to an empirical task analysis (Resnick and Ford, 1984).

Direct Subtraction 63 - 27 = .		Indirect Addition 27 + . = 63
63 - 20 = 43 43 - 7 = ? 43 - 3 = 40 7 consists of 3 and 4 40 - 4 = 36	N10	27 + 30 = 57 57 + ? = 63 57 + 3 = 60 60 + 3 = 63 30 + 3 = 33 33 + 3 = 36 'Overshoot' variant: 27 + 40 = 67 67 is -4 beyond 63 40 - 4 = 36
Variant 1: 63 - 3 = 60 60 - 20 = 40 7 consists of 3 and 4 40 - 4 = 36 Variant 2: 63 - 3 = 60 7 consists of 3 and 4 60 - 4 = 56 56 - 20 = 36	A10	27 + 3 = 30 30 + 33 = 63 3 + 33 = 36

Direct Subtraction 63 - 27 = .		Indirect Addition 27 + . = 63
60 – 20 = 40 3 – 7 = ?* 13 – 3 = 10 7 consists of 3 and 4 10 – 4 = 6 40 – 10 = 30 30 + 6 = 36	1010	20 + 30 = 50 7 + ? = 13 7 + 3 = 10 10 + 3 = 13 3 + 3 = 6 6 + 30 = 36 'Overshoot' variant: 20 + 40 = 60 7 + ? = 3* 40 – 10 = 30 or: 20 + 30 = 50 7 + ? = 13 7 + 3 = 10 10 + 3 = 13 3 + 3 = 6 6 + 30 = 36
Note. In these examples it is assumed that the problem solver is not yet able 'to go through ten' without splitting up the units into two parts (e.g. '7 consists of 3 and 4'). *Pitfall, often wrongly handled by answering with the well-known smaller-from-larger error, in this case '4' (Beishuizen, 1993).		

table 8: examples of procedures to solve a subtraction problem with trading

As can readily be seen from Table 8, the 1010 procedure is very bothersome, especially in case of indirect addition: there are several interim answers and many steps, some of them really risky (Beishuizen, 1993). Contrary to the 1010 procedure the A10 procedure requires only a few steps in case of indirect addition, whereas the N10 procedure also compares favorably with the 1010 procedure. There are several considerations in determining which one of the procedures is psychologically the most desirable. Probably not only the steps to be taken and the amount of information (such as interim answers) that have to be kept in the working memory (Wolters et al., 1990) are import factors, the generality of the procedures across addition and subtraction problems with and without trading seems to be important also. So, more research is needed to investigate the relative importance of these factors. For the moment it is important to know that the research of Beishuizen et al. (Beishuizen et al., 1994; Beishuizen et al., 1997) has shown that the children who solve a subtraction problem with the indirect addition strategy and the A10 procedure have a very low error rate. Moreover, Beishuizen et al. (1997) showed that children who switched to the A10 procedure performed much better than when they continued to use the 1010 procedure.

As said before, the 1010 procedure cannot be solved on a number line. This is because the 1010 procedure lacks the continuous movement from the starting number along the number line, which the N10-like procedures (including A10) as ordinally-oriented procedures do possess. This is a strong argument in favour of using the number line as a model for the mental solution. Even if the child never attains the level of an 'interiorised number line', he or she can quickly draw a raw sketch of the solution on the number line, as (originally) poorly performing children according to Beishuizen et al. (1994) indeed do.

In sum, first, we know that children, depending on the semantic and syntactic structure of the word problem, often solve subtraction problems spontaneously with indirect addition (Study 1 and previous research of others, e.g. Carpenter and Moser (1984), De Corte and Verschaffel (1987). Secondly, the first point also holds for subtraction up to 100 (Beishuizen et al., 1994; Klein and Beishuizen, 1994). Thirdly, when indirect addition is taught the children's performance with small numbers approaches the traditionally higher performance on addition problems (Fuson and Fuson, 1992). Fourthly, in the field of mental arithmetic subtraction problems with trading are more successfully solved by children who use N10-like procedures instead of the 1010 procedure (Study 2 and Beishuizen, 1993), notably the A10 procedure seems to result in very few errors (Beishuizen et al., 1997). Fifthly, context or word problems with an indirect addition-triggering quality are also triggering the most A10 procedures. Finally, using the number line is incompatible with the 1010 procedure. In view of these conclusions it seems worthwhile to consider the development of new methods in teaching children subtractions.

Building on the work of Klein, Beishuizen, and Treffers (1995) I propose that a mathematics instruction program will be developed and evaluated in which, especially for poorly performing children, the empty number line is used as a model to learn to use N10-like procedures such as A10. In this curriculum context and word problems should be used to trigger off indirect addition. It would be interesting to use the empty number line at the moment children are still counting, instead of postponing its use until the introduction of addition and subtraction of two-digit numbers. The number line used in Study 1 does not permit any conclusions regarding the effect of the empty number line, because the number line used was not really empty. If children are early acquainted with the use of the empty number line, perhaps they have the chance to link their ordinal solution strategies with small numbers smoothly to ordinal solution strategies (N10, A10) with larger, two-digit numbers.

However, there is pedagogical paradox. One would like to have the children use the effective combination of the indirect addition procedure and the A10 or N10 procedure. It would certainly be preferable that children apply an indirect addition strategy, even when the context is e.g. a Change 2 problem but with a small difference between the numbers (e.g. 63 – 57 =). However, one would also like that children would switch to direct subtraction when the subtrahend is small (e.g. 63 – 7 =). So while at first the context problems could be used as vehicles to provoke indirect addition, later on the students should be freed from the compelling nature of context problems and apply whatever procedure seems most effective to them.

How can we give students this freedom of strategy? Perhaps context or word problems can even play a role in solving the paradox. A problem such as the Combine 3 (see Table 1) can be used to trigger off indirect addition. Perhaps this effect can be strengthened by using a small difference between the two numbers. (Although Klein and Beishuizen (1994) failed to find such an effect with numbers up to 100.) After that the students are sufficiently proficient in solving these problems with an indirect addition procedure on the number line and also in solving other problems with the direct subtraction procedure on the number line, the students should get the possibility to exercise with transformations of the problems in order to put the compelling link between context or word problem into perspective. For example, children could practice in transforming a Combine 3 problem into a Combine 2 problem (see Table 1) and vice versa, and not only solving the original but also the transformed problem. Transforming these combine problems is only a matter of rearranging the first and second sentence (provided that the sentence with the unknown in the Cb3 is left out). Problems such as the Compare 1 problem (see Table 1 and Figure 1), which have no marked effect in either the direction of indirect addition or in the direction of direct subtraction, could also be used to help the student considering different solutions for subtraction problems.

Not only transformations from one context type into another could be practised. Also transformations from context problem into number sentence, which initially reflects the semantic or syntactic structure of the problem, could play a role. A number sentence that mirrors the semantic or syntactic structure of the context problem should be of the form $a + ? = c$ for e.g. the Change 3 and Combine 3 problem types and be of the form $c - a = ?$ for e.g. the Change 2 problems. Children who are not proficient in writing a number sentence as symbolic representation of a context or word problem can be helped to do so (Bebout, 1990; Van Lieshout and Pos, 1990; Stellingwerf and Van Lieshout, 1996). The children should practice in linking the different forms of a number sentence (canonical and noncanonical) to a variety of number line solutions. The variety in context problems could help them in writing down different number sentences. Hopefully this would give them a deeper insight into the nature of subtraction procedures.

There is still another pedagogical dilemma. The 1010 procedure is an efficient procedure for addition problems. Children who are skilled in arithmetic solve subtraction problems with the N10 procedure and addition problems with the 1010 procedure. Beishuizen (1993) has already discussed this problem extensively. For the moment I would tentatively opt for not paying too much attention to 1010 instruction, especially with poorly performing children. Instead, I would rather focus the instructional effort on N10-like procedures, without discouraging the 1010 procedure with addition problems, although a different viewpoint is possible (see e.g. Harskamp and Suhre, 1996).

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Young children's strategy choices for solving elementary arithmetic wor(l)d problems: the role of task and context variables

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1 Introduction

In Van Lieshout's chapter 'What can research on word and context problems tell about effective strategies to solve subtraction problems' two studies are reported. First I will give some specific comments on each study. Afterwards I will discuss some general issues about the role of task and context variables on young children's strategy choices for solving elementary arithmetic problems that are raised by Van Lieshout's chapter. In all parts of this reaction chapter, I will rely on past and ongoing work at the Leuven Center of Instructional Psychology and Technology.

2 Study 1: direct subtraction and indirect addition

Van Lieshout's first study presents a set of new findings about the effects of several task variables on children's strategy choice in elementary arithmetic word problem solving, thereby focusing on the question of how the choice for a direct subtraction (DS) or an indirect addition (IA) strategy is determined by (1) the semantic and (2) the syntactic structure of the problem statement. Using these two task variables, he arrives at the following hypothetical order 'from those (problem types) that should provoke mainly direct subtraction and hardly any indirect addition to those that should have the inverse strategy-provoking quality' (p. 8): Ch2, Cp2, Cp1, Cb3 and Ch3 (for an explanation of these abbreviations, see Van Lieshout's chapter). A second research question is whether the strategy choices of mildly mentally retarded (MMR) children are different from those of normal children, but here I could not find a hypothesis.

Van Lieshout's results can be summarized as follows. With respect to the task variables, the results correspond with the hypothesized sequence, in the sense that the proportion of IA decreases and the proportion of DS increases from Ch3 to Ch2. In his view, this can be interpreted in terms of the two task variables involved, namely the semantic and the syntactic factor. With respect to the subject variable, he found that the retarded children used the DS strategy more often than the normal children. Although Van Lieshout gives no clear explanation for it, he believes that it is due to the larger number of years of experience with mathematics education of the MMR children. Third, it was found that the use of IA strategies was associated with better performance on the task as a whole, and therefore Van Lieshout stresses in his conclusion 'the importance of indirect addition as a valuable alternative to direct subtraction in the domain of small numbers' (p. 11).

As a first comment, I generally agree with Van Lieshout's rational analysis of the strategy-eliciting characteristics of the different word problems involved in his study, but at a more specific level there are some unclarities with his definition of the notions 'semantic' and 'syntactic' in his chapter. According to Van Lieshout, the semantic factor relates to the question whether the semantics of the problem (e.g. the chronological description of a change in quantity) do or do not suggest the use of one particular strategy. From this definition one can derive that Van Lieshout expects that change problems will elicit other solution strategies than other semantic problem types like combine and compare problems, but it does not allow to make predictions about the mutual differences between these two other problem types or about the three problem types mentioned above, on the one hand, and equalize problems, on the other hand. For the other task variable, the syntactic factor, a precise definition is lacking too. In some previous studies, the syntactic factor has been operationalized in terms of the placement of the smaller and the larger of the two given numbers in the problem statement (see, e.g., Verschaffel and De Corte, 1990). If we use that definition of the syntactic factor, then we arrive at the following rational analysis of the strategy-eliciting 'power' of the different problem types involved in Van Lieshout's study.

Type	Semantic	Syntactic	Prediction[a]
Ch2	DS	DS	DS**
Cp2	-	DS	DS*
Cp1	-	IA	IA*
Cb2	-	IA	IA*
Ch3	IA	IA	IA**

a) The number of asterisks in the third column called 'Prediction' refers to the strength with which a particular strategy (DS or IA) is triggered by that problem type.

table 1: semantic and syntactic strategy-eliciting characteristics of the five different problem types involved in Van Lieshout's study

Starting from this rational analysis, it remains unclear why the Cb2 problem is placed after the Cp1 problem, – a placement that is supported by the empirical data. So, Van Lieshout must have operationalised these task variables somewhat differently. As far as I understood it, it is his definition of the syntactic variable which makes the difference: while in most previous analyses (including our own analysis in Verschaffel and De Corte, 1990, which will be presented below) the syntactic factor relates to the order in which the two given numbers are stated in the problem text, Van Lieshout's analysis takes into account the location of the two given numbers as well as of the unknown quantity. In the Cp1 problem the unknown quantity is mentioned at the end of the problem (after the two given numbers), but in the Cb2 problem the unknown quantity is introduced after the first and before the second given number. This may explain why Van Lieshout predicted (and found) more IA strategies for the Cb2 problem than for the Cp1 problem. As such, Van Lieshout's analysis provides additional insight into the possible role of subtle changes in the problem formulation on children's strategy choices.

Second, as Van Lieshout has stressed himself, it is important to note that his study was not especially designed to investigate the influence of several semantic and non-semantic factors on the solution process of elementary addition and subtraction word problems. Such a study requires another design in which the two task variables involved in the rational analysis (semantic and syntactic) are experimentally manipulated. By means of such a design the relative strength and the interaction of the two above-mentioned task variables can be investigated in a more appropriate and a more precise way.

In this respect I refer to a study which we have done some years ago (Verschaffel and De Corte, 1990). In that study we have analyzed the solution strategies of a large group of (beginning) second graders on versions of a subset of the fourteen problem types from the Riley, Greeno and Heller (1983) analysis beginning with the larger and the smaller given number (see Table 2). (All problems involved numbers below 20.) Hereafter, I will discuss only the hypotheses and findings with respect to the subtraction problems; a discussion of the results on the addition problems can be found in the original research report (Verschaffel and De Corte, 1990).

Structure	**Sequence**	**First number**	**Problem**
		Addition problems	
Ch1	Normal	Larger	Pete had 8 apples; Ann gave Pete 4 more apples; how many apples does Pete have now?
Ch1	Normal	Smaller	Pete had 4 apples; Ann gave Pete 8 more apples; how many apples does Pete have now?
Ch1	Inverted	Larger	Ann gave Pete 8 more apples; Pete started with 4; how many apples does Pete have now?
Ch1	Inverted	Smaller	Ann gave Pete 4 more apples; Pete started with 8; how many apples does Pete have now?
Cb1	-	Larger	Pete has 8 apples; Ann has 4 apples; how many apples do Pete and Ann have altogether?
Cb1	-	Smaller	Pete has 4 apples; Ann has 8 apples; how many apples do Pete and Ann have altogether?
		Subtraction problems	
Ch3	Normal	Smaller	First Pete had 4 marbles; now Pete has 13 marbles; how many marbles did Pete win?
Ch3	Inverted	Larger	Now Pete had 13 marbles; first Pete had 4 marbles; how many marbles did Pete win?
Cb2	-	Smaller	Pete has 4 car toys; Pete and Ann have 13 toy cars together; how many toy cars does Ann have?
Cb2	-	Larger	Pete and Ann have 13 toy cars altogether; Pete has 4 toy cars; how many toy cars does Ann have?

table 2: examples of problems used in the study of Verschaffel and De Corte (1990)

With respect to the subtraction problems, we first hypothesized that problems starting with the larger given number would elicit more DS and less IA strategies than would those in which the smaller number was given first. But we also hypothesized that the syntactic factor 'order of presentation of the given numbers' would interact with the semantic structure underlying the subtraction problem. More specifically, we expected that the implied joining action between the known start set and the unknown change set in Ch3 problems would elicit a large amount of IA strategies, even when the order of presentation of the given numbers favours a DS strategy. For Cb2 problems, on the other hand, the choice of either an IA or a DS strategy would be influenced more obviously by the position of the given numbers, because of the absence of an implied action in problems with a Combine structure.

The results were in line with the first prediction: we observed considerably more IA strategies (83%) and much less DS strategies (17%) for problems starting with the smaller given number than we did for problems in which the larger number was given first (67% IA strategies and 33% for DS strategies). This finding supports the hypothesis that the order of presentation of the two given numbers has an influence on the kind of strategies children use to solve subtraction problems. But, as shown in Table 3, the results showed also that Ch3 problems beginning with the smaller and the larger given number elicited almost the same percentages of DS and IA strategies: most children continued to apply IA strategies even when the larger number was given first. For the Cb3 problems, on the other hand, these percentages were much more different: Cb2 problems starting with the smaller given number elicited much more IA strategies than DS strategies, while for those beginning with the larger given number a much larger number of DS strategies was found. These findings confirm the hypothesis that the influence of the order of presentation of the given numbers is not the same for all semantic problem types.

Semantic Structure	First Number	% DS	% IA
Ch3	Smaller	16	84
	Larger	22	78
Cb2	Smaller	18	82
	Larger	43	57

table 3: percentages of DS- and IA-strategies on subtraction problems with the larger and the smaller given number (Verschaffel and De Corte, 1990)

Third, there is the possible role of other task variables besides the semantic and syntactic factor mentioned above, such as the size of the numbers. In Van Lieshout's study, this latter task variable was used as a control variable. In our study (Verschaffel and De Corte, 1990), it was another experimental variable. For every problem type involved in the study we constructed and used a version in which the difference between the two given numbers was (very) large (e.g., 'First Pete had 3 marbles and now Pete has 12 marbles. How many marbles did Pete win?') and another version in which the same problem was given with numbers with a small difference (e.g., 'First Pete had 9 marbles and now Pete has 12 marbles. How many marbles did Pete win?'). The hypothesis was that this number factor would also influence children's strategy choices, in the sense that the smaller the difference between the two given numbers the greater the chance that the corresponding problem would be solved by means of an IA strategy. Moreover, we expected an interaction effect between this number factor and the semantic factor. Contrary to our expectation, we did not find a main effect nor an interaction effect for this number factor. Apparently the number factor was too 'subtle' or too 'weak' – compared to the two other task variables involved in the study – to bear a significant influence on these children's strategy choices.

Fourth, there is Van Lieshout's finding that the older children from special schools applied the DS strategy more frequently and the IA strategy less frequently than the younger children attending the regular schools. Like Van Lieshout, I am not inclined to interpret this as evidence of more sophisticated and more flexible modelling and problem solving skills in the former group. A more plausible explanation is that these older pupils had already received much more experience with the (traditional) culture and practice of school arithmetic word problem solving, favouring the well-known 'caricatural' view of modelling and solving application problems as choosing and executing the correct (direct) operation with the numbers given in the problem statement. But in order to test this hypothetical explanation, one would have to compare in a more scrutinized and process-oriented way all cases wherein pupils from both groups reacted to the IA-triggering problems with a DS strategy. My hypothesis is that a significant amount of DS strategies from the older and retarded children on the IA-triggering problems were not the result of a mindful problem-solving process based on a meaningful representation of the problem situation, but on a mindless association of the operation of subtraction with certain superficial characteristics of the task.

3 Study 2: N10 and 1010 strategies

Van Lieshout's second study focuses on mental arithmetic procedures like N10, 1010 and A10 to solve problems with numbers from 20 to 100 (for an explanation of these procedures, see Van Lieshout's chapter). Three different kinds of addition and subtraction word problems with numbers in the 20-100 domain involving carrying (together with corresponding bare sum versions) were collectively administered to a group of third graders: two of these three types were expected to elicit the N10 procedure (i.e., the prestructured and the geometrical type), while the third was considered as neutral. The major research question was how the children solved these tasks and how children's strategy choices were related to their task performance.

First, in line with the hypothesis, Van Lieshout found that the N10 procedure was applied more frequently for subtraction problems than for addition problems, while the reverse was true for the 1010 procedure. A second interesting finding was that systematic N10 solvers performed almost equally well on addition and subtraction problems, while the performance of the systematic 1010 solvers dropped seriously from addition to subtraction problems. Finally Van Lieshout unexpectedly found no effect of context on use of N10, but in his view this was due to an intervention of the teacher who forced students to write down a number sentence (which may have made disappear the context effect on the pupils' strategy choices). Based on the results of his study, Van Lieshout concludes that 'he would tentatively opt for not paying too much attention to 1010 instruction, especially not for poorly performing children'. I want to make some comments with respect to this study too.

First, with respect to its design, it is unclear to me why the so-called geometrical problem type is defined as a N10-eliciting strategy. So, I am not so surprised that there was no context effect, at least not for that problem type.

Second, while the results of Study 2 indeed reveal that N10 strategies were applied more often on subtraction than on addition problems and 1010 strategies more on addition than on subtraction problems, one should not forget that on both problem types N10 was by far the dominant strategy. This general finding corresponds with the result of a recent study in which we investigated the development of a group of 60 second graders (from three different classes) during the school year (Van Eyck, 1995). A test consisting of four items of each of the eight problem types mentioned in Table 4 was individually administered to all children three times during the school year.

Example	Addition/ Subtraction	Direct/ Indirect	Carry/ No carry
14 + 55 = .	+	D	No
68 – 21 = .	–	D	No
21 + . = 64	+	I	No
69 – . = 52	–	I	No
28 + 43 = .	+	D	Yes
74 – 59 = .	–	D	Yes
38 + . = 57	+	I	Yes
75 – . = 18	–	I	Yes

table 4: overview of the problem types used in Van Eyck's (1995) study

The results of that study are very complementary to the findings reported by Van Lieshout. First, as in Van Lieshout's study, N10 was the dominant strategy (see Table 5 which contains the distribution of the correct solutions on the four direct problem types from Table 4 over the different kinds of solution strategies during each interview). Second, the proportion of direct addition problems with carrying correctly solved by means of 1010 is considerably greater than for the corresponding subtraction problems. Third, and in addition to what was found by Van Lieshout, we found that for direct addition and subtraction problems without carry, the difference in strategy choices was much smaller than for those with carrying.

	No carry		**With carry**	
	a + b = .	a − b = .	a + b = .	a − b = .
		Interview 1		
N10	18 (32%)	18 (58%)	17 (53%)	16 (94%)
10T	14 (25%)	1 (3%)	15 (47%)	1 (6%)
1010	24 (43%)	12 (39%)	0 (0%)	0 (0%)
		Interview 2		
N10	40 (30%)	50 (55%)	37 (33%)	38 (75%)
10T	28 (21%)	4 (4%)	59 (53%)	5 (10%)
1010	66 (49%)	37 (41%)	15 (14%)	8 (15%)
		Interview 3		
N10	55 (35%)	60 (57%)	51 (40%)	54 (83%)
10T	46 (30%)	1 (1%)	52 (41%)	9 (14%)
1010	54 (35%)	45 (42%)	23 (19%)	2 (3%)
		Total		
N10	113 (33%)	128 (56%)	105 (39%)	108 (81%)
10T	88 (26%)	6 (3%)	126 (47%)	15 (11%)
1010	144 (41%)	94 (41%)	38 (14%)	10 (8%)

table 5: distribution of correct solution strategies for the four direct problem types from table 4 during each interview

Third, it is somewhat surprising that Van Lieshout's analysis of the results of Study 2 does not involve a classification of children's solution strategies in terms of IA and DS strategies. Indeed, in solving the subtraction problems from Study 2 pupils do not only have to choose between a N10, a 1010 (and a 10T) procedure, but also between a IA and a DS strategy (which was the topic of the first study reported in his chapter). A reasonable hypothesis is that this other aspect of the strategy choice process will again be influenced by the three task variables discussed in Study 1 (namely, the semantic, the syntactic, and the number factor). Because Van Eyck (1995) worked with bare sums only (see Table 4), we can only report findings for such sums (and not for word problems). As Table 6 convincingly shows, we found a very strong effect of the task structure – whether it was a 'b – a = .' problem, a 'a + . = b' problem or a 'a – . = b' problem – on the kind of solution strategy (a 'direct subtraction' (DS), an 'indirect addition' (IA) or an 'indirect subtraction' (IS) strategy).

		DS	IA	IS
a – b = .	n %	274 97%	5 2%	4 1%
a + . = b	n %	62 25%	189 75%	1 0%
a – . = b	n %	141 48%	28 10%	122 42%

table 6: number and percentage of DS, IA and IS strategies produced by second graders for a – b = ., a + . = b and a – . = b problems in the study of Van Eyck (1995)

4 Children's awareness of strategy change

Besides these more specific questions and comments with respect to the two studies reported in Van Lieshout's chapter, I want to raise two additional points based on our own current research interests and activities at the Leuven Center of Instructional Psychology and Technology in the domain of children's strategy choices in the domain of elementary arithmetic.

First, over the past years many researchers have investigated children's strategy choices, and more specifically, the role of several task, subject and context variables on children's choice between a material and a verbal counting strategy, between a retrieval and a computation strategy, between a DS and a IA strategy, or between a N10 or a 1010 strategy. However, according to my knowledge of the research literature, we do not have yet a good psychological theory of the (meta)cognitive processes underlying children's strategy choices, and more specifically, about their awareness of the role of these different factors in their decision processes. In other words, it is difficult to say if children's adaptive use of different procedures or strategies on different tasks is the result of a deliberate and mindful decision process or if it happens more or less automatically and unconsciously. Therefore, further research is needed in which strategy choice in elementary arithmetic is investigated more intensively and systematically.

As Siegler (1988) has argued already several years ago, this research should consist of careful analyses of the consistency or flexibility of individuals rather than (purely) on averaged data over subjects. Moreover, this research should not restrict itself to the analysis of the strategies or procedures actually used by children on different arithmetic tasks, but also try to unravel the (awareness of the) reasons behind these adaptive choices e.g. by means of verbal reports.

In this respect I briefly refer to an ongoing study at our center which investigates the development of a clever strategy for estimation of numerosity from the theoretical perspective of 'strategic change' (Lemaire and Siegler, 1995). In that study a simple computer-based estimation task was used in which participants of three different age groups (20 university students, 20 sixth-graders and 10 second-graders) had to estimate 100 numerosities of colored blocks presented in a 10 x 10 rectangular grid (see Figure 1 for two example items from this estimation task).

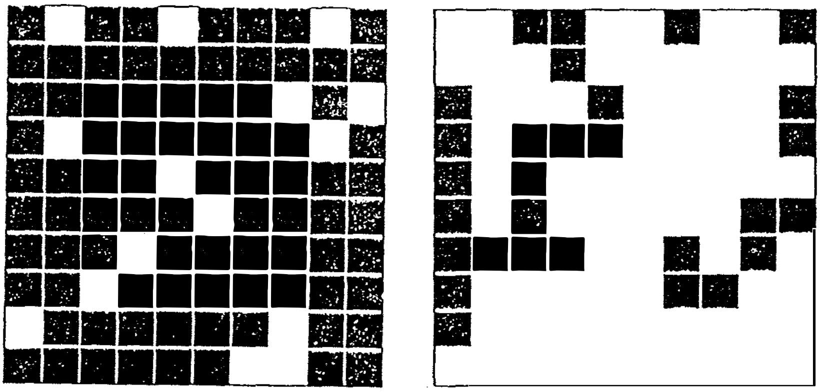

figure 1: two examples of estimation items used in the study of Verschaffel et al. (1996)

Generally speaking, this task allows two distinct estimation procedures: either repeatedly adding estimations of groups of blocks (= addition procedure) or subtracting the estimated number of empty squares from the (estimated) total number of squares in the grid (= subtraction procedure). A rational task analysis indicates that the most efficient overall estimation strategy consists of the adaptive use of both procedures, depending on the ratio of the blocks to the empty squares (= minimize or MIN strategy). To test to what extent subjects apply this clever and adaptive MIN strategy we analyzed the individual patterns of response times and of absolute deviations from the correct answer for the set of 100 items by means of Beem's (1995) 'segmentation analysis' (As an illustration, Figure 2 contains two patterns of response times from two sixth-graders: one of a pupil who performed extremely well on this estimation task and whose pattern – correspondingly – was perfectly in line with the one predicted by the MIN model, and one of a pupil with a considerably weaker performance on the estimation task and – accordingly – a pattern of response times that does not fit with the MIN model.) Information about the subjects' awareness of their overall estimation strategy was gathered through a semi-structured interview at the end of each session.

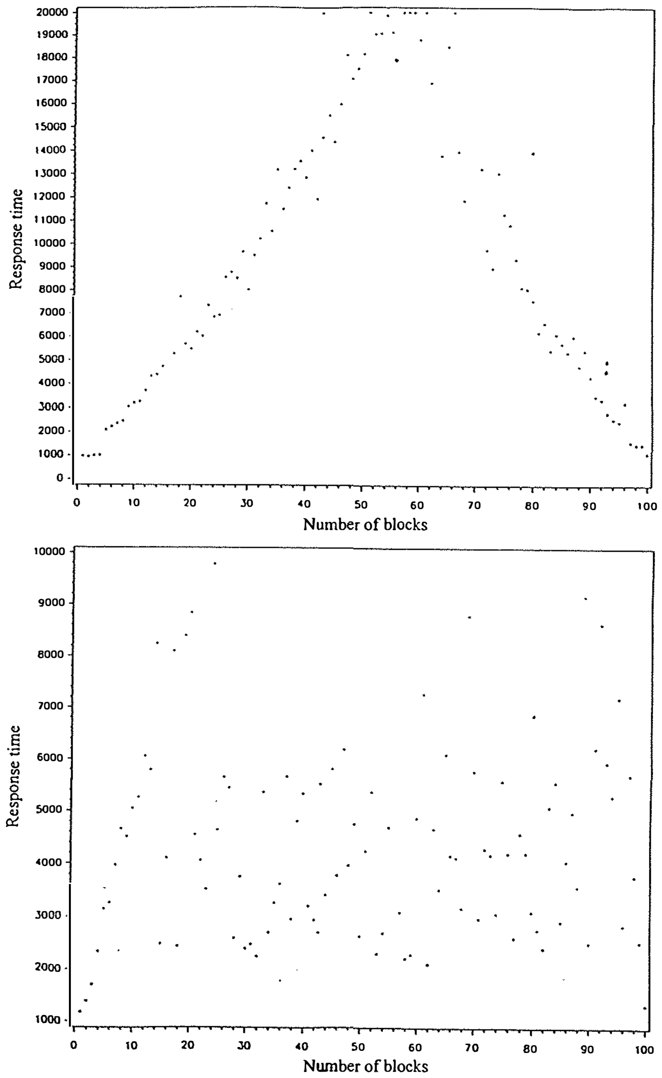

figure 2: two individual patterns of response times on the estimation task from the study of Verschaffel et al. (1996)

First, the data about the individual response time and error rate patterns on the 100 trials revealed that there was a developmental difference in the adaptive use of the two procedures, in the sense that older subjects tended to use the adaptive MIN strategy more frequently and more systematically than younger subjects. Second, the development towards the greater use of the adaptive MIN strategy, as evidenced by the response time and error rate data, was paralleled by a similar development in the capacity to verbalize the MIN strategy. Third, there was a strong relationship between the use of the MIN strategy, on the one hand, and subjects' overall performance on this estimation task, on the other hand (for a more detailed report of the study, see Verschaffel, De Corte, Lamote and Dhert, 1996).

5 Mathematical modelling of wor(l)d problems

Second, we should be aware that in studies like the ones reported in Van Lieshout's chapter (and most other word problem solving research of the past decade, including ouw own studies) the subject's task consisted of choosing between a set of possible solution strategies for solving a standard addition or subtraction problem (e.g. between an addition and a subtraction, between a DS and an IA strategy or between a N10 and a 1010 strategy). This implies that the word problems used in these studies confirm in a certain sense one of the heavily criticized characteristics of traditional instruction, namely that it may contribute to the development in pupils of a superficial or 'caricatural' view of mathematical modelling and problem solving as a process of selecting and performing the correct arithmetic operation with the numbers given in the problem, without any (critical) consideration of the meaningfulness of the problem and of their proposed solution.

Recently several researchers have demonstrated the omnipresence of such a superficial and meaningless approach (Greer, 1993, 1995; Reusser, 1995; Verschaffel, De Corte and Lasure, 1994; Yoshida, Verschaffel and De Corte, 1996). In all these studies large groups of pupils (10-13-years olds) were confronted with a set of word problems, half of which were standard items (S-items) that could be unambiguously solved by applying an obvious arithmetic operation(s) with the given numbers, while the other half were problematic items (P-items) for which the appropriate mathematical model was less obvious and less indisputable, at least if one seriously takes into account the realities of the context evoked by the problem statement. Examples of P-items used in these studies are 'John's best time to run the 100 meters is 17 seconds. How long will will it take him to run 1 km.' and 'Steve has bought 4 planks of 2.5 meters each. How many planks of 1 meter can he saw out of these planks?'. The analysis of the pupils' reactions to these P-items yielded an alarmingly small number of realistic responses or comments based on realistic considerations (e.g., responding the above-mentioned runner-item with 'This problem is unsolvable, because John will not be able to run constantly at his record speed' instead of the stereotyped reaction '17 × 10 = 170 seconds', or responding the planks-item with '8 planks' instead of the stereotyped response '10 planks', because in reality one can only saw 2 planks of 1 meter out of a plank of 2.5 meter. For instance, in the study of Verschaffel et al. (1994), only 17% of all answers given by a group of 75 fifth-graders to the 10 P-items of a collectively administered paper-and-pencil test, could be considered as 'realistic'.

At this moment we are setting up new studies focussing on particular kinds of modelling difficulties (see De Bock, Verschaffel and Janssens, 1995; Verschaffel, De Corte and Vierstraete, 1996). In one study, for example, we focus on additive problem situations wherein the simultaneous presence of 'ordinal' and 'cardinal' numbers makes it doubtful whether the result of adding or subtracting the two given numbers yields the appropriate answer or an answer that is 1 more or 1 less than the correct solution, like the problems in Table 7:

Example 1	In the summer of 1994 it was already 5 years ago that I met my friend John for the first time. When did I met my friend for the first time? (correct answer: 1994 – 5 = 1989)
Example 2	The first ticket sold at the cash desk of the museum today had n 421. The last ticket sold today has n 488. How many tickets have been sold today (correct answer: (488 – 421) + 1 = 68)
Example 3	Martha is reading a puzzle book. After reading page 215 she remarks that the next 4 pages are totally unreadable. From what page on can she start reading again? (correct answer: (215 + 4) + 1 = 220)

table 7: problem types used in the study of Verschaffel, De Corte and Vierstraete (1996)

The results of a large group of fifth- and sixth graders on a test involving a number of word problems about this topic revealed that the pupils produced a very small number of correct answers on those items for which an addition or a subtraction with the two given numbers does not yield the correct answer (like example 2 and 3). A qualitative analysis of the errors on these two categories of problems revealed that most of these errors (i.e., more than 80%) were so-called '+1 errors', i.e. errors that are either 1 more or 1 less than the correct answer. Second, as expected, the pupils' performance on the items with small numbers (50%) was significantly higher than on those with large numbers (32%). A detailed quantitative and qualitative analysis revealed that this number size effect was due to the more frequent use of informal solution strategies on the 'problematic' problems (like item 2 and 3 from Table 7) with small numbers.

The results of all these studies yield further evidence that by the end of elementary school many pupils have developed a tendency to approach school arithmetic word problems in a superficial and mindless way by choosing the correct operation with the numbers given in the problem statement, without any (critical) consideration of the meaningfulness or appropriateness of their proposed solution in relation to the realities of the problem context. Research on word problem solving should pay more attention at these aspects of mathematical modelling as a part of a genuine mathematical disposition.

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Development of mathematical strategies and procedures up to 100

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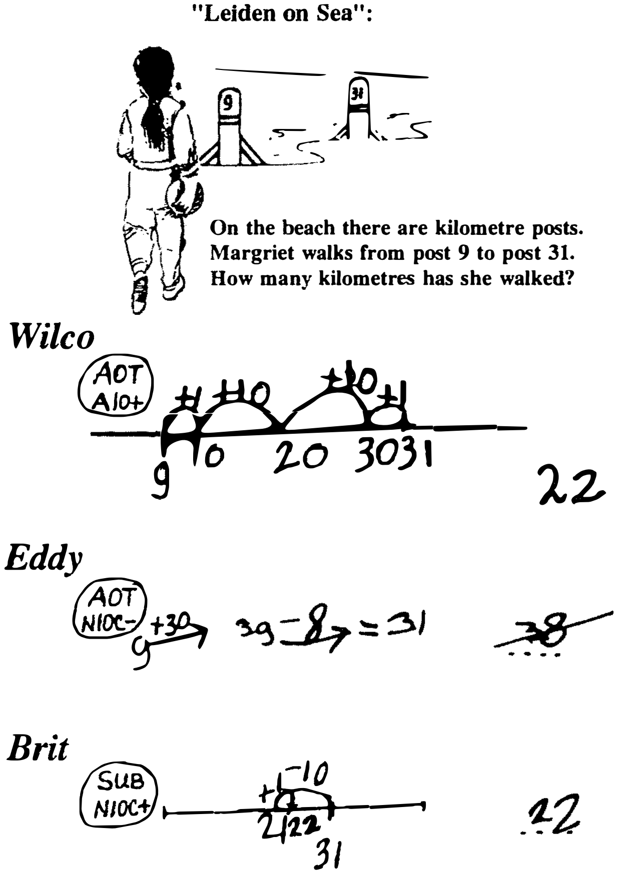
1 Introduction

In our study with the 'realistic' empty number line program for addition and subtraction up to 100 – developed together with Treffers, Freudenthal Institute – we focused on terminal solution behavior as measured on tests in April and June at the end of the 2nd grade (Klein, in press; Klein, Beishuizen and Treffers, in press). The collected data, however, provide a wider developmental perspective on strategies and procedures during a whole school year. We have previously described examples of pupils' work taken from tests and worksheets in two case studies (Beishuizen, Klein, Bergmans, Leliveld and Hoogenberg, 1996; Beishuizen and Klein, 1996). In this chapter we will report some of these developmental aspects in a more systematic way. In our research we felt the need to make a sharper distinction between (solution) strategy and (computation) procedure. Figure 1 illustrates how this pragmatic necessity works out in 'double' scoring of answers. The examples will be explained below. Another argument is a growing discontentment with the terminology in research literature where today *everything* seems to be called a strategy. In this chapter, however, we only raise the question of the distinction between strategy and procedure as a general issue. We do not go into details about the different characteristics of strategy (choice out of options related to problem structure) versus procedure (execution of computational steps related to numbers in the problem). We confine ourselves to some examples and some data to illustrate our arguments.

The first example is taken from unit 2.12 in our experimental empty number line program. We see in Figure 1 different solutions for the problem 'Leiden on sea', which context is deliberately chosen to invite an Adding-on-to (AOT) solution strategy. The first and second answer demonstrate such a strategy of two weaker pupils Wilco and Eddy (Beishuizen and Klein, 1996, p. 7), in combination with two quite different computation procedures A10 and N10C (cf. Table 2). The third answer shows another strategy which does not follow the contextual problem situation, but imposes a transformation into Subtraction (SUB). Probably this solution is evoked by the numbers 31 and 9 as non-semantic factors (cf. Verschaffel and De Corte, 1990) and the strategic preknowledge of this better pupil Brit.



"Leiden on Sea":

On the beach there are kilometre posts.
Margriet walks from post 9 to post 31.
How many kilometres has she walked?

figure 1: various solutions for the 'Leiden on Sea' problem in the experimental empty number line program

In the first and third answer we see a strong *correspondence* between solution strategy and computation procedure (AOT/A10 and SUB/N10C) so that in fact one label might be enough in these cases. The first answer is a typical (low level) modeling strategy with its own construction of several decadal landmarks between 9 and 31 for executing the (small) procedural steps. Given the visual and written support of jumps and numbers drawn on the empty number line it is not difficult to collect the correct answer at the end. On pure mental level, however, this A10 computation procedure would put a heavy load on working memory (Klep, 1996). For this higher mental level the third answer demonstrates a more elegant form of mathematization: an insightful short-cut strategy including transformation as well as compensation.

The level of the second answer is higher than that of the first one because of its use of arrows and larger procedural steps (+ 30) including compensation (– 8). Although our test results show many successful combinations of AOT/N10C (Fig. 3), this answer is incorrect (Fig. 1), which we interpret as a temporary ***misfit*** between strategy and procedure. This interpretation is inspired by the later development of both pupils as illustrated in Figure 5 (section 5). Wilco – like other weaker pupils – keeps on using the empty number line as a supporting model. Eddy, on the other hand, prefers the higher mental level of curtailed and integrated procedural steps A10/1010 as own constructions. From this latter perspective we perceive his error in Fig. 1 not as a fundamental misunderstanding of problem structure, but as a temporary 'developmental' phenomenon due to a new and unfamiliar combination of the AOT solution strategy and the N10C computation procedure. The interpretation of this example underlines, in our opinion, the necessity to discriminate between strategy and procedure.

Eddy learned earlier in the experimental program – like all other pupils in the study – to use the N10C (compensation) procedure for addition and subtraction problems. In combination with an AD or SUB strategy, however, the procedural N10C steps always lead to a last number which is the answer (cf. Table 1). In this case it seems likely that Eddy initially considered the last outcome, 31, as the answer (c.f. his notation '= 31') but then realised (perhaps looking at the picture again?) that this problem was different, in that he had to come up with something 'in between' the beach posts 9 and 31. Looking back at his written steps 'in between' 9 and 31, he saw two arrows with 30 and 8. Either overlooking the symbols + and –, or being unsure of how to cope with the directional dilemmas in the combination AOT and N10C, he decided that 38 (30 + 8) should be the final answer.

In summary, the 'Leiden on sea' example illustrates several solution levels and several combinations of strategy and procedure. Both in the case of correspondence or 'fit', and in the case that something goes wrong or leads to 'misfit', we felt the need in our number line project to discriminate between solution strategy and computation procedure. If not, we could not do the type of analysis as given above. In the following sections we will use this distinction for a further description of some aspects of the development of strategies and procedures up to 100.

2 Strategy or procedure?

In today's literature we see a widespread use of the term strategy. Since word problem research (De Corte and Verschaffel, 1987; Verschaffel and De Corte, 1993) and the realistic (Gravemeijer, 1994a, 1994b; Treffers, 1991), the constructivist (Cobb, 1994; Cobb, Yackel and Wood, 1992), and the problem-solving approach (Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier and Wearne, 1996) the concept '*solution strategy*' is much more in focus than '*computation procedure*'. And indeed the influence of (semantic) problem structure, of informal strategies and strategy choice as opposed to merely procedural computation and memorization of number facts, adds much to new insights in the solution behaviors of pupils.

However, the other side of this picture in our opinion is the inflated use of the term strategy in today's literature: almost everything is called a strategy. It seems as if authors have a preference for speaking of strategy and have an aversion to the term procedure. That is understandable given the shift in views on mathematics education as mentioned above. However from a psychological (process analysis) point of view there are many types of proceduralization. Using standard algorithms in a rigid way in the uppergrades is quite different from memorization of number facts in the lower grades. In the 1970s Realistic Mathematics Education (RME) argued against 'traditional' maths education's emphasis on much exercise and automatization. Today however, the RME view is making more subtle distinctions. For instance, memorization of number facts is now considered to be an important prerequisite for practising flexible mental arithmetic strategies (Van den Heuvel-Panhuizen, 1992, 1996).

Nevertheless the term strategy is often over-used, notably in cases where speaking of procedure would be more appropriate in our opinion. For instance Carpenter and Moser (1984) described the mathematical development in a longitudinal study from grade 1 through grade 3 as follows: 'Modeling strategies were gradually replaced with more sophisticated counting *strategies*' (p. 179). But our question is if it would not be more appropriate to call strategies like counting-on and number facts *procedures* in the sense of Anderson's (1982) psychological theory of proceduralization? Compare also Reys, Reys, Nohda and Emori (1995) describing their study as 'Mental computation performance and strategy use of Japanese students in grades 2, 4, 6 and 8'. After inspection of their examples we would conclude, however, that their categorization of 'strategies' (p. 318, cf. Table 3) comes very close to what we call our mental computation procedures 1010, 10s, N10, N10C.

Addition (with carry): 45 + 39	Subtraction (with carry): 65 – 49, 51 – 49
Sequential procedures: N10: 45 + 30 = 75; 75 + 5 = 80; 80 + 4 = 84 N10C: 45 + 40 = 85; 85 – 1 = 84 A10: 45 + 5 = 50; 50 + 34 = 84	*Sequential* procedures: N10: 65 – 40 = 25; 25 – 5 = 20; 20 – 4 = 16 N10C: 65 – 50 = 15; 15 + 1 = 16 A10: 65 – 5 = 60; 60 – 40 = 20; 20 – 4 = 16 A10: 49 + 1 = 50; 50 + 10 = 60; 60 + 5 = 65; 'answer: 1 + 10 + 5 = 16 (adding-on)[a] ∩[b]: 51 – 49 = 2 (because 49 + 2 = 51)
Decomposition procedures: 1010: 40 + 30 = 70; 5 + 9 = 14; 70 + 14 = 84 10s[c]: 40 + 30 = 70; 70 + 5 = 75; 75 + 9 = 84	*Decomposition* procedures: 1010: 60 – 40 = 20; 5 – 9 = 4 (false reversal) 20 + 4 = 24 (false answer) 10s[c]: 60 – 40 = 20; 20 + 5 = 25; 25 – 9 = 16

[a] See strategies in Table 2
[b] The Connecting Arc can only be used for subtraction problems.
[c] 10s is a sequential adaptation of 1010.

table 1: mental computation procedures for addition and subtraction up to 100

AD: Addition AOT: Adding-on-to	SUB: Subtraction TAT: Taking-away-to
Various combinations for the problem: 'Difference in price between ƒ 73 and ƒ 29 is?' SUB/N10: 73 – 20 = 53; 53 – 3 = 50; 50 – 6 = 44 SUB/N10C: 73 – 30 = 43; 43 + 1 = 44 TAT/A10: 73 – 3 = 70; 70 – 40 = 30; 30 – 1 = 29; answer 3 + 40 + 1= 44 TAT/N10: 73 – 40 = 33; 33 – 4 = 29; answer 40 + 4 = 44 TAT/N10C: 73 – 50 = 23; 23 + 6 = 29; answer 50 – 6 = 44 AOT/A10: 29 + 1 = 30; 30 + 40 = 70; 70 + 3 = 73; answer 1 + 40 + 3 = 44 AOT/N10: 29 + 40 = 69; 69 + 4 = 73; answer 40 + 4 = 44 AOT/N10C: 29 + 50 = 79; 79 – 6 = 73; answer 50 – 6 = 44	

table 2: mental solution strategies for addition and subtraction up to 100

Today Fuson (1990, 1992) is one of the few authors still using the term 'solution procedures', in which she discriminates increasing levels of proficiency in relation to underlying 'conceptual structures'. Her distinction between 'collected' and 'sequence' multiunit concepts/procedures – later renamed into 'separate-tens' and 'sequence-tens' (Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter and Fennema, 1997) closely resembles our distinction between the two main computation procedures for addition and subtraction up to 100: 1010 and N10.

This unclear use of the terms solution strategy and computation procedure might well reflect the growing research in the number domain up to 100, where the role of both strategy and procedure becomes more complex and less distinct in comparison with arithmetic under 20. Therefore we feel the need for a renewed discussion about the terms strategy or procedure, just as ten years ago fundamental discussions about '*conceptual* and *procedural* knowledge' (Hiebert, 1986) took place. Hiebert's well-known book could be said to mark the transition from structuralistic to cognitive views. Of course the perspective should now be different, emphasizing the development from informal to formal strategies, pupils' construction of meaning, etc. Nevertheless there is much in the Hiebert-book which is still relevant, both in its scheme and in its content. Compare for instance Carpenter (1986, p. 113): 'Procedural knowledge is characterized as step-by-step procedures executed in a specific sequence; conceptual knowledge involves a rich network of relationships between pieces of information, which permits flexibility in accessing and using information... the primary relationship in procedural knowledge is 'after', which is used to sequence subprocedures and superprocedures linearly. In contrast, conceptual knowledge is saturated with relationships of many kinds.' A renewed discussion should focus on strategies, in addition to conceptual and procedural knowledge. Sometimes it seems as if flexible strategy use today has many characteristics in common with the definition of conceptual knowledge in the past (cf. Carpenter's quotation above). Trying to clarify and to delineate these different meanings would be an important task for such a renewed discussion.

The *developmental* perspective also provides an argument for a better distinction between strategies and procedures. How else could we investigate and discuss the claim that strategies take over from procedures when problems become more complex in the middle and higher grades? For instance Greer (1987), summarizing research with multiplication and division problems, is focusing on semantic problem structure and 'whether or not the child can choose the correct operation' while in the tests 'carrying out the computation has usually not been required' (p. 65). Anghileri (1989, 1996) also investigated the variety of strategies for multiplication and division tasks. Mayer (1987, p. 347) emphasizes in his model for mathematical problem solving the first steps in the solution process, i.e. choice of an adequate problem schema and integration of the given facts into a correct problem representation. According to Mayer these first steps build on schematic and strategic knowledge, while the execution of the solution comes as a last step requiring only procedural knowledge and computational work.

From this developmental perspective *mathematical strategies* become more and more important because of an increasing variety and complexity of problems, whereas computation procedures become more and more routine. Of course in school practice there is the danger of early proceduralization, which means a development the other way round. Many authors have warned against one-sided 'school mathematics' (Nunes, Schliemann and Carraher, 1993). Recently, Kraemer (1996) has demonstrated this trend by presenting realistic test-items, designed to invite a variety of strategies (Van den Heuvel-Panhuizen, 1993, 1996) to pupils in grade 3, 4 and 5 of a school in The Hague. One of the items was the so-called Polar Bear problem, asking how many pupils together weigh as much as one polar bear of 500 kg. No instruction was given how to solve the problem, only the picture of a polar bear with '500kg' on the test form.

Kraemer's analysis of pupils' scrap-paper work illustrates how many 4th graders exhibited adequate solutions through repeated addition or repeated multiplication, using a realistic weight of a child (20, 30, 50) as a point of reference. Some of them used 4 x 25 = 100 as an insightful short-cut strategy. In the work of 5th graders, however, there is a dominating transition to written (vertical) computation procedures. Negative consequences of routine proceduralization become apparent: on the one hand hardly any strategic choice of easy reference points like 25 or 100, on the other hand cumbersome procedural computational work like 10 x 40 and 5 x 40 without applying the 0-rule or other possible short-cuts. Like other examples in literature, this longitudinal analysis clearly demonstrates the relevancy of the distinction between strategies and procedures – and their different development. However, the examples of pupils' work given by Kraemer (1996) underline the point that such a classification and discrimination is not always easy (and sometimes impossible). All the more reason, in our opinion, to work on a finer description and categorization of solution strategies and computation procedures.

3 Leiden research into N10 and 1010

Within the scope of this chapter we will summarize only some main points of our Leiden research into the two widely used computation procedures in the domain up to 100, namely sequential N10 and the split-method 1010 (Table 1; Beishuizen, 1993). We will not conceal, however, that just this research confronted us with the discussion question: strategy or procedure? For instance in a later publication (Beishuizen, Van Putten and Van Mulken, 1997) we came to speak about N10 and

1010 'when used as strategies' for solving indirect number problems. We acknowledge that the attentive reader might object here that we ourselves are not always consistent in our use of the terms strategy and procedure.

As a first and general remark we would underline the *sequential* character of mental arithmetic in the Netherlands, Germany and other European countries. It is not so that our pupils use sequential N10 – for horizontal computation – since the introduction of the hundredsquare or the empty number line. No, it's the other way round: N10 was there since long as a mental procedure in maths textbooks and didactic manuals. The hundredsquare and empty number line are of a much later date: they were introduced as models to support N10 in a more effective way than arithmetic blocks did in the past (Beishuizen, 1993). Arguments for mental arithmetic are voiced too by some authors in the U.S. like Baroody (1987) and McIntosh, Reys and Reys (1992), who argue that dealing with numbers 'of a piece' stimulates the development of number sense and understanding, that horizontal computation is more in line with pupils' informal strategies, that mental arithmetic is more susceptible to memory overload and therefore makes pupils more keen on finding short-cut and other flexible strategies.

From this viewpoint vertical column arithmetic is conceived as more *formal* and more complicated (place-value, carrying) and is therefore postponed until 3rd or 4th grades (age: 9 or 10 years) in Dutch classrooms. This school practice differs very much from the American maths curriculum, where place-value based (vertical) arithmetic is introduced in the 1st grade to support the conceptual (decimal) base of number structure and number operations (Fuson, 1990, 1992). In this approach arithmetic blocks play a central role as physical embodiments of tens and units and as concrete support for operations like taking-away or trading. In experimental lessons within the new constructivist (Cobb and Bauersfeld, 1995) or new problem-solving approach (Hiebert et al., 1996) we see again much use of arithmetic blocks or comparable configurations. Now, however, they are used to invite *informal* strategies, which means that we see much more horizontal and sequential computation procedures in the pupils' work.

However, in The Netherlands, both from the realistic theory (Gravemeijer, 1994b; Treffers, 1991) and from our empirical Leiden research (Beishuizen, 1993; Van Mulken, 1992), we make a sharper distinction between the linear or sequential model (number line) and the set-type or quantity model (blocks). We therefore have problems with Fuson's use of base-ten blocks for both 'sequence-tens' and 'separate-tens' conceptual structures as support for different computation methods (cf. N10 and 1010) in the number domain up to 100 (Fuson et al., 1997, Fuson and Smith, this volume). In Germany, Wittmann and Müller (1995) in their new didactic handbook emphasizing flexible mental arithmetic or 'Produktives Rechnen', also use the same ten-based blocks configuration for modeling both N10 or 'Schrittweise' and 1010 or 'Zehner zu Zehner' computation procedures.

In our experimental number line program we abandoned this didactic approach for reasons described elsewhere (Klein, in press; Klein, Beishuizen and Treffers, in press; Treffers and De Moor, 1990). First we introduce N10 on the empty number line, i.e. linking up with informal counting strategies of pupils and leveling them up to sequential counting by tens. At a later stage – to avoid early misconceptions – we introduce 1010 as a more formal and more complicated computation procedure, supported by a different set-type model (money or blocks). Not only the difference in conceptual number structure but also the difference in informal/formal and flexible characteristics of N10 compared to 1010 is an important argument in this Dutch 'realistic' viewpoint.

It is interesting to see how today in the U.K. several authors enter this same debate about the *discrepancy* between informal (mental) and formal (place-value) computation procedures (Deboys and Pitt, 1995; Thompson, 1997; Whitebread, 1995). Incidentally, some other British authors have expressed the same concern from a much earlier date. Hart (1989) criticized the gap between what she called 'Sums are sums and bricks are bricks' or between mental arithmetic on the one hand and solutions with materials support on the other hand. Plunkett (1979) criticized formal 'decomposition and all that rot' in 1010-like computation and suggested as an alternative the number line for sequential procedures like N10. In the CAN-project, implementing in schools the use of calculators without formal instruction, pupils spontaneously developed a great variety of informal mental strategies (Rousham, 1995; Shuard, Walsh, Goodwin and Worcester, 1991; Thompson, 1994).

A second remark bears upon our further research into N10 and 1010 'when used as strategies in problem solving situations' like indirect number problems of the type 27 + ... = 65 (Van Mulken, 1992). For this Van Mulken study (Beishuizen, Van Putten and Van Mulken, 1997) we selected two groups of 3rd graders (age: 9 years) within 11 classes, being consistent users of either N10 or 1010 as measured on pretests for addition and subtraction. The two groups had a comparable level of arithmetic competency and procedural knowledge of two-digit subtraction (including carrying). To our surprise many users of 1010 displayed a *change* to N10 when confronted with these indirect number problems. Adequate numerical adaptation of computation procedure – necessary for correct solutions – was mostly observed with the users of N10, and hardly with the (remaining) users of 1010. Many users of N10 adapted their computation procedure through anticipation (27 + 30 = 57; 57 + 8 = 65; answer 30 + 8 = 38), and some of them through compensation (27 + 40 = 67; 67 – 2 = 65; answer 40 – 2 = 38). In contrast, most remaining users of 1010 had problems with adapting their procedure or came up with inadequate and incorrect solutions.

From the analysis of written (scrap-paper) and recorded (interview) procedural steps we concluded that it is not only *decomposition* but also correct *recomposition* of separate solution steps, which makes the procedure 1010 so complicated. As Fuson (1990, 1992) observed with respect to American pupils, one could formulate the solution behavior of weaker users of 1010-like procedures even more negatively: driven by isolated number facts – such as split-off tens and units in two-digit numbers – without being able to re-combine and re-integrate the intermediate outcomes correctly within the total problem representation (Van Mulken, 1992). On the other hand, the easy and adequate adaptations of N10 not only confirm its picture of a smooth sequential procedure, avoiding barriers like carrying or borrowing by just jumping over-ten. These adaptations – both *anticipation* and *compensation* – also contribute to a picture of N10 as a more powerful representation model or cognitive schema (Mayer, 1987). It seems that on the linear model of the number row several problem types easily can be depicted. In this respect we should also mention Vergnaud (1982, 1989) who distinguishes more sharply than other authors between two different number concepts: 1) number 'as a quantification of magnitude' and 2) number 'as a quantification of relations or transformations' (Bednarz and Garnier, 1996, p. 133; cf. also Fuson, 1992, p. 247). We think this more extended number concept – including *number relations* – has more chance to develop for users of N10 than for users of 1010.

Dutch classrooms constitute the context of our research into N10 and 1010. As mentioned before our (realistic) maths textbooks are biased towards N10 as a mental procedure in the lower grades. The textbooks hardly pay attention to 1010 as a computation procedure. Nevertheless our research data (Beishuizen, 1993) as well as those of others (Harskamp and Suhre, 1995; Van der Heijden, 1993) indicate that only about 50% of Dutch lower graders use N10, while about another 50% use 1010. These outcomes underscore the initial difficulty of learning N10 (cf. Fuson, Richards and Briars, 1982), and the attractiveness of 1010 at first sight as a transparent conceptual structure and procedure (cf. Fuson, 1992). For this reason there are some Dutch researchers (Harskamp and Suhre, 1995; Kraemer, Nelissen, Jansen and Noteboom, 1995) as well as some textbook authors (Buys, Boswinkel, Meeuwisse, Moerlands and Tijhuis, 1996) who advocate that more attention should be paid to the 1010 number concept (blocks model) in addition to the N10 computation procedure (number line model).

It may be that our rather negative findings about 1010 as a computation procedure are biased by the Dutch context described above. Perhaps American pupils do better, because of a different instructional context favoring 1010-like conceptual structures as well as procedures (Fuson, 1990, 1992; Fuson et al., 1997). Since research with larger numbers in the domain up to 100 is growing (Carpenter, this volume; Cobb, 1995; Fuson and Smith, this volume; Hiebert et al., 1996), we expect more comparison and discussion of data, which may contribute to a better overall understanding in this new domain. In this respect we hope that studies from Germany will also join this research, because their tradition in mental arithmetic including a widespread use of N10 and 1010 could make German results very interesting as reference material (cf. examples of pupils' work given by Radatz, 1987, 1993; cf. Lorenz and Radatz, 1993; cf. Wittmann and Müller, 1995).

We will conclude this section with the remark that we would give a wider interpretation, within the scope of this chapter, to the Van Mulken study described above. In particular, we would interpret the outcomes as evidence for what we called in the introductory section the phenomenon of 'fit' and 'misfit' between strategies and procedures. Most pupils in our study solved the indirect number problems like 27 + ... = 65 by 'bridging' using an Adding-on-to or AOT strategy (Beishuizen, Van Putten and Van Mulken, 1997). So we could also view the better results of N10 as a proof for the *good fit* between AOT and N10, whereas the worse results of 1010 can be taken as a proof for the *misfit* between AOT and 1010. In the next section we will elaborate further on this discussion question for the conference.

4 More evidence about misfit in the solutions of 'difference' problems

Today we begin to see more 'difference' problems in experimental programs (Hiebert et al., 1996), because they invite more than addition and subtraction problems a spontaneous variation of Adding-on-to (AOT) and Subtraction (SUB) solution strategies. We also introduced them in our experimental number line program (Klein, in press; Klein, Beishuizen and Treffers, in press) to stimulate flexibility of strategy use. In a separate experiment we used 'Difference in age?' test items (Figure 2 and 3) to gather quantative data about the effects of these problem types (De Joode, 1996). This experiment was part of an explorative study by graduate Leiden students into word problems with larger numbers up to 100 (Hoogenberg and Paardekooper, 1995; De Joode, 1996; in collaboration with Verschaffel, personal communication).

The data we report here were collected at the end of four 3rd grade classes (age: 9 years; N = 94) working with the existing realistic textbook '*Wereld In Getallen*' (WIG or 'World In Numbers', edition 1981). Context problems are presented quite often in this textbook, and pupils have also some practice with writing number sentences to describe what is happening in a picture (as we asked them in the test items, cf. Figure 2). As in the Van Mulken study with indirect number problems, we used 3rd graders as experimental subjects, because for most of them computational work with numbers up to 100 (including carrying) would cause no problems. Since we were interested primarily in their solution strategies, they should feel at ease with the computational work in the test items.

