Speciale Relativiteitstheorie

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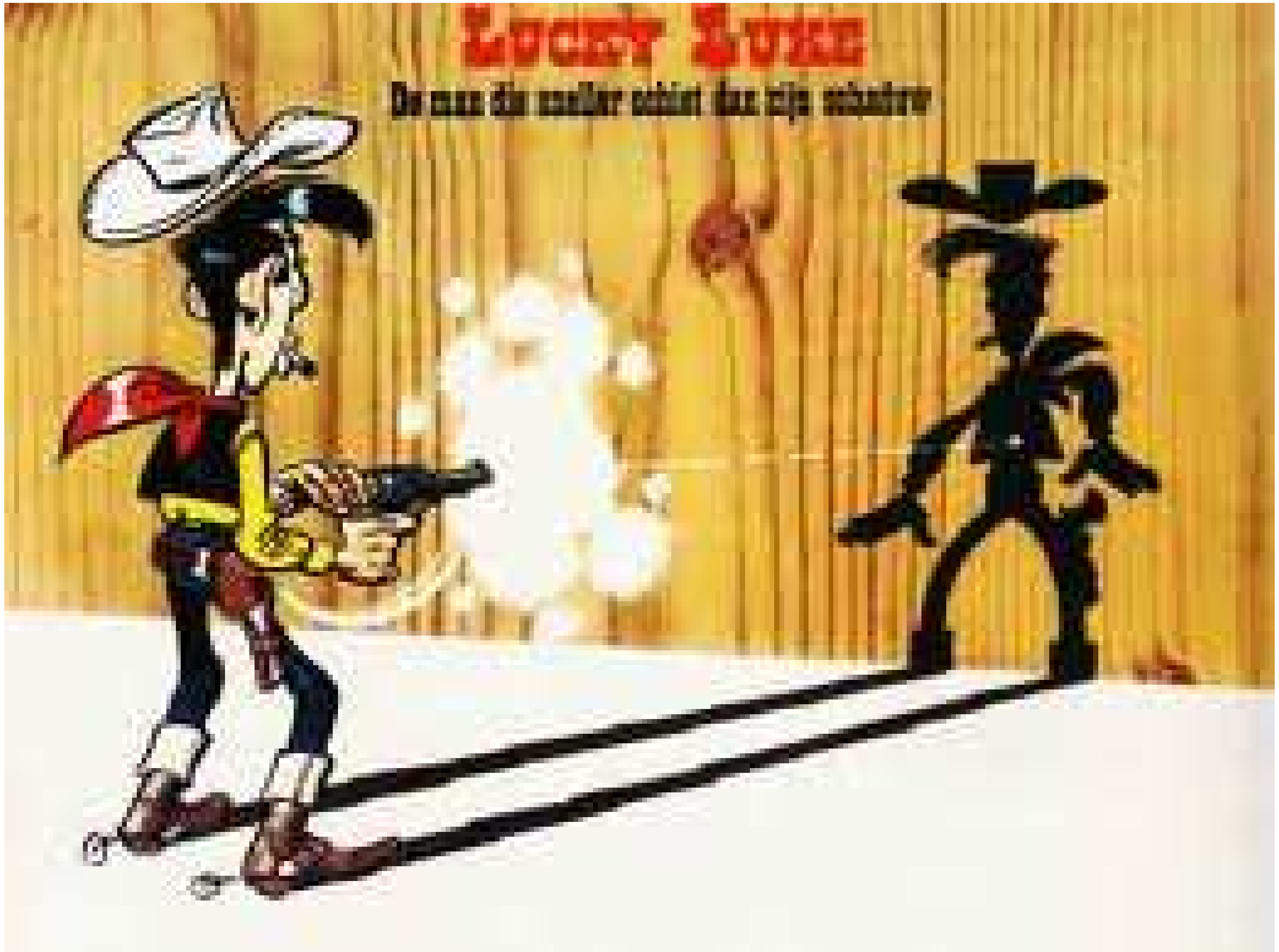
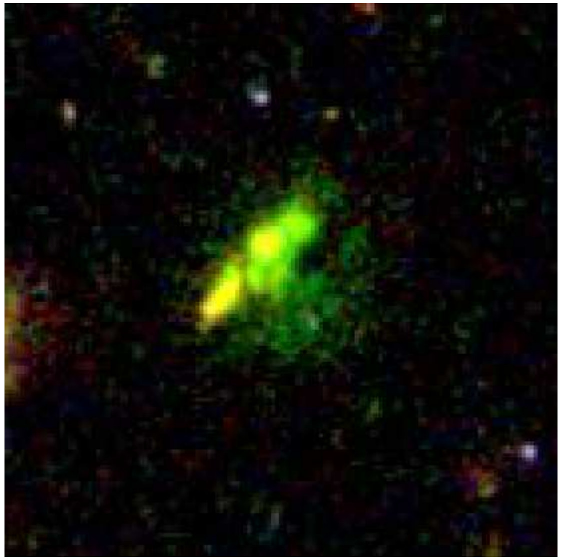


De man die sneller schiet dan zijn schaduw ….

…. de kogel is dus sneller dan het licht !!

Observatie aan ver verwijderde stelsels: roodverschuiving



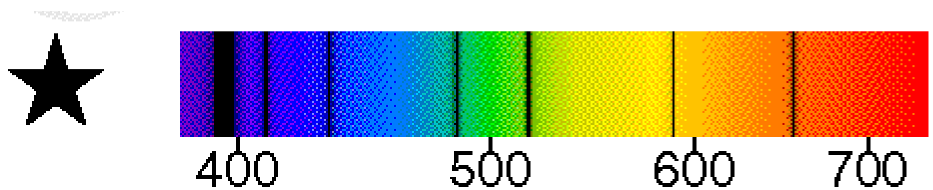
Baby Galaxy in Cluster TN J1338–1942
een radiostelsel op 13,5 miljard lichtjaar van de aarde

roodverschuiving

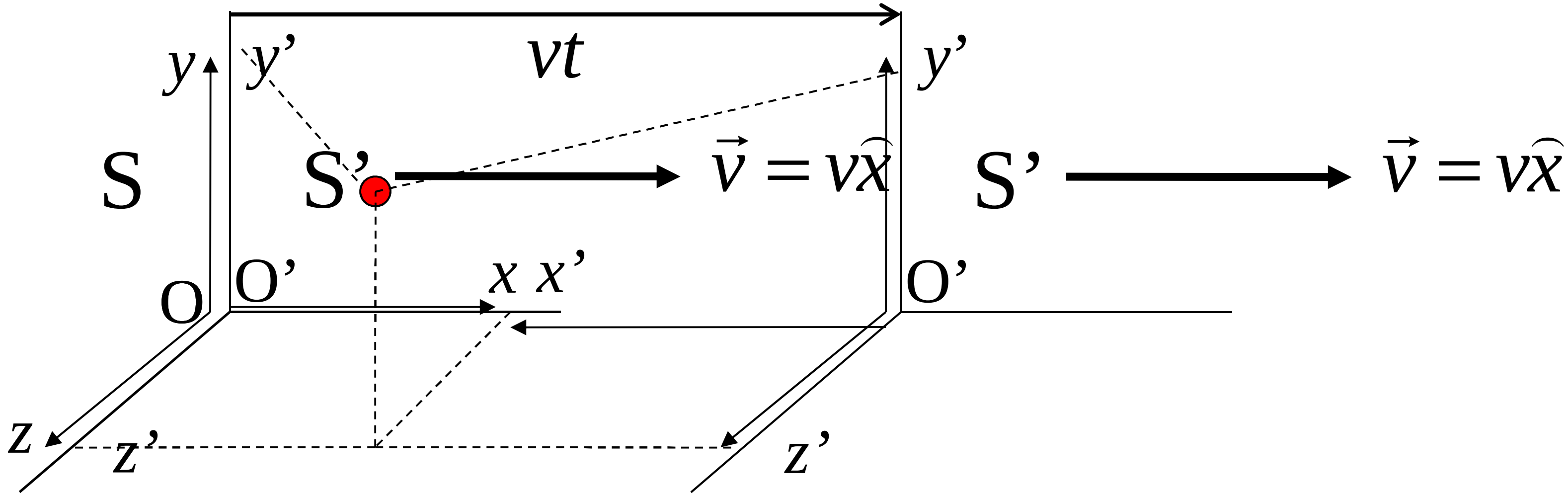
Meten aan een tov ons bewegend stelsel geeft afwijkingen te zien.

afstand / snelheid

Fraunhoferlijnen:
absorptie aan het oppervlak



Transformatie van coördinaten – klassiek (Galilei)



Op $t' = t = 0$ geldt $O'(0) = O(0)$

S → S'

S' → S

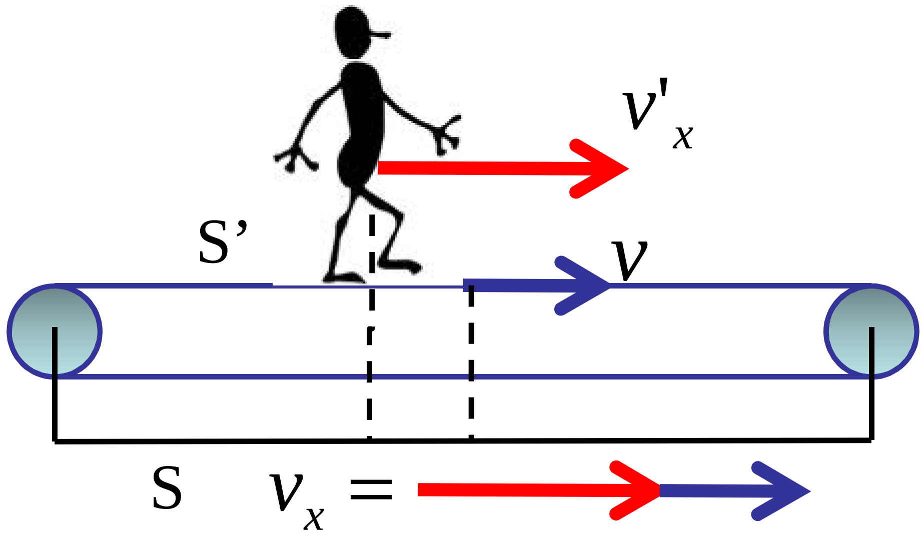
dan volgt:

$$\boxed{\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}} \quad \longleftrightarrow \quad \boxed{\begin{aligned} x &= x' + vt \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}}$$

Transformatie van snelheden - klassiek

$$x = x' + vt \qquad t = t \quad \longrightarrow \quad v_x = \frac{dx}{dt} = \frac{d(x' + vt)}{dt} = \frac{dx'}{dt'} + v = v'_x + v$$

loopband op Schiphol



Stel in S'wordt een foton uitgezonden

dus $v'_x = c \quad \longrightarrow \quad v_x = v'_x + v = c + v$

dan is **de snelheid van dat foton in S > c**

Geschiedenis

Klassieke mechanica (Principia, Newton: 1687)

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}$$

Klassieke elektriciteit & magnetisme

Maxwell-vergelijkingen

(Maxwell, 1865; 20 vergelijkingen

Heaviside en Gibbs, 1884; 4 vergelijkingen)

