Polygons, Triangles and Capes

Maths B-day

23 November 2007

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Introduction to the assignment

Example: on the inside or the outside?

You walk a long way from home and back again. You don't want boredom to set in so you make sure you don't pass the same point twice, excluding the start and finish. Nothing unusual, right? Actually there is because you have divided the world into two parts: the part inside the route you have walked and the part outside. Closed figures that do not go over the same point twice always have these characteristics. This maths B-day is about special closed figures that do not retrace points, figures composed of a finite sequence of straightline segments. You know them already of course, triangles, quadrilaterals, pentagons and polygons. The generic name is *simple polygon.* But they can also be quite big and complicated. Just like on the front page: a line with no ends and doesn't cross itself, a simple polygon. So there is an inside and an outside area. An obvious question could be whether the text **Maths B-day 23 November 2007** lies in the inside area or not.

Take a couple of minutes to think about it.

trial

With a bit of patience and a pencil you can work it out. You begin with the **M** and you draw a line that doesn't cross over the printed line. And then you can see if it is in the inside or outside area. Disadvantage: it takes too long and that is a waste of time.

or reasoning

Be clever: think of something else. Begin with the **7** and this time do cross the black line. Were you inside, then now you are outside or the other way round. If you continue to the right then you cross 13 lines and you are really on the outside. So you were at the **7** ..... inside the lines.

The goal of this Maths B-day assignment

The goal of this Maths B-day is that you don't just use the pencil method but that you think up strategies: a clear plan that you can also apply to other bigger and more complicated figures, give a skilful answer to a question: other questions than inside/outside! Considering even and uneven can for example, solve the inside/outside question. That was logical reasoning. Problem solved and then you know how to apply it to other more complicated figures. That is what today is all about.

And now?

In the rest of this assignment you will come across various other characteristics of simple polygons. You will investigate them using examples and guiding questions (indicated by **G**). This will mainly be in parts A and B, which form the basis for the important questions that you will find in the latter parts. From part C onwards, you will come across real research questions. They are in a grey box. These questions play an integral part in your research. They are also questions that you may not immediately know the answers to and can perhaps investigate them further during the Maths B-day afternoon. The guiding questions form as it were the basis of your research but you can really score points with the research questions. Show that you have bright ideas, set it in a new light, etc.

Something about the applications

Mathematicians don't talk very much about the practical applications of their work. But in this Maths B-day, the things you come across are often applied in different areas such as automation, robots, movement of machines, dividing up and monitoring secure areas, displaying photos in computer files, medical descriptions. Ah, what does it matter...
We are doing it to find good arguments and reasoning, just like the '**7**' on the front page.

The assignment

The assignment is divided as follows:

- Parts A and B comprise mainly questions designed to make you think and facilitate your ability to answer the questions in parts C to F. Make sure your team spends enough time on parts A and B and then divide up the work for parts C to F. In your report, do not list the answers to all the guiding questions in order. Make a sound decision as to which elements of the questions you will record in your final report!
- Parts C and F are linked to each other (with the research questions 1 and 4). ***The answer to these two research questions should form the core of your report.*** You must pay attention to these two subjects in your report!
- Parts D and E are interesting asides. The results (research questions 2a, 2b and 3) do not contribute to understanding the connection between parts C and F. But there are challenging activities that are well worth investigating! If you get round to these in your report then you will definitely earn extra points to put you on the way to getting the best results!

Show in your report that you have developed your insight in the basic understanding of simple polygons. You can do this by carrying out a good selection of the exercises, including drawings. Show, using the basic material and your own ideas how far you have come with the research assignments 1 and 4, where you have to use reasoning and strategies and express yourself clearly. It is to your advantage to supplement your report with findings from research questions 2a, 2b and 3!

To be handed in:
Make a continuous report of your work that is easy to read and can be understood by your fellow students even those who have never heard of simple polygons.

More tips

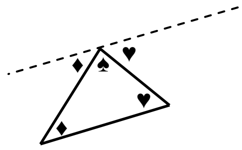
- For this assignment you will draw and try out things a lot. Often, your drawing will almost be right but you just need to change one small thing. Solution: lots of paper, a pencil and an eraser.
- Your work must be easy to copy so use black pens and drawing materials.
- Also think about how much time you have on this day. Make sure you start on time.
- Divide up the tasks when you have decided on your approach.

Finally

Since precise formulation is often needed, descriptions and definitions are given at various points in the assignment.The most important definitions and descriptions are given on a separate “defintions and descriptions” sheet which you received with the assignment.
Two worksheets have also been enclosed (for guiding questions **G25** and **G31**) where you can carry out the instructions given.

Good luck and enjoy this Maths B-day!

Part A Vertices of polygons



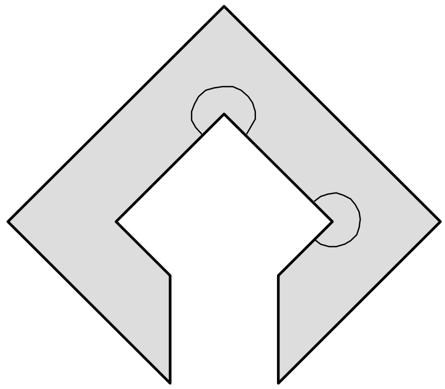
We assume that you know that the three internal angles of a triangle make up 180°.
In other words:

The sum of the internal angles of every triangle is 180°

The familiar proof can be seen in the figure on the right.