We took the opportunity to expand this experiment by including pupils from our first try-out schools with the realistic empty number line program. This experimental program had been implemented in four 2nd grade classes the year before. So, these pupils had now moved up to 3rd grades (N = 65), where they no longer worked with the number line but with the abacus as a model for the introduction of column arithmetic. However, their realistic textbook *'Rekenen and Wiskunde'* (R&W or 'Arithmetic and Mathematics') kept on practicing mental arithmetic in weekly exercises throughout the 3rd grade. Since we were interested in long-term effects of mental strategies as acquired and stimulated on the number line (NL), we included them in the experiment for a comparison with the realistic WIG-results (without number line model). All schools had about the same middle class social level. On a (Cito) test for General Arithmetic Achievement the two samples came out with comparable scores. However, the NL-pupils had more experience with the context problems of the new 'difference' type, which bias we have to keep in mind when we compare the test results.

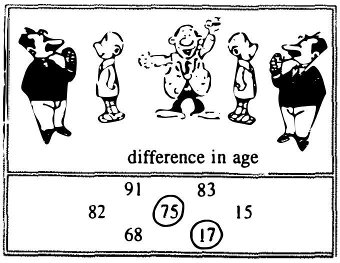
Figures 2 and 3 show representative examples of solutions as given by WIG- and NL-pupils. For a full understanding of the results it is necessary to explain the wording used in the presentation of the test. Since WIG-pupils had no or little experience with such difference problems, we anticipated questions like: 'Is it an add? Do I have to subtract?'. In order to avoid such questions in the classroom – which could influence pupils' solutions – we introduced and solved one example in a whole-class discussion. Therefore we started the test with the question: 'What is your age?' Mostly pupils gave answers like: '9 years'. We then went on: 'Do you have an elder brother or sister at school? What is his/her age?' A lot of answers were always given: 'Yes, he is 12, she is 11,' etc. Then we picked out a pupil and asked: 'Now, what is the difference in age between you and your brother/sister?' Almost immediately this pupil (or another) came up with the correct answer: '3 or 2'. And we went on: 'Now work out the differences of age in the test items in the same way. 'Grandfather is a little older ... (75) and the young boy is ... (17). What is the difference in age? You can do that. Go on and compute the difference of age in your own way.' We answered no further questions, and this instruction through a concrete example worked rather well. The lay-out of the test items (Figure 2 and 3) with circled random numbers – to avoid a prestructured order as in printed number and word problems – followed the realistic principles for design of test items as formulated by Van den Heuvel-Panhuizen (1993, 1996).

Figure 2 shows typical WIG-pupils' solutions. Frequencies of use of various procedures and strategies over all 4 problems are given in Table 3, corresponding percentages of correct answers in Table 5. In the whole WIG-group there is a rather flexible mix of SUB (56%) and AOT or TAT (35%) strategies, but individually most pupils showed a set-effect of the same strategy applied to all 4 problems. Pupil B (Fig. 2) is a flexible exception changing according to number size from SUB to AOT strategies. Pupil A has difficulties with finding the adequate solution strategy for such difference problems. Moreover, pupil A demonstrates how the 1010 procedure – used by many WIG-pupils – mostly leads to incorrect answers. This example of pupil A illustrates the discussion question of this chapter: the often reported *misfit* between the 1010 computation procedure and solution strategies like SUB or AOT. In both combinations SUB/1010 and AOT/1010 performance level was very low (.37 and .26 in Table 5) for the WIG-pupils.

WIG-pupil B, however, shows much more flexibility in procedural computation by changing from 1010 (on preceding problems) towards N10 as well as A10 (on difference problems in Fig. 2). Notice also the close correspondence or *fit* between strategy and procedure in the work of pupil B: SUB strategy and (units first) N10 procedure, AOT strategy and (curtailed/integrated) A10 procedure. Pupil C changes to the TAT strategy in combination with the N10C procedure, which however results in a considerable number of errors (Fig. 2). This example of WIG-pupil C raises the same discussion question as the introductory example of the pupil Eddy and his incorrect solution of the 'Leiden on sea' problem with AOT/N10C: fundamental misunderstanding of problem structure or temporary misfit of procedure as an indication of developmental progress? As can be seen in Table 5 performance of WIG-pupils with N10 (and N10C) was very good (.87) in combination with SUB strategies, but much lower (.55) in combination with AOT or TAT strategies. This constitutes an interesting discrepancy in results, to which we will return in the discussion below.

Figure 3 shows solutions typical for NL-pupils. Details are given in Table 4 and 6. In the whole group there is a somewhat greater mix of SUB strategies (43%) and AOT or TAT strategies (53%). Individually most NL-pupils also show a strong set-effect as WIG-pupils do. Pupil D (Fig. 3) is an exception, changing from SUB to TAT strategies through the test items. Like pupil E, about 20% of the NL-pupils spontaneously drew an (empty) number line to support their solutions of difference problems. They did *not* do so with the addition and subtraction problems in the preceding tests. Apparently they felt a greater need for modeling support when they tried to solve the (forgotten?) difference problems (Fig. 3). For us this is an interesting and relevant outcome with regard to the long-term effects of the NL-program in the 3rd grade: internalization of the number line model.

WIG-pupil A



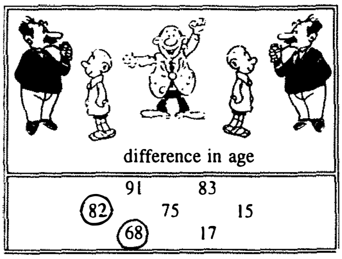
What is the difference in age?

Stapjes:

75+ . =10
10+60=70
5+2=7

62

AOT
1010-



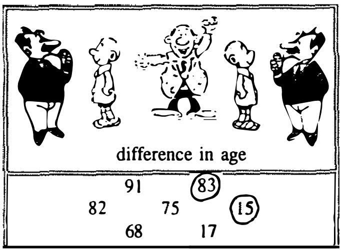
What is the difference in age?

Stapjes:

82-68=
80-60=20
68+14=82
60+20=80
2+6=8

26

SUB/AOT
1010+ ?



What is the difference in age?

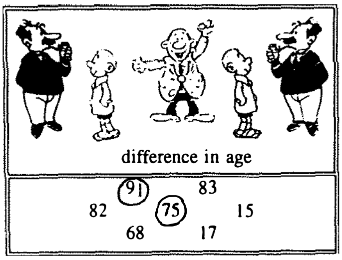
Stapjes:

83-15=

80-10
10+70=80
3+2=5

72

SUB
AOT
1010-



What is the difference in age?

Stapjes:

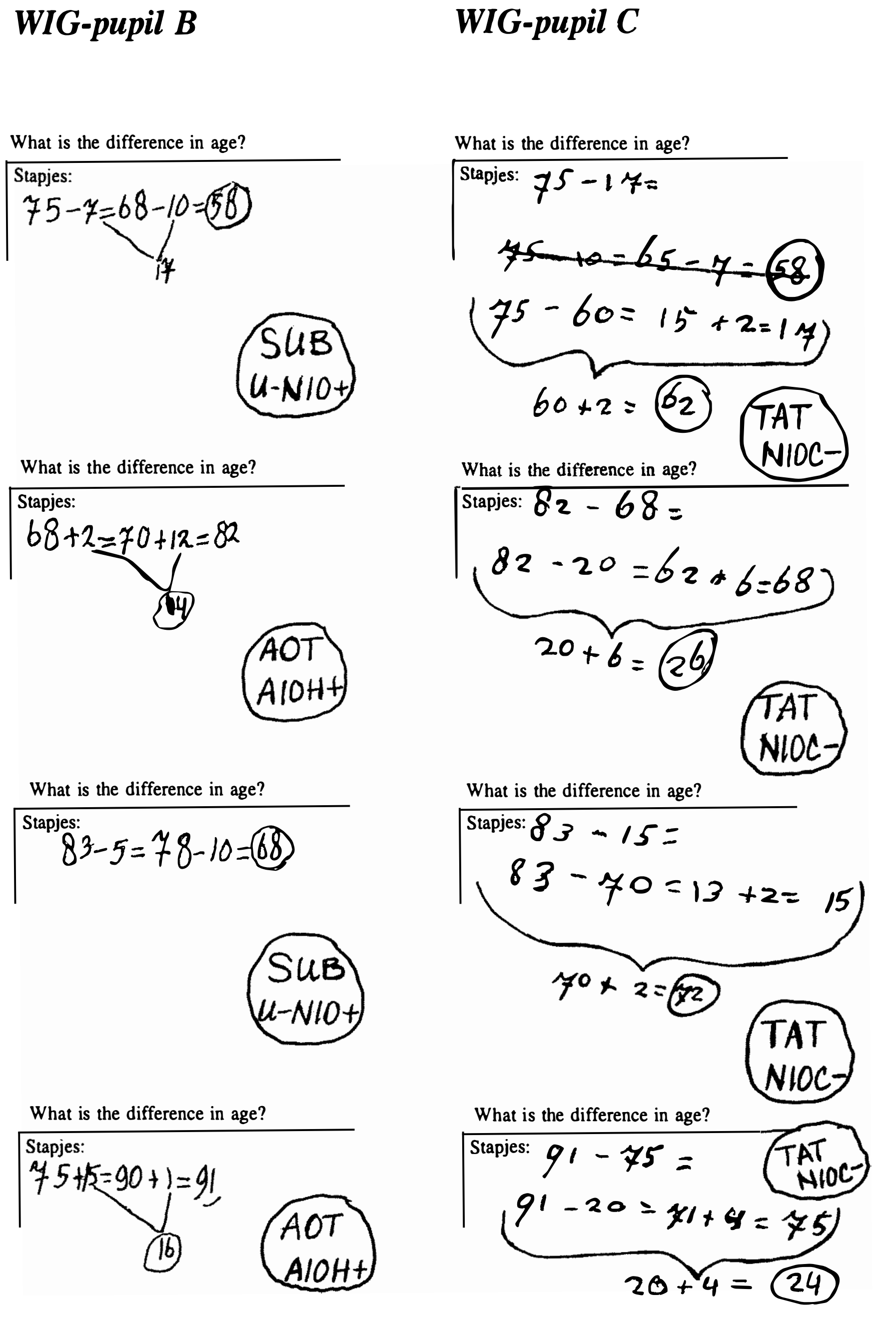
91-75=

70+20=90
1+4=5

24

AOT
1010-

figure 2: WIG-pupils’ solutions of ‘difference’ problems in a separate comparative study



WIG-pupil B

What is the difference in age?

Stapjes:

$75-7=68-10=58$

17

SUB U-N10+

What is the difference in age?

Stapjes:

$68+2=70+12=82$

14

AOT A10H+

What is the difference in age?

Stapjes:

$83-5=78-10=68$

SUB U-N10+

What is the difference in age?

Stapjes:

$75+15=90+1=91$

16

AOT A10H+

WIG-pupil C

What is the difference in age?

Stapjes: $75-17=$

~~75-10=65-7=58~~

$$\underbrace{(75-60=15+2=17)}_{60+2=62}$$

TAT N10C-

What is the difference in age?

Stapjes: $82-68=$

$$\underbrace{(82-20=62+6=68)}_{20+6=26}$$

TAT N10C-

What is the difference in age?

Stapjes: $83-15=$

$$\underbrace{83-70=13+2=15)}_{70+2=72}$$

TAT N10C-

What is the difference in age?

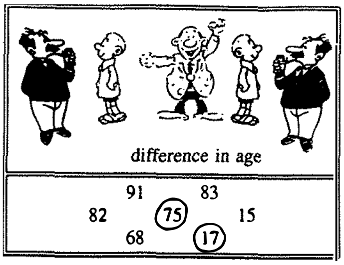
Stapjes: $91-75=$

TAT N10C-

$$\underbrace{(91-20=71+4=75)}_{20+4=24}$$

figure 2: WIG-pupils' solutions of 'difference' problems in a separate comparative study

NL-pupil D

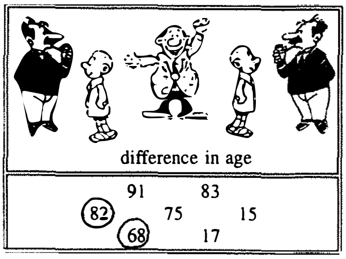


What is the difference in age?

Stapjes:

75-7= 68-10=58

SUB
u-N10+



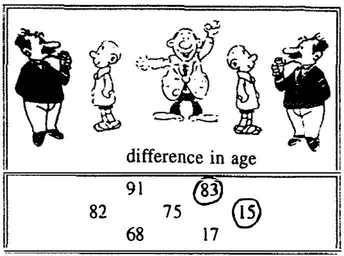
What is the difference in age?

Stapjes:

82-4=78-10=68

14

TAT
u-N10+

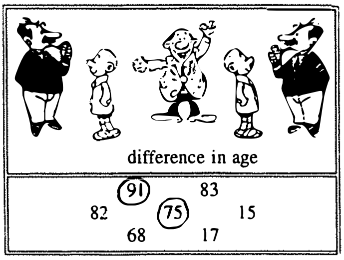


What is the difference in age?

Stapjes:

83-35=78-10=68

SUB
u-N10+



What is the difference in age?

Stapjes:

91-6=85-10=75

16

TAT
u-N10+

figure 3: NL-pupils' solutions of 'difference' problems in a separate comparative study

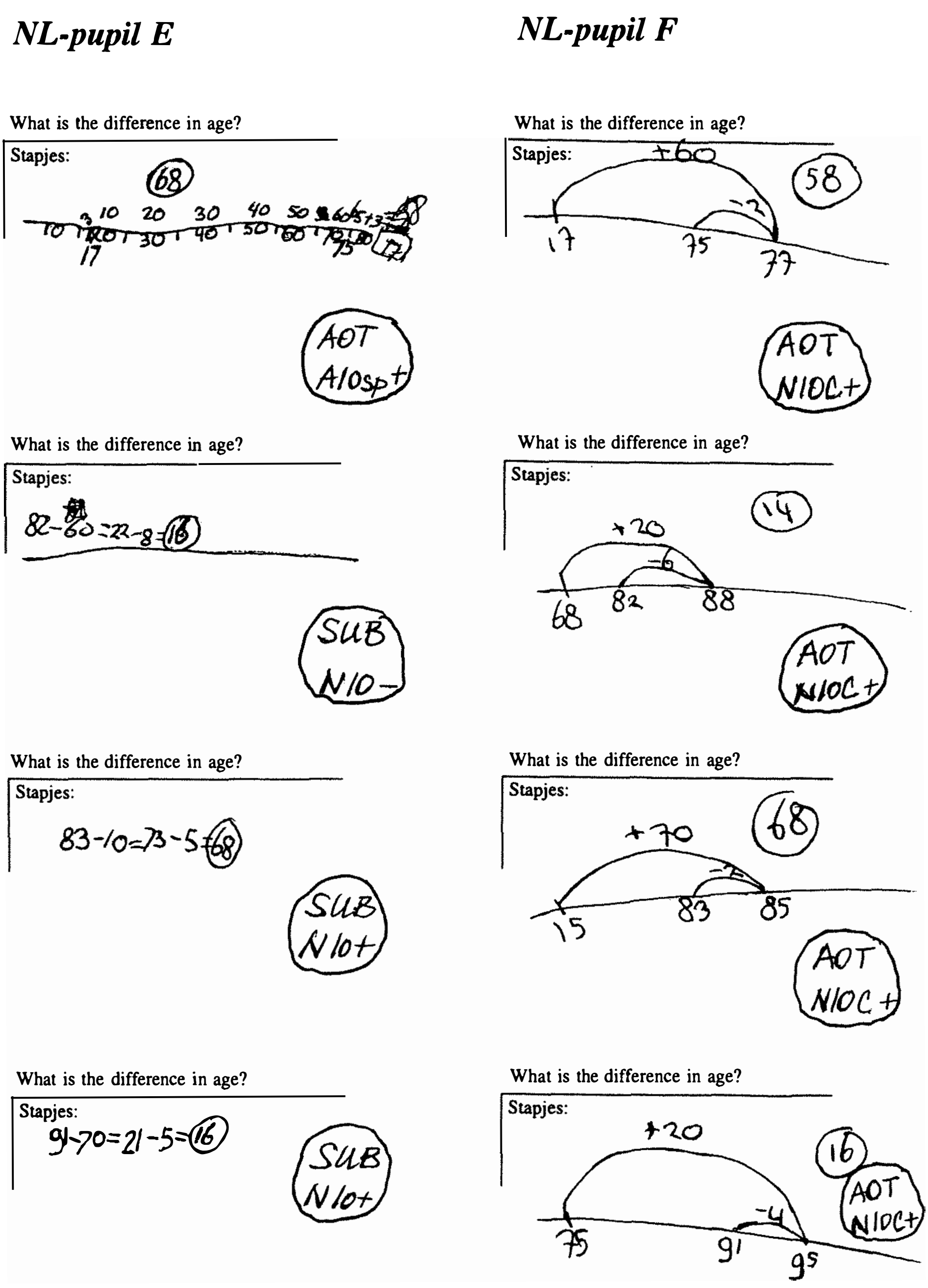


figure 3: NL-pupils' solutions of 'difference' problems in a separate comparative study

Solution Strategy	Computation Procedure					
	1010	N10	10s	A10	Other?	Total
Subtraction (SUB)	17	29	2	1	7	56
Adding-on-to (AOT) Taking-away-to (TAT)	6	16	-	2	10	35
Other?	2	1	1	-	6	9
Total	25	46	3	3	23	100

table 3: use of procedures by strategies (percentages) for 'difference' problems in WIG-group

Solution Strategy	Computation Procedure					
	1010	N10	10s	A10	Other?	Total
Subtraction (SUB)	2	36	-	1	4	43
Adding-on-to (AOT) Taking-away-to (TAT)	2	31	-	10	10	53
Other?	-	3	-	-	1	4
Total	4	70	-	11	15	100

table 4: use of procedures by strategies (percentages) for 'difference' problems in number line-group

Solution Strategy	Computation Procedure					
	1010	N10	10s	A10	Other?	Total
Subtraction (SUB)	.37	.87	.43	1.00	.85	.71
Adding-on-to (AOT) Taking-away-to (TAT)	.26	.55	-	.90	.39	.48
Other?	.00	.00	.00	.00	.00	.00
Total	.31	.75	.30	.85	.44	.56

table 5: percentage of correct answers broken down for procedures by strategies with 'difference' problems in WIG-group

Solution Strategy	Computation Procedure					
	1010	N10	10s	A10	Other?	Total
Subtraction (SUB)	.00	.95	-	1.00	.82	.89
Adding-on-to (AOT) Taking-away-to (TAT)	.25	.89	-	.96	.58	.82
Other?	.00	.00	-	.00	.00	.00
Total	.20	.87	-	.97	.63	.82

table 6: percentage of correct answers broken down for procedures by strategies with 'difference' problems in number line-group

The solutions of this Pupil E (Fig. 3) also demonstrate the interesting phenomenon of a start from lower level modeling *NL-support* in combination with a counting A10 procedure, followed by an increased incidence of higher-level mental solutions i.e. change to a SUB strategy in combination with a N10 procedure. Pupil F (Fig. 3) apparently also feels more comfortable with NL-support, and demonstrates how the combination of N10 or N10C with AOT or TAT strategies can be successful. We see in Table 4 how even after one year without NL experience N10 (or N10C) still is the dominating (mental) computation procedure (70%). Unlike the WIG-pupils, both combinations of N10 or N10C with SUB and with AOT or TAT strategies show a high level of performance for the NL-pupils (.95 and .89 in Table 6).

For the discussion question in this chapter concerning *fit* and *misfit* – we first emphasize as background information that the WIG-3rd-graders in this experiment – just as in the Van Mulken study – also demonstrated a spontaneous *change* from 1010 towards N10. In the preceding word problem tests – not reported here – they showed the commonly found distribution whereby 1010 was dominant (58%) and N10 less frequent (32%) for addition problems (De Joode, 1996). For subtraction problems the WIG-pupils showed a gradual change towards N10, culminating in a further change towards only 25% 1010 and a much higher percentage of 46% N10 with the 'difference problems' (Table 3). The low percentage of 25% 1010 might be an underestimation, because many (1010?) pupils had difficulties with writing out their solution steps for the difference problems. Such 'unclear' procedural steps have been scored in the category 'Other?', which increased to a high percentage of 23% (Table 3). Notice in Table 5 that all WIG-solutions in the category 'Other' were incorrect. We have already seen that most WIG-pupils using SUB/1010 or AOT/1010 did not succeed in solving the difference problems correctly (Table 5). So, 'Other' or 1010 solutions were both rather *unsuccessful* for the WIG-pupils. Using N10 or N10C gave them a much better chance when solving difference problems, but mainly in combination with the (familiar) strategy SUB and less in combination with the (unfamiliar) strategy AOT or TAT. See for examples Figure 2, but for underpinning this conclusion the scores in Table 5.

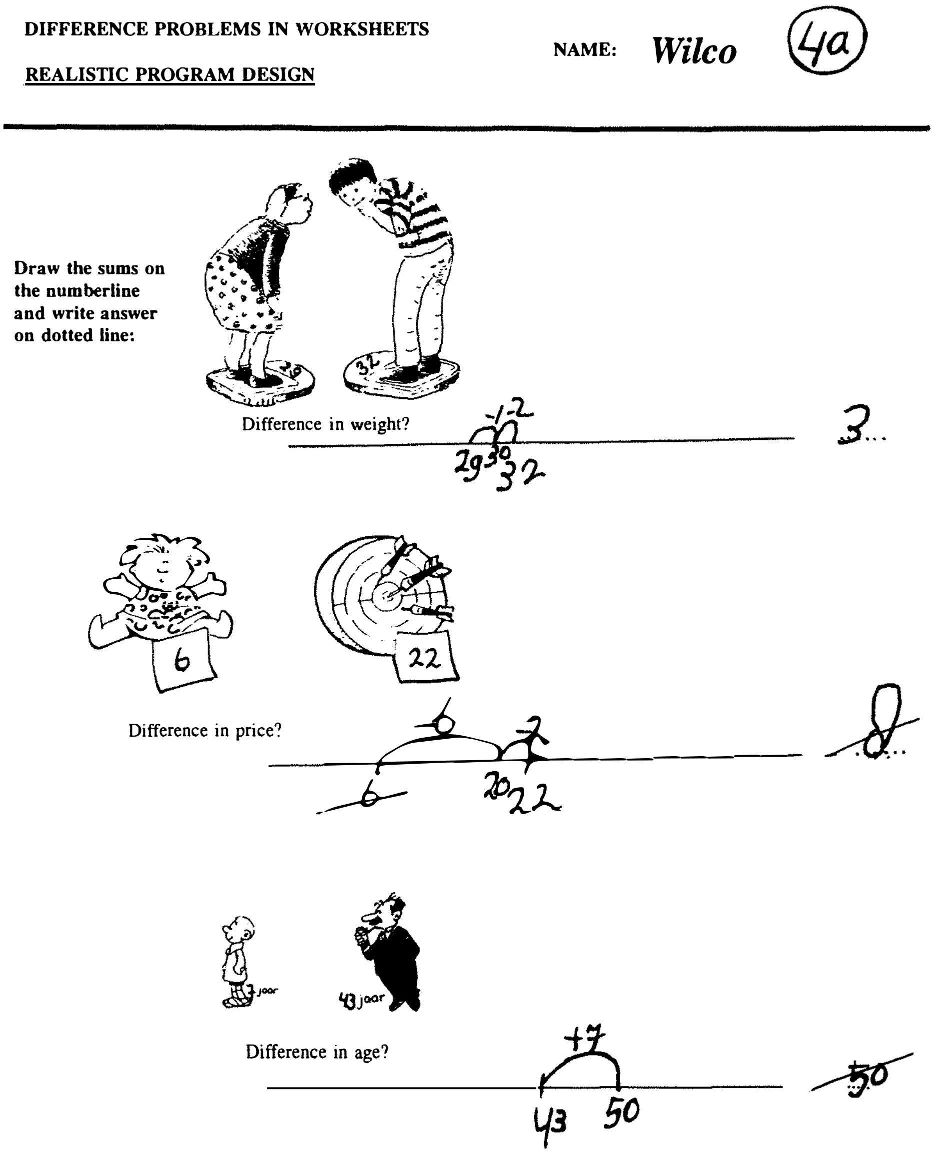
In summary, we see the outcomes of this experiment with 'difference' test items as new evidence for the deficiencies of the procedure 1010 as used in Dutch classroom practice. The data collected among average and 'realistic' WIG-pupils confirm the earlier findings in the Van Mulken study (Beishuizen, Van Putten and Van Mulken, 1997). In this WIG-study again, we see many users of 1010 change to N10 when they face subtraction problems and even more when they face (nonstandard) difference problems. Do they feel their usual 1010 procedural steps as a '*misfit*' or as too '*complicated*' with respect to (anticipated?) recomposition or memory-load or other pitfalls of the computation procedure 1010? Other WIG-pupils stick to 1010, but their low scores (Table 5) add further evidence of the widespread misfit phenomenon when the 1010 procedure becomes complicated in combination with the strategies SUB and OAT or TAT (Fig. 2). Therefore, we expected in the Van Mulken study that users of 1010 would come up with numerical adaptations such as 'sequential' 10s to meet the demands of procedural complexity (Beishuizen, Van Putten and Van Mulken, 1997). However, such numerical adaptations hardly showed up. In the present study we see this same low level of flexibility with WIG-pupils, where only 3% come to use 10s and only 3% to use A10 for solving nonstandard difference problems (Table 3). In recent American studies sequential adaptations like 10s are reported more frequently (Carpenter, this volume; Fusonet al., 1997), which can be taken as evidence for a greater emphasis in the US maths curriculum on both the 'separate-tens' and the 'sequence-tens' conceptual structures (Fuson and Smith, this volume).

The N10 procedure, on the other hand, comes out as more successful also in this experiment and shows a *better fit* to strategies in the combinations with SUB and OAT or TAT. This applies not only to the NL-pupils who have considerable experience with N10 (Table 6) but also to WIG-pupils having less experience with N10 (Table 5). An interesting outcome is the differential results of N10 for WIG-pupils, because they demonstrate that the procedure N10 is also susceptible to fit and misfit. Our interpretation is that the lower score (.55) of WIG-pupils when they use N10 or N10C in combination with the strategy AOT or TAT can be seen as *temporary misfit* due to lack of experience. For instance, inspection of the N10C-errors as made by WIG-pupil C (Fig. 2) remind us of Eddy's error type in the introductory example (Fig. 1). The first solution (SUB) of pupil C was correct but the 'difference' problem structure probably led her into thinking that an 'in-between' strategy like Take-away-to (TAT) might be more appropriate in this case. However, a TAT strategy causes a direction or compensation dilemma in the procedural steps to be carried out. A more experienced pupil – or an NL-pupil using modeling (NL) support like pupil F in Figure 3 – might handle such a direction dilemma correctly, but not so this WIG-pupil C in Figure 2.

However, WIG-pupils could learn with more experience. We know from our experimental NL-program that such errors with N10 or N10C can be overcome, as happened with pupil Eddy towards the end of the program (see section 5; Beishuizen and Klein, 1996). The good results of the NL-pupils (.89) in this experiment with similar combinations like TAT/N10 and AOT/N10C (Fig. 3) offer a promising perspective for the WIG-pupils too. It must be borne in mind that the NL-pupils also lacked experience with difference problems during the whole school year in the 3rd grade. Nevertheless many of them retrieved successful solutions – sometimes after some trials or with NL-modeling support like pupils E and F in Figure 3. From a developmental perspective these latter errors of temporary misfit (cf. above) can be interpreted as positive symptoms of increasing efficacy in the combination of strategies and procedures. Especially this seems the case for combinations like AOT/N10C or TAT/N10 (cf. above) when pupils increase from unexperienced towards experienced level in solving non-standard and standard problems. Our conclusion from this study is, that the NL-pupils open a *developmental perspective* which is also achievable for other pupils. The WIG-pupils were able to demonstrate flexible change to the strategies AOT or TAT and the procedures N10 or N10C, when confronted with other problem structures such as subtraction and 'difference' (Table 3). In our opinion this is an important first step of adaptive *strategy choice*. Now, as a second step, we expect that *procedural improvement* will follow for WIG-pupils, when they gain more experience with combinations like AOT/N10C or TAT/N10 as the NL-pupils already demonstrated in their test results.

5 Differential effects in the solutions of weaker and better pupils in the empty number line program

Difference problems were used among other context problems (addition, subtraction) in the realistic number line program (RPD). They elicited enhanced variation in strategies and procedures like the combinations AOT/N10C or TAT/N10 already discussed. Therefore it might be interesting – also from the developmental perspective in this chapter – to have a look at the terminal solution behaviors of our NL-pupils in the main study, specifically on 'difference' problems. Another reason is that we were not completely satisfied about the try-out results of the realistic NL-program as discussed above in section 4 (cf. Table 4). Even though many NL-pupils (53%) used AOT or TAT strategies for solving difference problems, change of com-



DIFFERENCE PROBLEMS IN WORKSHEETS

REALISTIC PROGRAM DESIGN

NAME: *Wilco* (4a)

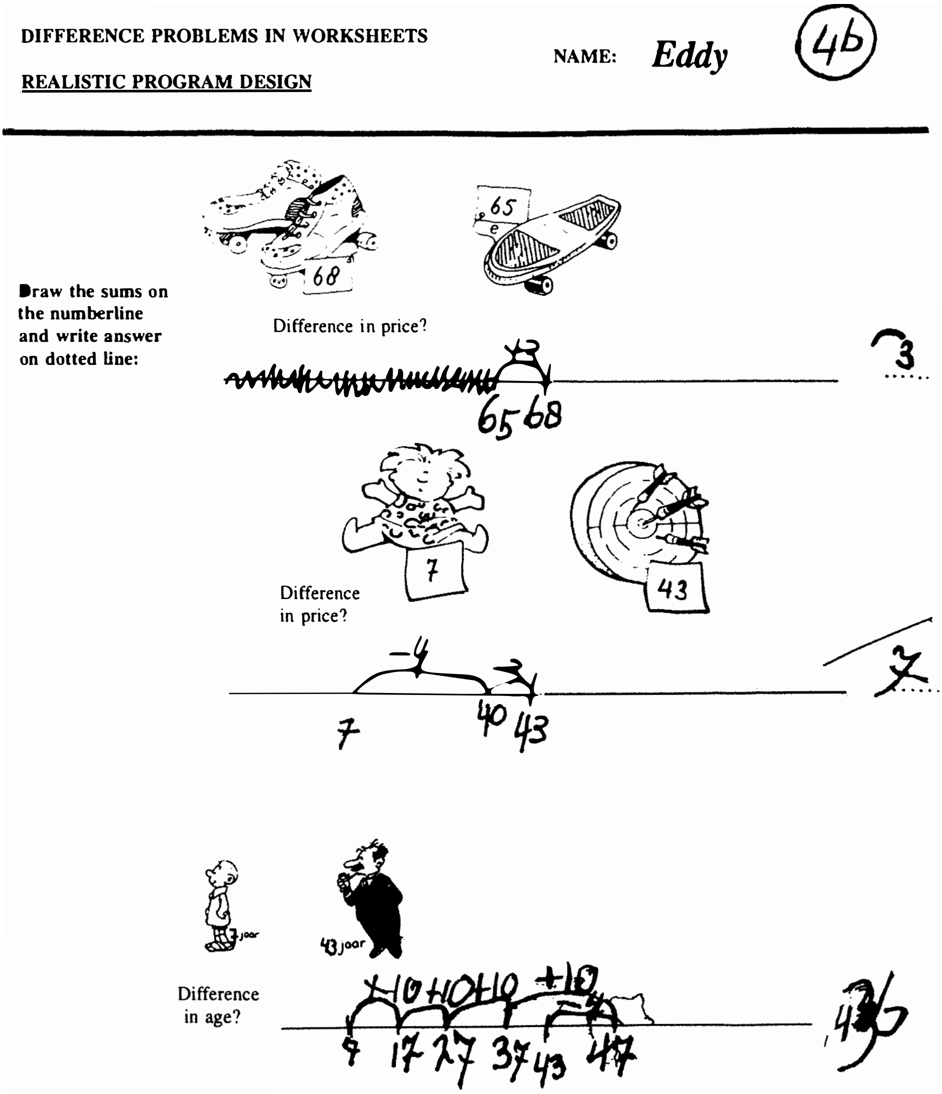
Draw the sums on the numberline and write answer on dotted line:

Difference in weight? ..3...

Difference in price? ..8...

Difference in age? ..~~50~~..

figure 4: Wilco and Eddy's solutions of 'difference' problems in the experimental empty number line program



DIFFERENCE PROBLEMS IN WORKSHEETS

REALISTIC PROGRAM DESIGN

NAME: *Eddy*

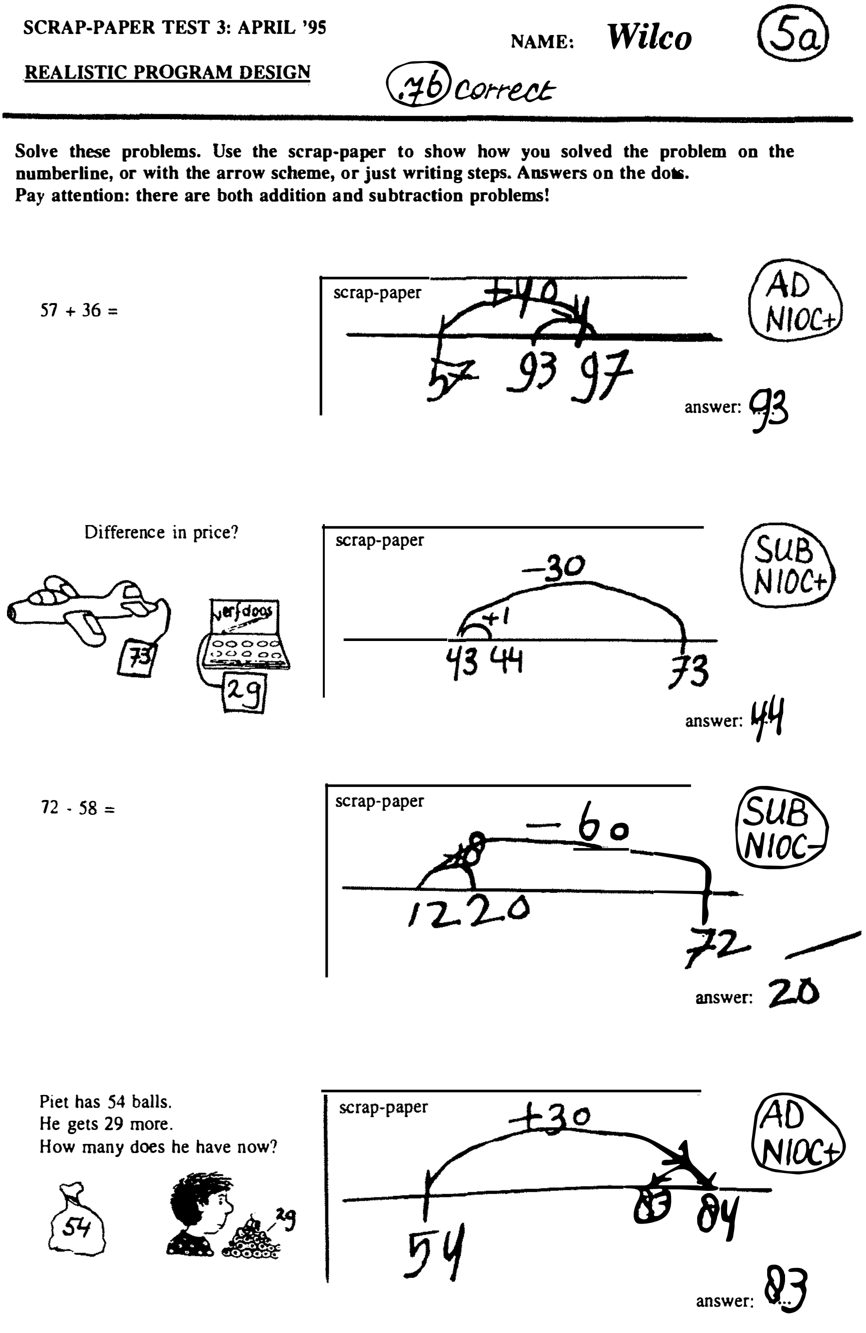
Draw the sums on the numberline and write answer on dotted line:

Difference in price?

Difference in price?

Difference in age?

figure 4: Wilco and Eddy's solutions of 'difference' problems in the experimental empty number line program



SCRAP-PAPER TEST 3: APRIL '95

NAME: **Wilco** (5a)

REALISTIC PROGRAM DESIGN

(76) correct

Solve these problems. Use the scrap-paper to show how you solved the problem on the numberline, or with the arrow scheme, or just writing steps. Answers on the dots.
Pay attention: there are both addition and subtraction problems!

57 + 36 =

scrap-paper

answer: 93

AD NIOC+

Difference in price?

scrap-paper

answer: 44

SUB NIOC+

72 - 58 =

scrap-paper

answer: 20

SUB NIOC-

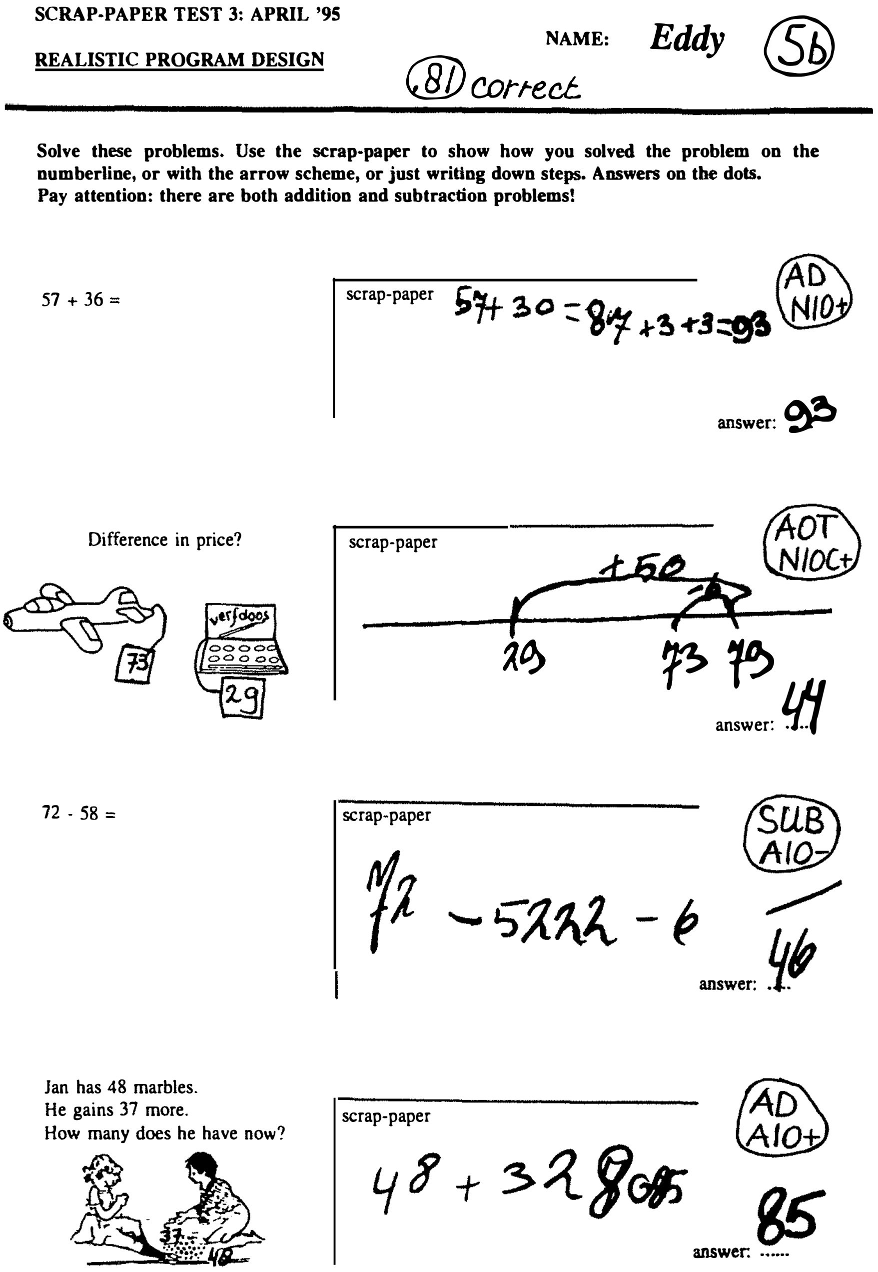
Piet has 54 balls.
He gets 29 more.
How many does he have now?

scrap-paper

answer: 83

AD NIOC+

figure 5: Wilco and Eddy's solution patterns in the Scrap-paper test ASPT in April



SCRAP-PAPER TEST 3: APRIL '95

REALISTIC PROGRAM DESIGN

NAME: *Eddy* (5b)

(81) correct

Solve these problems. Use the scrap-paper to show how you solved the problem on the numberline, or with the arrow scheme, or just writing down steps. Answers on the dots.
Pay attention: there are both addition and subtraction problems!

57 + 36 =

scrap-paper 57+ 30 = 87 +3 +3 = 93

AD N10+

answer: 93

Difference in price?

scrap-paper +50

29 73 79

AOT N10C+

answer: 44

72 - 58 =

scrap-paper 72 - 52 22 22 - 6

SUB A10-

answer: 46

Jan has 48 marbles.
He gains 37 more.
How many does he have now?

scrap-paper 48 + 32 80 85

AD A10+

answer: 85

figure 5: Wilco and Eddy's solution patterns in the Scrap-paper test ASPT in April

putation procedure didn't show the same flexibility. Too many NL-pupils (70%) persisted with N10 in the try-out, while only 11% of them used A10 in an adequate way (cf. Table 4).

Compared to the retention results reported in section 4 immediate test scores were not much better (Klein and Beishuizen, 1994). Therefore, we decided to improve the try-out version of the realistic NL-program through revision. We increased the variety of problems throughout the program. Additionally the use of 'labels' by pupils was stimulated, i.e. giving abbreviated names to their different ways of solution in worksheets and classroom discussions. The results of our main study (Klein, in press; Klein, Beishuizen and Treffers, in press) indicated *improved flexibility*, in particular on the difference problems. Use of A10 increased from 11% to 25%, while the distributions demonstrated in general a better spread over different procedures (Fig. 6; see also Klein, in press; Klein, Beishuizen and Treffers, in press). Therefore, the results of our main study give a better idea of the (possible) effects of difference problems than the WIG vs. try-out study we discussed in section 4.

Figure 4 presents some difference problems taken from worksheets completed by the two (weaker) pupils Wilco and Eddy which we return to as examples after having met them in the Introduction (Fig. 1). We see quite serious difficulties with problem structure, confusion with addition, etc. Difference problems were not easy to learn and remained relatively difficult in the tests, although in June mean correct answer percentage was over 75% (Klein, in press; Klein, Beishuizen and Treffers, in press). Figure 5 gives an idea of the Arithmetic Scrap-Paper Test (ASPT), taken on two occasions during the last three months, in April and June. The ASPT contained addition, subtraction and difference problems both in formula and context format in a mixed order (21 items total). Number characteristics were chosen to invite different strategies and procedures (following a balanced design). For instance the problem 'Difference in price between 73 and 29?' could be solved by SUB/N10C or AOT/N10C, while the context addition problems 54 + 29 and 48 + 37 differ more in inviting N10C and N10 respectively. The solutions of Wilco and Eddy illustrate these effects more or less, as well as the individual variation between pupils (Figure 5). Reliability of the ASPT-instrument was sufficient (Cronbach's alpha = .79)

At the outset of the main study all pupils undertook a *National Arithmetic Test* and an *Intelligence* test. In each of the five 2nd grade classes which worked with the realistic NL-program we selected about 5 weaker and 5 better pupils. We followed them closely through separate interview tests apart from the classroom tests. Thus, we had a sub-sample of about 25 weaker and about 25 better pupils, differing markedly on both Arithmetic and Intelligence level. The data we will discuss in this section are taken from the ASPT classroom tests of those two sub-groups.

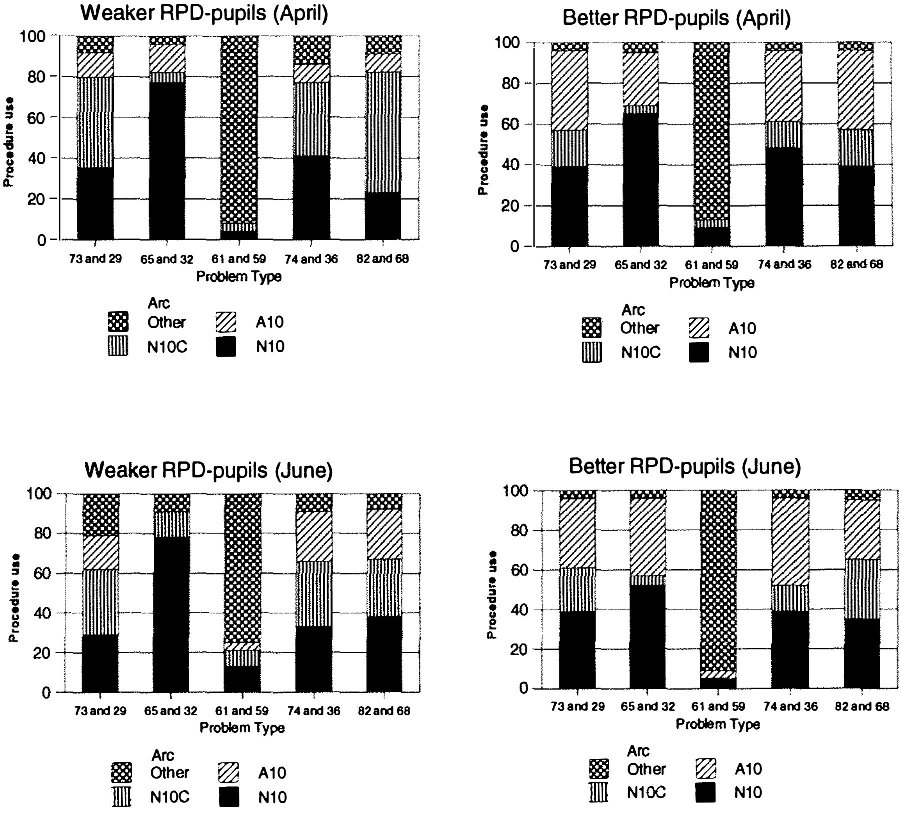


figure 6: type of procedure use (percentages) by weaker and better RPD-pupils for 'difference' problems in the ASPT in April and June

Figure 6 shows in diagrammatic form the varied use of computation procedure for weaker and better pupils when solving the 5 difference problems in the ASPT-test. The problem in the middle ('Difference between 61 and 59?') is a special context problem strongly inviting the specific ARC strategy (cf. Table 1 and 2), which we will leave out in this discussion. So, we concentrate on the other 4 difference problems whose structure is more 'neutral' or open in that they evoke both SUB and OAT/TAT strategies (cf. 'Difference between 73 and 29?' in Fig. 5). Strategy use is summarized for weaker and better pupils in Table 7 over the 4 difference problems (ARC problem excluded).

Strategy	Weaker	Better	
SUB AOT / TAT AD/?	41 56 2	16 80 4	**April test**
SUB AOT / TAT AD/?	34 55 11	22 75 3	**June test**

table 7: type of strategy use (percentages) by weaker and better RPD-pupils for 'difference' problems in the Scrap-paper test ASPT in April and June

We see in Table 7 how the AOT/TAT strategies are dominating over SUB strategies, but the distributions show a *differential* interaction with competency level. Weaker pupils stay more with SUB strategies (about 40%) while better pupils typically change to AOT or TAT (about 80%). These patterns of preference do not alter much from April to June. Figure 6 displays the varied use of computation procedures with difference problems. N10C and A10 are the most interesting procedures, because their patterns of use also *differ* markedly between weaker (more N10C) and better pupils (more A10). As for strategies (Table 7) this relative difference is greatest in April, but equalizes in June. During the last three months of the program the emphasis was on practicing flexibility i.e. choice of strategy and procedure for a great variety of problems, whole-class discussion of various solutions, naming them by 'labels', etc. As we see in Figure 6 the weaker pupils demonstrate the greatest change from April to June: a decrease in their preference for N10C (April) and some increase of their use of A10 (June). The better pupils show a similar but smaller trend, but the different patterns of preference between weaker end better pupils are significant in April ($chi^2 = 15.69$; N = 195; df 3; p < .001) as well as in June ($chi^2 = 9.32$; N = 191; df 3; p < .03).

These data, we think, also provide evidence for the main point of this chapter that strategy and procedure are not the same, and that we should make finer distinctions between the two. We will confine ourselves to some impressions of general trends. Of course for conclusions a more detailed (statistical) analysis is required. But for the purpose of this chapter raising some *discussion questions* will be enough. The addition and subtraction problems in the ASPT were mostly solved in the same way by all pupils: AD and SUB strategies in combination with mainly N10 and N10C procedures (Klein, in press; Klein, Beishuizen and Treffers, in press). The greater variance invited by the difference problems is stronger with the better pupils. Although many weaker pupils also change their strategy to AOT/TAT, their procedures do *not follow* i.e. they persist more with N10 and N10C (especially in April, Fig. 6).

So, it seems as if better pupils are more open to the influence of problem structure on their choice of strategy (AOT/TAT) and corresponding procedure (A10). However, both for better and weaker pupils computation procedure only *partly* follows a change of strategy. Here we could add that the need for a change of procedure might be less strong, because N10 fits very well with AOT and TAT. This is not always the case with N10C, for which procedure we have already seen some examples of misfit with AOT or TAT (Fig. 1 and 2). This latter combination N10C/AOT or N10C/TAT might well cause difficulties for some weaker pupils, so this raises the discussion question why they did not change their procedures towards A10 to a greater extent?

We have the impression – not only from these data but also from interviews – that most weaker pupils are focused more on procedure than on strategy. Could it be that they are more subject to *procedural constraints* and therefore prefer to stay with a computation procedure that has become routine for them? Does this also mean they are less inclined to changes of strategy and procedure? Literature about pupils with learning disabilities (cf. Ruijssenaars, 1992, 1994) emphasizes that aspects of information-processing such as memory-load and automatization are more critical for them. The choice of N10C gives interesting evidence for the bigger role of such psychological or nonceptual mechanisms of proceduralization (cf. Baroody and Ginsburg, 1996), with regard to our weaker pupils as well. Like many of them Wilco developed a preference for N10C to solve problems including carrying (Fig. 5). During the interviews he gave as his reasons – also voiced by other (weaker) pupils – that N10C is an easier alternative than N10 for crossing tens with units. In fact N10C avoids crossing tens and splitting up the units, because the computational steps stay in between the tens. For instance, in a problem like 57 + 36 (Fig. 5), the last step 7-4 (N10C) is much easier than 7 + 6 via 7 + 3 + 3 (N10). Similar difficulties with crossing tens had been already observed during the first half of the program with problems like 18 + 6 and 48 + 6 (Beishuizen et al. 1996). By the way, this seems a typical sequential (European) mental arithmetic learning barrier, as most American pupils would solve such problems following a place-value procedure (48 decomposed in 40 and 8, 8 + 6 = 14, 40 + 14 = 54; cf. Fuson, 1992). Most of our weaker pupils, however, made good progress in overcoming these crossing-ten barriers with the support of the empty number line model. Like Wilco and Eddy (Fig. 5) their performance scores were about 75% correct on the terminal ASPT.

A last remark about the *procedure A10*, more frequently used by the better pupils in our study (Fig. 6). This outcome seems at first sight incompatible with the comment mentioned in the introductory section that on mental level A10 is not a smooth procedure (Klep, 1996). Recollecting the separate steps of A10 is supposed to put a higher load on working-memory than for instance the sequential N10, where the last step is the answer. We agree that such an observation makes sense, and indeed we have noticed examples of such 'memory' or 'book-keeping' errors in our results, where the A10 steps were correct but the answer was wrong (one step forgotten, etc.). How should we interpret then the A10 solution behavior of our better pupils, when we see in Fig. 7 how they progress – especially in June – to the higher-level notation form of 'mental steps' instead of number line support?

Anyway, A10 comes out of this study as an interesting procedure, with great differences in levels of use or '*mathematization*'. For an illustration of this conclusion we have to look again at a first and a last example in this chapter. Wilco (Fig. 1) demonstrated in his solution of the 'Leiden on sea' problem the use of A10 on a low modeling level (supported by the number line). Eddy (Fig. 5), on the other hand, exhibits in his solution of the context problem 48 + 37 via 48 + 32 = 80 etc. the much higher level of A10 carried out as *mental* and '*integrated*' steps (Fuson et al., 1997). In the problem 72 – 58 Eddy, however, makes an inaccuracy mistake with his new 'reconstructive' A10 procedure (Fig. 5). His difference problem (Fig. 5) is not solved by AOT/A10 but by the less advanced solution AOT/N10C. Nevertheless it is our conclusion that Eddy – although his solution behavior is not yet stable – developed from a weaker to a better pupil during the NL-program. (In fact his intelligence test score turned out to be rather high, so after all it appeared he had begun the program as an underachiever, cf. Beishuizen and Klein, 1996).

Towards the end of the NL-program some other better pupils began to demonstrate similar curtailments of mental A10, mostly in the more usual form of 48 + 37 via 48 + 2 = 50, 50 + 35 = 85. When pupils attain such flexible and integrated solution behaviors the differences between computation procedures as we classify them in N10, A10 and 1010, in fact begin to disappear. Perhaps such a development of further mathematization stimulates the *correspondence* or even *integration* between strategy and procedure. In this respect the discussed increase of AOT/TAT strategies in combination with increased A10 procedures of our better pupils on the difference problems may be interpreted as an indication of this promising perspective. If problems like 'Difference between 73 and 29?' are solved via 29 + 1 = 30, 30 + 43 = 73, answer 43 + 1 = 44 (AOT/A10), there is such a close correspondence between strategy and procedure that a distinction no longer seems to make sense. Since we have already given in the introductory section of this chapter such an integrated (SUB/N10C) solution of a better pupil Brit for the 'Leiden on sea' problem, one might ask what this chapter was all about, and why actually we posed the discussion question: strategy or procedure?

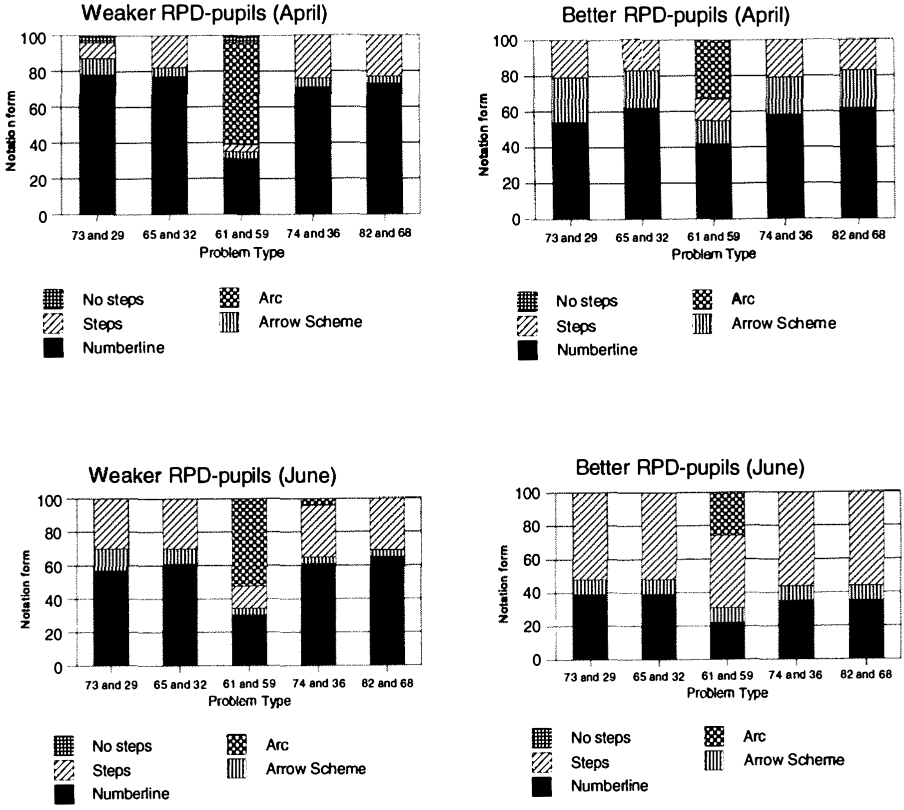


figure 7: type of notation use (percentages) by weaker and better RPD-pupils for 'difference' problems in the ASPT in April and June

We believe that our proposed distinction between solution strategy and computation procedure may aid a better understanding of integration processes as described above. From a developmental perspective, however, the earlier stages in the learning trajectory are even more important, when 'integration' is not yet there. In that respect the types of finer analysis as we described in this chapter may contribute to improved didactic design and improved guidance of pupils. For instance, during the experiment with our NL-program, difficulties like '*misfit*' between strategies and procedures became apparent in several errors. However, interpretation in some cases was a problem, for instance the discrimination between fundamental misunderstanding or temporary misfit (cf. Eddy's error discussed in the introductory section).

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Supporting multiple 2-digit conceptual structures and calculation methods in the classroom: issues of conceptual supports, instructional design, and language

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1 Introduction

In this chapter we will present theoretical descriptions of children's conceptual structures for 2-digit numbers and examine issues concerning how to support children in learning and using these conceptual structures in 2-digit addition and subtraction. We first briefly overview some aspects of our teaching approach in our current *Children's Math Worlds* project. We then summarize the UDSSI Triad Model of five conceptual structures for 2-digit numbers used by children who speak European languages. Next we describe an initial portion of our local instructional theory for helping children construct the three most advanced conceptual structures. Then we describe and discuss methods of 2-digit addition and subtraction and how these relate to problem situational structures and to 2-digit conceptual structures. We discuss classes of conceptual supports for 2-digit numbers and calculation, relationships between solution methods and conceptual supports, and six issues concerning implementing vertical mathematization and reflection in the classroom. We then briefly consider issues surrounding mental calculation.

We will use throughout the chapter the term 'method' rather than 'strategy' or 'procedure' for the way in which a child solves a 2-digit problem. 'Procedure' has for some readers a negative connotation of a rote method done without understanding. 'Strategy' implies some level of thoughtfulness and a choice of a method which may not be present for a given solution. We therefore prefer to use 'method' as a neutral term between these two extremes, and append adjectives if necessary for further definition.

2 Overview of Children's Math Worlds project

Children's Math Worlds is a project that is developing a mathematics curriculum by working initially and primarily in both English-speaking and Spanish-speaking urban Latino classrooms. We work simultaneously to develop understandings and models of children's conceptions of single-digit addition and subtraction, multidigit addition and subtraction, and word problem solving and to design effective teaching/learning activities that are based on children's understandings and are implementable in urban classrooms. The focus here is on our work in the multidigit domain; see Fuson, Hudson, and Ron (1996) for a summary of word problem work and Fuson, Perry, and Ron (1996) for an overview of the single-digit work.

Typically in the United States children are taught multidigit addition and subtraction without sufficient use of physical materials that help children construct concepts of multidigit numbers as consisting of groups of hundreds, tens, and ones. Instead many children view multidigit numbers as single digits placed beside each other (concatenated single digits); this view leads children to make many errors, especially in subtraction where they typically solve 72 – 28 as 56. In the *Children's Math Worlds* Project, we use various kinds of materials to help children construct conceptual understandings of numbers that they can use in computation. Because of large numbers of children who enter with little background in urban schools, and because of the long time it takes many children to construct robust conceptual multidigit structures, we focus heavily on materials that can help all children build methods that are generalizable to several digits. However, we also from the beginning of the multidigit work emphasize children's invention of mental (and sometimes also finger) methods for solving various problems and continue this focus on invention and exploration of different methods. Typically in our classrooms the top children invent a range of methods, middle children use quantities and then move to a written numerical method (often the traditional algorithms) that they can explain and understand, and lower children struggle to carry out correct methods using ten-structured quantities. With help, most of the lower children can come to general numeric methods they can explain, but some of them need to continue to use drawn quantities for long periods of time.

Nanci's Method

$49 + 25 = 6$

Four tens and two tens (writes 6).
(Looks at the ones; erases the 6.)
I can make another ten, and then you count the ones (fingers count 5 on to 9), writes 74.

Later she invents a way to record the new ten:

$49 + \overset{3}{\cancel{2}}5 = 74$

Cinthia's Method

$25 + 47$

I took three from the five and put it with the seven. Then I counted two plus four is six. Then there is another ten, so seven tens, and there are two left, seventy-two.

Later she invents

$48 + 2\overset{5}{\cancel{7}} = 75$

Viviana's Method

$48 + 23$

Forty and two tens makes sixty.
Eight in my mind. 68, 9, 10, 11, 71.

