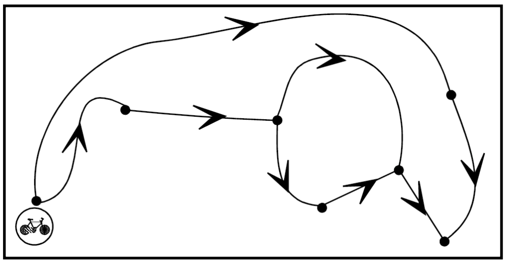
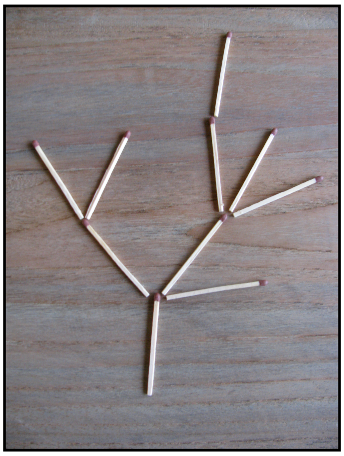
# MATHEMATICS B-DAY 2011

Friday, November 18, 9:00-16:00 hr



# The final move



Wiskunde B-Dag

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# Introduction to the assignment

This Mathematics B-dag assignment is all about games for two persons. These are games in which chance (for instance because what you can do is determined by rolling a die) plays no part. The rules of the game are prescribed in detail; as a result it's been determined which moves you can make in any situation in the game. Players take turns to make a move. If a player cannot make any move, he or she loses. But of course you want to win. So therefore you look for a plan to achieve that, if it is possible. Such a plan is called a **winning strategy.**

## Structure of the assignment:

**Part 1**

Four games are presented. These are intended to familiarize you with looking for a winning strategy. Of course, with four people in your group, you can divide the games between two pairs. In that case make sure to compare and share your results and ideas with each other! The text for **Part 1** has assignments, which you can recognize by the dot (•). These will help you on your way. Make sure to describe your findings of Part 1 in the final report.

**Part 2**

Part 2 offers you some theoretical grounding to analyze the various games. Be sure to link this theory to what you yourself came up with in **Part 1**. The technique that is proposed in part 2 is called ***Back-tracking***. Thinking back from the final position in a game to previous positions is very useful for finding a possible winning strategy. Your findings will be part of the final report

**Part 3**

Here you will analyze one of the games in more detail using an Excel file. With this file, you let the computer do the grunt work for you, so your thinking can be supported by the output from Excel. And again: describe your findings in the final report.

**Final assignment**

Finally you have to choose between a further analysis of the games from **Part 3,** or analyzing more difficult assignments for one of the other type of games in **Part 1** and **Part 2**. Here as well, the various questions – again marked with a dot (•) – serve as guidelines for your own research.

## The final product

The final report must be clearly understandable for someone who does not know already in advance what the topic of the assignment is. This means that you must describe the definition of the problem clearly, as well as what your team investigated. Of course you may follow the individual assignments from the text in your report. But it is also good – maybe even better – to hand in a report that is based on your choice for the **Final assignment**, including thorough argumentation, and of course using the elements that were mentioned in parts 1, 2 and 3, which you investigated.

Hand in a report that is easy to copy. If you include handwritten items, take care to use a black pen, because it will copy better than other colors.

## Suggestions for how to divide your time between 9 and 16:

| | |
|---|---|
| 9 - 11 hr | Part 1: Work out for yourselves how to play a game in a smart way. |
| 11 - 12 hr | Part 2: Familiarize yourselves with Back-tracking and its notation. |
| 12 - 13 hr | Part 3: Do your own research using Excel. |
| 13 - 16 hr | Do the final assignment and write the report. |

*Advice:* *Remember to take a break to rest, eat and drink.*

***Have fun, and good luck!***

# Part 1: 4 games for 2 players

Here are four games to be played in groups of two to get a grasp of ways to win.

## Game 1: The (1, 2, 3)-game

Materials needed: *A number of matches or other similar objects.*
Preparation: *Put a number of matches, for instance 17, in a stack on the table.*
Rules of the game: *A move consists of removing 1, 2 or 3 matches.*
*The players take turns to make their move.*
*Starting player: alternate.*
Win/lose rule: *The player who cannot make a move, loses.*

In every move at least one match is removed from the stack. In the end one of the players will end up losing: there are fewer matches on the table every time, so there will come a moment that no allowed move will be possible.

