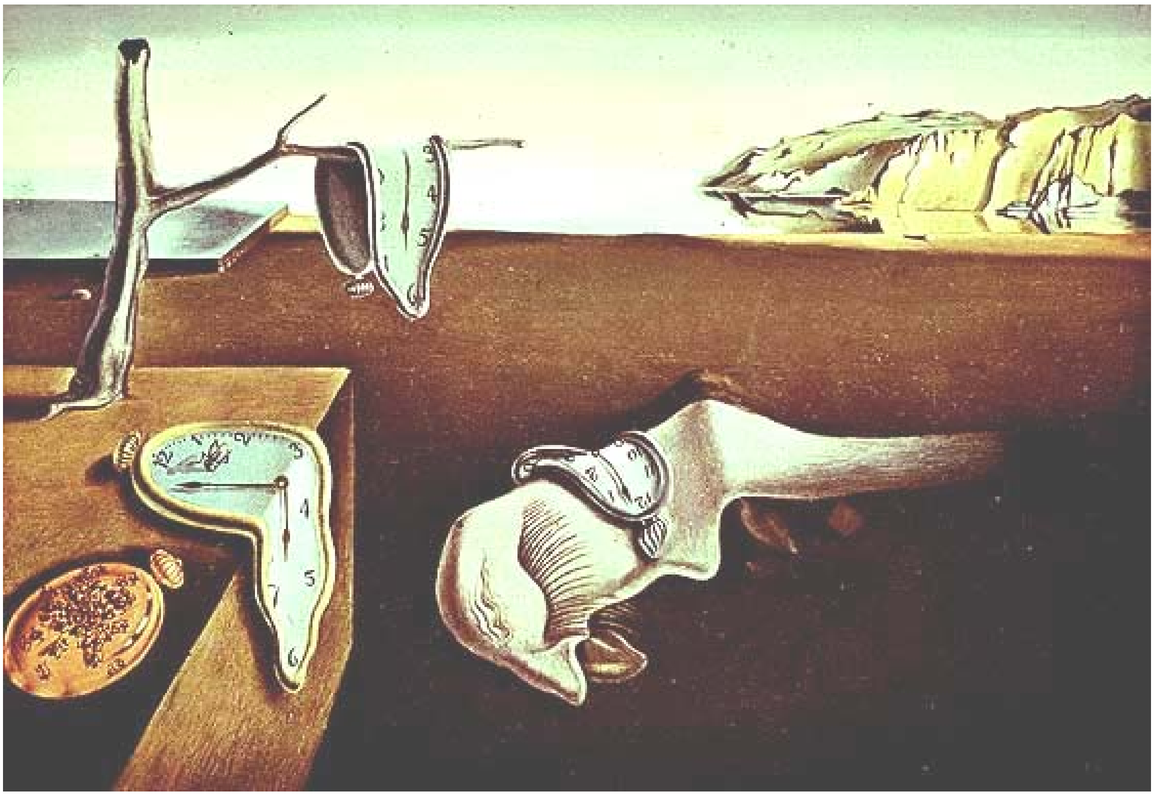
MATHS B-DAY 2006

Friday 24 November



IN THE HANDS OF TIME

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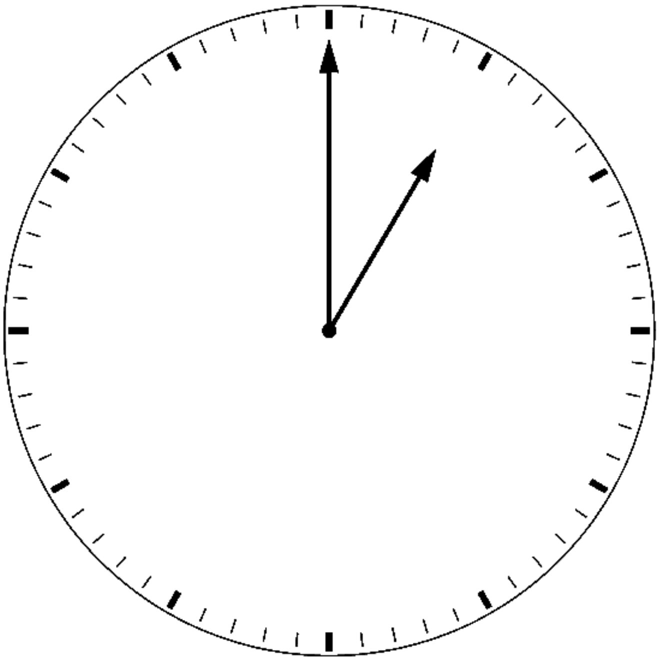
Introduction

The maths B-day assignment this year is totally focused on time.
Time as it is shown on a clock face with a little hour hand and a big minute hand and sometimes a third, rather quickly moving, second hand.
We mean a clock where the hand movement is perfectly fluid and regular. Not jolting and jumping like the one you might find at the train station where the second hand jumps round and then waits at the top until the minute hand moves on a step and then jumps on further.