Martha and Rufina's Methods

$\overset{1}{\underline{3}}7 + \underline{2}6 = 63$

$\underbrace{48 + 16} = 64$

$$\begin{array}{r} \overset{1}{2}5 \\ +\ 49 \\ \hline \end{array}$$

Jorge's Method

$\underset{\uparrow}{5}6 + \underset{\uparrow}{2}7 =$

I know these are tens.
50, 60, 70. Then I counted 7 (7 fingers up): 71, 72, 73, 74, 75, 76, 77. Then I counted 6 more (6 fingers up): 78, 79, 80, 81, 82, 83.

Karina's Method

$37 + 56$

Eight. Eighty seven
(counts on fingers, 6 fingers)
8, 9, 10, 90, 93

Methods of Marking Tens and Ones

$\overset{\text{TO}}{34} + \overset{\text{TO}}{19} = \_\_\ \_\_$

$4|8 + 1|6 = \_\_\ \_\_$

$\underline{4}7 + \underline{2}8 = \_\_\ \_\_$

$\underbrace{25 + 47} = \_\_\ \_\_$

figure 1: mental and written numeric 2-digit addition methods

To overview the early phase in multidigit addition, Figure 1 displays a few of the methods first graders near the end of the year invented when given horizontally presented tens and ones and challenged to see if they could find ways to solve them without drawing all the tens and ones. Notice that all the methods involve shifts between external and internal compositions of tens and ones. All at the very least involve written numbers serving as external memories that enable children to point out and focus on parts and then later return to other parts without 'forgetting' them, as they might for orally presented problems. Jorge (56 + 27) points to the 5 and the 2, emphasizes that they are tens, can and does count on tens from 50 keeping track internally (50, 60, 70), then uses 7 external finger counters to count on ones and then 6 more fingers to count on the rest of the ones in the problem. Nanci (49 + 25) composes 4 tens and 2 tens as 6 tens (internal fact), writes it (external token), looks ahead at the 9 + 5 ones, erases the 6 and says 'I can make another ten', increments the six tens to seven tens internally and then writes the 7, then uses fingers (external ones tokens) to count 5 onto a mental 9 ones. She also knows to avoid using the ten again when counting on 9 + 5 (because she already incremented her tens before finding out exactly how many ones there were). Cinthia (25 + 47) seems to make visual use of the written numbers, allowing her to take '3 from the 5 ones and *put* it with the 7 to make 'another ten', and still retain that there are '2 left' (from the 5). Knowing that 3 was needed to make ten, and being able to take 3 from 5 and know what is left, appear to happen on the fact level. Karina (37 + 56) also makes use of breaking the sum of ones into ten and ones left, but does quite a range of other methods as well (not shown in Figure 1). Viviana, except for use of the written number problem (48 + 23), seems to count tens and ones entirely internally: 'Forty and 2 tens makes sixty (note mixture of sequence tens and separate tens), 8 in my mind. 68 (mentally adjoining 8 ones), 9, 10, 11, (converting 11 into ten increment from sixty to seventy and one more) 71.'

Not only can different children work at different levels in the same classroom using external quantities, external tokens of them (fingers, written numbers, drawn tens and ones), and internal versions of quantities and words, but even advanced children find it helpful to weave methods across internal and external countable tens and ones, possibly to distribute processing burdens. The variation, even beyond affording accommodations to individual needs, seems to be helpful in stimulating thinking. Some children in this first-grade classroom preferred to continue working exclusively with quantities. Although we expose children to the challenge of composing tens and ones by media other than external tens and ones, we also let children do whatever they need to do to solve problems.

A major issue at this stage for many children was differentiating and remembering which of the numbers were tens and which were ones. Because we gave problems horizontally to force children to attend to this differentiation, children invented varied and elaborate scaffoldings to mark which were tens and ones. They underlined tens, drew loops and lines to connect the tens, drew separating lines between tens and ones, and labeled tens and ones. Rapid correct tens and ones interpretations are crucial to any internalization of multi-step 2-digit operations. Children have to chunk partial results and so have to know what to chunk with what. If they have to spend much attention on what goes with what, they can easily overload memory, lose track of what they are doing, and forget the numbers involved in the situation or their already obtained partial results.

3 Analysis of the mathematical domain

3.1 A model of conceptual structures used in the domain

Earlier literature identified three correct conceptions used by children in the United States: a unitary conception in which children count a 2-digit quantity by ones, a sequence conception in which they count by tens and then by ones, and a separate tens and ones conception in which the units of ten and the units of one are counted separately (see Fuson, 1990a, for a review of this literature). For example, if counting 3 bars each made from 10 unifix cubes and 2 extra cubes, children using a unitary conception would count all 32 of the unifix cubes (1, 2, 3, ..., 32), children using a sequence-tens conception would count '10, 20, 30, 31, 32,' and children using a separate-tens conception would count '1, 2, 3 tens and 1, 2 ones. 32.'

Children also use a concatenated single-digit conception in which the 2-digit number is thought of as two separate single-digit numbers. Because any single-digit number can be added to or subtracted from any other, this meaning cannot direct or constrain addition or subtraction methods. It leads to many well-documented errors (e.g. see VanLehn, 1986, for a discussion and examples). This concatenated single-digit meaning arises when insufficient opportunities are given to children to link accurate multi-digit quantity meanings to the written numerals in use in adding and subtracting.

In Fuson et al. (in press) and in Fuson, Smith, and Lo Cicero (in press), we extended this earlier work to a UDSSI Triad Model named for the five correct conceptions described in the model: unitary, decade, sequence-tens, separate-tens, and integrated conceptions. The UDSSI Triad Model is shown in the main part of Figure 2 (taken from Fuson, Smith and Lo Cicero, in press). Our view of these conceptions is that they involve a triad of relationships between quantities, number words, and written number marks. With single-digit numbers, there are three 2-way links in the triangle formed by these quantities, words, and marks (see the top left corner of Figure 2). Each 1-way link describes the numerical aspect initially seen or heard and the aspect that is linked to it; for example, I hear 'five' and think/see/can write 5 (bottom left-to-right arrow) or I see five birds and think/can say 'five' (left arrow from top to bottom). The user of the concatenated single-digit conception (top left) constructs these six relations for each of the pairs of single digits in a 2-digit number.

Children's early conceptual structures are shown on the outside of the big triangle in Figure 2. All children begin with a unitary conception that is a simple extension from the unitary triad for single-digit numbers. With this conception, the separate number words (e.g. twenty six) and the two digits (e.g. 26) do not have separate quantity referents. The whole number word (e.g. sixteen) or whole numeral (16) refers to the whole quantity. With time and experience, the first digit takes on a meaning as a decade in the decade and ones conception, and the second digit takes on a meaning as the extra ones in a decade. The number marks for this decade conception can be better understood if one thinks of the ones as written on top of the decade quantity (the arrow in Figure 2 shows the 3 going on top of the 0 in 50). This conception of a 2-digit quantity as a decade and some ones was identified by Murray and Olivier (1989). It leads some children to write number marks as they sound: as 50 and then a 3, so 503.

The sequence-tens and ones conception develops out of the decade conception as children become able to count by tens and to form conceptual units that are groups of ten single units (these may arise independently). Initially with the sequence-tens conception, there is no immediate knowing that there are five tens in fifty, though a user of this conception could find out by counting '10, 20, 30, 40, 50' while keeping track of the five counts.

Some children have experiences in which they come to think of a 2-digit quantity as composed of two kinds of units: units of ten and units of one. When adding or subtracting 2-digit numbers in this way of thinking, children count, add, or subtract the units of ten and then count, add, or subtract the units of one (or vice versa), leading to our designation of this way of thinking as the separate-tens and ones conception. In Figure 2, we show these units of ten as a single line to stress their (ten)-unitness, but the user of these units understands that each ten is composed of ten ones, and can switch to thinking of ten ones if that becomes useful.

Children's construction of the sequence-tens and separate-tens conceptions seems to depend heavily on their learning environment, though individuals in the same classroom may construct one or the other of these first. Which is first may partly depend on whether a child focuses on the words, which facilitate the sequence-tens conception, or on the written numerals, which facilitate the separate-tens conception.

Children may eventually construct both the sequence-tens and separate-tens conceptions and relate them to each other in an integrated sequence-separate conception (these connections are shown in Figure 2 as the double arrows). Children connect fifty and five tens, and the written marks 53 can take on either quantity meaning (fifty-three or five tens three ones).

Although we had originally conceptualized each of the 2-digit conceptual structures as a triad of six relations, it later became clear that only the separate-tens and ones conception has direct links between quantities and marks, and then only where the quantities of tens and ones are small enough to be subitized (immediately seen as a certain number of units) or are in a pattern. The other three conceptions must relate quantities to written marks via the number words by counting. Therefore the link between quantities and marks is not drawn in Figure 2 for these conceptions.

3.2 Learning to construct and operate on 2-digit quantities in Children's Math Worlds

There is a widespread cultural activity in which young children invest considerable efforts on their own, before entering school, that can serve as the point of departure for activities helping children to appropriate the grouping and counting processes at the core of the base-ten number system and multi unit arithmetic: Children try to extend their counting sequences. They initially try to do so by directly extending the chaining process by which they have memorized the first ten number names, and then they make use of what regularities are accessible to them. The number names give a pattern of x-ty 1 through x-ty 9 chunks (e.g. 21 to 29), then a shift to some new 'x-ty' word, to which a further x-ty 1 through x-ty 9 chunk can be appended. However, the English number names do not clearly signal the order of the decade words. 'Forty' is not obvious as a verbal abbreviation of 4 tens, and the rest of the first 5 decade names are either irregular (the teens) or offer even more obscure references ('twen' for 2, 'thir' for 3, 'fif' for 5). Children are reduced to attempting to memorize which 'ty' word comes next, amidst the interference of intervening x-ty 1 through x-ty 9 cycles. The errors children typically make reflect precisely these chunks, but with a confused decade list, e.g. 1 to 29, 50, 51 to 59, 30, 31 to 39, 20, 21 to 29, 40, etc. (Fuson, Richards and Briars, 1982). Cross-sectional data indicate that it takes children in the United States on the average about one and a half years to learn how the decades themselves are ordered (Fuson, Richards and Briars, 1982). Further, this learning is frequently interactively constructed as if it were a simple memorization task. The child counts one through twenty-niiiiiiiine, and pauses, searching for the next 'ty' word. The adult or child audience either immediately supplies the correct word to fill the pause or waits for the counter to make a guess and corrects if necessary. The child then marches rhythmically through the next x-ty one through x-ty nine chunk, to the next memory search for the next decade word.

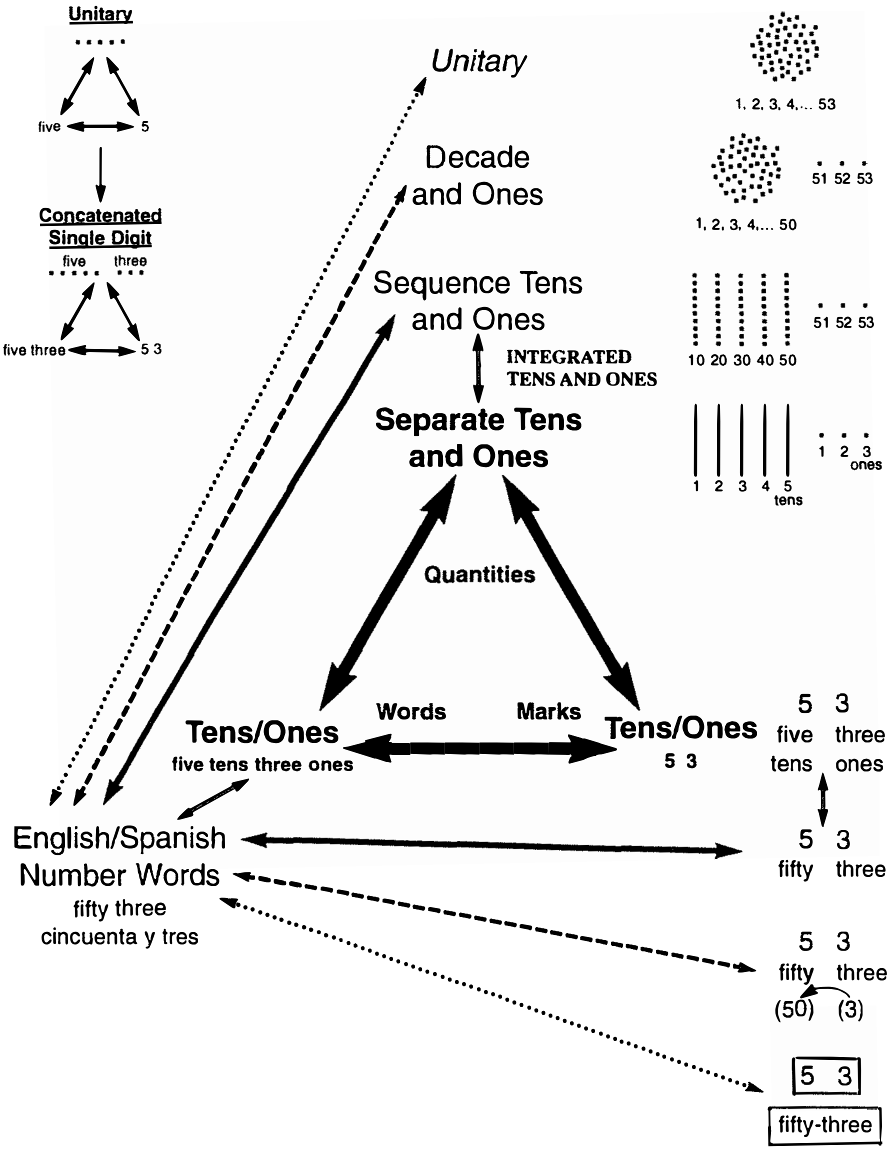

figure 2: a developmental sequence of conceptual structures for two-digit numbers: the UDSSI triad model

3.3 Helping children to build a generative core of base-ten quantity, number word, and number marks interrelationships

We have found that many urban first graders, some second graders, and a few third graders cannot count to 100. We therefore have had to design activities to help all children learn to count to 100. We begin by responding to children's search for the next decade word, not in the framework of memorization, but by showing them that they can learn to figure out what the next decade word is going to be. We try to make the relationships at each ten between the number of groups of tens (e.g. three tens), the name for how many things are in those tens (e.g. thirty), and the written number marks (30) accessible for learning and understanding. We simplify children's access to this network by centering the task as seeing and counting groups of ten ('one ten, two tens, three tens'). This is a simple extension of ordinary counting, and may even be, at least initially for some children, ordinary counting (i.e., the 'tens' are not yet units of ten or perhaps not even ten ones for them, initially). This count *of* tens then is linked to written number marks as telling how many tens (1̲0, 2̲0, 3̲0) and to number words (by counting all the things inside the tens to find out how many are in that many tens).

Figure 3 is simultaneously a model of the conceptual structures children need to construct in learning to count to 100 by tens and by ones using decade, sequence-tens, separate-tens, and integrated tens conceptual structures and a model of the teaching activities we design to help them do so. Children enter the network at 1a (using a mental grouping action to focus on the ten). The teacher then introduces the possibility of using the count of tens groups to figure out how to write and name the number (1b in Figure 3). We will describe that process below. The bottom half of Figure 3 is the later internalization of all or parts of the initially external counting activity, though, of course, many aspects of the top model are internal conceptual activities from the beginning of conceptual counting. Figure 3 shows at the top the ten groupings we used in the *Children's Math Worlds* project. These are strips of cardboard each showing ten pennies; on the back is one dime (the U.S. 10¢ coin). Below the penny strips is shown the beadstring used in the Dutch program. The activities we use in the classroom are designed to help children build sequence-ten and separate-tens conceptions and relate them to each other.

3.4 Counting by tens, counting the tens, number marks, and number names

We begin by building up tens grouping experience and exploring the relationship between tens groupings and count words. We ask, for example, 'How many groups of ten can you make from 30 pumpkin seeds?' and conversely, 'If you have 3 groups of ten pumpkin seeds, what number of pumpkin seeds do you have?' We finally discuss 'Is 3 tens the same number as 30?' Children are initially divided in their opinions on that question: The 'same number as' concept is as much under construction as it is a tool in early grouping discussions (Baroody and Ginsburg, 1985; Fuson, 1988), and counting as a criterion is vulnerable to a lack of counting expertise.

But systematically iterating the 'x tens -> what number?' question quickly builds up counting expertise and establishes the core of base-ten quantity, word, and numeral interrelationships (see 1b in Figure 3, 'the tens count process' for a model of this process). A specific account of one way of building up this core may be helpful. The teacher puts up 1 group of ten (we use strips of 10 pennies, but any objects grouped by ten will do), counts the pennies by ones with the class, discusses the writing of '10' below the penny strip in terms of its meaning as 'one ten' and (pointing to the zero '10') no ones extra.' Here, the links of the words to the written numerals and their left-right positions are crucial and are emphasized by gestures. The teacher then puts up another ten and asks 'How many tens now', eliciting the answer from the children and then modeling getting the answer by counting the tens, '1 ten, 2 tens', and again writes that answer, that number of tens counted (20), below the penny strip (again connecting the left-right positions to the number of tens and the number of ones). The goal of this part of the activity is to help children learn that if they can count the number of tens, they can write the correct 2-digit numeral: the 2-digit numeral *means* the number of tens. Finally the teacher asks, '2 tens is what number?' At this point the number name 'twenty' can be cued by a range of meaningful sources. More knowledge is pointing to it than just one (maybe memorized, maybe not) link in a verbal chain (19, 20). Some children may already have grasped the gist of the initial 'same number' discussion, or at least be cued by it. Some will have written numeral -> number name knowledge ('20' -> 'twenty') and are cued by that. Some first graders know that 'ten and ten makes twenty' and volunteer that. Some will have rapidly counted by ones while the teacher was posing the question and volunteer 'twenty'. This is a process of social elicitation: Some in this cultural pool invariably use one or another of these cues. The teacher can further establish this linkage by leading a choral count of the individual pennies to 'check' if 2 tens is 'really' twenty (most first graders can count to 20). Finally, a chaining mechanism of tens is evoked to establish a counting sequence. The count of tens built up so far is reiterated '1 ten, 2 tens' (while pointing to each strip) and also 'ten, twenty' (also while pointing to each strip). But this sequence of counting by tens now inherits the range of links that effectively cue it for that individual.

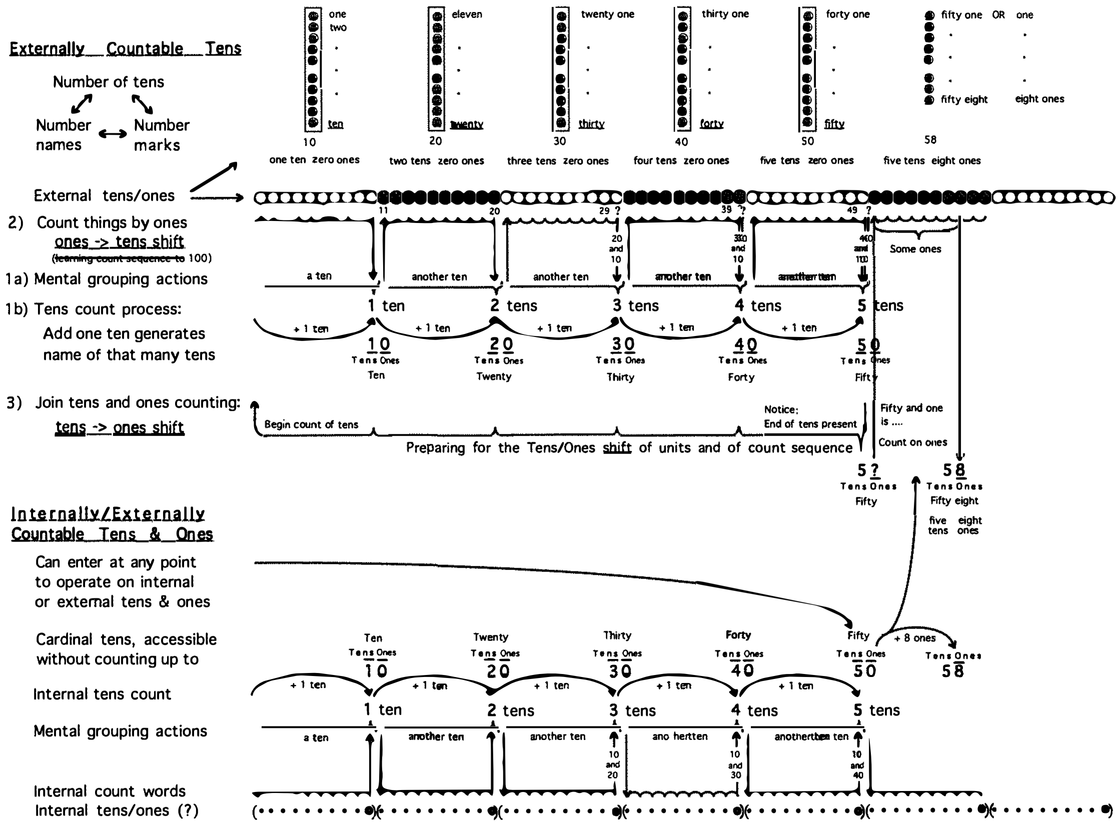

figure 3: a generative tens teaching / learning model for learning number relationships in counting to 100

For each remaining decade to 100, the mappings between the number of tens, number marks, and number words can then be elicited from the children themselves, almost without error, by adding 1 ten, then reiterating the questions referred to above: How many tens now? How do we write x tens? X tens is what number? With each added ten, the accuracy with which the number name is inferred increases (Smith, 1994). Over several such counts, most children learn to figure out the number name by this counting of the number of tens. A generative activity replaces memorization. Over a few days, many children learn to count tens both ways (count the tens and count by tens) and to map reversibly between each combination of the number of tens, written numerals, the number of ones, and number names. They can also quickly learn to add tens quantities (e.g. 60 + 30) because they can now construct and count them (if they have physical tens groupings with which they can work or which they can draw).

The teacher orchestrates further performances of various counting activities using the generative-tens model. Some children initially participate only with tens and ones words because those are the simplest. Increasingly, children come to fill in more and more parts of their own web of knowledge within counting activities. All the complex links involved are made by the teacher in different ways in different activities. The focus is continuously on helping children to link the number of tens, the written number marks, the number names (how many things in all), and the number of ones, and to negotiate the ones/tens and the tens/ones counting shifts. This focus continues throughout the 2-digit addition and subtraction activities, which are viewed as settings within which children can build up and use their 2-digit web of knowledge.

3.5 Counting things by ones to 100

For children to learn to use the tens count to support counting to 100 by ones (step 2 in Figure 3: the ones-to-tens shift) requires a further layer of interactive attentional direction, directly corresponding to an extra layer of complexity in the activity. Figure 3 portrays the extra layer of attentional directives involved in shifting from counting ones to using the grouping and incrementing tens process discussed above. This layer of attention is again (as with the tens count) established interactively in the classroom, via questions. The point is to help children replace pausing and searching memory for the next ten with pausing and framing a question. If a child is at twenty-nine, for example, and pauses, what frames the question is realizing that s/he now has counted up to twenty *and ten*, or another ten. This is the ones-to-tens shift. Once s/he asks the question, how many tens now?, s/he has shifted into the tens count process, discussed above and shown in 1a and 1b. The use of external tens groupings (see examples at the top level of Figure 3) is quite important here because the child can then simply count the number of tens so far, including the new ten. With more experience, a child may anticipate this process and abbreviate it, keeping track of the tens already counted, and knowing that one more ten is to be added: '2 tens, and one more ten makes three tens, thirty.' As that anticipation becomes consistent, so that a child is looking forward to the next ten (thirty) while counting up towards it in the twenties (see 3 in Figure 3), the counting sequence is becoming automatized, but now in the correct order, and with countable tens groupings embedded in it.

Through such activities, these countable tens are simultaneously building up rapid reversible mappings between quantity, number word, and written numeral uses, as sketched above. A number of tens can be produced in any of these forms (and added as well), given any other. Tens can be counted and produced as separate groupings as well as in sequence form.

3.6 Join tens and ones counting

But tens *and* ones cannot yet be counted or produced jointly to find any 2-digit number without a further layer of attentional directives. We have found that children can map a range of ten relationships, and be able to discriminate tens and ones, without being able to count them together. The tens count has a momentum once it gets going, and kids simply continue that tens count onto any ones present (Smith 1994; Fuson and Smith, 1995). Figure 3, at the 'Join tens and ones counting' level (step 3), portrays the layer of attentional directives involved. Children need to anticipate that some extra ones as well as some tens are to be counted, and maintain that anticipation sufficiently to monitor when all the tens have been counted, and then stop counting tens. They then, or earlier while tens-counting, need to prepare for the tens/ones shift of units (from units/groups of tens to units of ones) and of the counting sequence (from counting the tens or counting by tens to counting by ones from one or from the decade word). If they were counting by tens (ten, twenty, thirty, forty, fifty), they need to know that 'fifty and one more is fifty-one'. That is, they need to understand at the ones level a relationship of 'another entity' is '+1' is 'the next count word' within the unitary/decade count sequence. The counting by ones sequence also has to be familiar enough that children can easily negotiate this while remembering their tens count of 'fifty'; kindergarten children sometimes forgot their tens count result when initially attempting the tens/ones shift (Smith, 1994). If children were counting the tens ('one, two, three, four, five tens'), the tens/ones shift is easier: They just start counting from one again. But the cardinal joining of the tens and ones at the end of both the tens counts and the ones counts is simpler for the sequence-tens than for the separate-tens words because 'fifty eight' carries a joining from the unitary and decade conceptual structures. Managing everything in either tens/ones shift is demanding, and it takes most kids repeated efforts to do so consistently.

Again, this layer of attentional directives is established interactively, via questions, but a very simple question usually suffices to enable children to self-correct their errors: When a child misses the shift and counts ones as tens, a helper asks, 'Are those tens?'. This is often enough to engineer a shift to counting ones at that point, though multiple attempts across multiple sessions are often required to establish it consistently (see Smith, 1994, for a study of kindergartners negotiating this shift and Fuson and Smith, 1995, for a case study of first-grade peer tutoring with adult help). For some children, more scaffolding may be required, e. g., 'What are they?' (ones) 'So fifty and one makes...' (children can usually then continue the rest of the way to the next tens-ones shift).

With the ability to count jointly separate external tens and ones groupings into a whole number, or produce separate tens and ones from a verbal or written whole number, children have a minimal core of processes sufficient for adding 2-digit quantities with regrouping as well as other uses. We also help children develop right away other tens counting and adding processes to explore and use in multi unit arithmetic (this is often labeled 'mental addition': we will address this below). In particular, we develop counting on from 2-digit numbers as a basis for a broader development of 2-digit addition and later subtraction. Children keep track of such counting on with fingers or other means. Thus, we try to make available as rapidly as possible the whole range of solution methods to be discussed later by asking children to try to solve some problems without drawing all of the quantities, while of course allowing those who really feel that they need to do so to draw them.

Another naturally-occurring cultural activity begins to elicit internal tens and ones models: coin counting. Counting dimes and pennies extends counting-tens-and-ones experience to an activity in which children must overcome the perceptual influence of a dime as 'one' thing rather than as a visible ten (see Fuson and Smith, 1995, for a case study of such difficulties). Adding nickels to dimes and penny situations requires a child to construct a method for tracking the ones to be counted on: a nickel (5¢) does not offer 5 ones to count externally, but children can use 5 fingers or learn to count on by 5's. Another aspect of counting on, being able to start at any point in the counting sequence, is developed with other kinds of problems (e.g. A soda costs 48¢, but you also have to pay 4¢ tax. How much do you have to pay?). Being able to count on from any 2-digit number can be learned conceptually in the same manner as single-digit counting on (see Fuson, 1988, for a summary). The main constraint here is an insufficiently learned count-to-100 sequence. Such problems also give practice in counting over a decade word, which some children need.

3.7 2-digit addition

With experience building up in parallel in both counting external tens and ones and counting on that involves internalized abilities to start counting at any point in the sequence (without counting up to it) and the construction of tracking methods (e.g. fingers) in lieu of tens or of ones to count, children can construct a wide range of methods for 2-digit addition. However, some need to continue working with external tens and ones for a long time. Indeed, 2-digit addition with regrouping actually serves to consolidate competence at counting external tens and ones. Children who are still constructing the sequence-tens words can use tens and ones words and counts while in other activities working on their counting by tens.

Two-digit addition in this mode minimally requires constructing two sets of tens and ones, then counting the tens and ones together. Children don't even have to consciously regroup if they count the tens first: They can continue counting ones in the sequence across decades. Some first-grade children may at this point still be confusing which are tens and which are ones in 2-digit numbers (due to left-right confusion) or occasionally count ones as tens. The examples given earlier demonstrate the range of methods some children can construct while others are still struggling with basic place-value and 2-digit concepts. We view 2-digit addition problems in which the ones exceed ten (e.g. 38 + 26) as excellent activities within which children can continue to construct and use ten-structured 2-digit conceptions of numbers.

4 Two-digit addition, subtraction and unknown addend methods: Issues concerning problems, instructional sequences, conceptual supports, and number words

4.1 2-digit addition and subtraction methods and their developmental relationship to problem situation structure

Methods used by children to solve 2-digit addition, subtraction, and unknown addend problems are shown in Table 1; these methods were used by children in four projects that emphasized learning mathematics with understanding (Fuson et al., in press). Table 1 has been adapted from Fuson et al. (in press) to label in bold the methods identified by Beishuizen (Beishuizen, 1993, this volume; Klein, Beishuizen and Treffers, in press) that are used by Dutch children receiving instruction using traditional textbooks, using a Realistic approach with an empty number line, or using a Gradual approach with an empty number line. See Beishuizen (this volume) for more detailed descriptions of the methods in bold.

Begin-With-One-Number Methods: Increase or Decrease By Tens and Ones		
38 + 26 = □	64 - 26 = □	38 + □ = 64
Count on/add on tens then ones **AD/N10** [58+2=60; 60+4=64]	Count down/subtract tens then ones **SUB/N10** [44-4=40; 40-2=38]	Count up/add up tens, then ones **AOT/N10**
38,48,58,59,60,61,62,63,64 38 + 20 —> 58 + 6 —> 64	64,54,44,43,42,41,40,39,38 60 - 20 —> 44 - 6 —> 38 or 64 - <u>26</u>	like count on, keep track: count up 26 like add on, keep track: add up 26
Overshoot and come back^{a} **AD/N10C** 38 + 30 —> 68 - 4 —> 64	Overshoot and come back^{a} 44 **SUB/N10C** 38 64 - 30 —> 34 + 4 —> 38	Overshoot and come back^{a} **AOT/N10C** like addition, keep track: add up 26
Count on/add on to make a ten, count on/add on tens then rest of ones **AD/A10** [40+24=64]	Count down/subtract to make a ten, count down/subtract tens then rest of ones **SUB/A10**	Count up/add up to make a ten, count up/add up tens then rest of ones **AOT/A10**
38,39,40,50,60,61,62,63,64 38 + 2 —> 40 + 20 —> 60 + 4 —> 64	64,63,62,61,60,50,40,39,38 64 - 4 —> 60 - 20 —> 40 - 2 —> 38	like count on; keep track: counted up 26 like add on; keep track: added up 26
Mixed Methods: Add or Subtract Tens, Make Sequence Number With Original Ones, Add/Subtract Other Ones		
Count on/add on tens, add ones, count on/add on other ones **AD/10s**	Count down/subtract tens, add original ones, count down/subtract other ones **SUB/10s**	Count up/add up tens, add original ones, count up/add up other ones **AOT/10s**
30,40,50,58,59,60,61,62,63,64 30 + 20 —> 50 + 8 —> 58 + 6 —> 64	60,50,40,44^{b},43,42,41,40,39,38 60 - 20 —> 40 + 4^{b} —> 44 - 6 —> 38	30,40,50+8^{b}—>58,59,60,61,62,63,64;26 30 + 20 —> 50 + 8^{b} —> 58 + 6 —> 64; 26

Change-Both-Numbers Methods

Move some from one number to the other to make a tens number[c] (maintaining the total)
AD/A10

38 + 2, 26 - 2 —> 40 + 24 —> 64

Make subtracted number a tens number, change other to maintain difference

26 + 4, 64 + 4[d] —> 68 - 30 —> 38

Latin America, Europe: add a ten to both numbers

$$\begin{array}{r} 6^{1}4 \\ \underline{-\,^{1}2\,6} \\ 3\,8 \end{array}$$

Make initial number a tens number, change other to maintain difference

38 + 2, 64 + 2[d] —> 40 up to 66 —> 26

38 + 26	64 - 26

Decompose-Tens-and-Ones Methods: Add or subtract everywhere, then regroup

Add tens, add ones, **ADD/1010** make 1 ten from 10 ones

$$\begin{array}{r} 3\,8 \\ \underline{+2\,6} \\ 5\,0 \\ \underline{1\,4} \\ 6\,4 \end{array} \qquad \begin{array}{r} 3\,8 \\ \underline{+2\,6} \\ 514 \\ 6\,4 \end{array} \qquad \begin{array}{r} 3\,8 \\ \underline{+2\,6} \\ \cancel{5}4 \\ 6\;\; \\ \enclose{circle}{6\,4} \end{array}$$

or erase 5

Add ones, add tens make 1 ten from 10 ones

$$\begin{array}{r} 3\,8 \\ \underline{+2\,6} \\ 1\,4 \\ \underline{5\,0} \\ 6\,4 \end{array} \qquad \begin{array}{r} 3\,8 \\ \underline{+2\,6} \\ 5\,14 \\ \enclose{circle}{6\;\;4} \end{array}$$

Subtract tens, subtract ones, combine totals **SUB/1010**
This requires some notion of negative numbers of owing, or ones "in the hole"

$$\begin{array}{r} 6\,4 \\ \underline{-2\,6} \\ 4\,-2 \\ 3\,8 \end{array} \quad \text{so } 40 - 2 = 38$$

Subtract ones, subtract tens, combine totals
same as preceding method

4 - 6 = 2 is difficult
so 4 - 6 = 2 —> 42
is a typical error

Decompose-Tens-and-Ones Methods: Regroup, then add or subtract everywhere

Look to see if total ones ≥ 10, record or remember, then make 1 ten from 10 ones, add tens, add ones or make 1 ten from 10 ones, add ones, add tens.

$$\begin{array}{r} 38 \\ \underline{+26} \\ 64 \end{array} \qquad \begin{array}{r} 4\;\, \\ \cancel{3}8\cdot \\ \underline{+26} \\ 64 \end{array} \qquad \begin{array}{r} 1\;\, \\ 38 \\ \underline{+26} \\ 64 \end{array} \qquad \begin{array}{r} 38 \\ \underline{+26} \\ 64 \end{array}$$

Make 10 ones from 1 ten [open a ten], then subtract tens, subtract ones or subtract ones, subtract tens

$$\begin{array}{r} 5^{1}4 \\ \cancel{6}\;\cancel{4} \\ \underline{-2\;\;6} \\ 3\;\;8 \end{array} \qquad \begin{array}{r} 5^{1}0 \\ \cancel{6}\,4 \\ \underline{-2\;6} \\ 3\;8 \end{array} \qquad \begin{array}{r} 5\;\;\, \\ \cancel{6}^{1}4 \\ \underline{-2\;\;6} \\ 3\;\;8 \end{array} \qquad \begin{array}{r} 64 \longrightarrow 5\,0 \\ \underline{-2\,0} \\ 3\,0 \end{array} \qquad \begin{array}{r} 1\,4 \\ \underline{-\;6} \\ 8 \end{array} \quad \longrightarrow \quad \begin{array}{r} 6\,4 \\ \underline{-2\,6} \\ 3\,8 \end{array}$$

Decompose-Tens-and-Ones Methods: Alternate adding/subtracting and regrouping			
Alternate adding and making another ten:		Alternate subtracting and opening a ten:	
Add tens, look to see if there is another ten, add ones	Add ones, make 1 ten from 10 ones, add tens	Subtract tens, open a ten, subtract ones	Subtract ones, open a ten, subtract tens
3 8 or 3 8 look + 2 6 erase +2 6 before ~~5~~ 6 4 writing 6 4 tens	these look just like regroup, add ones, add tens methods above	1 4 6 ~~4~~ or 6 4 - 2 6 erase -2 6 ~~4~~ ← 3 8 3 8 open before writing tens	6 4 5 8 -2 6 - 2 0 - 2 3 8 [we have not seen this]

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[a] The reverse of this method is also used occasionally: increase the first number to make an easy addition/subtraction, add/subtract, and decrease the answer to compensate (38 becomes 40 + 26 = 66 - 2 = 64 or 64 becomes 66 - 26 = 40 - 2 = 38).

[b] Forgetting to add back in the original ones (the 4 from 64 or the 8 in 38) or subtracting them are (in a subtraction problem) frequent errors. The ones from the 26 sometimes are subtracted first and then the ones from the 64 are added back in; forgetting to add the 4 or subtracting it are also frequent errors.

[c] This is a different way to think of the method just above (sequence add on to make a ten) in which the rest of the number is added at once instead of in two steps as tens and ones.

[d] This step is difficult; the number is often subtracted rather than added, confusing what must be kept constant in addition (the total) and in subtraction (the difference).

Note. All methods in the table are for problems requiring regrouping (making another ten from ten ones or opening of ten to make 10 ones). This is done explicitly or by counting or adding/subtracting over a ten. Problems without regrouping are much simpler. All separate-tens-and-ones methods may also be written horizontally. Unknown addend methods for separate-tens and ones can be done by writing the addition problem with the second number empty and then adding up to the total to find that number. Single-digit subtraction for separate-tens and ones methods may be done as an unknown addend method (forward count up/add up methods). This table has been adapted from Fuson et al. (in press).

table 1: 2-digit addition, subtraction, and unknown addend methods using sequence-tens and/or separate-tens

Table 1 identifies three kinds of 2-digit methods: addition methods in which two 2-digit numbers are combined to make a total, subtraction (take-away) methods in which a 2-digit number is taken away from a larger 2-digit number, and adding-on unknown-addend methods in which the unknown addend is found by adding on from the known addend to get the known total (the number added on is the unknown addend). A fourth method can be used: taking-away unknown-addend methods in which the unknown addend is taken away from the known total to reach the known addend (the number taken away is the unknown addend). We did not include this method in Table 1 because it was rarely used by children in our projects (most children used adding-on unknown-addend methods instead).

The four classes of methods in Table 1 are taken from the literature about methods children use to solve single-digit word problems (see reviews of this literature in Fuson, 1992a, 1992b, 1994, where kinds of word problems are related to kinds of solution methods). The four classes of single-digit methods move through three (or four, depending upon details of classification) developmental levels of increasing abstraction and abbreviation. Children begin by directly modeling the problem situation with objects; they count out objects to show each number in the problem. They later begin to abbreviate initial modeling steps by embedding addends within totals, and they use the number words themselves to show numbers in the problem (counting on, counting back, counting up to, counting down to). Even later they chunk small numbers within other numbers to use derived facts (e.g. $6 + 7 = 6 + 6 + 1 = 12 + 1 = 13$). Finally, they may know number triplets so that they can immediately generate an answer.

In the first stage, children's solution methods directly follow the problem situation. At the number-word solution level, many children frequently solve a problem using a method that directly models the problem situation (e.g. counting up to for a Change-Add-To unknown change problem). However, some children begin to free themselves from the problem structure and select problem solution methods that differ from the problem situation (e.g. using counting up to for a Change-Take-From unknown result problem that formerly was solved by taking away). This freedom can be facilitated by instruction that discusses such alternatives. Until this independence of solution method from problem situation occurs, the solution methods in Table 1 also tend to describe the structure of the underlying problem situation that is solved by that method.

Many traditional methods of instruction have assumed that children have only two classes of methods: addition and subtraction. Children were to solve problems by deciding which class of solution methods to use (i.e., whether to add or to subtract). They usually were supposed to write a number sentence showing this method (e.g. $14 - 8 = ?$) and then carry out the operation shown in the number sentence. Research in the 1970's and 1980's (see Carpenter, Hiebert and Moser, 1983, or the above references for reviews) indicated that some children instead used number sentences to show the problem situation (e.g. 8 + ? = 14). Children forced to write a solution sentence that differed from the problem situation often solved the problem first and then wrote the solution sentence. Thus, word problem solving for children goes through at least four distinct levels of conceptualization (Fuson, Hudson and Ron, 1996). A solver first forms a *situation conception*: an initial conception of the problem situation in the world (I have some apples of which I eat some). The mathematical elements are then focused on to construct the *mathematized situation conception* (e.g. 14 take away 8 to make how many?). The unknown is then focused on to construct the *solution method conception* (e.g. 8 plus how many will give me 14?). That solution conception is then carried out by particular solution actions (e.g. counting from 8 up to 14 with fingers).

Problem situations involving 2-digit quantities can be solved by unitary methods just like the single-digit methods. Children's external models for the mathematized situation conception, the solution method conception, or the solution method itself may be at any of the developmental levels, though the final level of known fact is rare for 2-digit numbers except for special combinations such as 50 + ? = 100. However, many children also begin to develop conceptual structures for 2-digit numbers that enable them to carry out solution methods involving counting or adding/subtracting groups of ten entities. These more complex 2-digit solution methods fall into the same four classes of methods identified for single-digit numbers and outlined above with respect to Table 1. Because children construct these conceptual structures using groups of ten only after they have reached at least the second single-digit stage of counting on/counting up, they may already have some freedom from the problem structure in selecting a solution method. However, this issue of the extent of the freedom children can exercise in their choice of a solution method for given problem situations (what Beishuizen, this volume, called 'mismatches') has not been researched nearly as much as for single-digit numbers. Because the counting down methods tend to be so difficult, the counting down to methods may be even more so. Therefore in Table 1, we only emphasize counting up to, because it has been found to be simpler than counting down for single-digit subtraction (e.g. see literature reviewed in Fuson, 1992a, 1992b). Using for 2-digit numbers the whole range of word problem types (see details in Fuson, 1992a, 1992b) is one way to stimulate a wider range of 2-digit solution methods. Problems asking children to find the difference may be especially productive because different children interpret such problems in different ways (e.g. Beishuizen, this volume; Hiebert et al., 1996). Some work by Beishuizen (this volume), Van Eyck (1995), and van Lieshout (this volume) did indicate considerable dependence of solution method on problem structure. But little other work has been reported. Especially needed is work concerning various kinds of instructional supports on this issue.

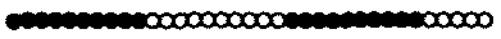
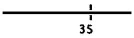
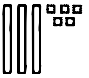
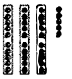
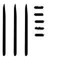
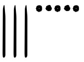
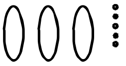
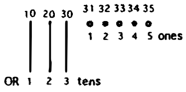
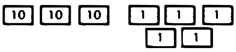
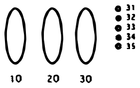
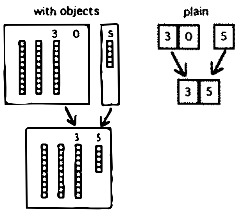
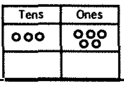
Typeof Embodiment	Embodiment Medium		
	Objects	Drawn Objects	Drawn Numbers and Objects
SIZE: Tens & Ones Cumulative length	 Beadstring 		Meter stick Number line Open number line  Thermometer
Cumulative area (folded length)	Gold bars  Bead Frames: ten rows of ten movable beads		100 number grid: 10 rows or columns of ten numbers each
Number of groups	Base-ten blocks or unifix cubes  Penny strips 	  	   
DECADE:	Any of the objects above can be thought of as decade objects: 30 + 5 where both are counted by ones.		Montessori cards 
PLACEVALUE:	Count fingers as ones Count fingers as tens Abacus Poker chip computer 		

table 2: classes of object and drawn conceptual supports for 2-digit numbers

There is a very complex intertwining of solution method, 2-digit conceptual structure, conceptual support introduced in the classroom, and number words. Many variations are possible. Before we discuss this intertwining further, we will turn to a brief classification of potentially meaningful conceptual supports for 2-digit numbers and addition and subtraction methods with such numbers.

4.2 Classes of conceptual supports for 2-digit numbers and calculation

Table 2 shows different classes of conceptual supports that can be used in classrooms to help children construct conceptual structures for 2-digit numbers. The major subclasses are size conceptual supports that show tens and ones, decade conceptual supports that show the decade and the ones, and place-value conceptual supports that show 1-digit numbers in different left-right locations (or taken from different left-right locations for finger ones and finger tens). Within each major subclass are shown three possible conceptual support media: objects, drawn objects, and numbers with drawn objects. Within the size conceptual supports are three subclasses. Considerable research has focused on differences in children's understanding according to these subclasses. In general, Dutch researchers (e.g. Beishuizen, Gravemeijer, Treffers) have tended to make sharper distinctions between the effects of these subclasses than have U.S. researchers (see chapter with Discussions at the conference).

Size conceptual supports can show tens and ones by cumulative length, by cumulative area (a sort of folded area with adjacent ten-lengths), and by the number of groups of tens and ones. Recent Dutch approaches have used cumulative length and cumulative area conceptual supports: the bead string, the open number line, and the 100 number grid. CGI (Carpenter, this volume; Carpenter et al., 1995) and the Hiebert and Wearne project (Hiebert and Wearne, 1992, 1996) used number-of-groups conceptual supports: base-ten blocks or Unifix cubes in groups of ten. Cobb projects have used the 100 number grid (Cobb, 1995) and number-of-groups conceptual supports: Unifix cubes in groups of ten and drawings of packages of ten candies and single candies (e.g. Cobb et al., in press). The South African project has used place-value conceptual supports (bead frames to build sequence-tens), and some classes have used Montessori cards without object drawings (Fuson et al., in press). In our Children's Math Worlds project, we have at various times used cumulative length conceptual supports (base-ten blocks as lengths; thermometers), number-of-groups conceptual supports (base-ten blocks, drawn blocks, and drawn blocks with numbers; penny/dime strips, drawn penny strips and pennies, drawn dimes and pennies with numbers; collections drawn as dots and ten-sticks of dots), decade conceptual supports (Montessori cards with objects), and place-value conceptual supports (finger ones and finger tens: using fingers to count by ones or count by tens).

Each of the types of conceptual support highlights particular aspects of the UDSSI Triad Model of conceptual structures for 2-digit numbers. The size cumulative length conceptual supports parallel the form of the unitary sequence of number words. As the tens within these conceptual supports come to be noticed and used by children (e.g. the alternating colors of tens beads; large 10, 20, 30, etc. markings on the thermometer), these conceptual supports can be used to construct and use the sequence-tens and ones conception. The cumulative area models foreground by their rows of ten the sequence-tens within a unitary sequence, but children can use these models at either level (unitary or sequence-tens). However, moving from unitary to sequence-tens methods on the 100 number grid, i.e., coming to see vertical jumps within the 100 number grid as a shortcut for counting all ten single jumps (as increasing or decreasing by ten in one jump) may be quite slow or be done without understanding (Cobb, 1995; Fuson, 1996). Drawing ten more squares after 36, and then 10 more squares, etc. rather than just counting them (or drawing on the number grid) might help children see the part of the ten up to the next decade and the rest of the ten within that decade. The size conceptual supports showing the number of groups of ten and the number of groups of one present the meaning of the 2-digit place-value numerals. The decade conceptual supports show the meaning of a 2-digit numeral as a decade plus some ones (30 + 8 = 38). The place-value conceptual supports look like the place-value numerals in that the tens and the ones look identical and are differentiated only by location. Some place-value conceptual supports use color as well as location to differentiate tens and ones. Because these introduce an extraneous meaningless feature that can be used by children instead of left-right position, color seems counter-productive. It seems better to use chips all of one color so that the poker chip computer looks maximally like written 2-digit numbers. However, because many children are counting on and counting up by the time they reach 2-digit computation, the need for objects to calculate sums or differences of tens or of ones does not seem imperative. Children could use fingers for such calculations (what we labeled finger ones and finger tens in Table 2). Therefore, a chip computer might only be used briefly to show the idea of place value (that numbers in positions look the same but they name different size groups). Even for children who know their sums and differences to 18, conceptual supports can be helpful in deciding how to combine, separate, or compare the tens and ones in multidigit numbers.

4.3 Relationships between solution methods and conceptual supports

The Dutch empty number line facilitates sequence counting solutions in which children begin with one number and move up or down the number-word sequence. The number-of-groups conceptual supports can be used for such sequence solutions (such solutions appear in our *Children's Math Worlds* classrooms and in CGI classrooms where such supports are used), but the number-of-groups supports especially facilitate decomposition methods in which groups of tens are combined separately from groups of ones. However, these grouping methods can be carried out using sequence-tens or decade words (e.g. fifty plus thirty is eighty) or separate-tens words (e.g. five tens plus three tens is eight tens). Many children in each project did use the methods consistent with their instructional supports.

However, within every project, some children used methods that were not so strongly facilitated by their instructional supports. This is probably due to several factors. First, each instructional support can be used for most methods even if one method is most obvious with that support (this position is not held by all conference participants; see discussion); therefore children can invent new methods even if they have not been discussed in a class. The empty number line or 100 grid can be used to do decomposition methods: A child could draw 30, then 20, then 8, then 6 to make 64 on the number line or on a 100 grid. Children can use number-of-groups conceptual supports to do methods that begin with one number: They can make the first quantity and then add on or take away and count as they do so. The methods using tens (A10) are especially clear with number-of-groups conceptual supports. The empty number line actually is no more facilitative of the N10 count/add on/up or back methods than the number-of-groups conceptual supports except for a general unitary up/down sense. With either kind of conceptual support, children must have some kind of learning experience to see and learn the regular pattern involved in counting on from 38 by tens (38, 48, 58). The number-of-groups conceptual support does afford the interpretation of 3 tens 8 ones plus 1 more ten is 4 tens 8 ones as well as a sequence word interpretation.

Second, a focus either on number words or on the written number marks may lead a child to a particular class of methods. Some children seem to be pulled by the number words and thus think predominantly (or at least initially) with sequence conceptions and use sequence-tens. Other children seem to be pulled by the number marks and think in terms of the groups of tens and single ones. Whether these preferences reflect more general dispositions toward oral versus visual thinking is not clear because so few data exist concerning these individual differences in uses of methods.

Third, the method used might vary with how well children can count to 100 by tens and by tens and ones. For children who cannot count to 100, using tens words and separate-tens conceptions initially is easier because one cannot begin to construct sequence-tens, or even the whole unitary sequence, until one knows the counting words of that sequence. Furthermore, we know that the counting sequence to 18 must be quite automatized for children to begin to count on/up within it. Therefore, it seems sensible that the count by tens list and the count by tens list embedded within the unitary count to 100 must be well automatized for children to use it in solution methods. This may be one reason why weaker Dutch pupils use the separate 1010 methods instead of the sequence N10 or A10 methods.

Fourth, the 2-digit conceptual web is very complex, and children have to construct it piece by piece. Early successes on the sequence path or the separate path may start a child down that path. Brief comments by a peer, sibling, or parent may be enough to facilitate an initial path. In a classroom where activities are designed to help children construct both sequence-tens and separate-tens, children still cannot do both simultaneously and so begin one path first.

Fifth, the sequence of problems given may affect the solution paths taken by individual children. The South African Problem Centered Mathematics Project found that teachers who gave many 2-digit addition problems before giving 2-digit subtraction problems had many more children who did incorrect mixed methods by incorrectly generalizing from the addition method: Add the decades and add both ones became subtract the decades and subtract both ones. In Beishuizen (1993) more of the weaker pupils might have used 1010 because 2-digit problems with no trades (regroupings, borrow/carries) were given for a long time before problems with trades appeared. The 1010 method is particularly easy for problems with no trades, but it requires thinking about the directionality of the subtraction of the ones to work for problems with trades.

Sixth, the number words used by children might facilitate one kind of method more than another. We have already discussed differences between using sequence words and tens words. Although each kind of word can be used with the opposite method, they do match the begin-with-one-number and decomposition methods better. Most classrooms do not emphasize the tens and ones words as much as we do in *Children's Math Worlds,* so many European number words would seem to suggest decade or sequence solution methods. However, Dutch and German reverse the decade and ones words for all number words between 10 and 100, not just for the teen words as in English. Hearing (internally or externally) a problem as 'eight and thirty plus six and twenty' seems to us to emphasize the separateness of the decade and ones portion of the number because the split is so obvious: The problem sounds as if you need to add four separate numbers. In contrast, hearing the same problem as 'thirty eight plus twenty six' sounds more like adding just two numbers. Therefore, Dutch and German words may predispose children to use decomposition methods, and this may be especially true for weaker children whose sequence-tens conceptions may be weaker and therefore less able to overcome the suggestion of the words themselves. English words (and others like them in which the decade and ones portions elide together to sound more like a single number) may support more those methods that begin with one number. This effect of number words may be one reason that Dutch researchers have used a sequence conceptual support (the empty number line) and U.S. researchers have used number-of-groups conceptual supports: Each is trying to support children to construct that which is less clear in the child's number words (participants at the conference had differing views on this suggestion, see chapter with Discussions). This contrast between methods suggested by the words and those suggested by the conceptual support then may be a reason that children in both countries invent and use both kinds of methods. It may, of course, be even more complex than this. The reversals in German and Dutch may make it easier for children to see or to say the pattern in jumping by tens: 'Eight and thirty, eight and forty, eight and fifty, eight and sixty' seems easier than 'thirty eight, forty eight, fifty eight, sixty eight' both conceptually and procedurally (you can elongate the 'eight' while thinking of the next decade word). The different intuitions at the conference about these issues of number words within the researchers from a given country may also indicate that individual differences exist in children in what the number words do and do not facilitate.

4.4 Vertical mathematization, reflection, and children's conceptual advancement

The Gravemeijer paper (this volume) summarizes the work in The Netherlands concerning vertical mathematization and reflective cycles in which models of situations become models for mathematical reasoning about methods and numbers. Our own work has rested on similar assumptions and was described briefly earlier. Here we would like to stress six aspects of such cycles at work in the classroom that can greatly facilitate children's movement through developmental levels. These aspects are especially necessary for the less advanced children in a class. Research conducted by one of us (Fuson) on a longitudinal study of one of the U.S. reform curricula, the *Everyday Mathematics* curriculum from the University of Chicago, has indicated that these are problematic aspects that need to be emphasized in a curriculum if teachers are to do them.

First, the models chosen by a curriculum or by teachers using a teaching approach such as CGI must be used in the classroom in such a way that they enable children to construct a model of the mathematical domain (in this discussion, 2-digit addition and subtraction). The *Everyday Mathematics* curriculum emphasizes the hundreds grid, and most teachers we visited had such a grid in the classroom. But the ways in which the grid was used did not enable some to many children in a given classroom to use it as a meaningful model of counting, adding, or subtracting tens and ones. Many children did not see the tens on the hundreds grid, especially when counting on from non-decade numbers such as 38. Second graders in Cobb (1995) also could not use the hundreds grid until they had constructed tens-units with some other model. The way the Dutch textbooks used the hundreds grid (reported in Beishuizen, 1993) seems particularly problematic. One square was darkened to show a quantity rather than darkening all of the squares up through that square; this is a counting rather than a cardinal quantity model.

Second, classroom activities need to be designed that move *all* children from unitary conceptions to sequence-tens and separate-tens (number of tens) conceptions.

Although, as Carpenter in his paper (this volume) points out, a class may collectively have a great deal of tens knowledge, the task of the teacher is to help every child have and use such knowledge. The *Everyday Mathematics* curriculum contains counting activities to practice counting by ones, tens, and fives (also other numbers) and to practice writing large numbers. But it does not contain a sustained sequence of activities that help children count or combine or separate tens and ones quantities. Consequently, the more advanced children in a class do invent mental methods, but the less advanced children have no methods except sometimes the standard algorithm they have learned somewhere (sometimes from teachers just before standardized tests). They use the latter with no connections to tens or ones quantities, and make many of the typical errors, especially in subtraction (Murphy, 1997). In the revised Dutch empty number line approach, there were explicit activities with the beadstring and empty number line to support construction of sequence-tens. Most CGI teachers and the teachers in the earlier Cobb projects (e.g. Cobb, Wood and Yackel, 1993; Cobb and Bauersfeld, 1995) did use models of tens and ones (often Unifix cubes stored in columns of ten), but few seem to have used systematic activities with such models to facilitate all children's construction of the generative tens conceptual structures. Some children in third grade (Lo, Wheatley and Smith, 1994) and in the fourth grade (Steinberg, Carpenter, and Fennema, 1994) were still using unitary methods. To us, this seems unnecessary and unacceptable. Having a period of exploration with models-of in first grade as outlined in Carpenter (this volume) seems fine as long as less advanced children are helped in some way to advance. But all second graders, even in urban schools, can come to use addition and subtraction methods using tens by several months into the school year if they have activities to help them construct the conceptual prerequisites for such methods (Fuson, 1996; Fuson, Smith and Lo Cicero, 1996).

A third aspect of using models-of that can support their use and reflection on such use is using drawn methods rather than physical objects. With drawn methods, a record of the whole problem-solving process is available after problem solving. This permits the teacher to examine children's methods and look for children's errors after a class is over. Such monitoring can provide daily feedback loops that permit teachers to select students to demonstrate particular methods or choose errors that would provide useful discussion for many children in the class. If students work at the board or on individual chalk boards (these are methods used frequently in the *Children's Math Worlds* project), their solution is available for reflection by their classmates. No waiting is necessary while children put their method on the board, a management consideration of importance in many schools. Working at the board seems to facilitate children helping each other more than working in a smaller scale on a piece of paper. It is also easier for a teacher to observe such helping. The empty number line is such a drawn method, and in Dutch classrooms one author has visited, seems to facilitate reflection and discussion much as our drawn ten-sticks and ones do. For the less advanced children, having their method physically present also seems to facilitate their explanation of their method. They can rely on gesture as well as on words, and the drawing helps them remember and sequence the steps of their explanation or learn to do so if they require help from the teacher or a peer.

Fourth, drawn methods can facilitate the linking of the model of the tens and ones to written numerals. For larger numbers, the goal of using models-of is so that the written mathematical symbols take on the mathematical meanings of the models-of. This process is facilitated greatly if the models-of are linked to the written mathematical symbols (Burghardt and Fuson, 1996; Fuson and Briars, 1990; Fuson, Fraivillig and Burghardt, 1992). Such linking can also be done in reverse by reading a numerical record of a method using models-of language. In Fuson (1986) reverse linking by asking children to 'think about the blocks' (they had used base-ten blocks to build their understanding of their subtration method) was sufficient for children to self-correct subtraction errors in problems with zeroes in the top number.

Fifth, 2-digit numbers within addition and subtraction problems need to be read as decade words or tens and ones words and not as concatenated single digits (e.g. 38 + 26 said as 'thirty plus twenty' or 'three tens and two tens' not as 'three plus two and eight plus six'). Many *Everyday Mathematics* teachers at least sometimes use concatenated single-digit language ('three plus two is five' for 38 + 26) themselves, and more allowed children to do so. We have found that using decade words and also saying the number of tens is necessary to keep all children in a class with the discussion. Initially, when many children are working on constructing a generative tens conceptual structure, some are thinking only with decade words (ordinary English or Spanish counting words), and others are thinking only with number-of-tens words (frequently these include the least advanced children, who do not yet know the decade words). Using both of these kinds of words can help each of these kinds of children understand any discussion, and later it helps children begin to construct the other related meaning. Coming to use and think with both kinds of words also gives children flexibility in understanding various kinds of addition or subtraction methods.

A sixth aspect that ties together all of these issues concerning vertical mathematization and reflection is the role of the teacher in relating a given child's described solution method to more advanced and to more primitive methods so that children at different levels of mathematization can understand that method. It is not necessary that a teacher do this for every method given by a student, but doing it frequently can help. For example, in the first-grade example from early in the year in Carpenter (this volume), the teacher was trying to help the child advance by asking her to describe her blocks method without using the blocks. Such experiences can help children to move from an objects method to an oral method. An important and natural step after such a verbal description without blocks present would have been to do the description again showing the oral actions with the blocks. This would have made the oral description accessible to most students. In the third-grade example in that paper, the reverse happened. The student said that she did not need blocks to describe her method, but the teacher insisted that the first description of the method be with blocks. This allowed the less advanced children to follow that method. Following that blocks description with an oral description not linked to the blocks might then have helped some of the listeners be better able to move away a bit from the blocks to using words. The third related method then moved to using the words as the objects, and fingers were used to keep track of the tens counted on and then of the ones counted on. Juxtaposing these methods in this fashion can help children see the relations between them and move ahead a level.

