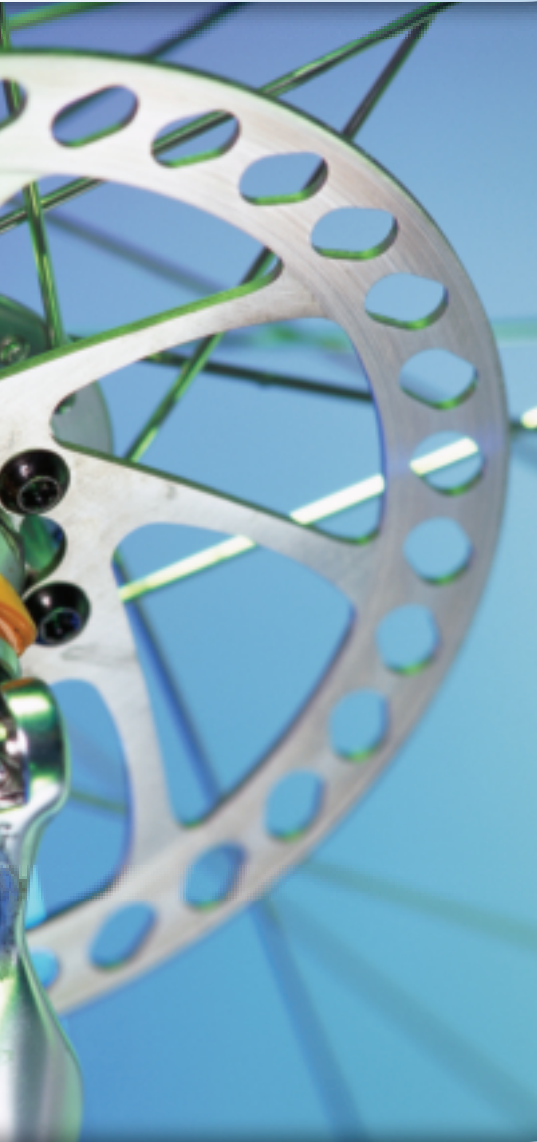


Packages and Polygons

Britannica®
Mathematics
in
Context

Geometry and
Measurement



ENCYCLOPÆDIA
Britannica®

Mathematics in Context is a comprehensive curriculum for the middle grades. It was developed in 1991 through 1997 in collaboration with the Wisconsin Center for Education Research, School of Education, University of Wisconsin-Madison and the Freudenthal Institute at the University of Utrecht, The Netherlands, with the support of the National Science Foundation Grant No. 9054928.

The revision of the curriculum was carried out in 2003 through 2005, with the support of the National Science Foundation Grant No. ESI 0137414.



National Science Foundation

Opinions expressed are those of the authors
and not necessarily those of the Foundation.

Kindt, M., Abels, M., Spence, M. S., Brinker, L. J., & Burrill, G. (2010). Packages and polygons. In Wisconsin Center for Education Research & Freudenthal Institute (Eds.), *Mathematics in context*. Chicago: Encyclopædia Britannica, Inc.

© 2010 Encyclopædia Britannica, Inc. Britannica, Encyclopædia Britannica, the thistle logo, *Mathematics in Context*, and the *Mathematics in Context* logo are registered trademarks of Encyclopædia Britannica, Inc.

All rights reserved.

No part of this work may be reproduced or utilized in any form or by any means, electronic or mechanical, including photocopying, recording or by any information storage or retrieval system, without permission in writing from the publisher.

International Standard Book Number 978-1-59339-941-2

Printed in the United States of America

1 2 3 4 5 C 13 12 11 10 09

The *Mathematics in Context* Development Team

Development 1991–1997

The initial version of *Packages and Polygons* was developed by Martin Kindt. It was adapted for use in American schools by Mary S. Spence, Laura J. Brinker, and Gail Burrill.

Wisconsin Center for Education

Freudenthal Institute Staff

Research Staff

Thomas A. Romberg
Director

Joan Daniels Pedro
Assistant to the Director

Jan de Lange
Director

Gail Burrill
Coordinator

Margaret R. Meyer
Coordinator

Els Feijs
Coordinator

Martin van Reeuwijk
Coordinator

Project Staff

Jonathan Brendefur
Laura Brinker
James Browne
Jack Burrill
Rose Byrd
Peter Christiansen
Barbara Clarke
Doug Clarke
Beth R. Cole
Mary Ann Fix
Sherian Foster

Fae Dremock
James A. Middleton
Jasmina Milinkovic
Margaret A. Pligge
Mary C. Shafer
Julia A. Shew
Aaron N. Simon
Marvin Smith
Stephanie Z. Smith
Mary S. Spence

Mieke Abels
Nina Boswinkel
Frans van Galen
Koenno Gravemeijer
Marja van den Heuvel-Panhuizen
Jan Auke de Jong
Vincent Jonker
Ronald Keijzer
Martin Kindt

Jansie Niehaus
Nanda Querelle
Anton Roodhardt
Leen Streefland
Adri Treffers
Monica Wijers
Astrid de Wild

Revision 2003–2005

The revised version of *Packages and Polygons* was developed by Mieke Abels and Martin Kindt. It was adapted for use in American Schools by Gail Burrill.

Wisconsin Center for Education

Freudenthal Institute Staff

Research Staff

Thomas A. Romberg
Director

David C. Webb
Coordinator

Jan de Lange
Director

Truus Dekker
Coordinator

Gail Burrill
Editorial Coordinator

Margaret A. Pligge
Editorial Coordinator

Mieke Abels
Content Coordinator

Monica Wijers
Content Coordinator

Project Staff

Sarah Ailts
Beth R. Cole
Erin Hazlett
Teri Hedges
Karen Hoiberg
Carrie Johnson
Jean Krusi
Elaine McGrath

Margaret R. Meyer
Anne Park
Bryna Rappaport
Kathleen A. Steele
Ana C. Stephens
Candace Ulmer
Jill Vettrus

Arthur Bakker
Peter Boon
Els Feijs
Dédé de Haan
Martin Kindt

Nathalie Kuijpers
Huub Nilwik
Sonia Palha
Nanda Querelle
Martin van Reeuwijk

Cover photo credits: (all) © Getty Images; (middle) © Kaz Chiba/PhotoDisc

Illustrations

1 Holly Cooper-Olds; **15, 17, 18** (top), **20** (bottom), **24** (top), **45, 47** (bottom), **52, 54** Christine McCabe/© Encyclopædia Britannica, Inc.

Photographs

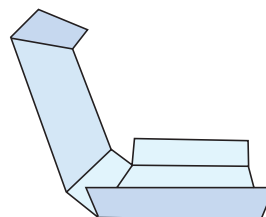
14, 15 Andy Christiansen/HRW; **23** Victoria Smith/HRW; **26** © PhotoDisc/ Getty Images; **27** © Comstock, Inc.; **34** Mark Haughton; **36** © Bettmann/ Corbis; **43** (top, bottom) Sam Dudgeon/HRW; (middle) Stephanie Friedman/ HRW; **44** Victoria Smith/HRW; **46** Sam Dudgeon/HRW; **47** Andy Christiansen/ HRW; **49** (top) Sam Dudgeon/HRW; (bottom) Victoria Smith/HRW; **50** Sam Dudgeon/HRW; **53, 56** © PhotoDisc/Getty Images

◆ Contents

Letter to the Student vi

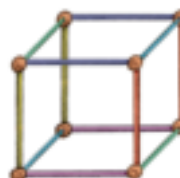
Section A Packages

Sorting Packages	1
Making Nets	3
Faces	8
Ne(a)t Problems	9
Summary	10
Check Your Work	12



Section B Bar Models

Making Bar Models	14
Stable Structures	16
Summary	20
Check Your Work	21



Section C Polygons

Put a Lid on It	23
Pentagon	26
Angles	27
Summary	28
Check Your Work	29



Section D Polyhedra

Special Polyhedra	31
Faces, Vertices, and Edges	34
Euler's Formula	36
Semi-regular Polyhedra	37
Summary	40
Check Your Work	41



Section E Volume

Candles	43
Finding Volume	45
The Height	48
Formulas for Volume	49
Summary	50
Check Your Work	51



Additional Practice 53

Answers to Check Your Work 59

Dear Student,

Welcome to the unit *Packages and Polygons*.

Have you ever wondered why certain items come in differently shaped packages? The next time you are in a grocery store, look at how things are packaged. Why do you think table salt comes in a cylindrical package? Which packages do you think are the most practical?



Geometric shapes are everywhere. Look at the skyline of a big city. Can you see different shapes? Why do you think some buildings are built using one shape and some using another?

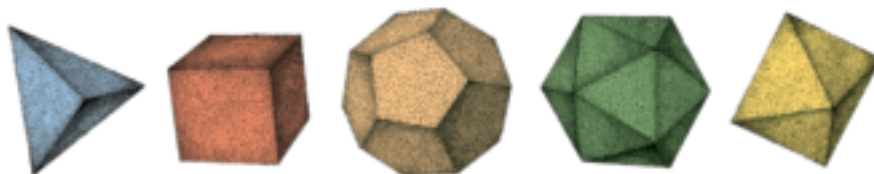
In this unit, you will explore a variety of two- and three-dimensional shapes and learn how they are related. You will build models of these shapes using heavy paper, or straws and pipe cleaners, or gumdrops and toothpicks. As you work through the unit, notice the shapes of objects around you.

Think about how the ideas you are learning in class apply to those shapes.

We hope you enjoy your investigations into packages and polygons.

Sincerely,

The Mathematics in Context Development Team



Packages

Sorting Packages

José did some shopping for a surprise party for his friend Alicia. When he got home, he put all the packages on the table. José has many different packages and decides to sort them.

1. Discuss the ways in which you might sort José's collection of packages. Choose at least two different ways and show how you would sort the collection.

Look around your home for some different-shaped packages. Select the shapes that you find the most interesting and bring them to class.

2. Select one package from your collection or José's collection. Write a reason why the manufacturer chose that shape for the package.

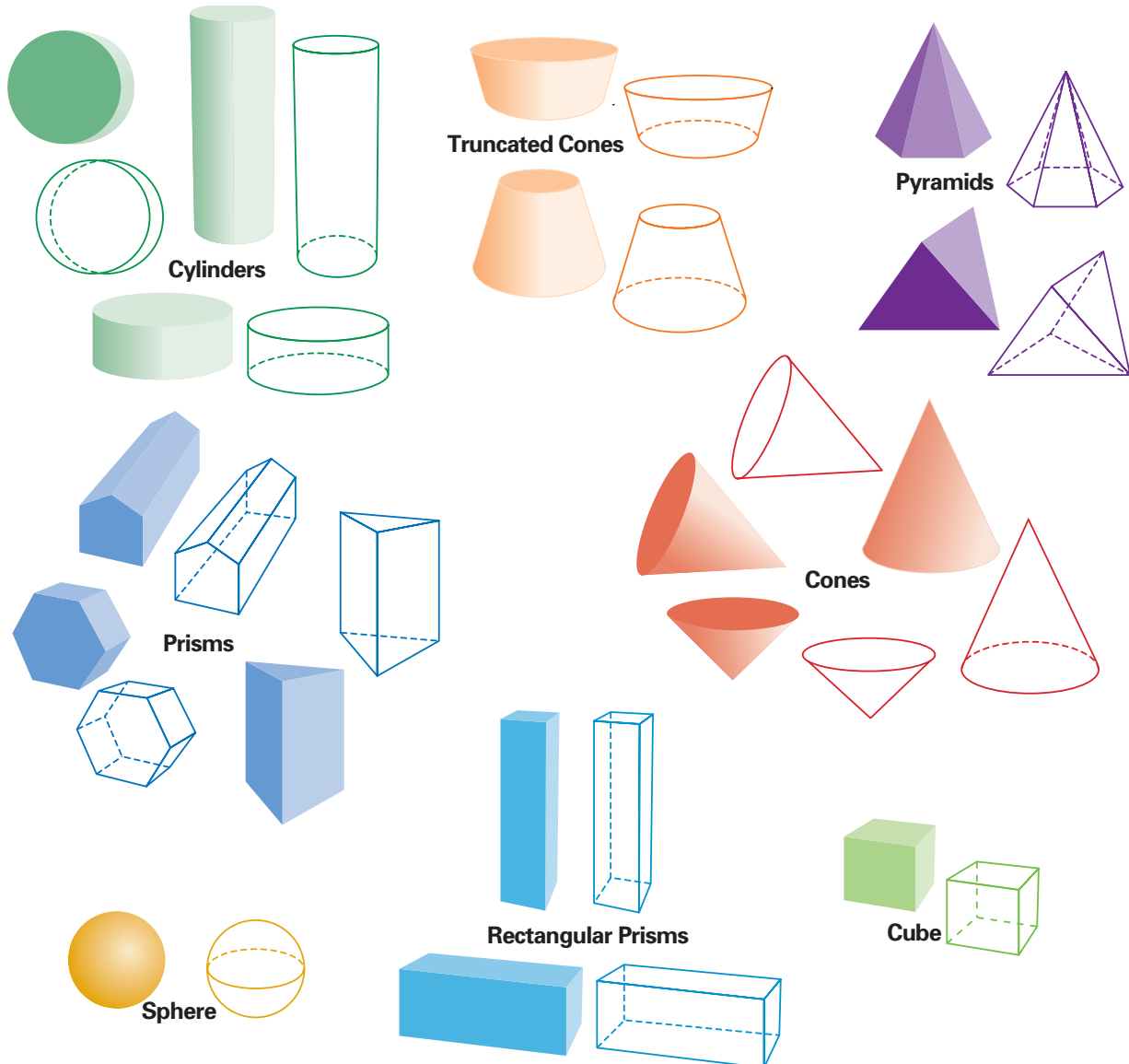



Look carefully at the shapes of your packages. Some shapes have special names. The models on this page highlight distinguishing features of the different shapes.

3. Classify each package in your collection and José's collection according to its special name.

- | | |
|----------------------|-------------------|
| a. rectangular prism | e. cone |
| b. cube | f. truncated cone |
| c. cylinder | g. prism |
| d. sphere | h. pyramid |

Models



4. Use the distinguishing features to answer these questions.
 - a. How are the three cylinders alike? How are they different?
 -  **b. Reflect** What do you think *truncated* means in “truncated cone”?
 - c. How are the prisms alike?
 - d. What are some differences between a prism and a pyramid?
 - e. Describe a difference between a cone and a pyramid.

Making Nets

Activity

Activity—From Package to Net

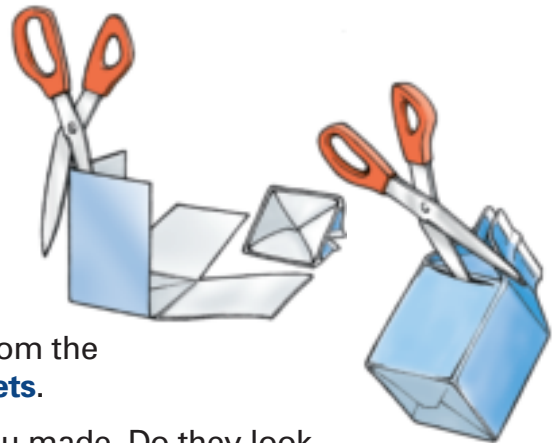
Find two packages such as a milk carton or a box.

Cut off the top of one package.

Cut along the edges of the carton so that it stays in one piece but can lie flat.

Cut the other package in a different way.

Open it up and lay it flat.

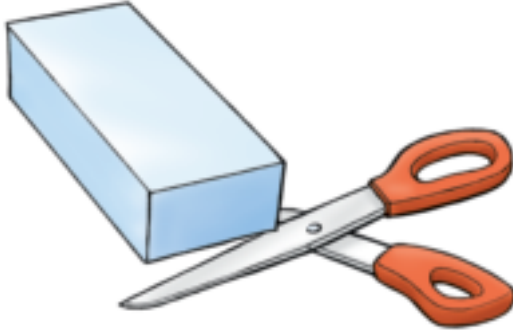


The flat patterns you made from the cartons or boxes are called **nets**.

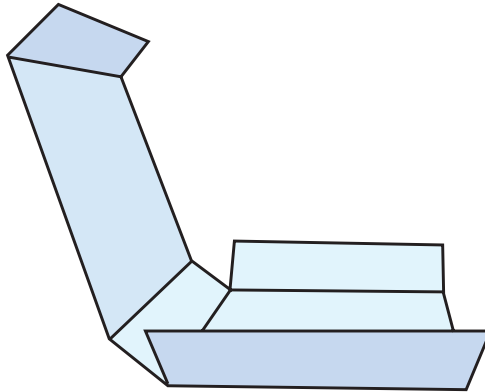
- Compare the two nets you made. Do they look the same? If not, what is the difference?
- Draw the two nets that you made. Make a sketch of the solid that produces each net.

Alicia decides to make a net of her box without cutting off the top. In pictures **i**, **ii**, and **iii** you see the steps she took.

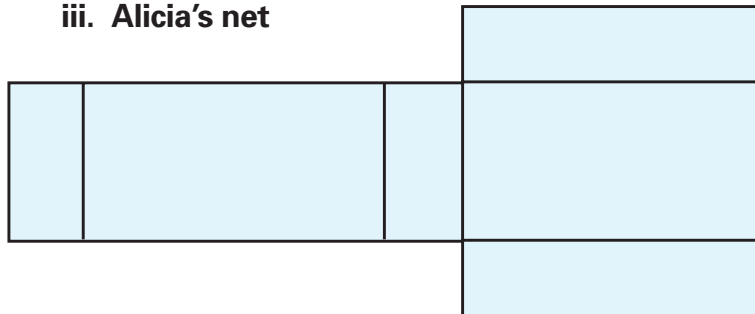
i.



ii.

