Teaching geometry through play

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Geometry begins with play.
Pierre van Hiele

ABSTRACT. In this short note we present an inquiry based approach starting from a puzzle playing activity for teaching the notion of equidecomposability and the Wallace-Bolyai-Gerwien theorem. This approach can be used from lower secondary school (13-14 years old students) to teacher's professional development courses in order to gain a deep understanding of the notion of area for polygons.

KEYWORDS: equidecomposability, area, Wallace-Bolyai-Gerwien theorem, inquiry based learning

MATHEMATICAL SUBJECT CLASSIFICATION: 97G30, 97U60

Introduction

In the last years the use of inquiry based approach in teaching mathematics and science was recommended by several professional forums (see [7], [8]) and also by high level educational policy makers (see [2]). Recent studies shows that this approach can be effective (see [4]) by using well designed teaching materials (see [9], [10]) and carefully chosen pedagogical methods ([11]). The aim of this paper is to sketch a possible approach for teaching the concept of area and also a common technique for planning IBL activities by acting more on the didactic milieu and less on the didactic contract (see [1], [5], [6]). The main focus is not on the introduction of the basic concepts, the connections with the measuring procedures or the major misconceptions that can appear but on the concept of equidecomposability. The polygons P_1 and P_2 are equidecomposable if both P_1 and $P2$ can be decomposed into the same finite set of polygonal pieces $PS_1, PS_2, \dots PS_k$ (see Figure 1). This means that both P_1 and P_2 can be fully covered with the pieces $PS_1, PS_2, \dots, PS_k$ (without overlapping). We wanted to create a context in which students can understand the Wallace-Bolyai-Gerwien theorem which states that for polygons the area is in fact a characterization of the equidecomposability (two polygons are equidecomposable if and only if they have the same area).

This property is a very important and very often neglected aspect of the polygon's area. On the other hand this property gives a background understanding for the basic formulas and proofs and can eliminate some common misconceptions related to areas. For a better understanding of the prerequisites we also list a few activities that can be used in order to acquire them.

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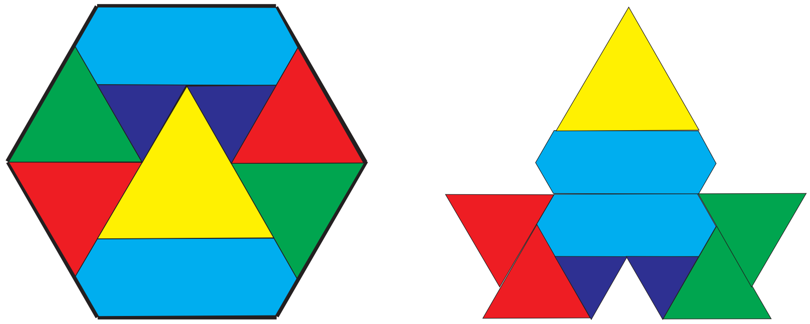

Figure 1: Equidecomposable polygons

Puzzles and playing with them

We designed a few puzzle type kits (see the appendix) made of decorative rubber in which the pieces can be manipulated separately. Each puzzle is in fact a decomposition of a regular polygon. We have 4 basic decompositions for a square (denoted by $S1, S2, S3$ and $S4$), 6 decompositions for pentagon ($P1, \dots, P6$), 2 decompositions for hexagon ($H1$, $H2$) and 2 decompositions for octagon[3] ($O1$, $O2$). These decompositions have the property that the pieces of each puzzle can be rearranged in several different ways to obtain different figures. From this point of view our puzzles are similar to the well known Tangram puzzle (and also to the puzzle presented in [3]), but for us it is essential (for educational aim) to reduce the number of rearrangements. For this reason $S3$ is obtained from the traditional Chinese Tangram by reducing the number of pieces (we joined two pieces of the Tangram). For each puzzle we designed a few figures that can be obtained by rearranging it's pieces. These figures are given by their contours (see the Appendix).

The basic context. We are living in the castle of Dream King who has a special magic room. The floor of this room is completely covered by a magic carpet which provides the power of the king. This carpet can be cut in pieces and moved to any other room, but it's magic power appears only if it covers the whole floor of the room, all the pieces are used and the pieces are not overlapped. In this context the pieces of a puzzle are representing the pieces of the carpet and the figures are representing the rooms. □

The students are working in groups and each group has several assignments on the following levels:

1. Level 1 : Finding the arrangement of pieces for a given puzzle and a given figure, if the figure and the pieces are given in full size.
2. Level 2 : Finding the arrangement of pieces for a given puzzle and a given figure, if the figure is not given in full size.
3. Level 3 : Move the carpet from one room to another room, if both floor maps are given in full size.
4. Level 4 : Move the carpet from one room to another room, if one of floor maps is given in full size, while the other is not.

[3] This is a basic collection, the full collection has more than 25 puzzles

5. Level 5 : Provide a guide to the employees, who are moving the carpet in order to help their work.

Figure 2: Finding the arrangement for a given puzzle and frame

Although level 1 is easy for most puzzles, some of them are easier than others ($H1$, $H2$ are the easiest while $O1$ and $O2$ are the most difficult), so teachers can choose the best puzzle for a group, they have some time control option by choosing the puzzles (during the activities) and also they can give additional assignments on the same level to the groups. This level is suitable for all ages.

Figure 3: Level 1 is easy for most puzzles

On level 2 some problems appear due to the fact that the figure is not full sized, so students can't use the side lengths in their attempts (or first they need a good estimation of the similarity ratio). These problems provide lot of opportunities to clarify important aspects about similarity (specially with younger students).

The third level is more complex and needs a crucial observation. In order to facilitate the learning process we need a good strategy in choosing the puzzles and the figures at the first two levels: each group has to play with at least two different puzzles labeled with the same first letter (two different decompositions of the same polygon) and from each puzzle they have to assemble the initial polygon itself and at least one different figure.

Figure 4: Level 2 is funny, but not so trivial

After this each team has to prepare a poster on which they illustrate the arrangement of pieces for two figures and the corresponding arrangement for the polygons. Working with younger students (12-14 years old) it is recommended to use the same polygon on each poster, while working with older students it is important only that some polygons should appear at least twice (with two different decompositions) on the posters and with each of these decomposition there should be another figure on the posters.