In our observations of *Everyday Mathematics* classes, teachers rarely carried out such supports for vertical mathematization, and the curriculum did not support them to do so. Most methods were described only orally. The methods that used objects were rarely acted out or described in detail; a brief 'I counted' or 'I used the hundreds grid' was accepted by the teacher. Thus, other children did not get to see concrete methods carried out, and oral methods were not related to quantities for those children at a lower level. Consequently, some to many children were not able to follow such descriptions (Murphy, 1997). A few teachers did record on the board in numerals whatever method a child described. This served the fourth aspect above of linking a method to written numerals and also helped memory because the whole method was there to be reflected upon after the description was completed. If descriptions of solution methods are to serve to do more than emphasize the individuality of methods and give children practice in describing their method (both worthwhile but limited ends), teachers need to link them to other methods within a vertical mathematization learning trajectory.

4.5 Developmental levels and 2-digit solution methods

The Gravemeijer paper (this volume) describes very well the goal of vertical mathematization as directing the design of instructional sequences. For 2-digit numbers, we see two concurrent kinds of vertical mathematization that specify the movement of individual students from using models-of a meaningful quantity context to using models-for mathematical reasoning. The first is similar to experiential levels for single digits: moving from the use of objects presenting quantities to the use of counting words presenting quantities (and for some problems to the use of recomposition change-both-number methods) to the eventual use of addition and subtraction facts at least for some parts of some problems and the use only of written numbers (and perhaps fingers) to record some method of 2-digit calculation. The second moves through the conceptual structures for 2-digit numbers: from a unitary conception to a decade conception to the sequence-tens or the separate-tens conception and eventually to an integrated-tens and ones conception that relates sequence-tens and separate-tens. These two involve different kinds of mathematical advancements by children. The first uses the models-of to models-for distinction. The second involves conceptual structures that become mathematically more sophisticated and perhaps should be given a different label; for convenience we will continue to call both of these kinds of vertical mathematizations.

The Dutch instructional sequence using the bead string and on to the empty number line does accomplish both of these kinds of vertical mathematization from objects up through drawn length and number methods using sequence-tens. The three US projects represented here at the conference all used size conceptual supports that presented the number of groups of tens and of ones, and they all accomplished at least parts of both vertical mathematizations. *Children's Math Worlds* used in different years two different instructional sequences: one went from grouped objects to drawn ten-sticks and dots with or without numbers to written numerical methods, and the other went from use of penny/dime strips to drawn ten-sticks and dots with or without numbers (and for some classes also to drawn coins) and from there to numerical methods, with mental methods used throughout for some classes. The CGI sequence reported by Carpenter (this volume) used base-ten blocks or Unifix tens and moved to number-word solutions without drawings. The Cobb and Yackel projects used at various times Unifix cubes stored in columns of ten, hundreds grids, drawn number balances, drawings of ten candies and single candies, and a computer program that allowed operating on (decomposing and composing) such drawings. Children moved at their own pace from using these conceptual supports to oral and number methods without such supports.

The solution methods in Table 1 have an interesting relationship to the developmental sequence of single-digit solution methods described earlier. The methods that decompose a 2-digit number into its tens and ones and then add or subtract those tens and ones are initially most like the single-digit Level 1 object methods. The adding and subtracting of tens and ones objects are similar to those used for single-digit numbers, and the tens and the ones are each counted by single-digit numbers. The new and difficult aspects of these 2-digit methods are knowing throughout the solution which are tens and which are ones and understanding how to deal with any needed trading (needing to make another ten from the ones or opening a ten to get more ones to subtract). The 2-digit methods that begin with one number are quite like the sequence number-word single-digit methods that count up or down the sequence. The 2-digit methods that change both numbers are like the single-digit Level 3 derived fact methods.

However, the 2-digit methods each also follow the single-digit developmental levels within themselves. The decomposition-into-tens-and-ones methods move from counting objects to counting on, down, or up or using known facts to find the total or difference of the tens or of the ones. The begin-with-one-number methods often start with concrete objects before they become sequence counting methods.

The methods that change both numbers may initially be done with concrete objects and then as sequence methods. Both for that reason, and because many children are at higher single-digit levels by the time they are working on 2-digit addition and subtraction, we would not expect the 2-digit methods to exhibit strong level effects among themselves. Rather, as discussed above, they are subject to several different kinds of instructional and individual influences.

Another aspect of 2-digit addition and subtraction methods that makes them different from single-digit addition and subtraction methods is that they are more complex multi-step methods that stretch memory very considerably. For this reason, it is frequently very useful to record results of some steps in the method. Numbers can be used to record such steps. Therefore numbers take on considerable importance, and numbers may be used to scaffold a multi-step method. Even when children are not allowed to record intermediate steps, the numbers facilitate computation, as indicated by the superiority of mental computation with problems with numbers visible rather than just presented orally (Reys et al., 1995; Reys, 1984).

5 Why mental computation?

Mental computation is stressed in some countries and by some researchers. The Dutch curriculum places considerable emphasis on mental computation for 2-digit numbers, delaying written methods until third grade. Reys et al. (1995) conclude their paper on mental computation in Japan by asserting:

> 'Finally, mental computation (when defined as self-developed strategies based on conceptual knowledge) should be a central focus of a computation curriculum. Whereas all agree on the importance of mental computation, surprisingly little is known about it in most countries. (p. 324).'

We think that it is very important to consider carefully the possible roles of mental computation in children's learning. If mental computation means moving directly from a problem presented orally or with written numbers to a method done completely internally, it seems clear that such mental computation should follow, and not precede, children's solution of such problems using some kind of quantity referent (objects or a familiar situation) for the written numbers in the problem. Such a quantity referent is necessary for children to have and use a meaning for the numbers with which they carry out a computation (e.g. for 2-digit numbers, for decimals, for fractions). Much research in many mathematical domains (e.g. see the reviews in Grouws, 1992) indicates that students must first construct meanings in these ways. Only after experience using the external referents do students construct robust enough internal conceptions to use these for mental computation. The recent research on the use of the beadstring and drawn open number line clearly recognizes that children need quantity referents initially.

Mental computation of carefully selected and sequenced problems can play an important role in helping children construct concepts for that mathematical domain and perhaps provide an impetus for vertical mathematization of concepts or of methods. As discussed earlier, in the *Children's Math Worlds* curriculum, we now use a sequence of different kinds of 2-digit problems (e.g. 40 + 10 initially, then 40 + 30, then 40 + 7, then 47 + 5, and then 40 + 36). These allow children to construct the major connections in Figure 2. The problems are given initially with external quantities (penny strips and pennies), and children are encouraged from the beginning to solve them mentally if they can. During the rest of the year, such mental computation questions are asked, and gradually more children become capable of solving them. But the major function of such work is to facilitate children's construction of the whole generative tens conceptual structure. We do eventually ask children to solve mentally the more difficult problems such as 38 + 26, but it is expected that only a few children will be able to do so initially. Most children need to use drawn objects and numbers to solve such problems initially.

We think that it is important that our present sequence of problems does not contain problems with no trades such as 32 + 36 or 46 - 25. In subtraction, such problems without trades can suggest the incorrect subtraction method in which the smaller top ones are subtracted from the larger bottom ones. Giving problems requiring trades immediately contributes to children's construction of tens-ones shifts within sequence-tens and/or separate-tens conceptions because children have to confront the issue of making another ten or opening a ten. These issues are easy to solve with drawn quantities.

It seems to us that an unanalyzed stress on mental computation is partially a result of traditional curricula in which children learned written number calculation with little understanding. Mental calculation in such cases meant either a child seeing that written method in his/her head or using a method the child had invented. In the latter case, such invented methods had a better chance of being conceptual than the taught algorithm. Therefore, mental computation was one way to encourage such inventions. For example, Reys et al. (1995) found that many Japanese children reported that they had invented the mental computation methods they used that were not just seeing written calculations. However, in classrooms in which all methods are based on understanding and no method is compulsory, mental computation no longer confers the advantage of more understanding. We certainly advocate 'self-developed strategies based on conceptual knowledge' but strategies do not have to be mental to meet this criterion.

Use of mental estimation is also related to mental calculation. Estimation is related to the elusive to characterize but desirable 'number sense'. Both estimation and number sense are desirable, but they involve different processes than exact mental computation. They both require a strategic analysis of the numbers in the problem, the result of which then directs the consequent process. These both are useful in real life and in checking (and sometimes directing) problem solving. They each require new ways to use a generative tens conceptual structure, and so help children extend their understanding. But these are both quite different from exact mental computation.

The roles mental calculation should play in the curriculum depend considerably upon what mental computation is considered to be. It might be considered to be the solution of problems presented orally without the use of any external action or object. This is a stringent definition that eliminates the use of the known numbers even as memory supports. The consistent and in many cases very large reductions in correct answers between problems presented orally and those presented in numbers for Japanese students (Reys et al., 1995) and for U.S. students (Reys, 1984) indicates how much the written numbers facilitate thinking about a solution.

Mental calculation might be the kind of calculation expected for written numeral problems. One bilingual staff member on the *Children's Math Worlds* project was taught in Argentina calculation methods for addition, subtraction, multiplication, and division in which every intermediate step was done mentally and only the answer was written (the answer could be written digit by digit as these were produced). However, as in the United States with traditional written algorithms, some children secretly used fingers to calculate.

When no intermediate step can be written, methods that carry along the answer-in-progress have an advantage because the steps already done are less likely to be forgotten. The methods that begin with one number do this. The methods (N10) that add/subtract the tens then the ones (or vice versa) are also particularly easy because each of those numbers is sitting there in the given problem number as a memory support. The other two kinds of begin-with-one-number methods require more remembering because the amount overshot (in N10C) must be remembered or the part of the ones not added or subtracted initially must be remembered (in A10). However, if intermediate steps can be written, these methods lose their advantage. Especially for less advanced children, an early stress on mental calculation seems like an unnecessary and perhaps even unfair choice. The focus in the Netherlands on 'mental calculation' in the second grade is actually more of an emphasis on using 'handy' numbers or methods that take advantage of the specific numbers in a given problem. The calculation does not have to be mental but may use objects or drawn supports such as the empty number line.

6 Conclusion

There now exist several examples of instructional sequences that support learning trajectories of children through vertical mathematization of 2-digit addition and subtraction methods while also affording a range of different solution methods carried out by different children. As more and more such classrooms exist, and as teachers attempt to use new curricular materials in these areas, we should be able to learn more about maximal learning trajectories and about how to support all children successfully to methods using tens. We also may learn more about the following important issues: How can discussions be used most profitably for the benefit of all children in the class? How much understanding of each method should all children have? Can understanding follow as well as precede learning a method? What are the relationships between autonomy and being helped to learn a method?

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Is mental calculation just strolling around in an imaginary number space?

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1 Introduction

When, in the spring of 1996, I came to visit Meindert Beishuizen and Ton Klein at Leiden University and talk about their project it was not the first time that I had been to Leiden. Some long time ago in my almost forgotten youth during a trip through the Netherlands, I had arrived at the central railway station and had taken a walk through the old part of town and to the campus and the university building.

This time I arrived by car. As I had no clear idea which the shortest way from the highway to my destination building was I took the second Leiden exit, which leads to the railway station. From there I went through some narrow streets (the main avenues were, like in Germany, as usually closed because of street work) to the university. I was rather astonished to learn that the university building is only a couple of minutes away from the first Leiden highway exit which I had passed almost an hour before.

What has this story to do with the problem of strategies and procedures? Well, there are several points of interest (see Clements, 1992, for a similar case of personal inconvenience):

- I actually did arrive at the Leiden University as my intended destination, but in a rather inefficient way, involving an annoying expenditure of time and gas for my car on the one hand and a marvellous sight of the old part of the town (and a less marvellous sight of torn up streets) on the other.
- The main reason for the inefficient route was the fact that I did not have a precise cognitive map of the relevant sections of Leiden. I was reduced to using literally a pedestrian strategy which, as I knew, would achieve its goal, though very slowly and in a rather cumbersome way.
- The act of driving to the main station and then to the university campus did not help me to extend my cognitive map of Leiden. I already knew how to do that.

- Even if I were driving with my wife and my ten year old son, who admittedly are both more intelligent than I am and with whom I could have discussed the problem, would not have helped as none of us would have had a plan of Leiden within his/her cognitive structure. No verbal information, elaborated visual memory or prior episodic experience was available to us which would have provided the basis for a more creative and elegant solution. With no basic information we were (and partly still are) plainly ignorant, i.e. we would have been unable to establish a useful (problem related) mental representation of Leiden. Even a family discussion would not have helped.
- But talking to a Leiden inhabitant like Meindert who did have a network of streets established in his mind, all of a sudden gave me insight into the links between the places and led to an overall map of how the streets were related to each other (oh, well, almost).

Walking or driving around in an unfamiliar town always reminds me of my early experience with numbers in preschool and primary grades and my personal tentative constructions of numbers. What I am referring to is 'number sense':

'Another way of capturing the idea of number sense is to liken it to ,road sense'. A person who lacks a great deal of road sense may know the location of a few places and even one way of getting there. What is missing is a mental picture of the terrain and flexibility in travelling around it. On the other hand, a person who possesses road sense, has an integrated mental picture and is able to visualize the direction and distance involved in travelling to a given point.' (Trafton, 1992, pp. 78-79)

So what I am trying to discuss in this chapter is the role of number sense in the development of arithmetic strategies and procedures, whether we can separate them, and primarily their relationship to 'space'. As it is rather hard to think about all this, at least for me, and even harder to talk about it, I require a lot of patience on the part of the reader and the audience. I will start with some observations from our project.

2 Some causes for deficiencies in arithmetic abilities: A case for mental operating

In my opinion and from my professional experience with primary school children who have disabilities in mathematical learning I regard the development of arithmetic strategies and procedures (to leave them unseparated for the moment) as necessarily linked to the emergence of an imagined *number space* (Lorenz, 1992). Arithmetic operations like addition and subtraction, multiplication and division are but (personal) movements in this space.

True, it is a very personal space which is not perceivable by others and even oneself as an adult has difficulties to describe it (or even communicate about it). Within this space singular numbers do not exist, they only exist in relation to other numbers. The number 17 can not be thought of in pure isolation but as something between 10 and 20 and closer to 20 than to 10 (and even closer to 15). Thus, the mental image of numbers

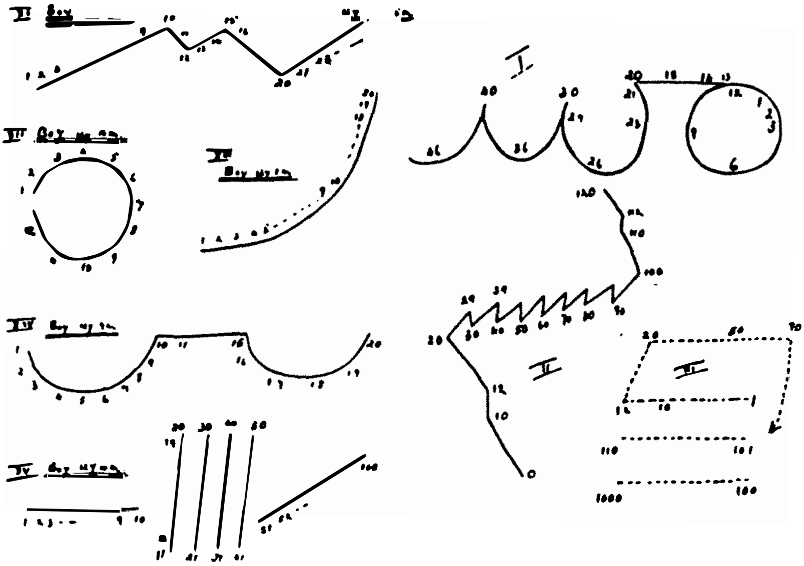
- is a personal construct,
- depicts the geometric relation between the numbers,
- forms a space with the properties of distance and nearness, surrounding and neighbourhood,
- allows movements which pictorially represent the arithmetic operations,
- and finally is built upon the personal (school) experience with numbers and especially operations and the manipulation of the corresponding school materials.

Children with mathematical learning disabilities whom we have met in our project very often show a cognitive disability to visualize (or at least seem to have a difficulty in using their visualization). This not only refers to the arithmetic content but to everyday situations as well (Lorenz and Radatz, 1993). Arithmetic disabled school children have difficulty in imagining objects and in particular in moving, turning, distorting, enlarging or reducing objects in their imagination or visualizing what it would look like when another or several other objects were added. As a consequence they remain dependent on the manipulatives provided by the school teacher and have to use them for their calculations. This is not unfamiliar as all school children undergo this phase of development. The problem for these children is that they cannot leave this stage as arithmetic operations, if done mentally, must *be operated in the imagination.*

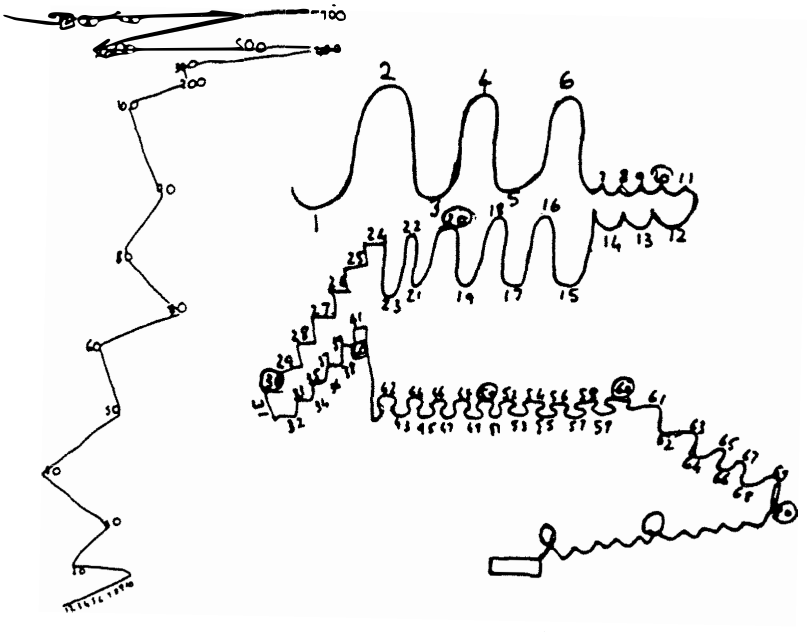
These children, like their classmates, are supposed to build or rather to construct their mental number space on the basis of the actions done with the manipulatives (the latter succeeding, the former failing to do so). Though the arithmetic material does not and cannot determine the structure of the imagined number space it aids its formation, at least that is why we use them in school.

Even for an adult, the idiosyncratic number space reminds one of (school) material, as several studies show (Carter, 1983; Galton, 1880a, 1880b; Morton, 1936; Oswald, 1960). The examples on the following page clearly demonstrate the close relation between certain materials and the final imagined number space constructed upon them and the corresponding actions:

- Most number spaces are number lines (!).
- Some have a similarity to the clock (and might have been constructed prior to schooling).
- One image resembles base-ten blocks.



MORTON, 1936, 70 f)



CARTER, 1983, 6)

- One image has a rhythmic structure (if it was a German child I would guess that it went to a Waldorff-Schule).
- There are no two equal number spaces or number images.
- All reveal a ten-structure

3 Number sense

The conviction underlying our project was that imagined number space is only partly and only in its roots influenced by the material, but its elaboration and its extension takes place while we are doing mental calculations. Well, early strategies (or procedures) like counting forwards or counting backwards do not lend themselves to the development of strong, powerful structures of 'number cities'. As in the example of my visit to Leiden it needs

- a lot of practice and moving around in the space, that is: exploration;
- the aid of an experienced companion, an expert who is on the same wave length;
- a continues refining of the cognitive map (which I hope I will get this time).

I do not want to stress the similarity between the 'road map' and the 'number space' too much but, as I said before, number sense mainly involves the geometric relationship between numbers: The distance between numbers, a sense for short cuts and fast high ways, the knowledge of different routes from one point to another according to the actual situation and demand. But from my point of view, developing number sense is the ultimate goal of teaching arithmetic:

> *Making the child feel at home with his or her numbers.*

Still this does not define explicitly what number sense means. So, what is it?

> 'It would seem appropriate, then, to begin with a definition of number sense. However, number sense is like problem solving – it is elusive and difficult to pin down, yet most people think they know problem solving – or number sense – when they see it exhibited.' (Sowder, 1992, p. 15)

That is the dilemma. So instead of a definition, I would like to begin with some characterizations of number sense that other people have given. And, according to plan, they underline my point of view. Let us first look at different characterizations put forward by mathematics educators and by psychologists.

A now classic characterization of number sense was given by Hilde Howden in her Arithmetic Teacher article (1989):

> 'Number sense can be described as a good intuition about numbers and their relationships. It develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms. Since textbooks are limited to paper-and-pencil orientation, they can only sug-

> gest ideas to be investigated, they cannot replace the 'doing of mathematics that is essential for the development of number sense. No substitute exists for a skilful teacher and an environment that fosters curiosity and exploration at all grade levels.' (Howden, 1989, p. 11)

And a psychologist stresses the point about geometric relationships:

> 'People with number sense know where they are in the environment, which things are nearby, which things are easy to reach from where they are, and how routes can be combined flexibly to reach other places efficiently. Mathematical features (of this environment) include different kinds of numbers, such as integers, rationals, and reals, and different quantitative domains, such as commercial transactions, cooking, and motions of objects. General mathematical features also include operations on numbers – addition, subtraction, multiplication, division, powers, roots – and operations on quantities – additive or multiplicative combinations of quantities, processes of exponential growth, and so on.
>
> (At a more local level, knowing the environment includes) knowing the locations of the various numbers to be represented, knowing which of them are near each other, knowing how to combine representations of various numbers to form representations of different numbers by combining, separating, expanding, and contracting other representations.' (Greeno, 1991b, pp. 185-186)

And similarly his colleague in cognitive psychology, L. Resnick:

> 'Number sense resists the precise forms of definition we have come to associate with the setting of specified objectives for schooling. Nevertheless, it is relatively easy to list some key features of number sense. When we do this, we become aware that, although we cannot define it exactly, we can recognize number sense when it occurs. Consider the following:
>
> - Number sense is nonalgorithmic.
> - Number sense tends to be complex.
> - Number sense often yields multiple solutions, each with costs and benefits, rather than unique solutions.
> - Number sense involves nuanced judgement and interpretation.
> - Number sense involves the application of multiple criteria.
> - Number sense often involves uncertainty.
> - Number sense involves self-regulation of the thinking process.
> - Number sense involves imposing meaning.
> - Number sense is effortful.' (Resnick, 1989, p. 37)

And to quote another math educator in a similar vain:

> 'I have compiled a list of behaviours that demonstrate some presence of number sense. While we must be careful in using any one of these behaviours to attribute number sense to an individual, the behaviours as a whole do help us recognize when number sense is present.
>
> 1 An ability to compose and decompose numbers; to move flexibly among different representations; to recognize when one representation is more useful than another.
> 2 An ability to recognize the relative magnitude of numbers.

3 An ability to deal with the absolute magnitude of numbers.
4 An ability to use benchmarks.
5 An ability to link numeration, operation, and relation symbols in meaningful ways.
6 The ability to understand the effects of operations on numbers.
7 An ability to perform mental computation through ‚invented' strategies that take advantage of numerical and operational properties.
8 An ability to use numbers flexibly to estimate numerical answers to computations and to recognize when an estimate is appropriate.
9 A disposition towards making sense of numbers.' (Sowder, 1992, pp. 18-20)

4 The Leiden research project (as far as I have understood it; please forgive me)

The research group around Meindert Beishuizen is trying, among other things, to

- analyse the strategies used by second grade children when solving addition and subtraction problems up to 100
- develop a curriculum unit which fosters powerful strategies by using the empty number line as material for children's use.

They have identified the relevant strategies (Beishuizen, this volume), but it is more than that. It is not sufficient to give a list of possible and observable strategies for a given problem like 19 + 27:

- Unifying the tens and the ones:
  $19 + 27 = 10 + 20 + 7 + 9 = 30 + 16 = 46$
- Adding the tens from the second addend, then the ones:
  $19 + 27 = 19 + 20 + 7 = 39 + 7 = 46$
- Starting with the second addend, adding ones and tens:
  $19 + 27 = 27 + 9 = 36 + 10 = 46$
  or tens first, then ones:
  $19 + 27 = 27 + 10 + 9 = 37 + 9 = 46$
- Subtracting 1 from the second number, add it to the first:
  $19 + 27 = 20 + 26 = 46$
- Add 20 to the second number then subtract 1:
  $19 + 27 = 27 + 20 - 1 = 47 - 1 = 46$
- Or any other strategy within your imagination (or beyond).

But what is a powerful strategy? Which one is to be fostered by instruction? The answer given by the Leiden research group is very simple and convincing: None, or all, or best of all: it depends!

No joking! It is like a walk through Leiden: the answer to the question about the best way is: it depends on where you start and where you want to go. In fact there is no best way for all intentions.

And that is why we should keep the road open to the development of a variety of strategies. But this is only half the truth. Of course, there are strategies that are better than others *for a given pair of numbers*!

We as trained calculation experts have no difficulty computing 81 – 2, and we do it differently than with the corresponding problem 81 – 79. In the first problem we go back, in the second problem we go from 79 to 81. It seems as if we are strolling around in our imagined number space and 'seeing from above' which way is the most appropriate for the actual task.

And we do it intuitively, without much ado and without much thinking either. Because it is plain 'seeing', it is 'obvious'. No strategical decision about a procedure, no painful and boring reflection like Hamlet about the various choices or whatsoever. We just do it! I will come to this point later.

Now to the second part of the Leiden research project: they use the empty number line as an aid for children to get them:

- to try out the different strategies/procedures and to test their usefulness for a given problem,
- to be actively engaged in the production of number relations,
- to build up a mental number space that is linear like most people's number space and continuously (re-)construct it, and
- to develop a means of communicating with others.

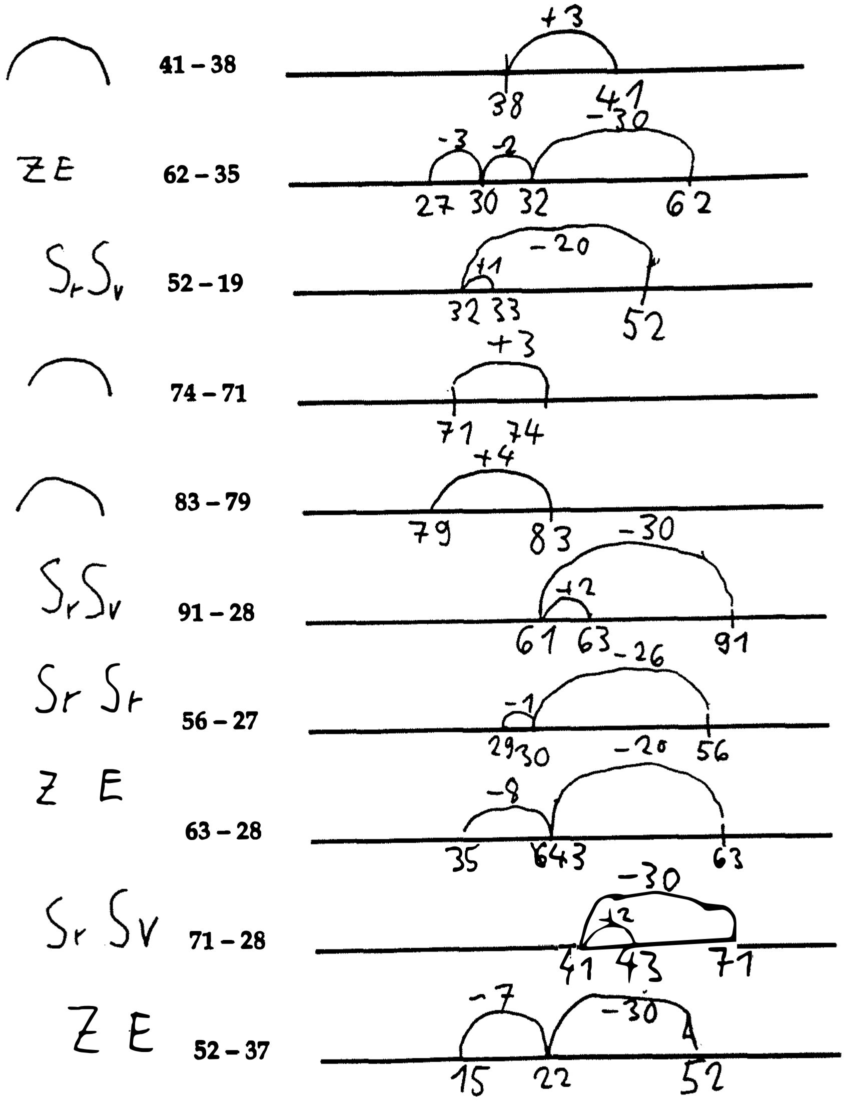
Influenced by this intriguing means for getting children to actively visualize the relationship between numbers and estimate the usefulness of a strategy in advance, we tried it with several school children in Germany (see next page).

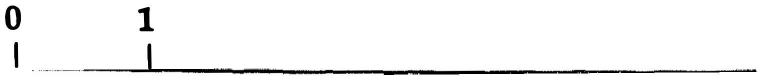
The abbreviations at the beginning of each line done by a third grade child need some explanation. Similar to the research and development project in Leiden we encouraged the children and the teachers to use names for the strategies. So an arch means 'it is pretty near, so I do not take away but add', ZE mens 'I jump in tens ('Zehner' in German) then in ones ('Einer'), Sr means a jump backward ('Sprung rückwärts') and similarly Sv is a jump forward ('Sprung vorwärts').

Of course, leaving the construction and discussion to the children does not guarantee the development of successful strategies, nothing can. But they have to try them out and test them in various tasks before making a decision for or against them.

'Strategies associated with mental computation should be developed explicitly throughout the years at school, and should not be restricted to the recalling of basic facts. People who are competent in mental computation tend to use a range of personal methods which are adapted to suit the particular numbers and situations. Therefore, students should be encouraged to develop personal mental computation strategies, to experiment with and compare strategies used by others, and to choose from amongst their available strategies to suit their own strengths and the particular context. ...less emphasis should be given to standard paper-and-pencil algorithms and, to the extent that they continue to be taught, they should be taught at later stages in schooling.' (Australian Education Council and the Curriculum Corporation, 1991, p. 109)

Of course, to use this means efficiently one has to start right from the very beginning in grade one stressing the point of numbers not only as quantities but as relations. At the moment we are trying out models of the empty number line up to 10 and 20 respectively. Most of our tasks involve the location of numbers as shown in the examples below.

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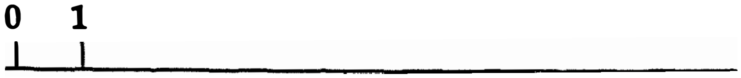
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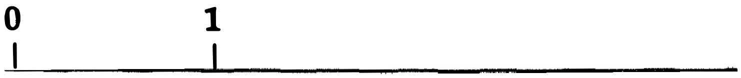
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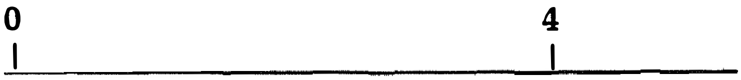
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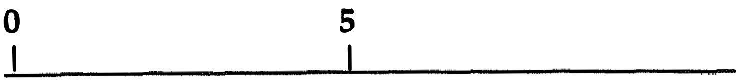
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5 Different manipulatives

In our project with school children with mathematical learning disabilities which, like the ‘math clinics’ at several American universities and teacher colleges, was an out-of-school institution, we made the mistake (?) of letting children use the particular materials which they knew from their classrooms. These materials included base-ten blocks, number line, hundred square, Cuisenaire rods, Montessori materials which are similar to ten-based blocks and so forth. To begin with we tried to figure out how the children thought and what their individual access to numbers and operations was. It was a perspective focused entirely on the individual student, attributing all successes and failures to his/her abilities, constructions, deficits or whatever label you like to use.

What we noticed was that a translation from one material to another is not only difficult for children but, at least for weaker students, almost impossible. Why? The materials all have the same structure as we know and see, don‘t they? That is why we use them in our classrooms. They lead to the same number system. Really?

I have come to doubt this in the meantime.Well, I do not doubt it with respect to written algorithms, but *with respect to mental operations.*

The difficulty in translating an arithmetic operation from one material to another stems from the differences in the actions associated with the materials. The merit of the base-ten blocks is the demerit of the number line and vice versa. And the actions (= movements of the hand which lateron become internal visual movements and operations) lead to strategies *connected with the material* (Lorenz, 1992).

Not surprisingly, the main strategy (or procedure) we observed when children used the base-ten blocks was 1010 (in the terminology of the Leiden group; Beishuizen, this volume; Beishuizen, Van Putten and Van Mulken, 1997). It is an intuitive action putting the ten rods together on the left side and the one cubes together on the right to get an immediate answer easily transformed into our symbolic notation form. True, it does not have to be this way and the actions with the base-ten blocks are not limited to this single, reduced format. Carpenter (this volume) gives an example of how this material can be used in a more sophisticated way leading to a N10-strategy:

$54 + 48 = 54 + 10 + 10 + 10 + 10 = 94, 94 + 8 = 102$

But the students'main actions even in this class had been 50 + 40, 4 + 8, 90 + 12= 102. I do not criticize this strategy, I am only trying to show the close relationship between the material at hand and the emergence of the arithmetic procedure. Of course it is possible, as Carpenter demonstrated, to overcome the 1010 strategy and replace it by the N10 strategy or one that is even more advanced and flexible. But it needs a special climate in the classroom open for discussion and children who come up with advanced strategies.

And what about strategies like

54 + 48 = 54 + 46 + 2,

i.e. the partitioning of hundred? I do not think that this is an obvious strategy with the base-ten blocks (and even less an action one would do with the hundred square).

The traditional number line has the disadvantage of giving children for too long the opportunity to count forwards and backwards because each number has its own mark on the line. In contrast, the empty number line forces the student to construct his/her own mental image of the ***number relation*** before starting the calculation. He is more actively engaged in the construction process because he cannot rely on the material.

The second difference is the conception of number as such. Whereas the base-ten blocks (and other materials as well) stress the cardinal number aspect and regard numbers primarily as a characteristic of a set of elements, the number line focuses on the linear, longitudinal aspect of number relations, which is not ordinal but comprises length and distance. On the number line, be it empty or traditional, a 1010-strategy does not make sense. It distorts the pre-given structure. The material lends itself rather to the N10-, N10C- or A10-strategy (cf. Table 1 in Beishuizen, this volume).

In fact, the dispute between the proponents and opponents of the two different materials revolves around the appropriate concept of numbers and arithmetic operations for children: set versus linearity. I think my own position is clear, I prefer a stroll on the number road, but the discussion is still wide open.

6 Strategies? Procedures? Or rather estimations?

As outlined above, most adults have come to construct an internal image of the number space in which arithmetic operations are slow walks (with a sight-seeing tour including the numbers in between), fast jumps or a combination of both. Each movement requires an estimation of the distance to be overcome (how far is it roughly from 54 to 100? Where do I end up approximately if I jump 28 backward from 72?). So a priori estimates play a central role in the constructions on the empty number line. This is in accordance with the suggestions of the NCTM-Standards:
‘(Teachers need to make wider use of ‘good estimation discussion questions’, NCTM-Standards), which share the following traits:

1 They present a natural problem-solving situation.
2 They can be solved in a variety of ways.
3 They encourage students to use approximate computational skills.

4 They can be used to help teachers better understand students‘ conceptions and misconceptions about numbers.

5 They furnish an opportunity for communication as students explain the processes and procedures used in making an estimate.

6 They stimulate different answers, and therefore offer an opportunity to discuss a range of reasonable results.’ (Reys and Reys, 1990, pp. 22 - 23).

After a long and perhaps boring and tiring detour I have now come to the point of dealing with the question Meindert Beishuizen has raised in his paper (Beishuizen, this volume): What is the distinction between strategies and procedures?

Well, to start with the everyday understanding of the term (and cognitive psychology does not seem to be far from it), *procedure* has the connotation of ‘automatic’, ‘fast’, ‘reliable’ and ‘always the same’, ‘leading to the desired result’ though not necessarily in an optimal way.

Strategy on the other hand implies something like ‘higher order cognitive processes’ (what ever that is), ‘decision making’, ‘choosing between different procedures according to relevant criteria’ and so forth. Thus, to use a strategy at least requires several distinctive procedures to be available from which to choose (if I know of only one I cannot make a decision). That is why we teach several computational procedures from which to choose.

But let us have a look at our own computational strategies. Do we have a strategy? Or several strategies? I have come to doubt it. As mentioned earlier, we compute 81 – 2 in a different way than 81 – 79. But does an adult reflect upon the various procedures at hand and, after an intelligent decision-making process comes up with the most appropriate way to solve the task? No, we just do it. It seems more like having a look from above down at the ‘number map’ which includes certain (individual) signposts like ‘tens’, ‘hundreds’, favorite or magic numbers etc. and seeing our right way in the familiar, intimate number surrounding. We have developed number sense! I am not advocating intuition against rationality. On the contrary, I am just trying to describe what is going on when we do mental calculation and which cognitive processes might be involved. In fact, there do not seem many cognitive processes participating in mental computation, *at least not strategies.*

We just do it. It seems more a kind of strolling around in the internal, imaginary number space and knowing where to start and where to go, which route or which short cut to take. The internal visual number space may be different for various aspects where numbers are involved. E.g. calculating the time the train needs between Bielefeld and Leiden, i.e. $8^{43} - 11^{47}$, could and should be different from $8^{43} - 11^{17}$. Our spacial representation of time and pure numbers may differ widely. But this does not affect the problem of strategies. Even though the structure of the ‘time space’ might be completely different, the principles of a spatial map like distances, fast routes etc. remain the same.

If this is true for adults or expert mathematicians, it does not necessarily hold for the students in our classes. They do not have a map of the number space, they do not know yet whether it is more convenient to take one route or the other, to go on a highway (as some children called the strategy ones-tens-ones or A10 for the task 82 – 36 solved as 82 – 2 = 80, perceived as 'highway entrance', 80 – 30 = 50 as 'fast highway' and 50 – 4 = 46 as 'highway exit') or take the faster subway and walk back a couple of steps, e.g. in the problem 82 – 19 = 82 – 20 + 1 (see Beishuizen, this volume, for more sophisticated and scientific labels).

My point is that it is not so much a problem of identifying various strategies or procedures from which to choose appropriately and in accordance with the demands of the task and the numbers given. But by using different strategies/procedures the student has the opportunity to refine his/her map of the number space, to perceive various relations between numbers, to explore the landscape and its peculiarities, to estimate distances and apply this knowledge to different problems.

To rephrase it more tentatively: perhaps the distinction between strategy and procedure as raised by Meindert Beishuizen is primarily a matter of language, a problem of the scientific model and the tools with which we try to describe the observable behavior of school children (and adults), but which do not necessarily correspond to the underlying cognitive processes.

But I confess that it is much more difficult and intriguing to pose intelligent questions as Beishuizen (this volume) did in his paper than to avoid straightforward answers and remain metaphorical as I have in mine.

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Stimulation of early mathematical competence

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Abstract

Research has shown that arithmetic difficulties later in life can be explained by an insufficient development of early mathematical competence: i.e. different aspects of early mathematical competence. The different aspects of early mathematical competence are derived from cognitive psychology and from a didactical point of view with an emphasis on seriation, correspondence, classification and the various phases of counting. This paper presents the results of a study into the development of early mathematical competence among young low arithmetic achievers in the 4 – 7 year age group.

After selection by way of an early mathematical competence test, low arithmetic attainers are presented with an additional program. This program consists of activities, embedded in real (daily) life themes, in which attention is paid to the different aspects of early mathematical competence. The program is given to the children with a guiding or a structural instruction form. The results of the study show that it is possible to stimulate the early mathematical competence development among young low arithmetic achievers. The way in which instruction is offered has no influence on achievement.

1 Introduction

The Education Act relative to primary education in the Netherlands came into force in 1985. With this act, a new school system was born. Prior to this time there were two kinds of schools for young children. Kindergarten for children from four to seven years with two grades, after which children went to the primary school until they were about twelve years old. The primary school consisted of six grades. After the introduction of the Education Act, these two schools were merged into one school, a so-called elementary school with eight grades. In the elementary school there would be less contrast between education in kindergarten and education in primary school starting with grade 1. This concept has led to a new point of view for people working in the field of reading, writing and arithmetic. The idea was to develop learning methods in such a way that the continuity of cognitive development was guaranteed. During their stay in elementary school, children would get their education according one particular educational method, for example in mathematics. This method would consist of successive courses throughout the eight grades of school.

In the field of arithmetic in particular, there has been much interest in the development of this kind of continuing courses. Newly developed courses for the youngest children, age four to seven years, were added to existing educational arithmetic methods so, in this way, one concept of educational arithmetic was secured for eight years. Unfortunately, for a number of reasons, in the first two years of elementary school it did not function as was planned. In reality, teachers in these two grades do have a role to play in the arithmetic method used in their school, but most of the time this part is shelved. Only when a teacher is looking for some new ideas for an arithmetic lesson does he or she use arbitrary information from the method, without following the concept of the method. Besides this, it takes a lot of time to control the ideas of all the lessons of a course. This is another reason for not using parts of the arithmetic method. A third and important reason for not using a method for young children is the idea that they do not have to be educated, but that they must play for the first two years of their school period. By playing they will learn enough to follow education as it is given from grade 1. And therefore, teachers will not use specially developed arithmetic parts of a method for teaching arithmetics.

For about 80% of the children in the first two years of the elementary school, playing in school is enough for learning some basic arithmetic skills and they do not need much structured help from an arithmetic method. However, for about 20%, this is not enough. These children need more formal education with specific rules and instruction because they do not have the capacity for learning by play. But if a teacher wants to use an arithmetic method for these children, he gets into trouble. The parts of the arithmetic methods do not pay specific attention to young children hampered by an arithmetic developmental lag. They are focused on the normal, average child and not on those children who need more. In addition, teachers have no idea what to do with children who are slow in their development of early mathematical competence and the basic arithmetic skills. So, it is of great importance that attention is paid to those children who show a lag in the development of the basic arithmetic skills. The department of Special Education of Utrecht University started a project in 1991 that is focused on these children.

2 Theoretical background

This paper discusses the concept of 'early mathematical competence' among young children from four to seven years old. Much has already been written about this subject and this paper has no pretention of being complete. The theories and research results important for our research are presented.

In the early seventies and eighties, emphasis was laid on the development of the Piagetian operations as prerequisites for the development of early mathematical competence. Although, according Piaget, number sense was a synthesis of seriation, classification and correspondence, to be able to conserve quantity under perceptual transformations was the criterium for number sense (Piaget, 1965). For example, preschool children do not know that to compare the lengths of two sticks it is necessary to align them at one end. They will also say that a row of objects includes more objects when it looks longer. Not until children are able to use certain rules (or strategies) to solve these kinds of problems, do they have, in the eyes of Piaget, knowledge of quantity. The Piagetian operations should develop in a sequential order, each operation being related to a certain age. McShane (1991) examined different studies on conservation and came to the conclusion that conservation plays a less important role in the development of number sense than Piaget suggested. Conservation-of-number seems to be the most important application of conservation and even children at the age of two are capable of solving a simple or daily conservation-of-number task correctly, according to McShane. A similar result was reported earlier by Siegler (1981) who found that conservation-of-number develops at an earlier age than other forms of conservation, like conservation-of-volume.

Counting would not contribute much to the development of number sense because it is just an acoustic action without any real meaning (Piaget, 1965). Later research, however, sheds a more detailed light on this rigorous opinion. Based on the results of his study on the effects of a training program for classification and seriation and a training program for counting skills, Clements (1984) concludes that neither seriation nor classification are prerequisites for counting. In his research he found a significant effect for training in counting skills in a test for counting, and also in tests for seriation and classification. Fuson, Secada and Hall (1983) found a similar result. Their study showed an effect of counting on the development of conservation-of-number.

As for the development of the counting skills, there is a great deal of consensus among researchers about the different counting skills and the order of development (see, for example, Frank, 1989; Fuson, 1988; Gelman and Gallistel, 1978; Ginsburg, 1977; Van den Brink, 1984). At the age of about three, children begin acoustic counting with little songs or rhymes. Then, at the age of about four, children begin to count, most of the time asynchronously. When the children are capable of counting and pointing to objects at the same time, they are able to count synchronously. One way to count synchronously is to arrange the objects while counting. At the age of about four and a half years, children begin to put into practice different options for arranging objects. At the age of five, children reach the stage of resultative counting. This means that they are aware of the fact that counting has to begin with the number one, that every object has to be counted once, and that the last number gives the total number of objects. Important in this phase is the fact that the children discover the one-to-one-relation between object and number. After resultative counting, children learn another strategy for counting, that is, shortened counting. From a number of objects, the children recognize the representation of, for example, the five on the dice, and they count on from this number: five, ...six, seven etc. Children at the age of about six should be able to deal with shortened counting.

3 Early mathematical competence

Reviews of developmental studies (Fuson, 1988; Geary, 1995; Pennings, Van de Rijt & Van Luit, 1995; Steffe & Cobb, 1988) and the results of, for example, the studies mentioned above, demonstrate that the Piagetian operations (seriation, classification, correspondence and conservation-of-number) are probably not prerequisites for counting skills, but that both types of skills are interrelated and that they constitute one early arithmetic skill.

For example, a study on the function of counting while solving a conservation-of-number task showed that there is a difference between children who use visible counting acts (like scanning with the eyes or pointing to the objects being counted) and children who are able to indicate immediately a number of objects with the correct number word (Steffe, Von Glasersfeld, Richards & Cobb, 1983). Children of the first group must perform different (visible or non visible) acts in order to give meaning to a number. Children of the second group do not have to go through these acts. A conservation-of-number task can be solved at different developmental levels of counting.

The results of the study by Clements (1984) are important. He compared the effects of a training program on classification and seriation with the effects of a training program on counting skills for children of four years old. Both training groups showed, on the post test for counting knowledge as well as on the post test for classification and seriation, a significantly better result in comparison with a group which had not followed any training program at all. On the post test for counting knowledge, however, the group who received the counting program, showed a significantly better result compared with the group who received the 'Piagetian' program. On the post test for classification and seriation, there was no significant difference between both training groups. Clements concludes that counting, seriation and classification are interdependent, but a training in counting is to be preferred because it has a stronger effect compared with a training in classification and seriation. This conclusion agrees with a conclusion of Hiebert and Carpenter (1982). Based on a critical analysis of some research studies, they conclude that success on Piagetian tasks is not necessary for success on other mathematical tasks. The conclusions of Clements and Hiebert and Carpenter are contradictive with respect to other research results (for example, Arlin, 1981). She concludes that reporting is consistent about the fact that there is a relation between the skills necessary to solve Piagetian tasks and later mathematical achievements.

Another issue concerns the relation between counting and correspondence. Although young preschool children count readily when asked how many objects are in a set (or row), they often do not count when asked about the correspondence between two sets (Saxe, 1977; Sophian, 1987). This has led several investigators to suggest that preschool children may not yet understand the quantitative significance of counting. However, there is also evidence that children know the important principles of counting even before they know the sequence of the counting numbers (Greeno, Riley & Gelman, 1984). An example of mathematical knowledge present before counting is mastered are the protoquantitative schemes (Resnick, 1989). During preschool years, children acquire a large store of nonnumerical quantity knowledge. They can assess size labels as big and more correctly, based on perceptual comparison.

This analysis and a study of the literature on the subject identified a number of the skills children develop in the period of birth until about seven years old. Even at the very young age of about seven months, children are capable of making comparisons between certain numbers (maximum four) of objects. Two processes are used to make these comparisons: subitizing and preverbal counting (Antell & Keating, 1983; Gallistel & Gelman, 1992; Saxe & Gearhart, 1988; Starkey & Cooper, 1980). Concepts of comparison is the first skill described by us as a skill leading to early mathematical competence.

From the delineation of the development of counting it appears that the acoustic use of number words forms a skill that develops at about the age of two. Using number words in a correct way also means the correct use of the cardinal and ordinal number word. Using number words is a second skill leading to early mathematical competence.

As can be concluded from the literature there seems to be a direct relationship between cardination and ordination on the one hand and classification and seriation on the other. Being able to classify and seriate are two other skills leading to early mathematical competence.

The relation between counting, classification, seriation and making correspondences is described above. By means of making one-to-one-relationships, children are able to determine whether sets of objects contain the same number of elements. However, while determining if there are the same amounts, it is not necessary for the total number of elements to be known (Fuson, 1988). To be able to count synchronously, however, is more important. Making correspondences is another skill leading to early mathematical competence.

The use of more and more number words leads to synchronous counting. Children must have mastered synchronous counting in order to be able to practice shortened counting or structured counting. Synchronous counting is therefore another important skill of early mathematical competence.

Shortened counting can be seen as a more adequate way of counting a number of objects compared with counting one by one. However, on the other hand, making use of shortened counting in an adequate way encourages the use of synchronous (shortened) counting, in a situation, for example, in which material is used and children are asked to count while skipping every second object or counting house numbers on one side of a street (they count: two, four, six, etc.). In this kind of situation they learn to point to objects in a synchronous (shortened) way (they may only point to every second object).

Resultative counting means that a child is able to give a correct answer to a 'How many are there?' question. The child knows how to determine a number of objects. Resultative counting is an important skill required for the development of early mathematical competence.

Children must learn to use the different skills mentioned above in simple daily situations in which elementary problems are presented. This is what we call a general knowledge of numbers and this forms the last skill of early mathematical competence.

In sum, study of the literature and research results provided a list of eight important aspects which contribute to the development of early mathematical competence. Given the presumed interrelation, we can describe this early mathematical skill in the terms of Werner (1957) as multi-linear and we speak of early mathematical competence when we discuss the development of the early mathematical skills in children in the 4 – 7 year age group. In figure 1 the eight aspects of early mathematical competence are presented.

EARLY MATHEMATICAL COMPETENCE
- Concepts of comparison
- Classification
- Correspondence
- Seriation
- Using counting words
- Structured counting
- Resultative counting
- General knowledge of numbers

figure 1: eight aspects of early mathematical competence

Children must be able to use concepts of comparison like greater, most, less, etc., in a correspondence task, a counting task and even in a seriation task. The relativity of the importance of the Piagetian operations, classification, correspondence and seriation, is mentioned above. However, in this research project, the names of the Piagetian operations are used, but the content of the task is more directed to the use of counting strategies because of the presumed interrelations. For example: a seriation task can also be solved by using a counting strategy.

The counting skills were divided by us into three categories: Using counting words (cardinal and ordinal aspects of counting and acoustic counting), Structured counting (counting synchronously, using, for example, the dice-five and pointing to objects while counting) and Resultative counting (children know that the last number word mentioned gives the total number of objects, and they can count the objects without pointing to them). When all eight aspects have developed in an expected, age-related way, children in the age group mentioned have achieved a normal (average) level of early mathematical competence.

The assumption of multi-linearity of the eight aspects of early mathematical competence formed the basis for the construction of the 'Utrechtse Getalbegrip Toets' [The Utrecht Early Mathematical Competence Test] (Pennings, et al., 1995; Van Luit, Van de Rijt & Pennings, 1994). In this test, the eight aspects are operationalized in eight parts, with five items each. After normalization and standardization of the test, norm and criterium scores were determined using the item response theory. Children with a score below the criterium (45% correct) can be seen as possible low arithmetic performers. From this test, three Early Mathematical Competence scales have been derived following the modern item response theory.

From our pilot studies with the test, it appeared that about 25% of the children in the 4 – 7 year age group can be seen as possible low arithmetic performers. Results of this part of the research showed that these children mostly experience difficulties with seriation, correspondence and several counting skills, like synchronous and resultative counting (Van de Rijt & Van Luit, 1994). With regard to seriation children with low early mathematical competence experienced more problems with seriation on two aspects compared with seriation on only one aspect. As Blevins-Knabe (1987) reported, consistent with the ideas of Piaget, children at the age of five should be able to make several serations. As for correspondence and the various counting skills, the children showed asynchronous counting, pointing to more objects than naming number words, and in this way not being able to make a one-to-one-relation between the objects in two rows in order to compare the number of objects in both rows.

The following concluding remarks concerning the theoretical background of our research can be made. First, the Piagetian operations and different counting skills like synchronous counting and resultative counting seem to be strongly interrelated. Given this hypothesis, we describe early mathematical competence as one arithmetic skill, consisting of eight different aspects: concepts of comparison, classification, correspondence, seriation, using counting words, structured counting, resultative counting and general knowledge of numbers. Children with low early mathematical competence development show a developmental lag in the several aspects.

Secondly, several studies have shown that after about the age of three, children must be able to understand the meaning of counting, the amount that has to be counted and the relation between the various numbers. This understanding is related to the ability of making seriations and classifications. Due to the fact that research has shown that it is possible to train children in the understanding and use of the skills needed for (aspects of) sufficiently developed early mathematical competence (e.g. Clements, 1984; Wynn, 1990) by offering them several contexts in which the different skills can be used, we take this as an assumption for our research to stimulate early mathematical competence development among young low arithmetic attainers.

4 The additional early mathematics program

In this project, we investigate the possibility of stimulating the development of the eight aspects of early mathematical competence among low arithmetic attainers. Van Luit and Van de Rijt (1995) have therefore developed a specific program for early mathematics. The program is called the Additional Early Mathematics (AEM) program and is based on the assumptions described below (e.g. Baroody, 1992).

The first assumption is based on the need for an additional program for our specific population: young children with a below average early mathematical competence. The second assumption is a constructivistic one and is related to the ideas of realistic mathematical education. A child constructs his own mathematical knowledge. By offering different situations and materials in a specific program, children are stimulated to think actively and learn together with other children. The third assumption is based on the idea that arithmetics, if possible, should take place in a real (daily), thematical context in which different arithmetic skills are presented in an integrated way. By doing this, the skills are more meaningful and the children experience when, for example, counting and correspondence can be used. The fourth assumption is an assumption concerning the content of the program. All the preparatory arithmetic skills mentioned above, like correspondence, synchronous counting and resultative counting, are offered to the children. Since our assumption was one preparatory arithmetic skill, that is early mathematical competence which consists of different aspects, we thought it important to present these different aspects. This, however, is done in an integrated way. The children learn to use skills and strategies in combination with each other. They will come to understand that it is possible to use several strategies to solve one arithmetic problem. Further, the program is constructed in such a way that the use of more adequate strategies is evoked and stimulated during the course of the program. An example of this is the use of the dice structure to stimulate shortened counting.

5 Instruction

In literature different forms of instruction can be distinguished. An example is the so-called 'drill and practice' method derived from the behavioristic point of view (Resnick & Ford, 1984). The thought behind this method is that practice followed by reward leads to learning. This form of instruction is criticized a lot at this moment. In this instruction no attention is paid to the underlying principles of the skills to be learned. From the student his point of view we can speak of 'habit learning' (Skemp, 1989). The contrasting form is 'intelligent learning' in which the teacher explains the use of a certain solving strategy, but also the possible use of other strategies. Other forms of learning are 'discovery learning' and 'guided discovery learning' (Resnick & Ford, 1984). In the first one the children must discover relations and rules in problem situations by themselves. In the second form the children also must discover these by themselves, however, they are guided by cues in the situation or cues given by the teacher.

It is obvious that there exists a strong relation between the form of instruction and the way children learn and that the different ways instruction can be given each have their own advantages. Literature shows that little research has been done on the effects of different forms of instruction in mathematical education. Research that has been done is mainly focused on the effects of direct instruction. Direct instruction means that the teacher mentions strategies for solving mathematical tasks himself and explains them to the children. This form of instruction is comparable with the structuring form of instruction described above. For example, Gersten and Garnine

(1984) have done some research to the effects of direct instruction on children from deprived situations. Their conclusion is that the use of direct instruction seems to result in a regression of the need for remedial and special education.

Because of the fact that no research is know to the effects of different instruction forms with young low achievers, in the project presented two forms of instruction are operationalized. The two forms of instruction were based on developmental psychological and information processing ideas (De Corte, 1980; Glaser, 1990; Resnick & Ford, 1984) and on the different forms of instruction mentioned above.

5.1 Guiding instruction

The first form of instruction is the guiding instruction method which is based on the idea that young children need to play for a long time and that some skills cannot be taught before the child is ready to learn. The meaning of educating with guiding instruction is to enable children to investigate different solving strategies and to choose the ones they prefer. Self discovery learning is promoted and the subject matter is being tuned to fit in with the children. The theoretical ideas are from developmental psychology in which the intrinsic motivation of the child is important. This means that children are looking by themselves for new experiences without explicit additional motivation being provided by the teacher. The task of the teacher is to observe the children in order to oversee the solving strategies used and to guide the child in this process. Based on this observation the teacher chooses the materials and activities he can offer a child which also fit in with the capabilities of the child.

In this kind of instruction verbalization is important. A concrete example of the steps in a guiding instruction form is given below:

- Presentation of the task
- Asking for solving strategies
- Asking if the task can be solved in different ways
- Giving the child the chance to explain how he has solved the problem
- Not explicitly presenting a solution to the child
- Giving hints or asking questions that can help the child to reach a solving strategy
- Offering materials without any structure

In this group, play has an important role. The task of the teacher is nothing more than one of guidance, showing the child a path it can follow on it's own way.

5.2 Structuring instruction

The second form of instruction is the structuring instruction form. A structured way of educating assumes an unambiguous strategy. The ideas of Gal'perin (1978) in particular on complete instruction are the foundation of this vision of instruction. For example, the teacher has to suggest the successive phases of solving the problem in order to achieve the goal that was set. After following this kind of instruction the child should have developed a cognitive scheme by means of which he can solve arithmetic problems by himself using the instruction learned.

The support of thinking by means of verbalization is also an influence from Gal'perin in this structuring instruction form. The child is 'forced' to put into words the arithmetic problem presented and his own solving strategy. The child learns to reflect his own actions. In a concrete situation the steps in a structuring instruction form can look like this:

- Presentation of the task
- Asking for solving strategies (how could you solve this?)
- Structuring the task
- Demonstration and explanation of the problem solving (for example with materials)
- Model-learning (this is finally used to find the correct algorithms if the child is not capable of doing this by himself. The teacher demonstrates the correct way of solving a specific problem. After the demonstration, the children and the teacher run over the algorithm together again and, after that, the children imitate the solution by themselves. Moreover, inherent in this treatment is a useful way of controlling the results of the problems.)

From these distinguishing features of the structuring instruction form it becomes obvious that the term 'structuring' does not mean that the child makes no contribution of his own nor that more solving strategies are not possible.

It is evident that the amount of interaction between teacher and child differs in both forms of instruction. The guiding instruction is a child-following way of providing education and is used when a child only needs some support. There is more interaction between child and teacher in a more or less informal way. The structuring instruction conducted by the teacher is mainly visible by the giving of instructions and reactions to mistakes. There is more one-way traffic coming from the teacher to the child compared with the guiding instruction.

6 Method

6.1 Subjects

Five hundred and five children in the 4 – 7 age group from twenty primary schools were tested on their early mathematical competence. Of these children, 136 participated in the experiment. They were selected on the basis of a below-criterium score performance of 45% correct on first Early Mathematical Competence Scale.

6.2 Materials

- The Early Mathematical Competence Scales. These three scales, derived from the item bank of the 'Early Mathematical Competence Test' [Utrechtse Getalbegrip Toets] (UGT) (Van Luit, et al., 1994), can be used to select young children with a possible developmental lag in early mathematical competence. The scales consist of eight parts: Concepts of comparison, Classification, Correspondence, Seriation, Using counting words, Structured counting, Resultative counting and General knowledge of numbers. Each of the scales has 24 items spread over the eight parts.
- The Additional Early Mathematics (AEM) program. The AEM program is intended for children (in the 4 – 7 age group) with difficulties in understanding and using the different aspects (preparatory arithmetic skills) which lead to early mathematical competence. The AEM program covers the Piagetian operations and counting skills with a slight emphasis on the development and knowledge of using the counting skills. Piagetian operations, however, are often incorporated into the counting tasks. The AEM program on the whole is an integrated program with a variety of activities and tasks concerning early mathematical competence. The AEM program consists of 26 lessons (each lesson lasts about half an hour) and involves the numbers 1 to 20. According to the ideas of realistic mathematics, the lessons are divided into several themes. The themes provide the children with a familiar background in which the activities become meaningful and useful. The themes in the AEM program are: family, party, post, shopping and animals. A cluster of numbers is dealt with in each theme.