↓

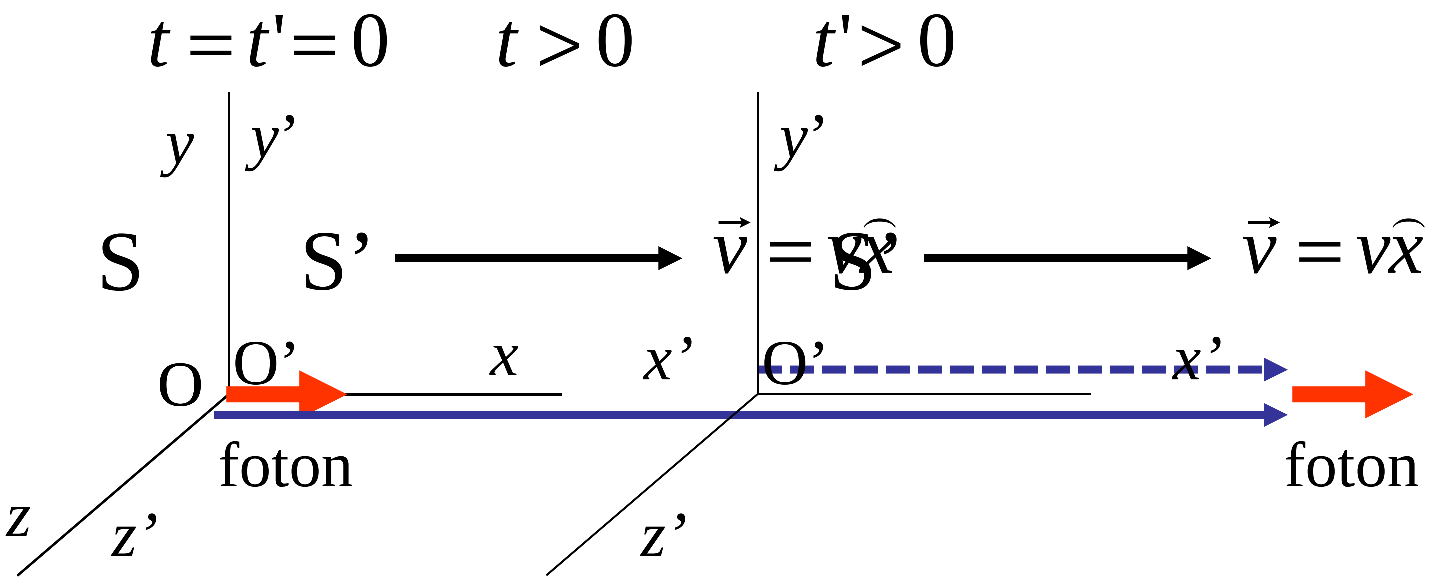
De snelheid van EM-golven in vacuüm tov de ether: $c = \frac{E}{B} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

Michelson en Morley, 1887; snelheid aarde tov ether is nul

Lorentz; alternatieve verklaringen (lokale tijd, lengtekrimp)

Einstein, 1905; speciale relativiteitsleer

in elk stelsel heeft de lichtsnelheid de waarde: **c = 299 792 458 m/s**



S: $x^2 = (ct)^2$

$x^2 - (ct)^2 = 0$

S':

algemeen $\underbrace{x^2 + y^2 + z^2}_{\text{kwadraat v.d. afstand - Pythagoras}} - (ct)^2 = 0$ $\quad x'^2 + y'^2 + z'^2 - (ct')^2 = 0$

Er moet gelden: $x^2+y^2+z^2-(ct)^2=0$

en $x'^2+y'^2+z'^2-(ct')^2=0$

Met de G-transformatie:

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\}$$

$$\begin{aligned} & x'^2+y'^2+z'^2-(ct')^2 \\ &= (x-vt)^2+y^2+z^2-(ct)^2 \\ &= \boxed{x^2+y^2+z^2-(ct)^2} - 2xvt + v^2t^2 \\ &= -2xvt + v^2t^2 \neq 0 \end{aligned}$$

Dat gaat dus fout

probeer:

algemene lineaire combinatie

$$\left.\begin{aligned} x' &= a_1 x + a_2 ct \\ y' &= y \\ z' &= z \\ ct' &= a_3 x + a_4 ct \end{aligned}\right\} \quad \boxed{\begin{aligned} x' &= \gamma(x-\beta ct) \\ ct' &= \gamma(ct-\beta x) \end{aligned}}$$

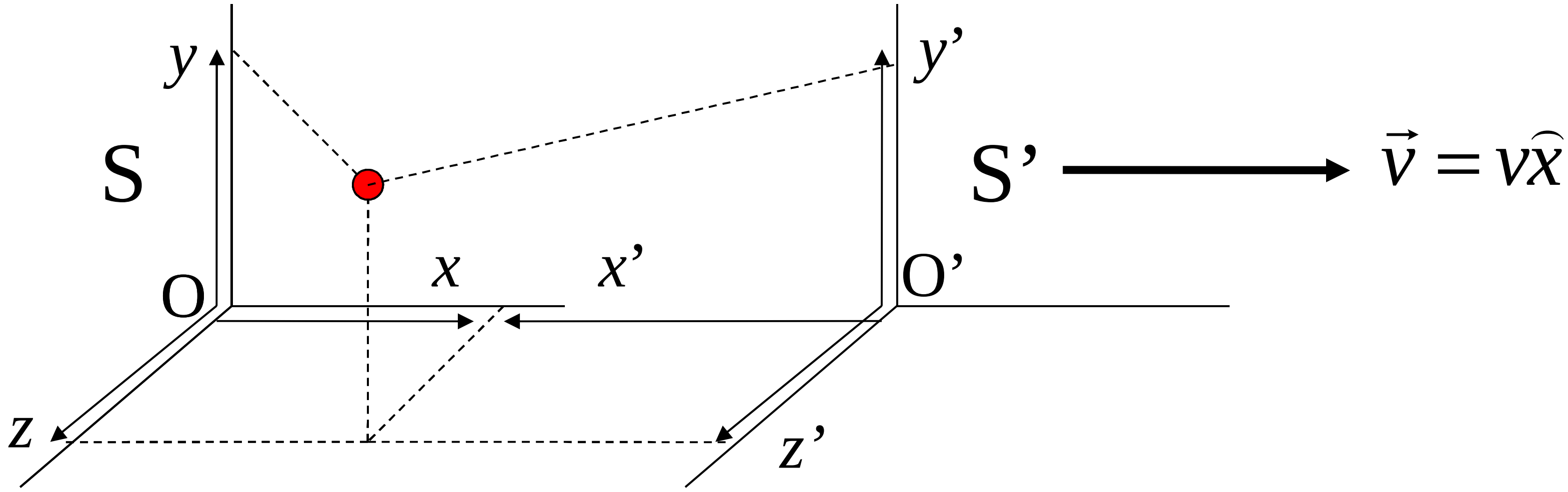
de enige ‘echte’ aanpassing

algemeen:

$$\boxed{q_1' = \gamma(q_1 - \beta q_2)}$$

met $\beta \equiv \dfrac{v}{c}$ en $\gamma \equiv \dfrac{1}{\sqrt{1-\beta^2}}$

Samenvattend



S → S'

$$x' = \gamma(x - \beta ct)$$
$$y' = y$$
$$z' = z$$
$$ct' = \gamma(ct - \beta x)$$

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Inverse L-transformaties

$S' \rightarrow S$

afleiding:

$$\boxed{\begin{aligned} x' &= \gamma(x-\beta ct) \\ ct' &= \gamma(ct-\beta x) \end{aligned}} \rightarrow \begin{aligned} x &= \frac{x'}{\gamma}+\beta \boxed{ct} \\ \boxed{ct} &= \frac{ct'}{\gamma}+\beta x \end{aligned} \rightarrow x = \frac{x'}{\gamma}+\beta\left(\frac{ct'}{\gamma}+\beta x\right)$$

$$\rightarrow (1-\beta^2)x = \frac{x'}{\gamma}+\frac{\beta ct'}{\gamma}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \rightarrow 1-\beta^2 = \frac{1}{\gamma^2}$$

$$\rightarrow \boxed{x = \gamma(x'+\beta ct')}$$

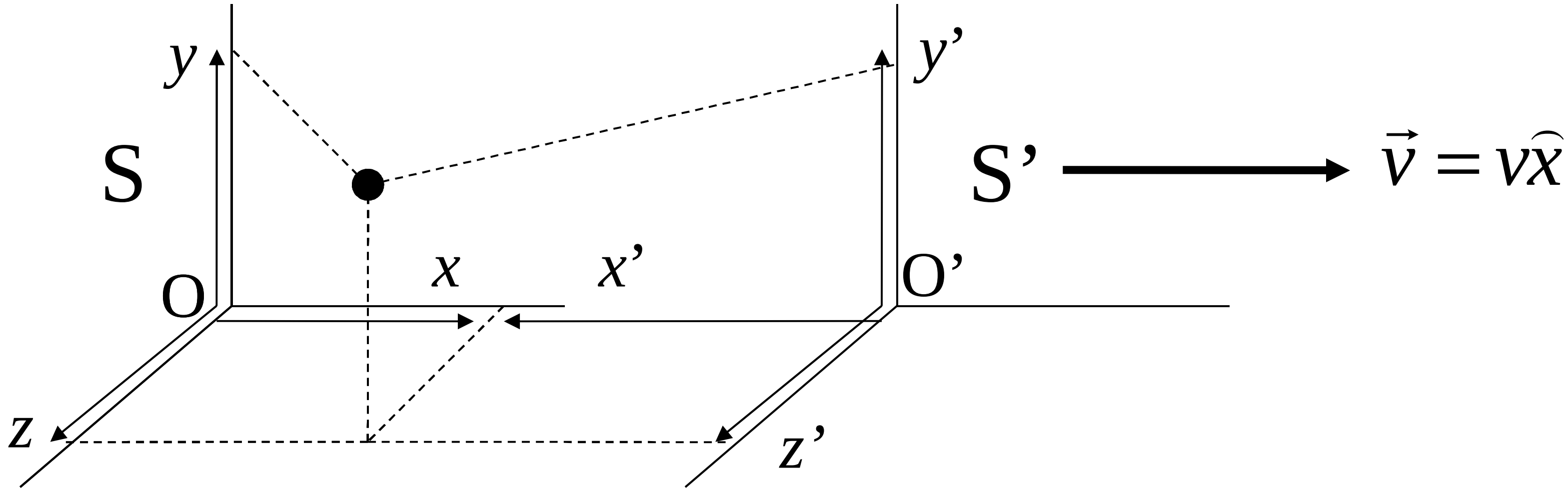
$$ct = \frac{ct'}{\gamma}+\beta x = \frac{ct'}{\gamma}+\beta\gamma(x'+\beta ct') = ct'\left(\frac{1}{\gamma}+\beta^2\gamma\right)+\gamma\beta x'$$

$$\boxed{= \gamma(ct'+\beta x')}$$

Maar …. het kan véél eenvoudiger:

vervang voor de inverse transformatie v door $-v$ en dus β door $-\beta$

Transformatie van coördinaten – relativistisch (Einstein)



S → S’

$$x' = \gamma(x - \beta ct)$$
$$y' = y$$
$$z' = z$$
$$ct' = \gamma(ct - \beta x)$$

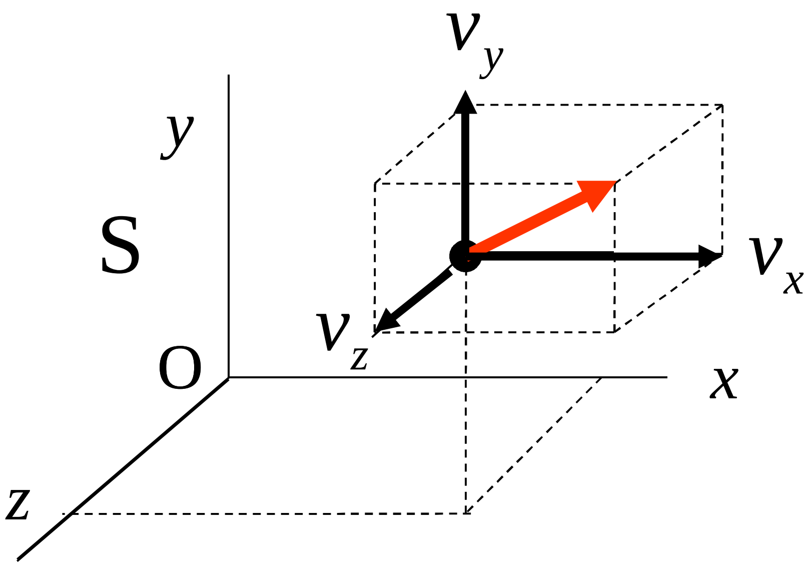
$$\beta = \frac{v}{c}$$

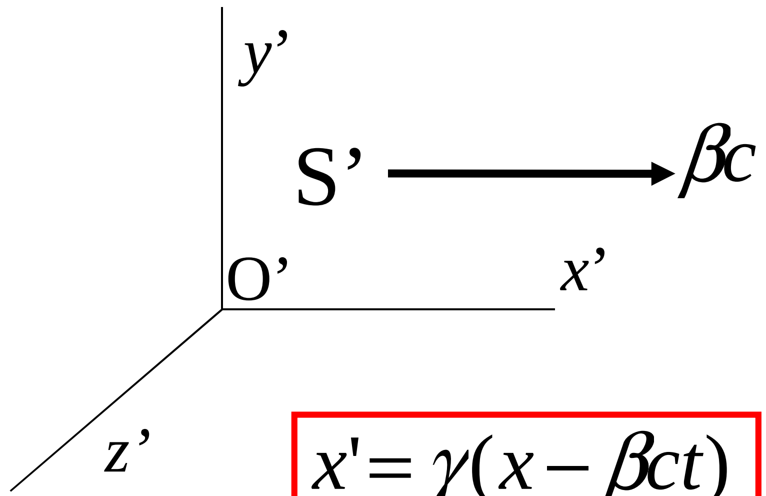
$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