- Play the game a few times. Also use other, smaller, starting amounts than 17.
  Continue playing until you know how you can win the game, if it can be won anyway ...
  Also think about starting amounts that make it impossible for you as first player to win, no matter how smart you play, and assuming your opponent is playing an optimal game.

*The intention is that you understand the game in such a way that you can indicate exactly which starting amounts allow the first player to win, no matter how well the opponent plays. Other starting amounts will let the second player win, no matter how well the first player plays.*
A way of playing that describes how you play the game is called **STRATEGY**.

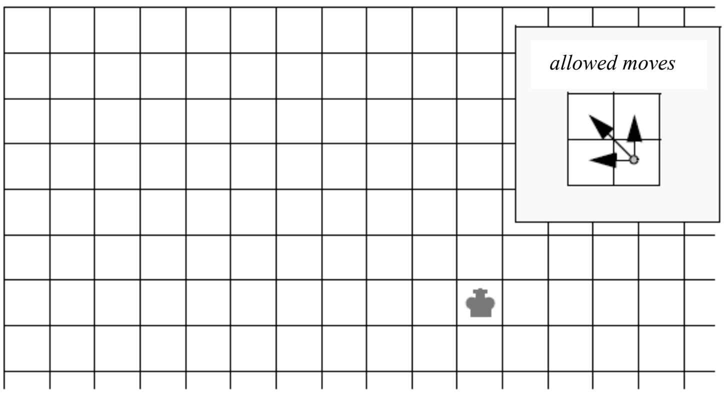
**Variant of game 1: the (3, 4)-game**

- Can you also think of a winning strategy for the taking away game if you are only allowed to remove either 3 or 4 matches from a stack of 65 matches?

## Game 2: A limited king on the chessboard

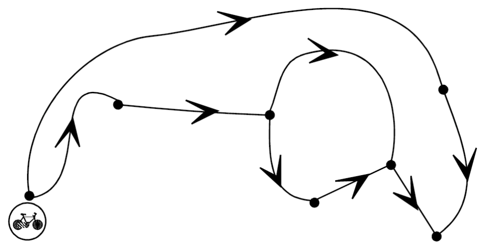
Materials needed: *Chessboard or (better) grid paper and a king; you can also use a coin or an eraser.*
Preparation: *Put the king in a square. That will be the starting point of the game.*
Rules of the game: *An allowed move consists of moving the king by one square: up, left or diagonally to the upper left.*
*The players take turns to move.*
*Starting player: alternate.*
Win/lose rule: *The player who cannot make a move, loses.*

The king is limited in his direction of movement: to the right or down are not allowed. Finally he will end up in the top left corner square, and he won't be able to move any further.
What you have to do is indicate precisely which starting points will allow the first player to win, no matter how well the opponent plays. Other starting points mean the second player wins, no matter how well the first player plays.




- Try to find the winning and losing starting points.

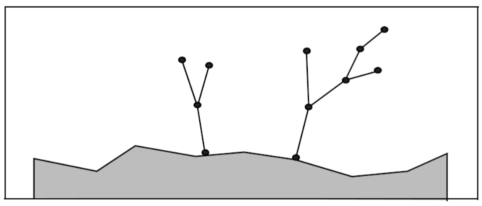
## Game 3. The one way game



In this game you take turns to move the bicycle by one stage, from the starting point that is marked with the little bicycle. A stage runs from dot to dot. Mind the arrows: the traffic is one way! The player who can make no more moves, loses!

- Can the first player win this game?