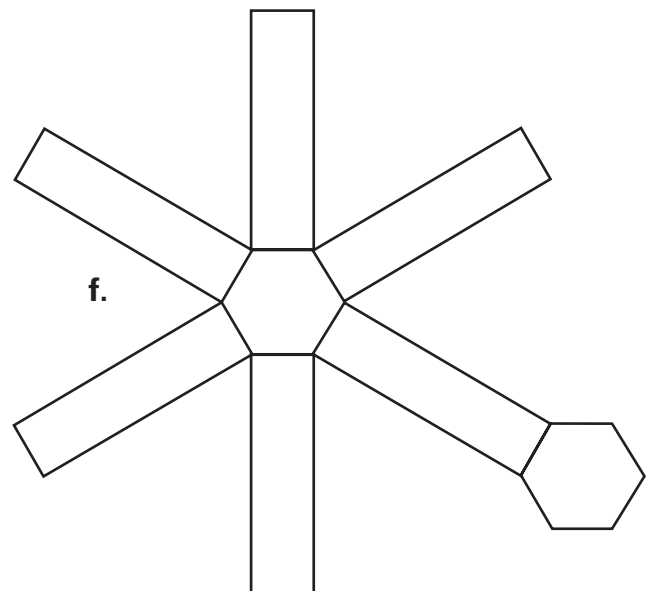
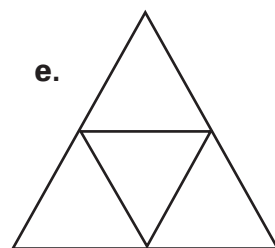
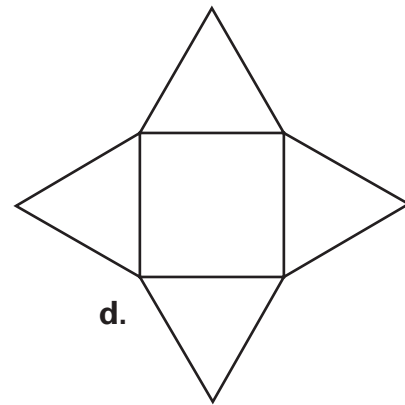
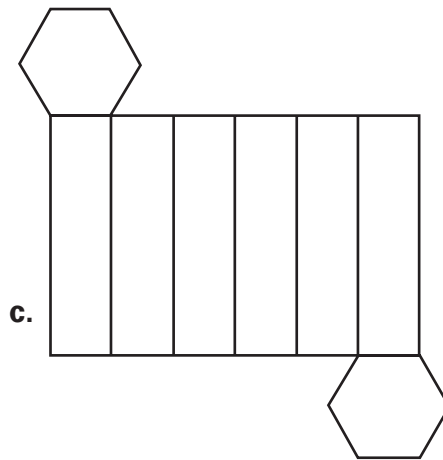
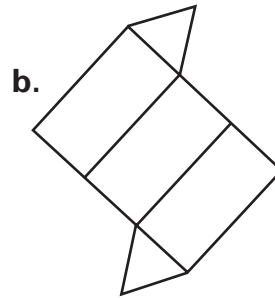
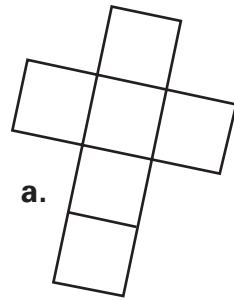


iii. Alicia's net

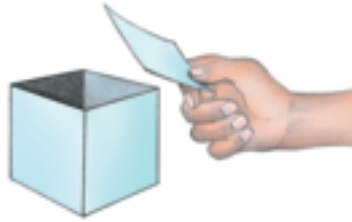


5. **a.** Describe how Alicia cut her box to end up with her final net.
- b.** The picture of Alicia's net is drawn as its actual size. What are the dimensions of Alicia's box? Use a centimeter ruler for measuring.

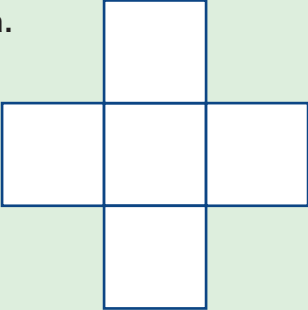
6. What shape would each net pictured below make if it were folded up? If you want, you can use **Student Activity Sheet 1**.

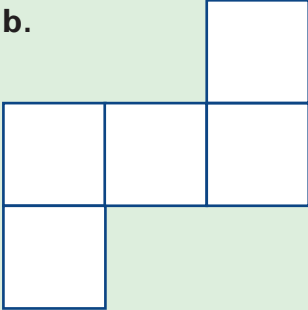


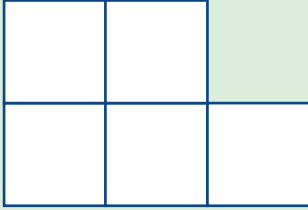
Here is a cube-shaped box without a top.

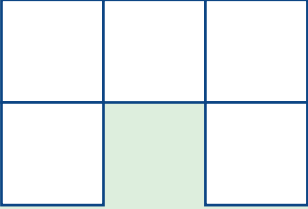


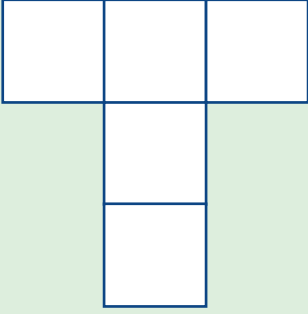
7. Which of these nets fold to make this box?

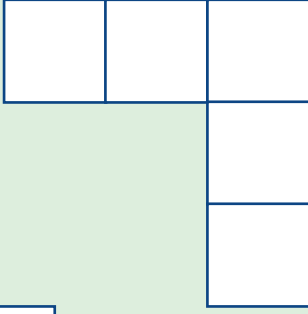
a. 

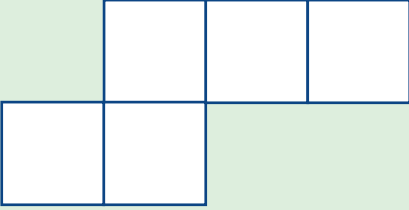
b. 

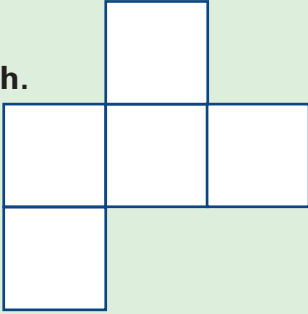
c. 

d. 

e. 

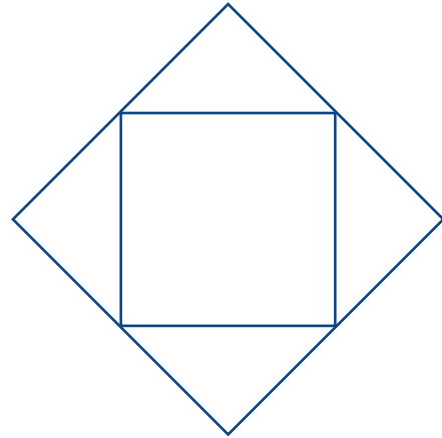
f. 

g. 

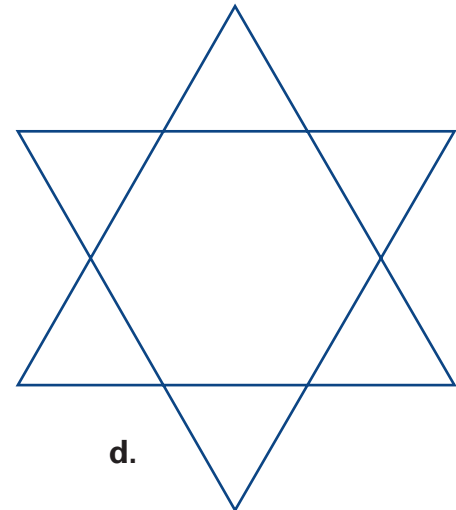
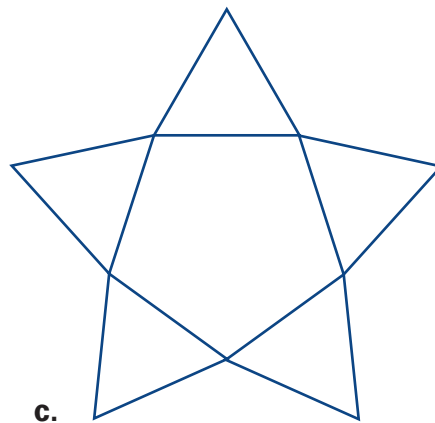
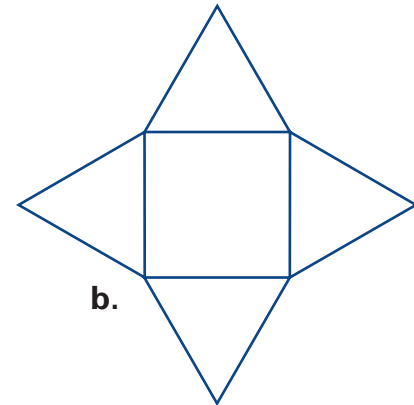
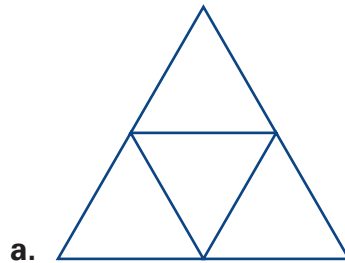
h. 


8. a. On a sheet of graph paper, trace each net that you selected for the previous problem. On each net, shade the bottom face, which is opposite the open top.
- b. It is possible to make three more nets for the open box. Draw at least one of these nets and shade the bottom face.

9. Explain what would happen if you folded this net.



10. Which of the following nets can be folded to make a pyramid? Explain.



-  11. a. **Reflect** Sandra says, “The pyramids you can make from the above nets all have the same height.” Is this true? Why or why not?
- b. Describe how you can compare the heights of these pyramids.

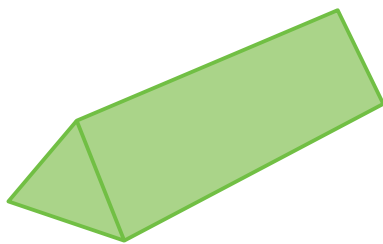
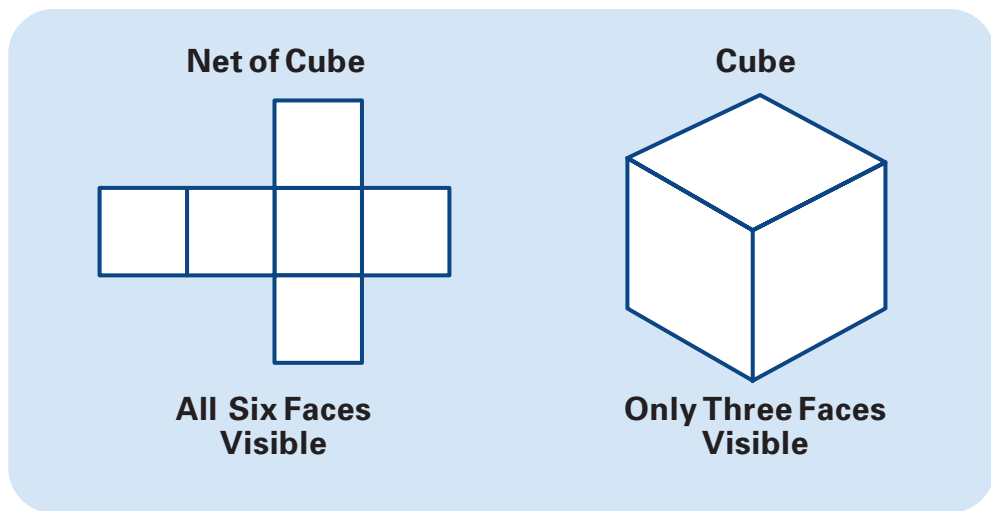
Faces

12. Use **Student Activity Sheet 2** to make a cube. You may want to trace the net onto heavy paper, and use that net instead.

Each flat side of a cube is called a **face**.

13. a. Hold the cube so that you see only one face. Draw what you see.
b. Hold the cube so that you see exactly two faces. Draw what you see.
c. Hold the cube so that you see exactly three faces. Draw what you see.
d. What happens when you try to hold the cube so that you can see four faces?

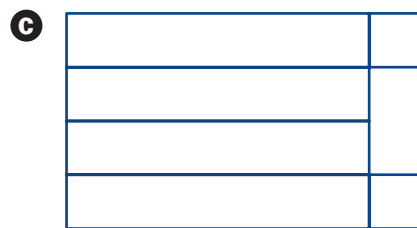
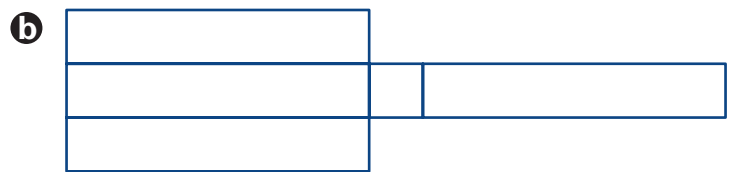
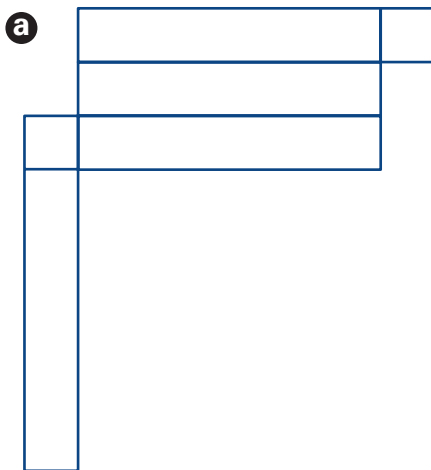
The only time you can view all faces of a shape at once is when you view the net of that shape.



14. Only two faces of the prism in the picture on the left are visible.
- How many faces are hidden?
 - Of all faces, how many faces are triangles? How many are rectangles?
 - Draw a net of this prism.

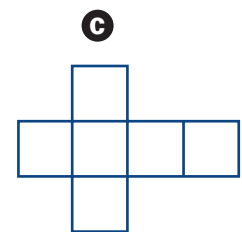
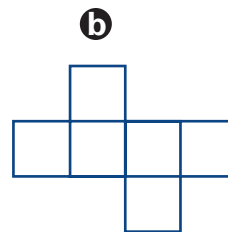
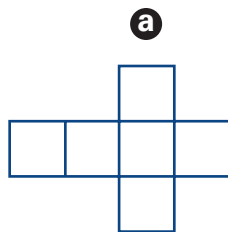
Ne(a)t Problems


15. a. Draw a net of a rectangular prism that is 3 inches long, 1 inch wide, and 2 inches high.
 b. Compare your net with the nets your classmates made. Are they all the same?
16. a. Which of these nets fold into a rectangular prism?



- b. Describe why some of the nets in part **a** cannot be folded into a rectangular prism.

Here you see three nets.



17. a. What do these nets have in common?
 **b. Reflect** How can you explain to someone that net A and net C are the same?
 c. A cube has 11 different nets. Try to find them all.

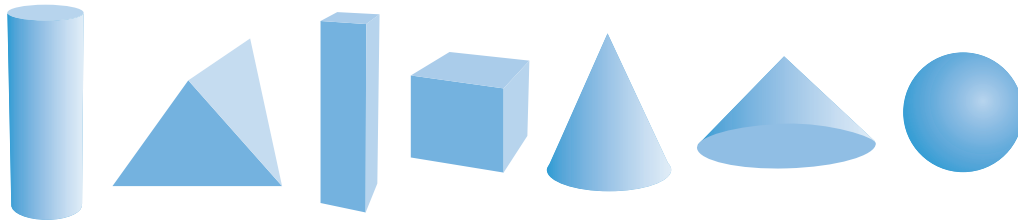


Summary



Shapes

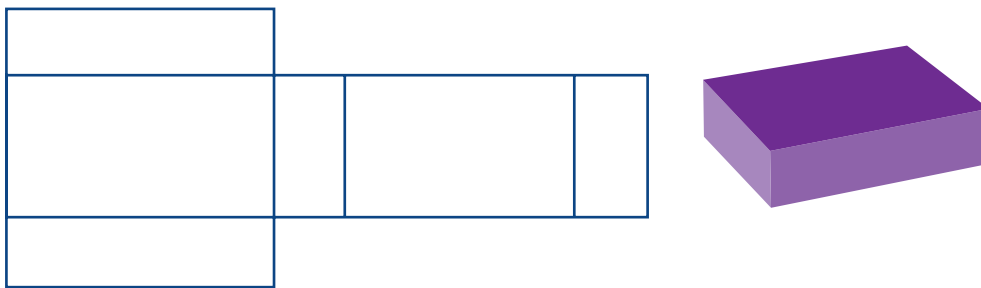
In this section, you studied several three-dimensional shapes. Special names for these shapes include: *cylinders*, *prisms*, *rectangular prisms*, *cubes*, *pyramids*, *cones*, and *spheres*.



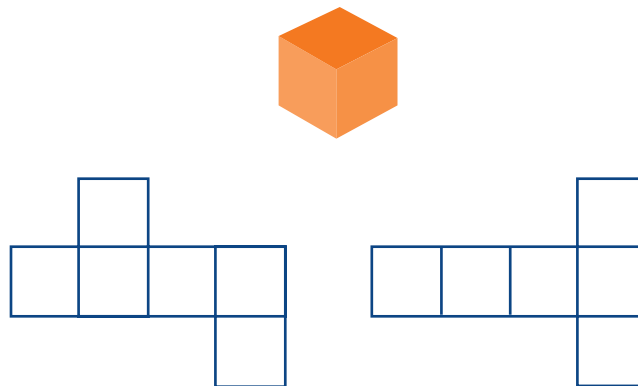
You investigated features that helped you distinguish among a prism, a pyramid, and a cone.