In the lessons 1 to 3, the numbers 1 to 5 are dealt with in the theme 'family'. Given the fact that pilot studies showed that children experienced the least problems, as expected, with these five numbers, only three lessons are spent on this cluster. In the lessons 4 to 8, in the party theme, attention is paid to the numbers 6 to 10. In the lessons 9 to 13, in the post theme, the numbers 1 to 10 are brought to the notice of the children with an emphasis on the row of numbers as a whole. The lessons 14 to 18 cover the numbers 8 to 12. In these lessons, in the shopping theme, attention is paid to going beyond the number ten. The lessons 19 to 23 focus on the numbers 11 to 15 and are dealt with in the animals theme. In lesson 24, a game is played using the numbers 6 to 15. The last two lessons of the program, lessons 25 and 26, form a kind of orientation on the numbers 16 to 20. Control of the arithmetic skills concerning these clusters of numbers, however, is not one of the goals of the program. In figure 2 the content of the program is represented.

LESSON	THEME	CLUSTER OF NUMBERS
1 - 2	Family	1 - 5
3	Game of goose	1 - 5
4 - 7	Party	6 - 10
8	Game of goose	6 - 10
9 - 12	The post	1 - 10
13	Memory	1 - 10
14 - 17	Shopping	8 - 12
18	Game of goose	8 - 12
19 - 22	Animals	11 - 15
23	Game of goose	11 - 15
24	Memory	6 - 15
25 - 26	Shopping	16 - 20

figure 2: content of the AEM program

The instructions are given in a guiding or structuring way. Two examples of an activity are given in figure 3 and 4 in order to provide a good impression of the differences between both forms of used instruction used.

Lesson 6: numbers 6 to 10
Theme: party: gifts

Exercise 6

The children were given cards with gifts on them, which were arranged in a dice structure. The children were asked to arrange the cards in a sequence from many to few gifts. This is a seriation activity in which the dice-structure can be used.

Guiding instruction

- The teacher asks the children if one of them has an idea how this problem can be solved. Is there a quick way of counting the gifts on the different cards? Do we have to count all the gifts one by one or is there another way of counting the gifts?

Structuring instruction

- The teacher gives the children some tokens the children can use when counting the gifts on the cards.
- The teacher presents an adequate solving strategy by selecting an arbitrary card. He counts the gifts on the card using the dice structure. Then he selects another card, counts the gifts on the card and compares the two numbers of gifts on both cards using the counting row.
- The teacher uses model-learning to lead the children to an adequate solving strategy. First he demonstrates the counting of the gifts using the dice structures. After the demonstration, both teacher and children repeat the solving strategy. After this, the teacher asks the children to perform the strategy by themselves.

figure 3: example of an activity in lesson 6

In figure 3 an activity is represented in which the children learn to count with, for example, the dice structure. Within a seriation task different counting strategies can be practised using the card belonging to the program. Learning to recognize the dice structure of five, for example, can be helpful when a great amount of objects must be counted. In that kind of situation the child can use the strategy of shortened counting starting with five and count on from five.

In figure 4 another example of an activity of the program is given in order to explain the possibilities and content of the program and how the materials can be used in a real (daily) life context familiar to the children.

Lesson 12: numbers 1 to 10
Theme: the post: mail delivery

Exercise 6

The children were given cards with the picture of a house on it. The numbers of the houses differ between 1 and 12 and are printed next to the front door. Next to the picture of the house the number of the house is represented in the dice structure. Also 15 postcards were given to the children and some lists on which was indicated by pictures how many cards must be delivered at a house with a certain house number. When a child must play the role of the postman he or she gets a cap of a postman and a list with the post he or she must deliver.

Guiding instruction

- The teacher asks if the child recognizes the number of the house looking at the symbol at the front door. If the child does not recognizes this the teacher can ask: how can you find out what the number of the house is? Can you tell me how many dots the dices have? How many cards must you deliver at that house? How do you know?

Structuring instruction

- The teacher gives the children some tokens the children can use when counting the cards.
- The teacher presents an adequate solving strategy by selecting an arbitrary list. He counts the cards on the list using the five structure. Then he looks at the number of the house next to the front door and names the number. Then he tell the child how many cards have to be delivered at that house with that particular number.
- The teacher uses model-learning to lead the children to an adequate solving strategy. First he demonstrates the counting of the cards using the five structure. After the demonstration, both teacher and children repeat the solving strategy. After this, the teacher asks the children to perform the strategy by themselves.

figure 4: example of an activity in lesson 12

As far as the material is concerned, we have tried to use concrete, existing materials as much as possible. On moments in the program when this was not possible, two-dimensional material was developed in the form of little carts with pictures on it. In figure 5 an example of a such a little card, belong to lesson 12, is given.

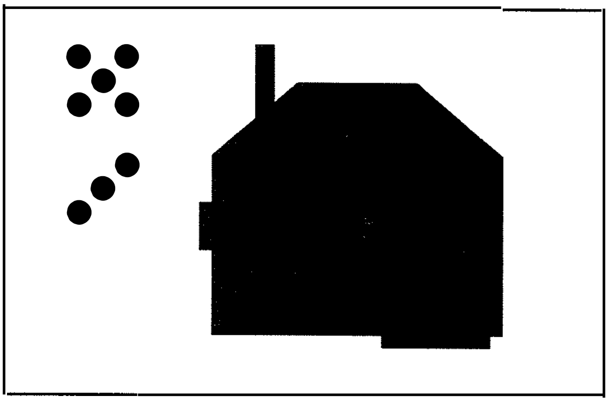

figure 5: example of a little card used in the AEM program

At the begin of the program it is clearly stated that the use of concrete material is preferred and that the little cards can be replaced by existing materials when available at school. The materials can be used at a 'Piagetian' way (for example: seriation from big to small) but mostly they can be used at a counting directed way (for example: seriation of the dice structures from less to much).

The program is constructed in such a way that it is suited for little groups of five children at the most. With this number of children the teacher can divide his or her attention among all the children in a sufficient way and he or she can take the individual differences of the children into account.

So, as far as the content is concerned, the program is identical for both instruction forms. Children who are offered the program with a guiding instruction received exactly the same program as children who are offered the program with a directing instruction.

7 Procedure

After selecting the children, with the first Early Mathematical Competence Scale administered in february, with a below-criterium score performance of 45% correct, 136 children were divided into four groups according to the matching principle. Matching variables were (in decreasing order of importance) (1) pretest score, (2) age and (3) sexe. Two of the four groups were experimental groups and the two other groups formed control groups. The two experimental groups differed in the kind of instruction used in the program: a guiding or structuring instruction. The two control groups differed in the way mathematics was presented, either according to one of the common Dutch arithmetic methods or according to a way of teaching arithmetic without use of a specific method. All children in each control group worked at least two times a week, half an hour each time, in mathematical tasks. Each group consisted of 34 children; the mean age of the children at the moment of pretest was 71 months. The children in the experimental groups as well as the children in the control groups received instructions in small groups of four or five children. After the 26 lessons of about 30 minutes each, two lessons a week, had been given, all the children participating in the experiment received the second Early Mathematical Competence Scale. After seven months they were tested by the third scale in order to examine the long-term effect of the AEM program.

8 Results

Data were analyzed with one-way analysis of variance. First of all, the effect of the kind of instruction used to present the program to the children was examined. To this end, a comparison between the two experimental groups was made at all three moments of measurement. The mean scores on the Early Mathematical Competence Scales are presented in table 1. The maximum total competence score on each scale is 100.

	Guiding instruction		Structuring instruction	
	M	*SD*	*M*	*SD*
pretest	53.26	7.19	52.58	5.47
'post-test	72.59	10.32	73.62	8.81
follow-up test	81.82	12.14	80.42	9.26

table 1: comparison of the two experimental groups

In order to test if there are differences between the two instruction groups, a one-way analysis of variance was used. The results showed no significant differences between the two experimental groups (guiding versus structuring instruction) on the pretest, $F(1,67) = .147, p = .70$; no significant differences on the 'post test, $F(1,66) = .068, p = .80$; and no significant differences on the follow-up test, $F(1,60) =.731$, $p =.40$. Both instructional groups achieved equal pretest mean total scores. This was not an unexpected result because the groups were formed according to the matching principle. However, on the 'post test as well as on the follow-up test the experimental groups also achieved equal mean total scores. These results indicate that the form of instruction by which the program was giving has no influence on the effect of the program.

The same analysis was carried out for both control groups in order to compare the effect of the way a regular arithmetic program is presented; in a spontaneous way or according to an arithmetic program, at least twice a week. The mean scores on the Early Mathematical Competence Scales are presented in table 2.

	Arithmetic education with regular method		Arithmetic education without regular method	
	M	*SD*	*M*	*SD*
pretest	51.38	6.23	52.83	8.53
'post test	62.29	9.45	59.72	9.29
follow-up test	72.83	10.52	76.83	8.12

table 2: comparison of the two control groups

In order to test if there were any differences between the two control groups, again a one-way analysis of variance was used. On the pretest, both control groups show equal mean total scores, $F(1,67) = .771, p = .38$. Again, this can be explained by the fact that the groups were formed according to the matching principle. However, on both post test and follow-up test, both control groups again showed equal mean total scores: post test $F(1,63) = .441$, $p = .51$; follow-up test $F(1,56) = .984$, $p = .33$. Based on these results, we conclude that the way the regular arithmetic program is given has no influence on the scores children get on the Early Mathematical Competence Scale.

Given the fact that the results presented in tables 1 and 2 showed no significant differences between the experimental groups and the control groups, they are merged, resulting in one experimental group and one control group. With these two groups, the effect of the program was investigated by means of a one-way analysis of variance. The mean scores on the Early Mathematical Competence Scales of both groups are presented in table 3.

	Experimental groups		Control groups	
	M	*SD*	*M*	*SD*
pretest	52.38	6.92	51.77	7.45
post test	72.94	10.26	60.23	9.00
follow-up test	80.74	11.10	75.09	9.72

table 3: the effects of the AEM program

The results of the one-way analysis showed a significant effect of the AEM program on the post test ($F(1,130) = 56.566$, $p = .000$) as well as on the follow-up test ($F(1,117) = 8.604$, $p = .004$) for the experimental groups. Although the results of a paired t test indicated that the control groups made significant progress from pretest to post test and from post test to follow-up test, the experimental groups achieved more progress, probably under the influence of the program. It appears that participation in the AEM program has a positive effect on children's performance.

Given the fact that the children who participated in the experiment (N = 136) were drawn from a sample of about five hundred, the remaining children (N = 369) formed a norm group. In order to see if the experimental groups reached the level of the norm group, the mean competence scores of the experimental groups and the control groups are presented in figure 6.

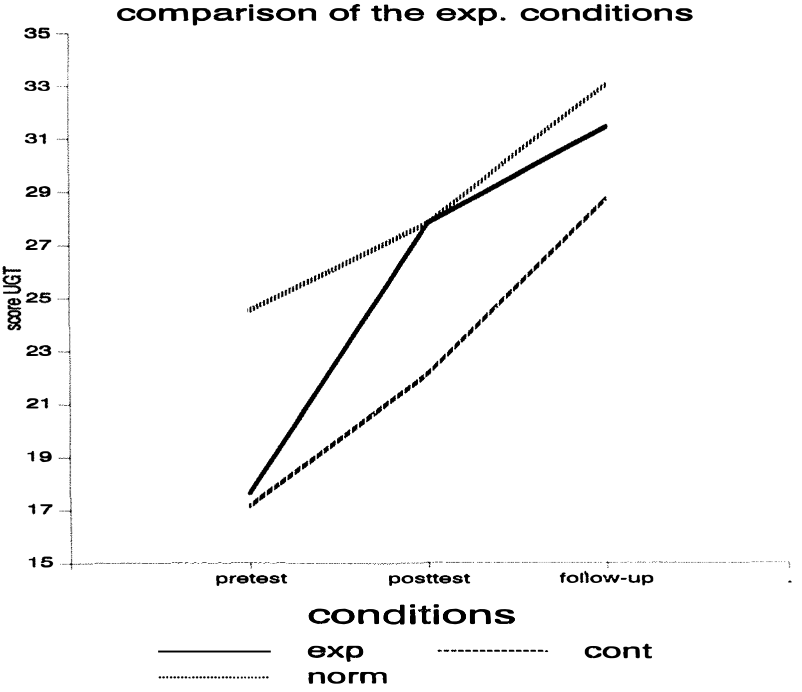

figure 6: comparison of the experimental and control groups with the norm group

When the results of the children who followed the AEM program, of those who were given the regular arithmetic program, and the results of those who formed the norm group, were all considered separately, each group tended to make significant progress in their mean total scores at each of the three moments of measurement. The progress made by the experimental groups which followed the AEM program however, was significantly greater than the control groups. In figure 6 it is clear that the experimental groups reach the level of the norm group. Results of the *t* test showed no significant differences between the experimental groups and the norm group on the post test or on the follow-up test. A comparison of the results of the control groups with the norm group showed significant different mean total scores on the post test as well as on the follow-up test.

9 Discussion and conclusion

The literature on the Piagetian operations and the different counting skills developing in the 4 – 7 year age group, reveals many comparable strategies in both kinds of cognitive abilities. Besides the fact that the use of the same strategies is possible to solve a Piagetian problem just as a specific counting problem, there is also the fact that both cognitive abilities seem to be strongly interrelated. They do not seem to develop in a sequential way, but more horizontally and vertically at the same time. We therefore like to speak of multi-linear development of the different aspects of early mathematical competence in the 4 – 7 year age group. We think, based on literature and our own pilot-studies (Van de Rijt and Van Luit, 1994), that eight aspects form the foundation of early mathematical competence and like to describe early mathematical competence as a prerequisite for later arithmetic, consisting of the following aspects: Concepts of Comparison, Classification, Correspondence, Seriation, Using counting words, Structured counting, Resultative counting and General knowledge of numbers. Given the fact that, according the results of our pilot-studies (Van de Rijt and Van Luit, 1994), about 25% of the children in the 4 – 7 year age group show a severe lag in the development of early mathematical competence, we therefore examined the effects of the AEM program for these low arithmetic achievers.

This experiment has attempted to assess the effect of two types of instruction in the AEM program on the development and learning of preparatory arithmetic strategies of young children with a developmental lag with respect to early mathematical competence. The two methods of instruction were a guiding and a structuring instruction. There were no differences found between these two ways of instruction. Children who received the AEM program with the guiding instruction performed at the same level as children who received the AEM program with a structuring instruction. These results do not give an answer to the question which form of instruction is most convenient for low performers in early mathematics. An explanation for these results is the fact that the teachers were not able to strictly maintain the guiding instruction. It happened that the children were not able to solve any problem without more direct instruction from the teacher. As a result, teachers in the group with the guiding instruction sometimes used a more structuring instruction so both groups tended towards more equal instruction. An alternative explanation was given by Wood, Wood and Middleton (1978). In an experiment on face-to-face teaching strategies with mothers and their 3 to 4 year old children, the mothers were rarely able to maintain the instruction they were instructed to use. The explanation they gave is that there is an effect of the mutual influence of mother and child. This may also be the case in our research. Not only can the teacher influence on the behaviour and thinking of the child, but the child, in turn, also evokes certain behaviour and thinking in the teacher, no matter what instruction group the child is in. However, this conclusion is purely based on quantitative data.

In order to get a more detailed picture of the way the children respond to the different forms of instruction, qualitative analysis can be done. Some comments of teachers who have used the guiding instruction:

- frustrating when a child does not find a solution
- tendency to help instead of guide
- children are not used to this kind of instruction
- good for the motivation of the children

Some comments of teachers who have used the structuring instruction:

- gives the children grip
- children feel themselves more independent
- the children are not flexible in their thinking strategies

This selection of comments shows that in both instruction groups the experiences differ; in both instruction forms teachers are satisfied to a greater or lesser extent with the interaction and the way children respond to the instruction. However, this qualitative interpretation has not yet been completely analyzed. In future research a systematic observation on the basis of video recordings of both groups would not only provide insight into the question as to whether the teacher was able to use the instruction in the way it was meant, but would also clarify the effect the instruction has on the interrelation between teacher and child.

Further, we can conclude that the program has a positive influence on the development of early mathematical competence, as described above, consisting of eight different aspects. The children learned, by means of the program, to apply the different strategies and skills which lead to well-developed early mathematical competence.

At the beginning of the program the children in both experimental groups had an average achievement of concepts of comparison and correspondence. The other six skills were achieved at a insufficient level. After following the program, the children achieved concepts of comparison, correspondence, seriation and using counting words at a good level. The skills of synchronous counting, resultative counting and general knowledge of number were achieved at an average level. So, it can be concluded that the program has a positive influence on the development and use of counting strategies which lead to early mathematical competence.

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Early training: who, what, when, why, and how?

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1 Early mathematical training

Mathematics is known to be hard to teach and difficult to learn. It is well known that many students never get real insight into important aspects of the subject. What kind of instructional support should be given to children and at what age level systematic instruction should start is discussed most controversially. In the Piagetian framework of cognitive development, minimal demands are made on systematic instruction and social support. It is widely believed that unless children do not grow up under extremely deprived conditions, their environment offers what they need to undergo the stages of cognitive development and thereby develop cognitive structures which are preconditions for abstract reasoning such as required in mathematics. Some researchers even question the purpose of structured mathematical instruction in elementary school (Kamii, 1985).

Hans van Luit and Bernadette van de Rijt (vL&vdR) clearly take an opposite view. They claim that starting systematic instruction in mathematics at the regular elementary school at the age of 6 may be too late at least for some of the children. Therefore the authors developed the impressive AEM program and presented it to the bottom third of a representative sample of five-year old children. The reported results clearly indicated positive training effects: the trained group performed better than an untrained group with a similar initial performance level. In fact, despite their poor initial performance level, the subjects of the trained group reached the performance level of the untrained upper two third group.

However, the difference between the control group and the training group declined in the follow-up test. The control group caught up, and the experimental group approached the ceiling of the test. When evaluating a training program one has to address the question of whether the trained group only is ahead of the control group for a certain period of time. To justify the costs of the training one has to prove that the training group outperforms the control group also in the long run. Only additional follow up-studies will allow further conclusions concerning the success of the AEM training program. Moreover, one has to prove that the superiority of the training group is due to specific components of the training rather than to general practice effects. In further evaluation studies subjects of the control group should be presented with an unspecific training program. Although currently only preliminary conclusions concerning the impact of the AEM program are possible, the paper of vL&vdR clearly provides an interesting basis for discussing principle questions concerning the effects of early training programs in mathematics.

The AEM program aims at compensating individual differences in cognitive preparedness which may be responsible for the huge variance in mathematical performance already observed when children enter school. Some children can hardly count to 10 while others already have acquired basic computing skills. In the following school years, tremendous achievement differences occur despite of rather homogenous learning environments. In order to justify an early applied training program, it is necessary, although not sufficient to prove stability of interindividual differences over time. Only if the children who had performed poorly at an early age level are still disadvantaged at a later age level, training programs such as the AEM can be considered as useful instruments to improve mathematical performance for children with disadvantageous prognosis. An appropriate application of training programs presupposes knowing in advance who will have particular difficulties with the acquiring mathematical competencies later on. From research on acquiring literacy we know that one can identify children at risk as early as preschool age. Children with underdeveloped phonological awareness can be expected to have particular difficulties with acquiring reading and writing later on (Schneider, in press). Moreover, offering these children exercises such as clapping syllables or recognizing rhymes already in preschool time facilitates later acquisition of reading and writing (Bradley and Bryant, 1985). However, in case of short resources it is only useful to train children who show symptoms of dyslexia, because the great majority of children can be expected to acquire reading and writing skills at school without particular difficulties. What dyslexia is in literacy is dyscalculia in mathematics. A small percentage of children can be expected to have particular difficulties with figuring out even simple calculation problems and with developing a factual network (Lorenz, 1992). The AEM, however, does not particularly focus on children who suffer from dyscalculia. Rather, AEM was applied to the bottom third of a representative sample and therefore not only aims at improving the performance of a small group of extremely disadvantaged children. Moreover, there are principle differences between the domains of mathematics and literacy. The main aim of literacy acquisition is skill-automatization, while the aim of learning mathematics is the acquisition of advanced concepts that can be used as tools of reasoning. Automatization required in reading and writing is acquired by deliberate practicing, and despite large individual differences in learning time, all learners who do not suffer from dyslexia become experts in automatized use of letters. Acquiring automatization, however, is only a subordinate goal of teaching mathematics. Running efficient computing procedures and developing numerical networks is necessary, but in no way sufficient for acquiring expertise in mathematics. The main purpose of elementary school mathematics is to prepare students for understanding advanced concepts such as fractions or decimals.

Infancy research suggests that humans are biologically prepared for understanding numbering and addition and subtraction when faced with small sets of elements (Gelman, 1991). With a minimum of instruction, these conceptual primitives guide activities based on the cardinal function of numbers, such as counting and modeling the exchange of sets by addition and subtraction. While humans are biologically privileged in the use of cardinal numbers, advanced mathematical reasoning is based on concepts which are the result of a long-lasting cultural development. Modeling static relationships between sets such as it is the case in quantitative comparison and measurement situations or the use of non-integers requires people to give up principles that guide the use of numbers as counting instruments (Staub and Stern, in press). Children's difficulties with modeling static relationships become apparent when they are faced with arithmetical word problems dealing with the quantitative comparison (Stern and Lehrndorfer, 1992; Stern, 1993). At the latest when faced with problems dealing with algebra, fractions, or decimals one has to overcome the idea that counting is the only function of numbers and that mathematical operations always correspond to concrete actions (Stern and Mevarech, 1996).

The main focus of this paper will be on the question of how children can be supported in extending their concepts of numbers and mathematical operations in the described sense. Number conservation, undoubtedly, is an important step in developing extended mathematical competencies because children have to understand that an obvious activity of changing the spatial arrangement elements has no effect on the more abstract dimension of quantity. In this sense, number conservation is a precondition for understanding the quantitative comparison. Quantitative comparison and number conservation are among the components to be trained in the program developed by Van Luit and Van de Rijt (1996). Therefore AEM can be expected to support an extended mathematical understanding already at an early age. The longitudinal studies to be discussed in the following investigate the impact of number conservation in preschool time on elementary school children's competencies in dealing with the quantitative comparison, and moreover, the effects and knowledge and the impact of elementary school knowledge on middle grade knowledge has been researched.

2 Longitudinal development of mathematical competencies

In order to research social, motivational, and cognitive development, the longitudinal studies *logic* and *scholastic* were run at the Max-Planck-Institute for Psychological Research in Munich from 1983 to 1993 (Weinert and Schneider, in press). Among other variables not discussed here children were presented with measures of numerical and mathematical competencies and general intelligence. The 186 children of the *logic*-study entered the sample at age 3-4 and were tested in individual sessions three times a year until they reached age 12-13. In 1988, when the *logic* children entered second grade of elementary school, the scholastic-longitudinal study started. In this study about 1200 elementary school children were presented with group tests in their classrooms four times a year from grade 2 to 4. 92 children of the *scholastic* sample also participated in the logic sample. 201 children of the *scholastic*-sample were also tested in fifth and sixth grade. These children were not part of the logic sample.

2.1 The impact of preschool performance on later mathematical competencies

The following analyses present data from the 95 children who participated in the logic sample as well as in the *scholastic* sample by considering the following measures:

Number conservation: Mastering the number-conservation task means to understand that verbal expressions such as 'more than' and 'less than' refer to the number of elements of a set rather than to the spatial expansion of the elements. Thus, the number conservation task might be an indicator of early quantitative reasoning rather than of a general cognitive level. At the age of 3-4 and 5-6, children were presented with number conservation problems.

Estimation of quantities: Another measure of early quantitative abilities was the test of estimating quantities, which is part of a German test of school readiness developed by Kern (1971). Children were presented with a set of 3-9 small cubes and had to tell the size of the quantity without counting. Although this test was developed long before sophisticated theories of knowledge representation had been developed, a post hoc theoretical explanation might be that the test measures the efficiency in transforming visual information into mathematical symbols.

Word problem solving: In the scholastic sample, children were presented two times a school year with mathematical word problems differing in complexity and in the underlying situational model. *Addition and subtraction* problems were presented in grades 2-4. The one-step problems were taken from the 14 standard problems mentioned in Riley, Greeno and Heller (1983). The multiple step problems were constructed from these problems. An example of a multiple-step comparison problem is:

> John has 5 marbles.
>
> He has 3 fewer marbles than Peter has.
>
> Peter has 2 more marbles than Susan has.
>
> How many marbles does Susan have?

The results reported in this paper are based on scores developed for each school year by considering the following problem-types:

- six one-step and multiple-step problems dealing with the *exchange* of sets;
- four one-step and multiple-step problems dealing with the *combination* of sets;
- six one-step and multiple-step problems dealing with the *comparison* of sets;
- six one-step and multiple-step *multiplicative* word problems in grades 3-4, which either required the multiplication or the division of numbers. Some of these problems were based on advanced understanding of multiplication and division, such as the cartesian product and multiplicative comparison.

The structure of the problems and the numbers were kept constant at all measurement points, while superficial features such as names and objects were changed. The problems were presented in a booklet with four problems on each side and the children were given sufficient time to work on all problems.

3 Results

For each school year the sum score of correctly solved word problems was developed. The following analyses were conducted: *Stability of word problem solving during elementary school time.*

The results proved high stability of performance in word problem solving during preschool time (correlation between second and third grade: $r = .64$, $p < .001$; correlation between second and fourth grade: $r = 62, p < .001$; correlation between third and fourth grade: $r = .75, p < .001$). These substantial correlations indicate that the sources of individual differences in word problem solving are already established in second grade. The results suggest that already in second stable individual differences in word problem solving are observed. Therefore, the following only presents results regarding the prediction of performance on word problem solving in second grade.

3.1 The impact of mastering number conservation on age level 3-4 on word problem solving

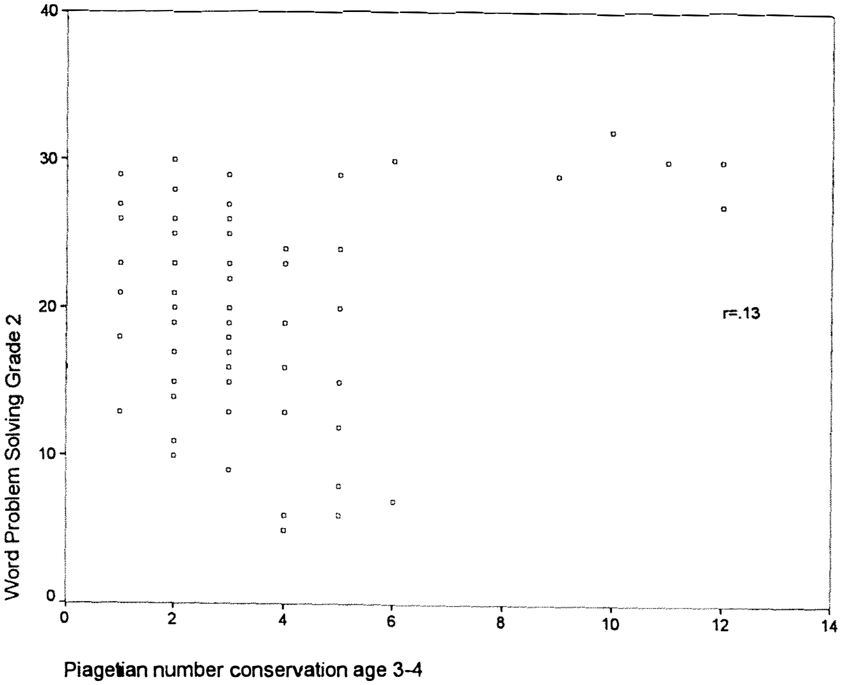

figure 1: the relationship between performance in the Piagetian Number Conservation Test at age 3-4 and word problem solving in grade 2

Figure 1 depicts the correlation coefficient and the scatter-plot between performance in word problem solving in second grade and number conservation at the age-level 3-4. The results suggest that at the age of 3-4 mastering the number conservation task is a sufficient although not a necessary precondition for high performance in word problem solving. The 5 children who had already mastered the number conservation task at this age level belonged to the group of the best word problem solvers and were ahead of their classmates during the whole elementary school time.

3.2 The impact of numerical competencies at age level 5-6 on word problem solving

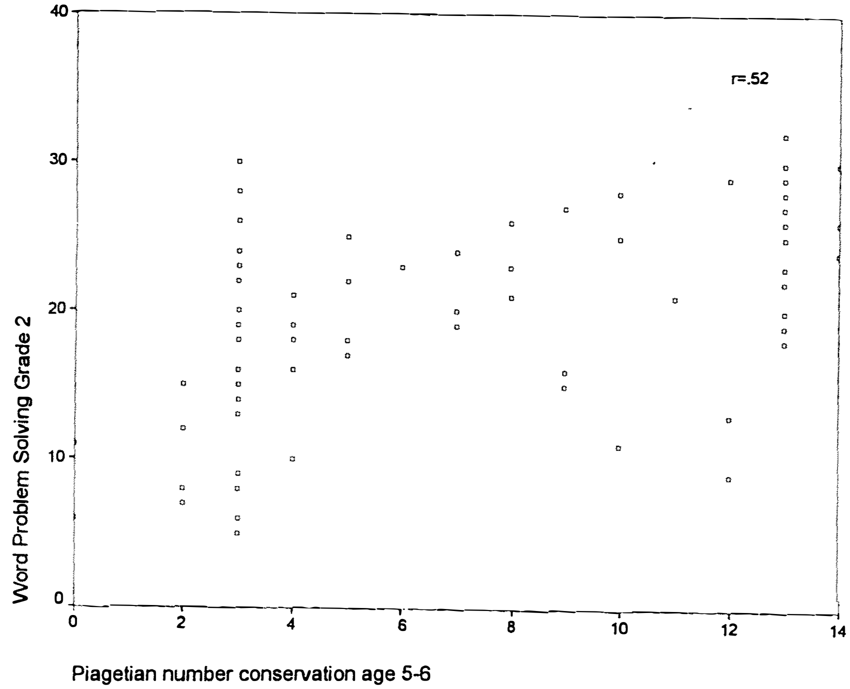

figure 2: the relationship between performance in the Piagetian number conservation test at age 5-6 and word problem solving in grade 2

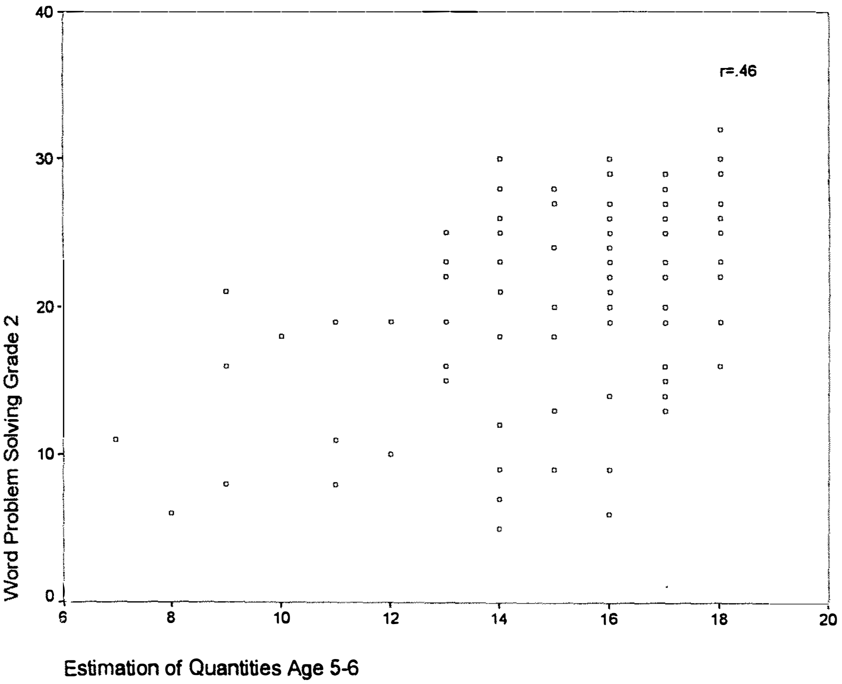

figure 3: the relationship between performance in the estimation of quantities test at age 5-6 and word problem solving in grade 2

Figure 2 and 3 depict substantial correlations between indicators of mathematical competencies at the age of 5-6 and word problem solving in second grade. However, as the plots also demonstrate, that there are many outliers. High numerical competencies do not guarantee high performance in word problem solving and many children who performed poorly in preschool measures showed above-average performance in word problem solving. None of the two measures can be considered as an appropriate indicator of identifying children at risk. However, as the correlation between the two measures is only moderate (*r* = .32, *p* < .05), combining both measures might allow to predict children at risk.

The multiple correlation between the preschool indicators of numerical competencies and word problem solving in second grade was $R = .60, p < .001$. A more detailed analysis showed that 86% of the children who were beyond average in **both** preschool measures also were beyond average in word problem solving in second grade. On the other side, 75% of the children of were above average in both preschool measures also were above average in word problem solving in second grade. The results suggest that before children enter school, individual differences of mathematical competencies are already quite stable. Elsewhere (Stern, in press) is has been shown that the high stability cannot be explained with the stability of measures of general intelligence, which were also administered in the longitudinal sample.

3.3 The impact of preschool numerical competencies of different types of word problem solving

Additional analyses were conducted to find out whether certain word problems are particularly affected by early numerical competencies. Addition and subtraction word problems dealing with the exchange, the combination and the comparison of sets presented in grade 2 and 3 were considered. To fulfil statistical preconditions, for each school year and each problem type the three problems closest to the solution rate of .50 were selected (defined criterium was .45-.55). Table 1 depicts the correlations between number conservation and estimation of quantities at age 5-6 and the three word problem types.

word problem type	number competencies	
	number conservation	estimation of quantities
comparison	.54**	.44*
exchange	.34*	.34*
combination	.36*	.29*

table 1: correlation between scores of word problem types in grade 2 and 3 and number competencies at age 5-6
**p <.001,*p <.05

Significance tests revealed that the correlation between comparison problems and number conservation was higher than the other correlations. The results suggest that performance in solving comparison problems is more affected by early number competencies than are combination and exchange problems.

Altogether the hitherto reported results suggest that early understanding of number conservation facilitates the acquisition of extended mathematical competencies in elementary school time, based on understanding mathematical symbols as instruments for representing static set relations. Children who lack basic numerical skills in preschool time can be expected to have difficulties with school mathematics.

4 The impact of mathematical competencies in elementary school on understanding advanced mathematical concepts in middle grades

The question to be addressed next concerns the stability of interindividual differences during school time. The results reported in the previous section suggest that the sources of individual differences in advanced mathematical understanding in elementary school time go back to preschool time. This paragraph analyzes the impact of performance in elementary school mathematics on extended mathematical understanding in middle grades. Students have to understand that numbers are not only used for counting but also to describe the relations between sets at the latest by middle grades. Understanding rational numbers requires giving up several principles that guided the understanding and use of natural numbers:

- While every natural number has a successor, this is not true for rational numbers. For natural numbers, there is a referent for the phrase 'the next number after one'. However, there is no referent for the phrase 'the next number after one half'.
- There is a smallest natural number but no smallest rational number.
- All natural numbers but not all rational numbers lying between two numbers can be enumerated.

From literature we know that in dealing with decimal numbers and fractions, children are particularly prone to errors and bugs (Hiebert and Wearne, 1986). By relying generally on the counting function of numbers, children conclude that larger numbers always refer to larger quantities and vice verse. Such results reflect children's difficulties with restructuring simple mathematical concepts into more advanced ones. Students who have attended mathematics instructions for years and who have acquired complex computing procedures and strategies have very restricted conceptual mathematical understanding because they have not overcome the cardinal function of numbers. However, long before being presented with problems containing fractions and decimals, children are faced with problem-situations based on number-concepts that go beyond counting. Understanding the quantitative comparison might be a first step in understanding that numbers are not only used as cardinal numbers but also as relational numbers. Therefore, word problems dealing with the comparison of sets might bridge the gap between understanding natural and non-natural numbers.

The following analysis intends to explain variance in conceptual understanding of non-natural numbers. Given that an early understanding of the quantitative comparison helps children to overcome the view that counting is the only purpose of numbers, high achievement in solving comparison problems at the beginning of elementary school is expected to be a valid predictor of later understanding fractions. To test the specific impact of knowledge genesis, measures of general intelligence presented were included in the analysis. To ensure that understanding the specific principles of quantitative comparison is not only an indicator of general mathematical abilities but does especially effect the later understanding of fractions, additional mathematical competencies were considered.

4.1 Subjects

Mathematical achievement measured in fifth grade was predicted by measures gained in second, third, and fourth grade. Two hundred and one children who entered the previously mentioned scholastic longitudinal study at the beginning of elementary school and participated until the end of sixth' grade.

4.2 Measures used as predictors

A test of non-verbal intelligence based on the culture free test of Cattell which was presented in second and fourth grade (Weiß and Osterland, 1979). The arithmetic word problems discussed in the previous section were presented. In addition, speed tests of arithmetic abilities were presented in grades 2-4. The subjects were presented with 20 problems presented on one page and were given one minute to solve as many problems as possible. The tests in grade 2 contained four pages with addition and subtraction problems with numbers up to 20, and the test presented in grade 3 and 4 contained four pages with multiplication and division problems with multiplicators and divisors smaller than 10, and addition and subtraction problems with numbers up to 100. The problems had either to be calculated or subjects had to mark whether given solutions were correct or not. By considering mathematical principles such as commutativity, performance could be improved dramatically in some problems.

4.3 Measures used as criteria

Fraction Understanding Test: This test was used to measure fifth graders' understanding of fractions. At this age level subjects had been taught some formal principles of fractional notation. The children were presented with two fractional numbers and had to choose the larger of the two (e.g. 6/7 or 6/8). Altogether, seven problems were presented and the children were allowed to work on the test for three minutes.

Multidigit Arithmetic Test: To examine the specificity of the predictors, an arithmetic test developed by Halford (1992) was presented in fifth grade. This test requires inserting the signs into numerical equations, such as '5 _ 8 _ 4 = 9'.

In order to pass this test, a rich numerical network is required that allows for the retrieval of the arithmetical relations between numbers. The children were presented with 13 problems and were given three minutes.

5 Results and discussion

Separate regression analyses were performed on the Fractions Understanding Test and on the Multidigit Arithmetic Test. The internal consistency of the predictors varied between .76 and .83. The mean solution rate as well as the variance of the Fraction Understanding Test ($M = .41$, $s = .26$) and the Multidigit Arithmetic Test ($M = .46$, $s = .20$) were alike. The purpose of the regression analysis was to the impact of general of the regression analysis are depicted in Table 2.

	Task	
Predictors	*Fraction*	*Arithmetic*
Intelligence		
Grade 2	n.s.	n.s.
Grade 4	2	n.s.
Arithmetic Tasks		
Grade 2	n.s.	25
Grade 3	n.s.	6
Grade 4	n.s.	n.s.
Word Problems Add. and Subtr.		
Exchange		
Grade 2	n.s.	2
Grade 3	n.s.	n.s.
Grade 4	n.s.	n.s.
Combination		
Grade 2	n.s.	n.s.
Grade 3	n.s.	n.s.
Grade 4	n.s.	n.s.
Comparison		
Grade 2	34	n.s.
Grade 3	9	n.s.
Grade 4	n.s.	2
Word Problems Mult. and div.		
Grade 3	n.s.	n.s.
Grade 4	4	n.s.

table 2: results of the regression analysis: percent of explained incremental variance ($p < .05$) for each predictor

In fact, the best predictor of the Fractions Understanding Test in grade 5 was the ability to solve comparison problems in grade 2. Fluid intelligence, although measured at the same time the criterium was measured, did not explain more variance than specific knowledge effects measured two years ago. The specificity of knowledge effects is supported because when predicting performance in the Multidigit Arithmetic Test, performance on comparison problems only played a minor role. Thus, understanding of comparison problems was not a general predictor of mathematical achievement, but rather was specifically related to the understanding of fractions. The results are in line with the claim that the understanding of fractions is guided by similar principles as the understanding of quantitative comparison problems. Therefore, early understanding of situations in which the function of numbers goes beyond counting facilitates later understanding of more advanced numerical concepts. It is a remarkable result that performance in comparison problems in grade 2 was a better predictor than performance on these problems in grades 3 and 4. This result suggests that children who extend their knowledge about numbers from cardinal use to relational use at an early age have a better chance to redescribe their number knowledge in a way that allows an understanding of rational numbers.

6 Final conclusions

What conclusions do the reported longitudinal results allow concerning the training program developed by Van Luit and Van de Rijt? The reported data contribute to the question of *why* it might be useful to train already preschool children in mathematical competencies. What the authors of the AEM program presuppose has been proved in the longitudinal results: before children receive structured mathematical instruction in regular first grade they already differ considerably in mathematical competencies, and these differences are amazingly stable. The results suggest that children's particular difficulties with mathematics in middle grade goes back at least partly to deficits in preschool time. Training programs that aim at compensating for individual differences at an early age level can be expected to facilitate the acquisition of school mathematics.

The reported longitudinal data revealed that number-conservation, which was a component of the training program, also was a good predictor of word problem solving in elementary school. This, of course, cannot be interpreted as a proof that training number conservation in preschool time guarantees better performance in word problem solving later on. The significant correlation between performance in two problems might go back to a common ability which itself might be rather unaffected by environmental factors. A significant interindividual stability over time is necessary although not sufficient for justifying a training program. The reported longitudinal results encourage to run additional training studies for further clarification.

Given that early training programs have long term effects the question arises of *what* to train. Van Luit and Van de Rijt have chosen eight components, some of them focussing more on the counting function of numbers, while others may support an extended understanding of numbers and mathematical operations. Further research is needed to find out, what training components are particularly helpful for raising mathematical achievement. This question cannot be addressed independent from the question of *who* needs an early training. Given most children's difficulties with mathematical problems that require going beyond the counting function of numbers and the action-based understanding of addition and subtraction, the bottom third of preschool children may not be the only ones who may profit from an early training program. While only few children might need help to master the counting function of numbers, the majority of children might gain from a training program that helps to overcome the view that counting is the only function of numbers. As in the Netherlands the majority of children enter preschool classes around the age of four, early training programs could be broadly applied. Broad application of a training program however, may be incompatible with the goal of compensating for individual differences because of the well known Matthew Effect of training programs. The Matthew Effect means that as a result of a training program variance increases because the higher the initial achievement level of a learner is the more s/he gains from a program. Therefore, when particularly aiming at the compensation for individual differences, training programs should give a start to those children who are expected to have particular difficulties with elementary school mathematics. This concerns the question of *when* - i.e. at what age level - to start with an early training program. In line with other findings, the reported longitudinal data suggest that poor performers in elementary school mathematics cannot be predicted before the age of five. Therefore, the application of compensatory training programs at an earlier age level makes no sense. However, given the reported longitudinal result according to which all children who mastered the number conservation task at a very early age level showed high achievement in elementary school mathematics, one could think about applying more number games to already to young children. In this case children who do not make appropriate progress could be identified more reliably and get a compensatory training.

The authors have developed two training programs which correspond regarding the content of the problems to be trained, but vary with respect to the method of teaching. Results revealed that the way of instruction had no effect on the mean performance rate. The question of *how* problems are presented seems to be subordinate. The authors discuss plausible reasons for why guided and structured instruction did not reveal different effects. The routine teachers have in running their own instruction style prevents them from following the instruction given by the scientists. It may be the content of the problems rather than the way of instruction that is crucial for improving mathematical competencies.

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Implementation and effect of realistic curricula

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1 Introduction

Inspired by Hans Freudenthal's ideas, a new approach of mathematics education, called realistic mathematics education (RME), has been developed in the Netherlands during the last 20 years. Basically these ideas can be traced back to the fundamental adagium that mathematics is a human activity. Among other things, this development has resulted in new mathematics textbooks, incorporating the principles of the realistic approach. These ideas were enthusiastically received by teachers. However research in the practice of mathematics education revealed that the reform was incomplete: although teachers adopted the new curriculum, their instructional practice hardly reflected the intended pedagogy.

The need to reform mathematics education is further legitimized by recent views on learning and instruction, emphasizing learning as an active and constructive process. When mathematics as a human activity is accepted as an essential characteristic, a number of problems accompany implementation. Basic problems that have been encountered in implementing new approaches concern how to realize different roles for students and teachers.

The idea of mathematics as a human activity is elaborated in Freudenthal's principle of guided reinvention. This principle however, comes with a tension between 'guidance' and 'invention'. Based on this tension we can discern an idealistic approach of enactment of realistic mathematics, and a pragmatic approach. The idealistic approach will allow as much room to invention as possible, combined with more indirect guidance, whereas the pragmatic approach emphasizes the role of direct guidance by the teacher. At this point, the contrast between a cognitivist and a more constructivist approach of learning is relevant. Starting from a cognitivist point of view a structured approach can be defended. This approach emphasizes the learners' active cognitive involvement, but, in this view, it is the teacher who can elicit and 'steer' this active cognitive involvement. Accepting a constructivist point of view may lead to an approach where the role of teachers and students will change more drastically. Although one cannot derive a pedagogy from a constructivist stance in a direct manner, sine the lemma that everyone constructs his, or her own knowledge, implies that this will happen in any instructional setting. However, this same point of departure forces one to consider, when we do want to call what the students construct 'mathematics' (see also Cobb, 1994; Gravemeijer, 1995). Following Freudenthal's notion of 'mathematics as a human activity', this would imply an emphasis on the students' intellectual authonomy. Such an elaboration of the enactment of RME, however, puts heavy demands on the teacher. To strive for such an innovation will in turn put heavy demands on teacher support. Moreover, this view on mathematics education will probably not resonate with the mainstream beliefs of teachers who may adhere a more cognitivist viewpoint.

In short, the idealistic approach may have all sorts of pro's, but it will take much more effort than a pragmatic approach. Therefore, we argue that research on the yields of both approaches is needed; to inform both mathematics educators, and administrators.

2 Background

More than twenty years ago the former Institute for the Development of Mathematics Education, IOWO, began developing what we now call realistic mathematics education (RME). Hans Freudenthal's ideas on mathematics education (cf. Freudenthal, 1973) inspired the developers to create an alternative to 'New Math', which had spread to Europe from the United States. The Wiskobas group approached their task on many fronts; they developed and researched curricula prototypes, published background articles, developed materials for teacher training and organized conferences and in-service teacher training courses. All these activities were part of a broad strategy of educational reform, which focused on the furtherance of expertise, material development, and consensus formation.

For primary school, these efforts resulted, among other things, in mathematics textbooks that have sufficiently incorporated the RME ideas (Jong, 1986). More than three-quarters of the Dutch primary schools have, in the meantime, acquired a realistic textbook series and a good number of these schools is now using such a textbook series up through the sixth grade. The reform set in motion by IOWO appears, therefore, to have been put into effect via the school textbooks.

3 Research into the implementation of realistic textbook series

The new mathematics textbook series that sprang from these circumstances now determine the look of the reform. Does this mean, therefore, that the desired reform has, indeed, been a success? This question is not so simple to answer. A number of studies shows favorable results for the realistic approach (Brink, 1989; Streefland, 1988; Nelissen, 1987) In a comparative study, however, Harskamp and Suhre (1986; Harskamp, 1988) did not find differences between 'traditional' and 'modern' methods. In contrast, a large-scale national assessment study on mathematics showed that realistic methods lead to better results (Wijnstra, 1988; Bokhove, Schoot, Eggen, 1996). Sources for differences are in the amount of control over the enactment of the innovation, and in the (kind of) test items that were used.

In a research project that was supported by SVO and carried out at Utrecht University (the MORE project; see Gravemeijer et al., 1993), the enactment and effect of the reform was investigated in more detail. Knowing more about the actual instructional practice seemed crucial, since the reform had primarily taken place through the introduction of new textbooks, whereas the reform itself assumes an adaptive use of the curriculum (Gravemeijer, 1994). This project focused on two issues. The first issue concerned the relative influence of textbook series and beliefs on the actual education. The second issue concerned the teachers' learning process. Both issues are connected to the work of Fullan, who points out that educational change takes place on three levels (Fullan, 1983). These levels have a bearing on changes in:

- use of materials
- educational activities and
- beliefs.

According to Fullan, true change is only possible when the beliefs of the teacher also change. In this context, he also speaks of the teacher's learning process. If we follow Fullan's train of thought, we reach the natural assumption that the curriculum document will be followed in terms of the subject matter, but that the differences will primarily manifest themselves in the teaching-learning process. It is in the interaction between teacher and student that implicit and explicit beliefs will be of decisive significance (see also Thompson, 1984). In association with Fullan's ideas, a distinction has been made in the research between the content of the instruction and the nature of the instructional practice. The content of the instruction is understood to be the subject matter and how it is constructed. The nature of the instructional practice concerns the character of the teaching-learning process. Along with a difference between subject matter content and teacher-learning process we also have here the difference between macro structure and micro structure. The content concerns primarily larger subject matter units and broad lines of subject matter sequence, while the instructional practice involves micro didactics. It is particularly on this micro-didactic level that beliefs may play an important role.

3.1 Nature and content

What characterizes a realistic approach of mathematics education? In the MORE project, Treffers' characterization of the realistic instruction theory (Treffers, 1987) has been used as a reference framework for assessing the intended 'idea-consistent' implementation of realistic mathematics education.

Starting point is that mathematics is seen as an organizing activity. Mathematics emerges as an organizing tool in a variety of fields of science, both in organizing everyday phenomena as in mathematics itself. Treffers (1987, p. 247) states that education has to

> 'put pupils in touch with the phenomena for which the mathematical structure is the organising tool in order to let them shape these tools themselves in a process of reinvention, and learn to handle and use these mathematical organising tools in concept formation'.

'Reinvention' refers to the situation where pupils base mathematical solutions on informal answers and a gradual refinement of these answers by discussion, reflection, and attempts to solve new problems. This 'organizing and structuring activity in which acquired knowledge and abilities are called upon in order to discover still unknown regularities, connections and structures' (Treffers, 1987; p. 247) is referred to as 'mathematizing'.

At this point, learning parallels the historic development of mathematical knowledge. Mathematics has been developed as an answer to real and concrete problems. Omitting the historical origins of mathematical knowledge deprives learners of the common sense roots of this knowledge. This is not to say that the learning process should somehow imitate or relive historic developments. Freudenthal (1991, p. 48) states that

> 'children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now'.

The consequence is that learners should be given the opportunity to think and perform as developers of mathematics instead of consumers of pre-developed rules and principles. This fits with Paul Ernest's (1991, p. 283) assertion, that 'the mathematical activity of all learners of mathematics – provided it is productive, involving problem posing and solving – is qualitatively no different from the activity of professional mathematicians'. Grounding reinventing mathematics in the historic development leads to the problem of how guidance can streamline the process of developing mathematical knowledge without imposing pre-defined structure on the learner. Simply preventing teachers from telling the solution doesn't solve the problem. Teaching aids should help pupils to construct mathematics, starting from their own perspective. The goal of learning and instruction can be described as 'to guide the student's construction of knowledge'. One tries to realize this via a process of progressive mathematization, to stimulate a learner-oriented learning process, capitalizing on learner-based initiatives, to stimulate a learning process that is based on reflection, discussion, and evaluation of various solutions, and to stimulate acquiring a mathematical attitude, to reduce competition within classrooms, to improve cooperation, and to increase motivation.

4 The status of reform

The MORE-project was aimed at investigating how successful enactment of the realistic approach has been realized, education in eight schools where the mechanistic textbook series 'Naar Zelfstandig Rekenen' (NZR) was used, was compared with education in ten schools which used the realistic textbook series Wereld in Getallen (WiG).
A textbook series analysis revealed clear-cut differences between the two textbook series. The following conclusions were drawn: NZR and WiG differ considerably in terms of supply and sequence of subject matter. The underlying instructional theories are expressed by a broader supply of subject matter in WiG (more attention to geometry and ratio, among other things) as well as a systematic integration of applications. There are also related temporal differences in subject matter planning. While NZR passes quickly through the subject matter, providing a narrow supply of subject matter and a one-sided focus on drill and practice, WiG chooses a broader and more gradual set-up.

For this research, twenty teachers and their students were followed from the beginning of grade 1 through the end of grade 3. As a first step, the relation between the content of the text books and the learning results was analyses. A quantitative comparison, however, revealed no overall differences, although more specific subject-matter differences were found.

At first sight, this may look disappointing. But, a more closer look reveals that these results tell us little about the effectiveness of the realistic approach. An analysis of the nature of the teaching/learning process showed that the use of a realistic textbook series did not result in the envisioned educational practice. Apparently the teachers adopted the ideas of the realistic approach on a global level, but not on a micro-didactical level. Hence, the intended enactment of RME was only partly released. Moreover, it showed that the teachers were not aware of the discrepancy between their more global, and their micro-didactical beliefs.

The MORE-project offers little support for the idea that teachers will come to grips with the enactment of RME on a micro-didactical level through a learning process that occurs simultaneously with the adoption and use of new textbooks. And this brings us back to the problem why the intended enactment of RME was only partly realized.

4.1 The need for reform

When trying to answer the question how to reform mathematics education, we may take a broader international framework of reference, to get a better handle on the problem of establishing a realistic approach. All over the world, a similar type of reform in mathematics education is widely endorsed, and experimented with. Traditionally much weight was put on transferring expert-knowledge to learners in most countries. An approach that is now referred to as a 'teaching by telling', or a 'transmission' model. In this 'teaching by telling' model, teachers take the position of experts, whereas students practice in imitating the experts' behaviour. In recent learning theories, however, there is a strong emphasis on learners' active (cognitive) involvement. Today, it is broadly accepted that the development of mathematical knowledge should be based on eliciting or fostering learner activities instead of teaching by telling. Basically, the argument refers to the role of teacher and student in the learning process. The theoretical argument is that learning demands learner-based activities, whereas simply practicing by imitating will not do. At the same time a vision on mathematics education emerges that emphasizes inquiry, problem solving, discussion, and communication (see for instance the recommendations of the National Council of Teachers of Mathematics (1989) and the National Research Council (1989), and the Cockcroft Report (1982)). At this point, the realistic curriculum can be linked with current ideas about learning and instruction.

Because of this uniformity in reform ideas, problems that are related with reform efforts can be compared. Why are changes towards problem centered mathematical education so difficult to realize?

4.2 Enactment of the intended reform

Research on educational practice has revealed a number of barriers that have to be surmounted before educational reform can be realized. Desforges and Cockburn (1987) describe how teachers that adapted a problem-oriented approach of mathematics education in theory, did not apply these in practice. The main reason was that students did not cooperate. By posing questions they forced teachers to reduce the multi-faceted tasks to more traditional one-track exercises. Apparently, most students do not like insecurity and would rather be told what to do. In practice, this means that students are constantly appealing to the teacher to tell them what they should do. Moreover, students will not automatically explain their solution, exemplify their approach, or reflect on proposed solutions (Jaworski, 1994). It is also clear that a class is much less manageable when the students are given problem-oriented instruction than when they can work in a more routine fashion. Teacher support will therefore have to focus on two objectives: the development of the micro-didactic knowledge on the one hand and, on the other hand, the development of the general pedagogical skills.

A qualitative analysis of lesson protocols, conducted in the framework of the MORE-project, revealed how demanding it really is to enact realistic mathematics education in the way it is intended (see also Streefland and Te Woerd, 1992). In mechanistic education, one can work according to a set plan. Moreover, the class follows a fixed routine of demonstrate-copy-practice which can be entirely planned beforehand. It is expected of the realistic teachers, by contrast, that they adapt the instruction to the students' contributions. At the same time, however, potential problems must be foreseen, and the teaching-learning process must be streamlined in such a way that the students can get the opportunity to deal with the mathematical issues embedded in the context problems. This requires not only pedagogical skills but also specific didactic know-how. The teacher must be able to construe what role the instructional designer envisioned for a given problem (or a given type of problem) in a certain course, what solutions are possible and how these relate to the various learning routes. In other words, the teacher must be able to constitute hypothetical learning trajectories (Simon, 1995) that fit with the local instruction theory (Gravemeijer, 1994) that underlies the instructional sequence.

4.3 Footholds for improvement

The main problem in enacting realistic mathematics instructional practice is the area of tension between 'letting the students (re)invent it themselves' and 'guiding the learning process'. Ideally, the guiding should be put into practice indirectly: by discussing solutions, clarifying solutions (or having them clarified), offering new problems, giving hints, posing critical questions, and so on (Goffree, 1979). The teacher has to integrate these elements in the construal, enactment, and adaptation of hypothetical learning trajectories that take into account what the students know and are capable of, an what the instruction in aiming for. This kind of guidance demands a great deal of micro-didactic knowledge on the part of the teacher. In the first place, the teacher must be aware of the potential (idiosyncratic) learning routes but, moreover, he or she must be able to recognize unclearly formulated or incomplete solutions. It should, in principle, be possible to impart such micro-didactic knowledge by way of courses for inservice teacher training. But, in addition to the fact that a rather extensive amount of specific knowledge is involved here, there is also the problem of the appropriation of theoretical knowledge. It would therefore be better if the teachers could develop this knowledge themselves. The teachers do already possess a great deal of informal knowledge that presumably has the potential to be developed further. This knowledge can be made more conscious through reflection on their own teaching practice, and through well-focused reflection on the context problems. Take, for instance, a problem like the following:

Dutch Cheese costs $1.20 per lb.
What does 0.75 lbs. cost?

This problem could provide the starting point for an assignment such as: Try and find as many different solution strategies as possible and use this knowledge to construe hypothetical learning trajectories. American students who were given a similar task produced a variety of solution procedures which, moreover, offered insight in possible learning trajectories (Gravemeijer, 1992). A number of solutions emerged which made use of the relationship between 0.75, and 'the ratio of 3 to 4'. One solution was to break up $1.20 into quarters and nickels and then remove three quarters and three nickels. The relation to money also affected the rising awareness that 0.75 corresponds to 3/4 (three quarters). Sometimes solutions were supported by a double number line or ratio table, such as: 'calculate the price of one and a half kilos and divide that by two' or, 'take the price of a kilo and of a half a kilo and calculate the amount in between'.

5 Approaches to realistic mathematics education

The necessary pedagogical skills also demand a learning process of the teacher that must take shape in the classroom. The great need for pedagogical skills springs from the above mentioned tension between 'guidance' (guiding the learning process) and 'invention' (letting students things invent themselves). On the one hand, the students themselves have responsibility and, on the other, the teacher is still in charge. This may lead to lack of clarity, which was perhaps the cause of the problems observed by Desforges and Cockburn (1987). In traditional classrooms it is clear how things stand: it's the teacher's to know and the student's to find out. The familiar question-answer pattern fits this situation, in which the teacher asks a question, the student answers, and the teacher determines whether the answer is correct (Voigt, 1985).

5.1 The idealistic approach

In realistic classrooms the situation is less clear. Teachers have to refrain from an imposing role, and take on a role that includes helping students develop productive small-group collaborative relationships, facilitating mathematical dialogue between students, and, above all, orchestrating a discussion around issues that are significant in view of the envisioned learning trajectories.

This focus on social dimensions relates to the development of a classroom culture that encourages and facilitates learning. Students should be willing to share views, to consider each other's solutions, being prepared to accept better solutions without an a priori acceptance of the teacher's view. Hence, this approach relies heavily on implicit agreements regarding the character of the teaching-learning process (see also Wijffels, 1993) – what Brousseau (1984, 1990) refers to as a 'didactic contract'. Compared with the situation in a teacher-directed learning process, something different is expected of the students in problem-oriented mathematics education. They have other obligations and they are to expect different things from the teacher. But do they know that? It is probable that a transition to problem-oriented education will require explicit attention to the change in obligations and expectations. The students must learn that 'the correct answer' is not the point, and that it's OK if they make mistakes. In addition, the students must adopt new obligations:

- the students are expected to justify their own solutions to themselves, and to explain and substantiate them to others
- the students are expected to try and understand the solutions of others and, when they do not, to discuss them.

Cobb, Yackel and Wood (1992) report that a change in didactic contract has to be explicitly designed and implemented. They refer to this aspect of didactical change as changing social norms. The point here is not to learn new rules of behavior by heart. It has to do with:

> '... establishing a culture in the classroom. A big piece of teaching for understanding is setting up social norms that promote respect for other people's ideas. You don't get that to happen by telling. You have to change the social norms – which takes time and consistency.' (Lampert in: Brandt, 1994, p. 26)

Social norms are not, after all, explicit agreements but, rather, indications of beliefs of both teacher and students. A change in social norms must be made visible by an actual change in behavior. Concrete situations can be used here to make the new norms explicit. Gradually, this will create a situation in which realistic mathematics education can flourish. In theory, a learning process can be initiated in which the teacher increasingly learns how to manage problem-oriented mathematics education. It may be possible to combine this practical learning process with a learning process in which the teacher expands his/her micro-didactic knowledge. This expansion can take place partly through studying teacher's guides, for instance, but, in the first place, by anticipating and analyzing the students' responses. The foundation for this learning process lies with the teachers themselves. The teachers, like the students, must 'gain respect for their own ideas' (see Lampert, ibid.). The teacher's own reflection on the instruction then becomes the motor for his/her own learning process (see also Clarke and Peter, 1993). The teachers must take up the role of researchers, who according to Steffe & Weigel (1992, 451), '(...) must be bold enough to make conjectures, hypotheses, or inferences about the mathematical reality of students, and we must be willing to test and refine them continually in interactive communication'. Based on these conjectures about the students' mathematical reality, the teachers must design (and revise) hypothetical learning trajectories.