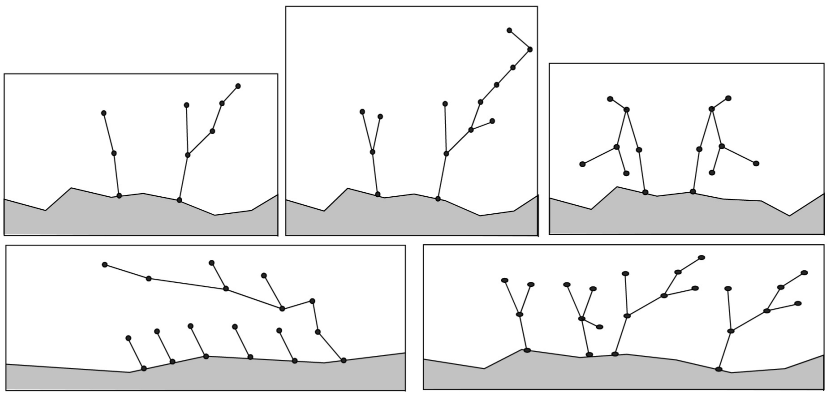
## Game 4. The pruning game



This game involves pruning trees. A tree contains twigs and knots. Each twig connects two knots and has no intermediate knots. Knots can be the meeting point for multiple twigs. The picture on the right is an example of a garden with two trees. There is exactly one flow of sap from the ground to that knot. A move consists of cutting (pruning) a twig right above the knot where it begins. So you can for example prune away a tree completely by cutting the twig directly above the ground. Or in the tree on the right in this garden you can cut the part above the second knot from the ground that goes to the right in one go by cutting the twig directly above that knot.

Materials needed: *Paper and pencil*
Preparation: *Draw a garden with several trees.*
Rules of the game: *A move consists of pruning a twig, along with everything connected to that twig.*
*The players take turns to move.*
*Starting player: alternate.*
Win/lose rule: *The player who cannot make a move, loses.*

- Play the pruning game with two players in these five gardens.



For the fifth garden (the one on the right in the second row) you can use a smart strategy! As your first move, you prune one of the twigs connected to the second knot from the ground in the second tree from the left.

With this first move you are certain to win in the end!

- How can you be certain? And using which strategy?
- Difficult! (*so it's okay to skip this*): with another first move your opponent can win. Is that always true?

# Part 2: Impartial games; the theory

## A Winning by backtracking with 0 and 1

In this part we discuss a technique that you probably used already in part 1. We will continue with that technique later, but here's some theory first!

***Looking back at the (1, 2, 3)-game***

You probably discovered the strategy for the (1, 2, 3)-game. If not, you are about to find out!
With every one of your moves, you leave a multiple-of-four amount of matches behind. With the first move, starting with 17, you take 1; you leave behind 16 matches. It's certain that your opponent cannot now make a move that leaves a multiple of four matches on the table. But you can in your next move, since your opponent will take 1, 2 or 3 matches, and you respond with removing respectively 3, 2 or 1 matches. So, the numbers of matches you leave behind starting from 17 are successively 16, 12, 8, 4 and 0.

**1.** We play the (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)-game, and there are 1000 matches on the table.
  **a.** Which move does the first player ***have*** to make to be certain of victory?
  **b.** After how many moves will the game be over, if the first player plays correctly?
  **c.** Can the first player win if the starting amount is 374 matches?
  Assume both players are using an optimal strategy.

***Thinking backwards: back-tracking in game 1***

How did you discover the multiples-of-four strategy in game 1? Undoubtedly it went something like this. You know, if there are 0 matches left on the table, the player whose turn it is has lost. If you want to win, you must not end up with 0 matches in front of you, but with 1, 2 or 3. That means that the person who has to take a turn with 4 matches, loses, because he has to leave one of the winning positions 3, 2, or 1 on the table. That in its turn means that situations with 5, 6, and 7 matches in front of you mean victory! Because those positions mean that you can leave your opponent with the losing 4 matches position. So that means that with 8 matches … etcetera.
What you do is to think *back* from the end of the game.
This strategy is called BACK-TRACKING. Using back-tracking we can do a good analysis of the games in part 1, and many other games as well.

***Winning and losing positions***

The theory of this kind of games talks about game positions, *positions* for short. These are the situations that the game is in. In the (1, 2, 3)-game the game position is the number of matches that is left on the table. In the one-way game it's the place where the bicycle is.
You have seen that the amounts 0, 4, 8, 12, etc are *losing positions* in the (1, 2, 3)-game. If it's your turn in a losing position, you can't win unless your opponent makes a mistake. The positions 1, 2, 3, 5, 6, 7, 9, etc are the *winning positions*. If you make no mistakes in those positions, your opponent hasn't got a chance.

