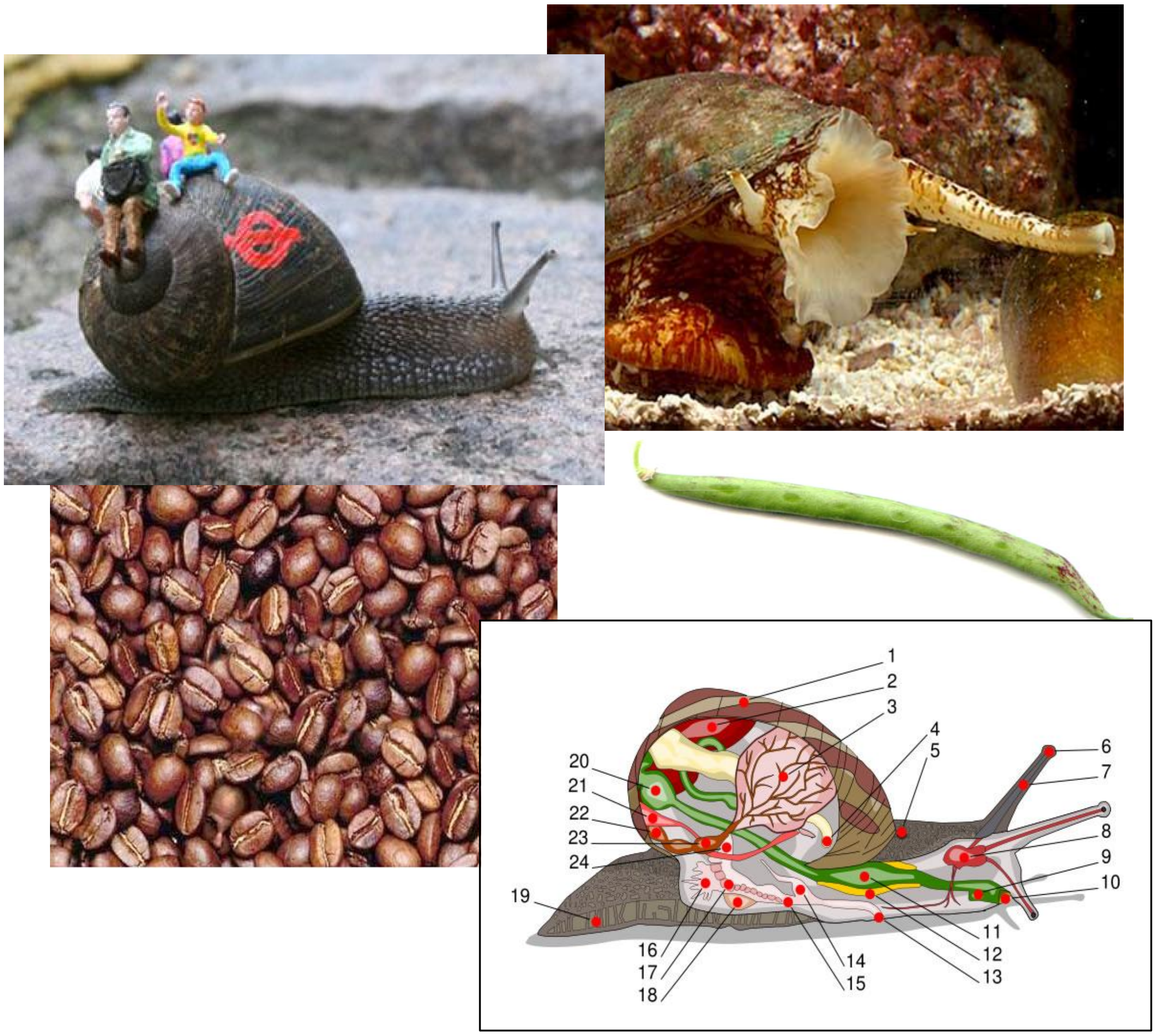
Maths B-DAY 2008

Friday 21 November



The snail and bean conundrum

Wiskunde B-Dag

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Introduction
This Maths B-day assignment comprises two parts.
In the first part, you will study the movement of a special sort of snail: snails made of cubical blocks.
The second part is inspired by a traditional African game which is usually played using seeds as counters and is known by many names such as: Kalaha, Oware, Wari or Awélé.
In this assignment the counters will be referred to as beans; we deviate from tradition anyway. You can use anything you like: coins, peas or bottle tops.
You don't have to use cubical blocks either for the snails, you could also use rectangular blocks or draughts pieces. However, you must be able to stack them.
The assignment is a conundrum looking at how the snail and the division of beans 'shape up'.
When you have finished your investigation you should formulate hypotheses and validate them where possible.

Timetable
Use the morning for the first part and the afternoon for the second part. This is just a guideline but you can be more flexible in your time planning.
The part about the snail contains quite a few tasks so you may not be able to finish them all. Don't worry: what you discover and how you report on it is more important.
You will need the knowledge and inspiration you gain in the first part to carry out the second part about the beans; there is less guidance in the second part so you will have to go it alone more.

Final report
Your final report should be easily understood by a lay person i.e. someone who has no prior knowledge of this subject. This means that you should describe clearly what it is about and what you have researched.