Figure 5: Preparing the posters

After the posters are prepared we can pose the most important questions. For the sake of simplicity we exemplify the questions and some possible answers by using the puzzles O1 and O2 (but this can be done with any pair of decompositions belonging to the same polygon). The main problem is how to move the magic carpet from one figure to the other (see Figure 6).

This is a very difficult problem if we attack it frontally, without preliminary steps. The trick is that by preparing the posters we construct a big part of the solution. We need only two crucial observation. Due to the posters we can see that pieces from both *a*) and *b*) can be rearranged to form an octagon. Moreover the two octagons are congruent,

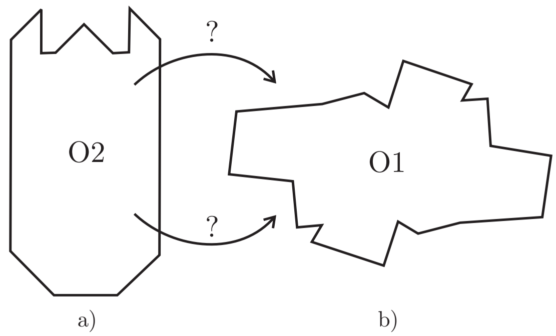

Figure 6: The main question: how to move the carpet from a) to b)

so it is sufficient to move the pieces from the first octagon to the second one eventually by cutting them into smaller pieces (this is the first crucial observation, which can be found also by a little help from teacher's side with well posed questions). The second main observation is that we can simply overlap the two decompositions and we have to cut along all the line segments in order to obtain a decomposition from which we can obtain both the pieces from $O1$ and $O2$ (practically we decomposed each piece of $O1$ and $O2$ into smaller pieces).

Remark 1. *At our activities (with 12-15 years old students at the "Florea Bogdan" Highschool in Reghin, with 15-18 years old students at the "Márton Áron" Highschool in Miercurea Ciuc, with elementary school teachers at the National Drama Teaching Days at Oradea, with preservice teachers and with highschool teachers at Mascil piloting actions at Cluj Napoca and Rimetea) both crucial observations were found by almost all the groups after the posters were made and posted in the classrooms. Moreover for teachers and a few groups of highschool students this was sufficient also to construct a proof of the Wallace-Bolyai-Gerwien theorem. For younger students and most of the older students, further additional steps were needed in order to reconstruct the Wallace-Bolyai-Gerwien theorem. These steps were the following:*

1. *transform a triangle into a rectangle;*
2. *transform a rectangle to a rectangle having one side of fixed length (we can suppose that this length is* 1*);*
3. *transform a polygon into a rectangle having one side of fixed length (we can suppose that this length is* 1*);*

Figure 7 shows the pieces from which we can construct both decompositions of the octagon and so both figures a) and b) from Figure 6. This gives a possible decomposition of these figures into the same set of pieces.

After this, student can go to level 3. For two given rooms they have to find (choose) a third room (a figure) with the following property: the carpet can be moved from both initial rooms to the third room. Here the main trick is that in fact they can choose any shape, hence it is convenient to work with simple shapes (triangles, rectangles, etc.). If the groups have problems with this level, teachers can help them by fixing the third room (a rectangular room). At this point a very simple question has to be answered: if from P_1 we

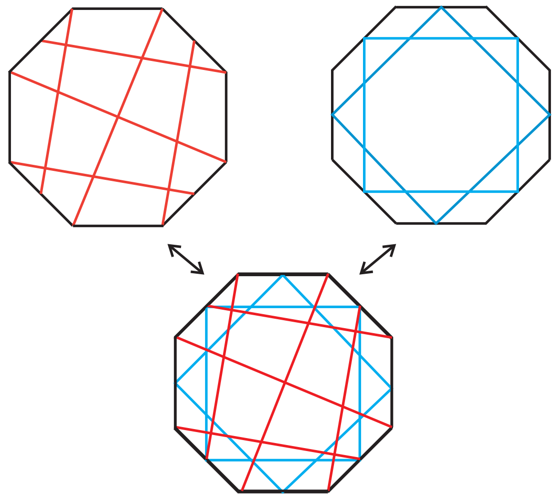

Figure 7: Overlapping two decompositions

obtain the rectangle R_1 and from P_2 the rectangle R_2, how can we transform R_1 into R_2? This shows that this is an important subproblem of the initial problem. It is possible that working with experienced problem solvers this problem appears at the beginning as part of usual heuristics. If all the sides are rational numbers (or are rational multiples of the same length s), than we can decompose both rectangles into the same set of rectangular pieces. This is guaranteed by the area measurement algorithm which has to be prepared at previous activities. To avoid further discussions about the nature of the side lengths we can give students additional auxiliary problems, or we can ask them to formulate such problems: how to transform an 1×2 rectangle into a square, how to transform a rectangle to a rectangle having a side of length 1.

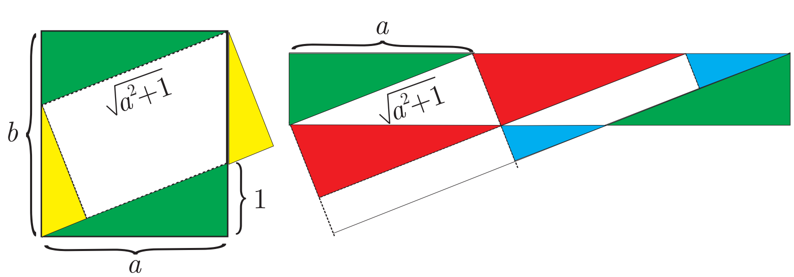

Figure 8: Equidecomposability of rectangles

After understanding that two rectangles are equidecomposable if and only if they have the same area we can formulate the following properties:

1. all polygons are equidecomposable with a rectangle having one side equal to 1;

2. all polygons that are equidecomposable with the rectangle $1 \times T$ are equidecomposable.