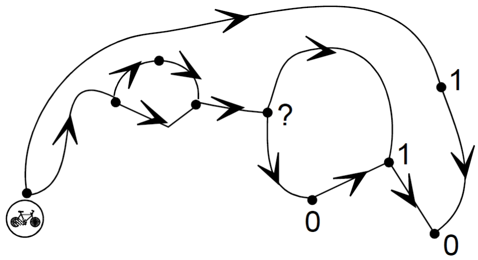
> A general definition of winning and losing positions in games where it's the case that "the player who cannot make a move, loses" is:
> - *Winning position*: there is at least *one* allowed move from this position to a losing position
> - *Losing position*: from this position the only allowed moves are to winning positions.

Please note that you can sometimes go from a winning position to a winning position, but there will always be at least one losing position to be reached. If you move to a winning position, you are making a bad mistake, because you hand over control of the game to your opponent!

### *Marking positions with the values 0 and 1.*

You can note the two positions on the 'playing field' with ***W***(in) and ***L***(ose), but in the literature about these games, you often see the values **1** (for a winning position) and **0** (for a losing position). Here, as an example of how to place the values 1 and 0, we will look at a game board for the one-way game with the bicycle.
It works like this:

- If there is an arrow leading from a position to a position with a 0, you put a 1 on that position. Of course, as the future winner you put your opponent in a losing position.
- If there is no arrow that leads to a position with a 0 at a position, you put a 0 on that position, since there is no possible move leading to a losing position for your opponent.




**2.** Use the figure on the right to see if you understand the above and can apply it, following questions **a**, **b** and **c**.

**a.** Check whether the 0's and 1's that have been given already are marked correctly.

The value of the position at the question mark (?) is 1, because you can present your opponent a losing position from there. You can also continue to a position with value 1, but that wouldn't be very bright …

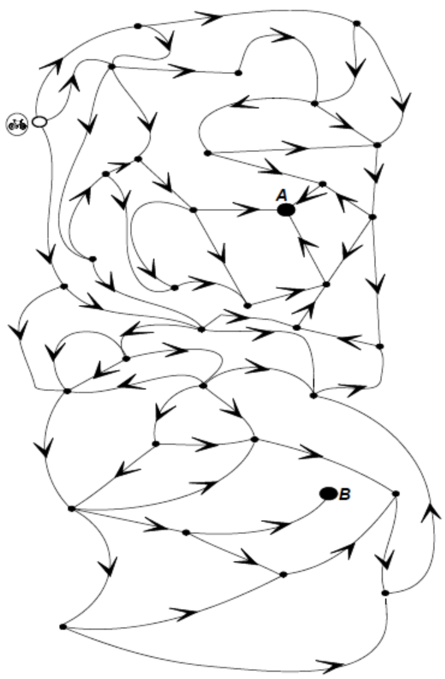
**b.** The position from where the arrow leads to the question mark, is given value 0. Because if that is your position .........

**c.** Who will win this game when the correct strategy is followed, the first or the second player?

**3.** Use your knowledge about placing the values 0 and 1 to examine the following, more complicated, one-way game board. The two possible final positions have been marked with *A* and *B*, so that is where you start by placing a 0.
An enlarged version of this game board is provided on a worksheet.