It is possible that you have discovered various things, through trial and reasoning, which you are convinced about but still haven't found any conclusive arguments (proof) for. You can also include these kinds of *hypotheses* in your report.
It's great if the reasoning that supports your hypothesis has a strong basis. It is not so great if your hypothesis is flawed and can be easily disproved with a simple example. Therefore you must test your hypotheses thoroughly!
There are several *reasoning tasks*. Reasoning is important for this Maths B-day so you must record the results of these tasks in your report.

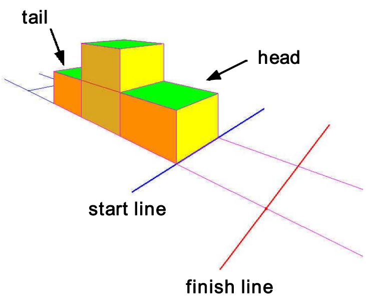
Experimenting in Excel
The experiments with snails and beans in this assignment can easily be carried out on the computer. Therefore, there are two Excel files available; one for each part of the assignment. It will be clearly indicated in the text when you can use these files.

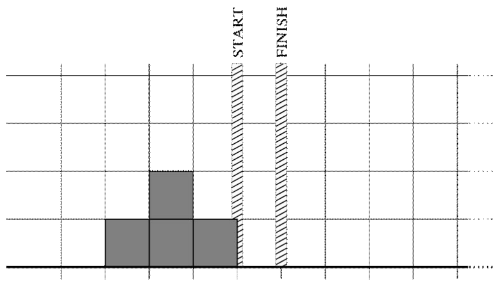
Good Luck!

Part 1: Snails

1A: Exploring movement and shape

You are going to work with rows formed by piles of blocks, see example:





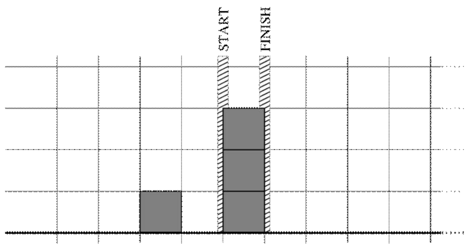
Side view of snail on squared paper

These rows of blocks are called a *snail*.
The snail drawn here is made of 1+2+1 = 4 blocks.
The far right pile is called the snail's *head*; the far left the *tail*.
A start and finish line have also been drawn. The snail is actually going to move.
At the beginning, the snail has its head against the start line. The snail moves step by step.
One *step* is made in the following way:

> *A block is taken from each pile. These blocks form a new pile that is placed directly in front of the head.*

As a result, the original piles become lower and a new pile is formed where the height is equal to original number of piles. The new head has moved a square further; the snail is on the move!

