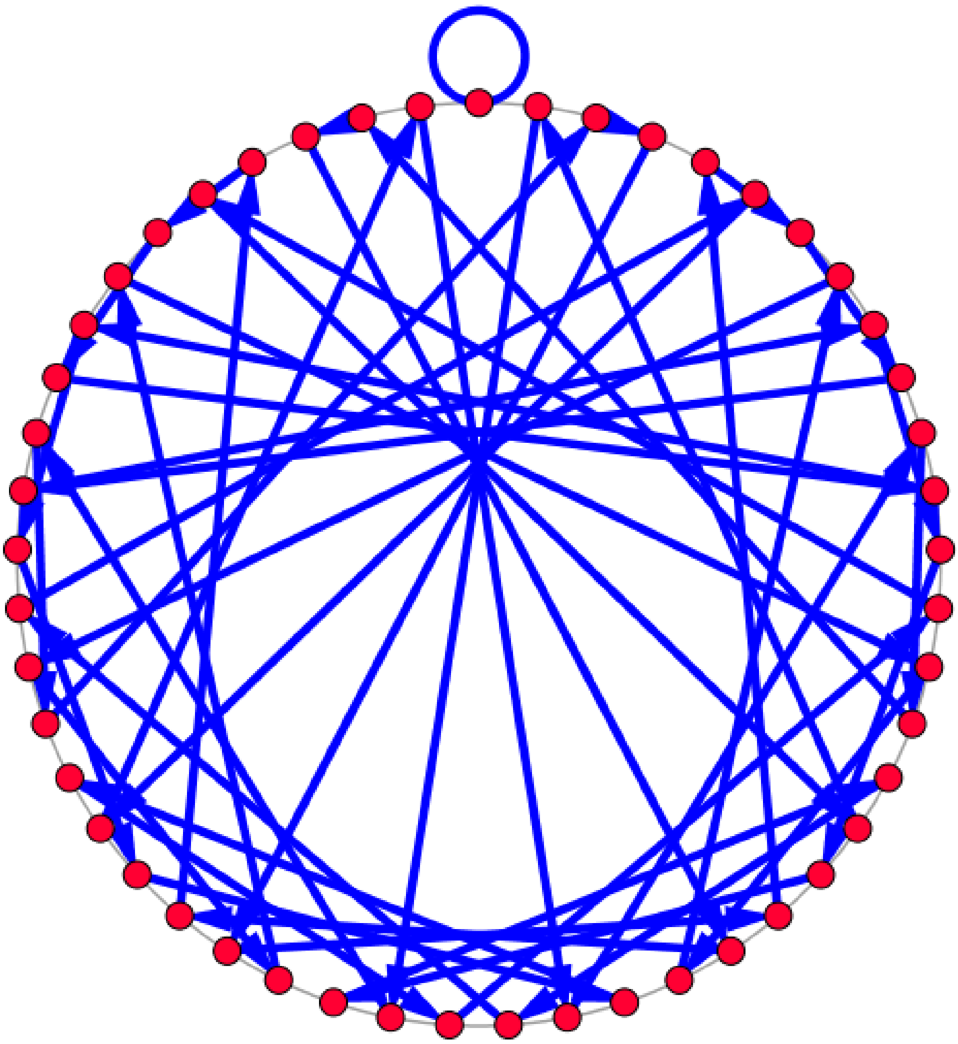
Arrow clocks



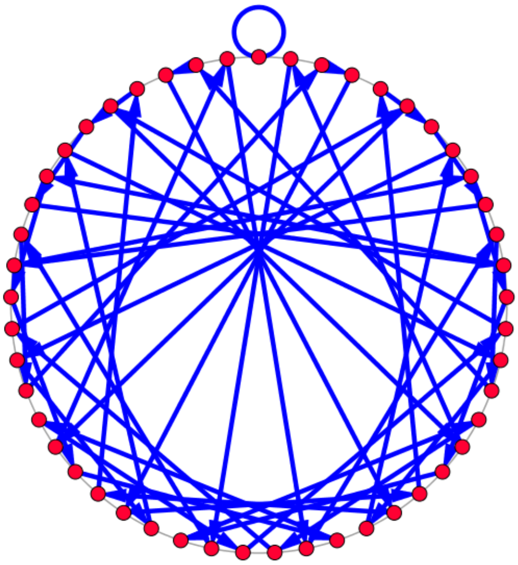
Mathematics B-day 2017

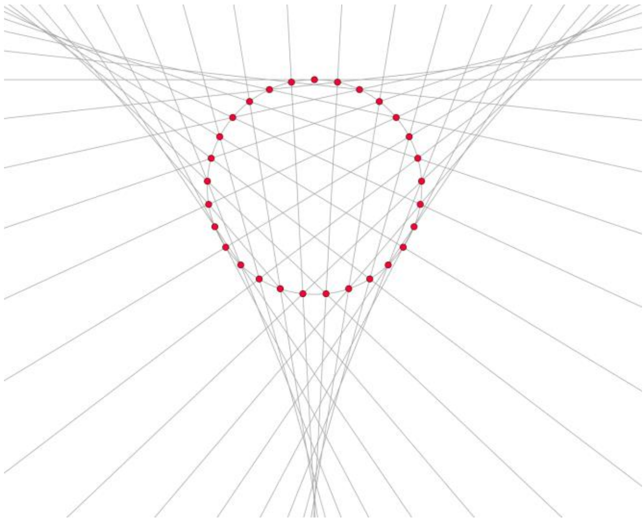
Mathematics B assignment 2017

Introduction

About the assignment

People (including you) are made to recognise patterns and structures. Mathematics turns this into a sport. In this assignment simple recipes lead to wonderful images which we call arrow clocks and line.





