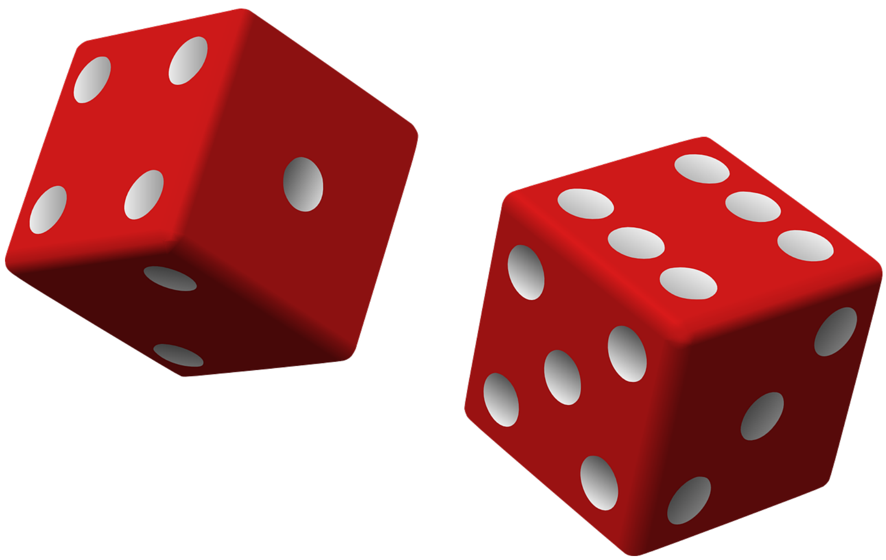
A nice set of dice



Mathematics B-day 2016, Friday 18 November, 9:00h-16:00h

Introduction

About the assignment

A soccer competition is often exciting until the end. Some strange results can occur along the way. For example, if Ajax beats PSV and PSV beats Feyenoord it does not imply that Ajax then beats Feyenoord. It is not unthinkable that Feyenoord still beats Ajax. In this case, which of these three teams is the strongest team remains unclear. But that's soccer... During the Mathematics B-day this year, we examine whether this also occurs when you are rolling dice. We study this using a simple game of dice: both players have a die, they throw it, and the one with the highest number of dots wins. Sets of dice where there is no strongest dice exist, and for the main exercise you will make such a set.

The structure of the day

This Mathematics B-day assignment consists of basic exercises, extra exercises and a main exercise. Try to spend approximately half the day on the main exercise.

What to hand in?

You will hand in a report by the end of the day. In this report you describe the results you found in the exercises and the main exercise. Tell your own story in such a way that it is clear and convincing. Of course, you will use relevant explanatory images as illustrations. Make sure you report is understandable for people who did not take part in the Mathematics B-day, but who do have sufficient understanding of mathematics. That means that you must take care to introduce the exercises clearly, and that, where needed, you must refer back to what you found in previous exercises.

In short: you write your own clear story, supported by mathematical arguments. The quality of your report will definitely play a part in the assessment!

It may be useful for your report to already start writing down the exercises and answers you found during the morning. Keep in mind that the whole report has to be handed in by 4 o'clock in the afternoon!

Basic exercises

Exercise 1 (More dots)

Susanne and Rogier are both playing with 'normal' six-sided dice with respectively 1, 2, 3, 4, 5 or 6 dots per face, they both throw it, and the one who throws the highest number of dots wins. Sometimes Rogier is the winner, sometimes Susanne is the winner. If they play often enough, they will, approximately, both win half of the games. Susanne doesn't like that...

a) Susanne draws an extra dot on the face with two dots. So now she's playing with 1, 3, 3, 4, 5, 6. Rogier still playing with 1, 2, 3, 4, 5, 6. In the long run, will Susanne now win more often, or doesn't it matter?

b) Rogier has found at that Susanne has put an extra dot on the face with two dots. Therefore, he is also going to add an extra dot on one of the faces. Which is the best face to draw an extra dot on to ensure he wins as often as possible, the one with 1, 2, 3, 4, 5 or 6? Why?