Nets

You made *nets*, two-dimensional drawings that when cut out can be folded into three-dimensional shapes.



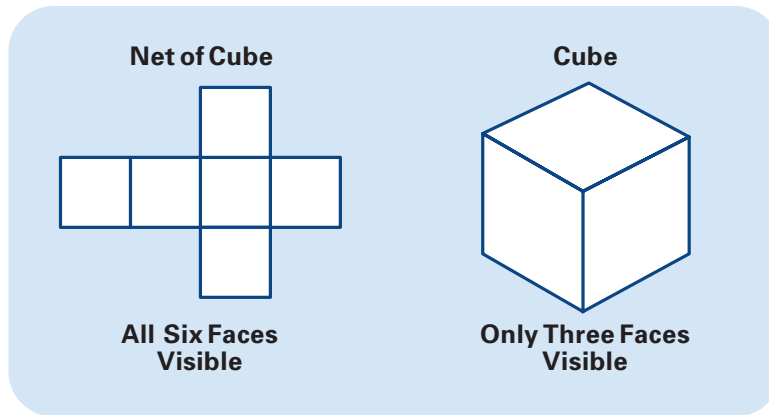
You made different nets for the same shape.



Faces

Each flat side of a shape is called a *face*.

The only time you can view all faces of a shape at once is when you view the net of that shape.

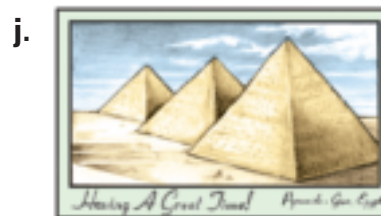




Packages

Check Your Work

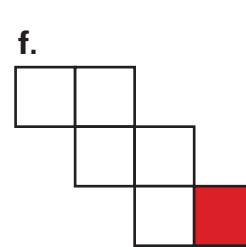
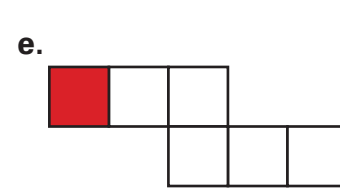
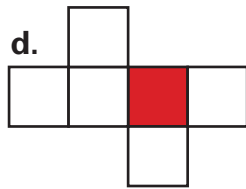
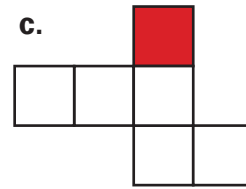
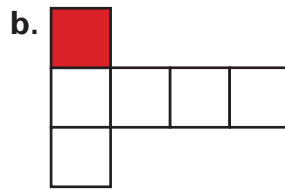
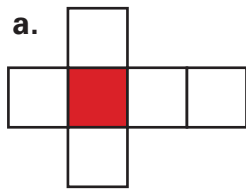
1. Look at the pictures below. What three-dimensional shape(s) do you recognize in each picture?



2. a. Select three different shapes you studied in this section. Write down a characteristic shared by two of the shapes, but not by the third.
- b. Repeat part a with three different shapes.

3. Draw a net of a rectangular prism with dimensions $4\text{ cm} \times 2\text{ cm} \times 3\text{ cm}$.

Tina decorated the faces of a cube using red, white, and blue paints. She painted opposite faces the same color. Here are six different nets for Tina's cube with only some of the red faces painted.



4. Copy each of these nets on graph paper and mark the color of all of the faces.



For Further Reflection

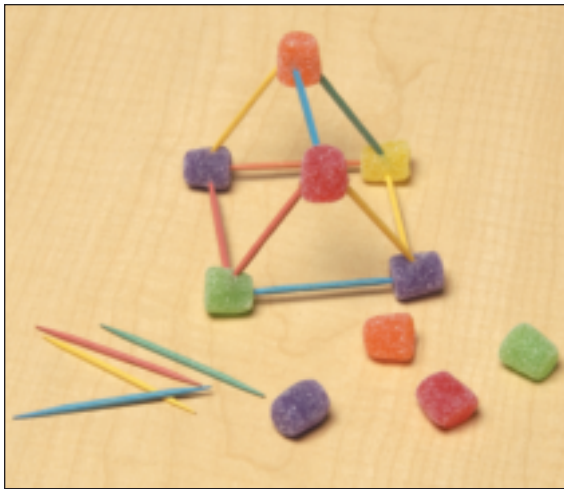
Which shapes are most commonly used for food packages? Explain why you think these are the most often used shapes.

B

Bar Models

Making Bar Models

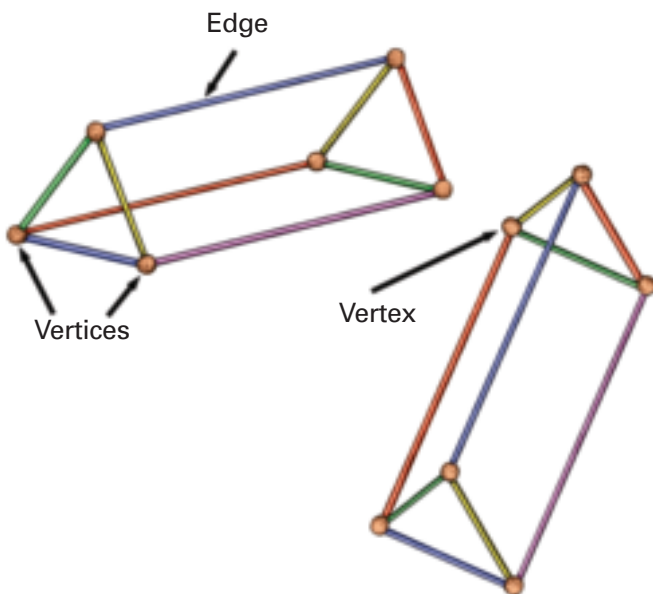
Activity



This is a **bar model** of a triangular prism.

You can make a bar model of a shape using drinking straws and pipe cleaners, toothpicks and clay, or toothpicks and gumdrops.

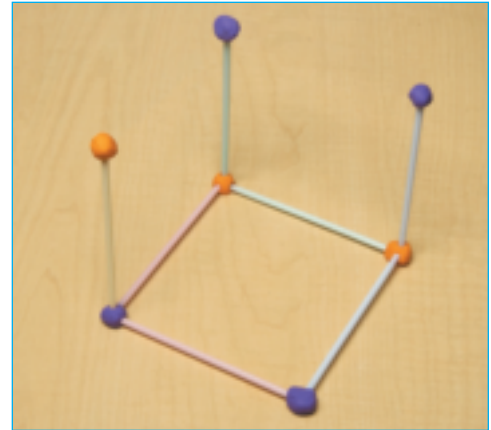
- Make a bar model of a cube.



The bars are the **edges** of the shape (the line segments that form the shape). A point at a corner where two or more edges meet is called a **vertex**. (Note: The plural of vertex is *vertices*.)

1. a. How many vertices and how many edges does the bar model above have?
b. How many vertices and how many edges does your bar model of a cube have?

Harold starts to make a bar model of a rectangular prism. When he is halfway, it looks like this:



2. On **Student Activity Sheet 3**, finish the bar model of Harold's rectangular prism by drawing the missing edges and vertices.

Tonya used a net to build this pyramid with a square base. Here is a drawing of her pyramid.



3. a. How many faces are in Tonya's pyramid?
 b. Use **Student Activity Sheet 3** to change Tonya's drawing into a bar model, showing all the edges and vertices.
 c. Use the second drawing on **Student Activity Sheet 3** to make a bar model of a different pyramid. What is the special name for this pyramid?

Here is Lance's drawing of his prism. Only three faces are visible.



4. a. How many faces are hidden?
 b. How many faces are rectangles?
 c. How many edges are hidden?
 d. How many vertices are hidden?
 e. Use **Student Activity Sheet 3** to change Lance's drawing into a bar model of the prism.

Peter is building a bar model of a rectangular prism that is 5 cm long, 3 cm wide, and 7 cm high.

5. **a.** How many straws will Peter need to build the bar model of the rectangular prism?
- b.** How many of each length does he need?

Yolanda wants to make a pyramid with nine edges.

6. Will she be able to make such a pyramid? Why or why not?
7. How many straws do you need to make a bar model of pyramid with a triangular base? Make a drawing to verify your answer.

Stable Structures

Activity

Building Structures

- Build a bar model of a pyramid with five vertices and eight edges.
- Build another bar model of a shape with six vertices and nine edges.

8. **a.** Which of the structures you made is more stable? How does the number of triangular faces affect the stability of the structure?
- b.** What could you do to make the other structure more stable?
- c.** Is it possible to build a structure with nine vertices and five edges? Explain.

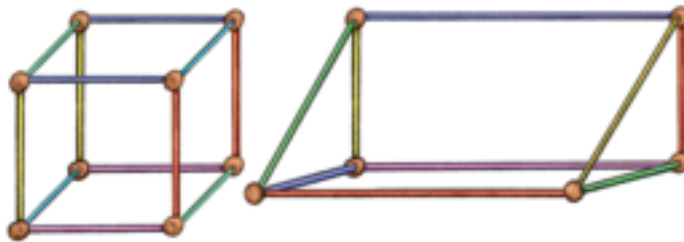


To make a frame rigid, designers use triangles to add strength and stability to the frame.

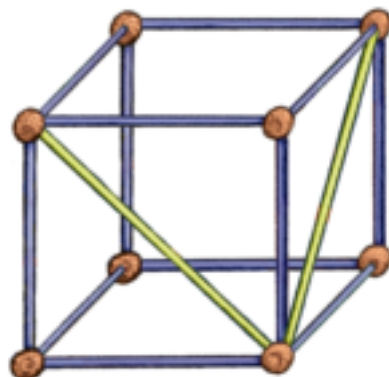
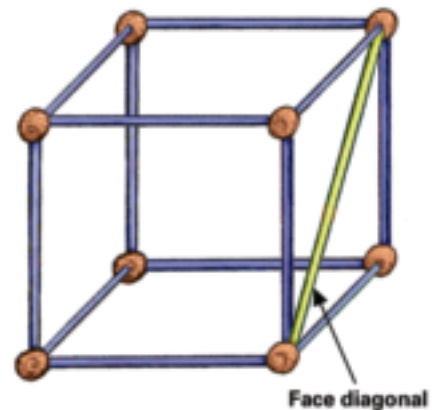
Theo designed this bookcase with three shelves.

- 9 a. Do you think this is a good design? Why or why not?
- b. Improve Theo's design by adding one piece of wood. Make a sketch of your improved bookcase.

10. Which bar model is more stable: a triangular prism or a cube? Why do you think so?



You can make the structure of a cube more stable by adding an extra bar. Here is an example of a **face diagonal**. This face diagonal is an extra bar on the face, connecting two vertices that are not next to each other.



In this cube, two face diagonals are shown.

11. How many different face diagonals are possible for any cube?

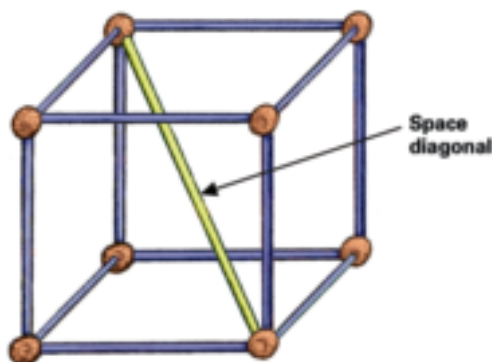
Stable Cubes

- Use the bar model of the cube you made in the activity on page 14.
- Add the minimum number of face diagonals to make the cube stable.

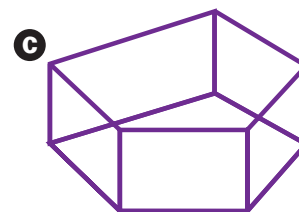
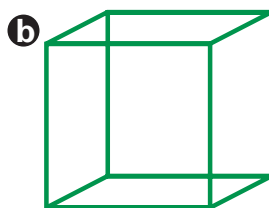
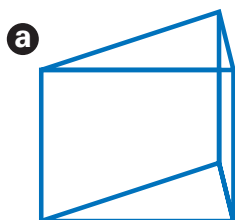
Another type of diagonal to consider is called the **space diagonal**.

The space diagonal goes through the inside or space of the cube, connecting two vertices.

It does not lie on any of the faces. In this drawing, the extra bar connects a vertex the upper back down to the vertex on the front bottom.



12. How many space diagonals does a cube have? Show them all on **Student Activity Sheet 4**.
13. Which of the shapes on page 2 do not have face diagonals?
Which of the shapes on page 2 do not have space diagonals?
14. Name each shape and find the number of face diagonals and the number of space diagonals.



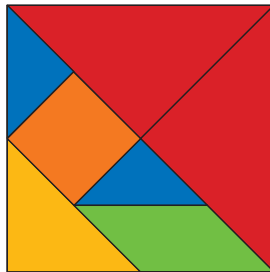
Math History

Mankind has made efforts to design three-dimensional puzzles. One of the most successful is the Soma Cube, invented by Piet Hein.

The Soma Cube is sometimes regarded as the three-dimensional equivalent of a tangram. Both types of puzzles are made up of seven pieces, and they can both be used to construct numerous shapes.

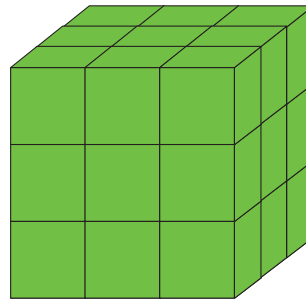
Tangram

Two-dimensional:

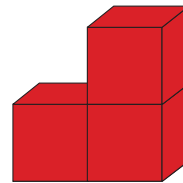


Soma Cube

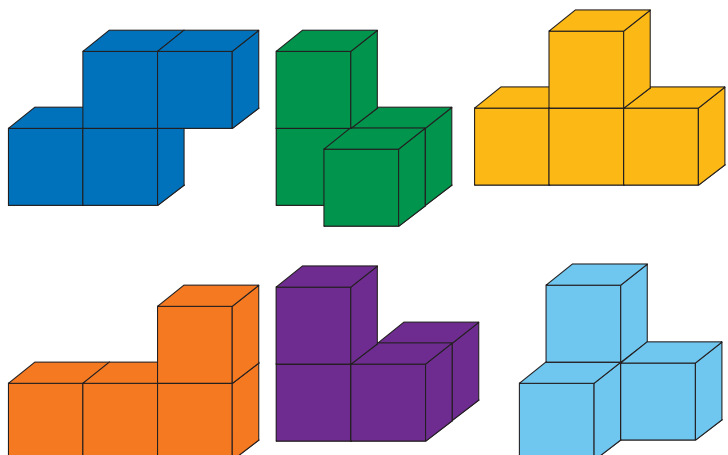
Three-dimensional:



All seven of the following pieces together can form a $3 \times 3 \times 3$ cube.



Can you figure out how they can be put together to form a cube?





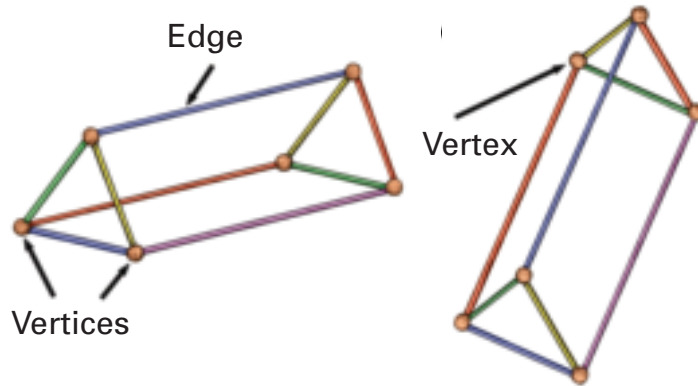
Summary



Bar Models, Edges and Vertices

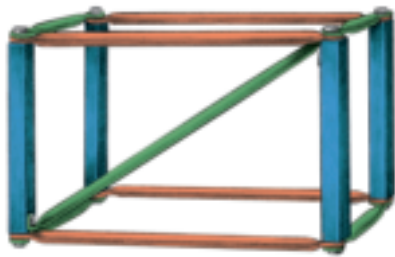
Drawing a net and folding the sides together is one way to make a model of a three-dimensional shape.

Another way is to make a bar model. The bars are the *edges* of the shape. The point where two or more bars meet is called a *vertex*.

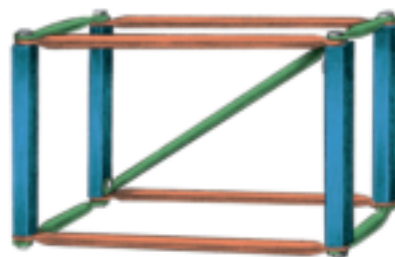


Stable Structures

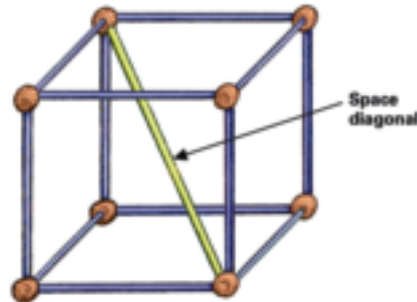
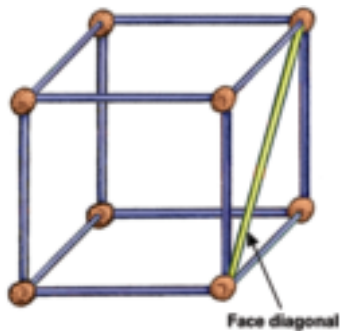
Triangles make a structure more stable. You can add additional bars to a structure to make it more stable.



Sometimes, the new bars are on the original face.