The challenge for you is to discover the patterns and structures underneath.

Structure of the day

This Mathematics B-day assignment consists of basic assignments, an extra assignment and a main assignment. Try to spend approximately half the day on the main assignment. It consists of an open investigation in which you will go on a journey of discovery. Experiment, ask questions, discover patterns and explain them. Each (group) can go its own way.
The complete report must be handed in by 16:00. Plan your time and divide the tasks among your group. It may be useful not to wait until the afternoon with writing out the details of your work on the introductory assignments.

What to hand in?

You will hand in a report by the end of the day. In this report you describe your solutions to the assignments, and in particular your results with respect to the main assignment. Use your own word to tell your story in such a way that it is clear and convincing.

Hints:

- Be clear: so that your work can be understood by someone who did not take part in the Mathematics B-day (but does have sufficient mathematical understanding). This means that you must formulate your text fully, and not refer to the text from the assignment.
- If you provide foundations, explanations or clarification, try to use *mathematical argumentation* as much as possible. A combination of clarity, conciseness and correctness would be great!

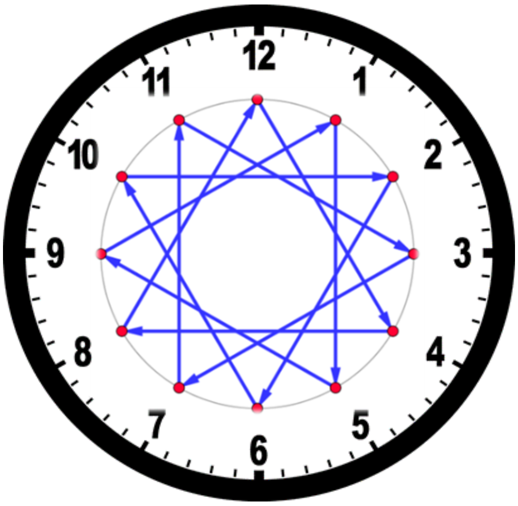
- Use a lot of figures to illustrate your ideas. Make use of, for instance, screen captures or a pair of compass and ruler.
- Draw up your planning and divide the tasks amongst the group.

Both the mathematical content of the report and the way it has been written down will play a part in the assessment!

Basic assignments

Assignment 1 Clock

Below you see a clock with twelve (red) dots at the numbers. Arrows have been drawn from each full hour to 4 hours later.

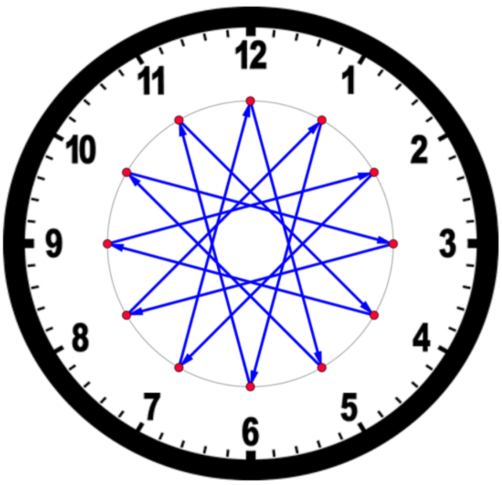


We can see four equilateral triangles with the previously mentioned (red) dots as their vertices.

(a) Draw a similar type of figure where the arrows point towards 9 hours later. What do you see now?

We call this kind of figure an arrow clock. We have now looked at arrow clocks where the arrows point 4 and 9 hours ahead. In both cases the blue arrows make polygons with the red dots as vertices. What would the arrow clocks look like if the arrows point at b hours later with $b = 1,2,3,4,5, \ldots, 10$ or 11?

Polygons could also be star-shaped. If $b = 5$ you will get the following figure



We say that the arrows form a dodecagon (or 12-gon).

(b) Investigate what kind of polygons the arrows form for the different values of b (1 to 11) and how many polygons there are then. You can use the provided work sheet.

On another planet the clocks have 15 hours instead of 12. You can do the same investigation for these clocks that you did in part b): for what values of b do the arrows form more than one polygon with red vertices. This involves the greatest common denominator of b and 15.

(c) Explain how this works.

(d) Find a clock (other than with 12 or 15 hours) which has only one polygon for each value of b.