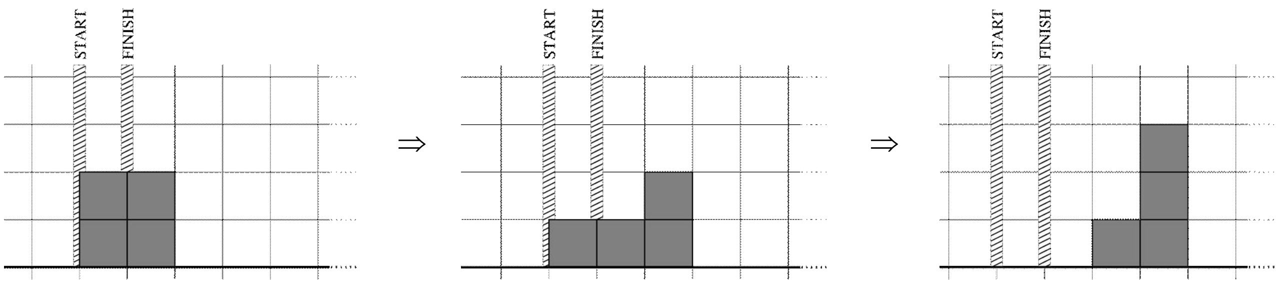
The first step is shown in the example:



step 1

There is now a gap in the snail which can happen with these kinds of snails. You can even begin with snails containing gaps.

The snail moves forward step by step in exactly the same way: remove a block from each pile and place it directly to the right of the snail forming a new head.



step 2 *step 3* *step 4*

You can see in the last figure that the whole snail is over the finish line. This snail needs four steps to pass the finish line.

Try out the forward movement with a couple of snails you have drawn yourself. You can use cubical blocks but draughts pieces are also very good.
Squared paper (or a draughtsboard) is very useful, especially for snails containing gaps.
Create a laying snail (all the blocks in a row, the height of the piles is then 1) and a standing snail (all the blocks in 1 pile) at the start line.
How do these beasts get on?
If a laying or standing snail is at the start line, do other laying or standing snails occur along the way?

Investigation 1.

Other snails made of four blocks can be also created.
First you are going to look at which four block snails are possible.
The start and finish remain the same as shown above.

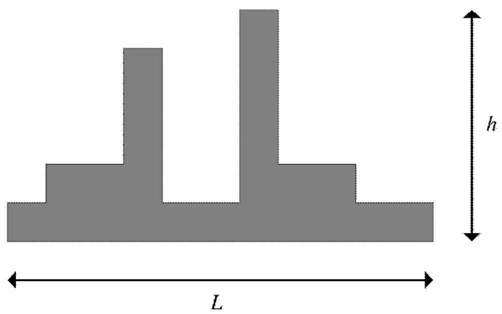
- **a** Can you create one that crosses the finish line in three steps?
- **b** Is it possible to make a four block snail that can move even faster?
- **c** Can you create a snail that needs five steps?
- **d** Is it possible to make one that moves even more slowly?

Investigation 2.

Consider snails made of 5 blocks.
What form should the snail have in the beginning so it is the quickest over the finish line?

Of course, a snail doesn't only have to be made of four or five blocks.
The total number of blocks is denoted by the letter M; $M = 4$ and $M = 5$ have already been considered.

Let's look now at a snail whose original shape is made of M blocks without any gaps.
Here is an example of a snail with length L (number of piles) and height h (number of blocks in the highest pile):



If the height of a snail is 5, you can say exactly how many steps are needed to pass the *start* line. That's not very difficult…
It is more difficult to determine when the snail is completely over the *finish* line.

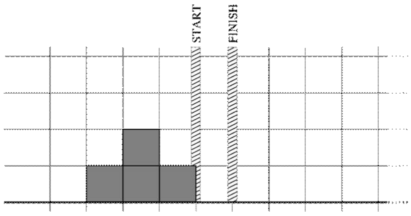
Investigation 3.

The finish line is still just one square from the start line and the original shape of the snail doesn't contain any gaps.

