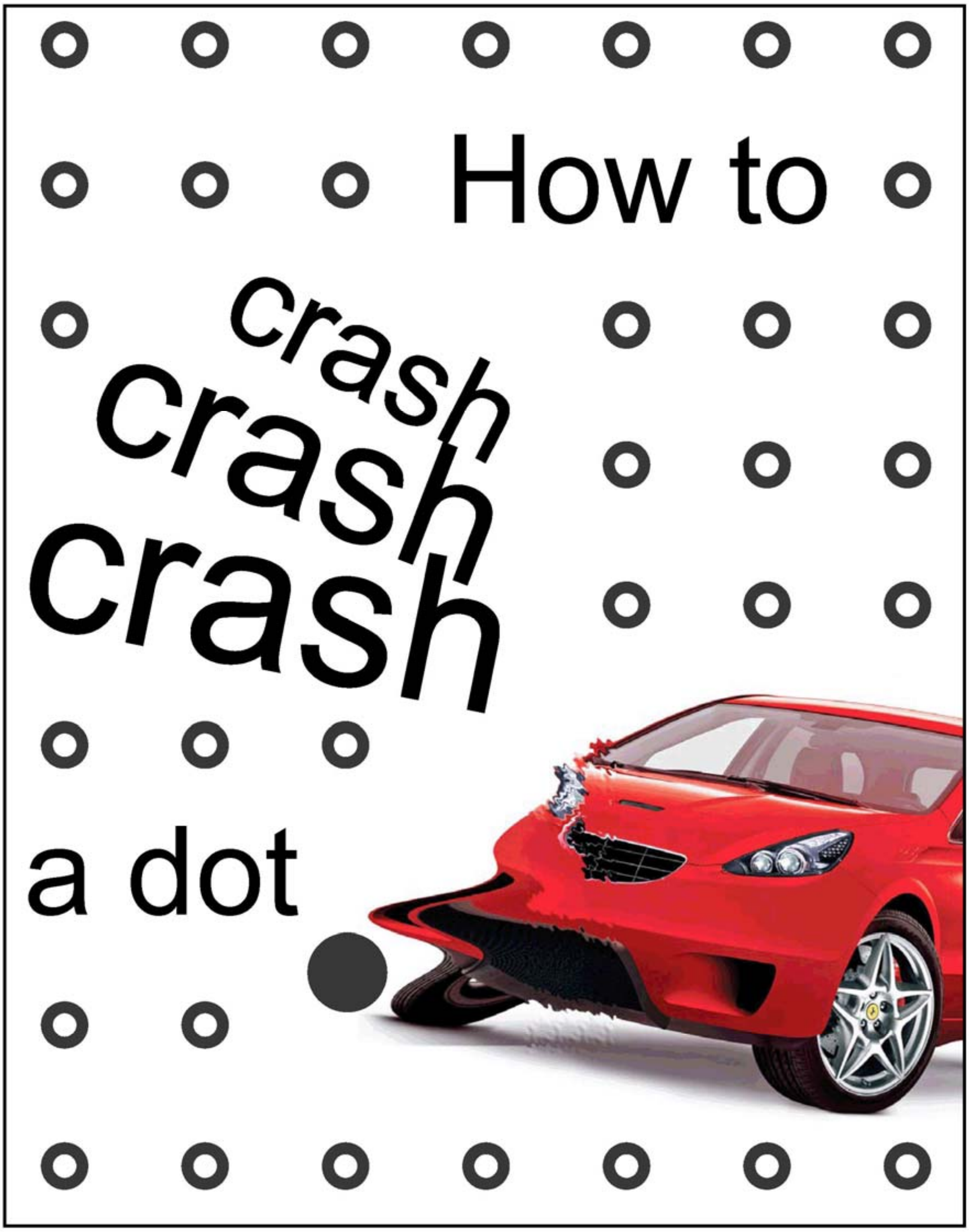
MATHEMATICS B-DAY 2009

Friday, November 20



Introduction

During this Maths B-day you will study a game that was invented in the 1970s. It was one of the first computer games. The game is very simple, since computers in those days were very limited in what they could do. So there's no point in looking for the game on-line, because it will no longer work on most computers, and it will not help you with the assignment!