S’ → S

$$x = \gamma(x' + \beta ct')$$
$$y = y'$$
$$z = z'$$
$$ct = \gamma(ct' + \beta x')$$

Transformatie van snelheden





$$\boxed{\begin{aligned} x' &= \gamma(x - \beta ct) \\ ct' &= \gamma(ct - \beta x) \end{aligned}}$$

klassiek: $\quad v'_x = v_x - v$

relativistisch: $\quad v'_x = \dfrac{dx'}{dt'} = \dfrac{\frac{dx'}{dt}}{\frac{dt'}{dt}} = \dfrac{\gamma(\frac{dx}{dt} - \beta c)}{\gamma(1 - \frac{\beta}{c}\frac{dx}{dt})} = \dfrac{v_x - v}{1 - \frac{vv_x}{c^2}}$

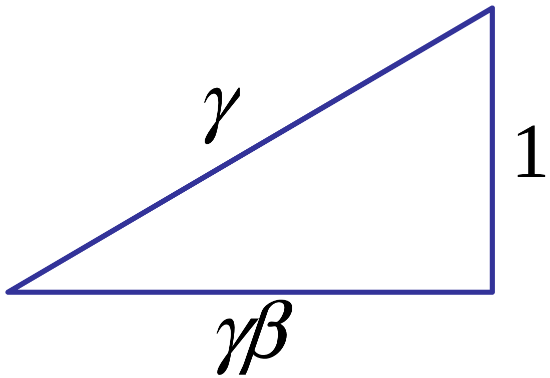
in y-richting: $v'_y = \dfrac{dy'}{dt'} = \dfrac{\dfrac{dy'}{dt}}{\dfrac{dt'}{dt}} = \dfrac{\dfrac{dy}{dt}}{\gamma(1 - \dfrac{\beta}{c}\dfrac{dx}{dt})} = \dfrac{v_y}{\gamma(1 - \dfrac{vv_x}{c^2})}$

conclusie:

$$\boxed{v'_x = \frac{v_x - v}{1 - \dfrac{vv_x}{c^2}} \qquad v'_y = \frac{v_y}{\gamma(1 - \dfrac{vv_x}{c^2})} \qquad v'_z = \frac{v_z}{\gamma(1 - \dfrac{vv_x}{c^2})}}$$

Nuttige formule:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \rightarrow \boxed{\gamma^2\beta^2 + 1 = \gamma^2}$$



klassieke limiet ($\beta \approx 0$) $\rightarrow$ γ = 1

relativistisch **klassiek**

$$x' = \gamma(x - \beta ct) \longrightarrow x' = 1(x - vt) = x - vt$$

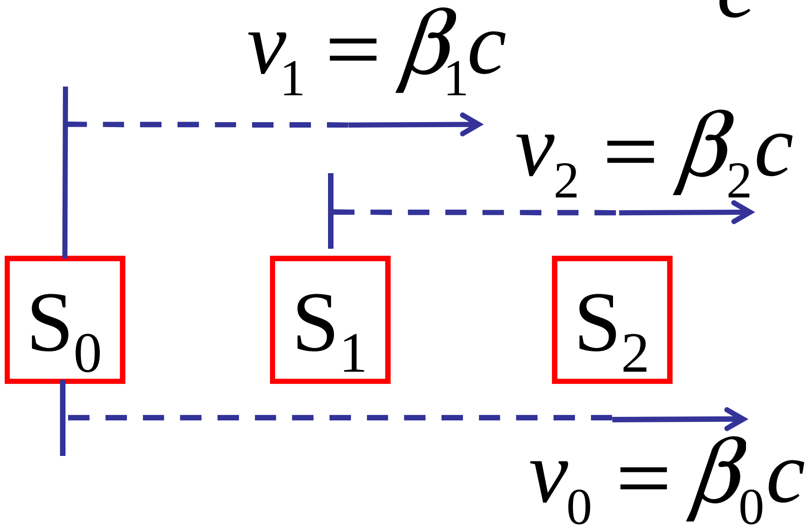
$$ct' = \gamma(ct - \beta x) \longrightarrow ct' = 1(ct - 0x) = ct \longrightarrow t' = t$$

$$v'_x = \frac{v_x - v}{1 - \frac{vv_x}{c^2}} \longrightarrow v'_x = v_x - v$$

$$v'_y = \frac{v_y}{\gamma(1 - \frac{vv_x}{c^2})} \longrightarrow v'_y = v_y$$

$$v'_z = \frac{v_z}{\gamma(1 - \frac{vv_x}{c^2})} \longrightarrow v'_z = v_z$$

S → S' $$v'_x = \frac{v_x - v}{1 - \frac{v v_x}{c^2}}$$ S' → S $$v_x = \frac{v'_x + v}{1 + \frac{v v'_x}{c^2}}$$



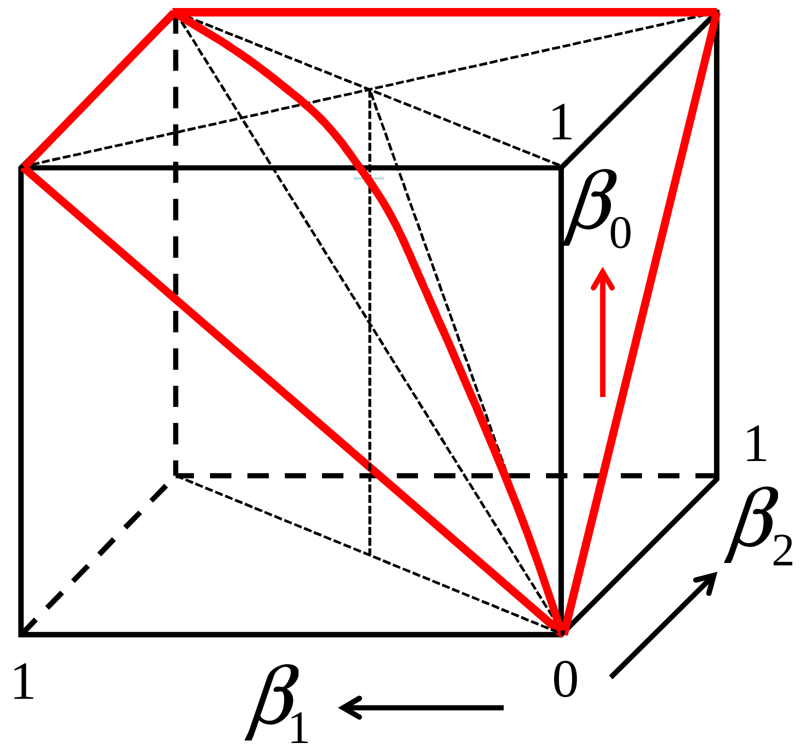
$$\boxed{\beta_0 = \frac{v_0}{c} = \frac{1}{c}\frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}}$$

als $\beta_2 = 1$ (S2 = foton) en $|\beta_1| < 1$ → $\beta_0 = \dfrac{\beta_1 + 1}{1 + \beta_1} = 1$

Het foton heeft dezelfde snelheid tov de systemen S en S_1;
dat is precies de vooronderstelling op grond waarvan de Lorentz-transformaties zijn afgeleid!

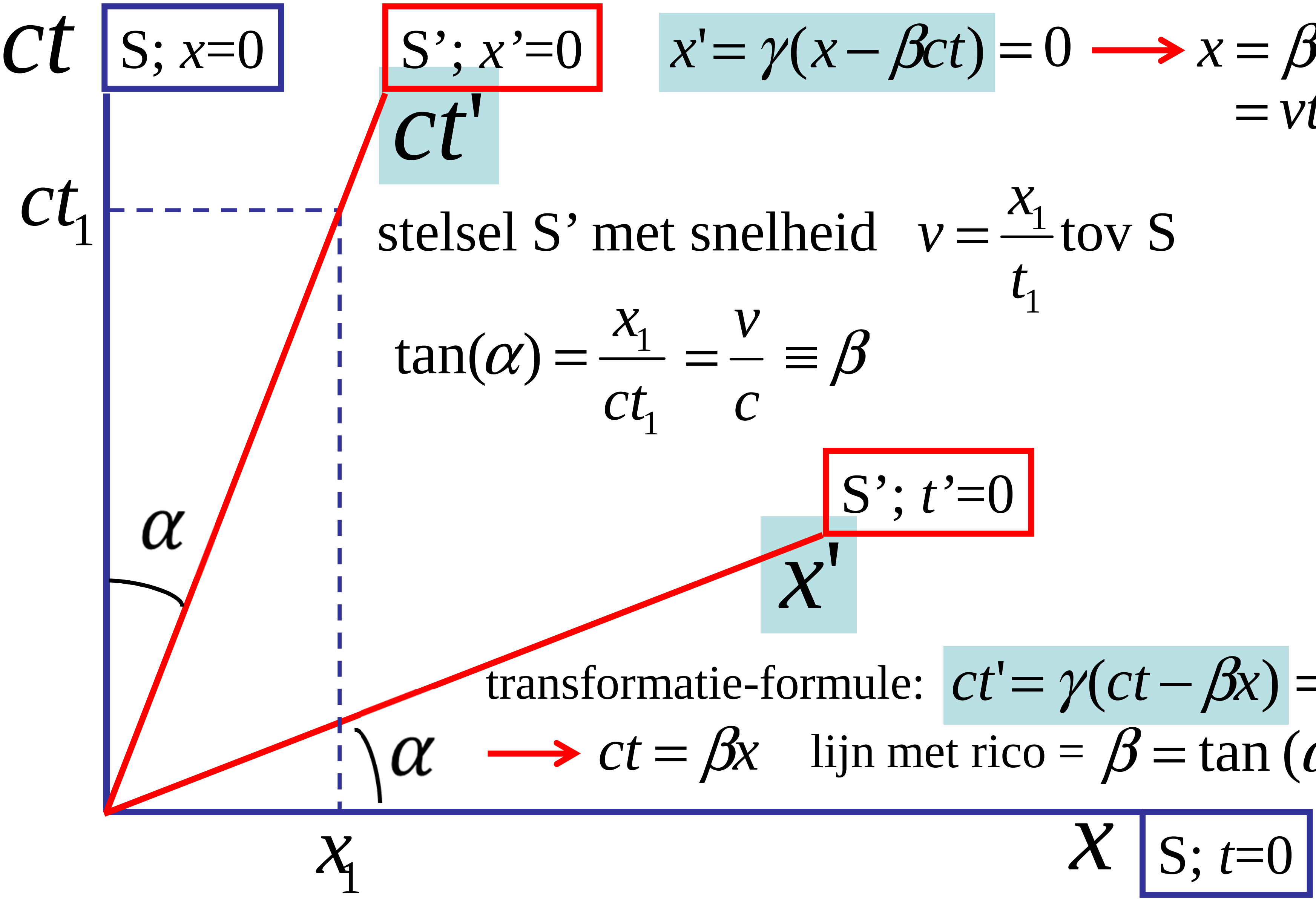
$$\beta_0 = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \quad \text{als} \quad \beta_1 \approx \beta_2 = \beta \approx 1 \longrightarrow \beta_0 = \frac{2\beta}{1+\beta^2} \approx \beta$$

Voor elke waarnemer bewegen alle snel-bewegende objecten snel; ook al is de waarnemer zelf een snel-bewegend object. Daardoor lijkt het alsof de waarnemer zich in het centrum van het heelal bevindt, immers alle snel-bewegende objecten vormen de rand van het heelal.



