Mathematics B-Day 2003

November 28 2003

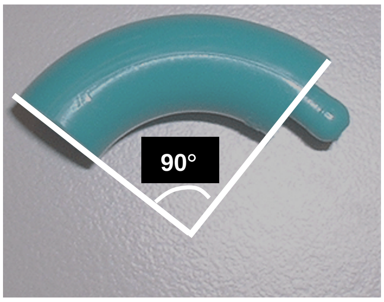
A ROUND MATHEMATICS

The Mathematics B-day is sponsored by **Texas instruments**

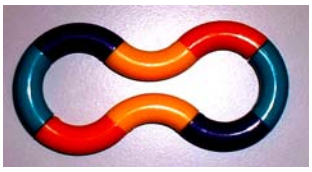
Introduction

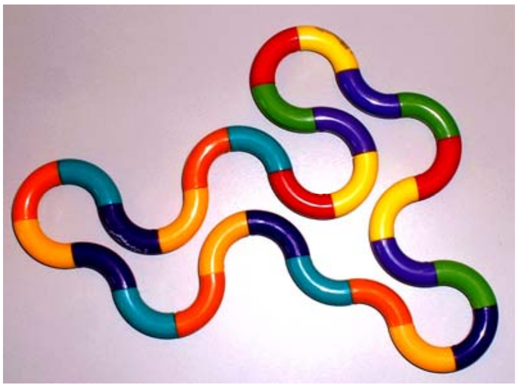
Sometimes mathematics comes up spontaneously, for example in a toy. You only have to pick it up and start constructing for hours as a basis to reason and calculate in a mathematical way.
Today we will do so with the use of a set of 18 so called 'ellbows'.

An elbow is a quarter of a circle. With one click the elbows can be linked together.

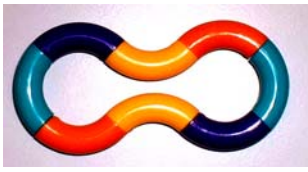


We will only consider closed links (without beginning or end). Such a closed link will be called a *circuit*.
In the pictures below you see some examples of circuits. There are four *flat* circuits with 8, 12, 16 and 28 elbows: flat, because they can be put on a table with all elbows completely resting on it.
But there is also a spatial circuit with 7 elbows; it is impossible to put it flat on a table.





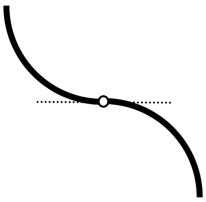
So, a circuit is only called flat if all elbows of the circuit are completely resting on the table. To avoid any misunderstanding: the following two pictures show a circuit of 8 elbows. The one on the left is a flat circuit; the one on the right is not flat, because two elbows are not completely resting on the table.



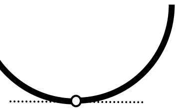
Mathematical representations of elbows.

An elbow can mathematically be represented as a quarter of a circle with radius 1. Using this mathematical representation, we neglect the thickness of the material of the elbows. In the connection points, the quarters of a circle are linked together having a common tangent.

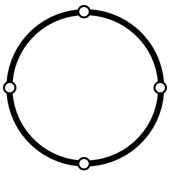
So this way:

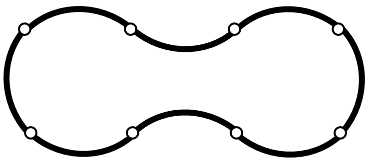


or this way:



Two examples of circuits, with 4 and 8 elbows, represented this way:





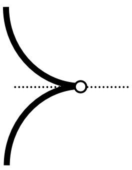
Sometimes a *plastic* circuit seems possible (with some wriggling), but if you try to represent it mathematically, it turns out to be impossible in a mathematical way. This kind of deformed closed link is not accepted as a "circuit".
For the concrete elbows, the tangent property of the mathematical representation means that the two linked elbows fit without any space left in between.