**a.** Can the first player win?

**b.** The loser will always end up at either *A* or *B*. Can the loser influence the game so that it will end in *B*?

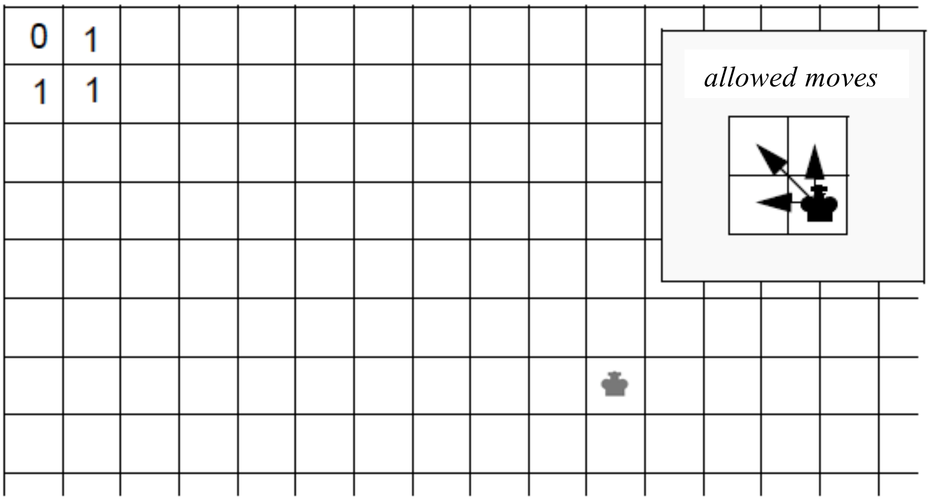


## B Doing more with 0 and 1: the other games

### *The "limited king" is also an arrow game!*

If we replace every middle of a square with a dot on the squared playing field of the game with the king, then you can put in arrows for all allowed moves, just as in the one-way track. It doesn't help a lot, since it's just as simple to keep the sketch of the three arrows handy. That is clear enough. It does make sense to put 0's and 1's in the squares of the playing field. To make it easier to refer to individual squares, we number them. The square at the top left is called (0, 0). The sixth square (from the left) on the third row from the top will then become square (2, 5).

**4.** Part of the playing field has already been filled in with 0's and 1's in the figure below. The square (0, 0) means the end of the game: no further moves are possible, so you put a 0 there. The three 1's are put in the squares (0, 1), (1, 0) and (1, 1) from which you can go directly to (0, 0). So these are winning positions!
From (0, 2) you can only go to the left, so only to the 1 in square (0, 1).




**a.** So why do you have to put a 0 in square (0, 2)?
**b.** Now you can continue with square (1, 2). Why must that be a 1?
**c.** Fill in squares until you can see a pattern. Try to explain the pattern.

Thus far the preparations. It's now your turn with the limited king in square (6, 10).

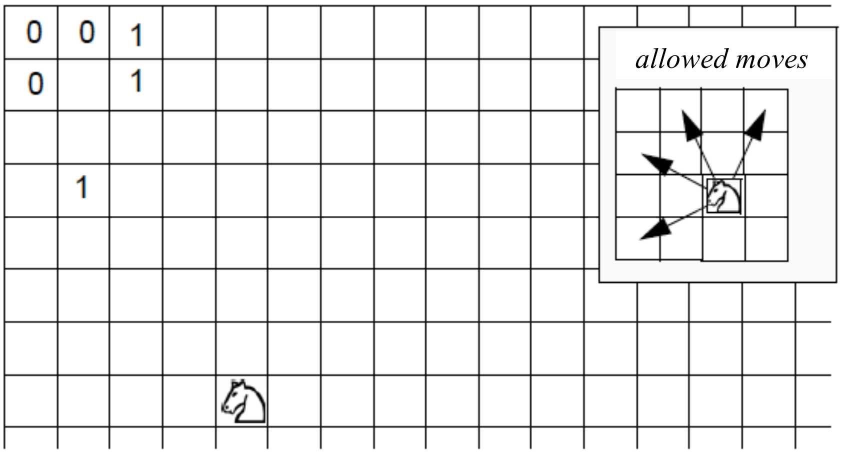
**d.** Can you win from that position? If yes, what is your first move?
**e.** Now put the king in square (1020, 389785).
Can you win from that position? If yes, what is your first move?

### *The game with a limited horse*

This game resembles the game with the king, but with different allowed moves. The four allowed moves are given in the figure on page 7.

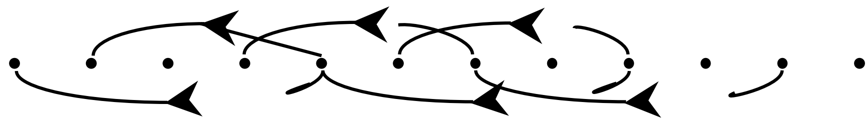
**5** Some 0's and 1's have already been placed in the playing field.

**a.** Continue the scheme, and say whether the first player will win from the indicated starting position (if he plays well).
**b.** Describe the losing fields in square-number language.