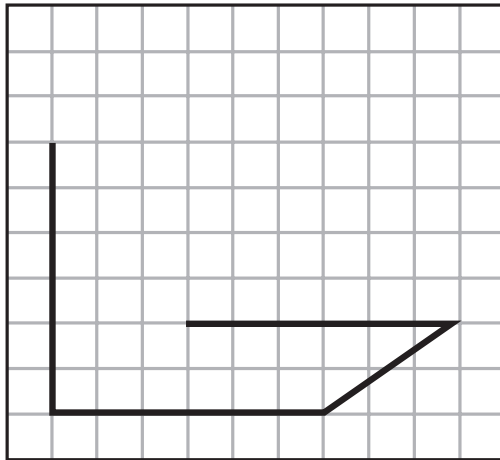


Sometimes, the new bars are within the solid, not on a face.



Check Your Work

Sylvia used graph paper to make a drawing of a bar model of a cube.



- Use **Student Activity Sheet 4** to finish Sylvia's drawing.
 - Use a colored pencil to draw two face diagonals.
 - Use a different colored pencil to draw one space diagonal.

Kelly has eight straws, each 10 cm long. She wants to build a bar model of a pyramid using all the straws.

- Make a drawing to show how she can do this.
 - Would this model be more stable than a pyramid built with six straws? Why or why not?

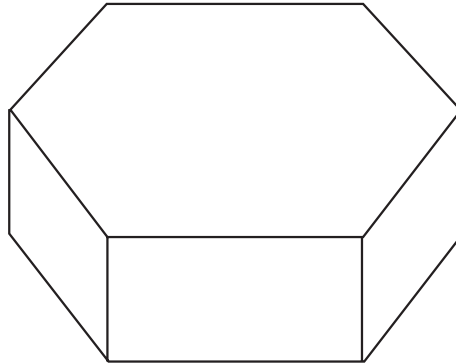
Maglio has six 5-cm straws and three 10-cm straws. Without cutting any straws, he wants to make bar models of prisms and pyramids.

- Which bar models can Maglio make with his straws if he does not have to use all nine of his straws at one time? Use drawings to support your answer.



Bar Models

Patti drew this prism, showing only four visible faces.



4.
 - a. How many faces are hidden?
 - b. How many vertices are hidden?
 - c. How many faces are rectangles?
 - d. Use **Student Activity Sheet 4** to change Patti's drawing into a bar model of the prism.
 - e. How many total faces, total vertices, and total edges are in Patti's prism?
 - f. How many face diagonals are possible for Patti's prism?



For Further Reflection

How many edges were attached to each vertex in the bar models you considered in this section? Why do you think this was? Do you think you could ever have a different number of edges per vertex? Why or why not?

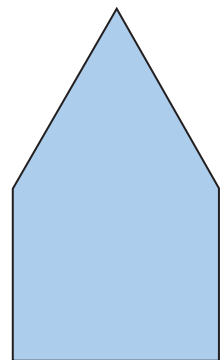
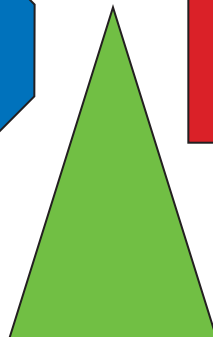
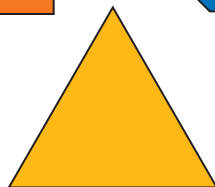
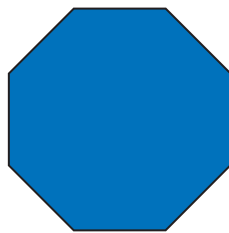
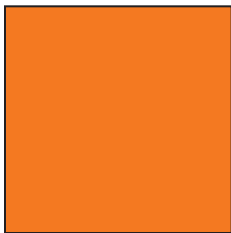
Polygons

Put a Lid on It

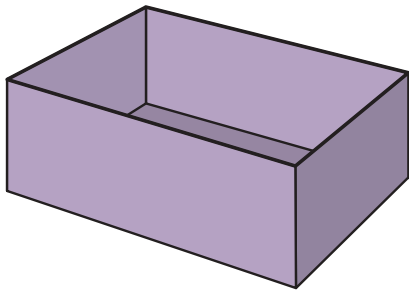


Susanne has a job after school making different shipping cartons. Most of the time, she must find the best carton and lid for packaging unusual sized items.

Here the lids that Susanne uses. All of the shapes are **polygons**.

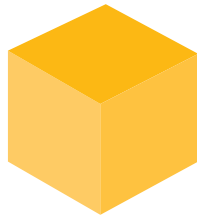


1. Study the shapes. Look for some similarities and write your own definition of a polygon.



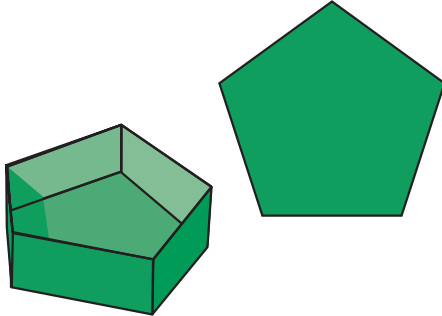
Today, Suzanne must find a lid for the carton on the left shaped like a rectangular prism.

2. a. What polygon will Suzanne use for the lid?
- b. How many different ways can she place the lid on the carton? Note that Suzanne also can put the lid on upside down.



Next, Suzanne must find a lid for a carton shaped like a cube.

3. a. What polygon will Suzanne use for the lid?
- b. How many different ways can she place the lid on the carton?



Suzanne found this lid for this carton.

4. a. In how many different ways can she place the lid on the carton?
- b. What is the name for the polygon used for the lid?

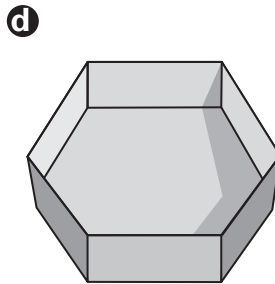
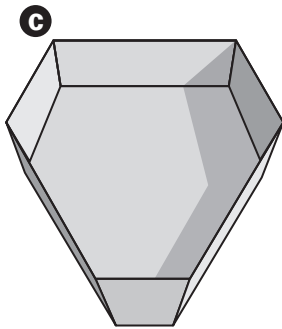
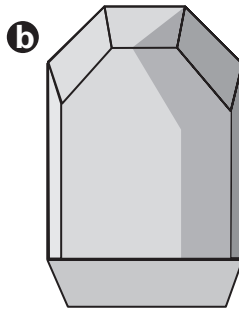
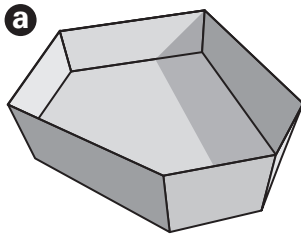
MONO = 1
DUO = 2
TRI = 3
QUADRA = 4
PENTA = 5
HEXA = 6
HEPTA = 7
OKTO = 8
NONA = 9
DEKA = 10

The word **poly**gon comes from the Greek word *polygonos*, which means "many angles."


A polygon with three angles is usually called a triangle, but could also be called a "3-gon."

A polygon with four angles has four sides. This 4-sided polygon could be called a "4-gon."

5. a. What is a more common name for a 4-sided polygon?
- b. Name the common name for a 5-gon, 6-gon, 7-gon, 8-gon, 9-gon, 10-gon, and 12-gon.



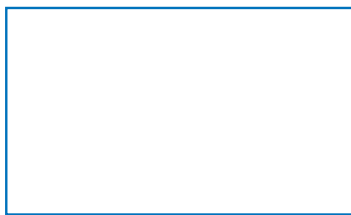
Susanne must find lids for these cartons. Use **Student Activity Sheet 5** to cut out the lids Suzanne selected.

-  **6. Reflect** Compare the four lids. How are the polygons the same? How are they different?

The lid for carton D is a special type of polygon; it is a **regular polygon**.

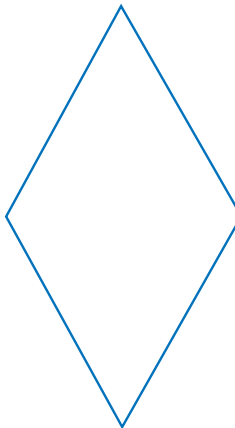
7. What do you think makes polygon D “regular”?
8. In how many different ways can Suzanne place the lid on each of the cartons A to D?

Figure A



Four Equal Angles

Figure B



Four Equal Sides

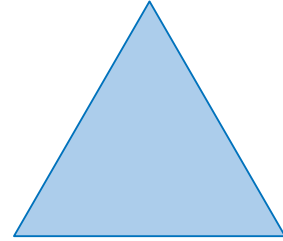
9. The two quadrilaterals at the left are not regular polygons.
 - a. What is irregular (not regular) about Figure A?
 - b. What is irregular about Figure B?
 - c. In how many ways can Figure A be folded in half so that it fits onto itself? Figure B?

10.
 - a. Draw a quadrilateral that can be folded in half in four different ways.
 - b. What is the name of this quadrilateral?
 - c. Is the quadrilateral you drew in part a regular? Why or why not?

A regular triangle has three equal sides and three equal angles. A regular triangle is more commonly called an **equilateral triangle**.

Here is a regular or equilateral triangle.


11. a. What is the measure of each angle? Explain the method you used to find this answer.
- b. Draw a regular triangle with sides 8 cm long.

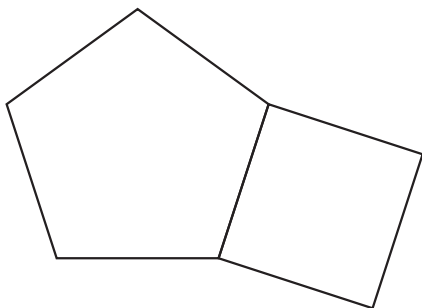


Pentagon

This is a photograph of the Pentagon building in Arlington, Virginia (outside of Washington, D.C.). The Pentagon is the headquarters of the U.S. Department of Defense and is one of the world's largest office buildings. The perimeter of the building is about 4,605 feet, and there are 17.5 miles of hallways. About 23,000 people work in the building.



-  12. **Reflect** Why do you think this building is named the Pentagon?

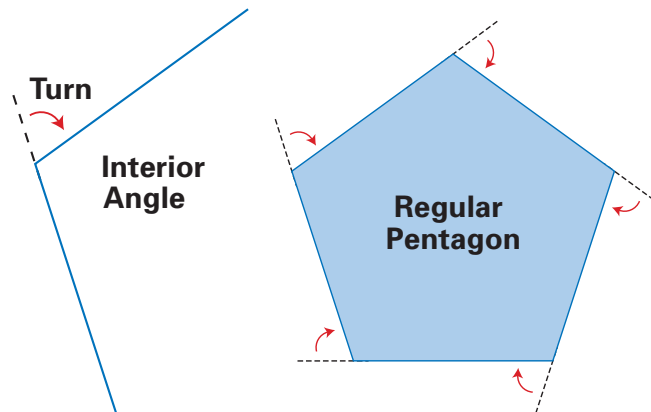


Kendra is designing a top view of the Pentagon. Here is the beginning of her drawing, showing the top view of the building.

13. Use **Student Activity Sheet 6** to complete Kendra's drawing of the entire building.

Angles

You can find the measures of the angles of a polygon by using turns. To do this, imagine yourself walking along the edges of a polygon. Picture the angle you make each time you turn a corner.



14.
 - a. How many degrees would you turn if you walked all the way around a regular pentagon? A square? An equilateral triangle?
 - b. If you walked all the way around any polygon, how many degrees would you turn? Why?
 - c. What is the relationship between the size of the turn and the angle inside the polygon?
15.
 - a. When you walk around a regular pentagon, you make five equal turns. How many degrees are in each turn?
 - b. How many degrees are inside each angle of a regular pentagon?
 - c. How can you use turns to find the measures of the inside angles of any regular polygon?



This is a picture of a bee in its honeycomb.

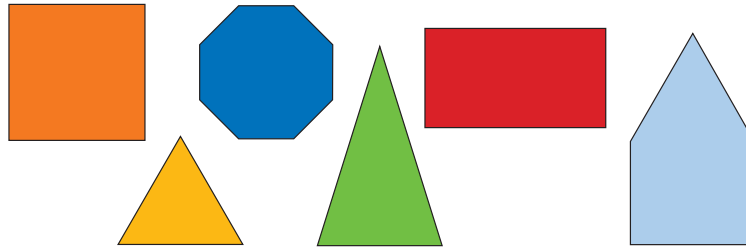
16.
 - a. What regular polygon do bees use in making their honeycombs?
 - b. What is the measure of one inside angle of this regular polygon?



Summary

Polygons

Polygons are two-dimensional closed shapes with three or more angles.



Polygons are named according to the number of sides or angles they have. Greek words are used for their names, for instance:

- a polygon with three sides is called a *triangle*.
- a polygon with four angles is called a *quadrilateral*.
- a polygon with five angles is called a *pentagon*.
- a polygon with six angles is called a *hexagon*.
- Others are called *n*-gons, or polygons that have *n* sides. For example, a nine-gon is a polygon with nine sides.

Regular Polygons

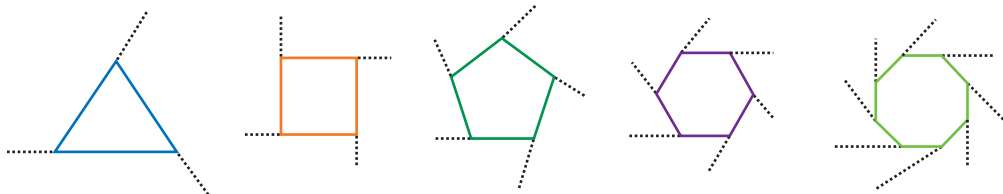
A regular polygon has equal sides and equal angles.

A regular triangle is more commonly named an *equilateral triangle*.

Angles of Regular Polygons

To walk completely around any regular polygon, you turn 360° .

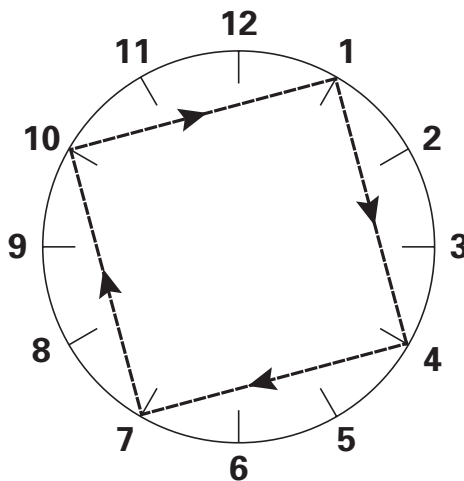
Depending on the number of sides, you can use this information to find the size of one turn. The size of one turn and the interior angle have a special relationship; they have a sum of 180° . You can use turns to find the measure of any interior angle of any regular polygon.



Check Your Work

- Draw two different shapes that have four equal sides. Was one of the polygons you made regular? Which one? Why?
 - Explain what happens to the size of the interior angles in regular polygons as the number of sides increases?
- If you connect the midpoints of the sides of an equilateral triangle, what kind of a shape will be formed? Explain your reasoning.
 - How you can find the size of the angles of a regular triangle without using a compass card or protractor?

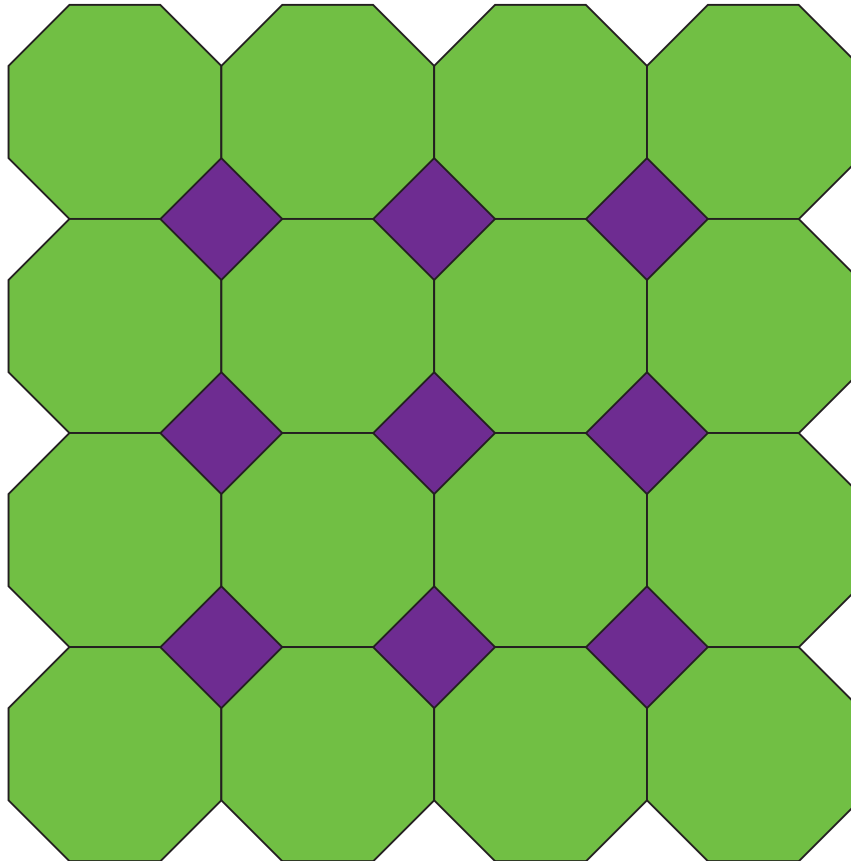
You can use a picture of a clock to create regular polygons. Starting at one o'clock, make jumps of three hours. After four jumps, you are back to where you started. The result is a regular quadrilateral inside the clock.



- Use **Student Activity Sheet 7** and a straightedge to make the following drawings. Be sure to count carefully.
 - Start at one o'clock and make jumps of two hours until you are back at one o'clock. What polygon did you make?
 - Do the same thing as in part a with jumps of four hours. What polygon did you make?
 - What polygon do you make with jumps of one hour?