Grafische voorstelling: Minkowski-diagram

Minkowski-diagram



transformatie-formule:

$$x' = \gamma(x - \beta ct) = 0 \longrightarrow x = \beta ct = vt$$

stelsel S' met snelheid $v = \dfrac{x_1}{t_1}$ tov S

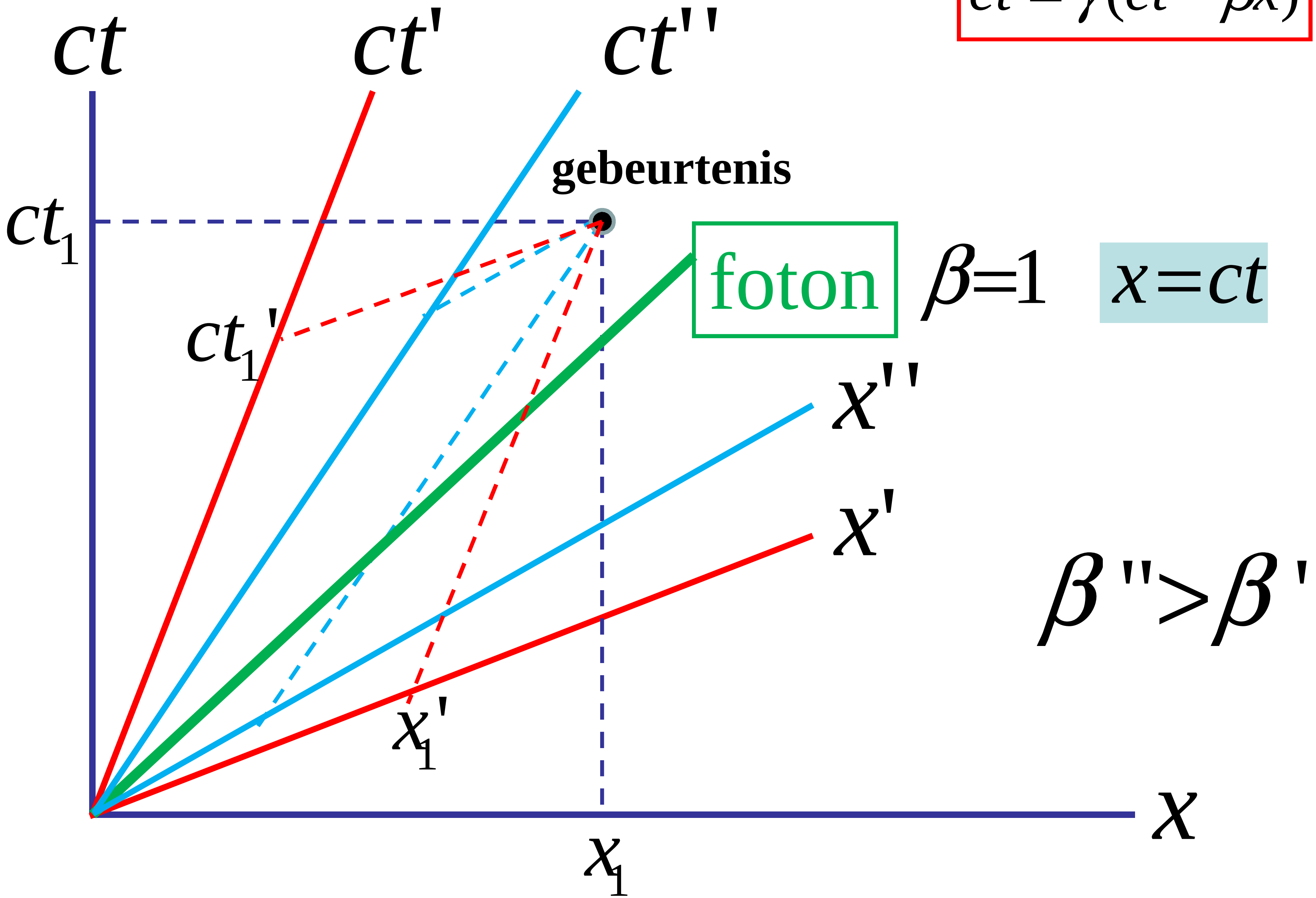
$$\tan(\alpha) = \frac{x_1}{ct_1} = \frac{v}{c} \equiv \beta$$

transformatie-formule: $ct' = \gamma(ct - \beta x) = 0$

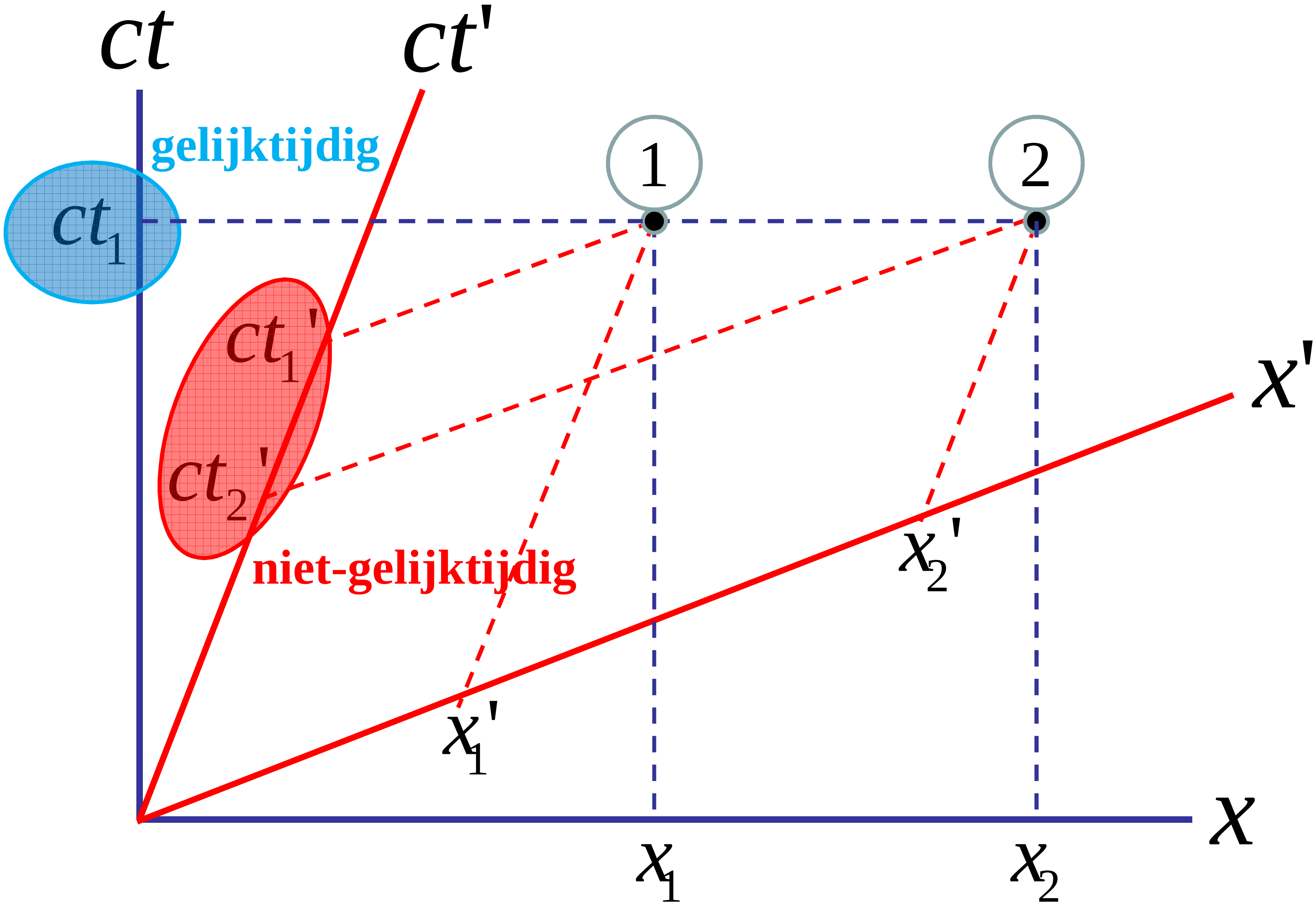
$\longrightarrow ct = \beta x$ lijn met rico = $\beta = \tan(\alpha)$

Minkowski-diagram - coördinaten

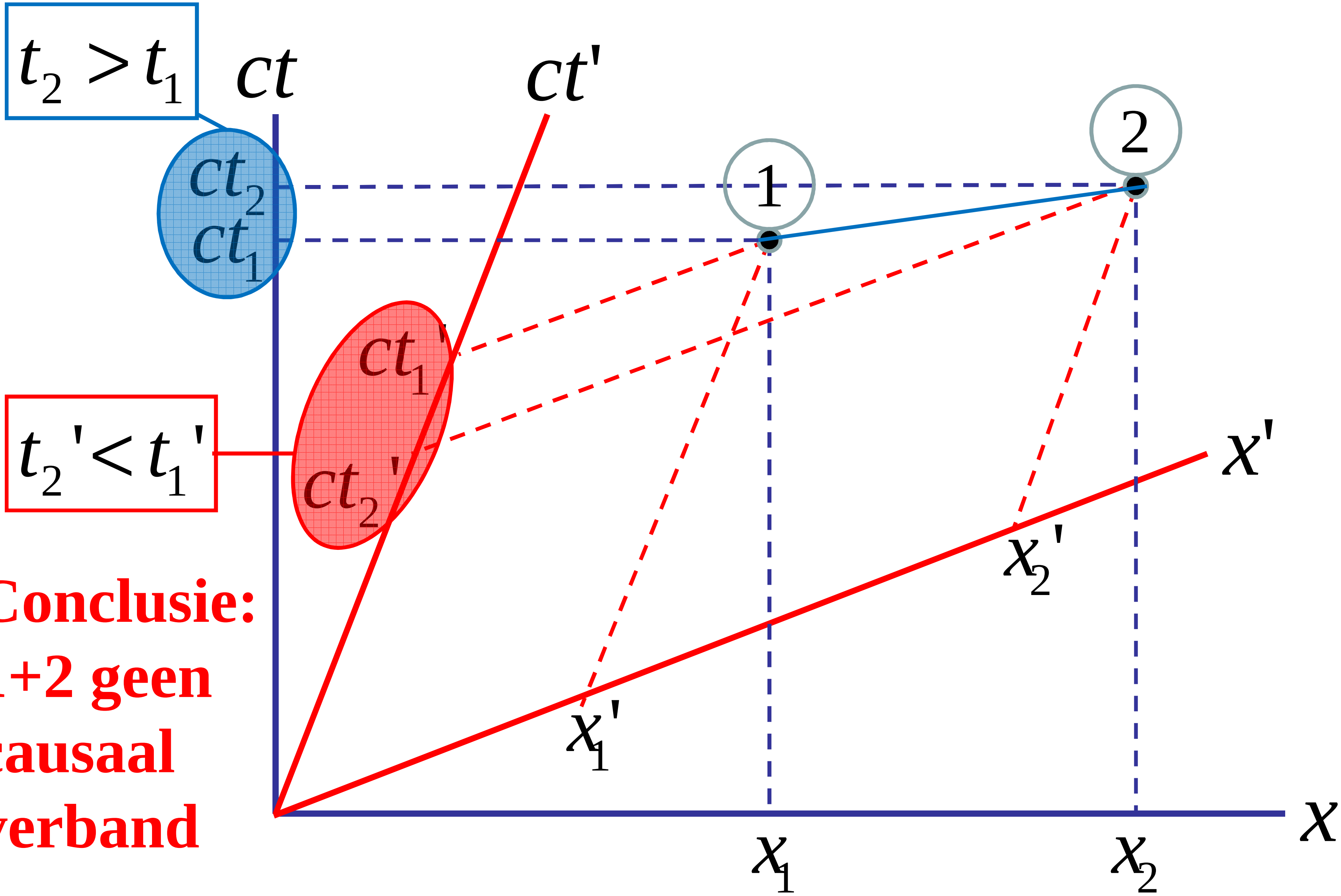
$$x' = \gamma(x - \beta ct)$$
$$ct' = \gamma(ct - \beta x)$$