Exercise 2 (Is more always better?)

Rogier is also caught and they stop the game. Then Susanne comes up with three new dice, which we will simply call T, U and V. Susanne and Rogier may not both play with the same die.

- $T : 1, 2, 3, 4, 5, 6$
- $U : 2, 3, 4, 5, 6, 7$
- $V : 1, 1, 1, 1, 1, 100$

They start another game of dice, and Rogier chooses a die first. After that, Susanne chooses a die.

a) Which die should Rogier choose, and why?

b) Say that Rogier chooses die U. Which die is then the best option for Susanne, and why?

Exercise 3 (Playing against the DiceBot)

It is time for you to play dice. You will do so digitally. You will play the variant of the game where you get to pick a die first. Next, your opponent DiceBot will choose a die. DiceBot is a computer program that is remarkably good at this game.

Go to `http://www.fisme.science.uu.nl/wisbdag/opdrachten/dice/`
(short URL: `http://bit.ly/2euSHM7`, or use the QR code shown below)

As you can see, there are three dice to choose from. Die A, for example, has 6 faces, and the faces have, respectively, 3, 3, 5, 5, 7 and 7 dots (so, there are two sides with 3 dots, etc). DiceBot is polite and lets you choose a die first (at the top, Player 1 is set to 'me'; leave this setting as it is).