Below is the kind of clock we are referring to. Stripped of all the decoration often added to make it more attractive to a potential buyer.



This clock shows the time as exactly one o’clock. Soon the big hand will overtake the little hand!
This will happen at just after 5 past 1.
The minute hand starts behind but moves so much faster than the hour hand that it soon takes over the lead from little hand.

As a warm up for theme of this day here are two introductory questions:

A. How often in a period of 12 hours do the minute and the hour hand lay exactly on top of each other?

B. At exactly what time does the minute hand overtake the hour hand on the clock above?

We talk about 'per 12 hours' and not 'per day' because the second part of the day of 24 hours (a whole day) is an exact repetition of the first part. So when studying a normal clock we limit ourselves to a **period of 12 hours**, beginning at exactly 12 o’clock until it is 12 o’clock again.

We do not take the second hand into consideration at the moment. It is only considered in the last part of this assignment.

What is this maths B-day really about?

The introductory problems already give an idea of what this assignment will be about:
The hands of a clock and various positions they can have.

Who is interested in the answers to the sort of questions given in the introduction? Not the train passengers and not the owner of an expensive designer watch, like the one pictured here.
But you and us from the maths-B-day: because maths is about surprising questions and perhaps even more surprising answers.
The answers are not the only thing that can be surprising but preferably the explanation too. An explanation such as "it is because you can see it if you calculate it" is not very clear.
A good explanation is like turning on a light in a dark treasure chamber. You blink your eyes and then you sigh: 'Of course, that's how it works. Cool!'

The content of the assignment.

Part ***A*** expands on the introductory question ***A***. There are several more examples of useful questions with interesting answers. Focus: observation and reasoning.
Part ***B*** is more theoretical and expands on introductory question ***B***, with exact calculation and reasoning. The use of graphs and algebra can help you with this. Focus: precision.
In part ***C*** the hands of the clock change places. Discover for yourself how you can apply the knowledge gained from parts ***A*** and ***B*** or find your own different approach. Focus: apply your knowledge from parts ***A*** and ***B*** to a new situation or be stubborn and follow your own strategies.
Part ***D*** is for the real time junkies. They get the chance to prove themselves!
Don't worry if you don't get that far. Focus in part ***D***: take your time and make space for your own ideas and approach.

The final product

You have looked at the problem of telling the time in a number of ways. You have followed the questions or (in your stubborn way) found your own path.
So the final product is:

A project where the hands of the clock display your puzzles where you put forward the solutions and where the answers compete for clarity and beauty.

Don't be tempted to only give answers to individual questions but provide a coherent report of your findings from this day that is easy to read for those that do not have the specific questions in front of them.

Tips

- Make sure that the work is easy to photocopy so use black pens and drawing materials.
- Also think carefully about how much time you have today! Putting together a report can take a lot of time.
  So start on time!
- Divide up the tasks if you are agreed on your approach.

Finally: good luck and have fun with this assignment

Part A Considering the position of the hands

Introductory question ***A*** can be expanded on with questions about other special positions of the two hands.

A1 If the two hands lay as if they are an extension of each other, for example at exactly 6 o'clock, then the two hands produce an angle of 180 degrees.
How often in a period of 12 hours are the hands separated by 180 degrees?

An angle of zero degrees or 180 degrees is unambiguous: you know exactly what is meant.
But if you were asked the question how often do the two hands make an angle of 90 degrees, you (should) answer with the question "what do you mean by an angle of 90 degrees?". There are actually always two angles to be seen. If one angle is 90 degrees then there is also an angle of 270 degrees.
In this assignment we agree that:

The angle between two hands is always the <u>smaller</u> of the two possible angles.

So at exactly 3 o'clock and exactly 9 o'clock the two hands make an angle of 90 degrees but there are still other positions of the hands which make an angle of 90 degrees.

A2 How often, in a period of 12 hours, do the two hands form an angle of 90 degrees, of 120 degrees, of 30 degrees?

If you want to clearly communicate the answers you find to the questions above with people who are not familiar with the assignment, then it is a good idea to think carefully how you are going to do it. You should consider illustrations with graphs, and a numeric notation or ..... fill it in yourself!

Some positions of the two hands are obviously possible and others are (almost) obviously impossible. The following question considers these (im)possibilities.