Minkowski-diagram – (on)gelijktijdigheid

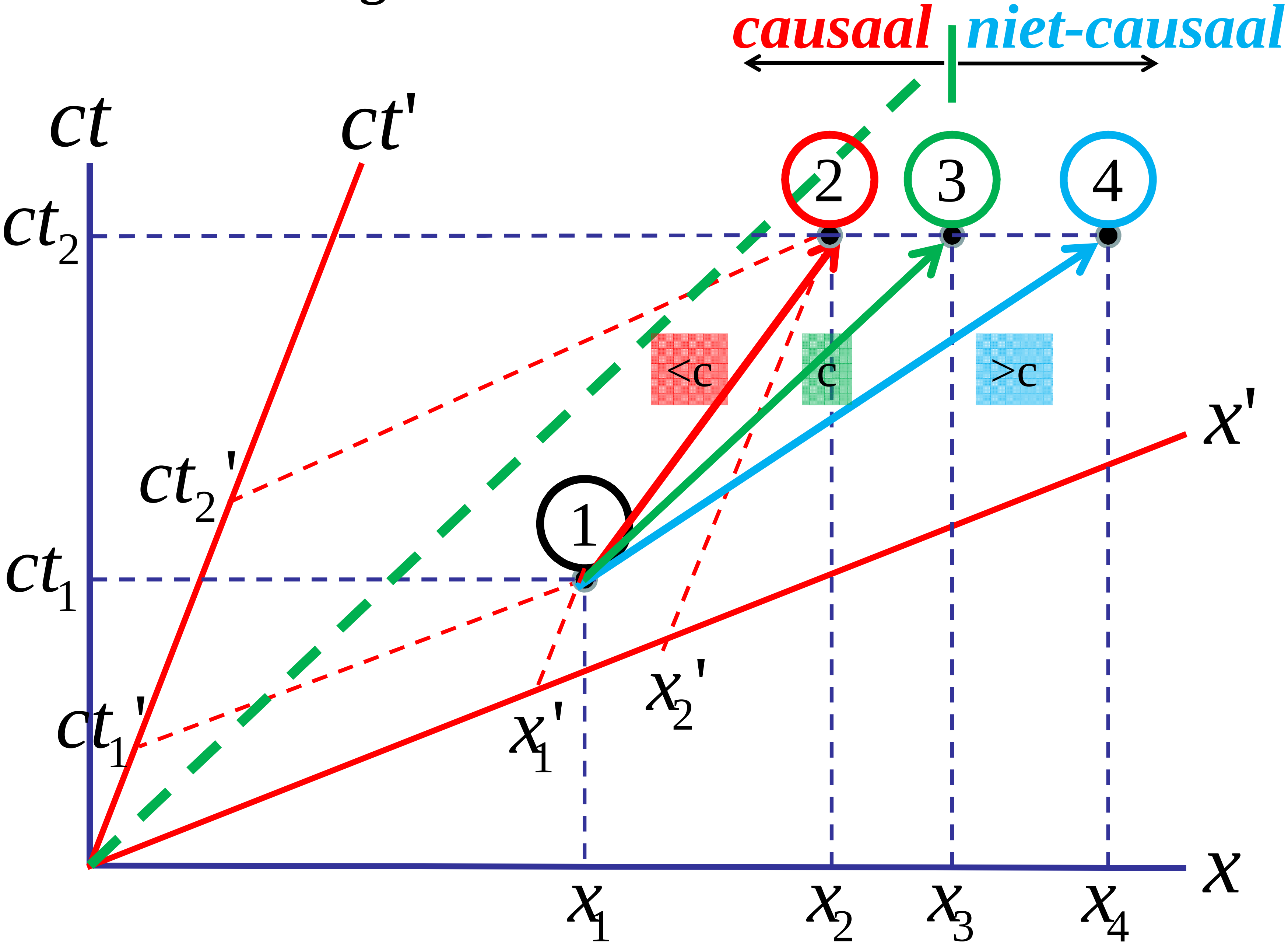


Minkowski-diagram – causaliteit

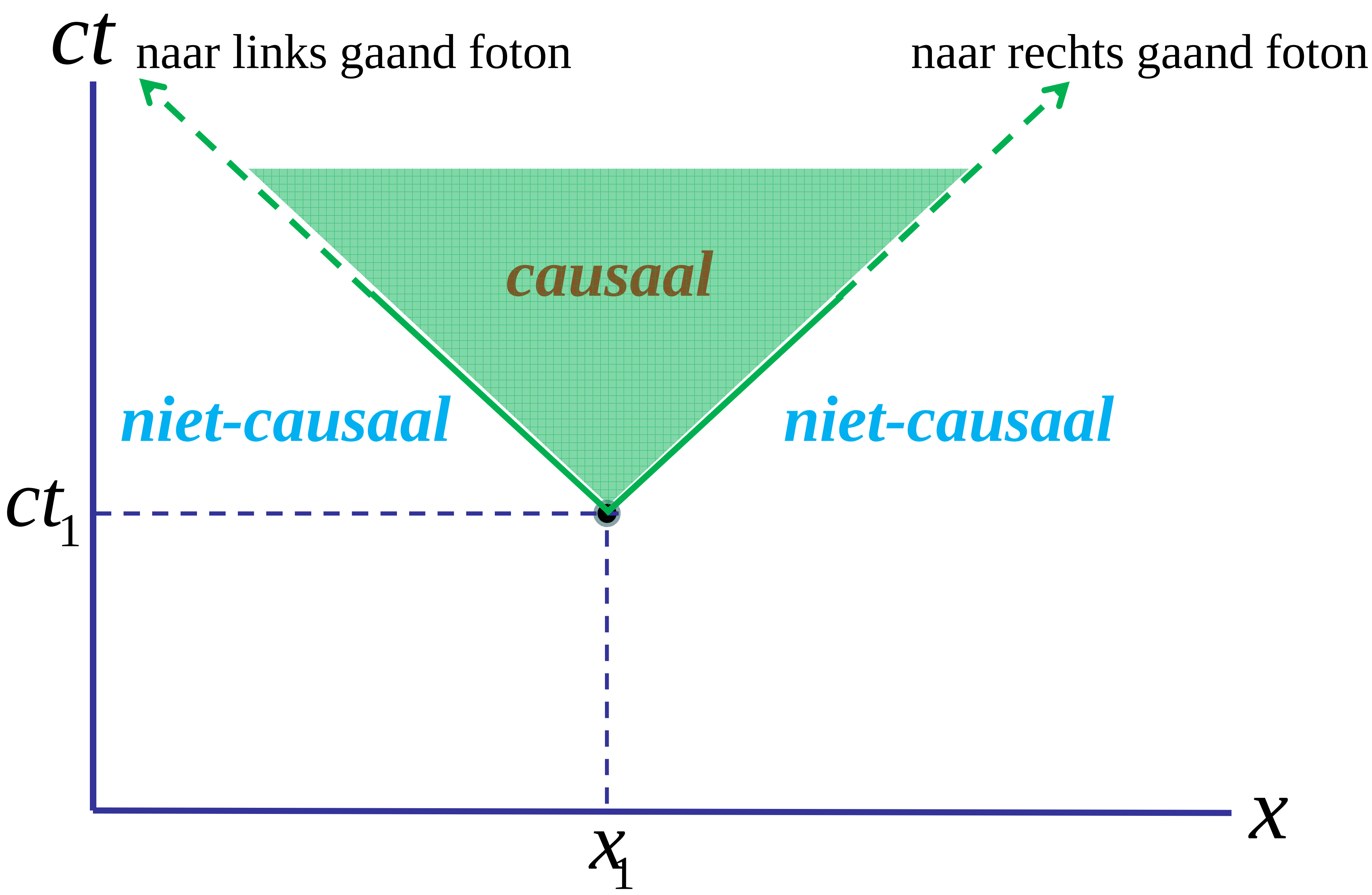


Conclusie:
1+2 geen causaal verband

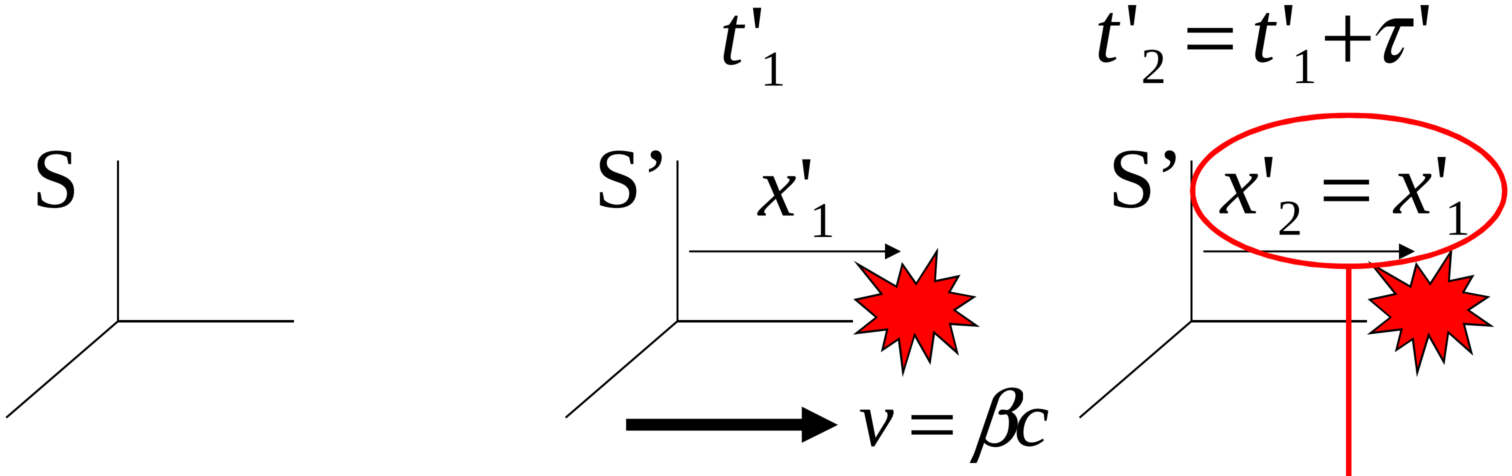
Minkowski-diagram – causaliteit



Minkowski-diagram – causaliteit



Tijdrek



$$\boxed{ct = \gamma(ct' + \beta x')}$$

$$ct_2 = \gamma(ct'_2 + \beta x'_2)$$

$$ct_1 = \gamma(ct'_1 + \beta x'_1)$$

$$-$$

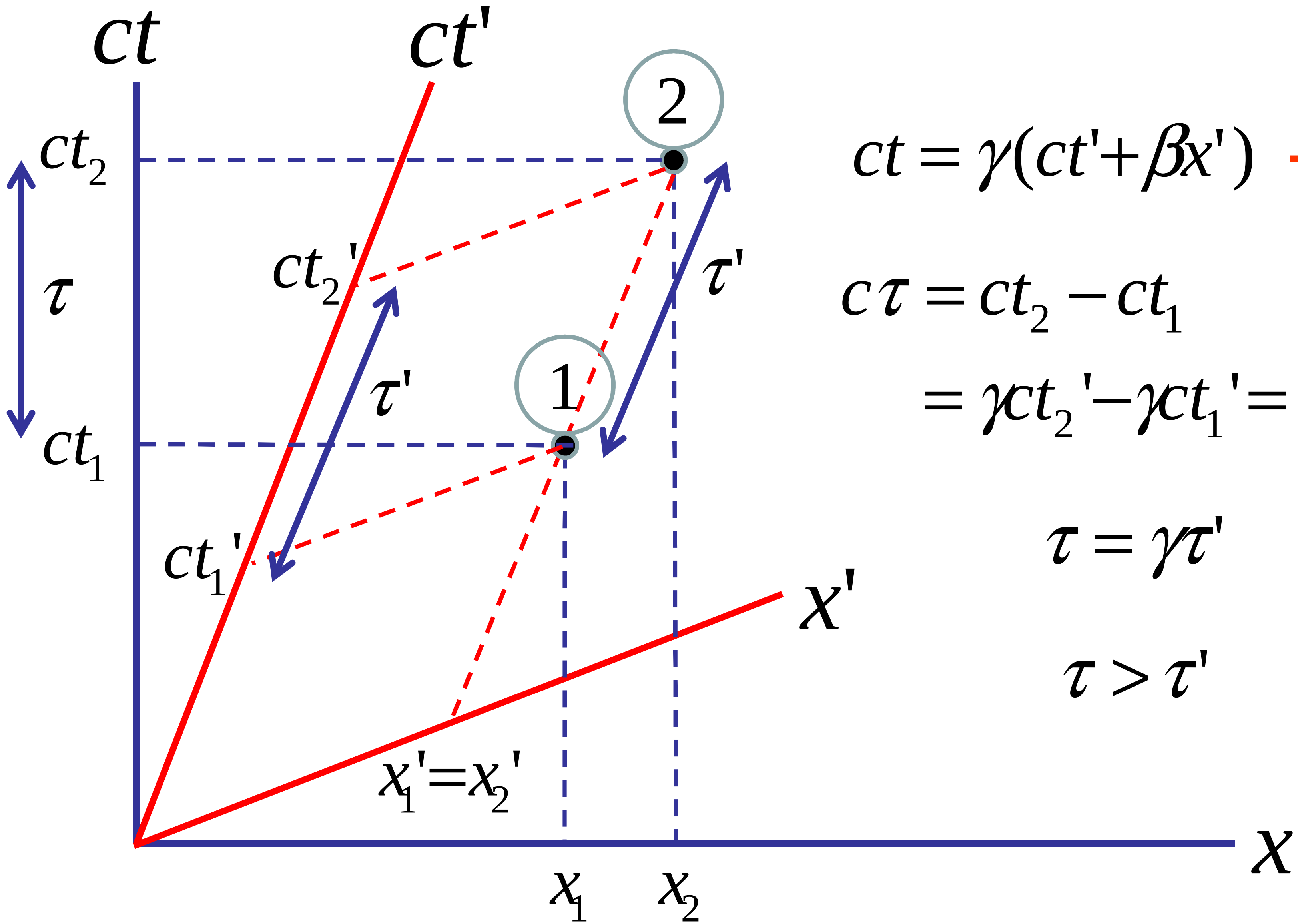
$$c\tau = c(t_2 - t_1) = \gamma[c(t'_2 - t'_1) + \beta(\cancel{x'_2 - x'_1})] = \gamma c\tau'$$

$$\tau = \gamma\tau' > \tau'$$

$$\gamma > 1$$

het proces lijkt in S langer te duren dan in S’