The last property is in fact the Wallace-Bolyai-Gerwien theorem and it shows that the number T can be interpreted as the area of the polygon if we assume that the area is invariant under decomposition and rearrangement of pieces.

Remark 2. *As a consequence we can observe that if we decompose a polygon into triangles, the sum of areas of triangles from the decomposition is not depending on the decomposition, hence the area of the polygon can be defined as the number obtained by adding up the areas of triangles from an arbitrary decomposition.*

In order to solve Level 4 it is necessary to understand how the area of a polygon changes if we apply a similarity transformation (zoom in or zoom out) to the polygon, hence this problem can be reduced to the previous one after resizing one of the figures. Level 5 was introduced in order to force the students to explain all their findings, observations, proofs and to focus also on how to simplify these explanations.

A deeper insight into our play

Before attacking this problem it is recommended to organize several preliminary activities about

1. estimating the area of complex shapes by constructing different coverings;
2. measuring areas for rectangles whose side lengths are natural and rational numbers;
3. equidecomposability of rectangles;
4. measuring the area of complex shapes by decomposition or completion;
5. similarity, the relation between similarity and area.

Remark 3. *Throughout these activities the concept of area can (and sometimes should) be avoided or even developed by referring to the measuring procedure.*

At the first two levels students are supposed to play with the puzzles. At the third level as we have described the milieu needs to be enriched by pedagogical techniques (producing and analyzing the posters) in order to move from the playing attitude towards a more scientific inquiring attitude. Starting from this point teachers should be prepared to facilitate students in focusing to the important problems and finding answers. This facilitation mostly depends on what students do, so teacher has to be prepared in advance with a large palette of additional questions, materials, he has to process in real time a lot of different viewpoints, arguments, otherwise the students' knowledge construction should easily fail or the whole activity will turn into a frontal teaching. As we mentioned two important aspects appeared: the estimation of sizes when the figures are not full sized. Most groups of teachers completely avoided this, they wanted to calculate effectively the similarity ratio (or some lengths) instead of giving a good approximation (Figure 9 shows a group of teachers using an estimation for the hexagon they had to construct). On the other side no group of students wanted to calculate the similarity ratio effectively (by calculating areas), they gave approximations for the length of sides, the diameter of figures etc. Most of the time these approximations proved to be more usable than the exact calculations (which were also exaggeratedly time consuming).

We observed also another major difference in how teachers (when they were acting as students) and students were handling situations. After the posters were posted and they had to move the carpet (remember Figure 6) most of the teachers tried to find some connection between angles and side lengths of the two figures or some kind of "principle" to describe where to move the pieces. On the other hand students simply started to move the pieces one by one and cutting the parts hanging out in order to rearrange these new parts at a later time (a kind of greedy algorithm in moving the pieces).

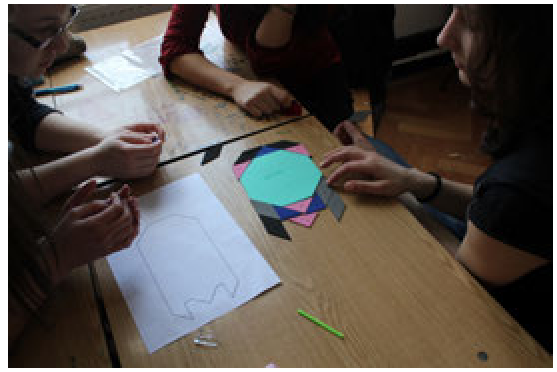

Figure 9: Estimation and similarity are important

The previous observations show that there is a major difference in how the students perceive situations, how they act in these situations and how teachers perceive, act in the role of students while facing the same situations. These differences usually represent a hindering factor in planning the IBL activities because during the planning phase teachers try to anticipate possible reactions of students. During the planning of a traditional lesson, where teachers usually act directly on the didactic contract, this problem is not so serious, but in an IBL framework teachers need to enrich the milieu in order to facilitate the learning process. The enriched milieu can rise a lot of additional problems, reactions and teachers have to handle these.

The problem of estimation for the size of figures (as in the first photo of Figure 9) and the problem of deciding whenever an arrangement is correct for a given figure (as in the second photo of Figure 9) are typical problems that arise because of the enriched milieu (in contrast to the equidecomposability of two rectangles, which appears in every context because it's related to an important step in the proof). These problems can be avoided if we only use full sized frames, during the activity and we skip level 2. If we think about

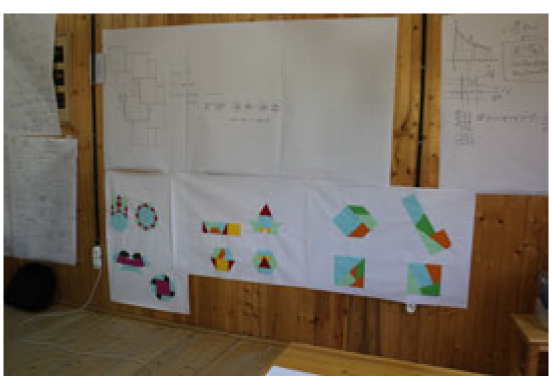

Figure 10: The posters and discussions about how to use the intermediate arrangements

Figure 11: Working with young students: the explications through language sometimes fail

the context this should imply that we avoid handling real situations, when the floor plans are given at a reduced size (the floor plans of a building are usually not full sized), so we lose a part of authenticity. In IBL activities these kind of compromises usually are not fruitful.

Working with different age groups we had to face different additional problems. Some 12-15 years old students had problems in explaining by words how they think the pieces can be rearranged, while they actually rearranged them. This points to a problem that is growing in the recent period, some mathematical and scientific ideas, observations, young students can understand are beyond their linguistic capabilities. This is also valid in the opposite way: sometimes they hardly understand ideas explained beyond their linguistic capabilities. The IBL activities based on manipulative artifacts can overcome a good part of these types of problems.

Conclusions

1. At such an activity the greatest problem is how to convince students to stop playing and to start doing mathematics. This has to be handled through the milieu.

