Hans Freudenthal and the Foundation of a Discipline of Mathematics Education

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Samenvatting

De grondslagen van de wetenschap 'wiskunde-onderwijs' en de rol van Freudenthal daarbij, vormen het onderwerp van dit artikel.
Allereerst wordt het belang aangestipt dat de wiskunde altijd centraal heeft gestaan – dus geen algemene theorieën –, in de tweede plaats dat Freudenthal de brug heeft weten te slaan tussen een wiskundige en een didactische uitspraak, en tenslotte hecht de auteur veel waarde aan de uiterst kritische rol van Freudenthal, essentieel voor het serieus nemen van 'wiskunde-onderwijs' als zelfstandige discipline.

Summary

The foundations of the discipline of mathematics education and the role of Freudenthal are discussed by the author.
Freudenthal emphasised that match education is subject-directed. Secondly he bridged the vital gap which separates a mathematical from a didactical statement; resulting in worked-out examples.
Finally he devoted much time to the criticism of the work of others: an activity necessary to lay foundations of a discipline.

Earlier this year we celebrated the birthday of Handel, another native of Germany who came to be treasured in his adopted country. A highpoint of the Handel celebrations in England was the singing of his *Utrecht Jubilate*. How appropriate it would be to have that same piece of music on this occasion, for here in Utrecht we have much to be joyful about today. I, and, I am certain, everyone else here must be grateful to have the opportunity to celebrate Hans Freudenthal's eightieth birthday and to pay homage to him and to his work in the field of mathematics education. When doing this, we must remember that this is but a fraction of Professor Freudenthal's work, for we might well be celebrating his enormous contributions to mathematics or the philosophy and history of science. Yet, to revert to my comparison with Handel, I see these last achievements as in some way comparable with Handel's concerti grossi, Italian operas and organ concertos. They are indeed, great achievements, but for me, tend to pale alongside his later, more original English dramatic oratorios and operas – the Messiah, Samson, Semele, ... Within mathematics and philosophy, Hans Freudenthal has carried on a great tradition. In mathematics education, however, he has done more; he has significantly altered our conception of a discipline and of how it should be practised.

Before developing the main theme of my talk, I should like to make a few remarks in my capacity as Secretary of the International Commission on Mathematical Instruction – ICMI.

There is no doubt of the extent to which Hans Freudenthal has affected the work of ICMI. He was a member of its Executive Committee from 1963 to 1974 and its President between 1967 and 1970. Even earlier, in the 1950's, he had a significant influence on ICMI through his criticism of what had become the Commission's traditional way of working. He objected to its preoccupation with the exchange of descriptions of organisation, administration and curricula. It was the duty of ICMI, he argued, to promote research and study of a scientific nature, to foster work in depth and to encourage school teachers to call upon their experiences – qualities which have always distinguished his own activities. Yet it is for his years as President that he will be best remembered, for these saw the first International Congress on Mathematical Education which he perceived as a showcase to display the best and most serious in mathematics education, and also the launching of *Educational Studies in Mathematics*. This journal, originally conceived within the context of ICMI, was in fact to appear as an independent publication. Its importance in fostering serious research and other work in mathematics education and in the establishment of standards is hard to overestimate.

This, then, seems a very appropriate point at which to interpolate a special birthday greeting to Professor Freudenthal from the current President of ICMI, Professor Jean-Pierre Kahane.

"Felix Klein, geometer, made a great and lasting contribution to the teaching of mathematics during the first half of this century. Hans Freudenthal, logician, has stamped his mark on mathematics education in the second half of the century. A great lesson to be learned from his work, concerns the links to be made between history, formalisation and application. Let me call upon a personal memory. I read in 1958, just after its publication, his short book in French on applications of mathematical logic, and I have very recently reread it. In it he develops in an original way the links between logic, and calculating machines including new mathematical problems suggested by computing. Thus a quarter of a century ago he anticipated the study which ICMI has recently launched on the influence of computers and informatics on mathematics and its teaching. As President of ICMI, I have much pleasure in greeting and saluting my illustrious predecessor."

Now to return to my theme on the foundations of mathematics education. First, what do we mean by 'mathematics education'? Here, I feel that we in England are disadvantaged. The term 'mathematics education' lacks precision. Does it simply mean 'mathematics teaching' and are, therefore, all mathematics teachers necessarily mathematics educators? This is not the interpretation I should wish to give to the phrase. Does it then mean 'didactics of mathematics' – the study of how mathematics is taught and learned? Again, my answer would be 'No', but this time with the qualification that 'didactics of mathematics' is a vital part of mathematics education, and that didacticians, indeed *all* who seriously study the learning and teaching of mathematics, can be termed mathematics educators.