Tijdrek - Minkowski-diagram



$$ct = \gamma(ct' + \beta x')$$

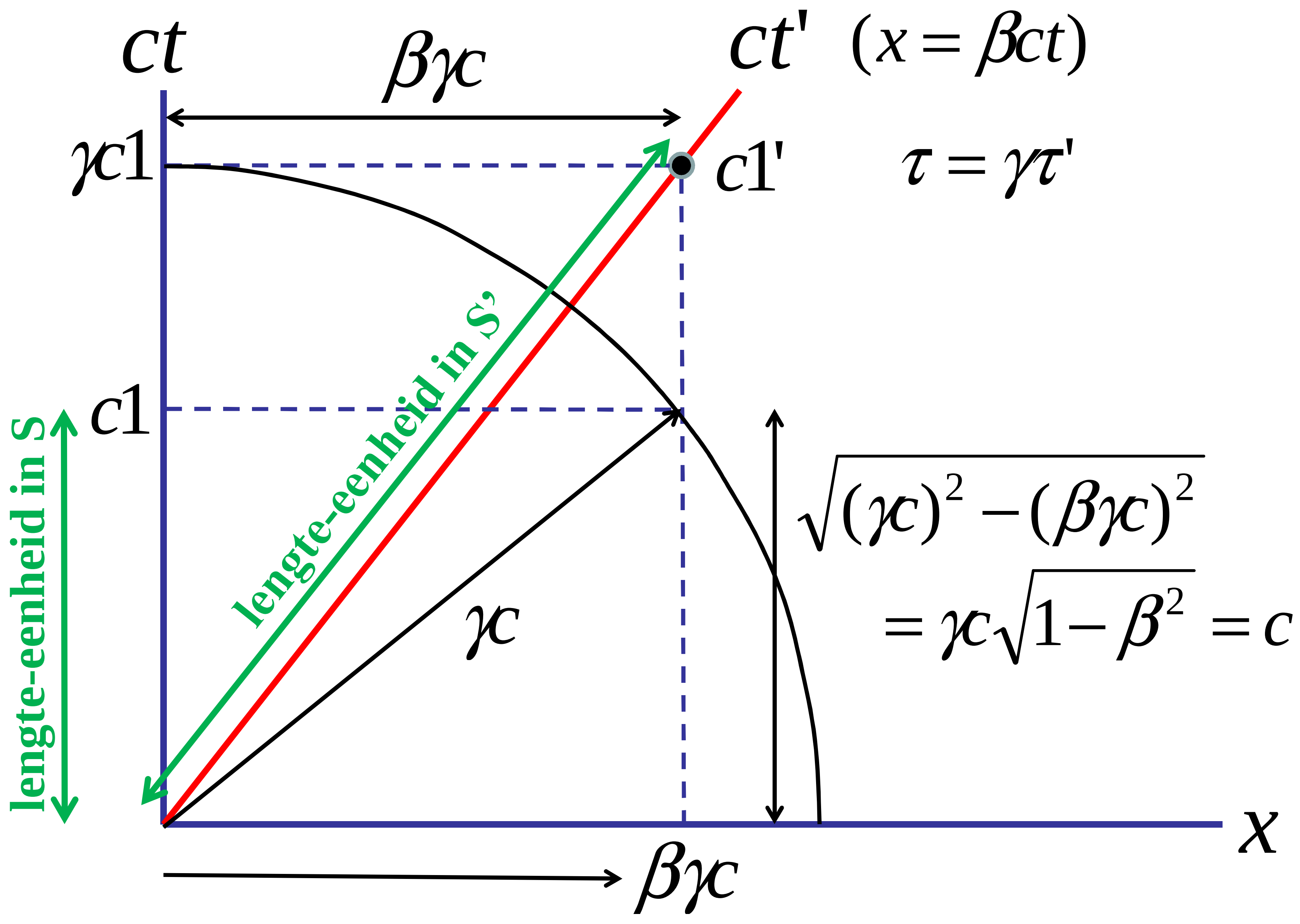
$$c\tau = ct_2 - ct_1$$

$$= \gamma ct_2' - \gamma ct_1' = c\gamma\tau'$$

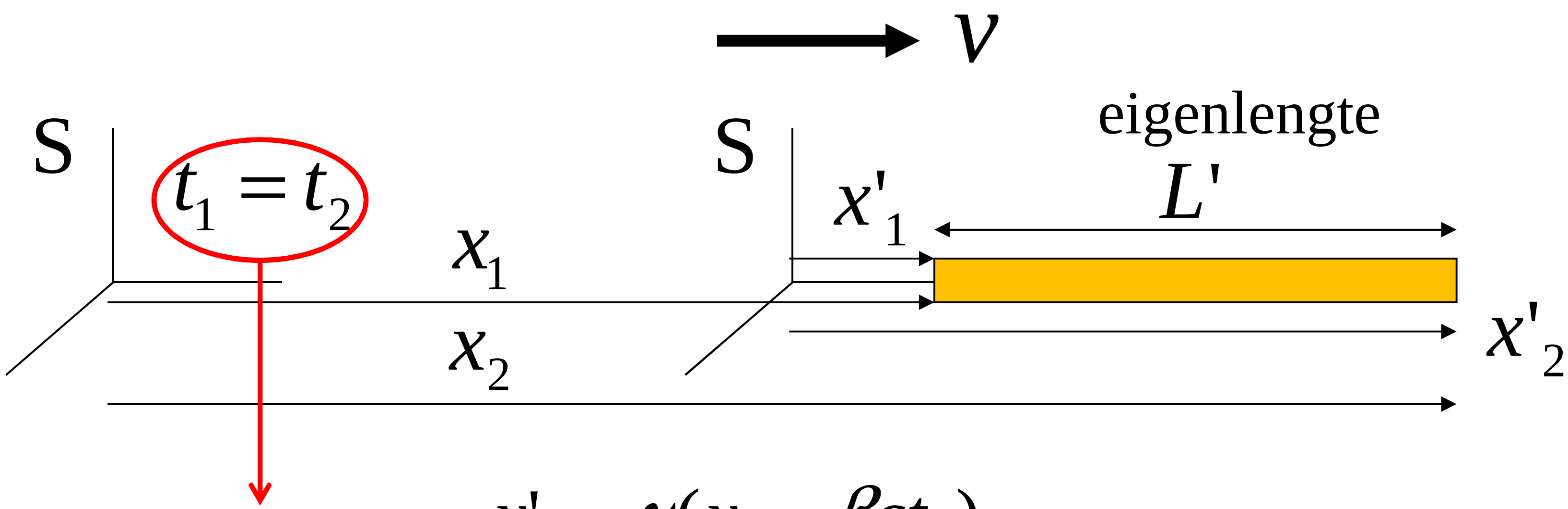
$$\tau = \gamma\tau'$$

$$\tau > \tau'$$

Tijdrek - Minkowski-diagram



Lengtekrimp



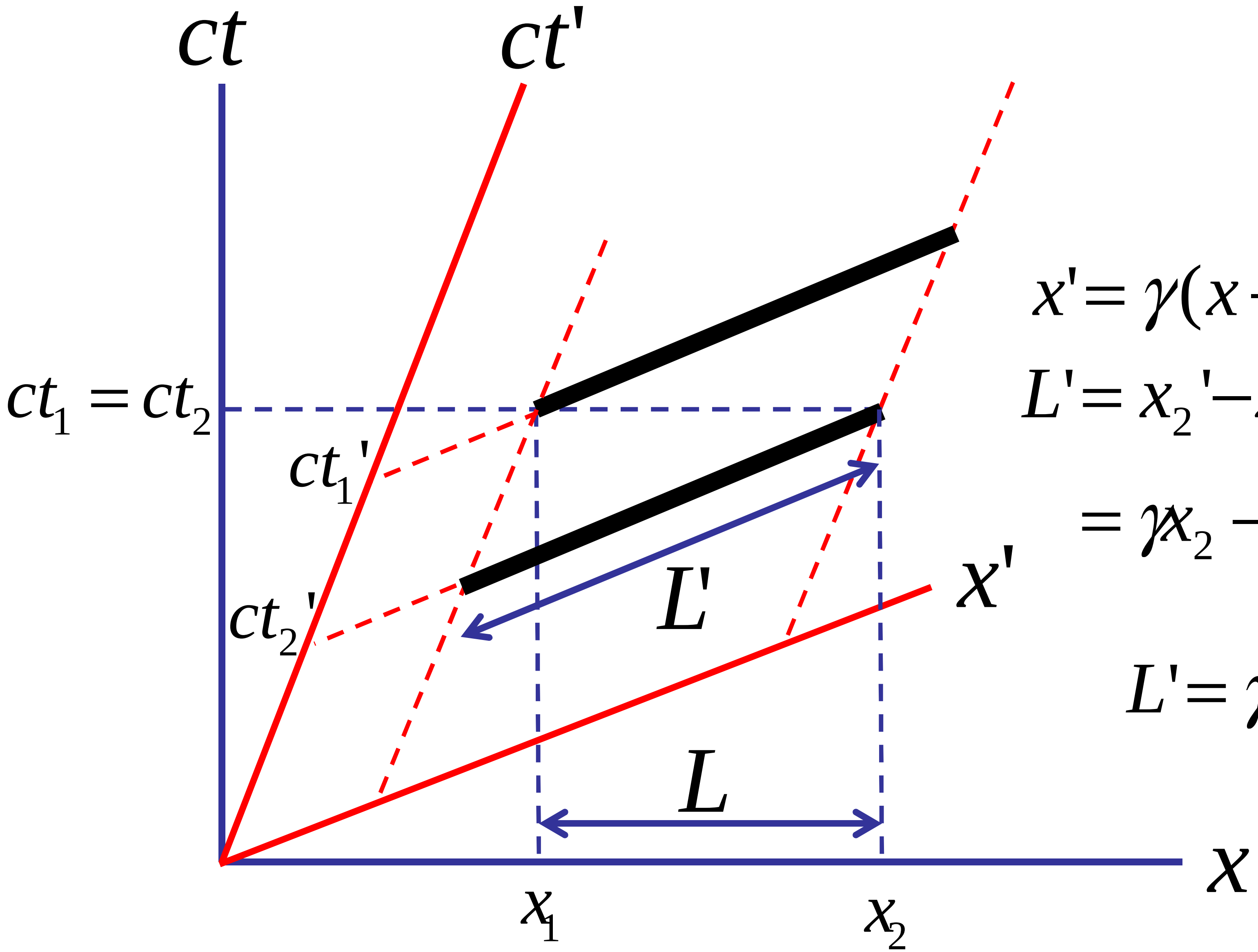
$$\boxed{x' = \gamma(x - \beta ct)}$$

$$
\begin{array}{l}
x'_2 = \gamma(x_2 - \beta ct_2) \\
x'_1 = \gamma(x_1 - \beta ct_1) \\
\hline
\end{array}
\quad -
$$

$$L' = x'_2 - x'_1 = \gamma(x_2 - \cancel{\beta ct_2}) - \gamma(x_1 - \cancel{\beta ct_1}) = \gamma(x_2 - x_1) = \gamma L$$

$$L = \frac{L'}{\gamma} < L'$$ de stok lijkt in S korter dan in S’

Lengtekrimp- Minkowski-diagram



$$x' = \gamma(x - \beta ct) \rightarrow$$

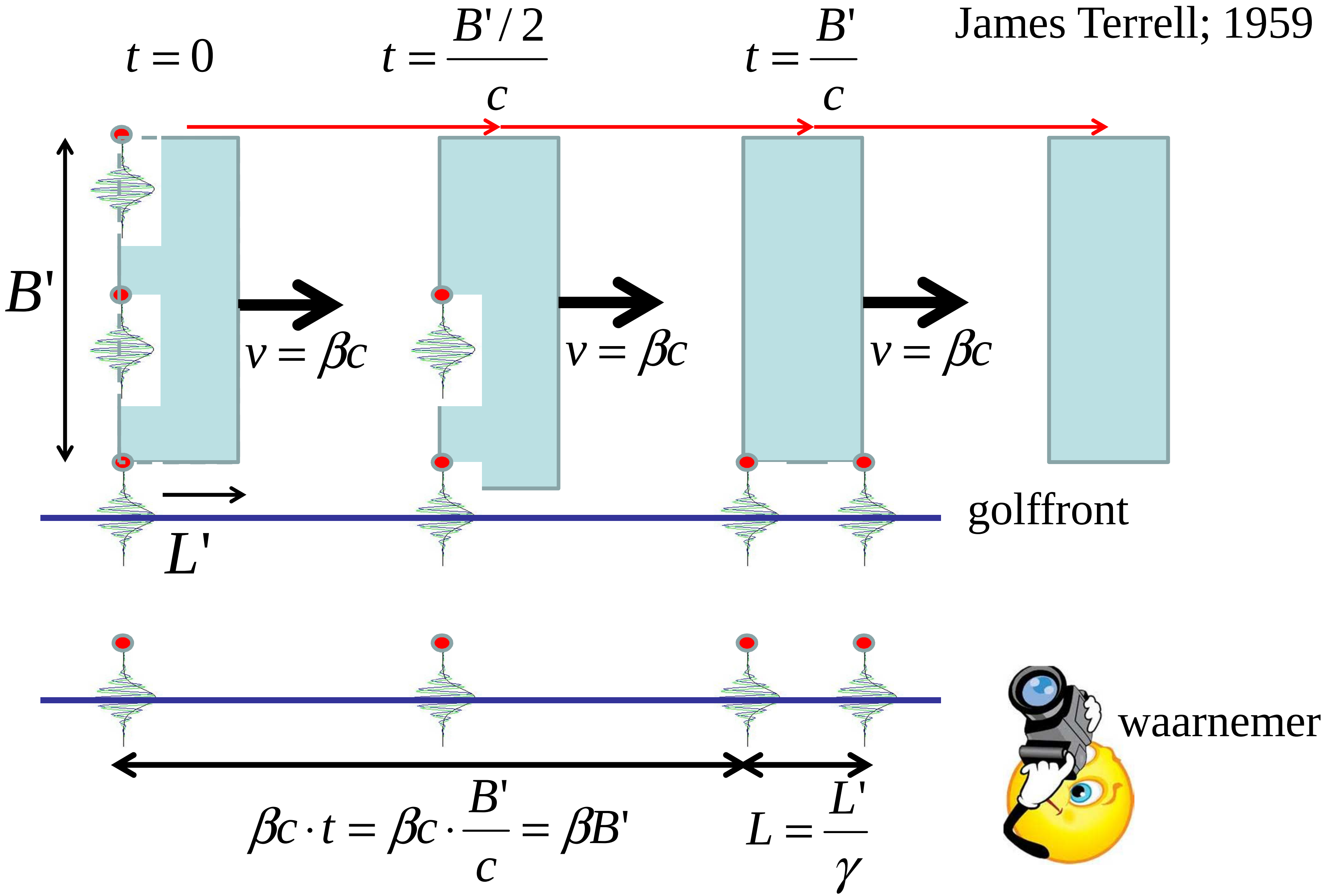
$$L' = x_2' - x_1'$$

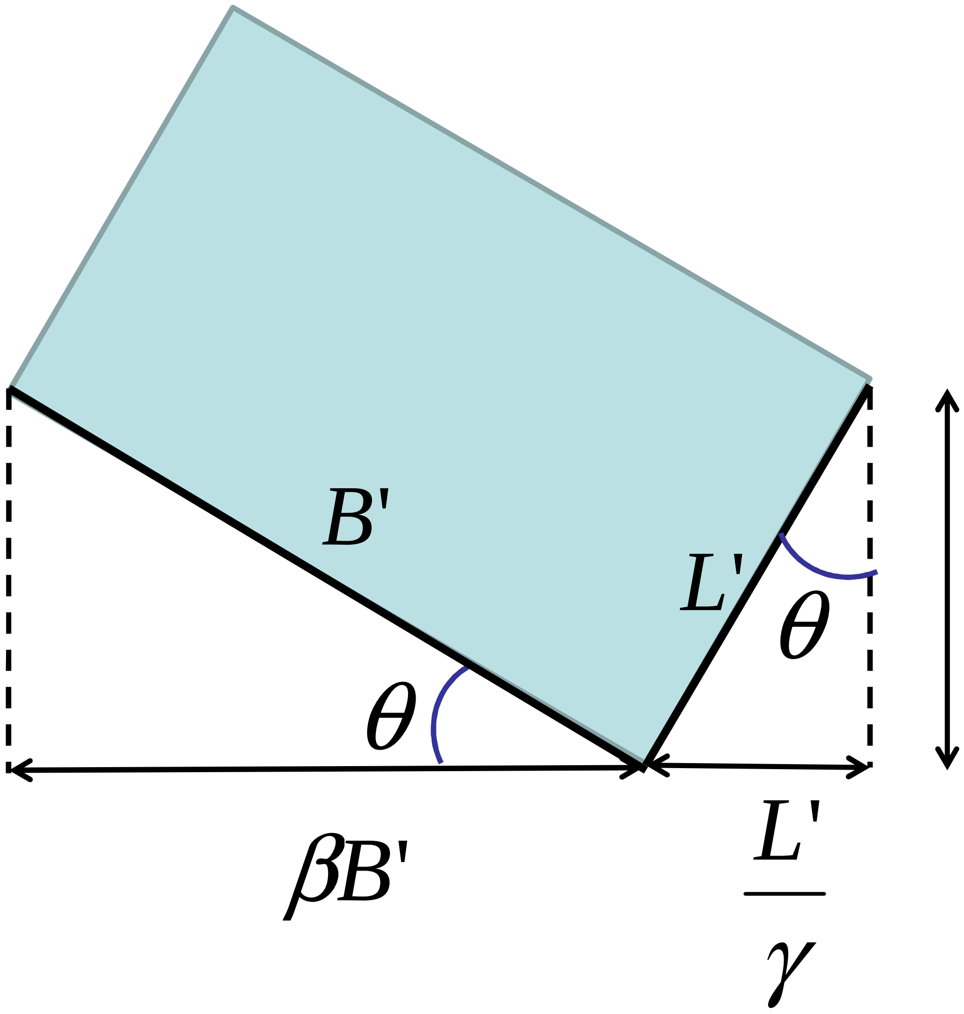
$$= \gamma x_2 - \gamma x_1 = \gamma L$$

$$L' = \gamma L$$

Lijkt een snel voorbij vliegend voorwerp korter?