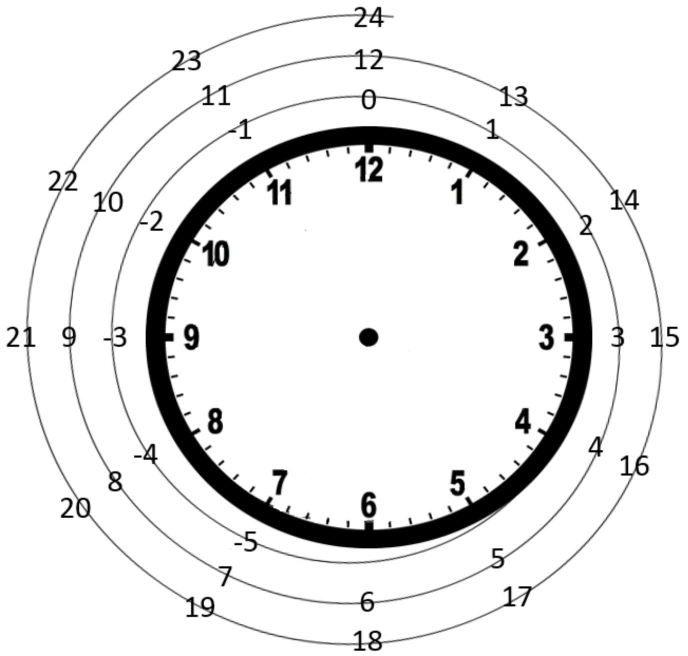
Modular arithmetic

The Figure above with the twelve hour clock contains arrows $1 \to 5$, $2 \to 6$, $3 \to 7$, etc. In general the arrows complied with the rule:

$$x \to x + 4,$$

with an important comment: past 12 you started at 1 again. So: $9 \to \cancel{13}\, 1$, $10 \to \cancel{14}\, 2$, etc. We write this as $13 \equiv 1 \pmod{12}$, $14 \equiv 2 \pmod{12}$ etc.
Another way to phrase it is that $a \equiv b \pmod{12}$ if the difference of a and b is a multiple of 12. For example, $38 \equiv 14 \pmod{12}$ and $100 \equiv 28 \pmod{12}$, but also $7 \equiv -5 \pmod{12}$ and $-11 \equiv 49 \pmod{12}$.

You can look at it as if you're winding the number line around the clock. The numbers -17, -5, 7, 19, 31, etcetera all end up on the 7 and are the same modulo 12.



The "mod" signifies "modulo" and using it for arithmetic is called "modular arithmetic".

If you want to know $92 + 74 \pmod{12}$, you can just do $8 + 2 \equiv 10 \pmod{12}$, because $92 \equiv 8 \pmod{12}$ and $74 \equiv 2 \pmod{12}$. This also works for multiplication

$$92 \cdot 74 \equiv 8 \cdot 2 \equiv 16 \equiv 4 \pmod{12}.$$

But be warned: it usually does not work for division!
Of course you can also do modular arithmetic with other numbers than 12.

Assignment 2 Polygon equation

In assignment 1 you looked into the rule of the form

$$x \to x + b$$

for whole numbers b. In the case of an equilateral triangle we see:

$$x \to x + b \to x + 2b \to x + 3b.$$

The final arrow should arrive back at the starting point, so $x + 3b \equiv x \pmod{12}$. Subtracting x on both sides, gives the equation $3b \equiv 0 \pmod{12}$.

We can solve this equation as follows. $3b \equiv 0 \pmod{12}$ means that $3b$ and 0 differ by a multiple of 12. So you can write

$$3b = 0 + 12k$$

for a whole number k. Dividing by 3 gives

$$b = 4k.$$

So we find that b is a quadruple, or

$$b = \ldots, -8, -4, 0, 4, 8, \ldots$$

Conclusion: for these values of b you will have an equilateral triangle… except for 0.

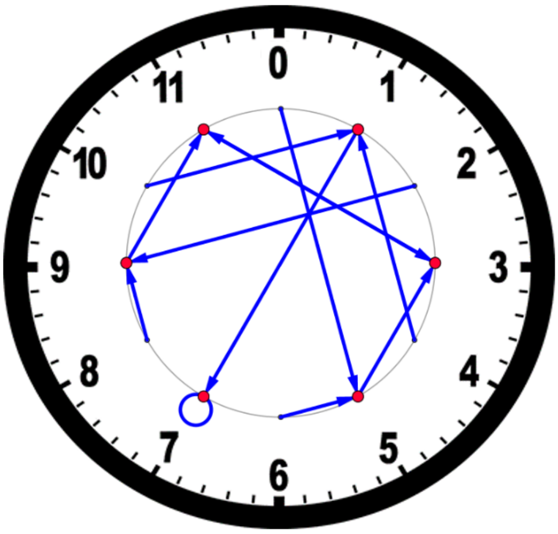
(a) Why do we also find $b = 0$, while we do not get a triangle?

You can investigate in the same way for what b the rule $x \to x + b$ leads to equilateral pentagons on a 15 hour clock.

(b) Provide the equation; explain how you found it; and show you can solve it.