2. We used this activity with several different groups of students and several different groups of teachers. At all the activities the main question (how to characterize all the rooms in which the magic carpet can be moved) arose. Moreover depending on the age group a lot of other aspects needed special attention:

   (a) with 12-14 years old students the similarity caused a lot of problems even for simple (but not trivial) puzzle kits;

   (b) the 14-18 years old students had problems with the estimation of sizes;

   (c) some university students and teachers wanted first to calculate the area of the polygons, and in some cases this led them to difficulties.

3. This approach was developed in order to give an opportunity to the learners to prove and to illustrate the transitivity of the equidecomposability relation. This goal has been attained at all the activities, all groups observed that by overlapping the two nets we can determine a common subnet, which gives a finite cutting set. After this

phase some groups were able to construct the proof of the Wallace-Bolyai-Gerwien theorem without any further hint, but most of the groups needed additional hints.

4. The activity we presented shows that in an IBL approach the content has to be completely restructured and adapted to the logic of the context we are using. The transitivity of the equidecomposability doesn't need the notion of area, hence we can use the presented activity with many different purposes: complementing the knowledge of our student after they have learnt about the area; building the concept of area using this approach and the activities for the measurement procedure in parallel or even building the concept of equidecomposability before teaching the concept of area. These purposes will determine the main course of the discussions and the actual structure of the content we are trying to build.

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0.1 Appendix - The puzzles

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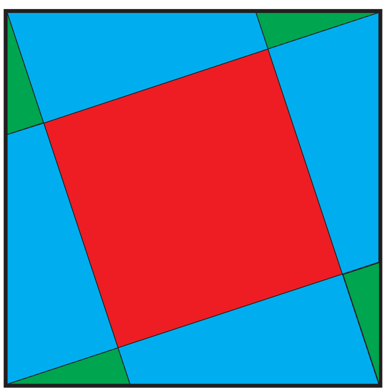

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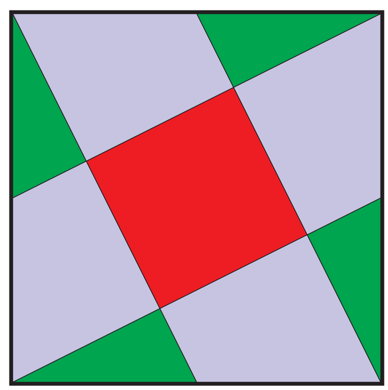

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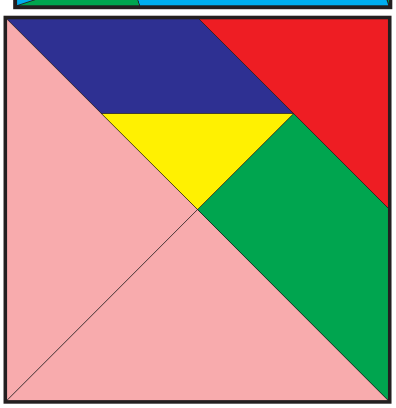

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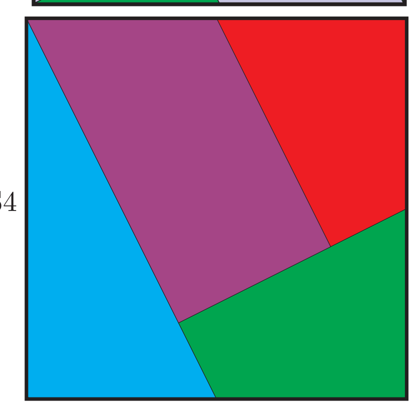

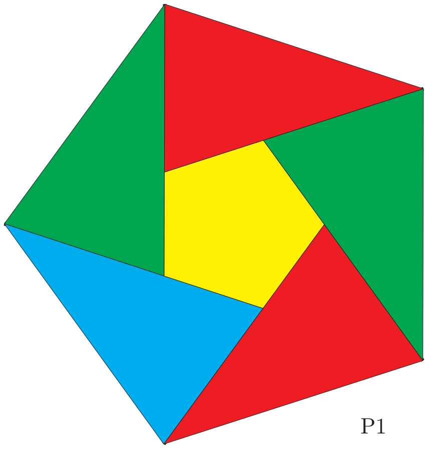
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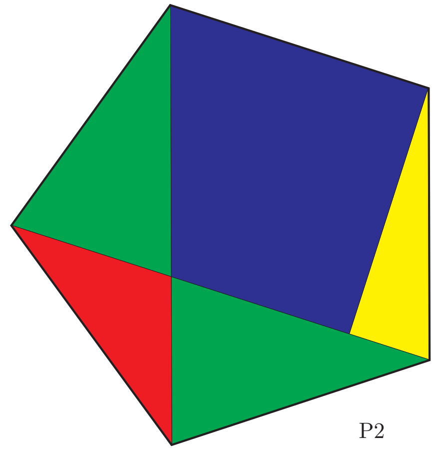
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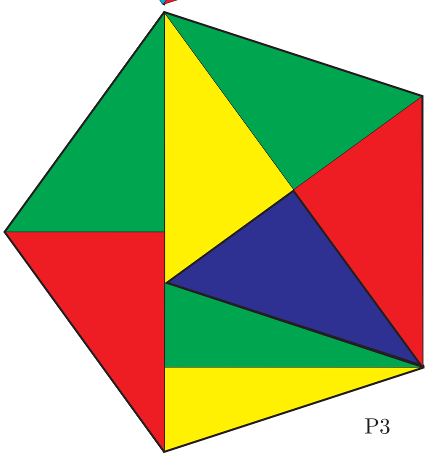
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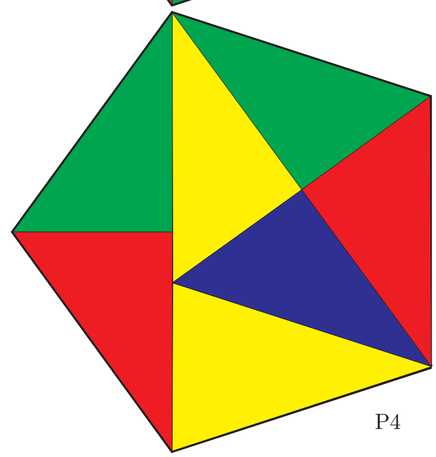
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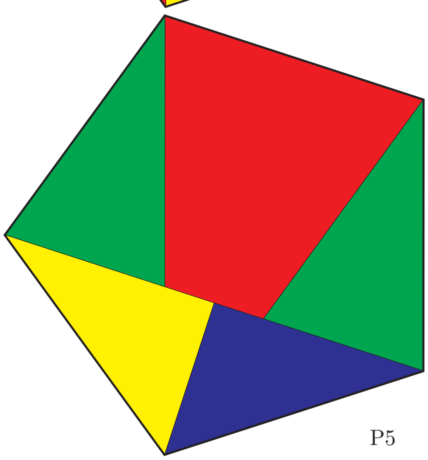
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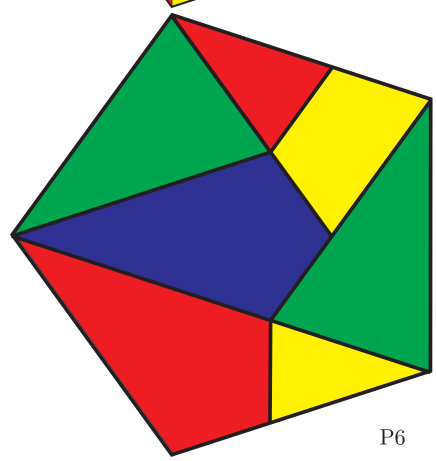
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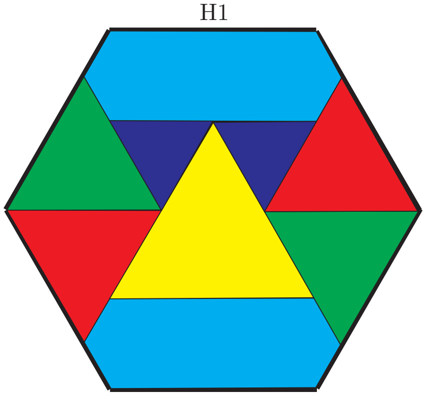