G1 **a.** Draw a quadrilateral and determine the sum of the internal angles. Is this true for all quadrilaterals also for those with an intruding angle?

b. Can you devise a formula for the sum of the internal angles of a polygon with *n* vertices and no intruding angles? (The formula must express the sum of the angles in the number of vertices *n* of the polygon).



In a 10-gon (10 angles - count them!) with an intruding part, as shown in the adjacent figure, you have to know which angles are concerned. They are the angles in the ***inside area*** of the polygon; in the grey area. There are two angles indicated that are 270°.

c. Check if the total sum of the angles complies with the formula in **b** again.

Maybe you divided the figures in **a** and **b** up into triangles. You could try that here as well.

d. Divide the 10-gon into triangles; the vertices of the triangles are also vertices of the 10-gon.

Definitions

This Maths B-day assignment is about various characteristics particular to polygons.
Exact use of language is essential so we have given the definitions of the most important concepts. These definitions can be found on a separate sheet but here are a few. We have kept to the regular mathematical terms. In several guiding questions we use the terms: watch out!

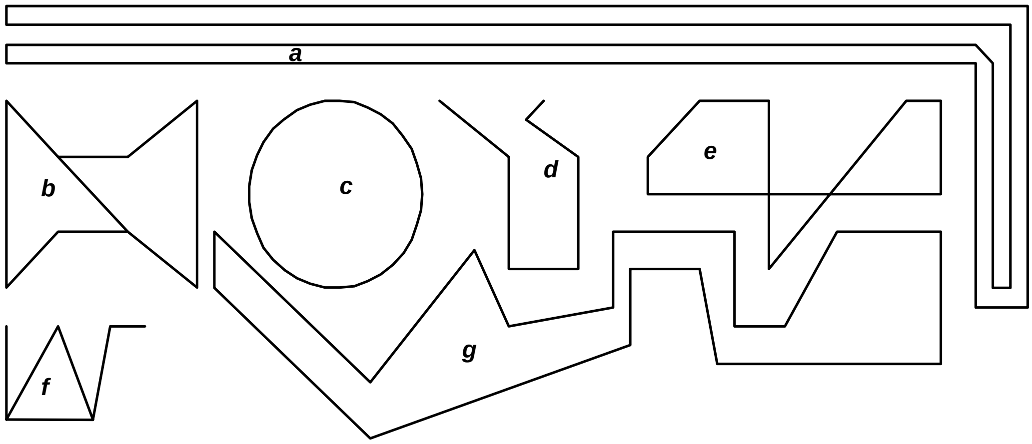
polygon with n vertices (n > 2)

A polygon is a closed figure that is made by n (thus a finite number) line segments.
There are n different vertices $p_1, p_2, \ldots, p_n$; these are the vertices of the polygon.
The polygon has n sides: $s_1 = p_1 p_2$, $s_2 = p_2 p_3$, ... , $s_{n-1} = p_{n-1} p_n$ *and* $s_n = p_n p_1$

Note:

- Vertices are usually denoted with a capital letter. In this exercise we use capital letters to indicate a polygon. The lower case letters (often with indices) are used to denote the vertices and the sides.
- $p_n p_1$ is the side of the polygon that joins the last vertex p_n with the start vertex p_1, so the polygon is closed.
- A polygon has the same number of sides and vertices.

G2. Which of the following figures are polygons?



A polygon can cross itself; this can be seen in one of these examples. We exclude this from our research.

simple polygon

A simple polygon doesn't have any sides that cross each other.

G3. Which of the above figures are simple polygons?

Whenever you come across the term '*polygon*' in the rest of this assignment then '*simple polygon*' is meant. You are already familiar with the next part.

Inside and outside areas

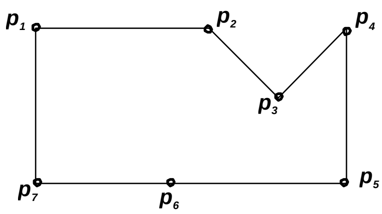
A simple polygon divides the surface into two interconnected parts: the inside and the outside area. Characteristic of the outside area: it is infinitely big. The polygon by definition does not belong to the inside or the outside area.

Looking at angles more closely

In guiding question **G1** you saw the following:

Angles of a simple polygon

The size of the angles of a simple polygon are determined by the angles in the inside area



On the right you can see a polygon with 7 angles:
Straight angles, such as vertex p_6, are also permitted!
Note the classification of angles:

protruding angle

A vertex is said to be protruding if the corresponding angle is between 0° and 180°

intruding angle

A vertex is said to be intruding if the corresponding angle is between 180° and 360°

Straight angle

A vertex is said to be straight if the corresponding angle is 180°

Note:

1. angles of 0° and 360° are therefore not permitted in simple polygons!
2. *angle and vertex* are different notions. The vertex of a polygon is the point where its sides intersect. These two sides make an angle between 0° and 360°.