This is a part of a tessellation. The tiles are regular polygons.



4. a. What are the names for the polygonal tiles?
b. What is the measure of one interior angle of a green tile?
Show your work.



For Further Reflection

Try to find a relationship between the number of sides and the number of ways a regular polygon can be folded in half.

D

Polyhedra

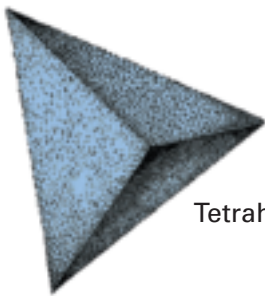
A **polyhedron** is a three-dimensional solid whose faces are all polygons.

The word polyhedron comes from the Greek words meaning “many bases.” You have already worked with many different polyhedra.

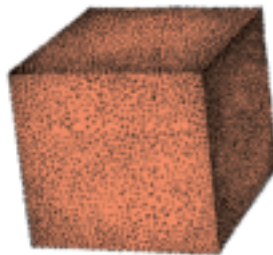
1. Refer to the geometric models on page 2. Which of the shapes are polyhedra? Which shapes are NOT polyhedra?

Special Polyhedra

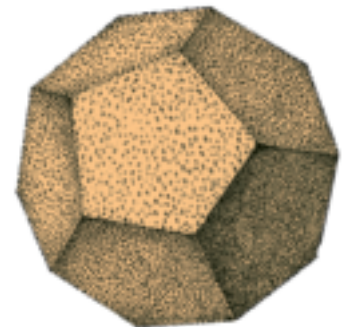
Some of the polyhedra are special, like these five.



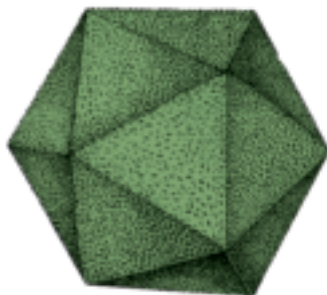
Tetrahedron



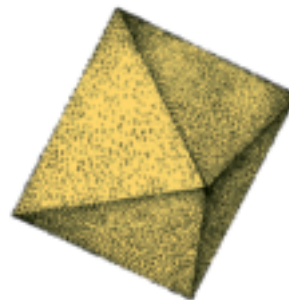
Cube



Dodecahedron



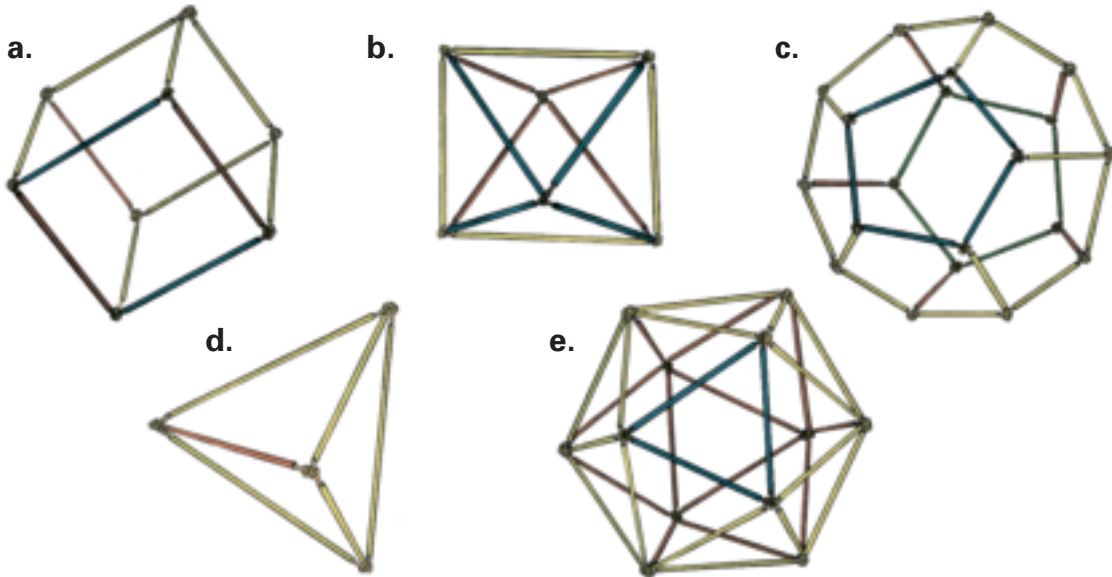
Icosahedron



Octahedron

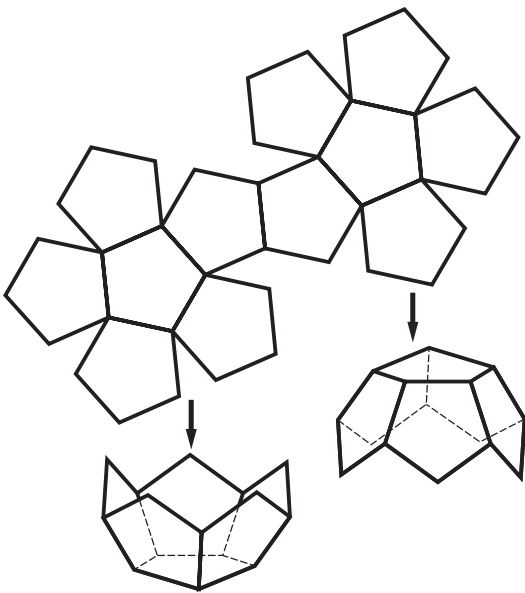
2. What is so special about these five polyhedra?

3. Here are the bar models for each of the five special polyhedra. Name each bar model.



Activity

- Cut the nets out on **Student Activity Sheets 8–12** to make the five regular polyhedra.

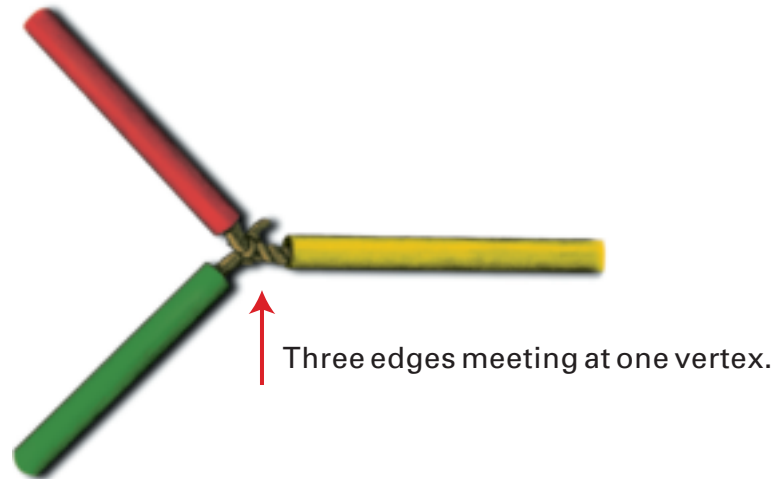


The drawings on the left illustrate how you can make a regular dodecahedron from the net. (Note: *Dodeca* means twelve.) You fit the two baskets together to make the 12-sided polyhedron.

Keep your models handy. They will help you answer questions throughout this unit.

You can make bar models using drinking straws and pipe cleaners or string. The pipe cleaner or string is threaded into each straw to make connections at the vertices.

Here is string connecting three straws, forming one vertex.



To make all five models, you need 90 straws.

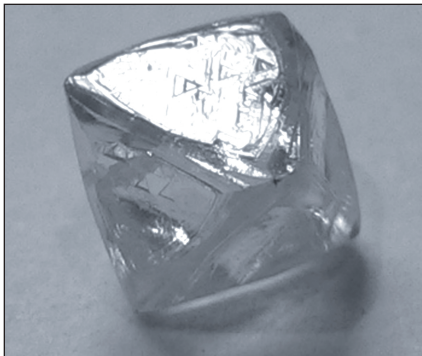
4.
 - a. How many straws does each bar model require?
 - b. Make bar models for the remaining four shapes. (You already made the bar model for the cube; you used 12 straws.)

In the previous problems, you made paper and bar models of the five famous **Platonic solids**.

The Platonic solids are named after the ancient Greek philosopher Plato. These five solids are the only **regular polyhedra** that exist. It is not possible to construct other regular polyhedra.

5.
 - a. Write the name of each regular polyhedron that has faces in the shape of an equilateral triangle.
 - b. Write the name of each regular polyhedron that has faces that are pentagons.

Faces, Vertices, and Edges



All of the Platonic solids occur in nature as crystals. The microscopic diamond crystal shown in this picture is in the shape of an octahedron.

The octahedron on the left calculated the number of his edges, even though he cannot see them.



I have 8 faces;
each face has 3 edges;
so I have 24 edges!

- 6 a. Do you agree with the octahedron? Explain.
- b. The cube seems to know better. In your notebook, complete his reasoning.



I have 6 faces; _____

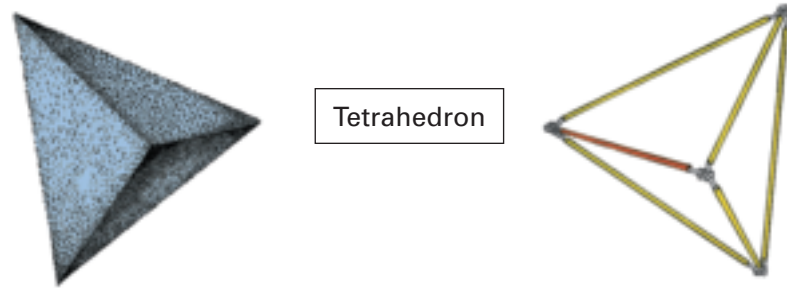
so I have _____ edges!






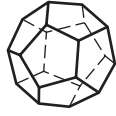

30 edges, YES I DO!

- c. The icosahedron says she has 30 edges. Is this true? How can you be sure?

The simplest Platonic solid is the **tetrahedron**. Remember that the prefix tetra- means “four” because it has four identical faces.

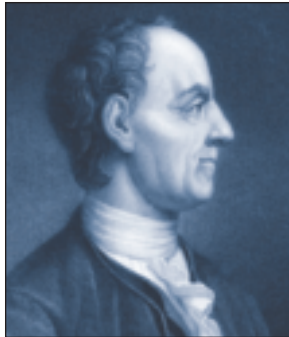


7. How many vertices and how many edges does a tetrahedron have?
8. a. Complete the table on **Student Activity Sheet 13**. Use the models you made to help you.

Name	Shape	Type of Face	Number of Faces	Number of Vertices	Number of Edges
Tetrahedron			4		
Cube					
Octahedron					
Dodecahedron					
Icosahedron					

- b. Study the numbers in your table. What patterns or relationships do you see?

Euler's Formula



Leonard Euler (pronounced "Oiler") was a Swiss mathematician who discovered the following formula around 1750. Euler's formula states that for any convex-polyhedron, the number of vertices and faces together is exactly two more than the number of edges.

You may have discovered a relationship between the numbers of faces, vertices, and edges of a polyhedron.

$$\begin{array}{c} \text{Number} \\ \text{of} \\ \text{Faces} \end{array} + \begin{array}{c} \text{Number} \\ \text{of} \\ \text{Vertices} \end{array} = \begin{array}{c} \text{Number} \\ \text{of} \\ \text{Edges} \end{array} + 2$$

The relationship is also written as:

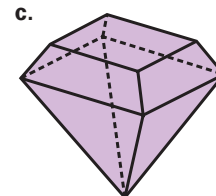
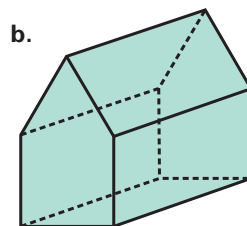
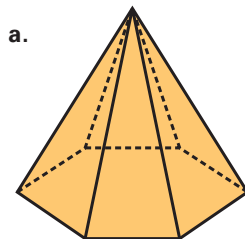
$$F + V - E = 2 \quad \text{where } F \text{ is the number of faces,}$$

$$V \text{ is the number of vertices, and}$$

$$E \text{ is the number of edges.}$$

9. a. Explain how both formulas say the same thing.
- b. Refer to your table from problem 9 and show that Euler's formula works for all the regular polyhedra.

Here are three solids that are polyhedra but are not regular polyhedra.



$F =$
$V =$
$E =$
$F + V - E =$

$F =$
$V =$
$E =$
$F + V - E =$

$F =$
$V =$
$E =$
$F + V - E =$

10. Does Euler's formula work for these polyhedra too? Show why or why not.

Semi-Regular Polyhedra

Activity

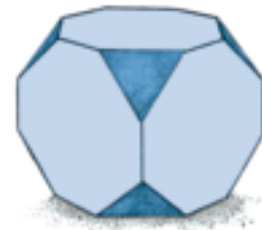
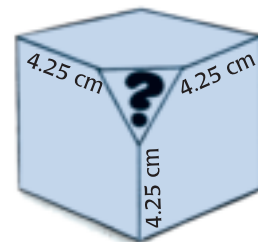
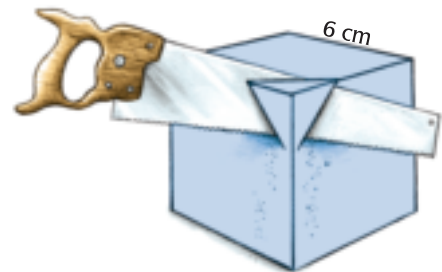
Cutting and Forming a Semi-Regular Polyhedron

Student Activity Sheet 14 has a net for a cube with 6-cm edges, but part of the cube has been cut off.

- Trace the net onto heavy paper and fold it into a cube. (Do not paste the tabs yet.)
- Describe your cube.
- Look through the hole in your cube and draw what you see. Unfold the cube and carefully cut the other corners off in exactly the same manner.
- Fold the net to make the solid.

This results in a polyhedron with two different kinds of faces.

- What are the names of these faces?

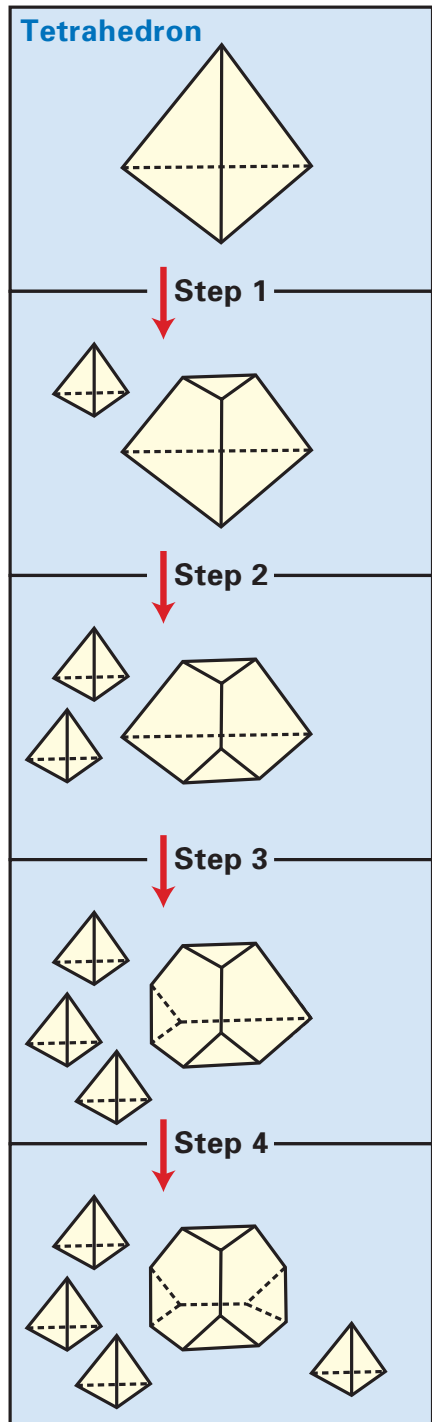


This new solid is called a **semi-regular polyhedron**.

- 11. a.** Describe the features of a semi-regular polyhedron.
 - b.** Verify that Euler's formula works for the semi-regular polyhedron created in the Activity above.

Euler discovered his formula around 1750, but it was not until 1794 that a French mathematician named Adrien-Marie Legendre proved it actually worked for all polyhedra.

The following is one way of explaining why Euler's formula works for all polyhedra.



Begin with any tetrahedron:
 It has four faces, four vertices, and six edges.
 Euler's formula states that: $F + V - E = 2$
 Verifying Euler's formula: $4 + 4 - 6 = 2$

Step 1: Cut off one tip of the tetrahedron.

This changes the number of:

- faces: F increases by 1.
- vertices: V increases by 2.
- edges: E increases by 3.

12. a. Explain the changes in Step 1.
 b. Why doesn't the formula $F + V - E$ change in Step 1?

Step 2: Cut off a different tip of the tetrahedron.

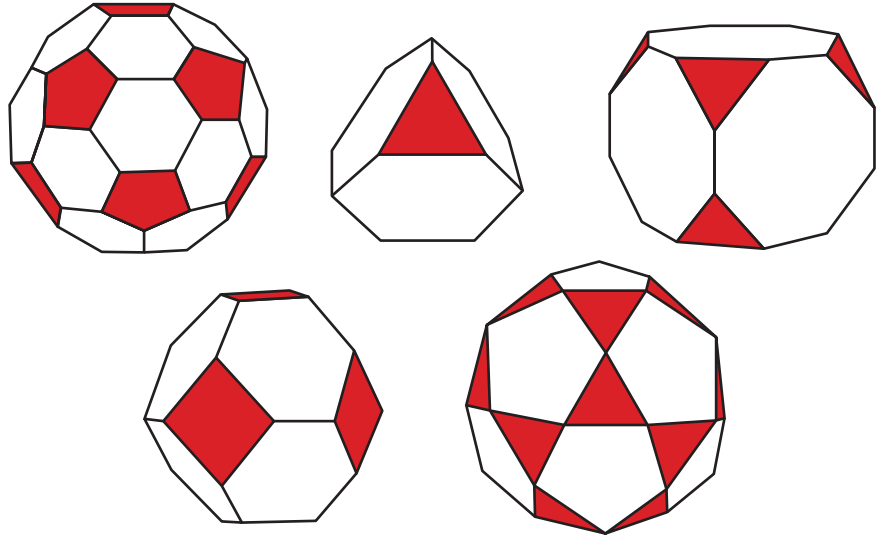
13. a. Describe the changes in the number of faces, vertices, and edges.
 b. Do the changes affect Euler's formula? Why or why not?