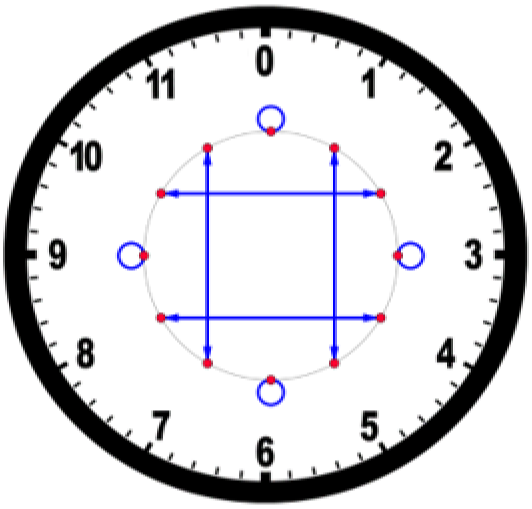
Assignment 3 Rules

We gain more variation by looking at a larger class of rules for the arrows, for example $x \to 2x + 5$. Note that $12 \equiv 0 \pmod{12}$, so you can also place a 0 at the top of the clock.



The loop at 7 indicates that this number will be connected to itself.

(a) Draw the arrow clock for the rule $x \to 3x + 2$ with a 12 hour clock.



(b) Provide a rule $x \rightarrow \ldots$ for the above arrow clock.

Now turn the arrows within the clock by $60°$ clockwise.

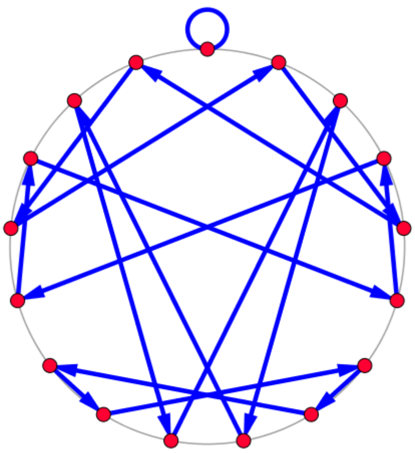
(c) What is the rule now?

This Mathematics B-day assignment involves arrow clocks like those you encountered in the assignments above.

Up to now we looked at arrow clocks based on 12 or 15 hours, but there is no reason to limit ourselves to those. We use n to refer to the number of hours on the clock.

Assignment 4 Target points

Below you see a nicely symmetrical arrow clock with $n = 17$. We will leave out the clock face from now on. The rule here is $x \rightarrow 4x$.



A (red) dot where the head of an arrow finishes is called the *target point*. A (red) dot where the tail of an arrow starts from is called the *starting point*.

(a) Calculate that every point is a target point, if the rule is $x \rightarrow 4x$ and $n = 17$.

How does that work? When are points target points or not?

(b) Why is 1 not a target point for the rule $x \rightarrow 2x$ and $n = 12$?

(c) How many arrows arrive in each target point for the rule $x \rightarrow 2x$ and $n = 12$?

For the rule $x \rightarrow 4x$ and $n = 15$, 1 is a target point, because $4 \rightarrow 1$.

(d) Use the fact that $4 \rightarrow 1$ to quickly find the starting point of the arrows with target points 2, 3, 4,…, 14.

(e) Use the same method to indicate with an equation what the starting point is for every target point 1, 2,3,…,44 for $x \rightarrow 4x$ and $n = 45$.

It is useful to involve a bit of number theory in reasoning about arrow clocks.

> Two numbers a and n are relatively prime if the only positive whole number that divides both is 1 (or in other words the largest common denominator of the numbers is 1, that is to say lcd$(a, n) = 1$).
>
> If this is the case, then the equation
>
> $$a\,x \equiv 1 \pmod{n}$$
>
> has exactly one solution modulo n.
>
> In the extra assignment you can find out why this is the case.

For example: 9 and 14 relatively prime; the equation $9\,x \equiv 1 \pmod{14}$ has $x = 11$ as its only solution modulo 14, because $9 \cdot 11 = 99 = 1 + 7 \cdot 14 \equiv 1 \pmod{14}$. If youwant, you can easily check that other values for x are no solutions.

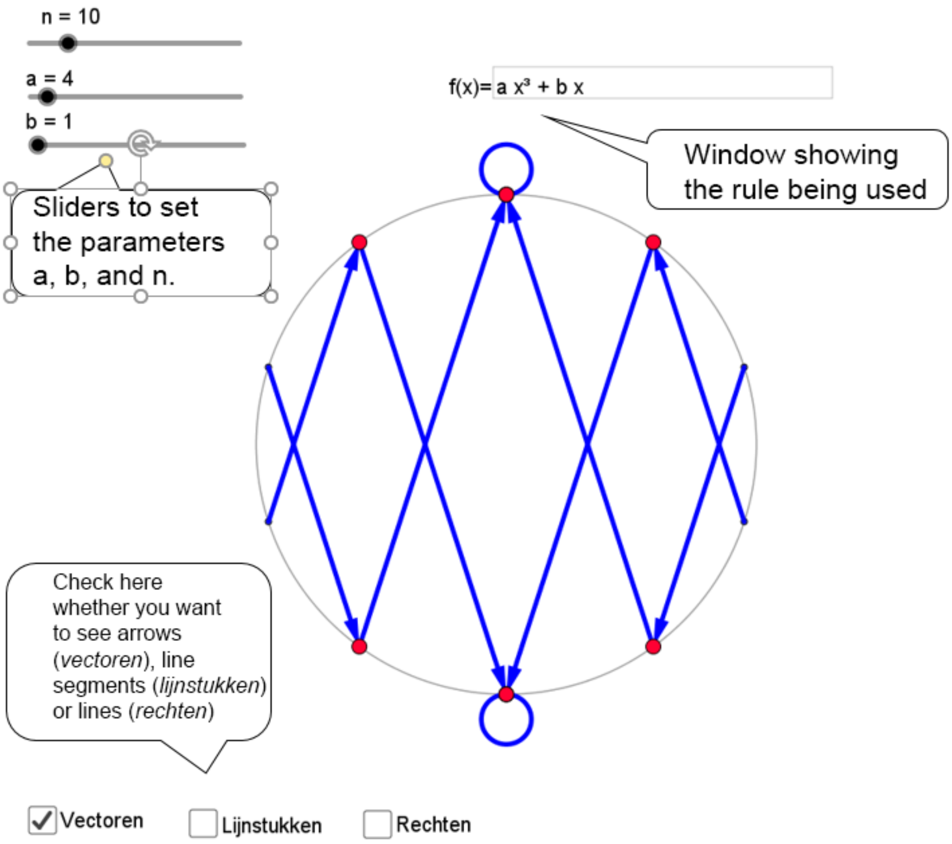
(f) Solve: $4x = 1 \pmod{9}$.