- **a** How many steps does it take for a snail of length L *and* height h to go over the finish?
- **b** Create an original shape of a snail for $M = 56$ that will pass the finish the quickest.
  Do the same for $M = 64$.
- **c** Can you say something about the original shape of the fastest snail with an arbitrary M?

Experimenting with snails takes a long time if you have to keep drawing the changing shape of the snail as it 'moves'. Therefore, we introduce a new notation:
A sequence of number where each number denotes the height of a pile.

The movement of the snail from the very first example



is shown as:

Start	1 2 1
Step 1	0 1 0 3
Step 2	0 0 0 2 2
Step 3	0 0 0 1 1 2
Step 4	0 0 0 0 0 1 3

You can give a snail a jagged original shape, e.g. 9 0 0 0 8 1 1, or even a very uniform shape, e.g. 5 4 3 2 1.
Once it is on the move, the number of shapes that a snail can adopt is limited.

The two shapes given above cannot occur during the forward movement. Why not?

During the movement, it is sometimes useful to look back at the shape the snail should have had before.

Investigation 4.

a Create a snail that, after 1 step, has the form: 4 4 4 4 4. Are there other possibilities?

b Create a snail that has the form 4 4 4 4 4 4 after 1 step, or explain why this cannot occur!

c Create a snail that has the form: 3 3 3, after 2 steps.

d Can the form 3 3 3 arise after 3 steps? and after 4 steps?

In general: once the whole snail has passed the start line, its length is limited. This is discussed in the next task.

Reasoning task 1.

Argue that the length of a snail cannot be longer than M after it has totally passed the start line.

1B: In the long run ...

In part 1A the start and finish line in the race are close together. This was in order to study the forward movement and the speed that the tail moves.

Now we focus the investigation on the development of the shape of a snail in the long run: How does a snail behave in a longer race? Then it is useful to observe all the snail's steps from the head, just like in the Tour the France where the leader of the peloton is filmed from an accompanying motorbike.

This can be done by aligning the snails: place them with their *heads* on top of each other and leave out all the zeros that are to the left of the tail:

Start	1	2	1
Step 1	1	0	3
Step 2		2	2
Step 3	1	1	2
Step 4		1	3

This notation is also used in the Excel file "snail" to be used with this assignment. The file has two sheets.

The first Excel sheet (called "snail race") can be used to calculate quickly how the shape of the snail develops in the long run. This is particularly handy for larger values of M.

You can use the second sheet (called "film") to follow the snail's change in shape step by step which is calculated very quickly in the first sheet.

It is a good idea to practise with this notation before you reach for the Excel file. First note the shape of a snail you have chosen yourself for a few steps.

In order to make it easier to communicate on paper, we separate the individual numbers in a row with commas; the row is written between brackets. So 1 1 2 will, from now on, be noted on paper as (1, 1, 2)

Investigation 5.

You are going to experiment with snails. You can also make use of the Excel file, "snail", provided.

- **a** If $M = 3$. Observe how the snails will look in the long run.
  What do you think of the hypothesis?
  A snail made of 3 blocks eventually forms the triangular shape (1, 2).
- **b** Experiment with $M = 6$. Do you always get a triangular shape?
- **c** Experiment now with $M = 4$ and formulate an hypothesis about the final shape of the snail.
- **d** Do the same for $M = 5$.
- **e** What about $M = 7$ and $M = 8$?

Snail (2, 2) develops as shown:

$$(2, 2) \Rightarrow (1, 1, 2) \Rightarrow (1, 3) \Rightarrow (2, 2) \Rightarrow (1, 1, 2) \Rightarrow (1, 3) \Rightarrow (2, 2) \Rightarrow ...$$

After three steps, the snail is back to its original shape (2, 2). Then it repeats what has already occurred. And the pattern keeps repeating itself.

We call the snail (2, 2) *periodic* with *period* 3.