***The (3, 4)-game is also one-way traffic***

Part of a one-way version of the (3, 4)-game, the variant of game 1, has been drawn here.
The dot on the left represents the losing final field 0; keep in mind that there are 66 dots in total, although they haven't all been drawn. Since we are back-tracking, you see the end of the game in the drawing! Some possible moves have been marked with arrows.



**6** **a.** You know you can only remove 3 or 4 matches. Some possible moves have been indicated here with one-way paths. Add some more one-way paths.

The winning and losing positions can be given the values 0 and 1.
You can see the start, from the losing position with 0 matches, in the table below:

| *Number of matches* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... | ... | ... | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *value* (0 or 1) | 0 | 0 | 0 | 1 | 1 | ... | ... | ... | ... | ... | ... | ... | ... | |

**b.** There are some 0's and 1's in the table already. Check whether they are right and complete the row from left to right.
**c.** Is there a pattern of 0's and 1's emerging in the second row of the table?
If yes, what does the pattern look like?
**d.** Does the first player win the (3, 4)-game with 65 matches as the starting amount, if both players play well?

A figure with points (or dots) and connecting lines is called a **graph**. If the connecting lines have a marked direction, it's called a **directed graph**. In the directed graphs (one-way traffic) that you have seen, you never return to a point you visited before. There are also graphs where it *is* possible to return to a point you visited before. That is like going around in a circle. In graph theory such a closed path is called a **cycle**. We don't want that in our games because it gives you the opportunity to circle around endlessly without ever reaching the end of the game.

# Part 3 Taking-away games: experimenting with Excel

***This is what you already know:***

In game 1 the player whose turn it is, must take away 1, 2 or 3 matches. At the start of the game there is a stack of for example 25 matches.
In the table below you see the number of matches that are left in the top row, and in the second row the corresponding 0's (the losing positions) and 1's (the winning positions) for the numbers of matches from 0 to as far as you want to go. The row starts from the left with the end of the game: 0 matches.

| *Number of matches* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Value* (0 or 1) | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | | | | | | | |

***The theory of the winning route***

The set {1, 2, 3} in the game is called the ***take-away set*** of the game; it's the collection of allowed moves. The row of zeroes and ones is called the ***winning route***. If you have a simple description of the winning route (like this one), you have a winning strategy for the first player, at least if the starting position has value 1.
What you're doing is taking 1, 2 or 3 steps to the left in the winning route; that is as if you're taking away 1, 2 or 3 matches from the stack.
As the future winner, you leave your opponent with a position in the winning route that has value 0. In the winning route you can always go left to the next 0. The loser can't, because there is no allowed move from one 0 to the next.
The row consists of a constantly repeating block 0111. The length of that repeating block is 4, therefore we call this winning route **periodical with period *4***.

Of course you will examine what happens with other take-away sets. These are the corresponding research questions:

- *What does the winning route look like?*
- *Is there a winning strategy for the first player?*

**7.** *Pre-practice*.
Practice in making winning routes for the following take-away sets.
If a periodic row emerges, note down the period as well.

**a.** {1, 2, 3, 4} with winning route 0 1 1 1 1 0 1 ....
**b.** {1, 2} with winning route …..
**c.** {1, 3} with winning route …..
**d.** {1, 2, 4} with winning route …..
**e.** {1, 3, 4} with winning route …..
**f.** {2, 3, 4} with winning route …..
**g.** {2, 4} with winning route …..
**h.** {3, 6} with winning route …..

**8.** *A conjecture with proof*.
**a.** The take-away sets in **g** and **h** consist of the 2- and 3-folds of the take-away set in **b**. Therefore the winning routes for **g** and **h** are related to the winning route for **b**.
How do you construct the winning route for {4, 8} from that for {1, 2}?
**b.** If you aren't entirely certain yet that your answer is generally correct, then try again with another take-away set and its multiples.
Examine **e** and its double {2, 6, 8}.
**c.** Perhaps you will look at more of such pairs of take-away sets. If you think you are certain of the relationship, you must be able to explain why the relation is generally valid. Such an explanation is a proof for the relationship that you found.

**9.** *Blocks of zeroes and blocks of ones in the winning route.*

**a.** In the take-away set {1, 2, 3} there is a maximum of 3 ones in a row in the winning route and at most one 0. 3 and 1 are exactly the largest and smallest number of the take-away set. Is this generally the case?