We will now look at the general case: the rule is $x \rightarrow ax$ and the clock has n hours.

(g) Explain how it follows from the above theorem in the frame that every number is a target point if a and n are relatively prime.

Assignment 5 Geogebra application

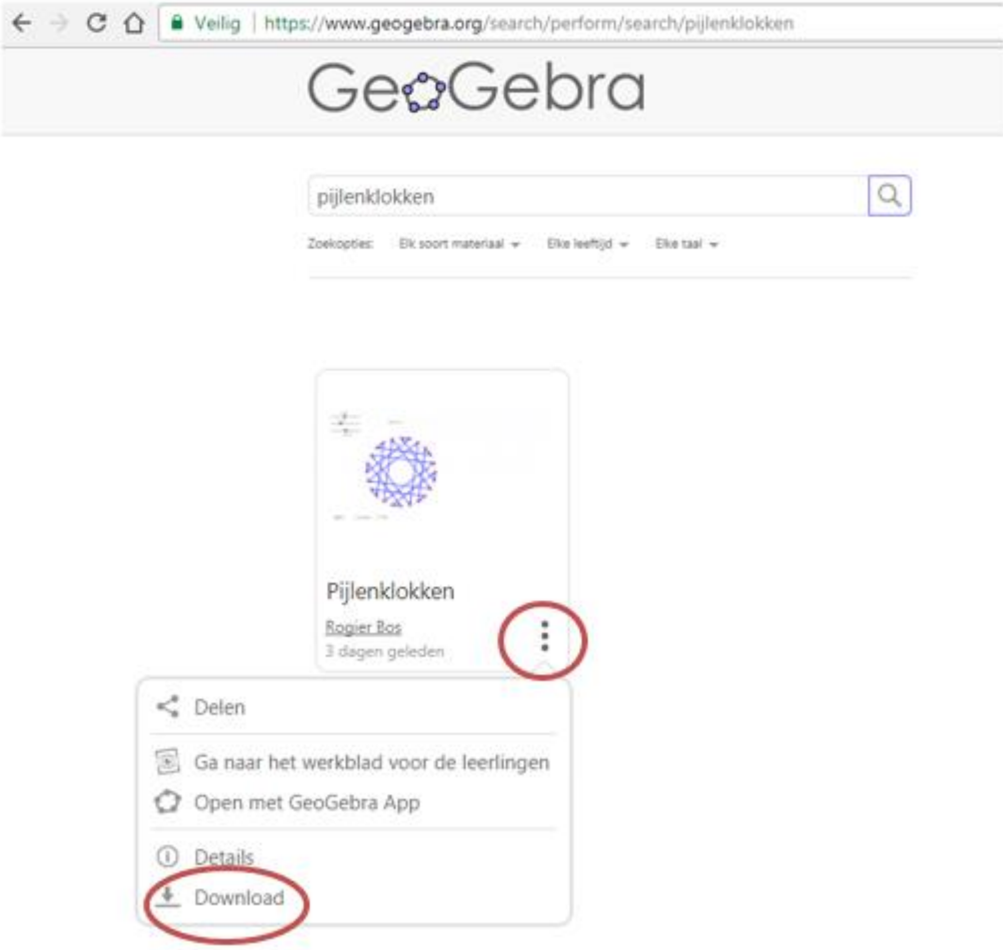
Go to https://www.geogebra.org/m/ZUDcUk2C in your browser. You will find an online Geogebra application that you can use to investigate arrow clocks for different rules.



You can use the sliders on the left at the top to vary the values of the parameters. Give it a try. At the bottom on the left you can check to select arrows (i.e. vectors) and/or line segments between the points and/or lines through the points.

Using the rule window you can set another rule. For instance, type x^2 or a x^3+b. The typed rule will appear in the rule window after you press *enter*. Keep in mind that the rule must have whole numbers as its output. You can use the parameters a and b in the rule, but you have to type them separately from the variables, so, so “a x” or “a*x”, not “ax”.