G4. You have polygons with and without intruding angles. Here is something to investigate! Use your creativity in the drawing assignment below. If you are completely sure that the assignment is impossible then you must explain why.

a. Draw a polygon with 4 protruding and 4 intruding angles; draw the intruding and protruding angles alternately.
b. Draw a polygon with 4 protruding and 4 intruding angles; draw first the intruding angles one after another and then the protruding angles.
c. Draw a polygon with 3 protruding angles and 2 intruding angles.
d. Draw a polygon with 2 protruding angles and 3 intruding angles.
e. Draw a polygon with 3 protruding angles and 20 intruding angles.
f. Draw a polygon with 12 straight angles and 3 angles of 60 degrees.

Just imagine ...

In the last example (**G4d**) you should have noticed that it is not possible. A good moment to find an argument that supports the following theorem:

All simple polygons have at least three protruding angles.

With this kind of theorem you can't just say: well that's logical because if you have less than three then it's impossible to draw. Because: maybe someone else can do it with a more complicated or even a simpler example.
You have to find an argument that shows that it is *never* possible. Such an argument may start like this:

Just imagine that there is a polygon with n vertices, and only 2 vertices are protruding. That is true for a certain value of n, you don't know which.

Now we are going to deduce outcomes. If we get stuck then we can from there conclude the premise by **just imagine** is not right!
First result from the *just imagine* theorem:
There are $n-2$ angles, that are intruding or straight.
Together they are at least $(n-2) \cdot 180$ degrees

G5. **a.** Explain why that is the case

b. What is the total of the angles according to the trusted formula?

c. **..**.. Finish the line of reasoning yourself.

From now on, you may use the following theorem:

the protruding angle theorem
All simple polygons have at least 3 protruding angles.

Very noteworthy: a simple polygon without any intruding angles is possible. But a simple polygon without any protruding angles is impossible
In other words: there is always a protruding angle in a simple polygon!
You need this simple result for part C.

Part B Diagonals and capes

In a simple polygon with n vertices p_1 , p_2 , ... , p_n each vertex can be attached to each of the $n - 1$ other vertices with a line segment.
The two line segments connecting that vertex with its ***adjacent vertices*** are ***sides*** of the polygon.
A connecting line segment that lies entirely inside the polygon we call a ***diagonal***.
All other connecting line segments (except for the vertices) are outside the polygon or partly inside and partly outside.

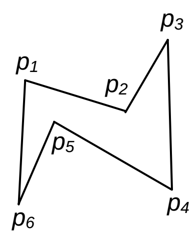
G6. Let's look at simple polygons for n = 4, n = 5 and n = 6.
For each of the following questions you can make assumptions and then test them by drawing examples and thinking about it.

a. How many connecting line segments (excluding the sides) does a simple polygon have in total?

b. Is it possible to draw a simple polygon with exactly one diagonal?
If that is not possible, what is the minimum number of diagonals?
And what is the maximum number of possible diagonals?

c. Can you divide up every simple polygon into non-overlapping triangles using diagonals? How many do you need?

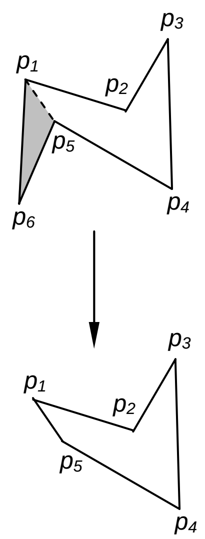
Each vertex p_i of a polygon has two adjacent vertices p_{i-1} and p_{i+1}. Vertex p_i is called a **cape** if the line segment that connects the two adjacent vertices is a diagonal.
Therefore all capes are protruding angles but not all protruding angles are capes!
The polygon drawn here as an example has 4 protruding angles (p_1, p_3, p_4 and p_6), two of them are also capes (p_3 and p_6).



G7. Look at other simple polygons than the one shown in the example with n = 6

a. Construct a 6-gon with exactly two capes. How many protruding angles are there? Are there more possibilities?

b. Construct a 6-gon with the same number of capes as protruding angles. Are there more possibilities?



A cape can always be removed from a simple polygon with n vertices, without drastically changing the appearance of the polygon. You trim away the triangle formed by the cape and its adjacent vertices so the diagonal that intersected the two adjacent vertices becomes a side of the new polygon.
The new polygon has one less vertex and one less side. The illustration on the right shows this procedure.
The trimming procedure is only possible with capes and not with other vertices!

The next question involves a small investigation into capes.

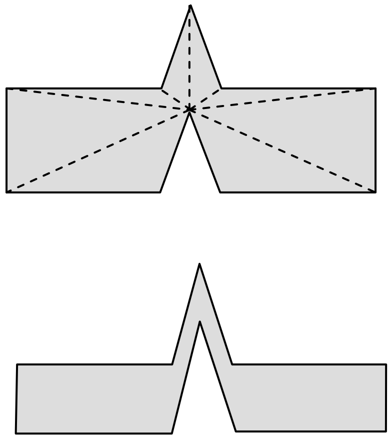
G8. In the illustration showing trimming a cape (triangle $p_1 p_5 p_6$) a new cape is automatically created at p_1. p_5 has also become an cape.

a. If you trim away another cape (p_1, p_3 of p_5) from the new polygon will another new cape be created in the remaining polygon?

b. Is this the result of the special example used in the illustration or is this always the case? In other words: investigate if removing an cape from a simple polygon always leads to a new cape being formed in the arising polygon.