**b.** Is this true?
If the length of the longest block of ones in de winning route is equal to the largest number in the take-away set, then the take-away set is a block from 1 up to and including that maximum number.

***Other conjectures about winning routes***

> ***Investigations with use of the computer: the program TakeAway***
> Because determining the winning route is the same every time, we will use the computer to do it from now on. Open the Excel file "***TakeAway***".
> At the top of the screen on opening the file is the take-away set {1, 4} with underneath the matching winning route. There is also information provided about the period and the position from where the periodicity starts.
> If you enter other numbers in the take-away set (with a maximum of 7), the program will generate the matching winning route.
> A short description about how to work with TakeAway you find in a seperate document.

In this part you are invited to do your own research, come up with conjectures and investigate them. We only provide some suggestions. What you investigate, the examples you select and the argumentations you come up with, it is all up to you...
That's why the suggestions haven't been numbered like questions, but only marked with dots.

## *Periodicity*

In the examples with short take-away sets of small numbers, you may have noticed that the same block is repeated again and again in the winning route.
We call the length of such a block the **period of the winning route**.
An objective argumentation will be appreciated!

- Determine for all values of $n$ ($n$ = 2, 3, 4, 5, …) the period for the take-away set {1, $n$}.
  Formulate a statement like: "For take-away set {1, n} the period of the winning route is …".

- Also do this for take-away sets of form {2, n}.

- For the two take-away sets {1, 3, 6} and {1, 2, 3, 8} you will find periods of length 9 in the winning route. But the block of 0's and 1's that repeat, isn't the same. Can you find more similar examples?

# Final assignment: taking-away games or chessboard

The choice is yours: do you want to continue analyzing taking-away games based on the Excel file "TakeAway" or do you rather want to investigate some more difficult variants of the limited king on the chessboard? Make your choice and show what you're worth!

## Final assignment 1 Taking-away games: Reasoning about periodicity

You will look further into the periodicity of winning routes in taking-away games. We give some suggestions for research questions that you could ask yourself. Of course you can also state and investigate your own conjectures.

- All the winning routes you have seen so far, are periodical.
  A first research question could be whether that is a general rule. We will immediately give away the answer to that question: yes, the winning route for every take-away set is periodical (for example in 0 0 0 1 1 0 0 0 1 1 0 0 0 1 1 0 0 0 1 1), with the possible exception of the start. Try to find a general proof for this.

*Here is a tip for proving that periodicity will always occur in the winning route.*
For take-away set {5, 13, 18} a computer found the following as the start of the winning route:

0 0 0 0 0 1 1 1 1 1 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 0 1 1 1 1 1 1 1 1
1 1 1 1 1 0 0 0 0 1 1 1 1 1 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 0 0 1 1 1
1 1 1 1 1 1 1 1 1 1 0 0 0 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 0
0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 1 1
1 1 1 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 0 0 0 0 0 .....

It's not that easy to spot the periodicity here, but it is there.
In the first 145 + 18 positions, the 18-number starting fragment returns after position 145 (both have been bolded below).

**0 0 0 0 0 1 1 1 1 1 0 0 0 1 1 1 1 1** 1 1 1 1 1 1 1 1 0 0 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 0 1 1 1 1 1 1 1 1
1 1 1 1 1 0 0 0 0 1 1 1 1 1 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 0 0 1 1 1
1 1 1 1 1 1 1 1 1 1 0 0 0 1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 **0 0 0 0 0 1 1 1 1 1 0**
**0 0 1 1 1 1 1**

- Does this mean that you can be certain that there is a 145 long period?
  Provide a good argumentation!

- Determine the length of the period for take-away set {*m*, *n*} for two natural numbers *m* and *n*. Additionally, you could look for how the length of the periodicity depends in take-away sets with more than two numbers.
  *Tip: It's a good idea to limit yourself to a specific type of take-away set at first, for instance one of the type {1, 2, other number }*

- Does the periodicity for a given take-away set always start right at the beginning?
  Is this for instance true: "In {*m*, *n*} the periodicity always starts at position 0"?
  Also consider take-away sets with more than two numbers, for instance {5, 11, 17} and {5, 13, 17}.