Yet there is more to mathematics education than didactics. Let me illustrate this by reference to a paper by Freudenthal in which he described changes in mathematics teaching in the Netherlands since the late 1960's [1978b]. In this he sets out the objectives of the CMLW (Commissie Modernisering Leerplan Wiskunde) (Commission for the Modernisation of the Mathematics Programme) as seen by the Staatssecretaris van Onderwijs, Kunsten en Wetenschappen, (Underminister for Instruction, Arts and Sciences). The first three objectives concerned the curricula which pupils of different types might follow. The fourth was of a different nature:

> 'to study the measures which should be taken to promote the orientation of teachers on the development of modern mathematics' (p. 261).

Considering desirable changes is then insufficient. One must also study how change is to be brought about. This introduces a new dimension to mathematics education in addition to 'didactics'. It is now, for example, necessary to study the various attitudes towards mathematics which are held by different interested parties – pupils, teachers, parents, employers, ... We become involved in the sociology of mathematics teaching – a vital part of mathematics education.

As Freudenthal points out, the two most influential factors in the changes in mathematics teaching in the Netherlands were:

> 'social change and change of our secondary educational system' (p. 262).

Reacting to such changes, which so greatly affected educational systems in the 1960's (and are being felt today in an even more intense form as national economies falter, unemployment rates soar and as attitudes of adolescents towards schooling become increasingly polarised), was not something to which traditional didacticians were accustomed, nor for which they were prepared. 'Mathematics education' then had to expand to meet these new challenges. Freudenthal was to stimulate study of such activities through his mounting of an ICMI review on how ideas for curriculum change had actually been realised during the 1960's and 1970's.

Here, then, we see how 'mathematics education' has a very significant political dimension. It is not only about teaching and learning mathematics, it is also about creating and changing political and social climates and opinion. If we do not realise this, and act accordingly, then we must be prepared to suffer the consequences. Let me give two examples. In the paper to which I have already referred, Freudenthal wrote:

> 'Though the government pays for all instructional needs, it is an axiom of educational policy that . . . the programme and the instructional methods are defined by those who run the school' (p. 263).

That would have been a true statement if made about England in 1970: it is no longer so. Axioms can be changed in educational systems as well as in geometry!

My second example is taken from a letter which Freudenthal sent to me in 1983. In this he explained the greatest mistake he had ever made in mathematics education. This was not a false theory he had accepted or pursued: rather it was his decision not to attend the Royaumont seminar in 1958 which effectively launched the modern math movement. As he wrote, he had attended many 'nonsense' meetings and assumed that this would be yet another in the series. Only too late, and when the damage has been done, did he realise that governments had been made to take a state in this piece of 'nonsense'. Here a deliberate attempt was being made to create a particular political climate. The community of mathematics educators must keep a close watch on such initiatives. We cannot ignore politics. Mathematics education of necessity must contain a strong social component which spills over into political considerations.

Sociological considerations are also important in mathematics education itself. Mathematics education should include the study of the sociology of itself. Freudenthal (1979a) has himself written about, and criticised, the codes which empirical researchers have established, to ensure that their work and that of their colleagues is given 'due' recognition. One outcome of the last twenty or so years has been the creation of a group of 'professional mathematics educators': a somewhat mixed blessing. There is no doubt that

mathematics education is a sufficiently serious subject to merit a person's full-time attention. However, if this cuts the person off from (or worse, if (s)he has never been strongly attached to) mathematics and its teaching (as an active teacher, not a passive observer), then there are great dangers. Not the least is the thought that this prevents the keen practising teacher from being a mathematics educator. For this should not be an exclusive club with entry determined through the post one holds; rather entry should be governed by a state of mind concerning mathematics and its teaching.

The defining quality of the teacher as mathematics educator comes through very strongly in all Freudenthal's work:

Explore what is known about learning and about what can happen in a classroom and try to use this knowledge in your teaching.

> Reflect on the means you use to teach mathematics.
> Reflect on the nature of mathematical activity.
> Determine upon a philosophy of mathematical activity and mathematics education and fit your instruction to this.
> (See, for example, van Bruggen and Freudenthal, 1977, pp. 230–231)

What then are the lessons we can learn from Freudenthal? Here I am not concerned so much with the content of what he has written, but rather with the manner in which he has approached the task.

First, there is the constant emphasis that mathematics education concerns the teaching and learning of *mathematics* – it is subject directed. This does not mean that we cannot look at wider educational issues – indeed, it is vital that we should never forget that the learning of mathematics should only form part of a much wider and rewarding educational experience. But Freudenthal never departs far from this primary concern. Of course, there will be few who can bring to mathematics education that wide knowledge of mathematics which Freudenthal possesses and that experience of creativity which can only come to so prolific a researcher in mathematics. Yet we can hope to match him in one respect, in our constant attempts to deepen our understanding of that mathematics we are called upon to teach.