James Terrell; 1959





$$\cos\theta = \beta$$

$$\sqrt{(L')^2 - \left(\frac{L'}{\gamma}\right)^2}$$

$$= \sqrt{1 - \frac{1}{\gamma^2}} \cdot L' = \beta L'$$

Conclusie: de rechthoek lijkt dus niet verkort, maar gedraaid!

Muon-verval

Verval van muonen - tijdrek

$t'_1 = creatie$

v

$x = v\tau'$

$\boxed{\tau' = levensduur\ = 2{,}2 \cdot 10^{-6}\,(\mathrm{s})}$

$t'_2 = t'_1 + \tau' = einde$

µ haalt het niet !!!

12 km

$x = v\tau' < c\tau' = 3 \cdot 10^5\,(\mathrm{km/s}) \cdot 2{,}2 \cdot 10^{-6}\,(\mathrm{s}) = 0{,}66\ \mathrm{km}$

maar ….. tijdrek

$\boxed{\tau = \gamma\tau' > \tau'}$

stel $\gamma = 20$

$\beta = 0{,}99875$

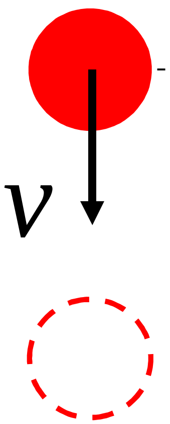
$v\tau \approx c\tau = c\gamma\tau' = 20 \cdot 0{,}66 = 13{,}2\ \mathrm{km}$

µ haalt het dus wel !!!

aardoppervlak

Verval van muonen - lengtekrimp

$t'_1 = creatie$



v

$x = v\tau'$

$\tau' = levensduur = 2{,}2 \cdot 10^{-6}\,(\mathrm{s})$

$t'_2 = t'_1 + \tau' = einde$

μ haalt het niet !!!