14. Describe Steps 3 and 4. Discuss any changes.

15. a. Describe the semi-regular polyhedra produced after Step 4.
 b. Verify Euler's formula for this solid.

When you cut a piece off each vertex of an icosahedron, you get a semi-regular polyhedron made of pentagons and hexagons.

16. Here you have five semi-regular polyhedra. Match each semi-regular polyhedron to its original Platonic solid.





Summary

Polyhedron

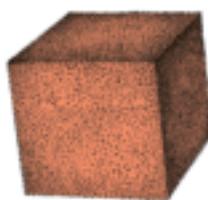
A *polyhedron* is a three-dimensional shape whose faces are all polygons. The word *polyhedron* comes from the Greek words for “many bases.”

Platonic Solid

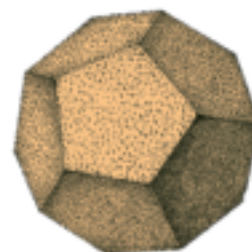
If all the faces of a polyhedron are the same regular polygon, then the polyhedron is a *Platonic solid*. There are only five Platonic solids; the *tetrahedron*, *cube*, *octahedron*, *dodecahedron*, and *icosahedron*.



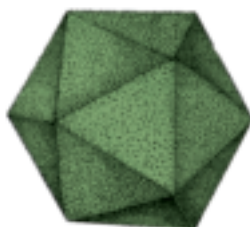
Tetrahedron



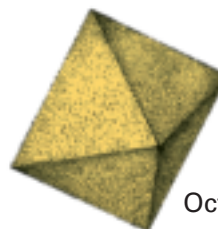
Cube



Dodecahedron



Icosahedron



Octahedron

Euler's Formula

There is a relationship between the numbers of faces, vertices, and edges of any polyhedron.

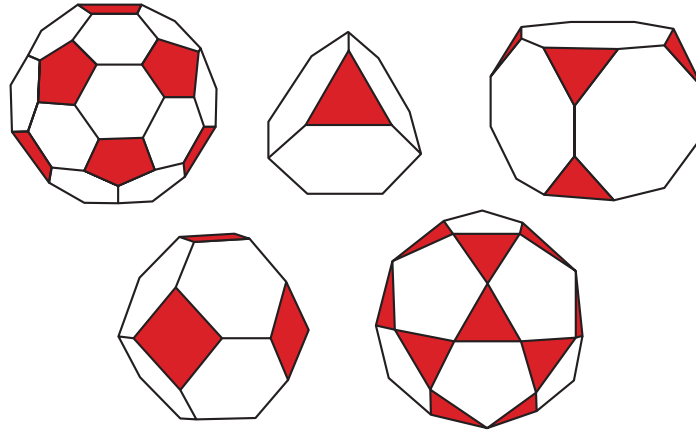
Euler expressed this relationship with the formula:

$$F + V - E = 2,$$

where F is the number of faces,
 V is the number of vertices, and
 E is the number of edges.

Semi-Regular Polyhedron

Semi-regular polyhedra have at least two different regular polygons as faces. You can make these solids by cutting off equal pieces from each vertex.

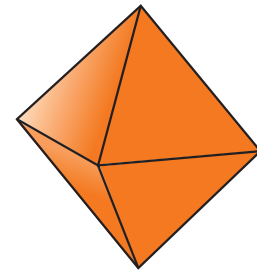


Check Your Work

1. Explain why the Platonic solids are regular polyhedra.

Jonathan has a strategy for counting the number of edges on the octahedron. He says, "I see eight edges in the picture, and the back side of the octahedron looks the same as the front side, so two times eight equals sixteen edges."

2.
 - a. What might Jonathan be thinking?
 - b. How can you change Jonathan's statement so that you get the correct number of edges?





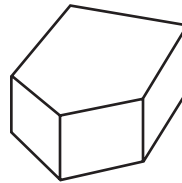
Polyhedra

This is a bar model of an icosahedron.
Toni wants to reuse the bars and vertices
of this model to make octahedra.

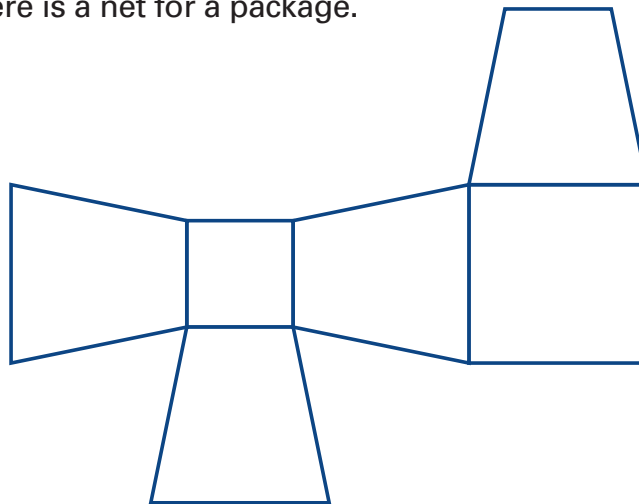


Icosahedron

3. How many octahedra can Toni make?
4. Does Euler's formula $F + V - E = 2$ work for a cube? Explain.
5. Verify that Euler's formula works for this polyhedron.



6. Here is a net for a package.



- a. Draw the shape of the assembled package.
- b. Is the formula $F + V - E = 2$ true for the package?



For Further Reflection

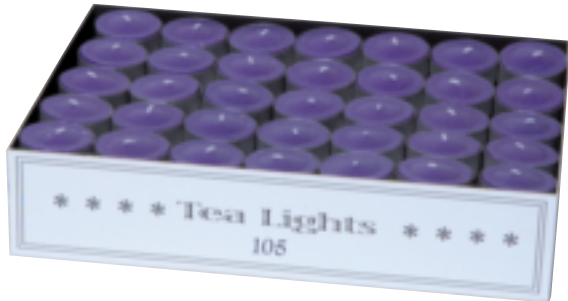
Which polyhedra have one or more faces in the shape of an equilateral triangle?

Which polyhedra have one or more faces in the shape of a hexagon?

E

Volume

Candles



In the previous sections, you studied several geometric solids; you learned their names, transformed solids into nets and nets into solids, and investigated properties about their faces, vertices, and edges.

In the unit *Reallotment*, you studied formulas for finding volume.

In this section, you will investigate the volume of a variety of geometric solids.

1. How many layers of tea-light candles are in the box? Explain how you know.

These tea-lights can be stacked in a different way, which means that other sized cartons can hold the same 105 candles.

2. a. Find all other possible arrangements for rectangular carton to hold 105 tea-lights. Each carton must be filled completely, except for the holes between the tea-lights.

- b. **Reflect** Is the volume of all these boxes the same? Explain how you know.

Candles come in a variety of shapes and colors.

3. What three-dimensional shapes do you recognize in the picture?



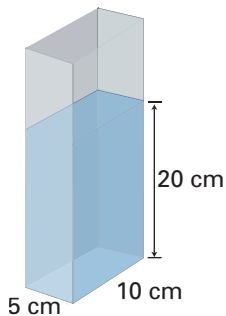
Lydia and Rosa plan to make their own candles. They have different molds, but they need to buy the wax. The directions for buying wax say that for one liter you need two pounds of wax.

Rosa looks at the molds and asks, "How much wax will we need?"



Lydia takes a one-liter bottle and fills it with water. She pours the water into the mold and says, "Well, for just this mold, we need more than two pounds of wax! The volume of this mold is more than one liter."

4. Explain Lydia's statement.



Rosa vaguely remembers something about a metric relationship between liquid measurements and volume.

Rosa suggests, "I cannot remember the precise relationship, but let's measure the dimensions of the liter of water in this mold!"

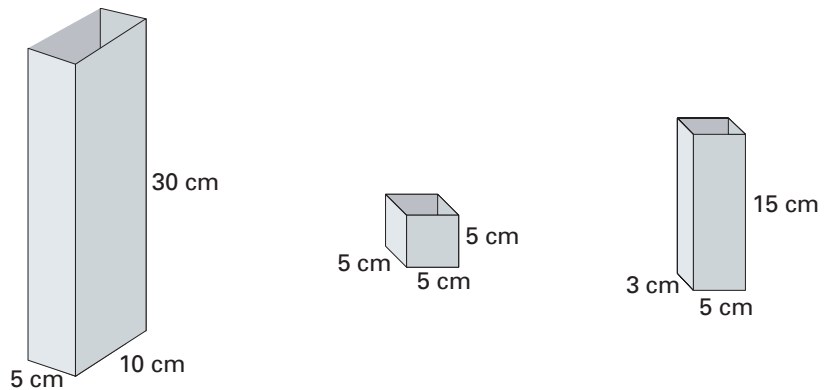
5. a. Use the measurements shown to find the volume of one liter of water.

b. Copy and complete the relationship in your notebook.

$$1 \text{ liter} = \underline{\hspace{2cm}} \text{ cm}^3$$

c. Find the dimensions of two other molds that would hold exactly one liter of water.

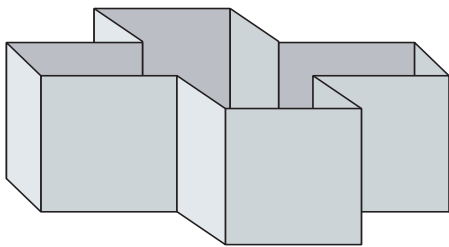
Rosa and Lydia decide to buy wax for these three molds.



6. How many pounds of wax will they buy to fill these three molds?
A ratio table may be helpful.

cm^3						
lbs						

Finding Volume



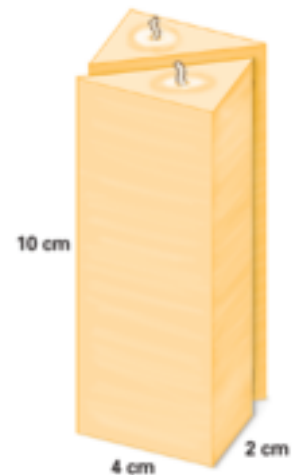
The bottom of this mold is in the shape of a cross.
The side faces are all squares with sides of 3 cm.

7. Find the volume of this mold.

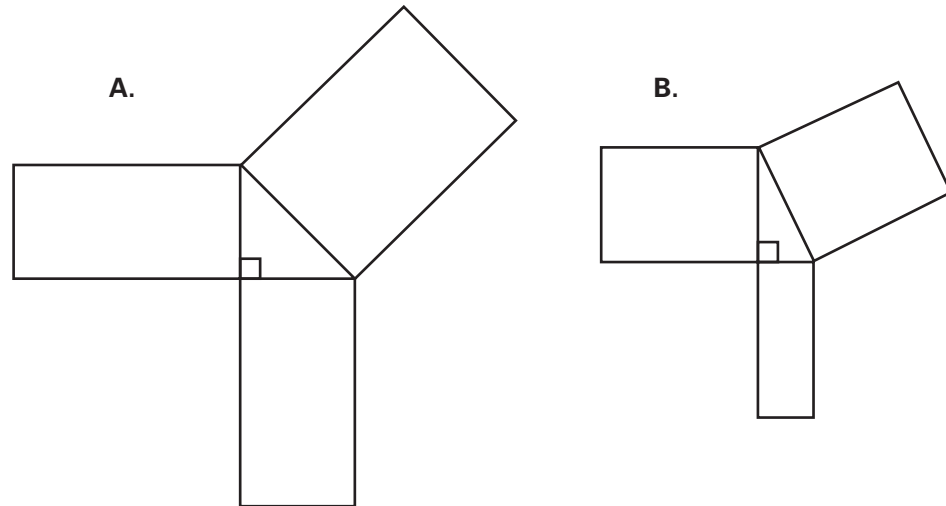
These two candles are made from the same mold. The height is 10 cm, and the base is a right triangle.

The two candles fit perfectly in a rectangular prism with dimensions 4 cm \times 2 cm \times 10 cm.

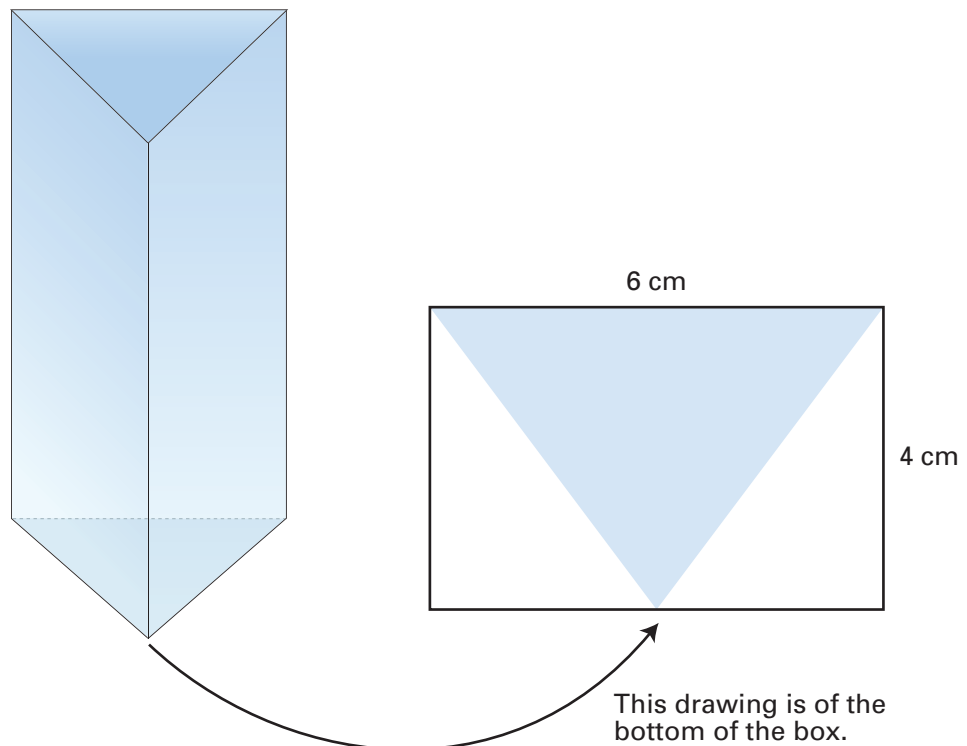
8. Find the volume of one candle.



9. a. Use the nets of mold A and mold B on **Student Activity Sheets 15 and 16** to find the volume of each mold.



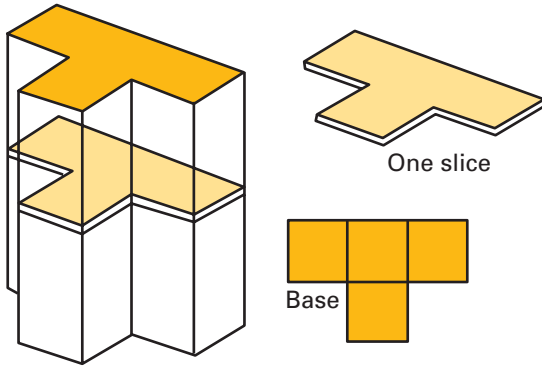
Suppose you have a candle the shape of a triangular prism. The candle is stored inside a rectangular prism with dimensions $4\text{ cm} \times 6\text{ cm} \times 15\text{ cm}$.



- b. Calculate the volume of the candle. Explain your strategy.

From the unit *Reallotment*, you may remember a general rule for finding volumes of three-dimensional shapes.

If a solid can be cut into identical *slices* that all have the same size, then the formula for the volume of the solid is:



$$\text{Volume} = \text{area of slice} \times \text{height}$$

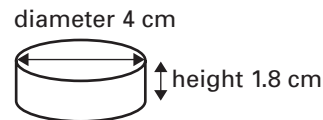
or,

$$\text{Volume} = \text{area of base} \times \text{height}$$

10. You can use this formula for only some of the shapes from page 2. Name the shapes for which you can use this formula and name the shapes for which you cannot use this formula. Explain.
11. Use the formula to check your answers to problems 7–9. Show your work.



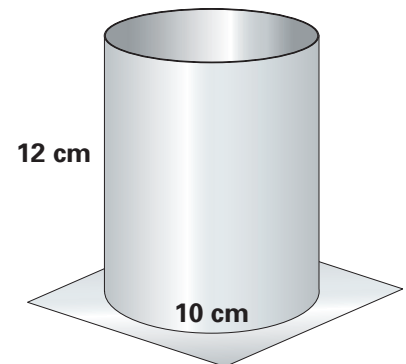
The volume formula also works for cylinders. In a cylinder, the shape of the base is a circle. Here are the dimensions of one tealight candle:



12. a. Find the area of the base of one tealight candle.
b. Find the volume of one tealight candle.

This candle mold has a diameter of 10 cm and a height of 12 cm.