A mathematical description of a circuit with *n* elbows:

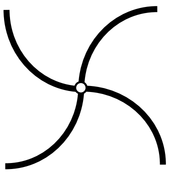
> **A *n*-circuit is a closed curve with *n* quarters of circles, with common tangents in all connection points.**

It seems superfluous (because the concrete elbows don't allow for it), but we have to exclude it mathematically: in a closed curve we do not accept "beaks" or double points.

beak:



double point:



The task

During this Mathematics B-day you are going to investigate (im)possibilities of flat and spatial *n*-circuits of elbows and their properties. The set of 18 plastic elbows is meant to be used for actual constructions of circuits to support thinking and reasoning about circuits in a more general way (also for $n > 18$).
The task is split into three separate parts.

In part A flat circuits are investigated. In part B you are going to study spatial circuits that have to fulfil some special conditions; you only have restricted space. But after that, in part C you will have freedom to use full space.

The numbered questions in parts A, B and C are meant to direct your investigations. They don't have to be addressed in the given way; the tasks can be allocated among the members of the team.
In every part you also find general tasks. These are the investigative tasks that can be used to stand out in mathematical depth and completeness.

The Assignment

You are asked to write a paper in which you report the results of your investigations. It has to be readable for a person who does not have the task at hand. The reader, who has a set of elbows at hand, should get a clear idea about (im)possibilities of flat and spatial circuits and their properties.
In your paper, the sequence of numbered questions does not play any role, but be aware that the findings of all these questions can be found in the report.

Finally:

Writing a paper takes a lot of time. It seems reasonable to reserve about two hours for that.

Have fun with your investigations and ... make something out of it !!

Part A: Flat circuits

It is clear that you need 4 elbows to construct the smallest flat circuit.
We call this a flat 4-circuit.

Flat *n*-circuits

So, a flat *n*-circuit is a closed link (without double points) of exactly *n* elbows, with all elbows flat on the table.
In this part, we start with investigations about possible flat *n*-circuits.
The set of 18 real elbows is meant for experiments. But you should realize that some of the questions are about values of *n* that exceed 18.

1. Make a flat 8-circuit. How many possibilites are there?
Also find the possible 12-circuits. Show that you found them all.

2. Constructing circuits is a question of trial and error. The set of 18 elbows will certainly help for that. But communicating a circuit on paper without always drawing such a circuit, is a different story. It is therefore useful to have a way of describing a circuit for answering questions. That can be done in many different ways. It is your task to find a handy way of describing a circuit, with which it is easy to communicate. Be sure that you lay down it exactly for the reader.

3. Many 16-circuits are possible. Find a systematic way to find them all and describe the system you used for that.

4. A flat circuit with an odd numer of elbows is impossible. Explain why.

5. Construct a link of three elbows. The endpoints and connection points are numbered 0, 1, 2 and 3 as shown here:

Fix points 0 and 1 (the first elbow). Describe where the endpoints of the consecutive elbows 2, 3, 4, ... can be located, including the direction in which a new elbow has to be be linked in such an endpoint.

6. Is a flat 6-circuit possible?

> *General question I:*
> For what values of *n* is a flat *n*-circuit possible? Are you able to show why?

Enclosed area of a flat *n*-circuit

We now only consider the mathematical representation of flat *n*-circuits. So material thickness is neglected and an elbow is a quarter of a circle with radius 1.
Therefore, the 4-circuit has an enclosed area of π.
Of course, the area of a flat *n*-circuit is related to the value of *n*, but the shape of the circuit also affects the enclosed area.

7. Show that the enclosed area for an 8-circuit is $\pi + 4$.

You see that π en $\pi + 4$ are possible measures for areas.