Part C Polygon Triangulation

When determining the sum of the angles of a polygon with *n* sides (in question **G1**) we made use of the fact that the polygon could be divided into *n – 2* triangles where no new vertices were needed. Actually we didn't check if that is always possible and that is a weak point in the line of reasoning!



G9. Look at the top example, in this figure (n = 10; 8 triangles) a possible division has been given.

a. The bottom figure (also n = 10) can also be divided into 8 triangles but not in the same way as the top figure! Divide it into 8 triangles, again without an extra vertex.

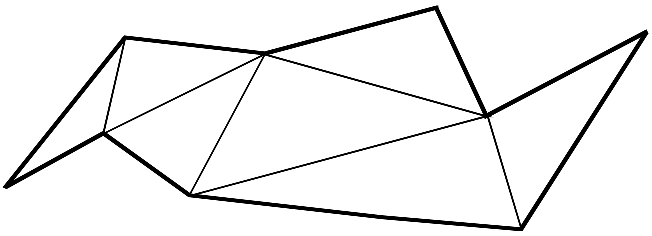
In these examples you can also show without dividing them up that the sum of the angles is 8×180 degrees because the upper point fits perfectly in the shape under it.

b. Now give an argument that conclusively proves that you cannot divide these figures into 7 triangles.

After this introduction, the following definition is easy to understand:

simple polygon triangulation

A ***triangulation*** *of a polygon P is the partition of its inside area into triangles that do not overlap each other and the vertices remain vertices of P.*



There is still an unanswered question: can all polygons be triangulated and are there always precisely $n - 2$ triangles needed?
This part of the assignment leads to the theorem to be proved:

Triangulation theorem (to be proven)

for all simple polygons with n vertices there is a triangulation with $n - 2$ triangles.

As an introduction we shall first look at some special cases: convex and semi-convex polygons.

convex polygons

The notion convex polygon plays an important role in a number of activities today. Normally a convex simple polygon would be defined as a polygon with no intruding angles. In this case straight angles are also permitted. For this maths B-day we prefer a stricter definition:

convex polygon

A simple polygon with only protruding angles is called a convex polygon

The nice thing about a convex polygon now is that you can draw diagonals from every vertex to another (except to the two adjacent vertices because then you get sides of the polygon).

G10. Give a *recipe* to show how to triangulate a convex polygon.

The recipe could begin like:

1. Given a convex polygon with *n* vertices.
2. Choose one of the vertices.
3. Draw...
4. …

a. finish the recipe

b. Include some examples.

semi-convex polygons

*A **semi-convex** polygon is a polygon where all, except one, intersecting line segments between non-adjacent vertices are entirely inside the polygon.*

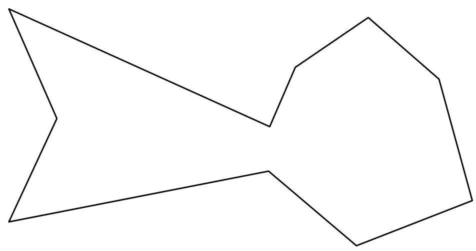
G11. It is also possible to devise a recipe for semi-convex polygons

a. Show that you have to, and are able to, adapt one of the steps in the recipe for convex polygons.

b. Give some convincing examples of semi-convex polygons and their triangulation.

Splitting and sticking!

It is also not difficult to triangulate the polygon shown here. It is possible to split the polygon into a convex and a semi-convex piece.



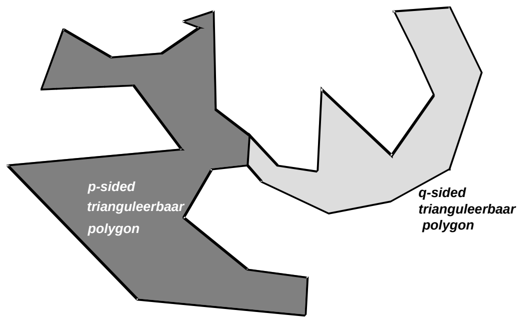
G12. a. Indicate the pieces on the figure

b. How do you know from your previous experience that the pieces can be triangulated separately?

The suggestion given here is an example of the ***divide and rule principle***:
If you can’t solve a problem directly then divide into two and solve both parts individually.

G13. Nevertheless, there is still something that you have to unravel! The polygon has 10 sides. If you divide it then you get two polygons, each with fewer sides. Both of the two part-polygons needs a number of triangles for triangulation. Does the total number of triangles you need still match that for a 10-gon?

G14. Consider a polygon with *n* sides that can be split into two polygons via a diagonal. One part-polygon has *p* sides, the other has *q*. Assume that both parts satisfy the triangulation theorem.



a. Express the number of triangles necessary for the triangulation of the part-polygons in *p* and *q*.

b. Show that there is a triangulation for the polygon, now merged together again, that satisfies the theorem.

Splitting is always possible!

In order to apply the divide and rule principle you must be able to find at least one diagonal in your polygon. It is not the case that the diagonals in a simple polygon are always there for the taking. In part B you practised a bit with diagonals. A simple polygon without any diagonals is not possible. But proof is necessary for the:

Diagonal theorem
every simple polygon has at least one diagonal.