> "There are so many kinds of understanding in mathematics. At every moment you may believe that you have just reached ultimate understanding of some subject, such that nothing is left to be desired. But no, there is no ultimate understanding in mathematics, you can understand any problem in an ever larger context, from an ever higher point of view; and finally – it looks the lowest of all, but perhaps it is the highest – you can learn to understand it in the perspective of the learning child." (Freudenthal, 1977, p. 374).

Some years ago Trevor Fletcher asked whether there was any special knowledge of mathematics which the mathematics teacher should have that distinguished him from other mathematicians. The answer surely lies in the deeper understanding of the fundamentals of which Freudenthal writes.

There is also the need for us to form a philosophy of mathematics, founded on a knowledge of how mathematics has developed, on which to base one's teaching – a philosophy of mathematics which can be translated into a philosophy for mathematical instruction. Of Freudenthal's philosophy there is no doubt:

> 'Mathematics is a natural and social activity which develops according to the growth and the growing needs of the individual in an expanding world. Mathematics is an attitude, a way to master this world cognitively, practically, emotionally' (Freudenthal, 1979b, p. 322)

> 'Our mathematical concepts, structures, ideas have been invented as tools to organise the phenomena of the physical, social and mental world'. (Freudenthal, 1938, p. ix)

Yet a knowledge of mathematical structures is insufficient for our needs as teachers. In addition we must have a knowledge of instruction with which to develop a didactical phenomenology. Professor Kahane, you will remember, drew our attention to the great influence which Felix Klein had on mathematics education earlier this century. His *Elementary Mathematics from an Advanced Standpoint* was intended to help teachers increase their understanding of basic mathematical structures. In a similar fashion, Choquet, say, was in the 1960's to help us see geometry in a new, 'modern' light. Yet, neither of these two mathematicians attempted to bridge the vital gap which separates a mathematical from a didactical statement. They provided what Georges Glaeser has termed 'pedagogy without pupils'. Freudenthal, in his books and at IOWO has, however, gone beyond this. He has illustrated most vividly that the special knowledge which we as teachers and mathematics educators must possess must include not only that deeper understanding of the mathematics we are called on to teach, but also an awareness of how we can, through a sound knowledge of didactics, foster our students' development of the underlying mathematical concepts.

At the level at which I am now writing, Freudenthal's message does not appear very original. One can give many examples of writers, from the early nineteenth century onwards, who made similar remarks. What Freudenthal has taught us to do, however, particularly in his recent writings, is to go beyond the stage of stating all-embracing principles, and to produce analyses and exemplifications. As teachers we cannot exist and progress on a diet of maxims: there is a need for worked-out examples which we can adopt or adapt in our teaching. These Freudenthal and IOWO have sought to provide. Of course, it is *safer* to stick to well-worn generalities, but Freudenthal has rarely opted for safety and appeasement. Indeed it is this which brings me to the third point which I wish to mention and where I believe Freudenthal has made a unique contribution to mathematics education; that is the time he has devoted to the criticism of the work of others.

A remarkable feature of the 1960's and 1970's was the number of mathematics educators who produced a variety of schemes, ideas, texts, etc. and who did so without any apparent recognition that others were doing so too and that there were remarkable differences in assumptions, goals and attempted solutions. In general there was an air of 'dog does not eat dog' – one educator did not criticise another, and everyone went on with his own work largely oblivious of the others. *This was not an atmosphere conducive to laying the foundations of a discipline.* One definition of 'discipline' given in my dictionary is 'training through correction and suffering'. Often in the 1960's and 1970's it was the children who suffered because of the lack of correction so far as the mathematics educators were concerned. Serious, detailed criticism is not easily carried out, and it takes one away from those activities which more readily gain general esteem. Freudenthal's biting criticism of the way in which much empirical research work in mathematics education is carried out and, more importantly, reported and quoted (Freudenthal, 1979a), must have taken hours and hours of work which in itself carried little personal reward or satisfaction. Yet that work had to be done in order to assemble a strong case against the system – a case presented by means of an argument which justified the word 'discipline'. Another example is the 1975 paper 'Pupils' Achievements Internationally Compared', a sixty page critique of the First International Study on Mathematical Attainment. Such critical work is, as I remarked earlier, unfashionable and certainly it is not guaranteed to make one many new friends, but without it there is no chance that we shall ever attain to that science of mathematics education which Freudenthal's *Weeding and Sowing* (1978a) sought to preface.