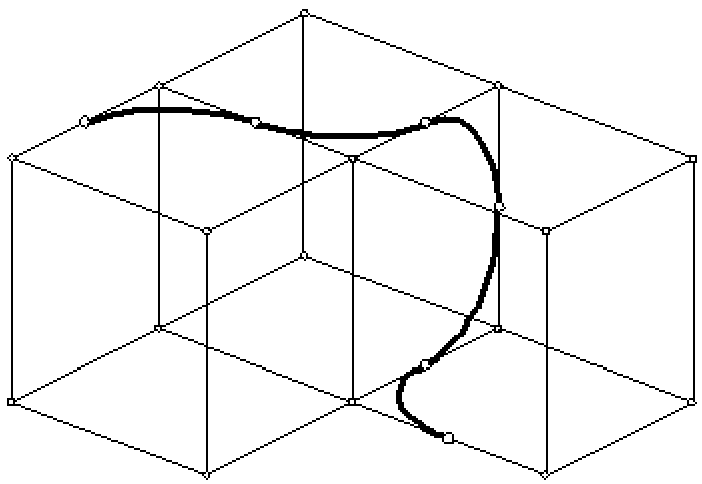
> *General question II:*
> What are possible measures of enclosed areas for flat 12-, 16-, ..., *n*-circuits ?

Part B: Restricted Space

The set of elbows can also be used to construct spatial circuits. In space, there are infinitely many constructions possible, because an elbow that is clicked to another one can be rotated for every possible angle. For that reason we first restrict the freedom of rotation in the following way:

> The elbows of a link are laying on the surface of a spatial cubic grid, with the endpoints of the individual elbows always in the middle of an edge of the cubes in the grid.

In the following figure, a small part of this cubic grid is drawn, with an example of 5 linked elbows that fit the restriction. The cubes in the cubic grid can of course be stacked.



8. Two different spatial 6-circuits that meet the stated restriction are possible. Try to construct them and describe them with use of the cubic grid.
Investigate which 8- en 10-circuits are possible, given the stated restriction.

9. Given the restriction, is it possible to construct a spatial n-circuit for odd values of n?

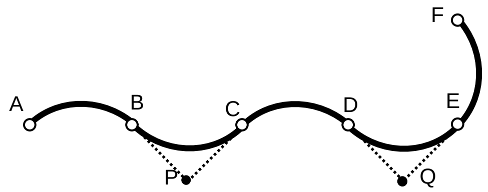
> *General question III*
> Given the restriction of the cubic grid, for which values of n is a spatial n-circuit possible?

Part C: Free Space

In this part, you will have full freedom in space. Because of this freedom of movement, it becomes much more complex. With full freedom it is also possible to construct spatial circuits for several odd values of *n*.
When experimenting with the plastic elbows, you should realize that the material allows for some deformation. Because of that, it is possible to construct some plastic circuits (by some wriggling) that are not accepted mathematical circuits. So always stick to te mathematical description of a *n*-circuit that was given on page 2.

A special case: n = 5

It is impossible, without wriggling, to construct a spatial 5-circuit. The following activity may eventually help to get an idea *why* it is impossible.
Put 5 connected elbows on the table. Keep the one in the middle (CD in the figure) fixed in one position and investigate how endpoint A moves in space when rotating elbows CB and BA. All possible positions of A in space appear to have something in common: they all are on a fixed distance from intersection point P of the two tangents in B en C. The same holds for all positions of point F in space: they all are on a fixed distance from point Q.



10. Show that the distance from A to P is fixed for every position of A in space. Calculate that distance.

The final task is again a general one and you have freedom of choice. The spatial circuits give cause for asking yourself questions. Some possibilities:

- Can you use the idea of question 10 to show that a 5-circuit is impossible?
- There are two spatial 6-circuits. One is flexible (it can be distorted without any wriggling). The other one is rigid and cannot be distorted in any way. Why? Are there more rigid spatial circuits?
- For what odd values of *n*, a spatial circuit is possible?

These kinds of questioning are certainly not easy to be answered, but maybe targeted experiments with concrete material can help to develop some good ideas.

General question IV
Experiment with spatial shapes and try to trace challenging problems that can be investigated with the set of elbows.
Even if you are not able to solve these problems yourself, you can describe them in your paper.

Finally:

Carry out the assigment that is described on page 3.
Remember that you are not expected to answer all numbered questions of part A, B and C. Make sure that you deliver a coherent report of all findings about flat and spatial circuits and don't hesitate to add some challenging problems in your paper.