The following observations give an idea how to find the proof.

- *Assume that the polygon has a cape. Then you are done! (Why? Where is the diagonal then?)*
- *But actually, we don't have any proof (yet) that all simple polygons have a cape.*
- *We do already know (see part A) that all simple polygons have a protruding angle. At a protruding angle h, that is not a cape, other pieces of the polygon come close to h, pieces that make it not possible for h to be a cape ...*

G15. Show: If h is a protruding vertex but not an cape, then there is a diagonal from h.

G16. Formulate your own proof for the diagonal theorem.

Research assignment 1

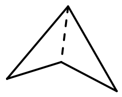
Using the above, you can now give conclusive proof for the previously mentioned triangulation theorem:

Triangulation theorem
For all simple polygons with n vertices there is a triangulation with $n-2$ triangles.

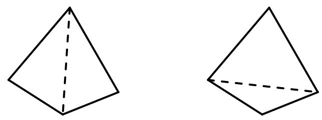
You should verify the proof that triangulation is possible for all simple polygons with the right number of triangles. For this proof you can use the ingredients introduced in parts A to C.

Part D Counting Triangulations

A quadrilateral can have one or two triangulations depending on the form:



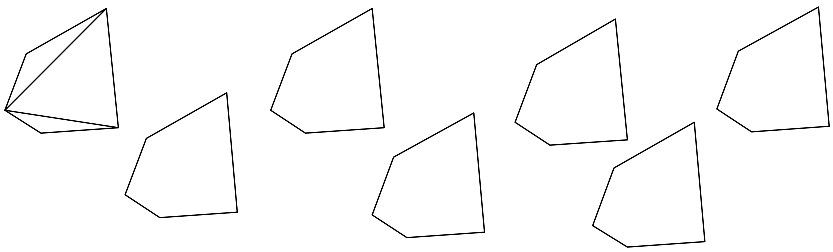
not convex: 1 triangulation



convex: 2 triangulations

Polygons with more vertices have even more possibilities!

G17. The same convex 5-gon has been drawn several times below.
Draw as many different triangulations as possible; one has already been given.



For the triangulation of convex polygons you can choose from all diagonals; we will investigate this case further.

Convex polygons: count the number of triangulations.

For convex polygons the number of possible triangulations is dependent on the number of vertices n. When n = 3 there is of course only one possiblity; the polygon is a triangle. Also when n = 4 there is no problem: 2 triangulations. Each of the two diagonals determines one. When n = 5 you have probably found 5 triangulations.

G18. Think of a simple reasoning for n = 5 that there can't be any more.

From now on, the number of triangulations for variable n is called T_n

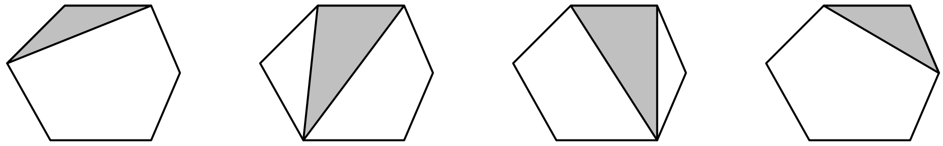
Work systematically

It is a good idea to follow a system for large values of n so you can be sure that you do not forget any possibilities and don't duplicate.

G19*.* Try to devise a system to find the possible number of triangulations for n = 6.

Here is an example of a systematic approach. It is of course possible that you already have another (better?) one.
There are 14 possible different triangulations of a convex 6-gon.
You have just investigated that. Did you not manage to find all 14 or did you have even more, then you can have another chance now!
You can group (or classify) these 14 possibilities.
The following classification uses something that resembles the divide and rule principle.

Here we work from only one side at a time. In the diagram below the top side has been chosen as the first side and the four possible triangles have been shaded grey.

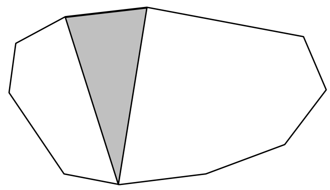


G20. Divide and rule: see what is still possible in each of the remaining white polygons. Obviously they have less than 6 sides.

G21. Explain that by using this method of division you definitely know:

a. That you get all the triangulations of the convex 6-gon

b. That you don't duplicate.



G22. As a lead up to bigger polygons we will look at the following interim question about this convex 10-gon.
How many triangulations of this 10-gon are there where this grey triangle appears?

The following table contains all the known values of T_n

n	3	4	5	6	7	8	9
T_n	1	2	5	14			

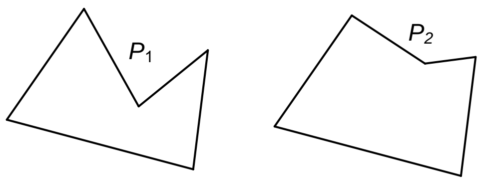
G23. Fill in the table for n = 7, n = 8 and n = 9.

Research question 2a
Formulate your method in the form of a recipe. You may process the recipe into a programme for the computer or GC. Then you will make more progress.
Challenge yourself with T_{17} and larger values of n.