A3 Four times the same question "What time is it?"
It is possible to find a time twice and twice it is impossible. Work it out...
You don't have to calculate the actual times, you need to consider *why* the position of the two hands is (im)possible.

a. the hands lay as if they are an extension of each other and the minute hand is quite close to the '11'.
What time is it?

b. the hands make an angle of 90 degrees and the minute hand is exactly horizontal.
What time is it?

c. the hands are symmetrical with regard to the horizontal line through the pivot point of the hands and the hour hand is near the '8'.
What time is it?

d. the hands are both horizontal.
What time is it?

The clock at the hairdresser's

At the hairdresser's you look in the mirror. If there is a normal clock behind you on the wall then you can see it but it will be the mirror image. Imagine that it is a clock without any numbers. Switch off your mind and look in the mirror and see exactly what you see without the interpretation from your brain to tell you how it is in reality.

A4 **a.** In the mirror you can see that it is about five past 2. What time is it actually?

b. Are there times of the day when it doesn't make any difference if you are looking at a real clock or a clock in the mirror?

c. If you look at the clock for a while, will you see the clock in the mirror go backwards. Strange. But if you just glance at the clock in the mirror you will always see a position of the hands that is also possible on a real clock. Explain.

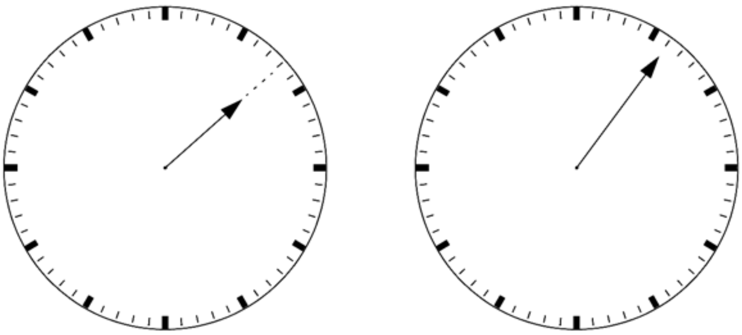
Part B Calculating the positions of the hands

In part **A** you have already come across the idea that not all the positions of the two hands of the clock are possible. The three clocks below demonstrate this.

If the minute hand is pointing to the 9 then the hour hand can never point exactly to the 10. We are only working with really good clocks today.
In these examples the explanation is simple: the hour hand is pointing exactly to a whole hour. So the minute hand must point exactly to the twelve.

But what is possible?

B1 Here you can see two different clock faces.



a. On the left hand clock you can only see the *hour* hand. It is pointing to the 8 minute mark. Write down the possible position(s) of the *minute* hand.
b. On the right hand clock you can only see the *minute* hand. It is pointing exactly to the 6 minute mark.
Write down the possible position(s) of the *hour* hand.

"The big hand is on the 6 minute mark" it means that it is 6 minutes past a whole hour. If the little hand is pointing to the 8 minute mark, then it has nothing to do with 8 o'clock. Therefore we need more clarity. The clock is divided into 60 minute marks; the twelve thicker marks are the hour marks (from 1 to 12). The combination of the position of the minute hand and a correlating position of the hour hand gives you the time.
We are going to look at possible *positioning of the hour* and *minute hands* and the corresponding time.
In order to differentiate between these two (*positions of the hands* and the actual time) we should agree that:

The positions of the hands are expressed in degrees with respect to the vertical line going upwards.

At precisely 10 past 3, the big minute hand is at 60 degrees and the little hour hand is a fraction more than 90 degrees.

B2 **a**. the angle between the two hands changes constantly. Which angles occur most? Which angles occur between
ten past three (3:10) and twenty to four (3:40)?
b. What is the angle between the two hands at twenty two minutes past four? (4:22)?