Abstinence of 'teaching by telling' is not enough: teachers have to develop ways to help their students to construct mathematical knowledge. The teacher is responsible for how this learning process will develop. However, guidance in a teaching-learning processes that allows for a maximum of student autonomy is a complex and subtle process. It is clear that direct instruction has to be rejected. But that leaves the question what type of guidance is left to the teacher.

In relation to this, the distinction Hiebert et al. (1996) make between functional understanding and structural understanding may be extended to imply a warning against emphasizing class activity as goal of instructional activities. Functional understanding means 'participating in a community of people who practice mathematics' (Hiebert et al., 1996; p. 16). They refer to the work of Brown, Collins, and Duguid (1989), Lave and Wenger (1991), and Schoenfeld (1988). Understanding, they fear, is defined to much in terms of classroom activities: the way students share perspectives, search for solutions, and evaluate methods. The teacher has a stimulating role, trying to elicit response by setting tasks and providing information. Structural understanding means 'representing and organizing knowledge internally in ways that highlight relationships between pieces of information' (Hiebert et al., 1996; p.17). In this view, emphasis is on what the students take with them from the classroom.

Teaching experiments, with RME, and RME-like, sequences in the US in an NSF-funded project 1 revealed three sorts of pro-active teacher guidance, which can be classified as 'pre-active', 'interactive', and 'retro-active'. [2]

Pre-active guidance of the reinvention process is most commonly realized in RME by offering a series of tasks that may give rise to a broad variety of solution strategies, which in turn may offer a starting point for progressive mathematization. Pre-active guidance can also take the form of a non-committal introduction of conventional (or didactical) forms of symbolization. The teacher may introduce these symbolizations in a casual manner, for instance to support a verbal expression. The teacher has to be aware that suggestions are non-committal, and students can accept these or not, depending on whether they accept its usefulness or relevance.

The second type of guidance is via the interaction between teacher and students. Part of the learning process is that students compare different solution procedures. As a results of comparison and discussion, the students may embrace the mathematically more sophisticated solution procedures. However, the notion of mathematical sophistication asks criteria that may be out of reach for the students. As a consequence, mathematically more sophisticated solution procedures may be rejected, and progress will be hampered. As an example, we may think of students who happen to prefer counting procedures over more sophisticated solution procedures. Somehow, the students have to develop norms that support progressive mathematization. These norms are referred to by Yackel and Cobb (1995) as 'socio-math norms'. These socio-math norms reflect the beliefs that students hold on what counts as a mathematical problem, and what count as viable mathematical solution. These norms also play an important part in the way students handle context-problems (Gravemeijer, 1992). Students are expected to take the reality, as implied in these contexts into account, however, not all solutions that are possible in reality (like 'go to the shop to buy an extra pizza') are accepted in a mathematics classroom.

An interesting instance of a socio-math norm that concerns mathematical sophistication, is given by McClain, Cobb, and Whitenack (1995). They describe a situation where different numbers of dots are presented on an overhead screen during brief moments of time. Students had to answer how many dots they counted, and they had to explain how they found this number. At a certain point, the teacher decides to show in her appreciation for 'different solutions', that she considers different counting strategies as one solution. In this manner, she contrasts the counting-based solutions with solutions that are based on grouping. In doing so, the teacher focuses attention on the grouping strategies, giving these an excess value.

Retro-active guidance can take place after students have explored and discussed several solutions. In this situation the teacher can suggest a way of symbolizing or a procedure that will match the mathematical conventions. For instance, when the students have been inventing and improving their own algorithms for addition and subtraction, the teacher may show them the conventional algorithm. Students are invited to analyse and discuss this algorithm in a way that is not different from how they discussed their peers' inventions. In this way, the students' autonomy is maintained, while the solution fits with accepted mathematical practice.

5.2 The pragmatic approach

The former approach depends heavily on a number of radical reforms in educational practice. This makes this approach expensive in terms of invested time and effort. Is it worth the effort? And are teachers willing to accept the demands that are put on them in the idealistic approach? Surely will a more pragmatic approach be more easy to implement. The problem, however, is, to what extent one can be pragmatic without getting in conflict with the basic assumptions of the realistic approach. Surely there will be a trade-off, but let us investigate what a pragmatic approach might look like.

How can a more pragmatic didactical approach be developed that is based on the principle of mathematics as a human activity? Although at a global level different arguments point in the same direction, at a more specific level different approaches can be defended. At a global level, the learning process is more learner-directed than the traditional learning situation. Teachers have to capitalize on students' informal solutions, have to elicit discussion and reflection, and are responsible for stimulating problems. Interaction and cooperation between students is stimulated, and students are invited to explicate and discuss various solutions.

As a starting point, learners' cognitive involvement is accepted as a condition for learning to take place. This requires a classroom which supports learner activities, rather than (passive) acceptance of facts. However, essential differences exist in character and role of learner activities. In the pragmatic approach, activities are primarily directed towards guided discovery and practice. Mathematical facts are what teachers determine them to be. This approach fits with a socio-cultural perspective where 'the teacher's role is characterized as that of mediating between students' personal meanings and culturally established mathematical meanings of wider society. From this point of view, one of the teacher's primary responsibilities [...] is to appropriate their actions into this wider system of mathematical practices' (Cobb, 1994; p. 15).

In the pragmatic approach, the teacher will choose an approach where 'guidance' is stressed. Freudenthal refers to this approach as the Socratic method:

> 'In a narrower sense I will assume, as Socrates did, that the teaching matter is reinvented or re-discovered in the course of teaching. [...] the students should be left with the feeling that the teaching matter arose while teaching, that it was born during the lesson, and that the teacher was in effect only a midwife...
> In the Socratic method 'reinvention' was not understood literally; it was simulated rather than being true reinvention. It could not have been otherwise, could it? The teacher's authority was still dominant... The initiative was only on the part of the teacher. Not only did he lead the student, he also showed him how rediscovery works, he rediscovered on behalf of the student.' (Freudenthal, 1973, 100-102).

Freudenthal rejects this approach as too narrow, and in conflict with any interpretation of mathematics as an activity. The pragmatic approach, however, can be more open. Within this approach, instructional activities are designed to give the students the opportunity to make their own inventions. The difference with the idealistic approach, however, is in the role of the teachers. They are expected to be much more clear in what they value. They are to make their view on the hierarchical order of solution methods explicit. They will actively participate in explaining and justifying solution methods that are suggested by the students. Moreover, they will bring unmentioned solution methods to the fore if they think these will be helpful for the students. A characteristic teaching strategy is to pick the students that are to present their solution in such a manner, that the solution methods will discussed in what the teacher – or the textbook author – sees as a hierarchical order. In this manner each student will get the chance to connect with a solution method on his or her level, and will get the opportunity to compare this solution method with the next-better method. Here too the teachers will assume an active role: students will be stimulated to try more advanced methods, and the advantages of better methods will be stressed.

Although there is ample room for student input in this pragmatic approach, the responsibility for the knowledge that is developed and accepted as valid rest with the teacher. Some argue for this approach (a) because an idealistic approach is not feasible, and/or (b) is not necessary. The key point, they argue, is that the students understand the mathematics they are learning, and insightful learning does not mean you cannot learn from others. The difference with the idealistic approach is in the authonomy of the students, and idealists will claim this influences what the students will take away from their mathematics education. This then opens the door for comparative research. Policy makers, for instance will be interested in the trade off between investment in educational change at one hand, and educational results at the other hand.

5.3 Evaluation

To evaluate the yield of different approaches, a number of outcome variables can be investigated. Globally these variables can be divided into cognitive variables, affective and motivational variables, and beliefs. Learning outcome is an important outcome variable. Mathematics education has to result in adequate mathematical knowledge. As an additional requirement the approach should be useful for students that differ widely in mathematical capacity. In addition, an effective approach should have a positive effect on affective and motivational variables that are relevant in the learning situation. Furthermore, students (as their teachers) enter the learning situation with different beliefs and expectations about their role in the learning process, about what mathematics is, and how learning is to be effected. A well-known belief is that learners consider achievement in mathematics as the outcome of capacity, not of invested effort. This belief will exert an effect on the learners' willingness to invest effort in working on mathematical problems, and in willingness to persist in cooperation when trying to solve challenging tasks, and taking part in discussions about possible solutions. The idealistic approach will aim at a non-competitive atmosphere, where gaining insight by personal effort will be experienced as rewarding. This approach therefore would result in a more positive self image. Moreover, the students will have a different view on what it means to do and learn mathematics.

6 Discussion

Teachers can probably develop the necessary micro-didactic knowledge themselves when they begin viewing individual solution strategies as research terrains. In-service teacher training and teacher support can be of assistance here, as can the teacher's guide for the textbook in question. Perhaps, with directed support, the teacher's learning process could get jump-started. The Dutch association for mathematics education, the NVORWO, is advocating a mathematics coordinator in every school for this purpose; this would be someone who could initiate such a learning process and support it over the long term (Dolk, 1993).

Both teachers and students have to adapt to reform. Both learn in interacting experiences to reform both social norms and their knowledge and beliefs about learning and teaching. In addition, teachers will have to understand their students' mathematical knowledge by reflecting on students' errors, by questioning, and by posing new problems that will reveal errors. This cannot be done by simply instructing teachers, but is the outcome of a long-term process.

It is evident that the pragmatic realistic approach is more in line with current teaching practice. At this point, teachers will be more easily convinced to apply this approach. The ideal approach demands teachers to give up a number of steering tools. In addition, for this approach to be successful, changes in the didactical contract are prerequisite.

As a consequence, the idealistic realistic approach will put heavy demands on the teacher, and implementation will be much more difficult. Capitalizing on students' initiatives carries without doubt a risk. Stimulating informal strategies should offer perspectives for a further progress of the learning process. Solution strategies that can be build upon actually have to be invented by the students, and the teachers will have to be able to recognize these as such.

In addition to abstinence of direct interference in learning situations, frustrating a default routine of teachers, they are also confronted with the problem that they have to develop knowledge on how students build up their mathematical knowledge. Learning to reflect on how students think must lead to decisions about posing questions and presenting problems that will allow students to progressively build their mathematical knowledge.

Realization of a huge change in mathematical didactics demands teachers who are willing to invest in new ways of teaching. But there are a number of additional conditions: proper learning materials that are in harmony with the new perspective on learning must be available. But also schools, parents, and politics have to change. Becker and Selter (1997) mention three conditions for change that have to be tackled: assessment, teacher education, and research. Emphasis on formal assessment stresses learning products, whereas assessment should also evaluate learning processes. Formal assessment should be extended with informal assessment (cf. Clarke, Clarke & Lovitt, 1990). Assessment that is informative about the student's learning process demands good problems (Heuvel-Panhuizen & Gravemeijer, 1993). Heuvel-Panhuizen (1996) provides a number of examples how problems can be more informative. As for teacher education, alternative curricula for teacher education must contribute to changing the prevalent perception of the teacher to a person who encourages children's mathematical activities. Teachers should learn to reflect on teaching/learning situations and processes.

In both approaches, one has to account for individual differences in learners in competence, motivation, and attitude towards learning. Here the pragmatic and the idealistic approach offer two different scenarios. In case of the pragmatic approach, the teachers may offer more direct help and scaffolding. In the idealistic approach the point of departure will be to help the students to learn to build on their own capabilities. At the same time, the teacher has to be sure that the instructional activities develop in such a manner that everyone can participate on his/her own level. However, at this point it is important that Cobb et al. (1991) describe how a socio-constructivist instructional approach that is more in line with the idealistic realistic approach leads to changes in students' beliefs and motivation. Students reported to be less ego-oriented, rejected the idea that success is caused by using the same solution as the teacher. The latter finding was confirmed by students' behaviour in a mathematics test: students in reformed classrooms more often applied other solutions than the standard algorithm. At the same time, teachers' pedagogical beliefs changed parallel with students' beliefs. Thus an idealistic approach may not only be argued for on base of educational principles and ideals, an idealistic approach may be demonstrable beneficial for the students. That is why we think that research on the yield of both approaches is called for. Based on such research, mathematics educators can decide upon their position. And in practice probably more important, research can inform administrators on what kind of innovation to support.

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notes

1 See Cobb & Yackel (1993).

2 See also Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., and Whitenack, J. (1997); Cobb, P., Boufi, A., McClain, K. & Whitenack, J. (in press); McClain, K. (1995); Whitenack, J. W. (1995).

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Instructional design and reform: a plea for developmental research in context

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In their chapter, Seegers and Gravemeijer (this volume) distinguish between what they call the pragmatic enactment and the ideal enactment of an instructional sequence. One of the central questions they raise is whether the extra effort required to make possible an ideal enactment can be justified in terms of the quality of students' mathematical learning when compared with a pragmatic enactment. I return to this issue in the final section of this chapter.

First, however, I attempt to further clarify what is involved in a so-called ideal enactment. To do so, I reflect on my own and my colleague's activity during a recently completed classroom teaching experiment. The purpose in doing so is to describe a particular way of enacting an instructional sequence in process terms by focusing on our ways of acting in the classroom. Particular attention is given to the planning of whole class discussions in which mathematically significant issues emerge as topics of conversation. The specific issues discussed include the focus on both individual students' meanings and the communal activities in which they participated, the framing of the overall intent of the instructional sequences in terms of Greeno's (1991) environmental metaphor, and the process of continually refining the conjectured learning trajectory in the course of the experiment.

The general approach that I describe clearly falls on the ideal(istic) side of the pragmatic-ideal dichotomy outlined by Seegers and Gravemeijer (this volume). This does not necessarily imply that it is exemplary. Instead, the account that follows is best viewed as a report from the field. It will serve a useful purpose if it constitutes a point of reference in the ongoing debate about the role of the teacher in reform classrooms.

1 Background

The classroom teaching experiment that serves as the basis for the discussion was conducted in a first-grade classroom for a four-month period between February and June 1996. Two closely related instructional sequences focusing on linear measuring and on mental computation with two-digit numbers respectively were enacted and refined in the course of the experiment. In previous discussions of the teaching experiment methodology (Cobb, in press; Yackel, 1995), we have distinguished between three general phases: planning for an experiment, experimenting in the classroom, and conducting a retrospective analysis. The primary focus in this discussion is on the second of these phases, experimenting in the classroom. Thus, the story begins after provisional instructional sequences had been outlined during the planning phase. The issues of interest are located at the micro-level and concern what Gravemeijer (1994) calls daily mini-cycles in which one conducts an ongoing analysis of classroom events and makes instructional decisions on that basis. In describing this process, I am in effect attempting to delineate aspects of our classroom-based practice of which we[1] have ourselves only recently become aware.

I should clarify at the outset that the first-grade teacher with whom we collaborated was a full member of the research and development team. We first began working with her in May 1993 when she recruited us to work in her classroom and conducted a year-long teaching experiment with her during the 1993-1994 school year. The relationship we had established with her by the beginning of the current teaching experiment was such that members of the research team could begin to co-teach with her at any point during a classroom session without prior arrangement. She, for her part, participated in the ongoing analysis of classroom sessions during both daily debriefing sessions and weekly project meetings conducted throughout the experiment. In addition, she made important contributions to the design of instructional activities.

One of the retrospective analyses conducted as part of the 1993-94 teaching experiment had focused on her role in proactively supporting her students' mathematical development (McClain, 1995). The significant aspects of her classroom practice that were identified included guiding the renegotiation of sociomathematical norms, facilitating the development of ways of symbolizing and notating, and initiating both reflective shifts in classroom discourse and the folding back of discourse. The account of the current teaching experiment begins after the teacher had guided the establishment of supportive social and sociomathematical norms in her classroom. Thus, a type of classroom microculture that characterizes what Seegers and Gravemeijer (this volume) term an ideal enactment constitutes the back drop against which I discuss three aspects of our practice of experimenting in the classroom.

2 Individual meanings and communal activities

In previous discussions of the teaching experiment methodology, we have emphasized the importance of analyzing students' mathematical activity as it occurs in social context. The particular approach we take involves coordinating constructivist analyses of individual students' activities and meanings with an analysis of the communal mathematical practices in which they participate. Our focus in the prior discussions has been on retrospective analyses that are conducted once the phase of experimenting in the classroom has been completed. It is only recently that we have come to view ourselves as coordinating-in-action these two perspectives as we experiment in the classroom. To describe this aspect of our practice, I first outline the classroom social arrangements.

The classroom sessions conducted in the course of the teaching experiment usually involved periods in which the students worked either in pairs or individually but with the proviso that they could move around the classroom to discuss their problem solving efforts with peers of their choosing[2]. The small-group or individual work was typically followed by a teacher-orchestrated whole-class discussion that focused on the students' interpretations and solutions. During the pair and individual work, the teacher usually circulated around the classroom to gain a sense of the diverse ways in which the students were attempting to solve the tasks. For our part, I and a graduate research assistant each observed and interacted with two students to document the process of their mathematical development throughout the teaching experiment. In doing so, we consciously attempted to infer the four students' individual mathematical interpretations on an ongoing basis.

Towards the end of pair or individual work, the teacher, the graduate assistant, and I 'huddled' in the classroom to discuss our observations and to plan for the subsequent whole-class discussion. In these conversations, we routinely focused on individual students' qualitatively different interpretations and meanings in order to develop conjectures about mathematically significant issues that might emerge as topics of discussion. In this opportunistic approach, our intent was to capitalize on the students' individual or small-group activity by identifying specific students whose explanations might give rise to substantive mathematical discussions that would advance our pedagogical agenda. At times, the discussions focused on one student's mathematical activity whereas, on other occasions, the discussions involved a comparison of two or more solutions. It is important to emphasize that our intent in proactively organizing discussions in this manner was *not* to confront solutions so that students who initially agreed with a solution classified as less sophisticated in some way would come to appreciate the superiority of the other solution. Instead, our justification for the discussions focused on their quality as social events and was cast in terms of participation. We contend that participating in discussions of issues that we judge to be mathematically significant constitutes a supportive situation for the students' mathematical development. The teacher's role in these discussions was therefore not to persuade or cajole the students to accept one particular interpretation, but was instead to orchestrate a conversation about issues judged to be mathematically significant per se.

In reflecting back on this process of planning whole class discussions, we have come to see that it involves coordinating the two perspectives that we had previously discussed when describing the retrospective analysis of classroom video-recordings. At the moment that we focus on individual students' qualitatively distinct interpretations and meanings, a psychological perspective comes to the fore and the communal practices in which the students are participating fade into the background. For example, during the measuring instructional sequence, the constructs that we used to account for the students' meanings were developed by drawing analogies with individualistic accounts of children's early number learning (Steffe, Cobb, and Von Glasersfeld, 1988). At this point in the planning process, both we and the students are in effect 'inside' the communal classroom practices.

This psychological perspective can be contrasted with that which we take when justifying the discussions we are attempting to organize. At this juncture, our focus is on the nature of the discussions as collective activities, and the students' individual interpretations now fade into the background. Our primary concern is with the quality of the social events in which the students will participate, and it is for this reason that we concentrate on the mathematically significant issues that might emerge from their explanations with the teacher's guidance. Once the discussion begins, we find ourselves monitoring both the nature of the discussion as a social event and individual students' qualitatively distinct contributions to it. In doing so, we attend to both the communal activity interactively constituted by the teacher and students, and to students' individual meanings as they participate in it.

This account of the way in which we plan for whole-class discussions clarifies how we currently attempt to cope-in-action with a tension endemic to teaching, that between the individual and the collective (cf. Lampert, 1985; Ball, 1993). The account does not describe instructional strategies, but is instead cast in process terms and deals with a way of acting in the classroom. It clearly indicates the importance of interpreting-in-action individual students' solutions and understandings (cf. Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). In addition, it suggests the value of locating the students' solutions in social context by focusing on the communal activities that constitute the social situations in which their mathematical development occurs. In such an approach, coordinating individual and communal perspectives is not merely an esoteric theoretical issue. Instead, it is an integral aspect of our classroom-based practice as we proactively attempt to support students' mathematical development.

3 Mathematical significance

In describing the process by which we plan whole-class discussions, I referred to issues that we judge to be *mathematically significant.* This way of talking is, of course, vague and leaves many questions unanswered. As a starting point, recall that a potential issue is judged as mathematically significant if it contributes to our pedagogical agenda. This agenda in turn takes the form of a conjectured developmental process that culminates with the global goals of an instructional sequence. In a very real sense, the conjectured learning trajectory serves to locate immediate, local judgements within a broader, more encompassing vision of the instructional process. To clarify what is meant by mathematical significance, I will therefore discuss both the global intent of an instructional sequence and the conjectured learning trajectory by which this intent might be realized in the classroom.

3.1 Instructional intent

One of the challenges when preparing for a teaching experiments is to clarify for ourselves the global intent of the instructional sequences we are outlining. In currently fashionable parlance, this involves specifying what are sometimes referred to as the *big ideas.* We have found that, for our purposes, the most useful way to explicate these big ideas is in terms of Greeno's (1991) environmental metaphor. In other words, we do *not* specify our instructional intent in terms of the observable solution methods or strategies that we hope students will develop. Neither do we specify particular internal concepts or cognitive mechanisms putatively located in students' heads. Instead, we attempt to articulate the nature of the mathematical environment in which we hope students will eventually come to act. It is against the background of a pedagogical agenda whose goals are stated in these terms that we make judgements about the potential significance of issues that might emerge as topics of conversations in whole-class discussions. The issues are *mathematically* significant if discussions centering on them contribute to our pedagogical agenda of making it possible for the students to eventually act in a particular type of mathematical environment.

It can be noted in passing that our use of Greeno's environmental metaphor is reflexively consistent with the description I have given of our classroom-based practice. The focus has been on our ways of acting and on the classroom as a pedagogical environment in which we act. Similarly, when attention turns to students' mathematical development, the focus is on their ways of acting in a mathematical environment. This approach is non-dualist in that it does not separate either our own or students' activity from the worlds in which we act. In each case, ways of acting and the world acted in are considered to be mutually constitutive and to co-evolve (Pea, 1993; Varela, Rosch and Thompson, 1991).

As an initial illustration of this way of framing the intent of an instructional sequence, consider first the relatively familiar case of the addition and subtraction of numbers up to 20 that was the focus of the prior eight-week year-long teaching experiment. Our global intent in this instance was that students would come to act in a quantitative environment structured by relationships between numbers up to 20. Observationally, this would be indicated by their flexible use of thinking or derived fact strategies to solve a wide range of tasks. For example, they might solve a task interpreted as 14 – ... = 6 by reasoning 14 – 4 = 10, and 10 – 4 = 6, so the answer is 8. Alternatively, they might reason that 7 + 7 = 14, so 14 – 7 = 7, and 14 – 8 = 6. It should be stressed, however, that the acquisition of these calculational methods was not itself the pedagogical goal. Instead, our intent was that the numerical relationships implicit in these and other observable strategies would be ready-to-hand for the students. In other words, they would not have to consciously figure out appropriate strategies to use. Instead, we hoped that the students would come to have the experience of directly perceiving relationships as they interpreted tasks. Needless to say, coming to act in such an environment is a major intellectual achievement that requires proactive developmental support.

In the case of the instructional sequence that dealt with measuring, our initial concern was that the students would come to interpret the activity of measuring as the accumulation of distance (cf. Thompson and Thompson, 1996). In other words, if the students were measuring by pacing heel-to-toe, we hoped that the number words they said as they paced would each come to signify the measure of the distance paced thus far rather than the single pace that they made as they said a particular number word. Further, our intent was that the results of measuring would be structured quantities of known measure. In other words, having paced a distance of, say, 20 steps, they could view this quantity as itself composed of two distances of ten paces, or of distances of five paces and fifteen paces as the need arose. By analogy with the case of addition and subtraction up to 20, we hoped that the students would come to act in a spatial environment in which distances are structured quantities whose measures can be specified by measuring. In such an environment, it would be self evident that while distances are invariant, their measures vary according to the size of the measurement unit used.

In the course of the teaching experiment, measuring with composite units also became an established mathematical practice. Initially, the students drew around their shoes and taped five shoe-prints together to create a unit that they named a footstrip. Later, in the setting of an ongoing narrative that appeared to be experientially real to the students, they used a bar of ten unifix cubes to measure. As a consequence of participating in these instructional activities, many of the students came to act in an environment in which distances with measures of up to 100 were composed of distances whose measure was ten. The students' activity in this environment subsequently served as the starting point for a second instructional sequence that focused on mental computation with two-digit numbers. In terms of Greeno's environmental metaphor, the intent of this latter sequence was that the students would come to act in a quantitative environment structured in terms of relationships between numbers up to 100. As was the case with addition and subtraction to 20, our immediate concern was *not* merely that students would acquire particular calculational methods. Instead, our intent was that they would come to act in an environment in which the numerical relationships implicit in these methods are ready-to-hand. This view shifts the focus from calculational strategies per se to the interpretations and understandings that make flexible strategy use possible.

The contrast I have drawn between stating the instructional intent in terms of observable solution methods and in terms of acting in an environment is analogous to the distinction that Thompson, Phillip, Thompson, and Boyd (1994) make between what they term a calculational orientation and a conceptual orientation. Whereas a calculational orientation is concerned with the calculational steps taken to produce an answer, a conceptual orientation is concerned with how the task is interpreted and understood – with why a particular calculation is performed in a particular situation. It is precisely this latter issue that is addressed by the environmental metaphor. This contrast between observable strategies and acting in an environment in no way plays down the importance of calculational proficiency. Instead, it involves a shift in focus from what Mackay (1969) terms the observer's perspective to the actor's perspective. When the instructional intent is cast in terms of observable strategies, the focus is on aspects of students' activity that can be documented by a detached observer. In contrast, when we adopt the actor's perspective, we attempt to understand students' activity from the their point of view rather than from that of a detached observer. The focus is then on the quality of their mathematical experience and on the tasks and situations as they understand them. This emphasis leads to a consideration not just of how students might calculate, but of *why* they might come to calculate in particular ways. An approach of this type is explicitly non-dualist in that to specify the mathematical environment in which students might come to act is to specify the intended nature of their mathematical experience.

3.2 Learning trajectories

The approach of formulating the instructional intent of a sequence in environmental terms provides what Thompson et al. (1994) call a conceptual orientation. However, the delineation of the global intent does not by itself give sufficient guidance for pedagogical decisions and judgements. As we have seen, local judgements in the classroom are made against the background of a conjectured learning trajectory. This trajectory takes the form of an envisioned developmental process by which students' current mathematical ways of knowing might evolve into the ways of understanding that constitute the intent of the sequence. This notion of a learning trajectory, which is taken from Simon (1995), is consistent with Gravemeijer's (1994) analysis of the process of instructional development. In Gravemeijer's account, the developer first carries out an anticipatory thought experiment in which he or she envisions both how the proposed instructional activities might be realized in interaction and what students might learn as they participate in them. As Gravemeijer notes, in conducting this thought experiment, the developer formulates *conjectures* about both the course of students' mathematical development and the means of supporting it. In other words, the rationale for the instructional sequence takes the form of a conjectured learning trajectory that culminates with students coming to act in a particular mathematical environment that constitutes the overall intent. As Seegers and Gravemeijer make clear, the means of supporting the conjectured developmental process include the development of particular ways of symbolizing. My concern in this chapter is not, however, with the viability of specific conjectures such as the model of/model for transition, but instead concerns the more general process of making judgements in the classroom.

A first issue that arises is to clarify who or what is the subject of the proposed developmental route. It clearly cannot be all of the students in a class because there will be significant qualitative differences in both their interpretations of the initial instructional activities in a sequence, and in the actual process of their individual development in the classroom. Descriptions of an instructional sequence written so as to imply that all students will come to reason in particular ways at particular points in the sequence appear to be untenable. To circumvent this difficulty, it could be argued that the conjectured learning trajectory is that of a fictional, idealized student. The limitation of this approach, however, is that it proves difficult to relate the conjectured trajectory to the reality of the classroom for the simple reason that no such student exists. In other words, the process of testing and revising the conjectures inherent in a sequence when experimenting in the classroom is problematic.

The approach that I and my colleagues have taken is to view the proposed learning trajectory as a conjecture about the mathematical development of the classroom community. In this view, a learning trajectory specifies both a possible sequence of classroom mathematical practices and the possible means of supporting the emergence of one from another. Elsewhere, we have discussed this notion of a classroom mathematical practice and have described its relation to the mathematical activity of the individual students who participate in it (Cobb and Yackel, in press). For my present purposes, it suffices to note that there is no implication that the individual students are acting and reasoning in identical ways. Instead, this notion acknowledges students' diverse ways of interpreting and solving tasks while, at the same time, treating them as members of a community that itself develops and evolves. In this approach, events that occur in the classroom over an extended period of time as an instructional sequence is enacted are analyzed in terms of the evolution of mathematical practices, thereby documenting the actual learning trajectory of the classroom community. Analyses conducted in these terms are reported by Bowers (1996), Cobb (1996), and Cobb, Gravemeijer, Yackel, McClain, and Whitenack

(1997). These analyses illustrate that a focus on the community can be complemented by a psychological focus on the diverse ways in which individual students participate in and contribute to the development of the collective practices. Analyses of this type can therefore serve to document the process of individual students' learning as it occurs in the social context of the classroom.

The divergence of the actual learning trajectory realized in the classroom from the intended learning trajectory envisioned at the outset is a product of pedagogical judgements made while the teaching experiment is in progress. The process by which we plan for whole-class discussions provides one illustration of this local decision making. I noted that our intent was not merely to encourage the students to explain their reasoning. A classroom discussion was justifiable only if the issues that emerged as topics of conversation were mathematical significant. We have seen that an issue is judged a mathematical against the backdrop of a conjectured learning trajectory. In other words, an issue is considered to be significant if it contributes to the realization of an envisioned developmental route for the classroom community. Metaphorically speaking, the learning trajectory might be said to constitute the big picture within which local decisions and judgements are made on a daily basis. The example of planning for whole-class discussions also illustrates that although learning trajectories are cast in the collectivist terms of classroom mathematical practices, these local judgements take account of the diverse ways in which individual students participate in those practices.

It is important to stress that in this way of working in the classroom, the relationship between the learning trajectory and the daily judgements is reflexive. On the one hand, daily decisions and judgements are framed by the learning trajectory. On the other hand, the envisioned learning trajectory itself evolves as a consequence of these local judgements. Thus, at any point in a teaching experiment, there are conjectures about the possible evolution of classroom mathematical practices and the means of supporting their emergence. In the case of the teaching experiment that focused on measuring, for example, we found it essential at the beginning of our weekly project meetings to talk through how the classroom mathematical practices might evolve during the remainder of the experiment. However, this conjectured trajectory itself continually changed as a consequence of local interpretations and judgements. For example, prior to the teaching experiment, our primary focus was in fact on two-digit mental computation. We initially viewed the proposed instructional activities involving measuring as precursors to those designed to support the development of mental computation. However, as a consequence of issues that arose once the teaching experiment began, measuring gradually became a focus of interest in its own right[3]. As a consequence, the actual learning trajectory came to diverge significantly from that which initially we envisioned.

The account we have given of the reflexive relationship between local judgements and the big picture is broadly compatible with Simon's (1995) discussion of what he calls the mathematics teaching cycle. This cycle is shown in simplified form in Figure 1.

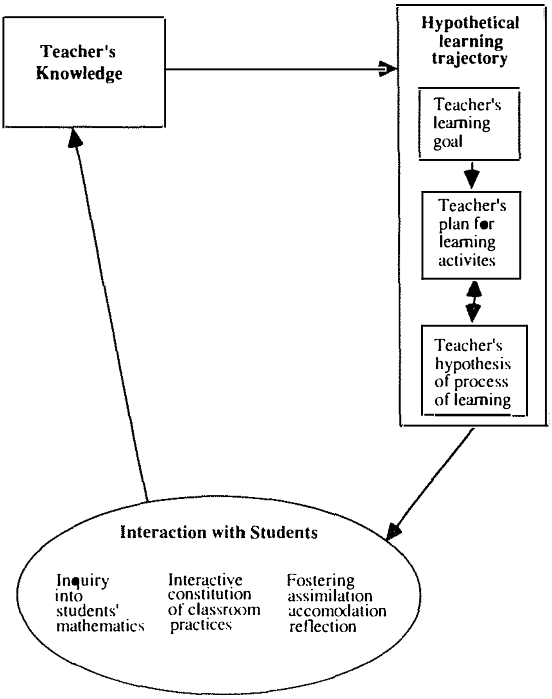

figure 1: a simplified version of Simons (1995) mathematics teaching cycle (reprinted with permission from Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26,* 136)

Simon stresses that his notion of a hypothetical learning trajectory 'is meant to underscore the importance of having a goal and rationale for teaching decisions and the hypothetical nature of such thinking' (p.136). At any point, the teacher has a pedagogical agenda and thus a sense of direction. However, this agenda is itself subject to continual modification in the act of teaching. Simon likens this process to that of undertaking a long journey such as sailing around the world.

> 'You may initially plan the whole journey or only part of it. You set out sailing according to your plan. However, you must constantly adjust because of the conditions that you encounter. You continue to acquire knowledge about sailing, about the current conditions, and about the areas that you wish to visit. You change your plans with respect to the order of your destinations. You modify the length and nature of your visits as a result of interactions with people along the way. You add destinations that prior to the trip were unknown to you. The path that you travel is your [actual] trajectory. The path that you anticipate at any point is your 'hypothetical trajectory'.' (pp.136-137)

As Simon observes, this way of acting in the classroom involves both a sense of purpose and an open-handed flexibility towards students' ongoing interpretations of activities.

The terms I have used to talk about instructional development and Simon to describe his activity as a mathematics teacher are generally consistent with Seegers and Gravemeijer's notion of *enacting* an instructional sequence. As Varela, Thompson, and Rosch (1991) emphasize, the idea of enactment implies that processes are 'inextricably linked to histories that are lived, much like paths that exist only as they are laid down by walking' (p. 205). In the case of an instructional sequence, the teacher and students lay down an actual learning trajectory as they interact in the classroom. This, it bears repeating, does not mean that classroom activities drift aimlessly. At any point, there is both an overall instructional intent and an envisioned means of achieving it. However, both the intent and the conjectured trajectory are subject to continual revision. Thus, to pursue Varella et al.'s metaphor, the path is laid down by walking even though, at each point in the journey, there is some idea of a destination and of a route that might lead there[4].

This enactivist view can be contrasted with the more traditional notion of *implementing* an instructional sequence. The latter metaphor casts the teacher's role as that of carrying out the plans and intentions of others, whereas the notion of enactment highlights the teacher's (and students') contributions to an instructional sequence as it is realized in the classroom. In addition, an enactivist view brings the teacher's learning to the fore. As Simon (1995) illustrates, teaching can be an occasion to deepen one's understanding of the big ideas that are the focus of classroom discussions, of students' reasoning, and of the means of supporting its development. These same comments apply to researchers who work in classrooms, and in fact constitute the primary reason why we conduct classroom teaching experiments. The deviation of the actual learning trajectory from that envisioned at the outset provides a general summative record of this learning while experimenting in the classroom.

4 Reflections

The parallels we have drawn with Simon's (1995) analysis indicate that teaching and classroom-based developmental research are closely related forms of activity. Both involve an intensive engagement with students that is motivated by a desire to support and organize their mathematical development. The various aspects of our classroom-based practice that I have discussed therefore inform a somewhat idealistic view of reform teaching. This view clearly emphasizes the importance of attempting to make sense of individual students' interpretations and solutions. It is therefore consistent with the generally accepted view that teaching should be informed by a relatively deep understanding of students' mathematical thinking. However, the discussion of developmental research also indicates the value of locating individual students' activity in social context by attending to the quality of the social events in which students participate. As we have argued elsewhere, students' participation in these events constitutes the conditions for the possibility of mathematical learning (Cobb and Yackel, in press). In the case of whole-class discussions, for example, this focus on activity in social context implies that pedagogical justifications should go beyond general claims about the role of interaction, communication, and discourse in mathematical development. Instead, particular classroom discussions should be justified in terms of their contributions to the fulfillment of a an evolving pedagogical agenda.

The summary comments made thus far concern the local level of pedagogical decision making. I have also attempted to clarify that the local judgements that we make when conducting a teaching experiment are situated within the broader context of a possible learning trajectory that involves specific conjectures about the means of supporting the evolution of classroom mathematical practices and thus the development of the students who participate in them. In addition, I discussed why we find it useful to state the big ideas that constitute the potential endpoints of these trajectories in terms of Greeno's environmental metaphor. Extrapolating to the activity of a teacher, these considerations indicate the importance of appreciating the pedagogical intent of an instructional sequence. This, it should be stressed, is not a separate 'piece of knowledge' that informs pedagogical decision making. The pedagogical intent involved an envisioned developmental process and thus involves the teacher's understanding of students' mathematical thinking. Further, it involves a relatively deep understanding of the mathematics under consideration (e.g., measuring) *in relation to students who are attempting to learn it.* Thus, it involves what Lampert (1990) terms a map of the mathematical territory in relation to students who might eventually come to act in such a mathematical environment. Finally, it involves specific conjectures about how the process of students' mathematical development might proceed in an instructional setting when proactive efforts are made to support their learning.

A detached analysis of the type of pedagogical activity that I have attempted to describe might objectify it and dissect it into components corresponding to a psychological theory of students' thinking, a theory about the sociology of the classroom, mathematical knowledge, and a domain-specific instructional theory (i.e., pedagogical content knowledge). Such an approach separates pedagogical knowing from the activity of teaching and treats knowledge as a commodity that stands apart from practice. It is precisely this separation that I have tried to resist by focusing on our ways of acting in the classroom. My primary concern has been with acts of knowing and judging that occur moment by moment as one attempts to support students' mathematical development. The perspective I have taken on pedagogical activity is therefore that of the actor rather than the observer.

The view of teaching that emerges from this account portrays teachers as professionals who continually modify their agendas even as they use instructional materials developed by others. It therefore goes some way beyond frequently made claims that reform should be fueled almost exclusively by either materials development or by teacher enhancement. It does, however, fall squarely on the idealist side of the distinction that Seegers and Gravemeijer (this volume) draw between pragmatic and ideal enactments of an instructional sequence. It could legitimately be argued that the form of practice I have outlined is unfeasible for any teacher working alone. In the case of a teaching experiment, for example, some members of the research team teach while others observe classroom events during instruction. This collective activity might best be viewed as a possibly unattainable ideal[5]. It constitutes a way of acting in the classroom to be aimed at, an aspect of a big pedagogical idea that can provide directionality to teacher development efforts. Given these considerations, the question that Seegers and Gravemeijer raise, that concerning what is feasible and practical, becomes significant. Although I cannot give a well formulated response to this question, I am convinced that the pragmatic option described by Seegers and Gravemeijer is entirely impractical. It is to this issue that I turn in the final paragraphs of this paper.

Seegers and Gravemeijer (this volume)describe a pragmatic approach in which the primary focus is on a hierarchy of solution procedures. The intent is that the teacher will structure the reinvention process by explicating this hierarchical order, both by introducing solution methods deemed important if the students do not come up with them on their own and by encouraging students to move from less-advanced methods for more advanced methods. The hope is that in spite of the teacher's explicit guidance the students will experience their progress towards the most advanced method as their own doing. Unfortunately, this hope is contradicted by detailed analyses of classroom interactions during mathematics instruction that involves a similarly ambiguous approach (Voigt, 1985). For example, Maier and Voigt (1989) demonstrate that interactions corresponding to the socratic method involve the elicitation pattern of interaction (Voigt, 1985). Initially, the teacher asks relatively open-ended questions to initiate students' contributions. However, unless a student fortuitously happens to give the response that the teacher has in mind, the teacher begins to give increasingly explicit cues, thereby funneling students to the desired response. The episode typically concludes with the teacher giving a reflective summary of what it was that students were supposed to have learned while participating in the discussion.

Voigt's (1985) analysis indicates that students can be entirely effective simply by waiting for the teacher to tell them what it is that they are now supposed to know. To me it seems that the pragmatic approach offers the students the same option of just waiting for the teacher to tell them what it is that they are supposed to think.

Note that in the teaching experiment I described earlier, the teacher's role was not to persuade the students to accept one particular solution method, but was instead to orchestrate a conversation about issues judged to be mathematically significant per se.

In describing the pragmatic approach, Seegers and Gravemeijer note that it avoids any conscious attempt to change the didactic contract or classroom social norms. This aspect of the approach seems to imply that the quality of an instructional sequence is unaffected by the didactical contract. This assumption is, in my view, completely untenable. When a developer performs an initial thought experiment while developing an instructional sequence, he or she necessarily assumes that a particular didactical contract has been established in the classrooms. In general, it is impossible to develop a conjectured learning trajectory without making implicit assumptions about the classroom participation structure. It is one thing to be unaware of such background assumptions and another to contend that issues relating to the classroom microculture can be ignored for practical purposes. An impressive body of research on the social and cultural aspects of mathematical learning has been conducted in a number of countries, including The Netherlands. This research demonstrates that enacted instructional sequences can differ radically from one classroom to another depending on the classroom microculture. As a consequence, the qualities that the developer sees in an instructional sequence as he or she envisions it might well not be realized when the instructional sequence is enacted in a particular classroom.

The arguments I have made about the importance of attending to the classroom microculture are corroborated by De Lange, Van Reeuwijk, Burrill, and Romberg's (1993) account of an experiment in which six American high school teachers used an instructional sequence that focused on data visualization (De Lange and Verhage, 1992). De Lange et al. made the following observations in their report.

> '[I]n five classes the teachers wanted to try some group work. A few found it difficult because it entails the cooperation of individuals, a new idea in American society[!] (p. 55).
>
> 'The students were getting used to the type of questions being asked and their group work was improving' (p. 79).
>
> '[I]n order to have groups function effectively with this new focus [on sharing ideas], we had to take time to work with students about the roles and responsibilities in a group and in the class; *we had to teach them more than mathematics.*' (p. 155, added emphasis).

These three observations deal with norms for collaborating in groups in order to learn. Working with students about their roles and responsibilities involves what I and my colleagues refer to as the explicit negotiation of obligations for one's own activity and expectations for others' activity in the classroom (Cobb, Yackel, and Wood, 1989).

In contrast to the negotiation of general classroom norms that are not specific to mathematics, other observations reported by De Lange et al. indicate that they also attended to the negotiation of sociomathematical norms (Yackel and Cobb, 1996). These norms are specific to students' mathematical activity and include what counts as a different, insightful, and sophisticated *mathematical* solution, and what counts as an acceptable *mathematical* explanation. For example:

> 'In the first and second week, the students had to learn how to respond to questions with complete answers. At the beginning, they *answered* Yes or No without explanation.' (p. 70)
>
> 'The teacher looked at the graphs and explanations [of the students] and found that the students had not given an explanation of the graph, but were simply describing what they saw in the graph. So she initiated a discussion in class about graphs. In the discussion it became clear what was meant by *explaining* a graph' (p. 75, emphasis in the original).

Numerous other observations relating to social and sociomathematical norms can be found in De Lange et al.'s report (e.g. pages 72, 79, 97, 119-120, 151, 157, and 160). The examples I have cited should, however, be sufficient to illustrate that the data visualization sequence as realized in these classrooms had some of the qualities envisioned by its developers only because De Lange et al. and the teachers with whom they collaborated 'taught more than mathematics.'

De Lange et al.'s analysis forcefully demonstrates the impracticality of a so-called pragmatic approach that aims at socratic dialogues and ignores the classroom microculture. An approach of this type might appear to have merit within the cloistered confines of an instructional development center. Its inadequacies become self evident as soon as one enters the classroom and attempts to support students' mathematical development for an extended period of time. In my view, an approach of this type that is divorced from the *reality* of learning and teaching mathematics is unjustifiable. If the mathematics education community has learned anything from current reform efforts, it is that sustainable reform involves materials development, teacher development, and broader policy considerations. A constructive response to Seegers and Gravemeijer's concern with what is feasible and practical requires that we take the social situation of students' mathematical development seriously. An approach of this type might focus on both the assumptions about the classroom microculture implicit in an instructional sequence as envisioned by its developers, and on the sequence as it is realized in different classrooms. In such an approach, the debate shifts beyond the confines of the instructional development center and is informed by analyses grounded in the reality of the classroom. Ironically, the blatantly idealistic account I have given of instructional practice would seem to have greater practical relevance in this regard than the pragmatic approach identified by Seegers and Gravemeijer.

acknowledgment

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notes

1 I use the first person plural to refer to the members of the research team who conducted the experiment. They were Beth Estes, Kay McClain, Koeno Gravemeijer, Maggie McGatha, Beth Petty, and Michelle Stephan.

2 This approach of allowing students to work with peers of their choosing allows them to actively contribute to the development of classroom participation structures (cf. Murray, 1992).

3 This example illustrates that the overall instructional intent can also evolve in the course of a teaching experiment. For ease of explication, I have somewhat misleadingly spoken as though the instructional intent is fixed from the outset.

4 This notion of enacting a learning trajectory is compatible with Nemirovsky and Monk's (1995) notion of trail making, Pirie and Kieren's (1994) recursive model of mathematical development, and with Lave's (1988) discussion of gap closing. At a more general level, it is consistent with Dewey's (1977) accounting of reflective intelligence.

5 The term 'ideal' is used here to acknowledge the idealistic nature of the vision of teaching that emerges from the analysis. It does not imply that we view our way of working in the classroom as ideal in the sense that it is beyond improvement.

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Discussions at the experts meeting

Meindert Beishuizen & Koeno Gravemeijer

This chapter gives an overview of the discussions at the experts meeting. Since most papers were available beforehand, the authors could do with a short explication of their main points. The following pages mirror the authentic style of the conversations. No attempt was made to further structure the discussion than only mentioning the related papers as section headings. This means that our summary sometimes may have a meandering or even redundant character. This style however, reflects how the participants tried to come to grip with each others' different cultures and different concept interpretations. This may help the reader, we think, to get a sharper awareness of the issues that played a role in this international discussion on primary mathematics teaching. We thank the participants for their comments on the wording of this account of the discussion. As one of them put it: 'It captures the spirit of our discussions very well'. We also thank the two Leiden students Jacqueline Besemer and Stephanie Juranek, who worked out the (audiotaped) conversations.

1 Discussion about the papers of Koeno Gravemeijer, Tom Carpenter and Christoph Selter

This discussion revolves around the need to provide externally developed teaching materials to support teachers. In relation to this, Simon's concept of a 'hypothetical learning trajectory' comes to the fore (cf. Gravemeijer, this volume). But first, the concept itself has to be clarified. Paul Cobb notes that the term learning trajectory seems to be used in different ways. He thinks that for some learning trajectory just means, having an idea of how this could develop or how the kids' thinking could evolve. While the way Koeno Gravemeijer is using learning trajectory also has a very strong instruction-theoretical aspect to it. It is not just a view of how the kids are going to learn. It is a specific conjecture of a means of pro-actively supporting that development. He adds, that for him, what is unusual about what happens in the Netherlands is, that they have developed over a period of time an instructional theory, about means of pro-actively supporting development that makes sense from an instructionalist point of view. He thinks, it is almost unique in that they have such a thing. So it is the learning trajectory of the kid in an instructional situation as it is pro-actively is supported. He suggests that maybe one way to bring out these differences, or get issues on the table, is to talk more about the specific conjectures of the means of supporting the process of development.

Karen Fuson suggests to introduce the term 'instructional support trajectory'. However, Paul Cobb objects that you can't separate conjectures on how the kids' thinking could evolve and the instructional activities. In other words you cannot describe a conjecture about the kids' thinking in a vacuum. In relation to this Tom Carpenter introduces the terms 'foreground' and 'background'. What are the kind of things you think about first? Do you think about the students thinking first, or do you think first about the instructional situation in terms of what the teacher's activity becomes in terms of the instructional sequence. The traditional way of thinking about teaching is in terms of the teacher's activity and that sort of becomes first and the learning is sort of background. Paul Cobb wonders himself, if that is the crucial phrase, the foreground and the background. Maybe one would want to say that we want foreground and background both. For example one used to think about research in early number, or place value or whatever, as if this was just the natural way of kids doing. Well it is the natural way of kids doing in particular problem situation, in a certain organised lore and practice and whatever. There is already support there. So we tend to just focus on the learning, but the support is there all along otherwise that development would not occur.

With this clarification, the question for the discussion can be framed as: Who is designing the hypothetical learning trajectory? Part of what the issue is, as Karen Fuson puts it, is the extent to which it is the individual teacher in the classroom, or to the extend to which someone else can develop it in teaching-learning materials, in textbook materials, and so forth. She refers to the Hiebert et al. (1996) paper (cf. Gravemeijer, this volume), of which she was a co-author. She recalls that, at the time which they were writing the paper, there was this pervasive contest around the world saying that the only worthwhile tasks in mathematical teaching are real-life tasks and anything else is very bad to do. So the Hiebert et al. paper was partly a reaction to that view. The authors were trying to emphasize: No, actually you can have more stripped down problem tasks that can be powerful learning stimulants. The important thing is for the tasks to be somewhat problematic for the students. But they maybe did not quite make this as clear as they could have. When kids were working with these more stripped down versions, they would already have had a history in which those are meaningful. Otherwise, she adds, kids can't engage in any work on that mathematical activity. She remarks that the authors were not intending to take any position on the issue of teachers constructing everything versus teachers using only teaching materials from the outside. That would have been hard given the varied background of the authors. In relation to this she argues that CGI (CGI = Cognitively Guided Instruction) very much ends up with teacher constructed materials, while she herself is trying out teacher-learning materials constructed from the outside.

With respect to CGI Karen Fuson comments that she thinks CGI has done really important powerful things but she has some concerns in that she thinks it works very well with better teachers and better students. Maybe some of those teachers weren't teachers who used textbooks. Maybe they already had their own classroom organisations, they already had constructed things and so they were already teachers who were doing that. She argues that weaker teachers, need support of careful instructional sequences. The challenge is to try to make the materials, the students will be working with, open enough so that teachers can adapt them to their own personal styles and to their own individual classrooms of kids. Because every classroom of kids every year is different. Even if a teacher is used to a particular kind of teaching-learning instructional sequence, the next year she may have different kids, and she will have to do a different adaptational process. So that is what she sees as an important other task: that one needs to think about as the basic question balancing between autonomy and teacher guidance. In the instructional materials we also have to have a balance between a sort of enough guidance for teachers so that they know what path they are going along, and enough flexibility in the materials so that teachers can do their adaptation for the given students that they have that year.

Tom Carpenter takes exception over the comment that CGI didn't work with all teachers with all kids. Although, of course, nothing works with all kinds of teachers. Also the American adaptation of the Dutch materials does not work with all the teachers, either, or with all kids. However, he claims CGI changed all teachers. One of the things that the CGI group was very successful in, is in getting a number of teachers to not feel, that when a kid was falling behind, that they all of a sudden had to forget about understanding. That, he adds, is really one of the things which separates the teachers who were more successful in the program. There were a number of the teachers that go to a point and say: 'Oh, it is the end of second grade and I have got these kids who really don't understand and I think I have got to do something, I have got to give them the way to solve this problem.' He stresses that we need to recognise there is huge variability. For some of the kids it takes a long time to develop the particular concepts and skills. The traditional mistake is to worry about the kids at the bottom, and to respond to that by falling back to very mechanistic ways to deal with those.

Karen Fuson agrees with that. That is what happens in a reform maths program she has been studying (Everyday Mathematics). Here the kids are supposed to invent their calculation methods in second grade and third grade. There are two problems: One is there is not enough sustained opportunity for kids to construct their methods, so only the better kids construct the methods. Two is: when they are discussing alternative methods it is all oral discussion. There is no drawing, most of the teachers don't do drawings, or write numbers. There is no point of concrete reference for the discussion, so the bottom half of the kids can't follow the discussion, only the top half. So it is a kind of 'the richer getting richer'. And then, just before the 'standardized' tests, the teachers all panic and they teach the kids the standard algorithm without meaning. You can't have a situation where you suddenly worry about the low kids and you teach them in a mechanistic rote way, because then what happens is that all the low kids start doing the top-from-bottom subtraction error. Then they really learn that error and it is very difficult to unlearn that. But there is another approach. That is that you do from the very beginning more sustained activities to support the low kids to do more advanced solution methods.

2 Discussion about the papers of Ernest van Lieshout and Lieven Verschaffel

The discussion opens with some specific remarks about the improved design of context formats that might (better) trigger N10 or 1010 strategies. Then Jens Lorenz broadens the discussion by raising the issue of the influence of problem structure versus number characteristics. Lieven Verschaffel answers that the differential effects of number characteristics are difficult to detect, because students often seem overwhelmed by the semantics of the (word or context) problem structure. In Leiden a study was done with word problems and rather extreme numbers like 82 – 79 with a difference of only 3 or 2. In that case, indeed, students changed to indirect addition. But in all other cases with normal numbers of 10 or more in between, like in 82 – 69, most students followed not the number characteristics (indirect addition) but the problem structure of the word problem (subtraction). Also in the Flemish study with bare number problems like 75 – . = 18 and 69 – . = 52 there was no significant effect of a large or small difference between numbers. The semantics of problem structure seems to exert a strong influence on the choice of strategy.

Elsbeth Stern then raises the question how we can determine that the strategies the students tell us, are really the strategies they used when solving the problem? Sometimes students tell you what they know as the most familiar strategy, or a solution that is common to the problem given. Especially with smaller numbers, when students arrive at developmental stages where number facts are going to play a role, it is difficult to be aware of what you have done precisely. Ernest van Lieshout answers there is mostly a relation between correctness of answers and scoring of the procedure, which may give you a clue. According to Tom Carpenter, videotaped recordings mostly contain enough clues about the used strategies. Moreover, if a pupil does not give a true answer, the given explanation will be close to the student's own strategy level. Stern adds that with larger numbers and multi-step solution procedures this question is much less a problem, because you carry out the steps more consciously and you are more aware of what you do. Lieven Verschaffel and Ernest van Lieshout agree this is true for strategies like N10, 10s and 1010 with two-digit numbers (cf. Table 1 in Beishuizen, this volume). However, this applies mainly in the case that correct solutions are given. If answers are incorrect or unclear the categorization of procedure can become much more difficult also for multi-digit numbers.

At the end of the discussion Koeno Gravemeijer and Paul Cobb pose questions about the relevancy of word problem research. Do the outcomes really mirror which strategy choices students consider in their heads? Or are the outcomes representative for a style of maths teaching where students learn mainly routine procedures for certain types of problems? Koeno Gravemeijer wondered how it is possible that number size seems not to influence the choice of strategies. Maybe because there has not been a whole-class discussion about several alternative approaches or solutions to a problem? Students could become more aware of the distinction between the semantic context structure and the syntactic number characteristics of a problem, and their role in the solution process. Ernest van Lieshout and Meindert Beishuizen add that such metacognitive pedagogy possibly could be built in a teaching program. For instance a program with the empty number line, which model is very suitable for drawing, demonstrating and discussing – by students themselves – various solution methods on the blackboard. Lieven Verschaffel agrees, but is, on the other hand, a bit concerned about the complexity of this strategy choice process. Are we really suggesting that in our teaching efforts we should emphasize so much this strategy choice process, so that our students may become deliberate strategy choosers taking into account all these task variables we discussed? Koeno Gravemeijer agrees that this is not a realistic expectation, but nevertheless introducing more whole-class discussions about strategy choice in relation to number and problem characteristics may help. Moreover, such interactive discussions are an important principle of RME (Realistic Mathematics Education) because students' informal strategies should be stimulated, and students should learn and get ideas from each other.

Paul Cobb comes back to the fundamental question about the representativeness of word problem research. He is wondering if the emphasis is not too much on the nature of word problems. Would it not be more realistic to take into account the typical 'instructional history' and typical 'classroom culture' of students in a given situation, which will influence their solution process? Paul Cobb thinks that from such a micro-analysis you get different patterns in different classrooms. Lieven Verschaffel points, however, to the fact that in so many countries with different students word problem research comes up with very similar results. In spite of these divergent cultures the recurrent similar trends in word problem research give evidence for the apparently strong influences of task variables as we discussed here. For Paul Cobb, however, those similar trends across countries and schools are symptoms of the stereotyped character not only of word problems but of many maths tasks and most classroom instruction. Let us change that type of instruction, is therefore Paul Cobb's concluding remark in this discussion.

3 Discussion about the papers of Meindert Beishuizen, Karen Fuson and Jens Lorenz

The discussion about these three papers first circles around the question whether linear images or dot configurations are a more natural representation of number. Young children mostly use dots because they have more experience with such quantity models.

According to Piaget this is an important concept for understanding the meaning of number, but many mathematicians (cf. Freudenthal) would pay more attention to the number line as a sequential counting model, including also number relations. In this context some doubt is expressed about Karen Fuson's hypothesis in her paper (this volume) that Dutch and German number words may elicit another computation procedure than English number words. Both Meindert Beishuizen and Christoph Selter think that the discussed differences between the quantity and linear models are more influential. They see a parallel in the diverging teaching and textbook practice, emphasizing in the U.S. calculation procedures with split-up two-digit numbers, and in Europe mental computation with whole numbers up to 100.

At this point some misunderstandings (and clarifications) enter the discussion. Both Tom Carpenter and Karen Fuson underline that maths teaching in their research projects (as well as in that of Paul Cobb) are atypical for the U.S. Indeed, early introduction of two-digit or column arithmetic is common practice in U.S. schools, which brings along the danger of 'concatenated' manipulation of isolated numbers. However, in the mentioned U.S. projects there is much emphasis on mental computation, as can be seen in the examples given in the papers of Tom Carpenter and Karen Fuson & Steven Smith (this volume). In the discussion Karen Fuson agrees that N10 appears to be a more efficient computation procedure than 1010, because intermediate outcomes can be hauled along (N10) and need not to be kept in working memory (1010). But Karen Fuson thinks this is true for experienced calculators and adults. For children, however, such mental strategies are more demanding to learn. For instance she foresees problems with positioning numbers correctly on a number line (length or point?) and she would therefore prefer a line of dots instead of an empty number line. From a wider point of view Karen Fuson would argue that instruction in mental strategies should come after practice with written strategies, which arguments are also given in the last pages of her paper (this volume).

Apparently, in the discussion, mutual misunderstandings of 'mental' and 'written' arithmetic are now playing a role. For instance Karen Fuson is saying that in her view students in Holland do not learn mental strategies first, because they start with a lot of written activities on the empty number line to build up N10 as a mental strategy. The Dutch experts object because in their view written jottings are allowed (as support) when doing mental computation (and do not transform the mental strategy into a written method). The Germans suggest that the labels mental and written might not express the right antithesis. They use in their country the label 'halbschriftlich' (half-written) as a sort of intermediate or transitional state in between. Many experts agree that the real (procedural) antithesis is the difference between mental strategies and columnwise algorithms. Mental strategies are mostly less standard and more varied, while (written) column algorithms are predominantly standard calculation procedures. Koeno Gravemeijer makes the remark that there is a language problem. The Dutch word 'hoofdrekenen' translates into 'mental arithmetic', but 'hoofdrekenen' includes also flexible mental strategies while 'mental arithmetic' has a more traditional connotation and a strong association with 'mental recall'. Karen Fuson suggests that the Dutch should translate their concept 'hoofd-rekenen' into 'flexible strategies' or 'strategies adapted to task characteristics'. In particular the descriptive expression of Treffers 'using your head strategies' she likes a lot.

Jens Lorenz draws the conclusion that at this moment the discussion is coming close to the 'cognitive map' interpretation in his paper (this volume), which values imagination and variety more than categorization of procedures and strategies. Elsbeth Stern, however, does not think that all students are making strategy choices or adapting to task/number characteristics when doing calculations. Many of them are sticking to one solution method because of routine or limited (pre)knowledge. Therefore, she supports the idea to distinguish between procedure and strategy to get a better description or interpretation of solution processes going on in the classroom. Jens Lorenz agrees that there is an interesting description of the development of the strategy/procedure AOT/A10 in Meindert Beishuizen's chapter (this volume). Meindert Beishuizen adds that it is sometimes overlooked in discussions about N10 versus 1010, that today's Dutch textbooks are putting emphasis on both N10 and 1010 to increase the level of mental flexibility. This applies not only to the Leiden number line program but also to the new 'Wis & Reken' textbook presented the previous day. For this didactic purpose a greater distinction is made between the linear or sequential (N10) representation and the quantity or place-value (1010) representation of number and number operation. He refers to Koeno Gravemeijer's characterization of the differences between the linear model (empty number line) and the set type model (blocks or money) in his JRME-article (1994). In this context Meindert Beishuizen makes a critical remark about Karen Fuson's conceptual structures (fig. 2 in Karen Fuson & Steven Smith, this volume), using similar block configurations (tens and ones) as models for both the 'sequence-tens' and 'separate-tens' concepts. In his opinion these two blocks models do not make much representational difference, in particular not for the sequence concept. In his view you better first underline the differences in modeling, in order to achieve in a later learning stage a higher level of 'integrated use' of N10 and 1010. Karen Fuson & Steven Smith mention in the last pages of their paper such a higher level of 'an integrated-tens and ones conception that relates sequence-tens and separate-tens', as an example of 'vertical mathematization' with reference to Koeno Gravemeijer (this volume).