Semi-convex polygons: count the number of triangulations

In a way, convex polygons are much more predictable in their behaviour than the non-convex. This is because non-convex polygons have more forms.

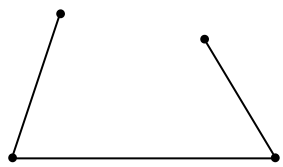
G24. Look at the two non-convex pentagons P_1 and P_2 shown here.



a. How many different triangulations does P_1 have?

b. How many different triangulations does P_2 have?

G25. Four of the five vertices of a 5-gon are shown here (the same figure is also on the **worksheet G25**). The position of the fifth point can still be freely chosen. Of course, it should be chosen so that a simple polygon is created (so not self-intersecting!).



Different numbers of triangulations of the complete 5-gon can be made depending on where that fifth point is positioned.

a. First choose which part the fifth vertex can lie. The sides may not cross any of the sides that have already been drawn!

For some points in the permitted area, the pentagon that is created by adding the fifth point can have five triangulations and for others three and so on.

b. Divide the acceptable area for the placing of the fifth point on the basis of the number of triangulations that is possible. Indicate clearly the different areas.

semi-convex polygons

Non-convex polygons are not as easy to get a grip on as convex polygons. You have noticed that by now. That's why we are only going to look at the previously mentioned semi-convex polygons.
Remember the definition:

> *A **semi-convex** polygon is a polygon where all intersecting lines (except one) of non-adjacent vertices are entirely in the polygon.*

You could say that a semi-convex polygon is a convex polygon where one vertex has been pushed inwards a little bit. So that only the intersecting line segment between the two adjacent vertices is crossed. Polynominal P_2 (see **G24** above) is semi-convex and P_1 is not.

G26. Determine how many different triangulations there are for semi-convex polynominals $n = 4$, 5, 6 and 7. Don't forget to use a system.

Here you also get the opportunity to improve your answer (if necessary!).
There is a relationship between the numbers T_n that you found in the convex case and the number in the semi-convex cases!
The number of different triangulations of semi-convex polygons we call $\boldsymbol{B_n}$
The B-numbers appear to be the difference between two successive T-numbers. More exactly:

$$\boldsymbol{B_n = T_n - T_{n-1}} \quad \textit{for } n > 3$$

Use this relationship to check or rectify the number found.
Of course it is nice that a relationship like this falls in your lap but a real mathematically minded person like you is not sastisfied with that! Of course you want to know *why* there is that relationship and you want to show that convincingly to the non-initiated!
In short:

G27. Show that the given relationship between these three numbers is valid.

It is possible that your curiosity has really been woken now.

What about a polygon that is convex apart from two small dents? Is it also possible to find a link between the number of possible triangulations and the T and B values?

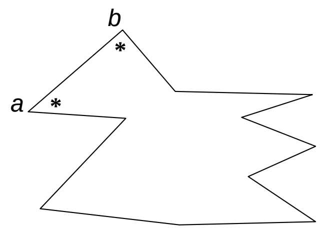
Research assignment 2b

Investigate cases where two intersecting line segments of non-adjacent vertices lie completely outside the polygon and all the other intersecting line segments lie completely in the polygon.
Can you find a general formula for this case?

Part E The two-capes theorem

You know that you can draw a polygon with lots of sides that still only has 3 *protruding* vertices. You can formulate a similar question about *capes*.

We distinguish capes into *overlapping* capes and *non-overlapping* capes. In the example given here you can see the difference.

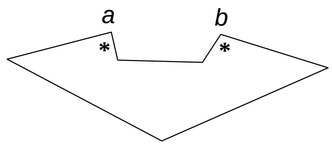


Overlapping capes

two capes a and b of a simple polygon are considered overlapping if ab is a side of the polygon.

Non-overlapping capes

two capes a and b of a simple polygon are considered non-overlapping if ab is not a side of the polygon.



G28. Try to draw the following polygons or give reasons why it is not possible.

a. a quadrilateral with exactly 3 protruding angles.
b. a quadrilateral with 2 protruding angles.
c. a quadrilateral with exactly 3 capes.
d. a quadrilateral with exactly 2 capes.
e. a quadrilateral with 2 non-overlapping capes.
f. a quadrilateral with exactly 2 overlapping capes.

G29. Try to draw the following polygons or give reasons why it is not possible.

a. A polygon with 5 sides and exactly two capes.
b. A polygon with 5 sides and exactly two capes, where the capes are overlapping.
c. A polygon with 12 sides and exactly two capes.

The following theorem is valid for capes, it resembles the theorem for 3 protruding angles.

two non-overlapping capes theorem

All simple polygons with more than three sides have at least two non-overlapping capes

This theorem can be proven in several different ways.
You can find your own way or use the following tip.

G30. Draw several different simple polygons to orientate yourself.

a. Triangulate these polygons. You know that there are always two triangles less than the number of sides.

b. Mark in the triangulations the triangles that have two sides on the polygon.

c. Show: at least one vertex of one of these triangles in question **b** is an cape. It is the vertex between the two sides that are also sides of the polygon.