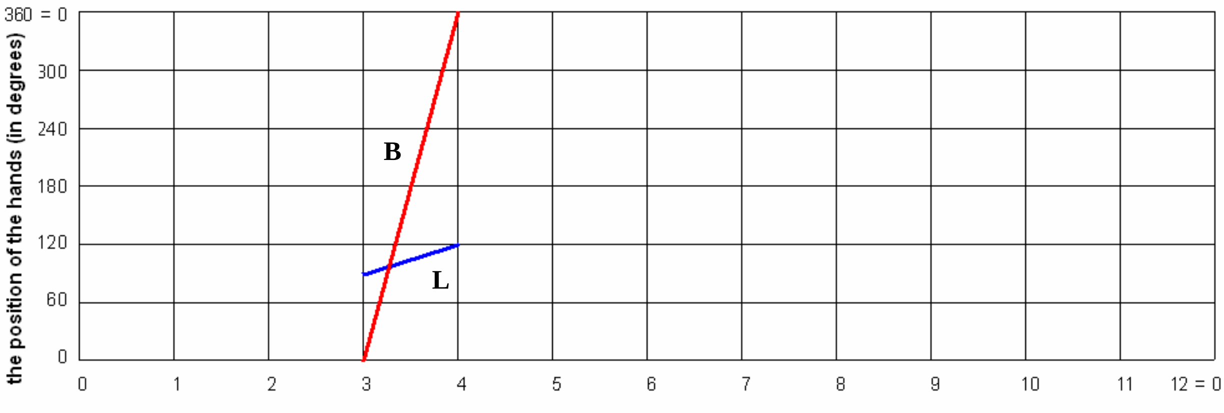
The big hand goes through a circle of 360 degrees in one hour and then starts again; the little hand needs exactly 12 hours before it can start again.
Therefore there is another difference that is important enough to note.
If we only look at the position of the big hand (in degrees with regard to the vertical line), then we are not taking into account how many circles of 360 degrees the hand has already completed. There is thus a difference between the ***position* of the *hand with regard to*** the ***vertical line*** and the ***angle it has covered from the starting time 12 o'clock (or: 0 o'clock)***. In the case of "10 past 3" above, the big hand has already covered more than three full circles.
For the big hand (the minute hand) we use the letter 'b' in two ways:

b is the angle covered by the big hand from the start time 0 o'clock;
B is the position of the big hand (in degrees) with regard to the vertical line.

In the case of "10 past 3" then: ***b*** = 1140 degrees and ***B*** = 60 degrees.

In a graph to show the positions of the hands, the position of the hands (in degrees) can be plotted against the time (in hours).
Below, the positions of the big hand ***B*** and the little hand ***L*** are shown for all times ***t*** from 3 o'clock until 4 o'clock.

Graph to show positions of the hands



Notice that the 12 (o'clock) on the horizontal axis and the 360 (degrees) on the vertical axis have both been substituted by the number 0. This shows that the time and the position of the hands have started on a new circle.

The following algebraic formula refers to the diagram:

$$B = 360t - 1080 \qquad \text{for} \quad 3 \leq t < 4$$
$$L = 30t$$

The complete graphs of *B* and *L* and all the associated algebraic formulae (for all values of *t* from *t* = 0 until *t* = 12) are useful when calculating the positions of the hands.
Of course you can also choose your own way. The condition being that you can defend your decision.

In part **A** you are asked how often certain positions of the hands appear during a 12 hour period.
A logical follow up question is:

B3 At precisely what times are
- the two hands lying exactly on top of each other?
- the two hands exactly opposite each other?
- the two hands at an angle of 90 degrees, 120 degrees, 30 degrees?

Note! When we say exact we don't mean "in minutes or in seconds or in tenths of a second precisely". You can also use fractions in an exact time, for example 2 o'clock and $12\frac{3}{7}$ *minutes.*

Graphs and algebra formulae can also be useful for other types of questions.
But, again, if you have your own way to answer these questions then use it but make sure you have a clear explanation of the choices you make.

B4 Here is a watch that is often shown in advertisements. The hands are at an exactly equal angle to the vertical axis through the pivot point of the hands, like arms opened invitingly to the buyer. Take note!

a. What is the exact time on this watch?

b. There are more symmetrical positions with regard to the vertical axis (even though they are less common in the advertisements).
What times is this the case?

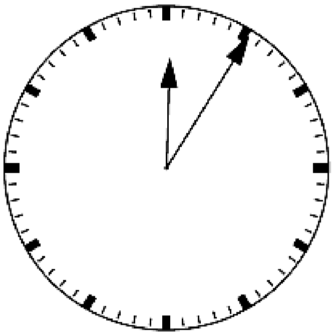
c. The hands are symmetrical with regard to the *horizontal* axis through the pivot point of the hands and the little hand is near the 8. What is the exact time?

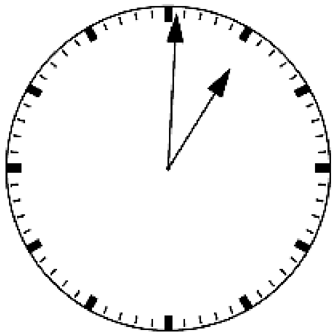
B5 Due to a defect the little hand begins to go the wrong way in the middle of the night at 0 o'clock (midnight). The big hand, however, continues moving in the right direction.
Calculate now what times the hands of this clock will lie exactly on top of each other.

B6 Think up at least two problems yourself that you can solve using the graph or algebraic formulae. Of course it would be nice if you could also put the solutions into words.