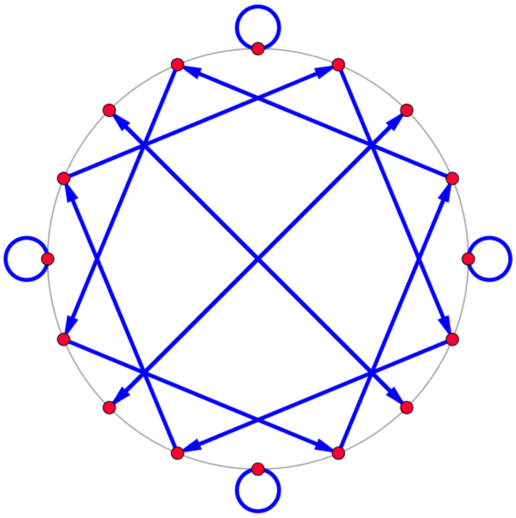
You can also use Geogebra offline if the software is installed on your computer. Download the .ggb-file via https://www.geogebra.org/materials/. Look for “Pijlenklokken”.

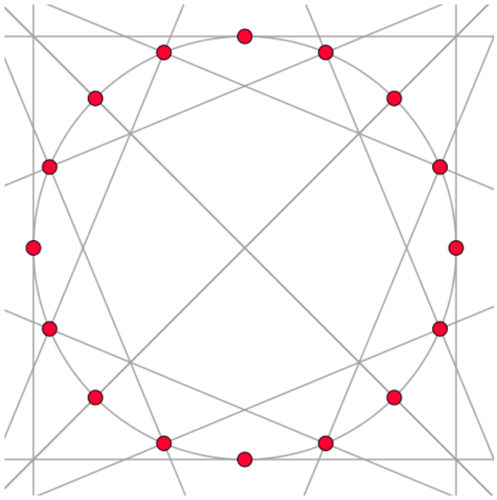


Click on the three small squares and then on “download”. You can then open the .ggb-file from Geogebra on your computer. You will probably find it in your download folder. If you are working offline, you cannot enter new rules with the rule window; you will have to use the entry window at the bottom. If you use that, your rule must always start with “f(x)=”.

Assignment 6 Lines

Instead of arrows, we can also draw lines: see below for the rule $x \to 5x$ and $n = 16$. We call the figure on the right a *line clock*.

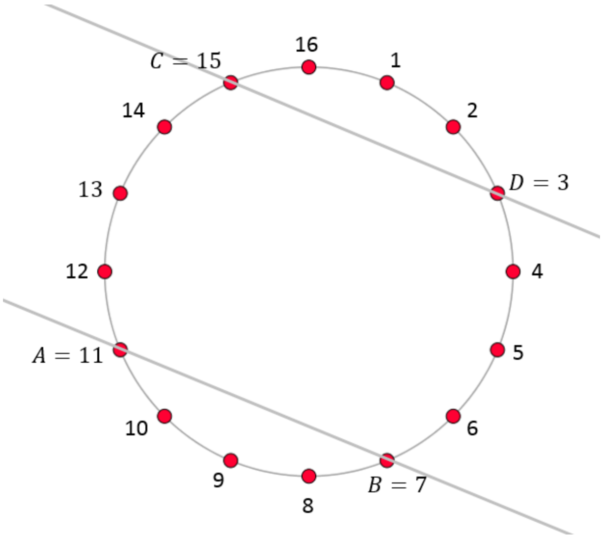




For the loops we draw a tangent line to the circle (why is that not such a bad idea?).

We can see many pairs of parallel lines. We will try to *explain* this in the following assignment.

Look at the following figure



Lines AB and CD are parallel.

(a) How can you quickly conclude from the numbers A, B, C, D that AB and CD are parallel? More formally: there is an equation in A, B, C, D that means precisely that AB and CD are parallel. Find that equation.

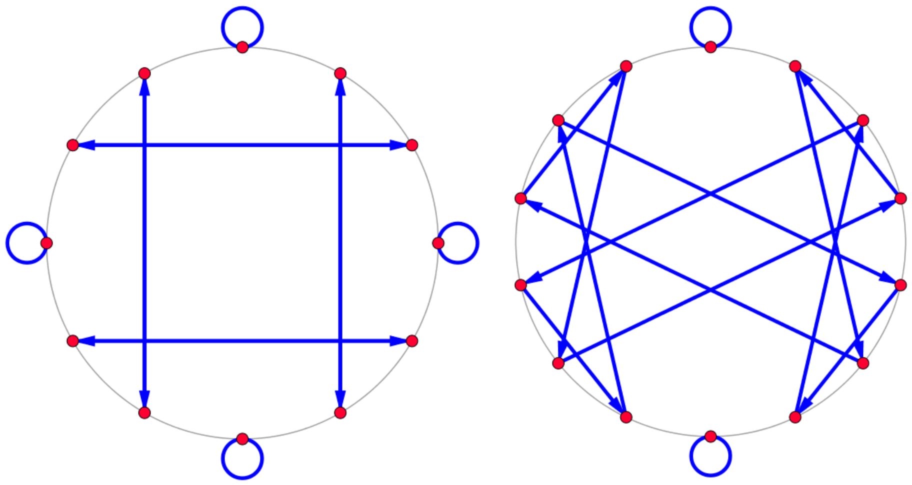
(b) Apply your finding from (a) to the rule $x \to 5x$ and $n = 16$. Can you find all pairs like this?