4 General discussion (including the audience) on Saturday the 14th December 1996

Julia Anghileri wants to come back to the discussion about N10 versus 1010 but now from the wider perspective of long term development of strategies. From her own research she knows that for multiplication many students use N10-like counting-up strategies, but for larger multi-digit multiplication 1010-like strategies are more efficient. And a bit further in the curriculum when students work on division problems, many of them revert back to N10-like repeated subtraction strategies. So, is it wise to concentrate mainly on N10 in the lower grades as the more powerful strategy? Meindert Beishuizen answers that apart from N10 and 1010, students develop other strategies like 10s and A10.

In his paper the development of A10 but also the change from 1010 to N10 is described from a long-term perspective. Some students like Eddy (figure 1 and 5 in Meindert Beishuizen, this volume) show an 'integrated' level of strategy use. Julia Anghileri agrees that making such connections and progressing to a flexible level of strategy use are important key factors to success in mathematics.

Karen Fuson then would shift a bit in the discussion from the two specific strategies N10 and 1010 to the underlying conceptual structures. We want children to have understanding of both the sequence structure and the quantity (decomposition) structure. Tom Carpenter is not so sure that it is clear what drives these strategies. Moreover, he thinks there is overlap between the 'sequence' and 'collected' (quantity) notion, for instance with multiplication and division were you get both of them operating as part of the solution. Tom Carpenter reminds us of a footnote in his paper (this volume) mentioning that in U.S. (CGI) school practice, it often is difficult to distinguish between N10 and 10s, for students slip back and forth between the two. For instance they seem to solve 35 + 40 = 75 by using N10, but then they will say: Well I know that 30 + 40 = 70, so + 5 the answer is 75. According to Meindert Beishuizen one sees these solutions (in Holland) in the third grade as a symptom of integrated strategy use, but according to Tom Carpenter U.S. (CGI) students demonstrate already in the first grade these integrated strategies.

Koeno Gravemeijer comes in with the remark that the linear and set concept may be too limited from the viewpoint of long-term perspective including multiplication. There are many other ways of seeing numbers as Jens Lorenz (this volume) argued in his paper. For instance when solving the subtraction 72 – 38, Koeno Gravemeijer would think of 72 as 2 × 36 and would take 2 more off (72 – 38) to get the answer 34. In the case of just the number 72 Koeno Gravemeijer would ask students not only the split into 70 and 2 (tens and units) but also to make a link with the closeby landmark of 75 (as three quarters of a hundred). So, he would like to involve broader conceptions of number structure than only the linear and set type. Meindert Beishuizen and Karen Fuson, however, see 1010 and N10 as more fundamental or general dimensions of number concept in that they apply to all numbers. Ian Thompson solves this little dispute by his remark that it is a matter of investment. In the early number curriculum you have to invest in two things: number and strategies or tools. If you invest in number sense and take your time for it, you get out a rich use of good tools. Of course one can throw away a tool a kid cannot handle, but it might be a better investment to learn the kid to handle that tool.

Jens Lorenz brings up another problem: the differences in language between English and German or Dutch for describing what students think; the different terms we use as for instance when we speak of 'digits'. In (American) English it is common to use expressions like two-digit or three-digit number problems while in Germany we never use these terms. You quite often speak of numbers in the domain from 1 through 100 respectively from 1 through 1000. Speaking of digits seems to have a different connotation emphasizing column arithmetic. Karen Fuson answers that these terms are not necessarily used with students and that they just give a description of the number size in problems, without the association of column arithmetic. Julia Anghileri throws in that in the U.K. teachers speak mostly of 'tens and units' and that, indeed, these terms T and U are used as column headings (between lines) for doing the vertical algorithms. Tom Carpenter agrees that this is also common practice in the U.S. and he agrees with Jens Lorenz that one may wonder if this implies some underlying (different) conception of number.

Karen Fuson too agrees that there might be some cultural differences. She admits having learned from this conference as well as from inspection of textbooks and from observation of classrooms in Holland that in comparison to the U.S. there is a more free disposition to do these things like building individual notions. Her own experience is that weaker students do not build up quantity meanings for multi-digit numbers and that they do not use those meanings when doing written arithmetic. She feels this as a major problem, so one has to do anything sensible to help them understand those quantitative concepts of number. The sort of flexible things and individual meanings, as discussed here, are indeed nice and important. But to Karen Fuson they are a sort of luxury. Like the early maths curriculum in Germany and Holland, where this long time of two years (all of 2nd grade and a lot of 3rd grade) is spent for developing all this very flexible knowledge about two-digit numbers. And then the written algorithms are introduced suddenly and there seems to be no attempt to make any meaning for these new three- and four-digit numbers? For Karen Fuson this instructional sequence is difficult to understand, because attaching meaning to multi-digit numbers is a major problem in the U.S. system. She would rather try to do both, but that is also an issue in terms of cost and time, a matter of investment as Ian Thompson said earlier in the discussion. Frans Moerlands gives some examples from his experience as author of the textbook 'Wis & Reken': how students can experience number structure in different ways by presenting varied context problems and representation models. For instance more open structures like 100 as 4 × 25, using bundles or cartons one sees in a supermarket. Always using the tens-and-units structure could make the instructional approach rather rigid. Karen Fuson is not sure of the latter conclusion because more evidence is needed. Many of us are trying to work from different approaches to bring more teachers and more students along to real understanding. And until we do more of that better, we are not able to answer that question whether or not a certain approach is more rigid.

Marja van den Heuvel-Panhuizen wants to come back to the discussion about N10 and 1010, in particular to what Julia Anghileri said about the further perspective with regard to multiplication and division. Marja van den Heuvel thinks you could broaden the distinction to mental calculation in general and from that perspective N10 is the more typical mental strategy while 1010 is more related to the tens-and-units written algorithms. So, by consequence, if we want to emphasize mental calculation N10 is more important than 1010. Tom Carpenter asks surprised why that is so? He understands the relationship between 1010 and vertical algorithms. But why should 1010 not also be a flexible alternative for mental calculation? According to Marja van den Heuvel that gives 1010 a different function as an alternative way of structuring numbers in bundles of 25. Ernest van Lieshout intervenes with the remark that the distinction between mental and column arithmetic is quite another thing, which means dealing with numbers as whole entities or numbers as split-up digits. However, as Ian Thompson comments, the 1010 procedure is not synonymous with written algorithms because 1010 is also treating numbers as entities. Marja van den Heuvel agrees, but it is in her opinion a small step from (mental) computation with 1010 to (mental) operating with only digits. Both Koeno Gravemeijer and Karen Fuson object that we should not turn the argument around and say that, because of the possible abuse, 1010 is a meaningless and rote computation procedure that should not be used. According to Karen Fuson it would cut off roots for students and teachers that could be very productive.

Karen Fuson agrees that both 1010 and N10 are important and that we all are wrestling with questions like the best conceptual support and how to lead students eventually to integrated and flexible use of various strategies. Julia Anghileri is concerned this claim might be too idealistic, because in school practice many students get stuck on 1010 and some others on N10 as the one and only procedure. Successful students can take into account all the things we were discussing earlier about semantics of problems and number size and the most appropriate way of solution. That is why mathematics becomes so easy for some students and so difficult for others. Julia Anghileri thinks it is crucial not only to develop strategies but to make connections between strategies. In her opinion we do not get that right yet in teaching arithmetic.

Julia Anghileri's last remark causes Meindert Beishuizen to return to the earlier discussed role of instructional development and long term (supporting) learning trajectories, as well as the role of textbooks. Some experts expressed quite different views on that topic. But with respect to N10 and 1010 the authors of the new realistic textbook 'Wis & Reken' emphasize an orientation on both strategies and on connections between the two. Frans Moerlands underlines that you have to pay attention to both N10 and 1010. If a student is using one of those wrongly for instance in subtraction, you have to reflect with the student on the error and not simply throw away what does not work. Kees Buys emphasizes as editor of 'Wis & Reken' that sequencing of N10 and 1010 was an important part of the developmental work. In the new textbook a lot is invested in number images and number structures first. Counting-on or counting-back are very basic strategies used by most students. Many students are inclined to go on with the jump method (N10) but in quite a primitive way. So, in terms of instructional support one has to abbreviate these

10-jumps (23, 33, 43, etc. and backwards) in order to make this method more economical and flexible (larger jumps). Thereafter when they have a more sophisticated knowledge of N10, and because of the rich conceptual base of number structure, the split method (1010) emerges in quite a natural way. Halfway the 2nd grade most students in the Wis & Reken try-outs, used N10, but in the 3rd grade they changed more to 1010. At the end of the 3rd grade there is a development to integrated use: in many solutions you can not make the distinction between N10 and 1010 anymore. Sequencing and split methods are then mixed up in a flexible way. In the beginning of the 4th grade when column algorithms are introduced the students understand and acquire these new procedures very quickly. Moreover, they still can think and discuss about alternative strategies in whole-class discussions. For instance 478 + 266 can be solved by column arithmetic but also by applying a mental strategy (478 + 200 = 678 + etc.).

Karen Fuson appreciates very much these descriptions of examples from try-out practice. She can imagine that in this cultural context the 'sequence' method is more natural for students, while the 'separate' method is more complex and laid on top. This, in her opinion, is also a result of the Dutch instructional sequence. This approach could be used in the U.S., but only for students with experience, because in some classrooms many of them are not yet able to count up to 100 in the 2nd grade. On the other hand, multi-digit numbers are introduced early with conceptual support, so these students can immediately engage in meaningful activities. They can use the tens-and-ones language quite early, without being able to count. In Karen Fuson's Children's Math World-project many students speak Spanish and there is the great confusing of number words for sixty and seventy ('sesenta and setenta'). In the classroom we can avoid this problem by having the students using language like 4 tens plus 3 tens is 7 tens. At least they can build their tens-and-ones conceptual structure, while at the same time they are trying to sort out their sequence structure. If you wait till students are good at their counting list first, and then build these 10-jumps on top of that, then it would be too much delayed for us. Karen Fuson concludes by saying that she just wanted to make clear the different cultural situations. So, we have to be careful with inferences about which are the most 'natural' solution methods of our students. Karen Fuson thinks that solution methods with smaller one-digit numbers are much less culturally varied, but once you start with multi-digit numbers you are getting into a bunch of different issues.

Ian Thompson argues that because in the U.K. too written algorithms are introduced very early, he is making a plea for 1010-like mental calculation first. The procedure 1010 is more similar to written algorithms (than N10) so this instructional sequence may lead to a better transition and transfer towards column calculation. However, in Ian Thompson's proposal the standard algorithm has to be changed into going from left to right, because then you still deal with whole numbers (like in mental calculation). For instance in the earlier example of 478 + 266 this 'left to right' method would proceed as 400 + 200 = 600, 70 + 60 = 130, so 600 + 130 becomes 730, etc. This similar 'left to right' procedure could be carried out vertically in column arithmetic. It would make the written algorithm a bit longer but fits in well with the mental 1010 strategy, and will support understanding of number and number operations more than in the past. Meindert Beishuizen, Ernest van Lieshout and Gerard Seegers react with references to the Dutch situation, where one sees the transition from N10 to 1010 (and vice versa), the role of estimation strategies, whole-class discussions about choices between N10 or 1010 (or another method) for given problem characteristics. These last remarks in the general discussion on Saturday give a good illustration of Karen Fuson's earlier conclusion that 'multi-digit numbers are getting you into a bunch of different issues'.

5 Discussion about the papers of Hans van Luit & Bernadette van de Rijt and Elsbeth Stern

This discussion (on Sunday morning the 15th December) began with questions about relationships between (possible) predictor variables like intelligence and word problem solving, which is a central issue in the longitudinal studies summarized by Elsbeth Stern (this volume). She underlined that correlations with such a global measure are significant but low mostly. More specific indicators of mathematical competence like 'number conservation' and 'estimation of quantities' as tested in preschool turned out to be better predictors for mathematical performance, for instance in grade 2. Some experts consider it as quite obvious that specific pre-knowledge contributes to later learning, while others expressed doubts about the predictive importance of intelligence or even of correlations. Elsbeth Stem, however, points to a relative high stability of performance in word problem-solving during preschool time in her studies. This makes it sensible to look for predictive factors (competences) which are trainable in an early stage in order to prevent or reduce later learning difficulties. In particular this is relevant for weaker students. Compared to reading problems, where lack of automatization is crucial, Elsbeth Stern considers lack of conceptualisation as more crucial for mathematical competences. That is also what she values in the training program developed by Hans van Luit and Bernadette van de Rijt (this volume): the aim to support students at risk in an early stage of preschooling.

Others, however, like Koeno Gravemeijer are less sure: it is a rather quick move from this research to advice about education. Jens Lorenz still sees the problem of delineating the meaning and significance of the so-called specific competences. According to him it is obvious that previous knowledge contributes to later knowledge attainment. He also is convinced that for instance 'estimation' is a predictive factor for later mathematical performance. But is 'estimation' a cognitive or a mathematical factor? Elsbeth Stern agrees that we have to figure out more about specific aspects, but nevertheless she wants to go on with training of predictive factors which have proved to be significant in stable correlations. In particular when positive program effects encourage you to go on.

Tom Carpenter then brings the discussion back to the general theme of how much instructional guidance should be given. In his opinion there is a difference between the less directive and more context driven programs described on the first day of the conference, and the more directive training programs of this morning. He would like a broader discussion on this issue. In his research experience learning-disabled children come with the same strategies as other students and they can follow interactive teaching as well. Tom Carpenter disagrees with the Special Education viewpoint where the emphasis is too much on direct instruction. Hans van Luit comments that the degree of direct instruction depends on the type of learning problems students have. There is no black-and-white contrast between direct and less direct instruction. From the analysis of the ways the two experimental programs (cf. his paper, this volume) were carried out it appeared that the teachers adapted their instruction to the apparent need of the students. In particular in the 'guiding' condition there were changes depending on the task. Sometimes all students understood the message of the learning task and the interaction, but sometimes with new tasks more structured modeling was needed because the students did not understand and gave no reaction.

Here Tom Carpenter responds that there are other ways of adaptation like try and find out what a child does know and to try to build up from there. Apart from misconceptions there are always some points of departure in the child's particular cognitive knowledge base. According to Koeno Gravemeijer there is a big difference between competences you train (and test) in a task-specific situation and competences students develop independently by re-invention and interaction. A difference between just following instructions or putting in and elaborating on own interpretations. The chosen approach depends also on what you think mathematics education should be. Paul Cobb joins in with the remark that as a constructivist he has difficulties with both options: giving guidance or direct instruction. He would prefer the approach Tom Carpenter mentioned of building up from what kids can do, or as it was stated the other day: pro-actively supporting the learning processes as constructive activities. Paul Cobb wonders whether the (artificial) black-and-white contrast is related to the connection between research and instructional conditions. The pure treatment conditions go back to the 1960s when U.S. research introduced experimental comparisons between discovery and expository instruction, and ATI-questions such as which one was better for which task or type of student. The current approach of developing instruction by analyzing what goes on, as it happens in the teaching/learning situation is quite different. Paul Cobb would like to raise this issue of the relation between research and instruction (instructional development) for the general discussion. He too – like Tom Carpenter – thinks that during these two days of conference, there have been two quite different conversations.

Karen Fuson relates the distinction to the type of task. For instance some children in her project have great difficulties with understanding word problems of the comparison type, because they lack the meaning of words like 'more' and 'less' as a frame of reference. As a teacher you have to go back to an instructional situation where you start with 'equal' amounts, and then you add 'more' so the extra amount is perceptible to them. The thing is to get the children started from where they are (like Tom Carpenter said): they have counting abilities, all right, but they lack many frames of reference for context and meaning, so you have to do a bit modeling as in the example given. The word direct instruction can have different meanings. Even in the direct instruction situation there is some adaptation to children's level of understanding and there is some space for feedback. According to Karen Fuson many people in the U.S. take direct instruction as a sort of training comparable to a 'fast train' that just runs over the children. Training is the fast train, and normal instruction is the slow train. She herself is always writing in terms of teaching/learning activities because she does not like the strong distinction in meaning between teaching and learning in the English language. In European languages like German the difference between 'lehren' and 'lernen' is not so strong and in some languages (e.g., Norwegian) the same word is used for both. She liked hearing many speakers during this conference saying: 'the teacher had to learn the children ..' In every teaching-learning interchange the roles are also reverse: the teacher is also a learner because s/he tries to understand about the person being taught, and students have to teach the teacher about their own knowledge. Karen Fuson thinks it would be helpful to refine the language we use for describing these different instructional situations a bit.

Jens Lorenz agrees that there are a lot of meanings for the word instruction. Like for instance in Germany where there is now a discussion going on about open instruction, and we seem to have twice as many open instructions as we have teachers! But it is also important that you see the philosophy behind the ideas, like there is a certain philosophy behind Paul Cobb's constructivist point of view. In connection to this remark Christoph Selter has some critical questions about the lack of specificity in the paper of Hans van Luit and Bernadette van de Rijt (this volume). Referring to a statement of Freudenthal that the proof of a theory is in its examples he badly missed examples of the two types of instruction and of the test in this paper. Now many sentences are difficult to understand. The problem is – also in the discussion of this morning – that as researchers we share some vocabulary on a general level, but we also know there are many different connotations and interpretations. Therefore we need, according to Christoph Selter, much more examples of given instructions and of students' work in order to be able to discuss general issues in a more productive way. Hans Van Luit agrees that he could have given more examples to make things clear, and he promises to do so now during the coffee break ...

6 Discussion about the papers of Koeno Gravemeijer & Gerard Seegers and Paul Cobb

Gerard Seegers pointed in his paper presentation to the fact that Freudenthal's principle of 'guided re-invention' includes a tension between guidance as a cognitivist aspect and invention as a realistic aspect. Now, he perceives another distinction compared to the viewpoint of Paul Cobb in his paper. The Realistic (RME) theory is, in the first place, a didactic theory with didactic claims about models and learning trajectories to be tested empirically. The Constructivist theory, however, has not such claims but has a different goal i.e. is interested in 'how it works'. Paul Cobb agrees that trying to understand is central in his focus. But he is working in classrooms and is also interested in changing things. He is not happy with the usual type of comparative research between instructional conditions, because such claims as well as effects are too general. Teachers need better tools like instructional sequences, not to follow step by step but as a guideline for students' mathematical development. According to Koeno Gravemeijer the issue is how to develop the optimum form of realistic mathematics education (RME) that teachers can handle in the classroom. How far can you go in adapting your principles to reality without losing the intent? For such a 'pragmatic' approach you need to do developmental research with teachers in classrooms. But there is a problem with this new type of research: it is difficult to get it funded in The Netherlands when the common criteria are applied to research proposals.

Lieven Verschaffel has a question about previous research into RME in The Netherlands, where it turned out that teachers did not follow the realistic approach in their teaching in the classrooms. Could you define this as a sort of pragmatic realistic approach of the teachers? Because it could be that the teachers have fully understood and tried out the realistic approach, but that they then as a result of their reflective thinking, came up with some kind of compromise between reality and the ideal situation? Koeno Gravemeijer reacts that this was not his impression from the data in the protocols. What he observed was the problem of the social maths norms. The teachers, although working with a realistic textbook and realistic guidelines, still went on valuing answers of students in a traditional way as right or wrong without looking for students' thinking. A Dutch problem is that now many (revised) realistic textbooks are in use in schools, but that in-service- training of present teachers lags behind. Sometimes they revert to traditional types of instruction and practice. According to Koeno Gravemeijer teacher consultation with feedback or teachers observing and discussing each other's lessons might help to start analyzing what students are thinking. Such a reflective process might develop towards interactive teaching and towards a really 'realistic' approach.

Karen Fuson comments that most teachers being rated on an observation scale in one of her projects, really moved towards an instruction style not imposing things, towards renegotiating social norms, towards having students explaining their thinking and being independent. The typical U.S. problem was that the tasks and the curriculum did not give them enough support. The teachers did not know about different solution strategies to problems and they had difficulties with organizing productive mathematical discussions in their classroom. Therefore, Karen Fuson proposes that internationally we share and discuss more whatever knowledge and experience we have about the innovation of mathematics teaching. She then puts a question to Paul Cobb about the instructional sequence dealing with measurement as described in his paper (this volume). For designing such a learning trajectory you need to know about students' thinking, so what was the hypothesis in this case? Paul Cobb answers that he did not describe the ideas for this 'measurement' learning trajectory in his paper. Moreover, these ideas developed further during experimental lessons and (audiotaped) reflective talks immediately after each lesson, which have not been analyzed yet. But he underlines that you cannot separate students' thinking from the instructional situation. And students' interpretations are important for the development of the instructional sequence. As an example Paul Cobb goes back to what he told about the 'arithmetic rack' the other day. On the first day of the conference the authors of the new realistic textbook 'Wis & Reken' demonstrated how students' activities using finger patterns for numbers up to 10, preceded the introduction of the arithmetic rack. To see this instructional sequence was a pleasant surprise for Paul Cobb, because that was precisely the problem when introducing the arithmetic rack in his U.S. experimental class. At first the model did not work because the students did not recognize the framework. You have to prepare the ground by pro-active support like the re-invention of finger patterns by the students. So, by observing and analyzing students' interpretations you can improve the instructional sequence, which is a central characteristic of developmental research.

7 General discussion (including the audience) on Sunday the 15th December 1996

A first theme in the discussion is the role of realistic problems and contexts. Tom Carpenter suggests that younger students tend to be more successful with realistic problems because they are still working in the reality of the situation. Erik De Corte agrees that younger students probably are less vulnerable to what you might call 'misbeliefs', because they have not yet been subjected to a kind of mathematics teaching that does not pay attention to real world knowledge. Many teachers accept answers to maths problems as (formally) correct although they are wrong (impossible) from a realistic point of view. Such classroom culture can indeed push the students in a direction of avoiding or neglecting real world knowledge as a result of traditional mathematics teaching. According to Lieven Verschaffel this question is a complex issue, because there always will be a gap between solving a mathematical problem in a real life situation outside school and solving context problems in a mathematical classroom lesson. Of course this has to do with socio-math norms, but how aware must we make our students of this problematic tension between reality and mathematics? On the other hand, some people will say that the very essence of mathematics is in abstracting, even in neglecting in a mindful way certain aspects of reality.

Karen Fuson suggests that such discrepancies could be solved by applying (as a teacher) the so-called 'if discourse', and by stripping down the mathematizing. But in the experience of Lieven Verschaffel this can lead to endless discussions using 'ifs' all the time. It then becomes more and more extreme as a sort of game, and moves the teaching/learning situation away from what is considered as valuable mathematical modeling. According to Lieven Verschaffel we really have not solved this problem. Paul Cobb agrees that there is a problem, but also that there ought to be a difference between a problem in the math classroom and in the real world. Sometimes, if it comes up, he discusses this with students. Sometimes an instructional situation is chosen with a didactical eye to lead the students to be aware of such a difference. Tom Carpenter adds that in reality it often happens that experts have to solve problems which have been abstracted from the context. They then have to negotiate over meaning as well. In mathematics teaching you get to a point where you can not go on with realistic problems, when it comes to abstract calculus or algebra, etc. The ability to deal with that kind of abstractions is a goal of mathematics too. Koeno Gravemeijer does not agree. He would not separate mathematics that much from the real world. He prefers the notion of 'experientially real' and he thinks part of the everyday world may not be experientially real for a student, while on the other hand mathematics itself can become experientially real for a student. Ernest van Lieshout and Hans van Luit mention examples from research, where students reacted not realisticly in a classroom situation, although they knew these problems from reality.

Koeno Gravemeijer and Paul Cobb immediately add that it is a matter of different expectations or different socio-math norms. In a given situation a person reacts as he is supposed to do. Marja van den Heuvel, however, comments that it also depends on the kind of problems and the way they are presented to students. For instance in the case of problems such as people to be transported by buses or balloons to be divided among children, it is rather unrealistic to come up with answers including a remainder or a decimal. Koeno Gravemeijer has an example the other way round with a problem like at a party, where there were 24 bottles of coke for 36 people. He remembers students reacting: 'Some people do not drink coke!' So, these students were not willing to solve the problem the way you want it to be solved (by proportional reasoning). Here Lieven Verschaffel cuts in with the remark that this latter example exactly illustrates the point he wanted to make earlier. At certain moments in the teaching/learning process you can appreciate such comments from students. However, in a next lesson you want to model multiplication or division as such and then you do not like such comments. How can we make this distinction clear to the students? In addition Tom Carpenter remarks that what we want students to do is: to examine the assumptions and to negotiate about the meaning of a problem situation. Sometimes, it is hard indeed to convince students what the rules of the game are in a given situation. The notion of shifting between sort of artificial solutions to ones that are more realistic.

Julia Anghileri has another comment with respect to realistic problems. The power of mathematics is in the pattern. One of the difficulties in giving real problems is that these do not exhibit the pattern. When young students start with problems embedded in contexts then at some point these models have got to expose the mathematical patterns, the relations between numbers. She reverts back to an example in the discussion the other day about the number 72, which you can see as composed of 70 and 2 or 60 and 12 (tens and units), but also as 2 × 36 (doubling) or as close by 75 (three quarters of a hundred). Seeing such rich patterns is what gives children the power to do mathematics. According to Julia Anghileri we should not discuss too much how students solve a particular problem, but how they make connections between (different) solution patterns. In her experience students stick too much to their own (idiosyncratic) understanding and to their own strategy or procedure. Many students have difficulties with understanding another strategy and with seeing links between the other and their own strategy. So, our focus should not be only on what is the best model for a problem, but also on how to get students to see and to make these connections and patterns.

Paul Cobb makes the remark that for him it makes a big difference whether you look at the pattern in a task as we see it, or whether you try to anticipate how kids might interpret a task. He thinks that also in the RME-view the source for instructional design is not the problem per se but the problem in relation to the child's interpretation. So, it is important to look at how problems or materials are used rather than what patterns we want to get out of it. Karen Fuson reacts that she had already started a conversation on this matter with Paul Cobb, because the other day he made the inference that she in her classification of conceptual structures (fig. 2 in Karen Fuson & Steven Smith, this volume) was not necessarily thinking of students' interpretations. To Karen Fuson this is a foreground-background problem, but also a communication problem because we sometimes get confused by our different use of the same terminology. We need a language that differentiates between description and analysis on different levels: the level of students' thinking and the level of instructional sequence design. Jens Lorenz then makes the remark that we also need a language in which students can communicate about their strategies. In his experience the explanations of students can be rather unclear. According to Koeno Gravemeijer such a language develops in a natural way along maths practices in the classroom. When certain things and procedures get accepted in the group there is no need for explanation anymore. Karen Fuson agrees that it is very important that students are discussing things in classroom. The teacher could assist by writing things as a referent on the blackboard for helping all students to clarify explanations.

In his research Paul Cobb is doing a lot of analysis of students' spontaneous communication during maths practices, which he calls 'reflective discourse' in a new JRME-article (1997). He mentions partitioning of small quantities as an example, for instance the different ways 6 monkeys can be sitting in 2 trees. When kids come with various answers the recording by the teacher is important. If somebody is saying 'I think we have found them all', the teacher can stimulate justification by asking: 'how can we know for sure?' Then some child might come up with the idea of a pattern: 6 and 0, 5 and 1, etc. They look through the table on the blackboard and go back to the original problem situation for checking. The pattern emerges out of the children's activities. Now it is the result of the math activity that is being organised or structured. You get a gradual shift from talking about what they are doing to the results which then become the subject of conversation. Karen Fuson adds that it not only becomes an object of discussion, it becomes an object of symbolization. And the 'monkeys' (context) model fulfills a bridge to the number symbols. If the children understand and can manipulate the 'monkeys' situation, they are ready for the transition to a number table. Paul Cobb agrees that symbolizing (into number symbols) is critical here. But it also an example of Koeno Gravemeijer's transition from 'model of' towards 'model for'.

Now the discussion comes to the role of 'numberwalls' in the German maths program published by the Dortmund University Mathematics Institute. Gerard Seegers is suggesting that one should start with authentic situations like the 'monkeys' and then as a second step go on to less authentic problems like the 'numberwalls' eliciting the analysis of patterns. Paul Cobb answers that from a constructivist point of view he would not use such particular rules as a basis for decision. It would depend on the students: not for first-graders but for teacher-students numberwalls could be a starting point. Christoph Selter comments that he can imagine the position of other experts assuming that numberwalls cannot be experientially real to first-graders. However, his experience is different and his position is that such assumptions should be figured out in empirical teaching experiments. Numberwalls can be experientially real to students. An important reason for the Dortmund group to use them are higher-order goals like giving arguments, explaining and justifying patterns, etc. The numberwall problems are suited to elicit these higher-order goals. His impression of RME problems, which he generally appreciates very much, is that the non-real-life ones address more procedural arithmetic skills like addition and subtraction than recognition of patterns. His observation is, for example, based on some examples given in the presentation of the realistic textbook 'Wis & Reken' during the conference.

Here, Elsbeth Stern comes in with the comment that what is typical of mathematics is that one can do things with mathematical symbols which one cannot do with reality. In her view mathematics is also a way to develop concepts that could not be developed otherwise. She illustrates this with some examples: we would not have a concept of end-lessness if we had no numbers, not a concept of intensive quantities such as speed if we had no numbers. Only, because one can do things with numbers which cannot be done with reality, one can develop such concepts. And that is what students learn by doing mathematics. They know that you cannot 'divide people' for example, but in the world of mathematics you can. What may be the result of realistic problems after a while is that students are aware of the fact that what can be done with mathematics is not always what can be done in reality. Sometimes you can do things in mathematics to extend your view of reality. Elsbeth Stern gives an example of an experiment with fourth graders which she called the 'paradox problem'. The students were asked: 'You have a birthday party and you serve orange juice. The first child comes for a glass, but you are afraid that you do not have enough orange juice for all children. Imagine that every next child will get only half of the glass of orange juice of the child that you served before. Will the orange juice in the end disappear?' In the experiment, until grade 6 the students answered in a realistic way that the juice would disappear. From grade 6 on the students gave differentiated answers. They said things like: from a realistic viewpoint it does not make sense because it will be so less that you cannot drink it anymore. But with numbers it is different because numbers can not disappear although they will become smaller and smaller. According to Elsbeth Stern this may be a bridge to understanding the nature of mathematics. The nature of mathematics is that it is a bridge between an abstract thing and reality, and that is what students have to become aware of.

Jens Lorenz responds that we may be again at the point where we attach different meanings to the concept of reality. Patterns indeed are not part of reality. We impose patterns on reality. It is a way to look at reality. Mathematics does not emerge out of reality. Not that just by looking at it will one see a pattern.

Paul Cobb thinks that these comments are very helpful and he gives a similar example from the 'candy shop' in his earlier teaching experiment with Koeno Gravemeijer. Quite a long time the children in the first grade classroom kept talking about candies (rolls of ten). But at a certain moment they are not the same candies anymore because they get mathematized. Then the candies signify quantity structures into units of ten. The reality, the real situation evolves itself through this process. For Paul Cobb it is a pragmatic decision – depending on the children and the situation – when you start to talk about patterns in terms of relationships between quantities or numbers. The same applies to things like numberwalls. If you have reason to believe kids can interpret them in terms of patterns and structures than you do. If not, then you have to build up to it. Of course you have to stimulate the transition to the awareness of number patterns through reflective discourse in the classroom. The Freudenthal Institute offers several design heuristics like the 'model of' – 'model for' transition, the arithmetic rack, the empty number line, which we found very helpful for putting together several activities into instructional sequences and for supporting the process of mathematizing.

Jens Lorenz reacts that you (and students) always think in terms of something: quantities, distances, measurements or whatever. But the inside number patterns are not so obvious. For instance you do not get the idea of 'Fibonacci' numbers by just looking at a sunflower – although they are there. So, from a certain point it is easier for students to study the numberwalls. You are looking for regularities within numbers and not within some reality. So, there seems to be a paradox. One constructs realistic situations to make something clear to a student, which would be more clear if you did it with numbers! According to Jens Lorenz some realistic problems bring you in unrealistic situations like dividing sandwiches by tables or people, which are ridiculous questions we would not solve ourselves. Paul Cobb reacts that it is crucial in what stage children are: do numbers immediately signify an experientially reality of numbers for them, then you can go on to the level of number patterns. But here you have to be careful too. For instance in the example of the 6 monkeys some kids will organise numbers only in patterns of doubling and halving (3 + 3), while others will grasp the idea of a wider system in table format (6 + 0, 0 + 6, 5 + 1, etc.). Ian Thompson interjects: do we not also want to get students to appreciate that not all mathematics has to be related to reality; that there are many people – mathematicians – who enjoy mathematics for its own sake? Mathematics as a collection of different ways and different tools to analyze complexities in reality, which we would teach them to use? Karen Fuson thinks this is true for older students, but for younger students numbers are not yet experientially real and that is the focus of this conference.

Koeno Gravemeijer wants to make a distinction between the concept 'realistic' in the RME-approach and 'realistic' in the sense of everyday reality. In his opinion this difference gets confounded all the time, and he admits: 'that, of course, is our fault by choosing this name'. The central RME-concept is that the starting point should be informal solution strategies. Working with young children you will use familiar situations which often will be real life context situations. Later the numbers and number relations itself will become experientially real. Then you can do the things from the Dortmund program and you can go even further and start doing algebra based on experientially real familiarity with numbers. So, it is just a matter of growth. At the same time, Koeno Gravemeijer thinks, the other argument for real life problems has to do with your goals of mathematics education. Do you want to develop a kind of pure mathematics or do you think it is more important to promote a kind of mathematical literacy. If the latter is your goal, you have to foster the relations with everyday life reality. Summarizing, Koeno Gravemeijer thinks there is not so much a difference in viewpoint with Christoph Selter in the starting points, but more in the long-term goals of mathematics teaching. Jens Lorenz and Ian Thompson both ask why RME does not stress both aspects? In Dutch realistic textbooks, they mainly have seen the second aspect of practical mathematics, but for instance not much investigative work related to the first aspect. Christoph Selter comments that in his opinion the distinction made by Koeno Gravemeijer is too suggestive: both aspects belong to mathematics and he also wants to stress them both. Lieven Verschaffel is surprised to hear about this difference between the RME and Dortmund approach. Koeno Gravemeijer is putting his remarks in a more relative perspective by saying that it is a matter of choice, a matter of goals more than a matter of didactics: 'it is a matter of how and when...'

At the end of the discussion, Julia Anghileri wants to come back to the role of the teacher. According to her, the teacher is there to expose the patterns and to explore the connections, not to teach the strategies. How teachers should do this using classroom discourse is in her opinion more important than discussing whether tasks should be more or less realistic. Koeno Gravemeijer reacts that this description of the role of a teacher sounds very much like the so-called Socratic discourse (questions and answers). He would prefer a greater role for the students starting with real life problems as described earlier. When Julia Anghileri asks what the role of the teacher is in this scenario, he refers to what Paul Cobb said about pro-active facilitating the learning process of students. The teacher also creates a classroom atmosphere with socio-math norms, where students can develop their own solutions. The teacher may bring in models like the empty number line at the moment this fits in the informal strategies of the students. A teacher also can bring in the mathematical conventions, after all kind of (informal) notations have been explored in the classroom. So, the role of the teacher is a mixture of bottom-up and top-down. Paul Cobb relates the question to the paper of Christoph Selter (this volume) about the development of teachers in a bottom-up way. He found the paper helpful, because in the U.S. there is an idea of what reform-teachers should be doing, but until now, not much thinking about how to build up such a teaching attitude has been done. The teacher has to create a classroom climate which is different from pure guidance leaving much to the students. It is important that the teacher clearly values certain types of answers more than others, so that the students get a sense of directionality. On the other hand, the teacher also has to create an encouraging atmosphere and opportunities for every student to participate at its own level. For instance in a first grade accept all the counting strategies given as solutions to a problem, but at the same time valuing more the grouping strategies given by some more advanced students. According to Karen Fuson that is also what Japanese teachers are doing a lot in their classrooms: highlighting or foregrounding some higher-level solutions and strategies. With this remark from an international perspective the discussion on the second day of the experts meeting is closed.

Instructional design and reform: a plea for developmental research in context

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In their chapter, Seegers and Gravemeijer (this volume) distinguish between what they call the pragmatic enactment and the ideal enactment of an instructional sequence. One of the central questions they raise is whether the extra effort required to make possible an ideal enactment can be justified in terms of the quality of students' mathematical learning when compared with a pragmatic enactment. I return to this issue in the final section of this chapter.

First, however, I attempt to further clarify what is involved in a so-called ideal enactment. To do so, I reflect on my own and my colleague's activity during a recently completed classroom teaching experiment. The purpose in doing so is to describe a particular way of enacting an instructional sequence in process terms by focusing on our ways of acting in the classroom. Particular attention is given to the planning of whole class discussions in which mathematically significant issues emerge as topics of conversation. The specific issues discussed include the focus on both individual students' meanings and the communal activities in which they participated, the framing of the overall intent of the instructional sequences in terms of Greeno's (1991) environmental metaphor, and the process of continually refining the conjectured learning trajectory in the course of the experiment.

The general approach that I describe clearly falls on the ideal(istic) side of the pragmatic-ideal dichotomy outlined by Seegers and Gravemeijer (this volume). This does not necessarily imply that it is exemplary. Instead, the account that follows is best viewed as a report from the field. It will serve a useful purpose if it constitutes a point of reference in the ongoing debate about the role of the teacher in reform classrooms.

1 Background

The classroom teaching experiment that serves as the basis for the discussion was conducted in a first-grade classroom for a four-month period between February and June 1996. Two closely related instructional sequences focusing on linear measuring and on mental computation with two-digit numbers respectively were enacted and refined in the course of the experiment. In previous discussions of the teaching experiment methodology (Cobb, in press; Yackel, 1995), we have distinguished between three general phases: planning for an experiment, experimenting in the classroom, and conducting a retrospective analysis. The primary focus in this discussion is on the second of these phases, experimenting in the classroom. Thus, the story begins after provisional instructional sequences had been outlined during the planning phase. The issues of interest are located at the micro-level and concern what Gravemeijer (1994) calls daily mini-cycles in which one conducts an ongoing analysis of classroom events and makes instructional decisions on that basis. In describing this process, I am in effect attempting to delineate aspects of our classroom-based practice of which we[1] have ourselves only recently become aware.

I should clarify at the outset that the first-grade teacher with whom we collaborated was a full member of the research and development team. We first began working with her in May 1993 when she recruited us to work in her classroom and conducted a year-long teaching experiment with her during the 1993-1994 school year. The relationship we had established with her by the beginning of the current teaching experiment was such that members of the research team could begin to co-teach with her at any point during a classroom session without prior arrangement. She, for her part, participated in the ongoing analysis of classroom sessions during both daily debriefing sessions and weekly project meetings conducted throughout the experiment. In addition, she made important contributions to the design of instructional activities.

One of the retrospective analyses conducted as part of the 1993-94 teaching experiment had focused on her role in proactively supporting her students' mathematical development (McClain, 1995). The significant aspects of her classroom practice that were identified included guiding the renegotiation of sociomathematical norms, facilitating the development of ways of symbolizing and notating, and initiating both reflective shifts in classroom discourse and the folding back of discourse. The account of the current teaching experiment begins after the teacher had guided the establishment of supportive social and sociomathematical norms in her classroom. Thus, a type of classroom microculture that characterizes what Seegers and Gravemeijer (this volume) term an ideal enactment constitutes the back drop against which I discuss three aspects of our practice of experimenting in the classroom.

2 Individual meanings and communal activities

In previous discussions of the teaching experiment methodology, we have emphasized the importance of analyzing students' mathematical activity as it occurs in social context. The particular approach we take involves coordinating constructivist analyses of individual students' activities and meanings with an analysis of the communal mathematical practices in which they participate. Our focus in the prior discussions has been on retrospective analyses that are conducted once the phase of experimenting in the classroom has been completed. It is only recently that we have come to view ourselves as coordinating-in-action these two perspectives as we experiment in the classroom. To describe this aspect of our practice, I first outline the classroom social arrangements.

The classroom sessions conducted in the course of the teaching experiment usually involved periods in which the students worked either in pairs or individually but with the proviso that they could move around the classroom to discuss their problem solving efforts with peers of their choosing[2]. The small-group or individual work was typically followed by a teacher-orchestrated whole-class discussion that focused on the students' interpretations and solutions. During the pair and individual work, the teacher usually circulated around the classroom to gain a sense of the diverse ways in which the students were attempting to solve the tasks. For our part, I and a graduate research assistant each observed and interacted with two students to document the process of their mathematical development throughout the teaching experiment. In doing so, we consciously attempted to infer the four students' individual mathematical interpretations on an ongoing basis.

Towards the end of pair or individual work, the teacher, the graduate assistant, and I 'huddled' in the classroom to discuss our observations and to plan for the subsequent whole-class discussion. In these conversations, we routinely focused on individual students' qualitatively different interpretations and meanings in order to develop conjectures about mathematically significant issues that might emerge as topics of discussion. In this opportunistic approach, our intent was to capitalize on the students' individual or small-group activity by identifying specific students whose explanations might give rise to substantive mathematical discussions that would advance our pedagogical agenda. At times, the discussions focused on one student's mathematical activity whereas, on other occasions, the discussions involved a comparison of two or more solutions. It is important to emphasize that our intent in proactively organizing discussions in this manner was *not* to confront solutions so that students who initially agreed with a solution classified as less sophisticated in some way would come to appreciate the superiority of the other solution. Instead, our justification for the discussions focused on their quality as social events and was cast in terms of participation. We contend that participating in discussions of issues that we judge to be mathematically significant constitutes a supportive situation for the students' mathematical development. The teacher's role in these discussions was therefore not to persuade or cajole the students to accept one particular interpretation, but was instead to orchestrate a conversation about issues judged to be mathematically significant per se.

In reflecting back on this process of planning whole class discussions, we have come to see that it involves coordinating the two perspectives that we had previously discussed when describing the retrospective analysis of classroom video-recordings. At the moment that we focus on individual students' qualitatively distinct interpretations and meanings, a psychological perspective comes to the fore and the communal practices in which the students are participating fade into the background. For example, during the measuring instructional sequence, the constructs that we used to account for the students' meanings were developed by drawing analogies with individualistic accounts of children's early number learning (Steffe, Cobb, and Von Glasersfeld, 1988). At this point in the planning process, both we and the students are in effect 'inside' the communal classroom practices.

This psychological perspective can be contrasted with that which we take when justifying the discussions we are attempting to organize. At this juncture, our focus is on the nature of the discussions as collective activities, and the students' individual interpretations now fade into the background. Our primary concern is with the quality of the social events in which the students will participate, and it is for this reason that we concentrate on the mathematically significant issues that might emerge from their explanations with the teacher's guidance. Once the discussion begins, we find ourselves monitoring both the nature of the discussion as a social event and individual students' qualitatively distinct contributions to it. In doing so, we attend to both the communal activity interactively constituted by the teacher and students, and to students' individual meanings as they participate in it.

This account of the way in which we plan for whole-class discussions clarifies how we currently attempt to cope-in-action with a tension endemic to teaching, that between the individual and the collective (cf. Lampert, 1985; Ball, 1993). The account does not describe instructional strategies, but is instead cast in process terms and deals with a way of acting in the classroom. It clearly indicates the importance of interpreting-in-action individual students' solutions and understandings (cf. Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). In addition, it suggests the value of locating the students' solutions in social context by focusing on the communal activities that constitute the social situations in which their mathematical development occurs. In such an approach, coordinating individual and communal perspectives is not merely an esoteric theoretical issue. Instead, it is an integral aspect of our classroom-based practice as we proactively attempt to support students' mathematical development.

3 Mathematical significance

In describing the process by which we plan whole-class discussions, I referred to issues that we judge to be *mathematically significant*. This way of talking is, of course, vague and leaves many questions unanswered. As a starting point, recall that a potential issue is judged as mathematically significant if it contributes to our pedagogical agenda. This agenda in turn takes the form of a conjectured developmental process that culminates with the global goals of an instructional sequence. In a very real sense, the conjectured learning trajectory serves to locate immediate, local judgements within a broader, more encompassing vision of the instructional process. To clarify what is meant by mathematical significance, I will therefore discuss both the global intent of an instructional sequence and the conjectured learning trajectory by which this intent might be realized in the classroom.

3.1 Instructional intent

One of the challenges when preparing for a teaching experiments is to clarify for ourselves the global intent of the instructional sequences we are outlining. In currently fashionable parlance, this involves specifying what are sometimes referred to as the *big ideas.* We have found that, for our purposes, the most useful way to explicate these big ideas is in terms of Greeno's (1991) environmental metaphor. In other words, we do *not* specify our instructional intent in terms of the observable solution methods or strategies that we hope students will develop. Neither do we specify particular internal concepts or cognitive mechanisms putatively located in students' heads. Instead, we attempt to articulate the nature of the mathematical environment in which we hope students will eventually come to act. It is against the background of a pedagogical agenda whose goals are stated in these terms that we make judgements about the potential significance of issues that might emerge as topics of conversations in whole-class discussions. The issues are *mathematically* significant if discussions centering on them contribute to our pedagogical agenda of making it possible for the students to eventually act in a particular type of mathematical environment.

It can be noted in passing that our use of Greeno's environmental metaphor is reflexively consistent with the description I have given of our classroom-based practice. The focus has been on our ways of acting and on the classroom as a pedagogical environment in which we act. Similarly, when attention turns to students' mathematical development, the focus is on their ways of acting in a mathematical environment. This approach is non-dualist in that it does not separate either our own or students' activity from the worlds in which we act. In each case, ways of acting and the world acted in are considered to be mutually constitutive and to co-evolve (Pea, 1993; Varela, Rosch and Thompson, 1991).

As an initial illustration of this way of framing the intent of an instructional sequence, consider first the relatively familiar case of the addition and subtraction of numbers up to 20 that was the focus of the prior eight-week year-long teaching experiment. Our global intent in this instance was that students would come to act in a quantitative environment structured by relationships between numbers up to 20. Observationally, this would be indicated by their flexible use of thinking or derived fact strategies to solve a wide range of tasks. For example, they might solve a task interpreted as 14 – ... = 6 by reasoning 14 – 4 = 10, and 10 – 4 = 6, so the answer is 8. Alternatively, they might reason that 7 + 7 = 14, so 14 – 7 = 7, and 14 – 8 = 6. It should be stressed, however, that the acquisition of these calculational methods was not itself the pedagogical goal. Instead, our intent was that the numerical relationships implicit in these and other observable strategies would be ready-to-hand for the students. In other words, they would not have to consciously figure out appropriate strategies to use. Instead, we hoped that the students would come to have the experience of directly perceiving relationships as they interpreted tasks. Needless to say, coming to act in such an environment is a major intellectual achievement that requires proactive developmental support.

In the case of the instructional sequence that dealt with measuring, our initial concern was that the students would come to interpret the activity of measuring as the accumulation of distance (cf. Thompson and Thompson, 1996). In other words, if the students were measuring by pacing heel-to-toe, we hoped that the number words they said as they paced would each come to signify the measure of the distance paced thus far rather than the single pace that they made as they said a particular number word. Further, our intent was that the results of measuring would be structured quantities of known measure. In other words, having paced a distance of, say, 20 steps, they could view this quantity as itself composed of two distances of ten paces, or of distances of five paces and fifteen paces as the need arose. By analogy with the case of addition and subtraction up to 20, we hoped that the students would come to act in a spatial environment in which distances are structured quantities whose measures can be specified by measuring. In such an environment, it would be self evident that while distances are invariant, their measures vary according to the size of the measurement unit used.

In the course of the teaching experiment, measuring with composite units also became an established mathematical practice. Initially, the students drew around their shoes and taped five shoe-prints together to create a unit that they named a footstrip. Later, in the setting of an ongoing narrative that appeared to be experientially real to the students, they used a bar of ten unifix cubes to measure. As a consequence of participating in these instructional activities, many of the students came to act in an environment in which distances with measures of up to 100 were composed of distances whose measure was ten. The students' activity in this environment subsequently served as the starting point for a second instructional sequence that focused on mental computation with two-digit numbers. In terms of Greeno's environmental metaphor, the intent of this latter sequence was that the students would come to act in a quantitative environment structured in terms of relationships between numbers up to 100. As was the case with addition and subtraction to 20, our immediate concern was *not* merely that students would acquire particular calculational methods. Instead, our intent was that they would come to act in an environment in which the numerical relationships implicit in these methods are ready-to-hand. This view shifts the focus from calculational strategies per se to the interpretations and understandings that make flexible strategy use possible.

The contrast I have drawn between stating the instructional intent in terms of observable solution methods and in terms of acting in an environment is analogous to the distinction that Thompson, Phillip, Thompson, and Boyd (1994) make between what they term a calculational orientation and a conceptual orientation. Whereas a calculational orientation is concerned with the calculational steps taken to produce an answer, a conceptual orientation is concerned with how the task is interpreted and understood – with why a particular calculation is performed in a particular situation. It is precisely this latter issue that is addressed by the environmental metaphor. This contrast between observable strategies and acting in an environment in no way plays down the importance of calculational proficiency. Instead, it involves a shift in focus from what Mackay (1969) terms the observer's perspective to the actor's perspective. When the instructional intent is cast in terms of observable strategies, the focus is on aspects of students' activity that can be documented by a detached observer. In contrast, when we adopt the actor's perspective, we attempt to understand students' activity from the their point of view rather than from that of a detached observer. The focus is then on the quality of their mathematical experience and on the tasks and situations as they understand them. This emphasis leads to a consideration not just of how students might calculate, but of *why* they might come to calculate in particular ways. An approach of this type is explicitly non-dualist in that to specify the mathematical environment in which students might come to act is to specify the intended nature of their mathematical experience.

3.2 Learning trajectories

The approach of formulating the instructional intent of a sequence in environmental terms provides what Thompson et al. (1994) call a conceptual orientation. However, the delineation of the global intent does not by itself give sufficient guidance for pedagogical decisions and judgements. As we have seen, local judgements in the classroom are made against the background of a conjectured learning trajectory. This trajectory takes the form of an envisioned developmental process by which students' current mathematical ways of knowing might evolve into the ways of understanding that constitute the intent of the sequence. This notion of a learning trajectory, which is taken from Simon (1995), is consistent with Gravemeijer's (1994) analysis of the process of instructional development. In Gravemeijer's account, the developer first carries out an anticipatory thought experiment in which he or she envisions both how the proposed instructional activities might be realized in interaction and what students might learn as they participate in them. As Gravemeijer notes, in conducting this thought experiment, the developer formulates *conjectures* about both the course of students' mathematical development and the means of supporting it. In other words, the rationale for the instructional sequence takes the form of a conjectured learning trajectory that culminates with students coming to act in a particular mathematical environment that constitutes the overall intent. As Seegers and Gravemeijer make clear, the means of supporting the conjectured developmental process include the development of particular ways of symbolizing. My concern in this chapter is not, however, with the viability of specific conjectures such as the model of/model for transition, but instead concerns the more general process of making judgements in the classroom.

A first issue that arises is to clarify who or what is the subject of the proposed developmental route. It clearly cannot be all of the students in a class because there will be significant qualitative differences in both their interpretations of the initial instructional activities in a sequence, and in the actual process of their individual development in the classroom. Descriptions of an instructional sequence written so as to imply that all students will come to reason in particular ways at particular points in the sequence appear to be untenable. To circumvent this difficulty, it could be argued that the conjectured learning trajectory is that of a fictional, idealized student. The limitation of this approach, however, is that it proves difficult to relate the conjectured trajectory to the reality of the classroom for the simple reason that no such student exists. In other words, the process of testing and revising the conjectures inherent in a sequence when experimenting in the classroom is problematic.

The approach that I and my colleagues have taken is to view the proposed learning trajectory as a conjecture about the mathematical development of the classroom community. In this view, a learning trajectory specifies both a possible sequence of classroom mathematical practices and the possible means of supporting the emergence of one from another. Elsewhere, we have discussed this notion of a classroom mathematical practice and have described its relation to the mathematical activity of the individual students who participate in it (Cobb and Yackel, in press). For my present purposes, it suffices to note that there is no implication that the individual students are acting and reasoning in identical ways. Instead, this notion acknowledges students' diverse ways of interpreting and solving tasks while, at the same time, treating them as members of a community that itself develops and evolves. In this approach, events that occur in the classroom over an extended period of time as an instructional sequence is enacted are analyzed in terms of the evolution of mathematical practices, thereby documenting the actual learning trajectory of the classroom community. Analyses conducted in these terms are reported by Bowers (1996), Cobb (1996), and Cobb, Gravemeijer, Yackel, McClain, and Whitenack

(1997). These analyses illustrate that a focus on the community can be complemented by a psychological focus on the diverse ways in which individual students participate in and contribute to the development of the collective practices. Analyses of this type can therefore serve to document the process of individual students' learning as it occurs in the social context of the classroom.

The divergence of the actual learning trajectory realized in the classroom from the intended learning trajectory envisioned at the outset is a product of pedagogical judgements made while the teaching experiment is in progress. The process by which we plan for whole-class discussions provides one illustration of this local decision making. I noted that our intent was not merely to encourage the students to explain their reasoning. A classroom discussion was justifiable only if the issues that emerged as topics of conversation were mathematical significant. We have seen that an issue is judged a mathematical against the backdrop of a conjectured learning trajectory. In other words, an issue is considered to be significant if it contributes to the realization of an envisioned developmental route for the classroom community. Metaphorically speaking, the learning trajectory might be said to constitute the big picture within which local decisions and judgements are made on a daily basis. The example of planning for whole-class discussions also illustrates that although learning trajectories are cast in the collectivist terms of classroom mathematical practices, these local judgements take account of the diverse ways in which individual students participate in those practices.

It is important to stress that in this way of working in the classroom, the relationship between the learning trajectory and the daily judgements is reflexive. On the one hand, daily decisions and judgements are framed by the learning trajectory. On the other hand, the envisioned learning trajectory itself evolves as a consequence of these local judgements. Thus, at any point in a teaching experiment, there are conjectures about the possible evolution of classroom mathematical practices and the means of supporting their emergence. In the case of the teaching experiment that focused on measuring, for example, we found it essential at the beginning of our weekly project meetings to talk through how the classroom mathematical practices might evolve during the remainder of the experiment. However, this conjectured trajectory itself continually changed as a consequence of local interpretations and judgements. For example, prior to the teaching experiment, our primary focus was in fact on two-digit mental computation. We initially viewed the proposed instructional activities involving measuring as precursors to those designed to support the development of mental computation. However, as a consequence of issues that arose once the teaching experiment began, measuring gradually became a focus of interest in its own right[3]. As a consequence, the actual learning trajectory came to diverge significantly from that which initially we envisioned.

The account we have given of the reflexive relationship between local judgements and the big picture is broadly compatible with Simon's (1995) discussion of what he calls the mathematics teaching cycle. This cycle is shown in simplified form in Figure 1.

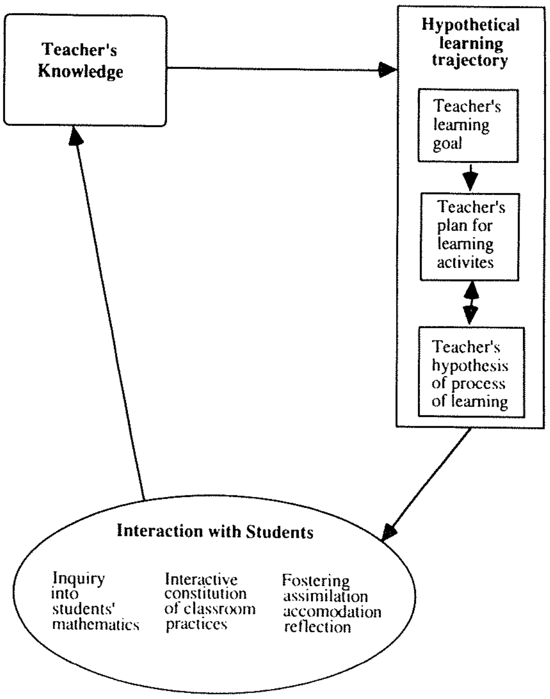

figure 1: a simplified version of Simons (1995) mathematics teaching cycle (reprinted with permission from Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26,* 136)

Simon stresses that his notion of a hypothetical learning trajectory 'is meant to underscore the importance of having a goal and rationale for teaching decisions and the hypothetical nature of such thinking' (p.136). At any point, the teacher has a pedagogical agenda and thus a sense of direction. However, this agenda is itself subject to continual modification in the act of teaching. Simon likens this process to that of undertaking a long journey such as sailing around the world.

> 'You may initially plan the whole journey or only part of it. You set out sailing according to your plan. However, you must constantly adjust because of the conditions that you encounter. You continue to acquire knowledge about sailing, about the current conditions, and about the areas that you wish to visit. You change your plans with respect to the order of your destinations. You modify the length and nature of your visits as a result of interactions with people along the way. You add destinations that prior to the trip were unknown to you. The path that you travel is your [actual] trajectory. The path that you anticipate at any point is your 'hypothetical trajectory'.' (pp.136-137)

As Simon observes, this way of acting in the classroom involves both a sense of purpose and an open-handed flexibility towards students' ongoing interpretations of activities.

The terms I have used to talk about instructional development and Simon to describe his activity as a mathematics teacher are generally consistent with Seegers and Gravemeijer's notion of *enacting* an instructional sequence. As Varela, Thompson, and Rosch (1991) emphasize, the idea of enactment implies that processes are 'inextricably linked to histories that are lived, much like paths that exist only as they are laid down by walking' (p. 205). In the case of an instructional sequence, the teacher and students lay down an actual learning trajectory as they interact in the classroom. This, it bears repeating, does not mean that classroom activities drift aimlessly. At any point, there is both an overall instructional intent and an envisioned means of achieving it. However, both the intent and the conjectured trajectory are subject to continual revision. Thus, to pursue Varella et al.'s metaphor, the path is laid down by walking even though, at each point in the journey, there is some idea of a destination and of a route that might lead there[4].

This enactivist view can be contrasted with the more traditional notion of *implementing* an instructional sequence. The latter metaphor casts the teacher's role as that of carrying out the plans and intentions of others, whereas the notion of enactment highlights the teacher's (and students') contributions to an instructional sequence as it is realized in the classroom. In addition, an enactivist view brings the teacher's learning to the fore. As Simon (1995) illustrates, teaching can be an occasion to deepen one's understanding of the big ideas that are the focus of classroom discussions, of students' reasoning, and of the means of supporting its development. These same comments apply to researchers who work in classrooms, and in fact constitute the primary reason why we conduct classroom teaching experiments. The deviation of the actual learning trajectory from that envisioned at the outset provides a general summative record of this learning while experimenting in the classroom.

4 Reflections

The parallels we have drawn with Simon's (1995) analysis indicate that teaching and classroom-based developmental research are closely related forms of activity. Both involve an intensive engagement with students that is motivated by a desire to support and organize their mathematical development. The various aspects of our classroom-based practice that I have discussed therefore inform a somewhat idealistic view of reform teaching. This view clearly emphasizes the importance of attempting to make sense of individual students' interpretations and solutions. It is therefore consistent with the generally accepted view that teaching should be informed by a relatively deep understanding of students' mathematical thinking. However, the discussion of developmental research also indicates the value of locating individual students' activity in social context by attending to the quality of the social events in which students participate. As we have argued elsewhere, students' participation in these events constitutes the conditions for the possibility of mathematical learning (Cobb and Yackel, in press). In the case of whole-class discussions, for example, this focus on activity in social context implies that pedagogical justifications should go beyond general claims about the role of interaction, communication, and discourse in mathematical development. Instead, particular classroom discussions should be justified in terms of their contributions to the fulfillment of a an evolving pedagogical agenda.