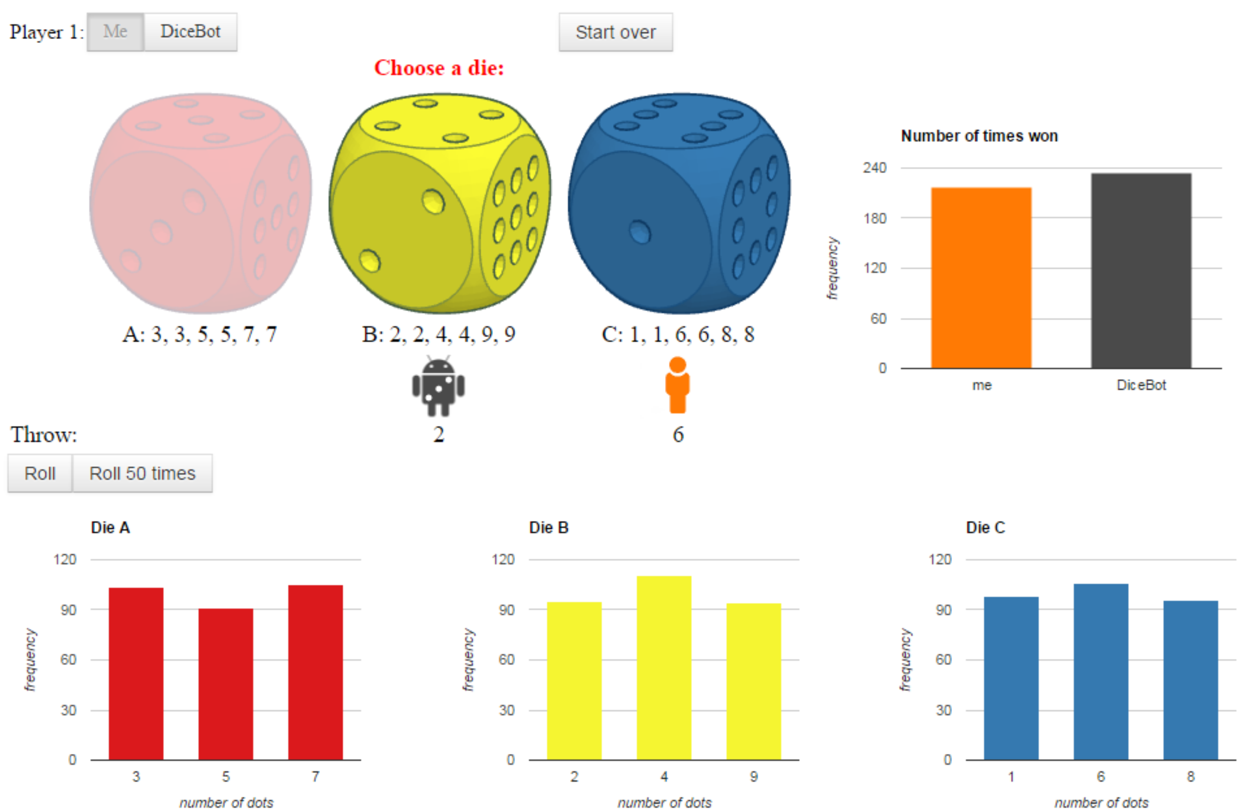


Figure 1: DiceBot after many games. The player has switched die a couple of times.

a) Choose a die to play with by clicking on it. DiceBot will now choose one of the remaining dice. Press the button 'Roll'.

You can now see your results and DiceBot's result. In the three graphs at the bottom the (cumulative) results for the individual dice are shown. In the graph in the top-right corner the (cumulative) results of the game are shown: how often you won (threw the highest number of dots) and how often DiceBot won.

b) Play against DiceBot at least 300 times. Switch die occasionally.

Quite likely, DiceBot outperformed you. See the graph in the top-right corner.

c) Now change Player 1 to DiceBot and press OK. Ensure that you win more often than DiceBot.

d) If you are player 1, which die is the best choice? Explain your answer.

Exercise 4 (Win probability)

How often does die A beat die B? In this exercise you will learn about the win probability of one die against another die.

We will start with the dice from the Dicebot simulation. In the table below you can write down when die A beats die B and vice versa. Part of the table has already been filled in.

Table 1: Who wins?

		A					
		3	3	5	5	7	7
B	2	A	A				
	2	A					
	4	B					
	4					A	
	9						
	9		B				

a) Finish the table given above.

We write down the **win probability** of die A when playing against die B as $w(A,B)$. The win probability $w(A,B)$ is equal to $\frac{\textit{the number of results where A beats B}}{\textit{the total number of results}}$

b) What is $w(A,B)$? And $w(B,A)$?

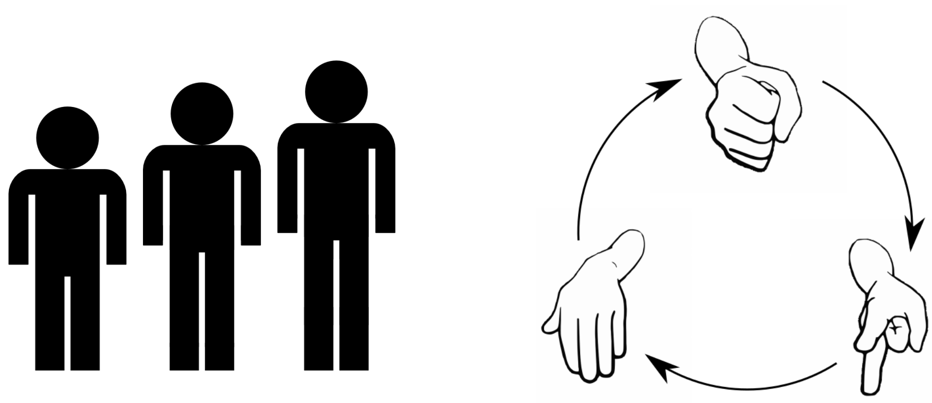
c) Also calculate $w(B,C)$ and $w(C,A)$ using a table.

We call a die A **stronger** than die B if $w(A,B) > w(B,A)$. We write this down as $A \rightarrow B$.

d) In the case of the game against the Dicebot, is there a die that is stronger than the other two dice? How is this related to what you discovered while playing against Dicebot?