After a while, snail (3, 1) also has a recurring pattern:

$$(3, 1) \Rightarrow (2, 0, 2) \Rightarrow (1, 0, 1, 2) \Rightarrow (1, 3) \Rightarrow (2, 2) \Rightarrow (1, 1, 2) \Rightarrow (1, 3) \Rightarrow (2, 2) \Rightarrow (1, 1, 2) \Rightarrow (1, 3) \Rightarrow (2, 2) \Rightarrow ...$$

The original shape (3, 1) of this snail never occurs again. We say that this kind of snail is *eventually periodic*, that is periodic from a certain point onwards. The period in this case is also 3.

Snails with a triangular shape, such as (1, 2, 3), are special: they have a period of 1.

A shape with period of 1 is also called a *stable shape*. For the triangular shape, it is essential that M is a *triangular number*, e.g. 6 (= 1+2+3) or 15 (= 1 + 2 + 3 + 4 + 5).

Investigation 6.

- **a** Argue that every triangular shape (1, 2, ..., n) is stable.
- **b** Are there other shapes that are stable apart from triangular shapes?

When M = 6 and M = 10, you can create snails that have a triangular form: (1, 2, 3) and (1, 2, 3, 4). Snails with values of M in-between (7, 8 and 9) can't have a triangular shape. What happens?

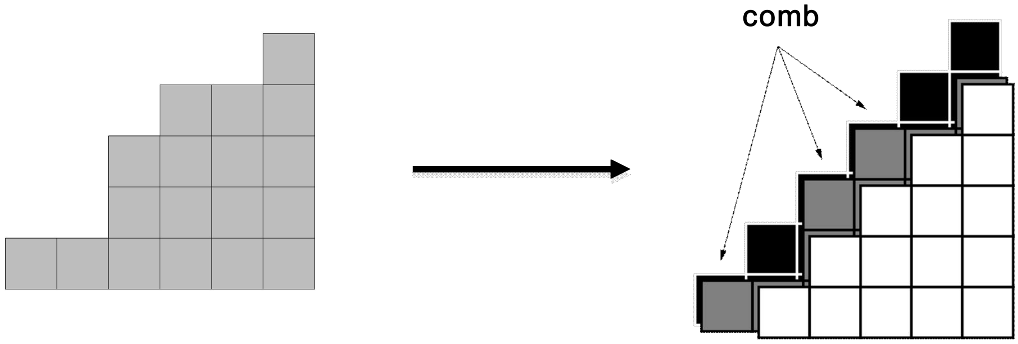
Investigation 7.

Consider M = 7 and M = 8 (from investigation 5e) and add M = 9 to your investigation.

- **a** How do these snails behave in the long run compared with the two triangular forms M = 6 and M = 10?
- **b** Can you deduce anything about the length of the period from the final shapes?

If M isn't a triangular number then it lies between two successive triangular numbers. Take the biggest possible triangular number a that is still smaller than M. Take the smallest possible triangular number b that is still bigger than M. Where $a < M < b$. We consider shapes containing M blocks where: the triangular shape with a blocks fits in completely, and the triangular shape with b blocks encloses it.
This is called a *comb-shape*.
For example: M = 18. the gray coloured snail below has a comb-shape.
In this case a = 1+2+3+4+5 = 15 (the white triangle below) and b = 1+2+3+4+5+6 = 21 (the almost totally concealed black triangle below).

The three blocks that are wedged between the smaller and the larger triangle are called the comb.

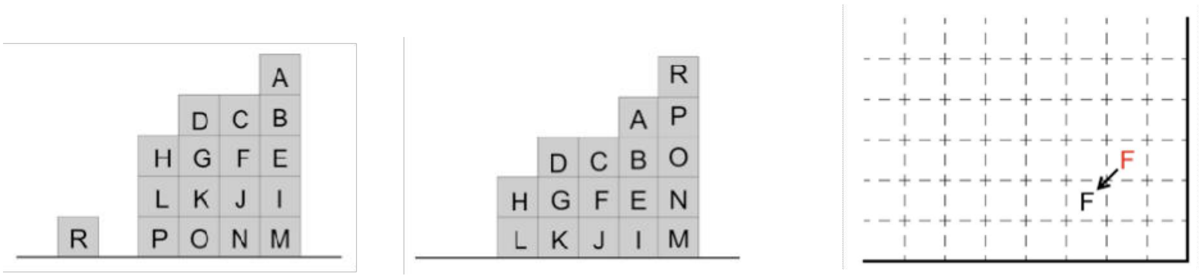


Investigation 8.