12 km

$x = v\tau' < c\tau' = 3 \cdot 10^{5}\,(\mathrm{km/s}) \cdot 2{,}2 \cdot 10^{-6}\,(\mathrm{s}) = 0{,}66\ \mathrm{km}$

maar ….. lengtekrimp

$$\boxed{x' = \frac{x}{\gamma}}$$

stel $\gamma = 20$

$$x' = \frac{x}{\gamma} = \frac{12\ \mathrm{km}}{20} = 0{,}6\ \mathrm{km}$$

μ haalt het dus wel !!!

aardoppervlak

Doppler-effect

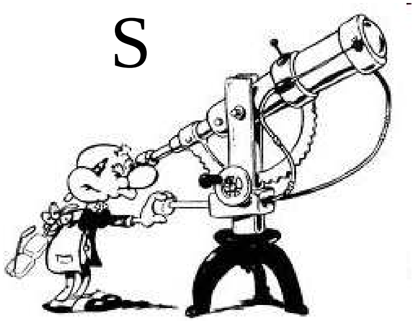
Doppler-effect

S'

$\vec{v} = \beta \vec{c}$

foton

S



t'_1 $\qquad$ $t'_2 = t'_1 + \tau'$

S' $\quad x'_1$ $\qquad$ S' $\quad x'_2 = x'_1$

$v = \beta c$

t_1 $\quad x_1$

t_2 $\quad x_2$

$$\boxed{x = \gamma(x' + \beta c t')}$$

aankomst 1e puls $\quad t_1 + \frac{x_1}{c}$

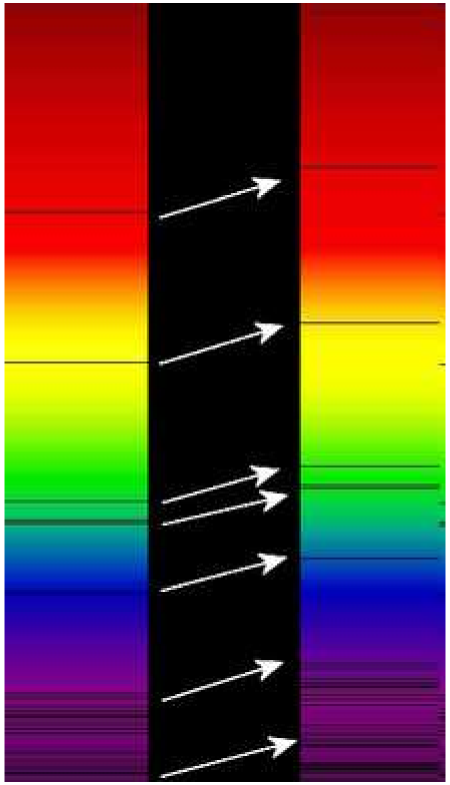
aankomst 2e puls $\quad t_2 + \frac{x_2}{c}$

$$\tau = (t_2 - t_1) + \frac{(x_2 - x_1)}{c}$$

$$= \gamma\tau' + \gamma\beta\tau' = \gamma(1+\beta)\tau' = \sqrt{\frac{1+\beta}{1-\beta}}\,\tau'$$

Roodverschuiving

$$\tau = \sqrt{\frac{1+\beta}{1-\beta}}\tau' \qquad \lambda = c\tau \;\rightarrow\; \lambda = \sqrt{\frac{1+\beta}{1-\beta}}\lambda'$$



$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 \qquad \beta = 0$$

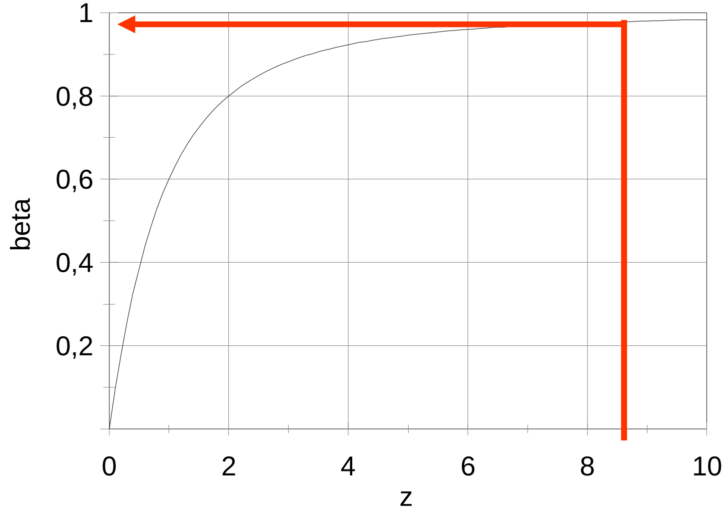
geen roodverschuiving: z = 0

Grootste waargenomen roodverschuiving: z = 8.6
melkwegstelsel UDFy-38135539

Wat is de snelheid?

$$z = \sqrt{\frac{1+\beta}{1-\beta}} - 1 = 8.6$$

$$\beta = 0{,}9785 \;\rightarrow\; v = 0{,}9785c$$



Appendix

Afleiding van de Lorentz-transformatie

$$x^2 + y^2 + z^2 - (ct)^2 = 0 \qquad x'^2 + y'^2 + z'^2 - (ct')^2 = 0$$

klassiek geldt:

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\}$$

$$\begin{aligned} & x'^2 + y'^2 + z'^2 - (ct')^2 \\ &= (x - vt)^2 + y^2 + z^2 - (ct)^2 \\ &= \boxed{x^2 + y^2 + z^2 - (ct)^2} - 2xvt + v^2t^2 \\ &= -2xvt + v^2t^2 \neq 0 \end{aligned}$$

probeer:

$$\left.\begin{aligned} x' &= a_1 x + a_2 ct \\ y' &= y \\ z' &= z \\ ct' &= a_3 x + a_4 ct \end{aligned}\right\}$$

$$\begin{aligned} & x'^2 + y'^2 + z'^2 - (ct')^2 \\ &= (a_1 x + a_2 ct)^2 + y^2 + z^2 - (a_3 x + a_4 ct)^2 \\ &= (a_1^2 - a_3^2) x^2 + y^2 + z^2 - (a_4^2 - a_2^2)(ct)^2 \\ &\quad + 2(a_1 a_2 - a_3 a_4) xct \end{aligned}$$

$$x'^2+y'^2+z'^2-(ct')^2$$
$$=(a_1^2-a_3^2)x^2+y^2+z^2-(a_4^2-a_2^2)(ct)^2+2(a_1a_2-a_3a_4)xct$$

er geldt: $x^2+y^2+z^2-(ct)^2=0$

dan is: $x'^2+y'^2+z'^2-(ct')^2=0$

als
$$\boxed{\begin{aligned}(a_1^2-a_3^2)&=1\\(a_4^2-a_2^2)&=1\\(a_1a_2-a_3a_4)&=0\end{aligned}}$$

$$x'=a_1x+a_2ct$$
$$y'=y$$
$$z'=z$$
$$ct'=a_3x+a_4ct$$

voor de oorsprong O geldt:

$$\left.\begin{aligned}&\text{in S:}\quad x=0\\&\text{in S':}\quad x'=-vt'\end{aligned}\right\}\quad\begin{aligned}x'&=-vt'=a_10+a_2ct\\ct'&=a_30+a_4ct\end{aligned}$$

$$\boxed{\frac{a_2}{a_4}=-\frac{v}{c}\equiv-\beta}$$

$$\boxed{(a_4^2 - a_2^2) = 1}$$

$$\left.\begin{aligned} &\boxed{(a_4^2 - a_2^2) = 1} \\ &\frac{a_2}{a_4} = -\frac{v}{c} \equiv -\beta \end{aligned}\right\} \quad a_4^2(1-\beta^2) = 1$$

$$a_4 = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma$$

$$a_2 = -\beta a_4 = -\beta\gamma$$

$$\boxed{(a_1 a_2 - a_3 a_4) = 0}$$

$$\left.\begin{aligned} &\frac{a_3}{a_1} = \frac{a_2}{a_4} = -\beta \\ &(a_1^2 - a_3^2) = 1 \end{aligned}\right\} \quad a_1^2(1-\beta^2) = 1$$

$$a_1 = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma$$

$$a_3 = -\beta a_1 = -\beta\gamma$$

$$\boxed{\begin{aligned} x' &= a_1 x + a_2 ct = \gamma(x - \beta ct) \\ ct' &= a_3 x + a_4 ct = \gamma(ct - \beta x) \end{aligned}}$$

$$\boxed{\begin{aligned} q_1' &= \gamma(q_1 - \beta q_2) \\ q_2' &= \gamma(q_2 - \beta q_1) \end{aligned}}$$

Afleiding van de relativistische Wet van Newton

Kracht *in de richting van de snelheid*

$$\vec{F} \equiv \frac{d}{dt}\vec{p} = \frac{d}{dt}\gamma\beta m\vec{c} = \frac{d\gamma}{dt}\beta m\vec{c} + \gamma\frac{d\beta}{dt}m\vec{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\frac{d\gamma}{dt} = \frac{d\gamma}{d\beta}\frac{d\beta}{dt} = \frac{d}{d\beta}\left(\frac{1}{\sqrt{1-\beta^2}}\right)\cdot\frac{d\beta}{dt} = \frac{\beta}{(1-\beta^2)^{3/2}}\cdot\frac{d\beta}{dt} = \beta\gamma^3\cdot\frac{d\beta}{dt}$$

$$\vec{F} = \beta\gamma^3\frac{d\beta}{dt}\beta m\vec{c} + \gamma\frac{d\beta}{dt}m\vec{c} = (\beta^2\gamma^2+1)\gamma\frac{d\beta}{dt}m\vec{c} = \gamma^3\frac{d\beta}{dt}m\vec{c}$$

$$\beta = \frac{v}{c} \longrightarrow \boxed{\vec{F} = \gamma^3 m\frac{d\vec{v}}{dt}}$$

Afleiding
van de
Kinetische energie

Kinetische energie

$$K = \int_0^v F \cdot dx = \int_0^v \gamma^3 m \frac{dv}{dt} \cdot v dt = \frac{1}{2} m \int_0^v \gamma^3 dv^2 = \frac{1}{2} mc^2 \int_0^v \gamma^3 d\beta^2$$

$$\gamma^3 = (1-\beta^2)^{-3/2}$$

$$\int_0^v \gamma^3 d\beta^2 = 2(1-\beta^2)^{-1/2}\Big|_0^v = 2[(1-\beta^2)^{-1/2} - 1] = 2(\gamma - 1)$$

$$\boxed{K = mc^2(\gamma - 1)}$$

klassieke limiet (β<<1) $\quad \gamma = \dfrac{1}{\sqrt{1-\beta^2}} \approx \dfrac{1}{1-\dfrac{\beta^2}{2}} \approx 1 + \dfrac{\beta^2}{2}$

$$K = mc^2(\gamma - 1) \approx mc^2\left(1 + \frac{\beta^2}{2} - 1\right) = mc^2 \frac{\beta^2}{2} = \frac{1}{2} mv^2$$

Afleiding van

$$E^2 - p^2c^2 = m^2c^4$$

$$p = \gamma\beta mc = \beta\frac{E}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \rightarrow \beta^2 = 1 - \frac{1}{\gamma^2}$$

$$E = \gamma mc^2$$

$$p^2c^2 = \beta^2E^2 = (1-\frac{1}{\gamma^2})E^2 = E^2 - \frac{E^2}{\gamma^2} = E^2 - m^2c^4$$

$$\boxed{E^2 - p^2c^2 = m^2c^4}$$

of

$$p^2 - \frac{E^2}{c^2} = -m^2c^2$$

Transformaties
voor
energie en impuls

Transformatie van energie en impuls

$$\left.\begin{aligned} x^2+y^2+z^2-(ct)^2&=0\\ x'^2+y'^2+z'^2-(ct')^2&=0\end{aligned}\right\}\qquad\left\{\begin{aligned} p_x^2+p_y^2+p_z^2-\frac{E^2}{c^2}&=-m^2c^2\\ p'^2_x+p'^2_y+p'^2_z-\frac{E'^2}{c^2}&=-m^2c^2\end{aligned}\right.$$

$$\boxed{\begin{aligned} x'&=\gamma(x-\beta ct)\\ ct'&=\gamma(ct-\beta x)\end{aligned}}\qquad\boxed{\begin{aligned} p'&=\gamma\left(p-\beta\frac{E}{c}\right)\\ \frac{E'}{c}&=\gamma\left(\frac{E}{c}-\beta p\right)\end{aligned}}$$

NB. Het teken van β en p wordt bepaald door hun richting: positief naar rechts; negatief naar links