(c) Reason, with the aid of the rule in (a), for which values of n we can find pairs of parallel lines for the rule $x \to 5x$.

(d) Explain why your equation from part (a) yields parallel lines.

Assignment 7

In assignment 3 you encountered the arrow clock on the left below.



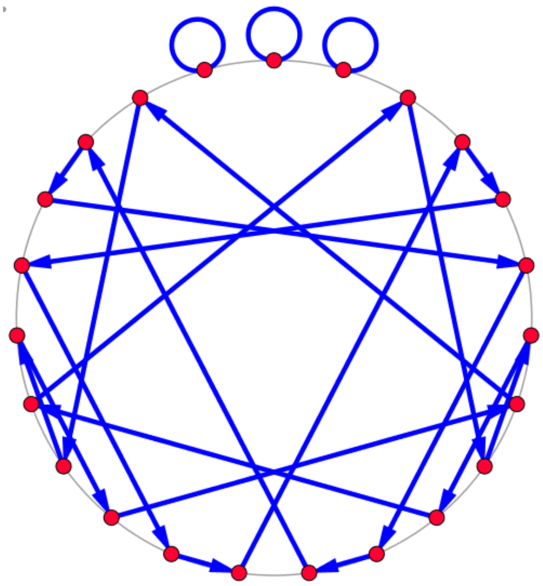
You see four loops in this clock. In the arrow clock on the right, that was made with the same rule, but with $n = 14$ you see two.

(a) Examine rules that have the form $x \to ax$ and vary a and n. Find a rule/equation for the number of loops as a function of a and n.

(b) Explain this rule/equation.

Assignment 8

(a) Play a bit with the different values for n with the rule $x \to x^3$ and write down some things you notice.



You may have noticed that all arrow clocks have axial symmetry on the vertical line through the middle.

(a) Explain why the arrow clocks for the rule $x \to x^3$ have axial symmetry on the vertical axis for all values of n.

(b) Give more rules where the arrow clock has axial symmetry on the vertical line through the middle. Explain, if you have used a mathematical argument in your search.

(c) Describe a rule where the arrow clock has axial symmetry fors ome values of n, but not for other values of n. Explain, if you have used a mathematical argument in your search.

Extra assignment

Theorem: if lcd$(a, n) = 1$, then the equation

$$a\,x \equiv 1 \pmod{n}$$

will have exactly one solution modulo n.

You will prove this by explaining how, if we vary the value of x from 0 to $n-1$, $a\,x$ will assume all values from 0 to $n-1$ exactly once modulo n.

(a) How does the theorem follow from this?

Assume that

$$a\,x_1 \equiv a\,x_2 \pmod{\mathrm{n}}$$

for two whole numbers x_1 and x_2.

(b) Explain how it follows from this that

$$a\,(x_1 - x_2) = k \cdot n$$

for a whole number k.

(c) Derive from this that n divides the difference $x_1 - x_2$.

(d) Complete the proof.

Final assignment

If you vary the rules and sliders, you end up with beautiful arrow and line clocks.

We challenge you to study them closely.

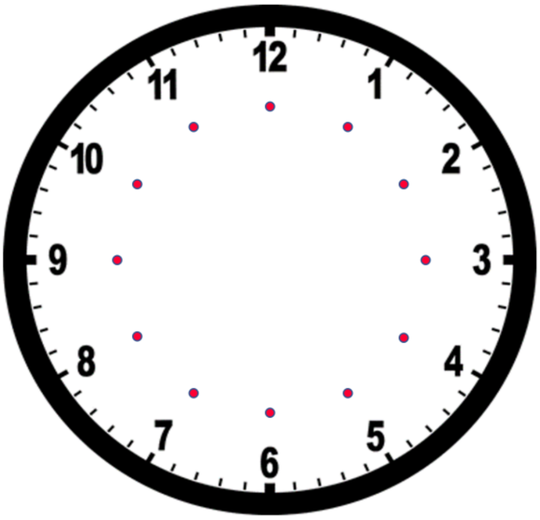
- Describe what geometrical phenomenon you are seeing: think of the mutual placement of the lines or arrows, rotational or axial symmetry, (the number of) target points, (the number of) loops, polygons, …
- Find patterns. For example: if n is a multiple of three, then…
- Explain the patterns, for example using division properties of numbers.