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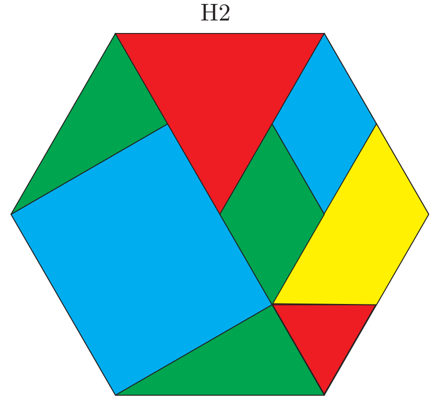

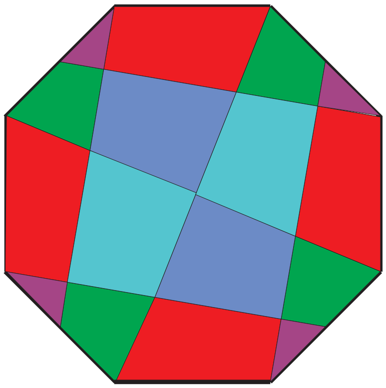

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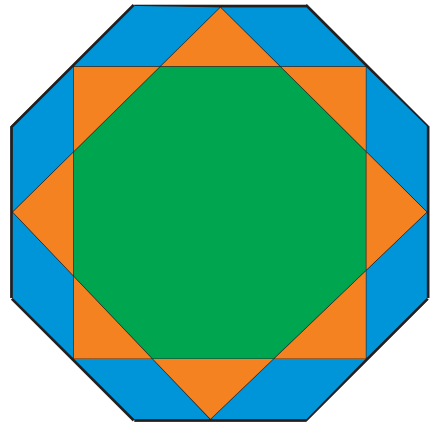