Reasearch assignment 3:

Prove the theorem:

two non-overlapping ears theorem

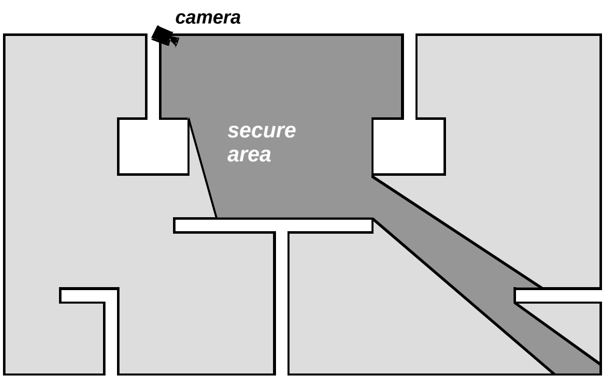
All simple polygons with more than three sides has at least two non-overlapping capes

Part F Guarded polygons

This section of this maths B day is of great practical importance.
It is about the surveillance of spaces. You want to place cameras to be able to view everything but also no more than are really necessary. They can turn all ways. Exhibitions with a lot of side-sections and corners and expensive things are good examples. The floor plan of the exhibition space is a polygon.
In this part

- *The area to be watched is always in the inside area of a simple polygon,*
- *The cameras are only allowed to be mounted on the vertices of the polygon.*

In the actual application this is also often the case. The polygon will move along the various sides of the walls as shown in the example.
In this example one camera has already been mounted and the area that can be watched is shaded.
An obvious question: how few cameras can be used to overview the whole area?



G31. This floor plan is also on **worksheet G31**, without the camera. Find the minimum number of cameras needed.

The minimum number of cameras depends on the shape of the area inside the polygon and the number of vertices. A number of simple assertions are not that difficult to explore or refute.

G32. Prove or give a counterexample:

a. All (semi-)convex polygons can be viewed by 1 camera.

b. All simple 5-gons can be viewed by 1 camera.

c. All polygons can be viewed by placing cameras on all the protruding angles only.

d. All polygons can be viewed by placing cameras on all intruding angles only.

General theorem about guarding polygons
A simple theorem about a sufficient number of cameras is:
all polygons with n angles can be guarded by $n-2$ cameras.

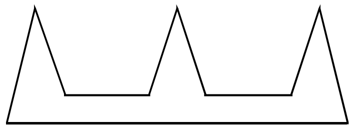
This is not an impressive clear cut result, but it is an assertion that links the number of angles and the number of cameras. The proof is also characteristic:
Triangulate the polygon. That is possible with $n-2$ triangles. All triangles can be guarded with 1 camera from a vertex.

A much better theory but more difficult to prove is the following.

The guarded polygon theorem.
A simple polygon with n vertices can always be guarded by p cameras that are placed in the vertices, where p is the smallest whole number for which $p > \frac{1}{3}n - 1$

It is clear that there are many polygons that can be guarded with a lot less cameras. But the general theorem can not be improved!

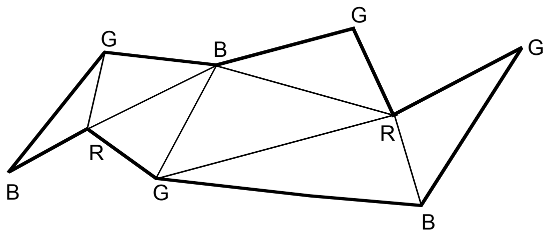
G33. Show that $p=\frac{1}{3}n-1$ is really too small when n = 9 for a polygon like the one drawn here, also for other values of n.



Two other representations of polygons may help with the investigation:
- colouring a polygon
- using a dual graph to represent a triangulated polygon.

Colouring a polygon

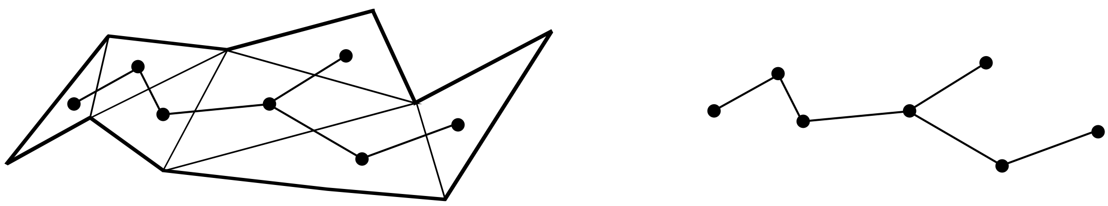
In the example below you can see the vertices of a polygon coloured with three different colours (**B**lue, **G**reen and **R**ed). The rules for colouring are that two vertices connected via a side or a diagonal must have a different colour. It should be possible with three different colours.



You can show that it is not possible to colour it only with two colours and that more than three colours is never necessary. In order to be certain, perhaps you need the following representation of the triangulated polygon.