In the set of dice in the game against Dicebot, for every die there is another die that is stronger. That is quite remarkable! You will examine dice sets like these further today,

In a **cheating set** every die is an attractive option, since it is stronger than another die or (even better) several other dice. But each die is also weaker than at least one other die in the set. This is exactly what you need to have an advantage if you are Player 2.



(a) Height has no cycles. (b) Rock-paper-scissors does.

Figure 2: What is a cycle?

In every cheating set you find at least one 'circle' of dice where each die is stronger than the next (you can attempt to prove this in exercise 16). If that is a circle of three dice A, B and C, you will for example have $C \to A$, $A \to B$, $B \to C$. We call such a circle of three or more dice a **cycle**.

If you're familiar with the game of **rock-paper-scissors**, you will know that rock, paper and scissors also from a type of cycle: there is a stronger (and weaker) counterpart for each choice. Together they form a circle.

It should be obvious by now that you can sometimes make your classmates look like a fool with a cheating set. If you're lucky, you can win their pocket money off them playing dice, hence the term 'cheating'.

Exercise 5 (Another cheating set)

There are more cheating sets. We consider the following dice:

- $D : 1, 1, 7, 7, 8, 8$
- $E : 2, 2, 3, 3, 9, 9$
- $F : 4, 4, 5, 5, 6, 6$

a) Calculate the win probabilities and confirm that this is a cheating set.

Susanne and Rogier will play another game of dice.

b) Rogier has to choose first, and he carefully considers his choice. He takes die D. Why?

Exercise 6 (Tables and diagrams for dice)

Consider the two dice $K : 1, 1, 3, 5, 5, 6$ and $L : 2, 2, 2, 4, 5, 6$. If you are going to play dice with them, a draw is also possible. We write that down as $-$ in the table.

Table 2: Dice K and L: who's the winner?

		K					
		1	1	3	5	5	6
L	2	L	L	K	K	K	K
	2	L	L	K	K	K	K
	2	L	L	K	K	K	K
	4	L	L	L	K	K	K
	5	L	L	L	$-$	$-$	K
	6	L	L	L	L	L	$-$

It is clear that $w(K,L) + w(L,K) < 1$.

a) Consider two dice with the following property: if a specific number of dots is on one of the faces of A, it can not also occur on dice B. Explain that $w(A,B) + w(B,A) = 1$

Making a table can be a lot of work. If you like, you can download an Excel file in which you can quickly compare two six-sided dice. If you want, you could modify this file so it can compare more dice or dice with a different number of faces.

`http://www.fisme.science.uu.nl/wisbdag/opdrachten/dice/table.xlsx`
(short URL: `http://bit.ly/2fv0oEH`, or use the QR code shown below)

We add die $M : 1, 2, 3, 4, 5, 6$.

b) Using diagrams, show that this set of dice KLM is not a cheating set.

Exercise 7 (A set of four dice)

Susanne has acquired a taste for dice sets and comes up with a set of four dice. Rogier has to choose the first die.

- $W : 1, 1, 1, 5, 5, 5$
- $X : 2, 2, 2, 2, 6, 6$
- $Y : 0, 0, 4, 4, 4, 4$
- $Z : 3, 3, 3, 3, 3, 3$

a) What is the sum of all the dots on each of the dice? Does this make you suspect that one of the dice is stronger than all other dice?

b) Is $WXYZ$ a cheating set?

With more dice you can also have larger cycles. With a cycle of four dice, 1 is stronger than 2, 2 is stronger than 3, 3 is stronger than 4 and 4 will be stronger than 1.

c) Can you find a cycle of four dice in this set?

d) How many cycles of three dice are there in this set?

Exercise 8 (Comparing cheating sets)

Susanne and Rogier are playing dice again. Rogier has learned from the previous game; Susanne has to choose a die first. They both have calculated all the win probabilities. Susanne can choose with which set they are going to play with: ABC (exercise 4), DEF (exercise 5) of $WXYZ$ (exercise 7).

a) Which is the best set for Susanne to choose?