13. Is the volume of this candle mold more or less than one liter? Show your work.

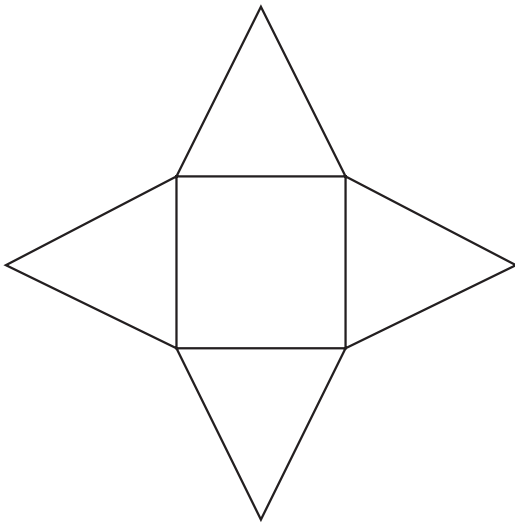


The Height

Activity

Pyramids to Make a Cube

Use the net for a pyramid on **Student Activity Sheet 17**.

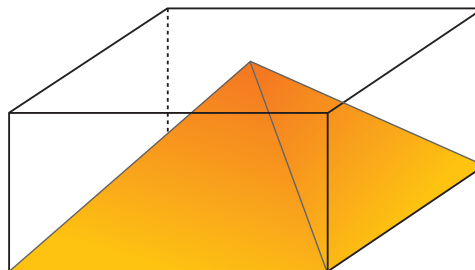


- Cut the net out and fold it into a pyramid.
- Discuss with your classmates how you can determine the height of your pyramid.
- Check the height of your pyramid. It should be 3 cm.

Six of these pyramids can form a cube. Work with five partners to verify this.

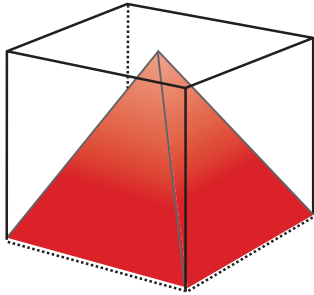
- Find the dimensions of the cube formed with six pyramids.
- Find the volume of one pyramid.
- Check with other classmates to verify your group's results.

Here is a drawing of one pyramid from the previous activity enclosed in a rectangular prism. The prism has the same base and height as the pyramid.



14. a. What is the volume of this prism? Explain.
- b. What is the relationship between the volume of this prism and the volume of the pyramid?

Formulas for Volume



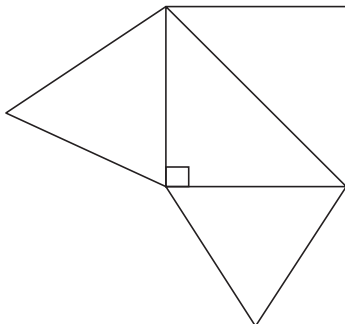
In the previous problems, you discovered a relationship between the volume of a pyramid and the volume of a rectangular prism that had the same base and height as the pyramid. The rule for this relationship is:

The volume of a pyramid is $\frac{1}{3}$ of the volume of a prism with the same base and height.



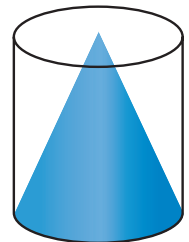
This candle mold at the left is a square pyramid. The height of the mold is 6 inches, and the base is a square with side lengths of 4.5 inches.

15. a. Find the volume of a prism with the same square base and height.
- b. Calculate the volume of this square pyramid.



16. a. Make the pyramid using the net on **Student Activity Sheet 18**.
- b. Find the volume of a triangular prism that has the same base and height as the pyramid. You may measure the height of your pyramid.
- c. Find the volume of the pyramid.

The volume of a cone relates to the volume of its corresponding cylinder in the same way the volume of a pyramid relates to the volume of its prism.



17. a. Write a rule for finding the volume of a cone.
- b. Show how you can use your rule to find the volume of a cone with a height of one foot and a base diameter of 8 inches.



Pastry chefs use pyramid molds to decorate culinary presentations. The molds are made of stainless steel and often used to shape ice cream.

18. Find the volume of one square pyramid with an edge length of three inches.



Volume

Summary

The liter is a unit for volume. You can also measure volume in cubic centimeters (cm^3). The relationship is:

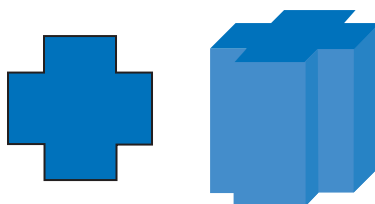
$$1 \text{ liter} = 1,000 \text{ cm}^3$$

One liter of water completely fills up a cube of $10 \text{ cm} \times 10 \text{ cm} \times 10 \text{ cm}$, which is also one cubic decimeter (1 dm^3).



Formulas for Volume

If a solid can be cut into identical *slices* that all have the same size, then the formula for the volume of the solid is:



$$\text{Volume} = \text{area of slice} \times \text{height.}$$

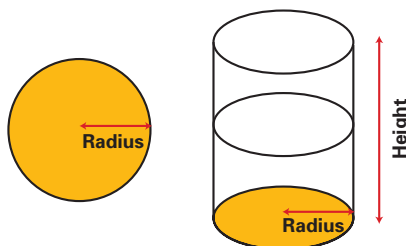
or

$$\text{Volume}_{\text{prism}} = \text{area of the base} \times \text{height}$$

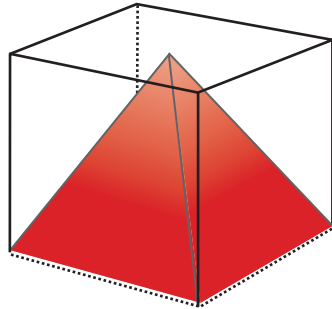
The formula works also for cylinders

$$\text{Volume}_{\text{cylinder}} = \pi \times \text{radius} \times \text{radius} \times \text{height}$$

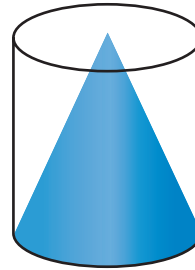
Area of the base,
which is a circle.



The slices are not identical for pyramids and cones, so this formula does not work. For these shapes you can use:



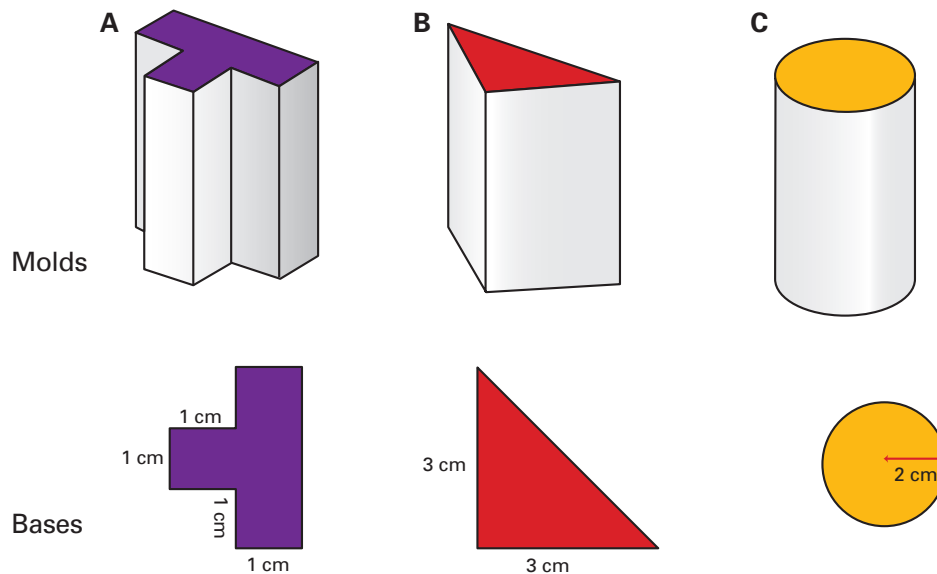
$Volume_{pyramid} = \frac{1}{3}$ of the volume of a prism with the same base and height.



$Volume_{cone} = \frac{1}{3}$ of the volume of a cylinder with the same base and height.

Check Your Work

Here are three different molds. Each mold has a height of 4 cm.



1. Use the drawings of their bases to calculate the volume of each mold.



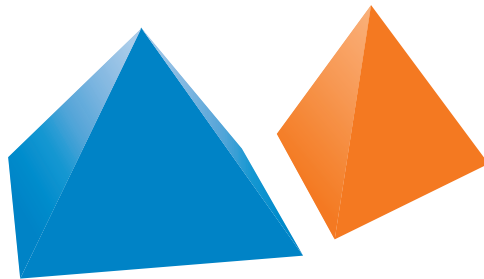
Volume

This juice can is 18 cm high and measures 8.5 cm in diameter.

2. Is the volume of the can more or less than one liter?



Here are two molds.



One pyramid mold has a square base with side lengths of 4 cm.

The base of the other pyramid mold is an equilateral triangle with side length of 6 cm.

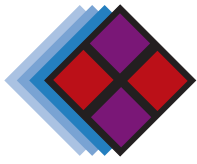
The height of both molds is 6 cm.

3. Which pyramid mold has the larger volume? How do you know?



For Further Reflection

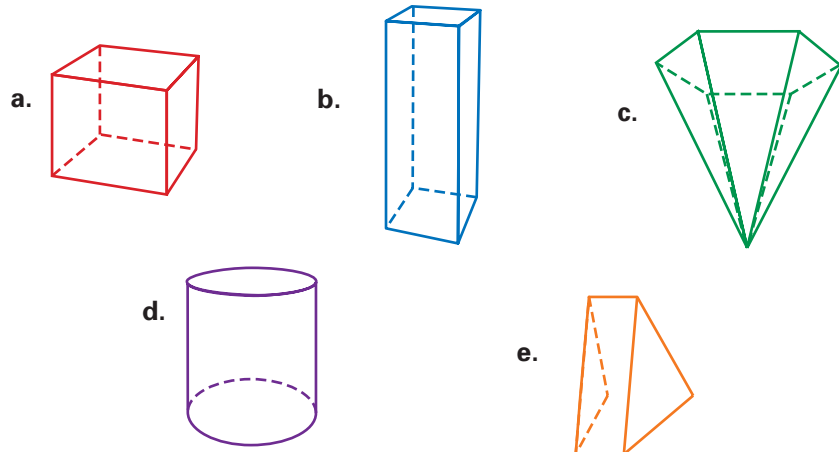
Think of three different shaped food containers that come in different sizes for different volumes. How are the dimensions of the container different when the volume is increased?



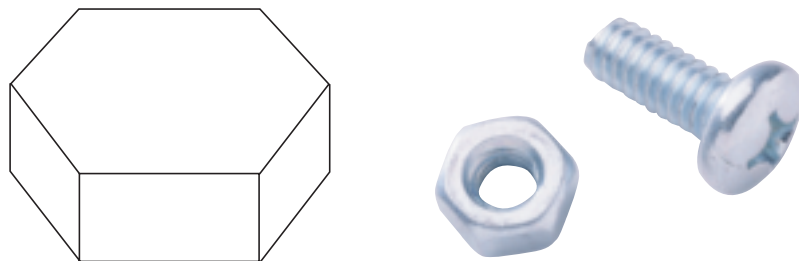
Additional Practice

Section A Packages

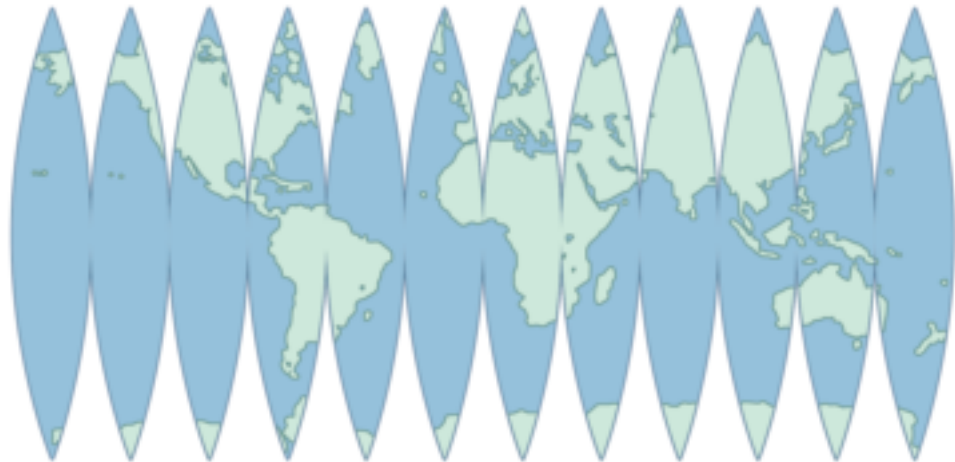
1. Name a familiar object that has the following shape.
 - a. sphere
 - b. rectangular prism
 - c. cone
 - d. cylinder
2. Use the proper names for the shapes in your answers to these questions.
 - a. What features are common in **a**, **b**, and **d**? What are different?
 - b. What features are common in **c** and **e**? What are different?



This is a prism. A bolt looks like a prism with a circle cut out of the interior.



3.
 - a. How many faces are hidden?
 - b. Draw a net of this shape.



4.
 - a. Suppose you cut this out and fold it into a shape. What will you get?
 - b. Will the shape become one of the shapes that are on page 2? Explain.
 - c. Which shapes on page 2 have only flat faces?

Section **B** Bar Models

Maha has six bars: three are 6 cm long, and three are 4 cm long. She makes a triangle with one of the long bars and two of the shorter ones.

1.
 - a. As carefully and accurately as you can, draw this triangle.
 - b. Draw a triangle with sides 4 cm, 6 cm, and 6 cm long.

Maha tries to make a prism with her six bars but finds it impossible.

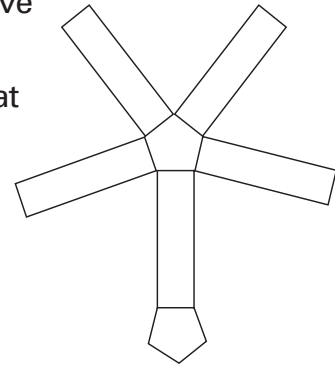
2.
 - a. Why can't Maha make a prism using just six bars?
 - b. How many more bars does she need?
 - c. What are possible lengths for the additional bars? Include a drawing to explain your answer.
 - d. Would the prism you drew for part c be stable? Explain.



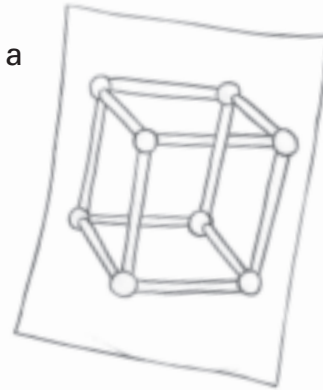
Kim has a piece of wire that is 100 cm long. She will cut the wire into pieces to build a pyramid. She wants the bottom of the pyramid to be a square. She also wants to use all the wire.

3. a. How many pieces of wire does she need to build the pyramid?
- b. Many lengths are possible for Kim's pieces of wire. Describe two of these possibilities.

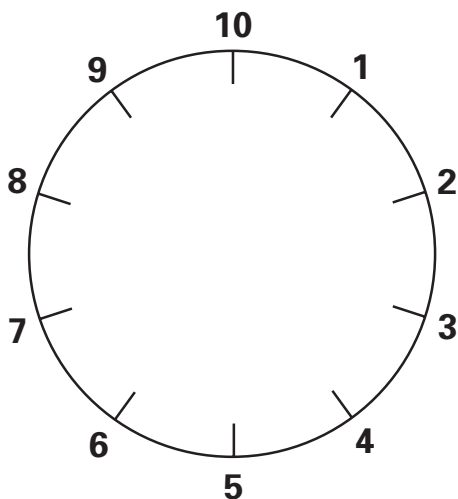
4. a. How many faces does the shape have that can be folded with this net?
- b. And how many vertices in the net at the right? And how many edges?
- c. What is the name of this shape?
- d. How many face diagonals will the shape have?
- e. And how many space diagonals? Show how you found your answer.



5. Rajeev makes this drawing of a bar model of a cube. Can you build this model? Explain.

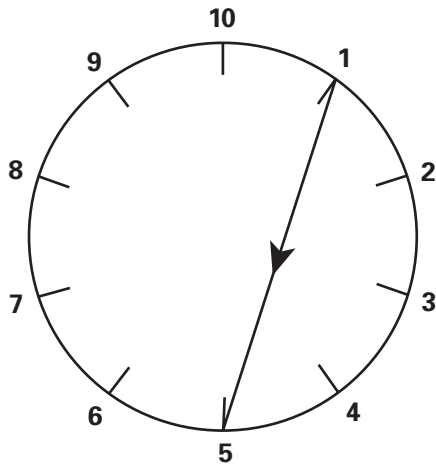


Section G Polygons



Use **Student Activity Sheet 19** for the following problems.

1. Which polygons can be made with equal-size jumps from one whole number to another on this diagram?
2. Do you see a relationship between your answer to problem 1 and the numbers on the diagram? If so, describe the relationship.



3. a. Use **Student Activity Sheet 19** and a straightedge to make the following drawing.

Start at number one and make jumps of four hours, going around the dial, until you are back at number one. The first jump is already drawn. The result is a star with exactly five points.

- b. What polygon do you see inside the star?
- c. How many degrees are in each angle that forms the points of the star? Explain how you found your answer.
4. a. Start again at number one and make equal-size jumps so you will get a star with a different number of points.
- b. How many degrees are in each angle that forms the points of this star? Explain how you found your answer.

Section **D** Polyhedra

1. Picture a pyramid that has between 100 and 200 edges.
- a. Choose a number of edges for your pyramid (between 100 and 200). Is Euler's formula still valid for the pyramid you chose? Explain.
- b. Is Euler's formula valid for a pyramid with any number of edges between 100 and 200? Explain.



A soccer ball is similar to a sphere with a diameter of about 22 centimeters. The ball in the photograph is made of black and white pieces of leather.

There are differences, besides color, between the black and the white faces.