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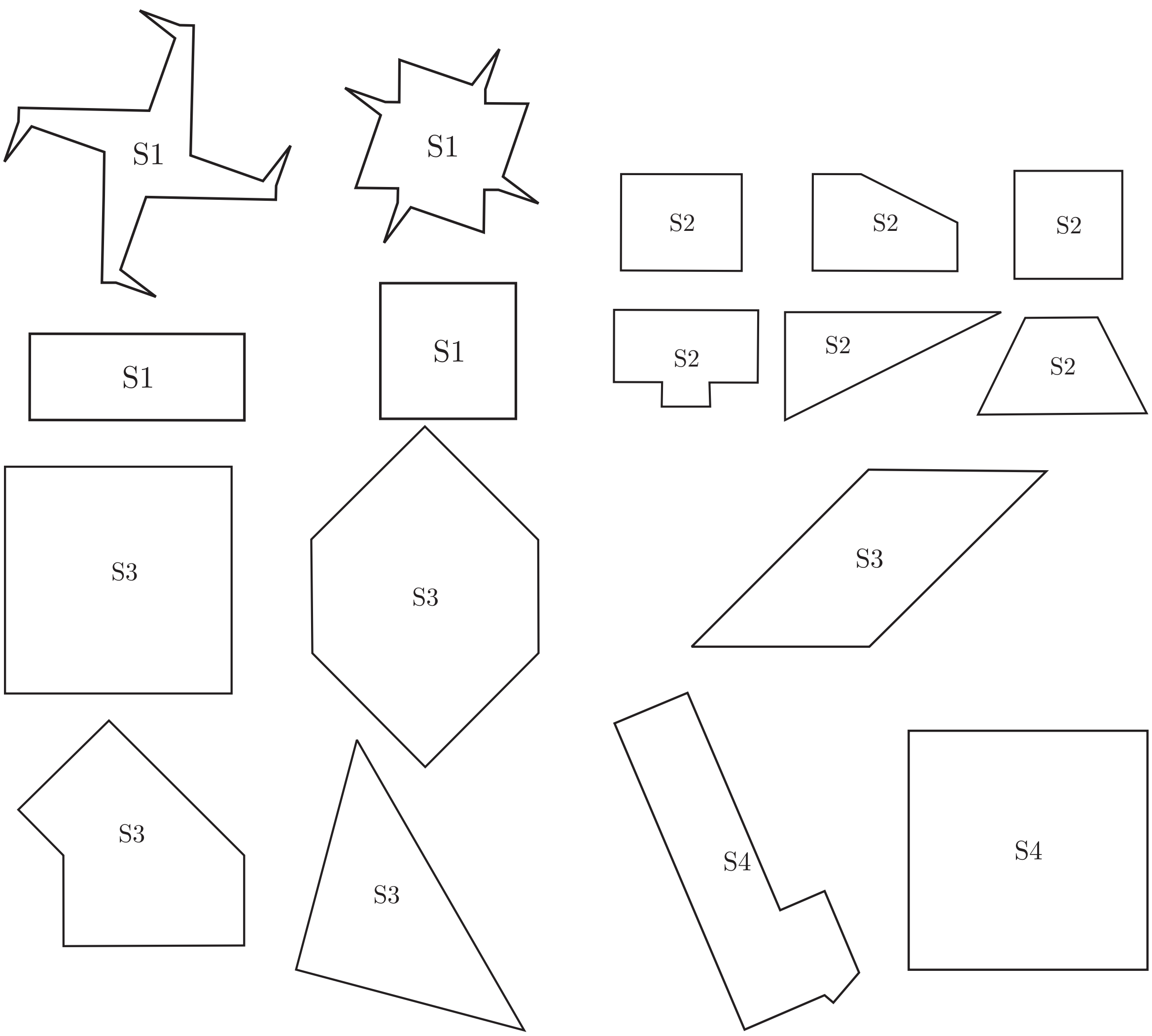
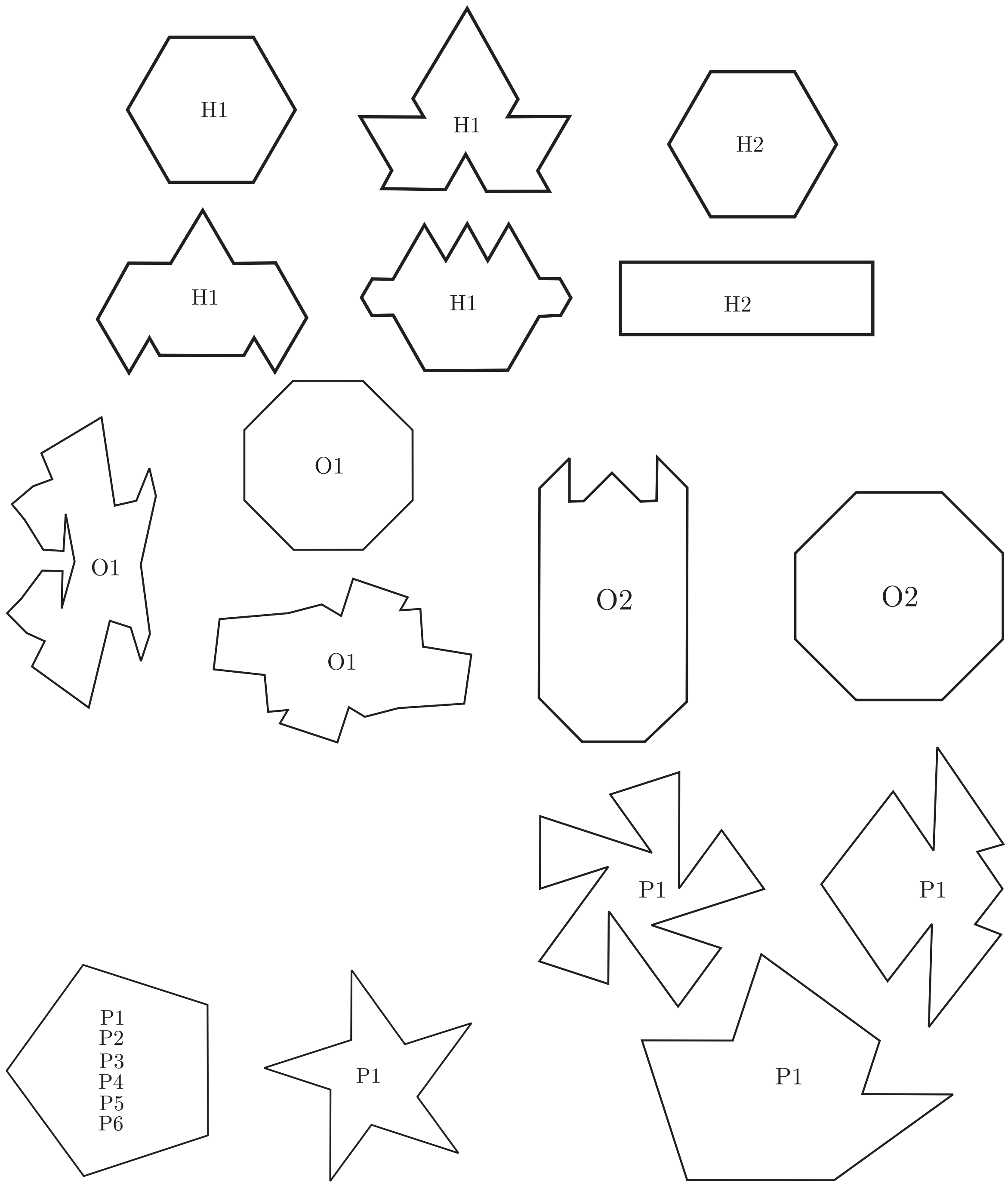
0.2 Some possible figures - the frames