What if Rogier can choose which set to play with (and Susanne still has to choose the first die).

b) Which set should he choose?

Exercise 9 (Multi-sided dice)

After Rogier has lost to Susanne many times, he's about to try something different. Not all dice are six-sided. There are for example four, eight, twelve and twenty-sided dice. And with double pyramids you can even make seven- or thirteen-sided dice!

Figure 3: Multi-sided dice

To confuse Susanne, Rogier comes up with a very strange set of dice:

- $Q : 3, 3, 6$
- $R : 2, 2, 5, 5$
- $S : 1, 4, 4, 4, 4$

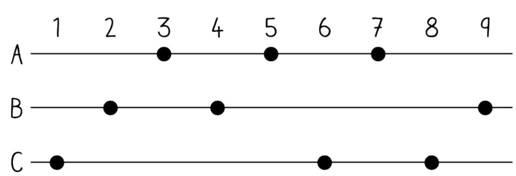
Is this a cheating set?

Exercise 10 (Shifting in a dot diagrams)

Up to this point you have been analysing existing sets of dice. But how can you make a cheating yourself? In this exercise we describe a possible approach.

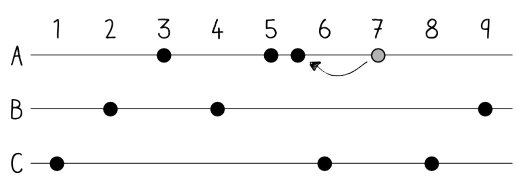
Let's take another look at Dicebot's three dice: $A = 3, 3, 5, 5, 7, 7$; $B = 2, 2, 4, 4, 9, 9$; $C = 1, 1, 6, 6, 8, 8$. Below we've represented these three dice in a different way using a **dot diagram**.

Because all numbers occur exactly twice, you can simplify the situation to three-sided dice $A = 3, 5, 7$; $B = 2, 4, 9$; $C = 1, 6, 8$ without the win probabilities changing.



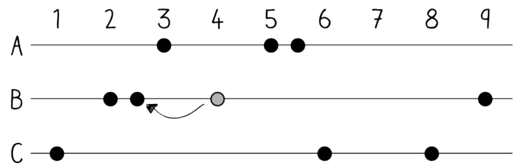
a) Explain how, using a dot diagram, you can calculate the values for $w(A,B)$ and $w(B,A)$ (exercise 4b)

You can treat a dot diagram as a starting point. By shifting dots, you can try to increase the win probabilities. For example, start with the diagram above and for die A, shift the dot at 7 to the left:

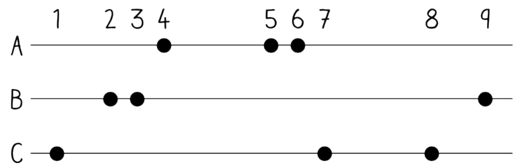


b) What are $w(A,B)$, $w(B,C)$ and $w(C,A)$?

We can do another shift:



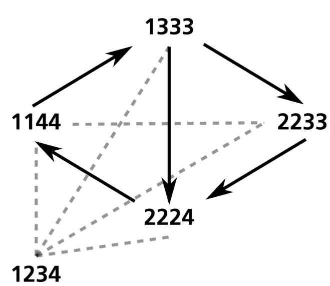
If it annoys you that the 2 and the 5 now occur twice, you can shift all the numbers a bit, for example, like this:



You can see that we have now created the dice from exercise 5.

We will look now at a set of five four-sided dice, with the sum of the dots on each of the dice 10 and the number of dots per face is 1, 2, 3 or 4. With these constraints, exactly five dice are possible. For simplicity's sake we no longer write down the dice with a letter, but with the number of dots without commas: 1234, 1333, 2233, 2224 and 1144. Is this set of dice a cheating set?