2. a. What other differences do you see?
- b. On the soccer ball in the picture, you can see six black pieces. How many black pieces do you think are on the whole soccer ball?
- c. How many white pieces are on the whole soccer ball?



This shape is an icosahedron.

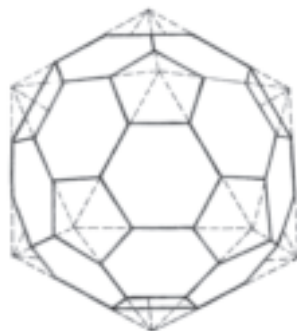


3. a. Explain how a soccer ball is related to an icosahedron.

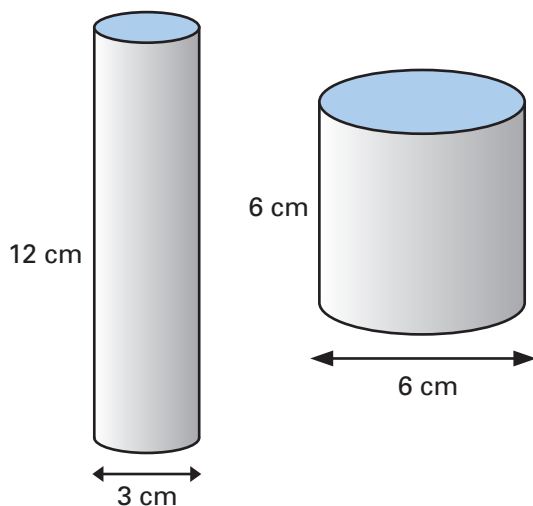
Tim reasons as follows:

A soccer ball has 12 black pentagons. Each pentagon borders five white hexagons. Therefore, the number of hexagons has to be $12 \times 5 = 60$.

- b. What is Tim's mistake?
 c. Investigate to see whether Euler's formula is valid for a soccer ball.



Section E Volume



Here are two molds. The second mold is twice as wide, but half as high as the first one.

Jesse thinks that both molds have the same volume.

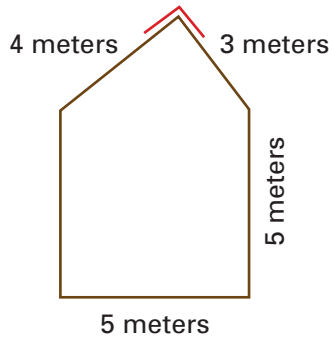
1. Is Jesse right? Explain.



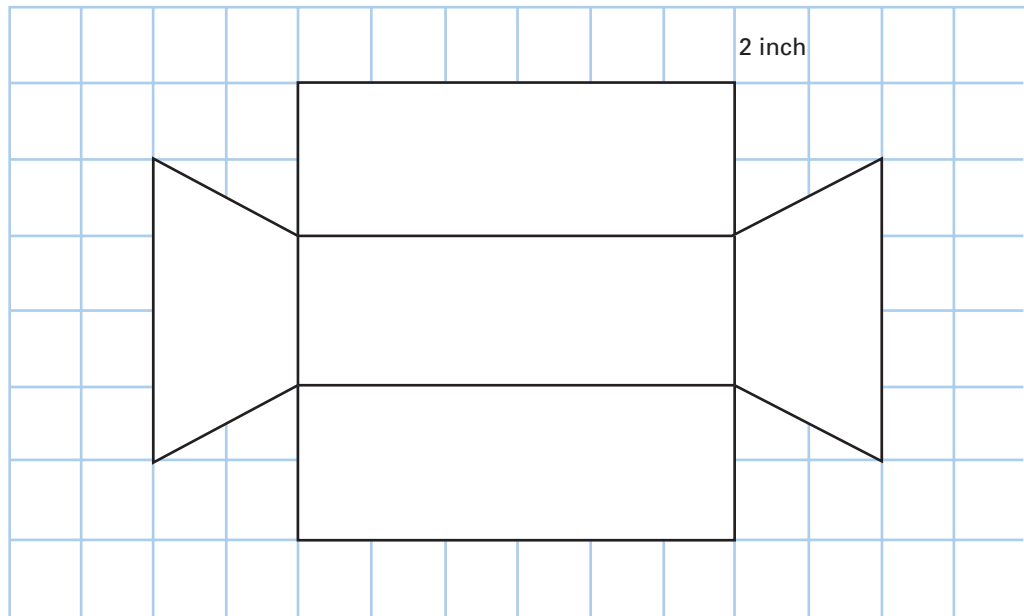
Additional Practice

Ms. Berkley wants to have a cooling system in her summer house. In order to buy a system that will be efficient for her house, she needs to calculate the volume of the house.

Here you see the measurements of her house.



2. Calculate the volume of the house in cubic meters.
3. Find the volume of the shape that can be built with this net.



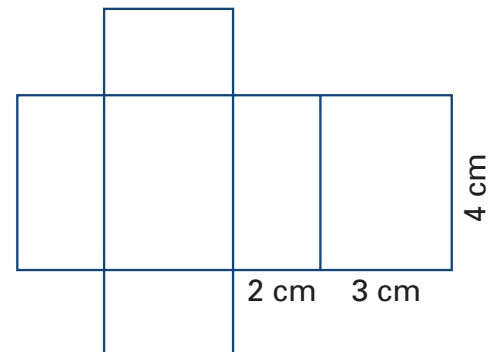


Section A Packages

1. Descriptions may differ, but the mathematical names for the shapes should be the same as you see below.
 - a. baseball—sphere
 - b. suitcase—rectangular prism
 - c. donut box—prism
 - d. domed building—cylinder; half a sphere on top
 - e. wedge of Swiss cheese—prism
 - f. barrel—cylinder
 - g. sugar container—truncated cone
 - h. party hat—cone
 - i. pizza box—prism
 - j. Egyptian pyramids—pyramid
2. Discuss your answers with a classmate. You might choose to say something like the following:
 - a. A pyramid, a prism, and a sphere—the pyramid and the prism have edges that are straight lines, and the sphere does not have any straight edges.
 - b. A cube, a prism, and a cylinder—the cube and the prism have sides that are made out of straight lines while the cylinder has one part that is made out of a circle.
3. Your net should have two rectangles of 4 cm by 2 cm, two rectangles of 4 cm by 3 cm, and two rectangles of 2 cm by 3 cm.

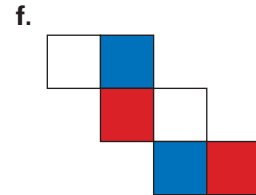
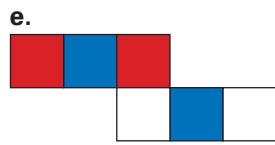
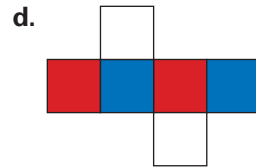
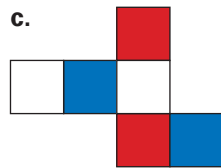
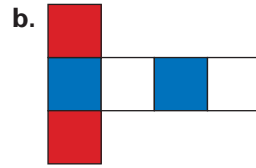
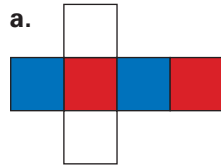
Here is one example of a net you may have drawn; you could have different ones as well.

You can check your design if you cut it out and fold it into a rectangular prism.





4.

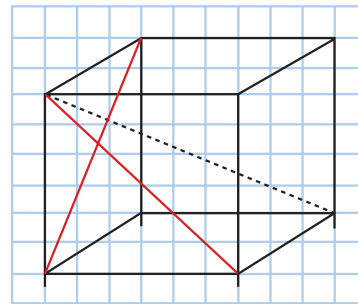


Other solutions can be found by exchanging blue and white.

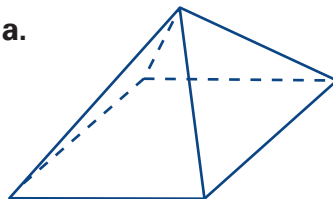
Section B Bar Models

1. a.-c.

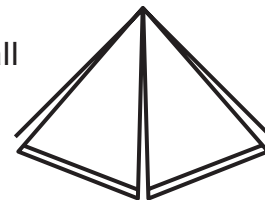
(Note that the space diagonal is drawn as a dotted line.)



2. a.



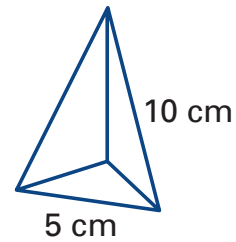
b. The pyramid built with six straws has all triangles, so it will be more stable than the pyramid with one square as face. The bar model shown in a could be folded flat if the vertices are flexible.



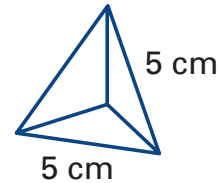


3. He can make three different models:

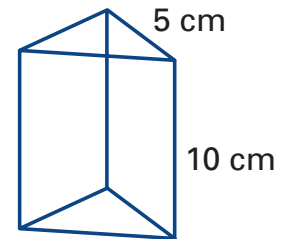
The first one is a pyramid with a triangular base with sides of 5 cm. The other three edges are 10 cm.



The second model is also a pyramid with a triangular base, but now all edges of the pyramid are 5 cm.



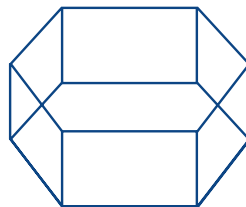
The third possible model is a prism. The top and bottom faces are triangles with sides of 5 cm, and the edges of the rectangles are 10 cm.



Note that a pyramid with a triangular base with sides of 10 cm is not possible!

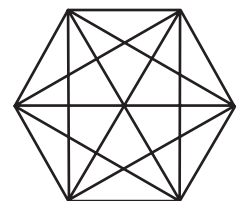
4. a. The prism has eight faces in total. Four faces are hidden.
 b. Two vertices are hidden.
 c. Six faces have the shape of a rectangle.

d.



e. Faces: 8; vertices: 12; edges: 18.

- f. Hint: To find the number of diagonals in the top face, you can draw the shape and find all diagonals. Count them while you are drawing!

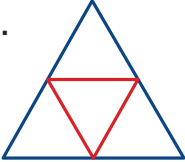


The total number of face diagonals of the prism is 30. The top and bottom face have nine diagonals each (2×9), and the other faces have two diagonals each (6×2), so in total: $6 \times 2 + 2 \times 9 = 30$ diagonals.



Section C Polygons

1. a. Show your answer to a classmate. You should have drawn a square with four equal sides and four right angles. This one is a regular polygon, because the sides and angles are equal. The other shape (a rhombus) has four equal sides, but the angles are not equal.
- b. As the number of sides increases so does the size of each interior angle.

2. a.  An equilateral triangle. You may have used "all sides are equal," or "all angles are 60 degrees," in your explanation.

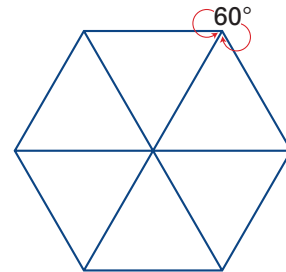
- b. You may have used one of the strategies below.

- Using turns:

$$360^\circ \div 3 = 120^\circ$$

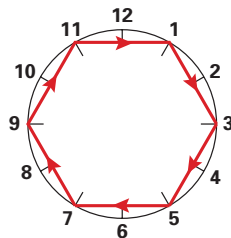
$$180^\circ - 120^\circ = 60^\circ$$

- Using six regular triangles:

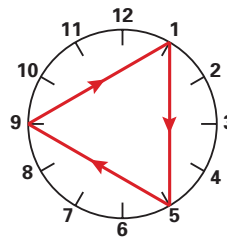


$$360^\circ \div 6 = 60^\circ$$

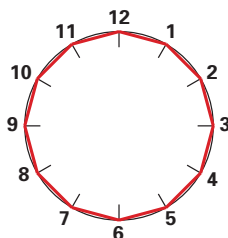
3. a. A hexagon is formed.



- b. A triangle is formed.



- c. A dodecagon is formed.



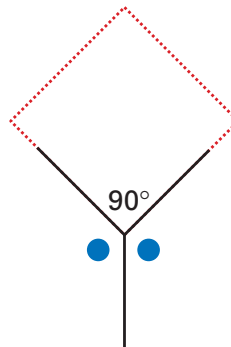


4. a. There are square tiles and octagons.
 b. 135° . Your strategy may differ from the strategy shown here.

- The number of equal turns you make walking around an octagon is eight. Now divide 360° by $8 = 45^\circ$ to find the angle of each turn. The interior angle equals 180° minus the angle of the turn.
- The sum of all the angles where the three polygons meet is 360° . The angle in the square tile is 90° .

So the two other angles together are $360^\circ - 90^\circ = 270^\circ$.

These two angles are equal, so each of them is $270^\circ \div 2 = 135^\circ$.



Section D Polyhedra

1. Hint: Look at the pictures of the five Platonic solids in the Summary.

For each Platonic solid, you can say that the faces are regular polygons, and an equal number of edges must meet at each vertex. The faces of three Platonic solids are all regular triangles. The other two have faces that are all squares or faces that are all regular pentagons.

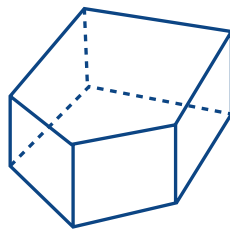
2. a. He is thinking the back and the front are the same but is forgetting that four of the edges that are visible in the picture are shared by the back of the octahedron. He counts these four edges twice.
 b. Since he counted four edges twice, just subtract four of the sixteen edges that Jonathan got.

So $16 - 4 = 12$ is the number of edges of the octahedron.



Answers to Check Your Work

3. First you have to find out how many vertices and edges an icosahedron has.
One icosahedron has 12 vertices and 30 edges.
An octahedron requires six vertices and 12 edges, so two octahedrons require twice as many, or 12 vertices and 24 edges.
Now there are six edges left but no vertices, so she cannot make more than two octahedra.
Answer: Toni can make two octahedrons from the bar model of one icosahedron
4. Yes:
A cube has six faces, so $F = 6$
Eight vertices, so $V = 8$
12 edges, so $E = 12$
 $F + V - E = 6 + 8 - 12 = 2$.
5. Hint: To find the number of vertices, edges, and faces, you can change the drawing so it shows a bar model of the shape or so that it shows the invisible edges, faces, and vertices.



Now you can see that this shape has:

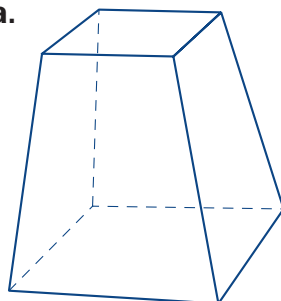
Seven faces, so $F = 7$

$2 \times 5 = 10$ vertices, so $V = 10$

$3 \times 5 = 15$ edges, so $E = 15$

$F + V - E = 7 + 10 - 15 = 2$. YES!

6. a.



b. Yes, the formula holds.

$F = 6, V = 8, E = 12$

$6 + 8 - 12 = 2$



Section **E** Volume

1. **A:** The volume is 16 cm^3 .

You can use different strategies to calculate the volume.

- One strategy is that you split up the shape into four rectangular prisms with a square base with sides of 1 cm and a height of 4 cm.

The volume of one prism is $1 \times 1 \times 4 = 4 \text{ cm}^3$. So the volume of the mold is $4 \times 4 \text{ cm}^3 = 16 \text{ cm}^3$.

- Or you calculate the area of the base of the mold, which is 4 cm^2 .

The volume of the mold is area of the base \times height, which is $4 \times 4 = 16 \text{ cm}^3$.

B: The volume is 18 cm^3 .

Maybe you used one of the following strategies to find the volume:

- One strategy is that you calculate the volume of a prism with a square base with sides of 3 cm and a height of 4 cm. Then the volume can be calculated with $3 \times 3 \times 4 = 36 \text{ cm}^3$. And then you take half of this volume for this mold. So the volume of the mold is 18 cm^3 .
- Or you can calculate the area of the base of the triangular prism, which is 4.5 cm^2 . The volume is area of the base \times height, which is $4.5 \times 4 = 18 \text{ cm}^3$.

C: The volume is about 12.6 cm^3 .

The area of the base is $3.14 \times 1 \times 1 \approx 3.14 \text{ cm}^2$.

The volume is area of the base \times height, which is $3.14 \times 4 \approx 12.6 \text{ cm}^3$.

2. The diameter of the can is 8.5 cm, so the radius is 4.25 cm.

The area of the base is $3.14 \times 4.25 \times 4.25$, which is about 63.6 cm^2 .

The volume of the can is area of the base \times height, which is about $63.6 \times 18 \approx 1,144.8 \text{ cm}^3$. This is more than one liter because 1 liter is $1,000 \text{ cm}^3$.



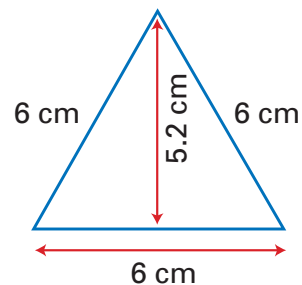
Answers to Check Your Work

3. The pyramid with the square base has the larger volume. Your strategy may differ from the strategy shown here. Share your strategy with the class if you found another one.

Sample strategy: The volume of a pyramid is $\frac{1}{3}$ of the volume of a prism with the same height. Both pyramids have the same height, so the only important part is the surface area of their base. The area of the base of the pyramid mold with the square base is the easiest to find:

$$4 \times 4 = 16 \text{ cm}^2.$$

For the pyramid mold with a triangular base you could make a drawing of the triangular base in its actual size and measure the height. (Note that this drawing is to scale.)



The area is about $\frac{1}{2} \times 6 \times 5.2$ or 3×5.2 , which is 15.6 cm^2 .

So the volume of the pyramid with the triangular base is less than the volume of the pyramid with the square base.