- **a** Consider the snail (1, 2, 3, 4, 9). This is a triangular shape with blocks on the head: (1, 2, 3, 4, 5 **+ 4**). Will this snail get a comb-shape?
- **a** Argue the following general hypothesis: *Once a snail has a comb-shape then it always keeps a comb-shape.*
- **b** Formulate a hypothesis about the shape of the comb and the corresponding period.

Up until now you have looked at the snails as piles of blocks and rows of numbers.
Most likely, you have chosen the easiest way when making the steps: removing the top block and stacking it in front of the snail.
The following is more difficult to carry out in practise but possible on paper: take the *bottom* block away from each pile and place them in the same order in front of the snail. This gives, of course, exactly the same result!
The advantage of this method of working is that it is possible to follow the progress of an individual block in the consecutive steps.

In the example, you can see a snail where the blocks now have letters stuck on them. The bottom row of blocks is removed and placed upright at the front. After this step, all the blocks (except for M) have a different position in the snail. The bottom row (R PONM) is now in a vertical pile at the front. The rest of the blocks move one position to the left and drop down a row. The movement of block F is shown separately.



Follow the position of F in the snail a few more steps. How do the other lettered blocks behave? How does R behave differently?

Periodic and stable snails are particular examples of eventual periodic snails.
It sometimes takes a long time before a snail becomes periodic, for example snail (3, 5, 0, 0, 3, 6, 9, 2, 4, 5).
The period only begins from step 47:

Start	3	5	0	0	3	6	9	2	4	5
			....							
step 47		0	1	2	3	4	5	7	7	8
step 48		0	1	2	3	4	6	6	7	8
step 49		0	1	2	3	5	5	6	7	8
...		0	1	2	4	4	5	6	7	8
		0	1	3	3	4	5	6	7	8
		0	2	2	3	4	5	6	7	8
		1	1	2	3	4	5	6	7	8
		0	1	2	3	4	5	6	7	9
		0	1	2	3	4	5	6	8	8
step 56		0	1	2	3	4	5	7	7	8

The last task in part 1 is by no means easy! As a team you can really distinguish yourselves here.

Reasoning task 2.

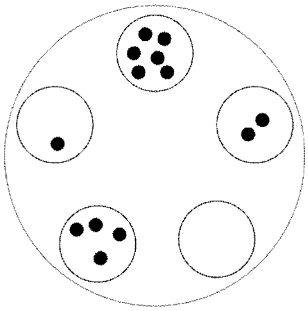
Give sound argumentation to the following claim:

All snails, regardless of their original shape, are eventual periodic.

Part 2: Sowing Beans

In part 1 you looked at the change in shape of piles of blocks when you carried out a certain process. You came across a number of hypotheses and you were probably able to prove a number of hypotheses. In this part you are going to study another process: sowing beans in a limited number of containers. The task is to think up hypotheses (by experimenting) and try to prove them. The experience you have gained with the snail is useful because you will also come across things like triangular numbers, periodicity and stability. It seems that snail racing and sowing beans are similar!