Part C Nightmare?

You wake up just after midnight and you see on your alarm clock that it is just after twelve. You fall asleep again but after about an hour you wake up. You are startled! The time seems to have stood still. Luckily you quickly realise that the big and the little hand have changed places.





It is clear that when the two hands change places that the positioning of the hands appears to be correct. Look at the positions drawn above. The question is, however, when the hands change places, does the new position also produce an exact position of the hands, and if so, how should it be calculated.

The following train of thought will hopefully help you on your way to working this out.
Look at the positions of the big and the little hand on the left hand clock above (let's call them $\boldsymbol{B}_1$ and $\boldsymbol{L}_1$ for now).
Since the big hand goes round 12 times as fast as the little hand, then the angle covered by the hands in degrees is: $\boldsymbol{b} = 12\,\boldsymbol{L}$.
Since $\boldsymbol{b}_1$ is a bit more than 30 degrees, therefore $\boldsymbol{L}_1$ has to be a bit more than 2.5 degrees.
On the right hand clock it has to be the other way round: $\boldsymbol{B}_2$ is a bit more than 2.5 degrees and $\boldsymbol{L}_2$ a bit more than 30 degrees. But the minute hand has completed more than a whole circle of 360 degrees! So $\boldsymbol{b}_2$ must be a bit more than 362.5 degrees.
A first attempt to find the exact time using an intial estimation can be shown schematically as follows. With a initial estimation of
$\boldsymbol{L}_1$ = 2,52 for the left hand clock, the other values can be calculated:

Just after *12 o'clock:* *but just after 1 o'clock (the two hands change places)*

$$\begin{tikzcd}
\boldsymbol{L_1} = 2{,}52 \arrow[d, "\times 12"] & \boldsymbol{L_2} = 30{,}24 \arrow[d, "\times 12"] \\
\boldsymbol{b_1} = 30{,}24 \arrow[ur] & \boldsymbol{b_2} = 362{,}88
\end{tikzcd}$$

So the position of the big hand just after 1 o'clock is thus $\boldsymbol{B}_2$ = 2,88 degrees.
Not bad but it could be better! The gap between $\boldsymbol{L}_1$ and $\boldsymbol{B}_2$ could be made smaller.

Look very carefully at the calculation in this diagram and try again using another initial estimation for $\boldsymbol{L_1}$ on the left hand clock so the value of $\boldsymbol{B_2}$ on the right hand clock is the same as $\boldsymbol{L_1}$.
Now look at how you carried out the calculation.
Try to describe it on a more general level and use that to answer the two key questions:

C1 How many exact positions of the hands in a period of 12 hours can be found when the two hands are exchanged and still produce an exact correct hand position?

C2 Calculate some of these exact positions of the hands.

Part D Hours, minutes and seconds: a challenge for the *real* time junkies

The second hand has not played any role so far. However, in this part we are going to include it!
The second hand takes one full minute to go all the way around the clock. In a period of 12 hours it completes 720 full circles.
So for the second hand, like the minute hand, it is necessary to differentiate between the *position of the hands with regard to the vertical line (going up)* and the *angle it has covered from 0 o'clock*.
We agree that:

s is the angle covered by the second hand from the starting time of 0 o'clock;
S is the position of the second hand (in degrees) with regard to the vertical line.

At ***t*** = 0 (12 o'clock) all three hands lay on top of each other.

D1 Find out if there are more points in time when all three hands lay exactly on top of each other. You can of course make use of your results from part **B**.

An interesting question:

D2 Are there times when the hour hand, the minute hand and the second hand are each separated by an angle of 120 degrees?

Another division of time; new opportunities?

Until now we have used the standard division of time:

In 1 full circle of the hour hand, the minute hand goes through 12 full circles;
In 1 full circle of the minute hand, the second hand goes through 60 full circles.

This division of time you could call the (12, 60) division.
Perhaps you are a bit disappointed by the answers you found to ***D1*** and ***D2*** in this division of time.
Nice positioning of the three hands does not occur very often.
But maybe another division of time offers the solution to this problem!

We define the **(*p*, *q*)-division** of the time as:

1 full circle of the hour hand is the equal to *p* full circles of the minute hand;
1 full circle of the minute hand is equal to *q* full circles of the second hand.

So for the (10, 10)-division:

In 1 full circle of the hour hand, the minute hand goes through 10 full circles;
In 1 full circle of the minute hand, the second hand goes through10 full circles.

The final question of this Maths B-day assignment is:

D3 In which (*p*, *q*)-division of the time can you find times where the three hands are each separated by an angle of 120 degrees?