We can make the following visual representation of the strengths of the dice in comparison to each other.



We place the die spread out over the page. Next we draw an arrow from die A to die B if $A \to B$ (recall: this means that die A is stronger than die B, i.e. $w(A,B) > w(B,A)$). We draw a dotted line if $w(A,B) = w(B,A)$. This kind of representation is called a **graph**. In a graph for a cheating set there is at least one arrow *to* and at least one arrow *from* each of the dice.

It's easy to look for cycles in the graph: there is one cycle of three dice and one of four!

There are no arrows to or from die 1234. The die is **neutral**. This means that the set is not a cheating set. If you leave out the neutral die, it is a cheating set.

Exercise 11 (The graph for a set of dice)

Draw the graph for the dice from exercise 7.

Exercise 12 (Playing algebraic dice)

The dice

- $G: 1{,}4{,}4$
- $H: 3{,}3{,}3$
- $I: 2{,}2{,}5$

form a cheating set. The probabilities are $w(G{,}H) = \frac{2}{3}$, $w(H{,}I) = \frac{2}{3}$ en $w(I{,}G) = \frac{5}{9}$. Rogier wants to change the set so that that lowest probability ($\frac{5}{9}$) becomes a bit higher! He does that by making the number of faces 21. Die G has n ones and $21-n$ fours: $G: 1{,}1{,}\ldots{,}1{,}4{,}4{,}\ldots{,}4$. Dice H has 21 threes, $H: 3{,}3{,}\ldots{,}3$. Dice I has n fives en $21-n$ twos. For example, $n=3$ gives the set $G: 1{,}1{,}1{,}4{,}4{,}\ldots{,}4{,}4$, $H: 3{,}3{,}3{,}3{,}\ldots{,}3{,}3{,}3{,}3$ en $I: 2{,}2{,}\ldots{,}2{,}2{,}5{,}5{,}5$.

a) Express the win probabilities $w(G{,}H)$, $w(H{,}I)$ and $w(I{,}G)$ in n.

Two of the three probabilities are the same.

b) Which number should Rogier choose for n, such that the lowest of the probabilities $w(G{,}H)$, $w(H{,}I)$ and $w(I{,}G)$ is as large as possible?

c) And how about when the dice are 30-sided?

Extra exercises

These extra exercise are not obligatory and not necessary for the main exercise. They can help to generate ideas and increase your understanding of cheating sets. Only do the extra exercises if you have time for them in the morning.

Exercise 13 (Extra practice: a special graph)

Say that you have the following set of five six-sided dice: 222777, 116666, 055555, 444449, 333388. This set has several special properties.

a) Draw a graph for this set of five dice.

b) Make an overview of all cycles in the graph.

Exercise 14 (The neutral die)

We will only look at six-sided dice for which the sum of the dots (dotsum) is 21 and the number of dots per side is 1, 2, 3, 4, 5 or 6. One example is the 'normal' die $O : 1, 2, 3, 4, 5, 6$.

a) Make a number of dice A with the dotsum 21 and for each of them calculate $w(A,O)$ en $w(O,A)$.

Just as 1234 was neutral against the dice with dotsum 10, O is neutral against dice with dot sum 21.

b) Prove that $w(O,A) = w(A,O)$ for each die A with dotsum 21.

If the dotsum of A isn't necessarily 21, but the dots are still only 1, 2, 3, 4, 5 or 6, there is a more general outcome.

c) Prove that $w(A,O) = \frac{dotsum(A)-6}{36}$ en $w(O,A) = \frac{36-dotsum(A)}{36}$

d) In line with this, prove a more general outcome for dice with dots $1, 2, 3, \ldots, n-1, n$ for a positive whole number n.

Exercise 15 (Dual dice)

We will now consider five-sided dice that have a dotsum of 15 and for which the numbers of dots are 1, 2, 3, 4 or 5, for example 12255.

a) List all options for this type of die.