Yet, this kind of work can have its effect. I know personally how much the Second International Mathematics Study was influenced by Freudenthal's criticism of its predecessor.

Again, every teacher and mathematics educator cannot be expected to criticise with the weight and authority of Freudenthal. It is vital, though, that we all see criticism and the weighing of evidence as essential parts of our work – we must learn not to accept glib explanations; not to be overawed by a smokescreen of long words, complicated prose or pseudo-scholarship; not to accept wild extrapolations based on what a gifted teacher has found possible on a limited scale and often in privileged conditions; or to take optimistic claims of curriculum developers, publishers and textbook authors at face-value. Such a critical appraoch is not just for the Freudenthals of this world, it should be adopted by all.

Criticism is an essential part of the establishment of standards. It is a mark of Freudenthal's efforts that not only *Education Studies in Mathematics* but now other periodicals in mathematics education subject submitted papers to a strict refereeing system.

Yet criticism must be based on knowledge and understanding – the person who whistles or boos at a theatre is not necessarily acting in the critical manner that we should wish to encourage, or that is associated with a science or discipline.

It is fashionable in some quarters of mathematics education to dismiss the notion of 'gurus' to whom we should listen with rapt attention. Instead there are arguments in favour of discussions and 'participatory activities' in which all persons are equal. This would seem to me to undermine the idea of mathematics education as a discipline, for it discounts the possibility of accumulated knowledge. All men are not equal in a science, although all practitioners may have something valuable to contribute. Can one have a discipline without there being some basic knowledge which all those who wish to work in it, or one of its sub-disciplines, should possess? Here, I believe Freudenthal has played his part in demonstrating clearly that there are those who have accumulated such knowledge as to command our attention and respect – although, of course, not necessarily always our full agreement. In his books, too, he has tried to identify and explore that basic knowledge on which we can build a true discipline. And here we must not gloss over the difficulties. As I have already emphasised, mathematics education is greatly affected by political, social, technological and mathematical changes. We cannot then expect the 'basic knowledge' of a mathematics educator to remain constant: it is a function of time and position. Yet in some senses we can compare it to the wheel. Technological change has demanded new types of wheel – from bicycle tyres to those for jumbo jets. Yet designers have built on existing knowledge and techniques: they have modified existing technology and developed new ideas. What they have not done is kept reinventing the wheel! Yet we in mathematics education seem to have fallen into the habit of doing this. Perhaps, soon, building on Freudenthal's work, we shall more clearly identify that foundation knowledge of mathematics education.

As I remarked earlier one can study the sociology of mathematics educators, and I have already mentioned the creation of ritual codes. But groups, in order to win esteem and power, create not only codes of behaviour but also new languages, their own professional jargon. Some expert language is essential, for we need new words for new concepts: much, I suspect, is not. Here, I should like to quote Freudenthal again: "what is true may be said in plain language" (1983, p. 179). No doubt I shall now be accused, along with Freudenthal, of falling prey to "simplistic reductionism." However, there is a clear message for mathematics educators and teachers here. It is our job to communicate – and widespread communication, in or out of the classroom, depends upon the choice of intelligible, simple language. The value of incomprehensible gurus is very limited.

I, however, realise only too well how I am failing to live up to my own maxim. Today you are having to make the effort to comprehend in a foreign language. How I envy Freudenthal's command of language! It must be reassuring to answer Dieudonné, as Freudenthal did on one occasion, by remarking "Don't shout at me, because I can shout as loudly as you *and in more languages*!"

That brings me to another point: although Freudenthal treats mathematics education in a serious way, it is

always clear in his writings that there is much else in his world, that this is no blinkered, short-sighted specialist, but a broadly educated man – who retains a twinkle in his eye. We must all have our favourite Freudenthal stories – those simple, deflating remarks that drive a swordpoint through windbags. I recall an English reviewer who compared him with Don Quixote: one who never missed the opportunity to go out of his way to tilt at a windmill. I don't think the comparison was entirely apt, but I can see the reviewer's point – and I rejoice that Freudenthal should have so often yielded to the temptation. It has brought me, and I am certain others, much enjoyment: "Hegel, Husserl, Heidegger, Habermas – Is it by accident that the names of the most pretentious producers of unintelligible talk in the German philosophy start with an H? (1983, p. 28)"

I must draw this paper to an end, otherwise I dread to think of what Freudenthal will say about Englishmen whose names begin with an H. In closing, may I refer again to those qualities which we associate with him and which we can all try with profit to acquire: the capacity for increasing understanding, for observing, reflecting, criticising (both self and others), experimenting, learning, and explaining. Then we can, in his beautiful and simple phrase, go not 'back to basics', but rather 'forward to basics'.

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