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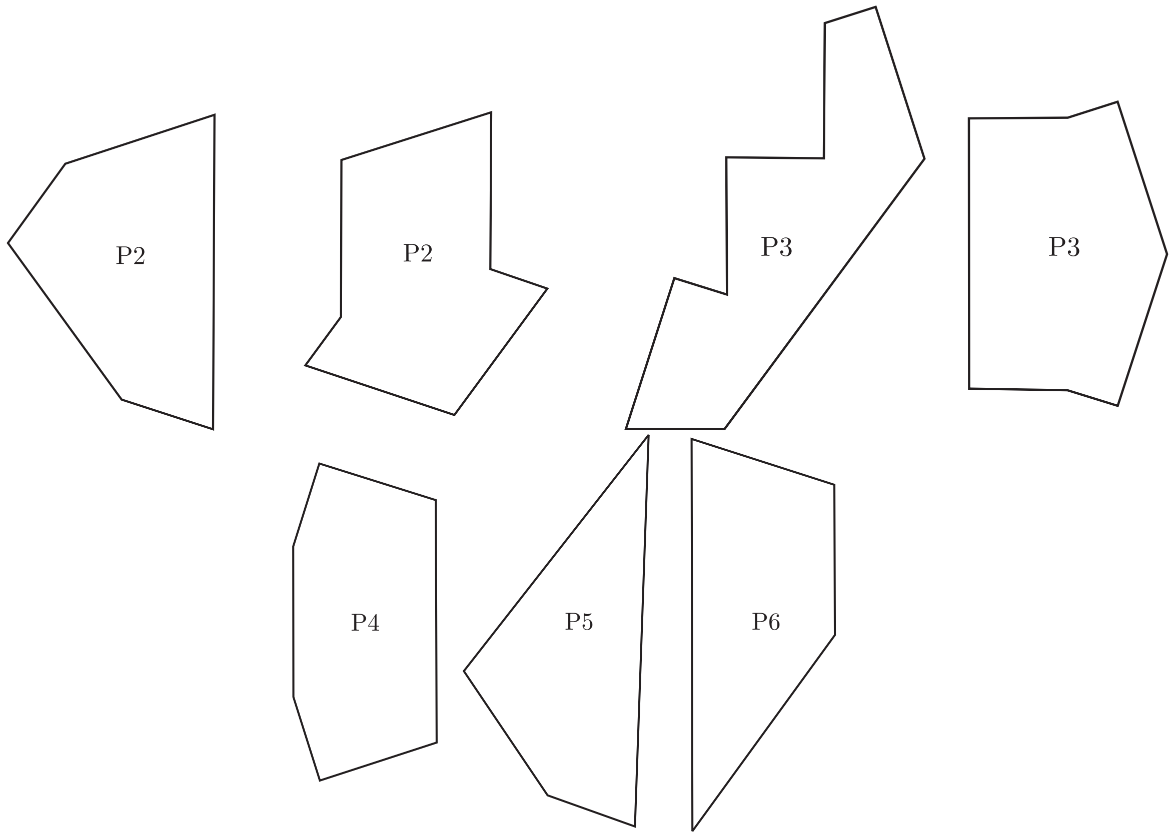
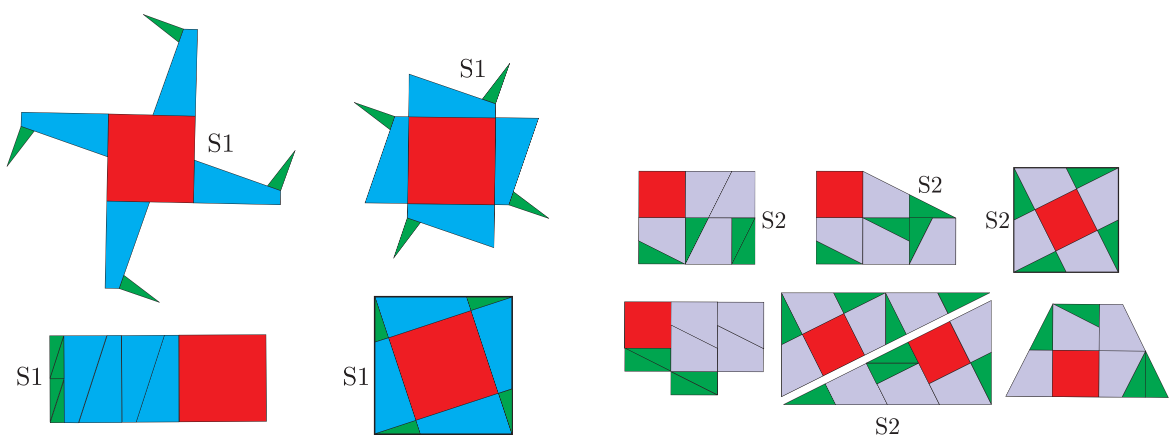
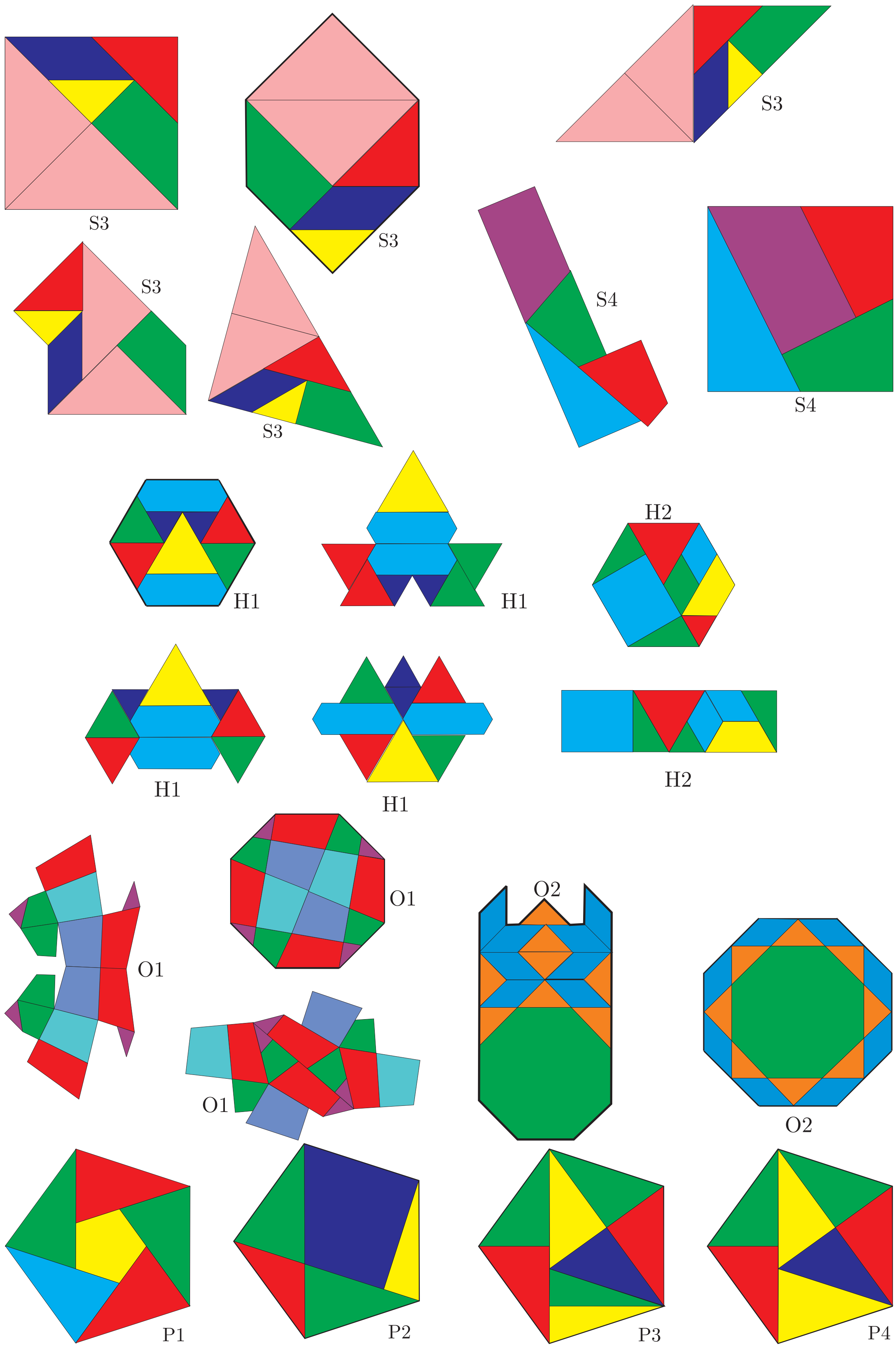