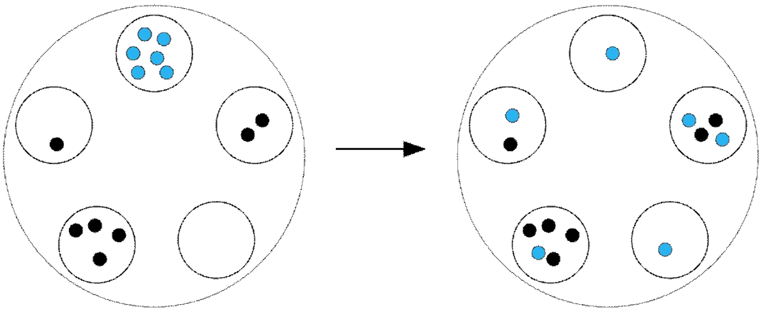
You start with n containers and M beans. Place the containers, evenly spaced, in a circle and distribute the beans among the containers. The distribution doesn't have to be fair: containers can be totally empty or nearly full of beans. Here is an example where $n = 5$ and $M = 13$:



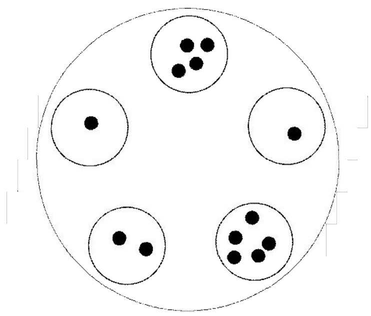
A *sowing step* is as follows:

1. You begin with the uppermost container. Remove all the beans and hold them in your hand (movement 1).

2. Then drop one bean into the next container to the right. Then drop one bean in the next container and every other container in turn in a clockwise direction until your hand is empty. This is movement 2.
In the example, the result of movement 2 is:

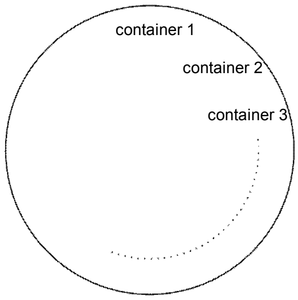


3. The last movement (movement 3) is to move the containers one place in a counter clockwise direction:



This last movement ensures that the distribution, starting with movement 1, always begins with the contents of the uppermost container.

For practise, draw the next two sowing steps.
Just like the snails, it is very time consuming to keep drawing. Therefore, a notation system is also used here. Number the containers from 1 to n as follows:



You can now indicate the number of beans per container using a row of numbers.

Therefore, (6,2,0,4,1) means that there are five containers (n = 5); the first container holds 6 beans, the second 2, the third 0, the fourth 4 and the fifth container 1 bean. This is the example you began with. The next six steps are:

Start situation:	(6,2,0,4,1)
After the first step:	(4,1,5,2,1)
After the second step:	(2,6,3,2,0)
After the third step:	(7,4,2,0,0)
After the fourth step:	(6,4,1,1,1)
After the fifth step:	(6,2,2,2,1)
After the sixth step:	(4,3,3,2,1)

There is also an Excel file ("sowing") available for the sowing where you can run through the sowing processes.

Note, when using the Excel sheet, the amount of numbers you enter for ‘start situation:’ also determines the value of n.

Effect: when you enter the start row (4, 5, 3, 2) you get a different value for n than when you enter (4, 5, 3, 2, 0). In the first instance, n equals 4, in the second it equals 5.

To help you on your way, here is a exploratory task for a sowing process with four containers (so n = 4). Don’t forget, you will have to think of hypotheses yourself soon.

Investigation 11.

- **a** The start situation is (4,3,2,1). What do the next steps look like? (Use Excel if necessary)
- **b** The start situation is (8,6,4,2). What do the next steps look like?
- **c** Give a *stable* start situation for when n = 4 and M is a multiple of the fourth triangular number (4+3+2+1).
- **d** An example of such a number is $M = 2 \cdot (4+3+2+1) = 20$. What happens in the corresponding start situation (20,0,0,0) and (2,9,9,0)?
- **e** Is there stability in the start situation (21,0,0,0)?
- **f** Add one or two more beans. What happens?

Investigation 12.

Investigate the sowing process for 5 containers (so n = 5) in a comparable way.

Main investigation in part 2.

How do the number of beans in the containers progress in the sowing process with n containers and M beans? Formulate hypotheses and try to prove these hypotheses.

THE END