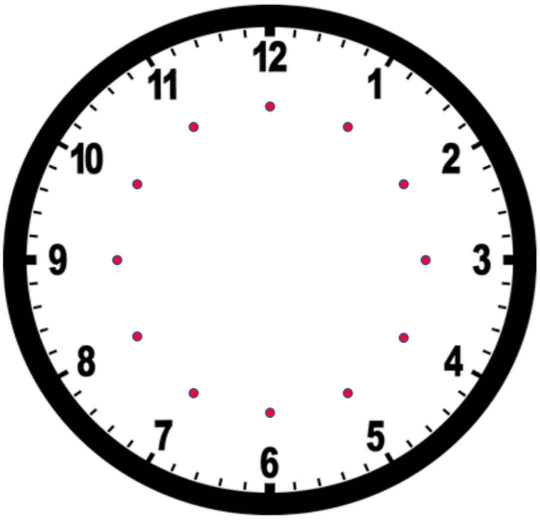
You do not have to go through all three steps each time. You may in some cases settle for naming a beautiful phenomenon or describing a pattern without explaining it.

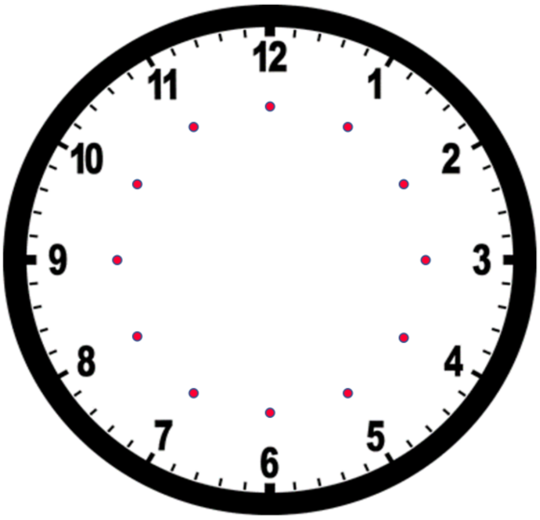
We advise to start with the rule $f(x) = a\,x + b$. This will already lead to any number of discoveries. Then try if you find the same type of phenomena for rules like $f(x) = a\,x^2$ or $f(x) = a\,x^3$ or even more complicated ones. Of course you can then also look for new phenomena for these complicated rules.

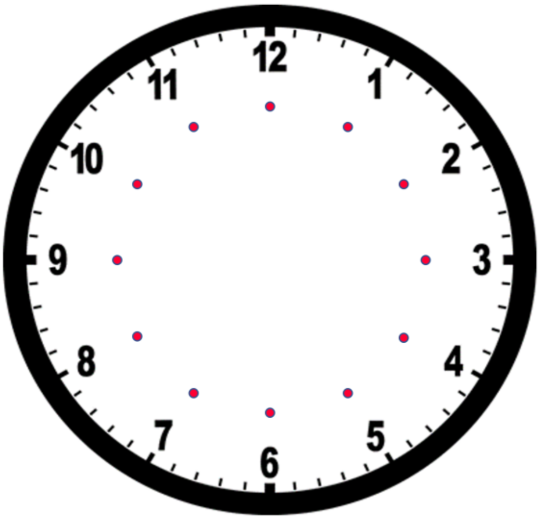
Sources
Clock: https://www.kamogo.com

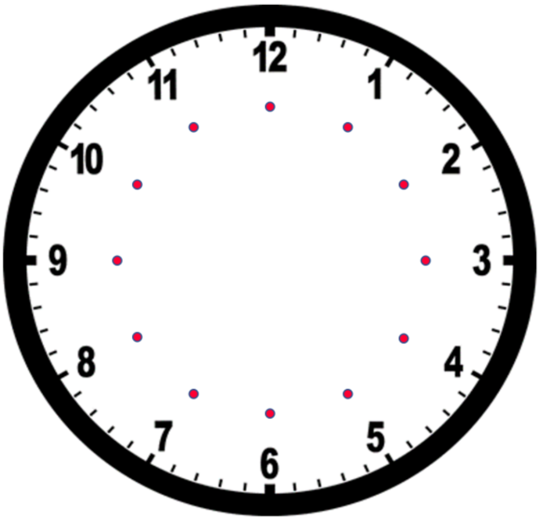
Work sheet

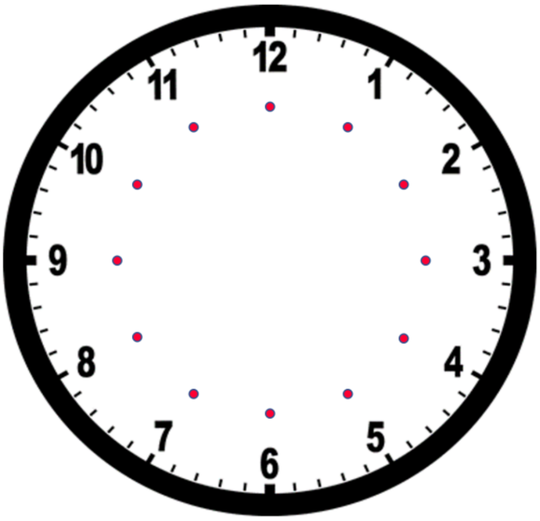

12
1
2
3
4
5
6
7
8
9
10
11


12
1
2
3
4
5
6
7
8
9
10
11


12
1
2
3
4
5
6
7
8
9
10
11


12
1
2
3
4
5
6
7
8
9
10
11


12
1
2
3
4
5
6
7
8
9
10
11


12
1
2
3
4
5
6
7
8
9
10
11