Each of these dice A has a **dual** A^*. You can find the dotsums of A^* by doing 6 minus the dot numbers of A. For example, the dual die of 12255 is 6-5, 6-5, 6-2, 6-2, 6-1 is 11445.

b) Explain that for each of these dice A the dotsum of the dual A^* is also 15.

c) Prove that $w(A,B) = w(B^*,A^*)$

A die A is **self-dual** if $A = A^*$.

d) Prove that the following is true for self-dual dice A en B: $w(A,B) = w(B,A)$

e) Say that A is self-dual. Prove: if $w(B, A) > w(A, B)$, then $w(B^*,A) < w(A, B^*)$

Exercise 16 (Cycles)

Prove that every cheating set has a cycle.

Exercise 17 (An upper boundary for the sum of the probabilities)

Suppose the dice A, B and C form a cycle $A \to B$, $B \to C$, $C \to A$, then we have been computing the win probabilities $w(A,B)$, $w(B,C)$ and $w(C,A)$. Of course these win probabilities are all ≤ 1. Can they all be equal to 1? Or is there an upper boundary? In this exercise you are going to prove that

$$w(A,B) + w(B,C) + w(C,A) \leq 2.$$

If you want to compute the win probabilities, you compare the sides of the dice using tables. To reason with the information in these tables we need some notation. We suppose that all dice have six sides. The numbers on the sides of A are denoted by $A_1, A_2, A_3, \ldots, A_6$. For example, in exercise 4 $A_1 = 3, A_2 = 3, A_3 = 5, A_4 = 5, A_5 = 7$ and $A_6 = 7$. We use the same notation for the numbers on the sides of the dice B en C; For example $B_5 = 9$.

We denote $A_1B_2 = 1$ if the number on side 1 of A is greater than the number on the side 2 of B (dus als $A_1 > B_2$), and $A_1B_2 = 0$ if this is not the case (if $A_1 \leq B_2$). Similarly, for $1 \leq i,j \leq 6$.

$$A_iB_j = 1$$

if $A_i > B_j$ and else $A_iB_j = 0$.

a) Explain that A_1B_1, B_1C_1 and C_1A_1 cannot all be 1.

b) More generally: explain that A_iB_j, B_jC_k and C_kA_i cannot all be 1 (for $1 \leq i,j,k \leq 6$).

c) Explain that

$$w(A,B) = \frac{A_1B_1 + A_1B_2 + \ldots + A_6B_5 + A_6B_6}{36}$$

d) Explain that the sum with 216×3 terms

$$\begin{aligned}
&A_1B_1 + B_1C_1 + C_1A_1\\
&+A_1B_1 + B_1C_2 + C_2A_1\\
&+A_1B_1 + B_1C_3 + C_3A_1\\
&+\ldots\\
&+A_6B_6 + B_6C_5 + C_5A_6\\
&+A_6B_6 + B_6C_6 + C_6A_6
\end{aligned}$$

is equal to

$$216(w(A,B) + w(B,C) + w(C,A))$$

e) Explain that from (b), (c) and (d) follows

$$w(A,B) + w(B,C) + w(C,A) \leq 2.$$

f) Formulate a generalisation to general number of dice with a general number of sides.

Main Exercise - cheating sets

The main exercise for the Mathematics B-day 2016 is for your group to make a cheating set of dice. We are giving you three objectives. Objectives 1 and 2 are more important than objective 3. Make a cheating set that meets as many of the objectives as possible.

Objectives:

1. Make sure that the probability for player 2 to win is as high as possible. Assume that player 1 has calculated all win probabilities and is playing a smart game.
2. Make sure that the set has a large cycle.
3. Make sure that the set has several cycles.

Explanation for objective 1: You want a good set for when player 1 (who chooses the first die) is playing smartly (he will choose the die which has the lowest win probability for you). Therefore, that lowest win probability should be as large as possible.