A dual graph for a triangulated polygon

A triangulated polygon can be characterised by a dual graph. This is a collection of nodes where some are directly connected to each other by a line segment. In the polygon a node is drawn for every triangle in the triangulation. The nodes from the triangles that have a side in common are connected. The following dual graph is produced by the polygon above (first drawn in the polygon and then without the polygon):



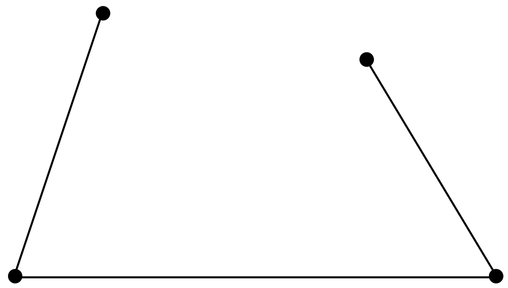
From a dual graph that represents the triangulation of a simple polygon you can show that
- never more than three line segments leave one node,
- the number of nodes is always one more than the number of line segments,
- there are always at least two nodes with only one connecting line segment,
- it is never possible to return from one node via connections with other nodes to the starting point.

Research assignment 4

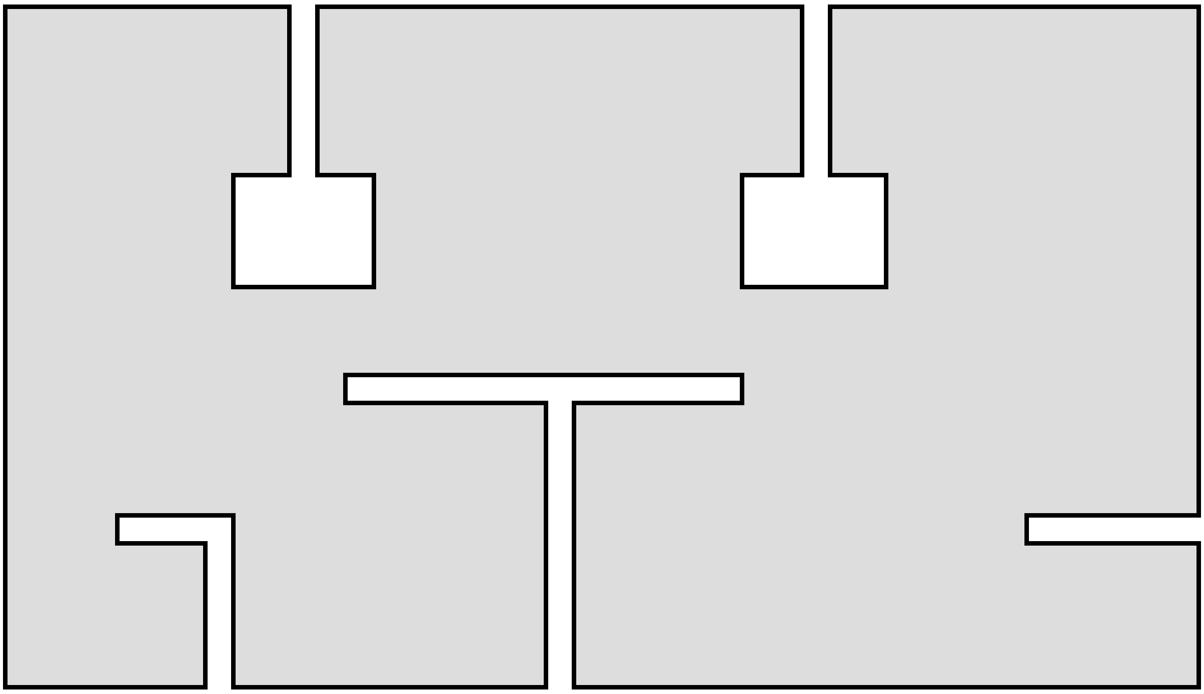
Prove the **_Guarded polygon theorem._**

A simple polygon with n vertices can always be guarded by p cameras that are mounted in the vertices, where p is te smallest whole number for which $p>\frac{1}{3}n-1$

Worksheet question G25



Worksheet question G31



Definitions and descriptions

Inside and outside area of a simple polygon.

A simple polygon divides the surface into two connected parts: the inside and the outside area. Characteristic of the outside area: it is infinitely big. The polygon by definition does not belong to the inside or the outside area.

Adjacent and non-adjacent vertices

Each vertex of a polygon has two adjacent vertices, the line segment from a vertex to its adjacent vertex makes up one side of the polygon. All other vertices are non-adjacent vertices.

Convex polygon

A simple polygon with only protruding angles is called a convex polygon.

Diagonal of a simple polygon

An connecting line segment that lies entirely inside the polygon is called a diagonal.

Straight angle

A vertex is said to be straight when the corresponding angle is 180°.

Intruding angle

A vertex is said to be intruding if the corresponding angle is between 180° and 360°

Cape

A vertex p_i *of a polygon is called a cape if* $p_{i-1}\,p_{i+1}$ *is a diagonal.*

Polygon with n vertices (n > 2)

A polygon is a closed figure that is made by n (thus a finite number) line segments. There are n different vertices $p_1, p_2, \ldots, p_n$*; the vertices of the polygon. The polygon has n sides* $s_1 = p_1 p_2$, $s_2 = p_2 p_3$, ..., $s_{n-1} = p_{n-1} p_n$ *and* $s_n = p_n p_1$.

Simple polygon

A simple polygon doesn't have any sides that cross each other.

Triangulation of a simple polygon with n sides

A triangulation of a polygon P is the partition of its inside area into triangles that do not overlap each other and the vertices remain vertices of P.

Protruding angle

A vertex is said to be protruding if the corresponding angle is between 0° and 180°.