The summary comments made thus far concern the local level of pedagogical decision making. I have also attempted to clarify that the local judgements that we make when conducting a teaching experiment are situated within the broader context of a possible learning trajectory that involves specific conjectures about the means of supporting the evolution of classroom mathematical practices and thus the development of the students who participate in them. In addition, I discussed why we find it useful to state the big ideas that constitute the potential endpoints of these trajectories in terms of Greeno's environmental metaphor. Extrapolating to the activity of a teacher, these considerations indicate the importance of appreciating the pedagogical intent of an instructional sequence. This, it should be stressed, is not a separate 'piece of knowledge' that informs pedagogical decision making. The pedagogical intent involved an envisioned developmental process and thus involves the teacher's understanding of students' mathematical thinking. Further, it involves a relatively deep understanding of the mathematics under consideration (e.g., measuring) *in relation to students who are attempting to learn it.* Thus, it involves what Lampert (1990) terms a map of the mathematical territory in relation to students who might eventually come to act in such a mathematical environment. Finally, it involves specific conjectures about how the process of students' mathematical development might proceed in an instructional setting when proactive efforts are made to support their learning.

A detached analysis of the type of pedagogical activity that I have attempted to describe might objectify it and dissect it into components corresponding to a psychological theory of students' thinking, a theory about the sociology of the classroom, mathematical knowledge, and a domain-specific instructional theory (i.e., pedagogical content knowledge). Such an approach separates pedagogical knowing from the activity of teaching and treats knowledge as a commodity that stands apart from practice. It is precisely this separation that I have tried to resist by focusing on our ways of acting in the classroom. My primary concern has been with acts of knowing and judging that occur moment by moment as one attempts to support students' mathematical development. The perspective I have taken on pedagogical activity is therefore that of the actor rather than the observer.

The view of teaching that emerges from this account portrays teachers as professionals who continually modify their agendas even as they use instructional materials developed by others. It therefore goes some way beyond frequently made claims that reform should be fueled almost exclusively by either materials development or by teacher enhancement. It does, however, fall squarely on the idealist side of the distinction that Seegers and Gravemeijer (this volume) draw between pragmatic and ideal enactments of an instructional sequence. It could legitimately be argued that the form of practice I have outlined is unfeasible for any teacher working alone. In the case of a teaching experiment, for example, some members of the research team teach while others observe classroom events during instruction. This collective activity might best be viewed as a possibly unattainable ideal[5]. It constitutes a way of acting in the classroom to be aimed at, an aspect of a big pedagogical idea that can provide directionality to teacher development efforts. Given these considerations, the question that Seegers and Gravemeijer raise, that concerning what is feasible and practical, becomes significant. Although I cannot give a well formulated response to this question, I am convinced that the pragmatic option described by Seegers and Gravemeijer is entirely impractical. It is to this issue that I turn in the final paragraphs of this paper.

Seegers and Gravemeijer (this volume)describe a pragmatic approach in which the primary focus is on a hierarchy of solution procedures. The intent is that the teacher will structure the reinvention process by explicating this hierarchical order, both by introducing solution methods deemed important if the students do not come up with them on their own and by encouraging students to move from less-advanced methods for more advanced methods. The hope is that in spite of the teacher's explicit guidance the students will experience their progress towards the most advanced method as their own doing. Unfortunately, this hope is contradicted by detailed analyses of classroom interactions during mathematics instruction that involves a similarly ambiguous approach (Voigt, 1985). For example, Maier and Voigt (1989) demonstrate that interactions corresponding to the socratic method involve the elicitation pattern of interaction (Voigt, 1985). Initially, the teacher asks relatively open-ended questions to initiate students' contributions. However, unless a student fortuitously happens to give the response that the teacher has in mind, the teacher begins to give increasingly explicit cues, thereby funneling students to the desired response. The episode typically concludes with the teacher giving a reflective summary of what it was that students were supposed to have learned while participating in the discussion.

Voigt's (1985) analysis indicates that students can be entirely effective simply by waiting for the teacher to tell them what it is that they are now supposed to know. To me it seems that the pragmatic approach offers the students the same option of just waiting for the teacher to tell them what it is that they are supposed to think.

Note that in the teaching experiment I described earlier, the teacher's role was not to persuade the students to accept one particular solution method, but was instead to orchestrate a conversation about issues judged to be mathematically significant per se.

In describing the pragmatic approach, Seegers and Gravemeijer note that it avoids any conscious attempt to change the didactic contract or classroom social norms. This aspect of the approach seems to imply that the quality of an instructional sequence is unaffected by the didactical contract. This assumption is, in my view, completely untenable. When a developer performs an initial thought experiment while developing an instructional sequence, he or she necessarily assumes that a particular didactical contract has been established in the classrooms. In general, it is impossible to develop a conjectured learning trajectory without making implicit assumptions about the classroom participation structure. It is one thing to be unaware of such background assumptions and another to contend that issues relating to the classroom microculture can be ignored for practical purposes. An impressive body of research on the social and cultural aspects of mathematical learning has been conducted in a number of countries, including The Netherlands. This research demonstrates that enacted instructional sequences can differ radically from one classroom to another depending on the classroom microculture. As a consequence, the qualities that the developer sees in an instructional sequence as he or she envisions it might well not be realized when the instructional sequence is enacted in a particular classroom.

The arguments I have made about the importance of attending to the classroom microculture are corroborated by De Lange, Van Reeuwijk, Burrill, and Romberg's (1993) account of an experiment in which six American high school teachers used an instructional sequence that focused on data visualization (De Lange and Verhage, 1992). De Lange et al. made the following observations in their report.

> '[I]n five classes the teachers wanted to try some group work. A few found it difficult because it entails the cooperation of individuals, a new idea in American society[!] (p. 55).
>
> 'The students were getting used to the type of questions being asked and their group work was improving' (p. 79).
>
> '[I]n order to have groups function effectively with this new focus [on sharing ideas], we had to take time to work with students about the roles and responsibilities in a group and in the class; *we had to teach them more than mathematics.'* (p. 155, added emphasis).

These three observations deal with norms for collaborating in groups in order to learn. Working with students about their roles and responsibilities involves what I and my colleagues refer to as the explicit negotiation of obligations for one's own activity and expectations for others' activity in the classroom (Cobb, Yackel, and Wood, 1989).

In contrast to the negotiation of general classroom norms that are not specific to mathematics, other observations reported by De Lange et al. indicate that they also attended to the negotiation of sociomathematical norms (Yackel and Cobb, 1996). These norms are specific to students' mathematical activity and include what counts as a different, insightful, and sophisticated *mathematical* solution, and what counts as an acceptable *mathematical* explanation. For example:

> 'In the first and second week, the students had to learn how to respond to questions with complete answers. At the beginning, they *answered* Yes or No without explanation.' (p. 70)
>
> 'The teacher looked at the graphs and explanations [of the students] and found that the students had not given an explanation of the graph, but were simply describing what they saw in the graph. So she initiated a discussion in class about graphs. In the discussion it became clear what was meant by *explaining* a graph' (p. 75, emphasis in the original).

Numerous other observations relating to social and sociomathematical norms can be found in De Lange et al.'s report (e.g. pages 72, 79, 97, 119-120, 151, 157, and 160). The examples I have cited should, however, be sufficient to illustrate that the data visualization sequence as realized in these classrooms had some of the qualities envisioned by its developers only because De Lange et al. and the teachers with whom they collaborated 'taught more than mathematics.'

De Lange et al.'s analysis forcefully demonstrates the impracticality of a so-called pragmatic approach that aims at socratic dialogues and ignores the classroom microculture. An approach of this type might appear to have merit within the cloistered confines of an instructional development center. Its inadequacies become self evident as soon as one enters the classroom and attempts to support students' mathematical development for an extended period of time. In my view, an approach of this type that is divorced from the *reality* of learning and teaching mathematics is unjustifiable. If the mathematics education community has learned anything from current reform efforts, it is that sustainable reform involves materials development, teacher development, and broader policy considerations. A constructive response to Seegers and Gravemeijer's concern with what is feasible and practical requires that we take the social situation of students' mathematical development seriously. An approach of this type might focus on both the assumptions about the classroom microculture implicit in an instructional sequence as envisioned by its developers, and on the sequence as it is realized in different classrooms. In such an approach, the debate shifts beyond the confines of the instructional development center and is informed by analyses grounded in the reality of the classroom. Ironically, the blatantly idealistic account I have given of instructional practice would seem to have greater practical relevance in this regard than the pragmatic approach identified by Seegers and Gravemeijer.

acknowledgment

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notes

1 I use the first person plural to refer to the members of the research team who conducted the experiment. They were Beth Estes, Kay McClain, Koeno Gravemeijer, Maggie McGatha, Beth Petty, and Michelle Stephan.

2 This approach of allowing students to work with peers of their choosing allows them to actively contribute to the development of classroom participation structures (cf. Murray, 1992).

3 This example illustrates that the overall instructional intent can also evolve in the course of a teaching experiment. For ease of explication, I have somewhat misleadingly spoken as though the instructional intent is fixed from the outset.

4 This notion of enacting a learning trajectory is compatible with Nemirovsky and Monk's (1995) notion of trail making, Pirie and Kieren's (1994) recursive model of mathematical development, and with Lave's (1988) discussion of gap closing. At a more general level, it is consistent with Dewey's (1977) accounting of reflective intelligence.

5 The term 'ideal' is used here to acknowledge the idealistic nature of the vision of teaching that emerges from the analysis. It does not imply that we view our way of working in the classroom as ideal in the sense that it is beyond improvement.

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Discussions at the experts meeting

Meindert Beishuizen & Koeno Gravemeijer

This chapter gives an overview of the discussions at the experts meeting. Since most papers were available beforehand, the authors could do with a short explication of their main points. The following pages mirror the authentic style of the conversations. No attempt was made to further structure the discussion than only mentioning the related papers as section headings. This means that our summary sometimes may have a meandering or even redundant character. This style however, reflects how the participants tried to come to grip with each others' different cultures and different concept interpretations. This may help the reader, we think, to get a sharper awareness of the issues that played a role in this international discussion on primary mathematics teaching. We thank the participants for their comments on the wording of this account of the discussion. As one of them put it: 'It captures the spirit of our discussions very well'. We also thank the two Leiden students Jacqueline Besemer and Stephanie Juranek, who worked out the (audio-taped) conversations.

1 Discussion about the papers of Koeno Gravemeijer, Tom Carpenter and Christoph Selter

This discussion revolves around the need to provide externally developed teaching materials to support teachers. In relation to this, Simon's concept of a 'hypothetical learning trajectory' comes to the fore (cf. Gravemeijer, this volume). But first, the concept itself has to be clarified. Paul Cobb notes that the term learning trajectory seems to be used in different ways. He thinks that for some learning trajectory just means, having an idea of how this could develop or how the kids' thinking could evolve. While the way Koeno Gravemeijer is using learning trajectory also has a very strong instruction-theoretical aspect to it. It is not just a view of how the kids are going to learn. It is a specific conjecture of a means of pro-actively supporting that development. He adds, that for him, what is unusual about what happens in the Netherlands is, that they have developed over a period of time an instructional theory, about means of pro-actively supporting development that makes sense from an instructionalist point of view. He thinks, it is almost unique in that they have such a thing. So it is the learning trajectory of the kid in an instructional situation as it is pro-actively is supported. He suggests that maybe one way to bring out these differences, or get issues on the table, is to talk more about the specific conjectures of the means of supporting the process of development.

Karen Fuson suggests to introduce the term 'instructional support trajectory'. However, Paul Cobb objects that you can't separate conjectures on how the kids' thinking could evolve and the instructional activities. In other words you cannot describe a conjecture about the kids' thinking in a vacuum. In relation to this Tom Carpenter introduces the terms 'foreground' and 'background'. What are the kind of things you think about first? Do you think about the students thinking first, or do you think first about the instructional situation in terms of what the teacher's activity becomes in terms of the instructional sequence. The traditional way of thinking about teaching is in terms of the teacher's activity and that sort of becomes first and the learning is sort of background. Paul Cobb wonders himself, if that is the crucial phrase, the foreground and the background. Maybe one would want to say that we want foreground and background both. For example one used to think about research in early number, or place value or whatever, as if this was just the natural way of kids doing. Well it is the natural way of kids doing in particular problem situation, in a certain organised lore and practice and whatever. There is already support there. So we tend to just focus on the learning, but the support is there all along otherwise that development would not occur.

With this clarification, the question for the discussion can be framed as: Who is designing the hypothetical learning trajectory? Part of what the issue is, as Karen Fuson puts it, is the extent to which it is the individual teacher in the classroom, or to the extend to which someone else can develop it in teaching-learning materials, in textbook materials, and so forth. She refers to the Hiebert et al. (1996) paper (cf. Gravemeijer, this volume), of which she was a co-author. She recalls that, at the time which they were writing the paper, there was this pervasive contest around the world saying that the only worthwhile tasks in mathematical teaching are real-life tasks and anything else is very bad to do. So the Hiebert et al. paper was partly a reaction to that view. The authors were trying to emphasize: No, actually you can have more stripped down problem tasks that can be powerful learning stimulants. The important thing is for the tasks to be somewhat problematic for the students. But they maybe did not quite make this as clear as they could have. When kids were working with these more stripped down versions, they would already have had a history in which those are meaningful. Otherwise, she adds, kids can't engage in any work on that mathematical activity. She remarks that the authors were not intending to take any position on the issue of teachers constructing everything versus teachers using only teaching materials from the outside. That would have been hard given the varied background of the authors. In relation to this she argues that CGI (CGI = Cognitively Guided Instruction) very much ends up with teacher constructed materials, while she herself is trying out teacher-learning materials constructed from the outside.

With respect to CGI Karen Fuson comments that she thinks CGI has done really important powerful things but she has some concerns in that she thinks it works very well with better teachers and better students. Maybe some of those teachers weren't teachers who used textbooks. Maybe they already had their own classroom organisations, they already had constructed things and so they were already teachers who were doing that. She argues that weaker teachers, need support of careful instructional sequences. The challenge is to try to make the materials, the students will be working with, open enough so that teachers can adapt them to their own personal styles and to their own individual classrooms of kids. Because every classroom of kids every year is different. Even if a teacher is used to a particular kind of teaching-learning instructional sequence, the next year she may have different kids, and she will have to do a different adaptational process. So that is what she sees as an important other task: that one needs to think about as the basic question balancing between autonomy and teacher guidance. In the instructional materials we also have to have a balance between a sort of enough guidance for teachers so that they know what path they are going along, and enough flexibility in the materials so that teachers can do their adaptation for the given students that they have that year.

Tom Carpenter takes exception over the comment that CGI didn't work with all teachers with all kids. Although, of course, nothing works with all kinds of teachers. Also the American adaptation of the Dutch materials does not work with all the teachers, either, or with all kids. However, he claims CGI changed all teachers. One of the things that the CGI group was very successful in, is in getting a number of teachers to not feel, that when a kid was falling behind, that they all of a sudden had to forget about understanding. That, he adds, is really one of the things which separates the teachers who were more successful in the program. There were a number of the teachers that go to a point and say: 'Oh, it is the end of second grade and I have got these kids who really don't understand and I think I have got to do something, I have got to give them the way to solve this problem.' He stresses that we need to recognise there is huge variability. For some of the kids it takes a long time to develop the particular concepts and skills. The traditional mistake is to worry about the kids at the bottom, and to respond to that by falling back to very mechanistic ways to deal with those.

Karen Fuson agrees with that. That is what happens in a reform maths program she has been studying (Everyday Mathematics). Here the kids are supposed to invent their calculation methods in second grade and third grade. There are two problems: One is there is not enough sustained opportunity for kids to construct their methods, so only the better kids construct the methods. Two is: when they are discussing alternative methods it is all oral discussion. There is no drawing, most of the teachers don't do drawings, or write numbers. There is no point of concrete reference for the discussion, so the bottom half of the kids can't follow the discussion, only the top half. So it is a kind of 'the richer getting richer'. And then, just before the 'standardized' tests, the teachers all panic and they teach the kids the standard algorithm without meaning. You can't have a situation where you suddenly worry about the low kids and you teach them in a mechanistic rote way, because then what happens is that all the low kids start doing the top-from-bottom subtraction error. Then they really learn that error and it is very difficult to unlearn that. But there is another approach. That is that you do from the very beginning more sustained activities to support the low kids to do more advanced solution methods.

2 Discussion about the papers of Ernest van Lieshout and Lieven Verschaffel

The discussion opens with some specific remarks about the improved design of context formats that might (better) trigger N10 or 1010 strategies. Then Jens Lorenz broadens the discussion by raising the issue of the influence of problem structure versus number characteristics. Lieven Verschaffel answers that the differential effects of number characteristics are difficult to detect, because students often seem overwhelmed by the semantics of the (word or context) problem structure. In Leiden a study was done with word problems and rather extreme numbers like 82 – 79 with a difference of only 3 or 2. In that case, indeed, students changed to indirect addition. But in all other cases with normal numbers of 10 or more in between, like in 82 – 69, most students followed not the number characteristics (indirect addition) but the problem structure of the word problem (subtraction). Also in the Flemish study with bare number problems like 75 – . = 18 and 69 – . = 52 there was no significant effect of a large or small difference between numbers. The semantics of problem structure seems to exert a strong influence on the choice of strategy.

Elsbeth Stern then raises the question how we can determine that the strategies the students tell us, are really the strategies they used when solving the problem? Sometimes students tell you what they know as the most familiar strategy, or a solution that is common to the problem given. Especially with smaller numbers, when students arrive at developmental stages where number facts are going to play a role, it is difficult to be aware of what you have done precisely. Ernest van Lieshout answers there is mostly a relation between correctness of answers and scoring of the procedure, which may give you a clue. According to Tom Carpenter, videotaped recordings mostly contain enough clues about the used strategies. Moreover, if a pupil does not give a true answer, the given explanation will be close to the student's own strategy level. Stern adds that with larger numbers and multi-step solution procedures this question is much less a problem, because you carry out the steps more consciously and you are more aware of what you do. Lieven Verschaffel and Ernest van Lieshout agree this is true for strategies like N10, 10s and 1010 with two-digit numbers (cf. Table 1 in Beishuizen, this volume). However, this applies mainly in the case that correct solutions are given. If answers are incorrect or unclear the categorization of procedure can become much more difficult also for multi-digit numbers.

At the end of the discussion Koeno Gravemeijer and Paul Cobb pose questions about the relevancy of word problem research. Do the outcomes really mirror which strategy choices students consider in their heads? Or are the outcomes representative for a style of maths teaching where students learn mainly routine procedures for certain types of problems? Koeno Gravemeijer wondered how it is possible that number size seems not to influence the choice of strategies. Maybe because there has not been a whole-class discussion about several alternative approaches or solutions to a problem? Students could become more aware of the distinction between the semantic context structure and the syntactic number characteristics of a problem, and their role in the solution process. Ernest van Lieshout and Meindert Beishuizen add that such metacognitive pedagogy possibly could be built in a teaching program. For instance a program with the empty number line, which model is very suitable for drawing, demonstrating and discussing – by students themselves – various solution methods on the blackboard. Lieven Verschaffel agrees, but is, on the other hand, a bit concerned about the complexity of this strategy choice process. Are we really suggesting that in our teaching efforts we should emphasize so much this strategy choice process, so that our students may become deliberate strategy choosers taking into account all these task variables we discussed? Koeno Gravemeijer agrees that this is not a realistic expectation, but nevertheless introducing more whole-class discussions about strategy choice in relation to number and problem characteristics may help. Moreover, such interactive discussions are an important principle of RME (Realistic Mathematics Education) because students' informal strategies should be stimulated, and students should learn and get ideas from each other.

Paul Cobb comes back to the fundamental question about the representativeness of word problem research. He is wondering if the emphasis is not too much on the nature of word problems. Would it not be more realistic to take into account the typical 'instructional history' and typical 'classroom culture' of students in a given situation, which will influence their solution process? Paul Cobb thinks that from such a micro-analysis you get different patterns in different classrooms. Lieven Verschaffel points, however, to the fact that in so many countries with different students word problem research comes up with very similar results. In spite of these divergent cultures the recurrent similar trends in word problem research give evidence for the apparently strong influences of task variables as we discussed here. For Paul Cobb, however, those similar trends across countries and schools are symptoms of the stereotyped character not only of word problems but of many maths tasks and most classroom instruction. Let us change that type of instruction, is therefore Paul Cobb's concluding remark in this discussion.

3 Discussion about the papers of Meindert Beishuizen, Karen Fuson and Jens Lorenz

The discussion about these three papers first circles around the question whether linear images or dot configurations are a more natural representation of number. Young children mostly use dots because they have more experience with such quantity models.

According to Piaget this is an important concept for understanding the meaning of number, but many mathematicians (cf. Freudenthal) would pay more attention to the number line as a sequential counting model, including also number relations. In this context some doubt is expressed about Karen Fuson's hypothesis in her paper (this volume) that Dutch and German number words may elicit another computation procedure than English number words. Both Meindert Beishuizen and Christoph Selter think that the discussed differences between the quantity and linear models are more influential. They see a parallel in the diverging teaching and textbook practice, emphasizing in the U.S. calculation procedures with split-up two-digit numbers, and in Europe mental computation with whole numbers up to 100.

At this point some misunderstandings (and clarifications) enter the discussion. Both Tom Carpenter and Karen Fuson underline that maths teaching in their research projects (as well as in that of Paul Cobb) are atypical for the U.S. Indeed, early introduction of two-digit or column arithmetic is common practice in U.S. schools, which brings along the danger of 'concatenated' manipulation of isolated numbers. However, in the mentioned U.S. projects there is much emphasis on mental computation, as can be seen in the examples given in the papers of Tom Carpenter and Karen Fuson & Steven Smith (this volume). In the discussion Karen Fuson agrees that N10 appears to be a more efficient computation procedure than 1010, because intermediate outcomes can be hauled along (N10) and need not to be kept in working memory (1010). But Karen Fuson thinks this is true for experienced calculators and adults. For children, however, such mental strategies are more demanding to learn. For instance she foresees problems with positioning numbers correctly on a number line (length or point?) and she would therefore prefer a line of dots instead of an empty number line. From a wider point of view Karen Fuson would argue that instruction in mental strategies should come after practice with written strategies, which arguments are also given in the last pages of her paper (this volume).

Apparently, in the discussion, mutual misunderstandings of 'mental' and 'written' arithmetic are now playing a role. For instance Karen Fuson is saying that in her view students in Holland do not learn mental strategies first, because they start with a lot of written activities on the empty number line to build up N10 as a mental strategy. The Dutch experts object because in their view written jottings are allowed (as support) when doing mental computation (and do not transform the mental strategy into a written method). The Germans suggest that the labels mental and written might not express the right antithesis. They use in their country the label 'halbschriftlich' (half-written) as a sort of intermediate or transitional state in between. Many experts agree that the real (procedural) antithesis is the difference between mental strategies and columnwise algorithms. Mental strategies are mostly less standard and more varied, while (written) column algorithms are predominantly standard calculation procedures. Koeno Gravemeijer makes the remark that there is a language problem. The Dutch word 'hoofdrekenen' translates into 'mental arithmetic', but 'hoofdrekenen' includes also flexible mental strategies while 'mental arithmetic' has a more traditional connotation and a strong association with 'mental recall'. Karen Fuson suggests that the Dutch should translate their concept 'hoofd-rekenen' into 'flexible strategies' or 'strategies adapted to task characteristics'. In particular the descriptive expression of Treffers 'using your head strategies' she likes a lot.

Jens Lorenz draws the conclusion that at this moment the discussion is coming close to the 'cognitive map' interpretation in his paper (this volume), which values imagination and variety more than categorization of procedures and strategies. Elsbeth Stern, however, does not think that all students are making strategy choices or adapting to task/ number characteristics when doing calculations. Many of them are sticking to one solution method because of routine or limited (pre)knowledge. Therefore, she supports the idea to distinguish between procedure and strategy to get a better description or interpretation of solution processes going on in the classroom. Jens Lorenz agrees that there is an interesting description of the development of the strategy/procedure AOT/A10 in Meindert Beishuizen's chapter (this volume). Meindert Beishuizen adds that it is sometimes overlooked in discussions about N10 versus 1010, that today's Dutch textbooks are putting emphasis on both N10 and 1010 to increase the level of mental flexibility. This applies not only to the Leiden number line program but also to the new 'Wis & Reken' textbook presented the previous day. For this didactic purpose a greater distinction is made between the linear or sequential (N10) representation and the quantity or place-value (1010) representation of number and number operation. He refers to Koeno Gravemeijer's characterization of the differences between the linear model (empty number line) and the set type model (blocks or money) in his JRME-article (1994). In this context Meindert Beishuizen makes a critical remark about Karen Fuson's conceptual structures (fig. 2 in Karen Fuson & Steven Smith, this volume), using similar block configurations (tens and ones) as models for both the 'sequence-tens' and 'separate-tens' concepts. In his opinion these two blocks models do not make much representational difference, in particular not for the sequence concept. In his view you better first underline the differences in modeling, in order to achieve in a later learning stage a higher level of 'integrated use' of N10 and 1010. Karen Fuson & Steven Smith mention in the last pages of their paper such a higher level of 'an integrated-tens and ones conception that relates sequence-tens and separate-tens', as an example of 'vertical mathematization' with reference to Koeno Gravemeijer (this volume).

4 General discussion (including the audience) on Saturday the 14th December 1996

Julia Anghileri wants to come back to the discussion about N10 versus 1010 but now from the wider perspective of long term development of strategies. From her own research she knows that for multiplication many students use N10-like counting-up strategies, but for larger multi-digit multiplication 1010-like strategies are more efficient. And a bit further in the curriculum when students work on division problems, many of them revert back to N10-like repeated subtraction strategies. So, is it wise to concentrate mainly on N10 in the lower grades as the more powerful strategy? Meindert Beishuizen answers that apart from N10 and 1010, students develop other strategies like 10s and A10.

In his paper the development of A10 but also the change from 1010 to N10 is described from a long-term perspective. Some students like Eddy (figure 1 and 5 in Meindert Beishuizen, this volume) show an 'integrated' level of strategy use. Julia Anghileri agrees that making such connections and progressing to a flexible level of strategy use are important key factors to success in mathematics.

Karen Fuson then would shift a bit in the discussion from the two specific strategies N10 and 1010 to the underlying conceptual structures. We want children to have understanding of both the sequence structure and the quantity (decomposition) structure. Tom Carpenter is not so sure that it is clear what drives these strategies. Moreover, he thinks there is overlap between the 'sequence' and 'collected' (quantity) notion, for instance with multiplication and division were you get both of them operating as part of the solution. Tom Carpenter reminds us of a footnote in his paper (this volume) mentioning that in U.S. (CGI) school practice, it often is difficult to distinguish between N10 and 10s, for students slip back and forth between the two. For instance they seem to solve 35 + 40 = 75 by using N10, but then they will say: Well I know that 30 + 40 = 70, so + 5 the answer is 75. According to Meindert Beishuizen one sees these solutions (in Holland) in the third grade as a symptom of integrated strategy use, but according to Tom Carpenter U.S. (CGI) students demonstrate already in the first grade these integrated strategies.

Koeno Gravemeijer comes in with the remark that the linear and set concept may be too limited from the viewpoint of long-term perspective including multiplication. There are many other ways of seeing numbers as Jens Lorenz (this volume) argued in his paper. For instance when solving the subtraction 72 – 38, Koeno Gravemeijer would think of 72 as 2 × 36 and would take 2 more off (72 – 38) to get the answer 34. In the case of just the number 72 Koeno Gravemeijer would ask students not only the split into 70 and 2 (tens and units) but also to make a link with the closeby landmark of 75 (as three quarters of a hundred). So, he would like to involve broader conceptions of number structure than only the linear and set type. Meindert Beishuizen and Karen Fuson, however, see 1010 and N10 as more fundamental or general dimensions of number concept in that they apply to all numbers. Ian Thompson solves this little dispute by his remark that it is a matter of investment. In the early number curriculum you have to invest in two things: number and strategies or tools. If you invest in number sense and take your time for it, you get out a rich use of good tools. Of course one can throw away a tool a kid cannot handle, but it might be a better investment to learn the kid to handle that tool.

Jens Lorenz brings up another problem: the differences in language between English and German or Dutch for describing what students think; the different terms we use as for instance when we speak of 'digits'. In (American) English it is common to use expressions like two-digit or three-digit number problems while in Germany we never use these terms. You quite often speak of numbers in the domain from 1 through 100 respectively from 1 through 1000. Speaking of digits seems to have a different connotation emphasizing column arithmetic. Karen Fuson answers that these terms are not necessarily used with students and that they just give a description of the number size in problems, without the association of column arithmetic. Julia Anghileri throws in that in the U.K. teachers speak mostly of 'tens and units' and that, indeed, these terms T and U are used as column headings (between lines) for doing the vertical algorithms. Tom Carpenter agrees that this is also common practice in the U.S. and he agrees with Jens Lorenz that one may wonder if this implies some underlying (different) conception of number.

Karen Fuson too agrees that there might be some cultural differences. She admits having learned from this conference as well as from inspection of textbooks and from observation of classrooms in Holland that in comparison to the U.S. there is a more free disposition to do these things like building individual notions. Her own experience is that weaker students do not build up quantity meanings for multi-digit numbers and that they do not use those meanings when doing written arithmetic. She feels this as a major problem, so one has to do anything sensible to help them understand those quantitative concepts of number. The sort of flexible things and individual meanings, as discussed here, are indeed nice and important. But to Karen Fuson they are a sort of luxury. Like the early maths curriculum in Germany and Holland, where this long time of two years (all of 2nd grade and a lot of 3rd grade) is spent for developing all this very flexible knowledge about two-digit numbers. And then the written algorithms are introduced suddenly and there seems to be no attempt to make any meaning for these new three- and four-digit numbers? For Karen Fuson this instructional sequence is difficult to understand, because attaching meaning to multi-digit numbers is a major problem in the U.S. system. She would rather try to do both, but that is also an issue in terms of cost and time, a matter of investment as Ian Thompson said earlier in the discussion. Frans Moerlands gives some examples from his experience as author of the textbook 'Wis & Reken': how students can experience number structure in different ways by presenting varied context problems and representation models. For instance more open structures like 100 as 4 × 25, using bundles or cartons one sees in a supermarket. Always using the tens-and-units structure could make the instructional approach rather rigid. Karen Fuson is not sure of the latter conclusion because more evidence is needed. Many of us are trying to work from different approaches to bring more teachers and more students along to real understanding. And until we do more of that better, we are not able to answer that question whether or not a certain approach is more rigid.

Marja van den Heuvel-Panhuizen wants to come back to the discussion about N10 and 1010, in particular to what Julia Anghileri said about the further perspective with regard to multiplication and division. Marja van den Heuvel thinks you could broaden the distinction to mental calculation in general and from that perspective N10 is the more typical mental strategy while 1010 is more related to the tens-and-units written algorithms. So, by consequence, if we want to emphasize mental calculation N10 is more important than 1010. Tom Carpenter asks surprised why that is so? He understands the relationship between 1010 and vertical algorithms. But why should 1010 not also be a flexible alternative for mental calculation? According to Marja van den Heuvel that gives 1010 a different function as an alternative way of structuring numbers in bundles of 25. Ernest van Lieshout intervenes with the remark that the distinction between mental and column arithmetic is quite another thing, which means dealing with numbers as whole entities or numbers as split-up digits. However, as Ian Thompson comments, the 1010 procedure is not synonymous with written algorithms because 1010 is also treating numbers as entities. Marja van den Heuvel agrees, but it is in her opinion a small step from (mental) computation with 1010 to (mental) operating with only digits. Both Koeno Gravemeijer and Karen Fuson object that we should not turn the argument around and say that, because of the possible abuse, 1010 is a meaningless and rote computation procedure that should not be used. According to Karen Fuson it would cut off roots for students and teachers that could be very productive.

Karen Fuson agrees that both 1010 and N10 are important and that we all are wrestling with questions like the best conceptual support and how to lead students eventually to integrated and flexible use of various strategies. Julia Anghileri is concerned this claim might be too idealistic, because in school practice many students get stuck on 1010 and some others on N10 as the one and only procedure. Successful students can take into account all the things we were discussing earlier about semantics of problems and number size and the most appropriate way of solution. That is why mathematics becomes so easy for some students and so difficult for others. Julia Anghileri thinks it is crucial not only to develop strategies but to make connections between strategies. In her opinion we do not get that right yet in teaching arithmetic.

Julia Anghileri's last remark causes Meindert Beishuizen to return to the earlier discussed role of instructional development and long term (supporting) learning trajectories, as well as the role of textbooks. Some experts expressed quite different views on that topic. But with respect to N10 and 1010 the authors of the new realistic textbook 'Wis & Reken' emphasize an orientation on both strategies and on connections between the two. Frans Moerlands underlines that you have to pay attention to both N10 and 1010. If a student is using one of those wrongly for instance in subtraction, you have to reflect with the student on the error and not simply throw away what does not work. Kees Buys emphasizes as editor of 'Wis & Reken' that sequencing of N10 and 1010 was an important part of the developmental work. In the new textbook a lot is invested in number images and number structures first. Counting-on or counting-back are very basic strategies used by most students. Many students are inclined to go on with the jump method (N10) but in quite a primitive way. So, in terms of instructional support one has to abbreviate these

10-jumps (23, 33, 43, etc. and backwards) in order to make this method more economical and flexible (larger jumps). Thereafter when they have a more sophisticated knowledge of N10, and because of the rich conceptual base of number structure, the split method (1010) emerges in quite a natural way. Halfway the 2nd grade most students in the Wis & Reken try-outs, used N10, but in the 3rd grade they changed more to 1010. At the end of the 3rd grade there is a development to integrated use: in many solutions you can not make the distinction between N10 and 1010 anymore. Sequencing and split methods are then mixed up in a flexible way. In the beginning of the 4th grade when column algorithms are introduced the students understand and acquire these new procedures very quickly. Moreover, they still can think and discuss about alternative strategies in whole-class discussions. For instance 478 + 266 can be solved by column arithmetic but also by applying a mental strategy (478 + 200 = 678 + etc.).

Karen Fuson appreciates very much these descriptions of examples from try-out practice. She can imagine that in this cultural context the 'sequence' method is more natural for students, while the 'separate' method is more complex and laid on top. This, in her opinion, is also a result of the Dutch instructional sequence. This approach could be used in the U.S., but only for students with experience, because in some classrooms many of them are not yet able to count up to 100 in the 2nd grade. On the other hand, multi-digit numbers are introduced early with conceptual support, so these students can immediately engage in meaningful activities. They can use the tens-and-ones language quite early, without being able to count. In Karen Fuson's Children's Math World-project many students speak Spanish and there is the great confusing of number words for sixty and seventy ('sesenta and setenta'). In the classroom we can avoid this problem by having the students using language like 4 tens plus 3 tens is 7 tens. At least they can build their tens-and-ones conceptual structure, while at the same time they are trying to sort out their sequence structure. If you wait till students are good at their counting list first, and then build these 10-jumps on top of that, then it would be too much delayed for us. Karen Fuson concludes by saying that she just wanted to make clear the different cultural situations. So, we have to be careful with inferences about which are the most 'natural' solution methods of our students. Karen Fuson thinks that solution methods with smaller one-digit numbers are much less culturally varied, but once you start with multi-digit numbers you are getting into a bunch of different issues.

Ian Thompson argues that because in the U.K. too written algorithms are introduced very early, he is making a plea for 1010-like mental calculation first. The procedure 1010 is more similar to written algorithms (than N10) so this instructional sequence may lead to a better transition and transfer towards column calculation. However, in Ian Thompson's proposal the standard algorithm has to be changed into going from left to right, because then you still deal with whole numbers (like in mental calculation). For instance in the earlier example of 478 + 266 this 'left to right' method would proceed as 400 + 200 = 600, 70 + 60 = 130, so 600 + 130 becomes 730, etc. This similar 'left to right' procedure could be carried out vertically in column arithmetic. It would make the written algorithm a bit longer but fits in well with the mental 1010 strategy, and will support understanding of number and number operations more than in the past. Meindert Beishuizen, Ernest van Lieshout and Gerard Seegers react with references to the Dutch situation, where one sees the transition from N10 to 1010 (and vice versa), the role of estimation strategies, whole-class discussions about choices between N10 or 1010 (or another method) for given problem characteristics. These last remarks in the general discussion on Saturday give a good illustration of Karen Fuson's earlier conclusion that 'multi-digit numbers are getting you into a bunch of different issues'.

5 Discussion about the papers of Hans van Luit & Bernadette van de Rijt and Elsbeth Stern

This discussion (on Sunday morning the 15th December) began with questions about relationships between (possible) predictor variables like intelligence and word problem solving, which is a central issue in the longitudinal studies summarized by Elsbeth Stern (this volume). She underlined that correlations with such a global measure are significant but low mostly. More specific indicators of mathematical competence like 'number conservation' and 'estimation of quantities' as tested in preschool turned out to be better predictors for mathematical performance, for instance in grade 2. Some experts consider it as quite obvious that specific pre-knowledge contributes to later learning, while others expressed doubts about the predictive importance of intelligence or even of correlations. Elsbeth Stern, however, points to a relative high stability of performance in word problem-solving during preschool time in her studies. This makes it sensible to look for predictive factors (competences) which are trainable in an early stage in order to prevent or reduce later learning difficulties. In particular this is relevant for weaker students. Compared to reading problems, where lack of automatization is crucial, Elsbeth Stern considers lack of conceptualisation as more crucial for mathematical competences. That is also what she values in the training program developed by Hans van Luit and Bernadette van de Rijt (this volume): the aim to support students at risk in an early stage of preschooling.

Others, however, like Koeno Gravemeijer are less sure: it is a rather quick move from this research to advice about education. Jens Lorenz still sees the problem of delineating the meaning and significance of the so-called specific competences. According to him it is obvious that previous knowledge contributes to later knowledge attainment. He also is convinced that for instance 'estimation' is a predictive factor for later mathematical performance. But is 'estimation' a cognitive or a mathematical factor? Elsbeth Stern agrees that we have to figure out more about specific aspects, but nevertheless she wants to go on with training of predictive factors which have proved to be significant in stable correlations. In particular when positive program effects encourage you to go on.

Tom Carpenter then brings the discussion back to the general theme of how much instructional guidance should be given. In his opinion there is a difference between the less directive and more context driven programs described on the first day of the conference, and the more directive training programs of this morning. He would like a broader discussion on this issue. In his research experience learning-disabled children come with the same strategies as other students and they can follow interactive teaching as well. Tom Carpenter disagrees with the Special Education viewpoint where the emphasis is too much on direct instruction. Hans van Luit comments that the degree of direct instruction depends on the type of learning problems students have. There is no black-and-white contrast between direct and less direct instruction. From the analysis of the ways the two experimental programs (cf. his paper, this volume) were carried out it appeared that the teachers adapted their instruction to the apparent need of the students. In particular in the 'guiding' condition there were changes depending on the task. Sometimes all students understood the message of the learning task and the interaction, but sometimes with new tasks more structured modeling was needed because the students did not understand and gave no reaction.

Here Tom Carpenter responds that there are other ways of adaptation like try and find out what a child does know and to try to build up from there. Apart from misconceptions there are always some points of departure in the child's particular cognitive knowledge base. According to Koeno Gravemeijer there is a big difference between competences you train (and test) in a task-specific situation and competences students develop independently by re-invention and interaction. A difference between just following instructions or putting in and elaborating on own interpretations. The chosen approach depends also on what you think mathematics education should be. Paul Cobb joins in with the remark that as a constructivist he has difficulties with both options: giving guidance or direct instruction. He would prefer the approach Tom Carpenter mentioned of building up from what kids can do, or as it was stated the other day: pro-actively supporting the learning processes as constructive activities. Paul Cobb wonders whether the (artificial) black-and-white contrast is related to the connection between research and instructional conditions. The pure treatment conditions go back to the 1960s when U.S. research introduced experimental comparisons between discovery and expository instruction, and ATI-questions such as which one was better for which task or type of student. The current approach of developing instruction by analyzing what goes on, as it happens in the teaching/learning situation is quite different. Paul Cobb would like to raise this issue of the relation between research and instruction (instructional development) for the general discussion. He too – like Tom Carpenter – thinks that during these two days of conference, there have been two quite different conversations.

Karen Fuson relates the distinction to the type of task. For instance some children in her project have great difficulties with understanding word problems of the comparison type, because they lack the meaning of words like 'more' and 'less' as a frame of reference. As a teacher you have to go back to an instructional situation where you start with 'equal' amounts, and then you add 'more' so the extra amount is perceptible to them. The thing is to get the children started from where they are (like Tom Carpenter said): they have counting abilities, all right, but they lack many frames of reference for context and meaning, so you have to do a bit modeling as in the example given. The word direct instruction can have different meanings. Even in the direct instruction situation there is some adaptation to children's level of understanding and there is some space for feedback. According to Karen Fuson many people in the U.S. take direct instruction as a sort of training comparable to a 'fast train' that just runs over the children. Training is the fast train, and normal instruction is the slow train. She herself is always writing in terms of teaching/learning activities because she does not like the strong distinction in meaning between teaching and learning in the English language. In European languages like German the difference between 'lehren' and 'lernen' is not so strong and in some languages (e.g., Norwegian) the same word is used for both. She liked hearing many speakers during this conference saying: 'the teacher had to learn the children ..' In every teaching-learning interchange the roles are also reverse: the teacher is also a learner because s/he tries to understand about the person being taught, and students have to teach the teacher about their own knowledge. Karen Fuson thinks it would be helpful to refine the language we use for describing these different instructional situations a bit.

Jens Lorenz agrees that there are a lot of meanings for the word instruction. Like for instance in Germany where there is now a discussion going on about open instruction, and we seem to have twice as many open instructions as we have teachers! But it is also important that you see the philosophy behind the ideas, like there is a certain philosophy behind Paul Cobb's constructivist point of view. In connection to this remark Christoph Selter has some critical questions about the lack of specificity in the paper of Hans van Luit and Bernadette van de Rijt (this volume). Referring to a statement of Freudenthal that the proof of a theory is in its examples he badly missed examples of the two types of instruction and of the test in this paper. Now many sentences are difficult to understand. The problem is – also in the discussion of this morning – that as researchers we share some vocabulary on a general level, but we also know there are many different connotations and interpretations. Therefore we need, according to Christoph Selter, much more examples of given instructions and of students' work in order to be able to discuss general issues in a more productive way. Hans Van Luit agrees that he could have given more examples to make things clear, and he promises to do so now during the coffee break ...

6 Discussion about the papers of Koeno Gravemeijer & Gerard Seegers and Paul Cobb

Gerard Seegers pointed in his paper presentation to the fact that Freudenthal's principle of 'guided re-invention' includes a tension between guidance as a cognitivist aspect and invention as a realistic aspect. Now, he perceives another distinction compared to the viewpoint of Paul Cobb in his paper. The Realistic (RME) theory is, in the first place, a didactic theory with didactic claims about models and learning trajectories to be tested empirically. The Constructivist theory, however, has not such claims but has a different goal i.e. is interested in 'how it works'. Paul Cobb agrees that trying to understand is central in his focus. But he is working in classrooms and is also interested in changing things. He is not happy with the usual type of comparative research between instructional conditions, because such claims as well as effects are too general. Teachers need better tools like instructional sequences, not to follow step by step but as a guideline for students' mathematical development. According to Koeno Gravemeijer the issue is how to develop the optimum form of realistic mathematics education (RME) that teachers can handle in the classroom. How far can you go in adapting your principles to reality without losing the intent? For such a 'pragmatic' approach you need to do developmental research with teachers in classrooms. But there is a problem with this new type of research: it is difficult to get it funded in The Netherlands when the common criteria are applied to research proposals.

Lieven Verschaffel has a question about previous research into RME in The Netherlands, where it turned out that teachers did not follow the realistic approach in their teaching in the classrooms. Could you define this as a sort of pragmatic realistic approach of the teachers? Because it could be that the teachers have fully understood and tried out the realistic approach, but that they then as a result of their reflective thinking, came up with some kind of compromise between reality and the ideal situation? Koeno Gravemeijer reacts that this was not his impression from the data in the protocols. What he observed was the problem of the social maths norms. The teachers, although working with a realistic textbook and realistic guidelines, still went on valuing answers of students in a traditional way as right or wrong without looking for students' thinking. A Dutch problem is that now many (revised) realistic textbooks are in use in schools, but that in-service- training of present teachers lags behind. Sometimes they revert to traditional types of instruction and practice. According to Koeno Gravemeijer teacher consultation with feedback or teachers observing and discussing each other's lessons might help to start analyzing what students are thinking. Such a reflective process might develop towards interactive teaching and towards a really 'realistic' approach.

Karen Fuson comments that most teachers being rated on an observation scale in one of her projects, really moved towards an instruction style not imposing things, towards renegotiating social norms, towards having students explaining their thinking and being independent. The typical U.S. problem was that the tasks and the curriculum did not give them enough support. The teachers did not know about different solution strategies to problems and they had difficulties with organizing productive mathematical discussions in their classroom. Therefore, Karen Fuson proposes that internationally we share and discuss more whatever knowledge and experience we have about the innovation of mathematics teaching. She then puts a question to Paul Cobb about the instructional sequence dealing with measurement as described in his paper (this volume). For designing such a learning trajectory you need to know about students' thinking, so what was the hypothesis in this case? Paul Cobb answers that he did not describe the ideas for this 'measurement' learning trajectory in his paper. Moreover, these ideas developed further during experimental lessons and (audiotaped) reflective talks immediately after each lesson, which have not been analyzed yet. But he underlines that you cannot separate students' thinking from the instructional situation. And students' interpretations are important for the development of the instructional sequence. As an example Paul Cobb goes back to what he told about the 'arithmetic rack' the other day. On the first day of the conference the authors of the new realistic textbook 'Wis & Reken' demonstrated how students' activities using finger patterns for numbers up to 10, preceded the introduction of the arithmetic rack. To see this instructional sequence was a pleasant surprise for Paul Cobb, because that was precisely the problem when introducing the arithmetic rack in his U.S. experimental class. At first the model did not work because the students did not recognize the framework. You have to prepare the ground by pro-active support like the re-invention of finger patterns by the students. So, by observing and analyzing students' interpretations you can improve the instructional sequence, which is a central characteristic of developmental research.

7 General discussion (including the audience) on Sunday the 15th December 1996

A first theme in the discussion is the role of realistic problems and contexts. Tom Carpenter suggests that younger students tend to be more successful with realistic problems because they are still working in the reality of the situation. Erik De Corte agrees that younger students probably are less vulnerable to what you might call 'misbeliefs', because they have not yet been subjected to a kind of mathematics teaching that does not pay attention to real world knowledge. Many teachers accept answers to maths problems as (formally) correct although they are wrong (impossible) from a realistic point of view. Such classroom culture can indeed push the students in a direction of avoiding or neglecting real world knowledge as a result of traditional mathematics teaching. According to Lieven Verschaffel this question is a complex issue, because there always will be a gap between solving a mathematical problem in a real life situation outside school and solving context problems in a mathematical classroom lesson. Of course this has to do with socio-math norms, but how aware must we make our students of this problematic tension between reality and mathematics? On the other hand, some people will say that the very essence of mathematics is in abstracting, even in neglecting in a mindful way certain aspects of reality.

Karen Fuson suggests that such discrepancies could be solved by applying (as a teacher) the so-called 'if discourse', and by stripping down the mathematizing. But in the experience of Lieven Verschaffel this can lead to endless discussions using 'ifs' all the time. It then becomes more and more extreme as a sort of game, and moves the teaching/learning situation away from what is considered as valuable mathematical modeling. According to Lieven Verschaffel we really have not solved this problem. Paul Cobb agrees that there is a problem, but also that there ought to be a difference between a problem in the math classroom and in the real world. Sometimes, if it comes up, he discusses this with students. Sometimes an instructional situation is chosen with a didactical eye to lead the students to be aware of such a difference. Tom Carpenter adds that in reality it often happens that experts have to solve problems which have been abstracted from the context. They then have to negotiate over meaning as well. In mathematics teaching you get to a point where you can not go on with realistic problems, when it comes to abstract calculus or algebra, etc. The ability to deal with that kind of abstractions is a goal of mathematics too. Koeno Gravemeijer does not agree. He would not separate mathematics that much from the real world. He prefers the notion of 'experientially real' and he thinks part of the everyday world may not be experientially real for a student, while on the other hand mathematics itself can become experientially real for a student. Ernest van Lieshout and Hans van Luit mention examples from research, where students reacted not realisticly in a classroom situation, although they knew these problems from reality.

Koeno Gravemeijer and Paul Cobb immediately add that it is a matter of different expectations or different socio-math norms. In a given situation a person reacts as he is supposed to do. Marja van den Heuvel, however, comments that it also depends on the kind of problems and the way they are presented to students. For instance in the case of problems such as people to be transported by buses or balloons to be divided among children, it is rather unrealistic to come up with answers including a remainder or a decimal. Koeno Gravemeijer has an example the other way round with a problem like at a party, where there were 24 bottles of coke for 36 people. He remembers students reacting: 'Some people do not drink coke!' So, these students were not willing to solve the problem the way you want it to be solved (by proportional reasoning). Here Lieven Verschaffel cuts in with the remark that this latter example exactly illustrates the point he wanted to make earlier. At certain moments in the teaching/learning process you can appreciate such comments from students. However, in a next lesson you want to model multiplication or division as such and then you do not like such comments. How can we make this distinction clear to the students? In addition Tom Carpenter remarks that what we want students to do is: to examine the assumptions and to negotiate about the meaning of a problem situation. Sometimes, it is hard indeed to convince students what the rules of the game are in a given situation. The notion of shifting between sort of artificial solutions to ones that are more realistic.

Julia Anghileri has another comment with respect to realistic problems. The power of mathematics is in the pattern. One of the difficulties in giving real problems is that these do not exhibit the pattern. When young students start with problems embedded in contexts then at some point these models have got to expose the mathematical patterns, the relations between numbers. She reverts back to an example in the discussion the other day about the number 72, which you can see as composed of 70 and 2 or 60 and 12 (tens and units), but also as 2 × 36 (doubling) or as close by 75 (three quarters of a hundred). Seeing such rich patterns is what gives children the power to do mathematics. According to Julia Anghileri we should not discuss too much how students solve a particular problem, but how they make connections between (different) solution patterns. In her experience students stick too much to their own (idiosyncratic) understanding and to their own strategy or procedure. Many students have difficulties with understanding another strategy and with seeing links between the other and their own strategy. So, our focus should not be only on what is the best model for a problem, but also on how to get students to see and to make these connections and patterns.

Paul Cobb makes the remark that for him it makes a big difference whether you look at the pattern in a task as we see it, or whether you try to anticipate how kids might interpret a task. He thinks that also in the RME-view the source for instructional design is not the problem per se but the problem in relation to the child's interpretation. So, it is important to look at how problems or materials are used rather than what patterns we want to get out of it. Karen Fuson reacts that she had already started a conversation on this matter with Paul Cobb, because the other day he made the inference that she in her classification of conceptual structures (fig. 2 in Karen Fuson & Steven Smith, this volume) was not necessarily thinking of students' interpretations. To Karen Fuson this is a foreground-background problem, but also a communication problem because we sometimes get confused by our different use of the same terminology. We need a language that differentiates between description and analysis on different levels: the level of students' thinking and the level of instructional sequence design. Jens Lorenz then makes the remark that we also need a language in which students can communicate about their strategies. In his experience the explanations of students can be rather unclear. According to Koeno Gravemeijer such a language develops in a natural way along maths practices in the classroom. When certain things and procedures get accepted in the group there is no need for explanation anymore. Karen Fuson agrees that it is very important that students are discussing things in classroom. The teacher could assist by writing things as a referent on the blackboard for helping all students to clarify explanations.

In his research Paul Cobb is doing a lot of analysis of students' spontaneous communication during maths practices, which he calls 'reflective discourse' in a new JRME-article (1997). He mentions partitioning of small quantities as an example, for instance the different ways 6 monkeys can be sitting in 2 trees. When kids come with various answers the recording by the teacher is important. If somebody is saying 'I think we have found them all', the teacher can stimulate justification by asking: 'how can we know for sure?' Then some child might come up with the idea of a pattern: 6 and 0, 5 and 1, etc. They look through the table on the blackboard and go back to the original problem situation for checking. The pattern emerges out of the children's activities. Now it is the result of the math activity that is being organised or structured. You get a gradual shift from talking about what they are doing to the results which then become the subject of conversation. Karen Fuson adds that it not only becomes an object of discussion, it becomes an object of symbolization. And the 'monkeys' (context) model fulfills a bridge to the number symbols. If the children understand and can manipulate the 'monkeys' situation, they are ready for the transition to a number table. Paul Cobb agrees that symbolizing (into number symbols) is critical here. But it also an example of Koeno Gravemeijer's transition from 'model of' towards 'model for'.

Now the discussion comes to the role of 'numberwalls' in the German maths program published by the Dortmund University Mathematics Institute. Gerard Seegers is suggesting that one should start with authentic situations like the 'monkeys' and then as a second step go on to less authentic problems like the 'numberwalls' eliciting the analysis of patterns. Paul Cobb answers that from a constructivist point of view he would not use such particular rules as a basis for decision. It would depend on the students: not for first-graders but for teacher-students numberwalls could be a starting point. Christoph Selter comments that he can imagine the position of other experts assuming that numberwalls cannot be experientially real to first-graders. However, his experience is different and his position is that such assumptions should be figured out in empirical teaching experiments. Numberwalls can be experientially real to students. An important reason for the Dortmund group to use them are higher-order goals like giving arguments, explaining and justifying patterns, etc. The numberwall problems are suited to elicit these higher-order goals. His impression of RME problems, which he generally appreciates very much, is that the non-real-life ones address more procedural arithmetic skills like addition and subtraction than recognition of patterns. His observation is, for example, based on some examples given in the presentation of the realistic textbook 'Wis & Reken' during the conference.

Here, Elsbeth Stern comes in with the comment that what is typical of mathematics is that one can do things with mathematical symbols which one cannot do with reality. In her view mathematics is also a way to develop concepts that could not be developed otherwise. She illustrates this with some examples: we would not have a concept of end-lessness if we had no numbers, not a concept of intensive quantities such as speed if we had no numbers. Only, because one can do things with numbers which cannot be done with reality, one can develop such concepts. And that is what students learn by doing mathematics. They know that you cannot 'divide people' for example, but in the world of mathematics you can. What may be the result of realistic problems after a while is that students are aware of the fact that what can be done with mathematics is not always what can be done in reality. Sometimes you can do things in mathematics to extend your view of reality. Elsbeth Stern gives an example of an experiment with fourth graders which she called the 'paradox problem'. The students were asked: 'You have a birthday party and you serve orange juice. The first child comes for a glass, but you are afraid that you do not have enough orange juice for all children. Imagine that every next child will get only half of the glass of orange juice of the child that you served before. Will the orange juice in the end disappear?' In the experiment, until grade 6 the students answered in a realistic way that the juice would disappear. From grade 6 on the students gave differentiated answers. They said things like: from a realistic viewpoint it does not make sense because it will be so less that you cannot drink it anymore. But with numbers it is different because numbers can not disappear although they will become smaller and smaller. According to Elsbeth Stern this may be a bridge to understanding the nature of mathematics. The nature of mathematics is that it is a bridge between an abstract thing and reality, and that is what students have to become aware of.

Jens Lorenz responds that we may be again at the point where we attach different meanings to the concept of reality. Patterns indeed are not part of reality. We impose patterns on reality. It is a way to look at reality. Mathematics does not emerge out of reality. Not that just by looking at it will one see a pattern.

Paul Cobb thinks that these comments are very helpful and he gives a similar example from the 'candy shop' in his earlier teaching experiment with Koeno Gravemeijer. Quite a long time the children in the first grade classroom kept talking about candies (rolls of ten). But at a certain moment they are not the same candies anymore because they get mathematized. Then the candies signify quantity structures into units of ten. The reality, the real situation evolves itself through this process. For Paul Cobb it is a pragmatic decision – depending on the children and the situation – when you start to talk about patterns in terms of relationships between quantities or numbers. The same applies to things like numberwalls. If you have reason to believe kids can interpret them in terms of patterns and structures than you do. If not, then you have to build up to it. Of course you have to stimulate the transition to the awareness of number patterns through reflective discourse in the classroom. The Freudenthal Institute offers several design heuristics like the 'model of' – 'model for' transition, the arithmetic rack, the empty number line, which we found very helpful for putting together several activities into instructional sequences and for supporting the process of mathematizing.

Jens Lorenz reacts that you (and students) always think in terms of something: quantities, distances, measurements or whatever. But the inside number patterns are not so obvious. For instance you do not get the idea of 'Fibonacci' numbers by just looking at a sunflower – although they are there. So, from a certain point it is easier for students to study the numberwalls. You are looking for regularities within numbers and not within some reality. So, there seems to be a paradox. One constructs realistic situations to make something clear to a student, which would be more clear if you did it with numbers! According to Jens Lorenz some realistic problems bring you in unrealistic situations like dividing sandwiches by tables or people, which are ridiculous questions we would not solve ourselves. Paul Cobb reacts that it is crucial in what stage children are: do numbers immediately signify an experientially reality of numbers for them, then you can go on to the level of number patterns. But here you have to be careful too. For instance in the example of the 6 monkeys some kids will organise numbers only in patterns of doubling and halving (3 + 3), while others will grasp the idea of a wider system in table format (6 + 0, 0 + 6, 5 + 1, etc.). Ian Thompson interjects: do we not also want to get students to appreciate that not all mathematics has to be related to reality; that there are many people – mathematicians – who enjoy mathematics for its own sake? Mathematics as a collection of different ways and different tools to analyze complexities in reality, which we would teach them to use? Karen Fuson thinks this is true for older students, but for younger students numbers are not yet experientially real and that is the focus of this conference.

Koeno Gravemeijer wants to make a distinction between the concept 'realistic' in the RME-approach and 'realistic' in the sense of everyday reality. In his opinion this difference gets confounded all the time, and he admits: 'that, of course, is our fault by choosing this name'. The central RME-concept is that the starting point should be informal solution strategies. Working with young children you will use familiar situations which often will be real life context situations. Later the numbers and number relations itself will become experientially real. Then you can do the things from the Dortmund program and you can go even further and start doing algebra based on experientially real familiarity with numbers. So, it is just a matter of growth. At the same time, Koeno Gravemeijer thinks, the other argument for real life problems has to do with your goals of mathematics education. Do you want to develop a kind of pure mathematics or do you think it is more important to promote a kind of mathematical literacy. If the latter is your goal, you have to foster the relations with everyday life reality. Summarizing, Koeno Gravemeijer thinks there is not so much a difference in viewpoint with Christoph Selter in the starting points, but more in the long-term goals of mathematics teaching. Jens Lorenz and Ian Thompson both ask why RME does not stress both aspects? In Dutch realistic textbooks, they mainly have seen the second aspect of practical mathematics, but for instance not much investigative work related to the first aspect. Christoph Selter comments that in his opinion the distinction made by Koeno Gravemeijer is too suggestive: both aspects belong to mathematics and he also wants to stress them both. Lieven Verschaffel is surprised to hear about this difference between the RME and Dortmund approach. Koeno Gravemeijer is putting his remarks in a more relative perspective by saying that it is a matter of choice, a matter of goals more than a matter of didactics: 'it is a matter of how and when...'

At the end of the discussion, Julia Anghileri wants to come back to the role of the teacher. According to her, the teacher is there to expose the patterns and to explore the connections, not to teach the strategies. How teachers should do this using classroom discourse is in her opinion more important than discussing whether tasks should be more or less realistic. Koeno Gravemeijer reacts that this description of the role of a teacher sounds very much like the so-called Socratic discourse (questions and answers). He would prefer a greater role for the students starting with real life problems as described earlier. When Julia Anghileri asks what the role of the teacher is in this scenario, he refers to what Paul Cobb said about pro-active facilitating the learning process of students. The teacher also creates a classroom atmosphere with socio-math norms, where students can develop their own solutions. The teacher may bring in models like the empty number line at the moment this fits in the informal strategies of the students. A teacher also can bring in the mathematical conventions, after all kind of (informal) notations have been explored in the classroom. So, the role of the teacher is a mixture of bottom-up and top-down. Paul Cobb relates the question to the paper of Christoph Selter (this volume) about the development of teachers in a bottom-up way. He found the paper helpful, because in the U.S. there is an idea of what reform-teachers should be doing, but until now, not much thinking about how to build up such a teaching attitude has been done. The teacher has to create a classroom climate which is different from pure guidance leaving much to the students. It is important that the teacher clearly values certain types of answers more than others, so that the students get a sense of directionality. On the other hand, the teacher also has to create an encouraging atmosphere and opportunities for every student to participate at its own level. For instance in a first grade accept all the counting strategies given as solutions to a problem, but at the same time valuing more the grouping strategies given by some more advanced students. According to Karen Fuson that is also what Japanese teachers are doing a lot in their classrooms: highlighting or foregrounding some higher-level solutions and strategies. With this remark from an international perspective the discussion on the second day of the experts meeting is closed.