0.3 Frames with solutions

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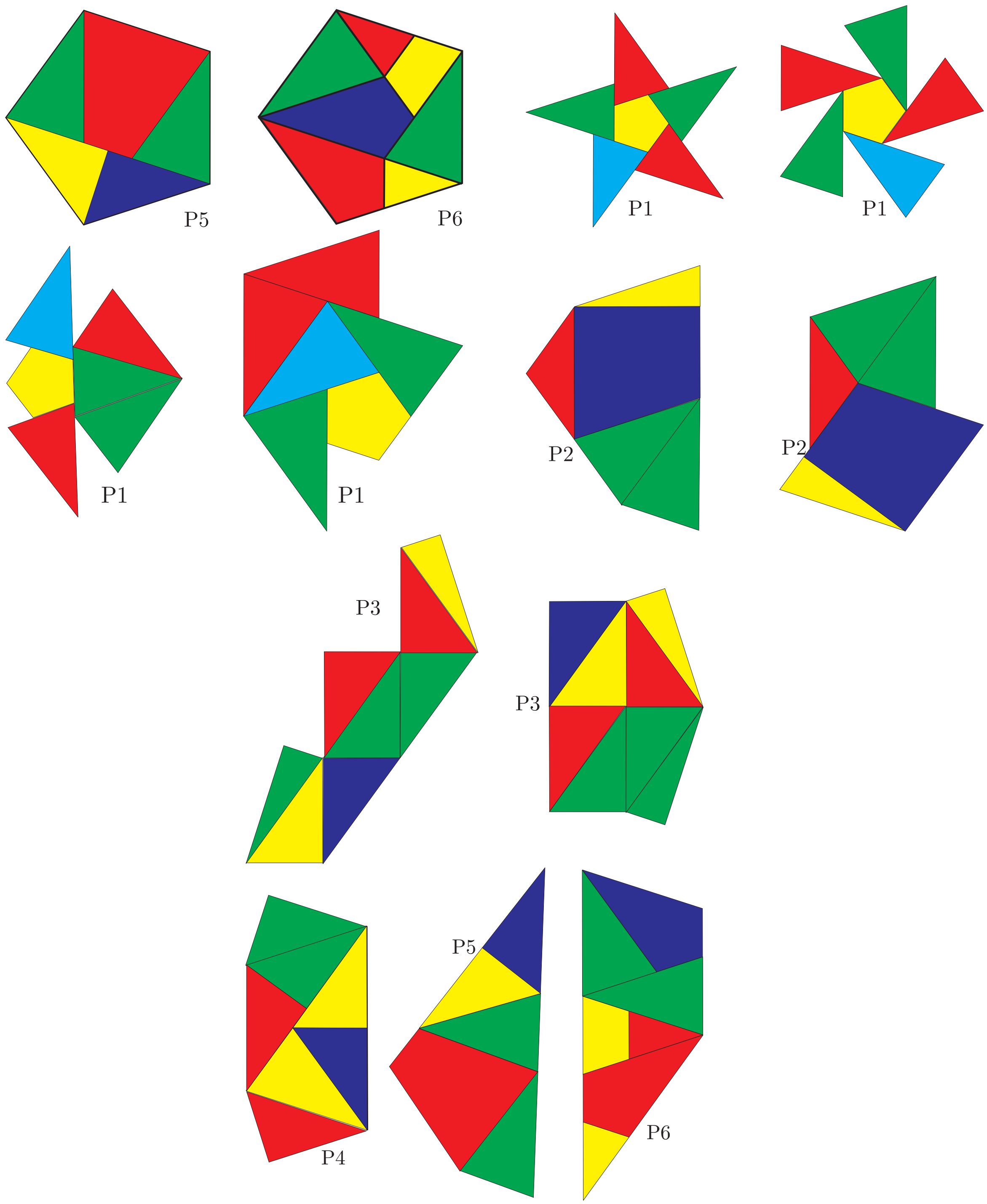
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