- Look at the periodic row with period six: 0 1 0 1 1 1 0 1 0 1 1 1 0 1 0 1 1 1…
  Is there a take-away set for which this is the winning route?
  The same question for the periodic row with period seven: 0 1 0 1 1 1 1 0 1 0 1 1 1 1 0 1 0 1…
  Can you formulate general rules (and prove them) for when a periodic row that starts with a 0 can occur as a winning route?

## Final assignment 2 Other games on the chessboard

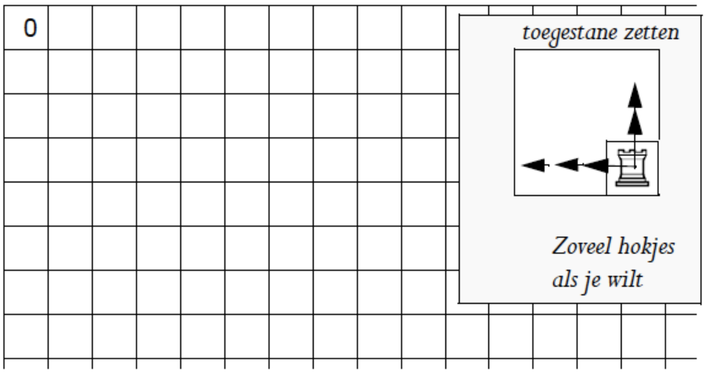
You already played on squared paper with the limited king and the limited horse before. Here are two similar games. Characteristic is that the piece you are using doesn't only make limited moves, but also as far as you want. In both cases you can find out the distribution of 0's and 1's over the squares that is needed to play an optimal game. It may be difficult to find the optimum in these games. You can let yourself be challenged to find a general rule for the possible losing fields, but you can also try to reach a more limited goal.

### The game with the North-West rook

The rook may move an arbitrary number of squares, as long as it is up (north) or left (west).

*Unlimited North or West*

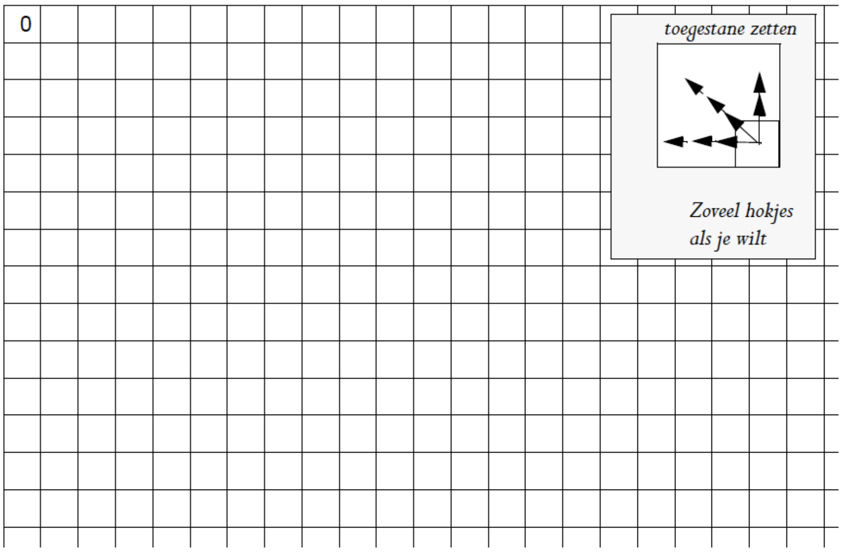
- Determine the winning and losing positions on the whole field.
- Is there a winning strategy for the first player?




### The game with the North-West queen

The queen may move an arbitrary number of squares up (north), diagonally to the top left (northwest) or left (west).

*Unlimited North, Northwest or West*




- To begin with, put 1's in the squares from where you can reach the 0 in one move.
- Now you can indicate some more losing fields.
- Work out how to back-track further.
- Play the game with the queen starting from square (20, 40)
- Again, determine the winning and losing positions for the whole field.
- Perhaps you can find a characteristic description of the losing fields. That would be excellent!
- It would also be nice if you can indicate the winning move for the following starting squares: (15, 31); (20, 21); (100, 200).
- The more you can say about the pattern of the losing squares, the better!