The structure of the assignment

This Maths B-day assignment contains the following parts:

Part 1 Start
In this part, the rules of the game that let you move a dot along grid points are formulated and you have the opportunity to learn how to use the rules. That is very important to make sure you have a good idea of the problems in the later parts of the assignment, but the questions (we call them "starts") in this part are not really worth including in your report.

Part 2 Racing along a straight horizontal line
In this part you explore the case of movements in one dimension. How do you use the rules to make a dot move so that it obeys your actions? The results of this part are certainly useful to include in your report!

Part 3 Racing in two dimensions
In this part we look at movement in both horizontal and vertical direction. You can make use of the results you found in part 2. The results from this part are also worth including in your report.

Part 4 The final assignments
In this part three different final assignments are formulated, letting you make a choice between:
- doing all three with the risk that you can only report superficially on them,

or

- making a wise choice and reporting fully and carefully.

Timetable

It's a good idea to take the time to thoroughly understand the rules (part 1) as a team. After that, parts 2 and 3 are intended to fully get to grips with the mathematical aspects of the game. What works well is dividing the tasks within your team, but you must also take care to discuss your results among yourselves. Finally, the challenge is to find a good, mathematically grounded route for one or more of the final assignments (part 4) that is, in the opinion of your whole team, the best route. Make sure you have enough time for the final assignment.

Generally speaking our advice is:
- Take an hour or so to go through part 1 in detail and discuss it among yourselves;
- Use the time between 10 and 12 to do parts 2 and 3 and to discuss them;
- Work on part 4 and decide which assignment/assignments you want to include in the report;
- Remember you have to hand in the report at 16:00 and plan your thinking and work time with that in mind!

Resources for this assignment

You will probably need a lot of graphing paper. Especially at first it helps to make a lot of drawings to get to know the game wel.
You won't need a (graphical) calculator!
From part 3 onwards there is an Excel file which you can use to draw the routes you write down in 'button language'. You can use it, but it isn't obligatory!
The Excel file is explained further in an appendix.

The final product

Your final report must be easily legible for someone who didn't know in advance what the assignment is about. That means you have to describe clearly what the assignment is about, including the rules of the game and what your team researched.
Of course you can follow the individual tasks in the report, but you may also hand in a report based on the final assignment(s) you selected in part 4, with a clarification using the elements that were brought up in parts 1, 2 and 3 and which you examined.

Further advice:

- Hand in a report that's easy to copy. If you include handwritten materials, use black ink, because it copies better than other colours.
- Remember that your report must be clear to a reader who doesn't have the assignment at hand.

Finally…

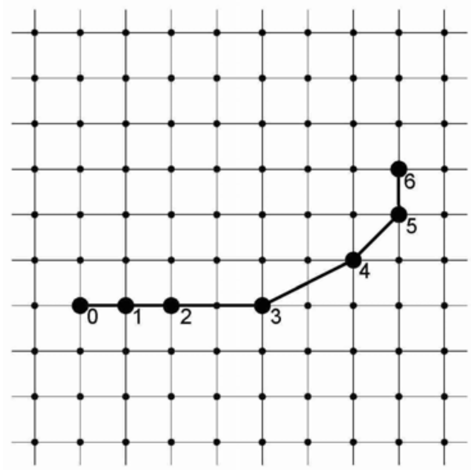
We hope you and your team will enjoy working on this assignment, so from the design team:

Have fun and good luck!

Part 1 Start and: your turn

In this first part you will mainly practise the rules of the game. It is a starting point for the parts that follow. And because the game consists of making moves, it is constantly "your turn".

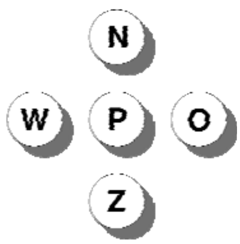
You can play the game by yourself (though it's also fun to race someone else).
The playing board consists of a grid of horizontal and vertical lines.
You are somewhere in this area on an intersection of gridlines, so on a gridpoint.
During the game, the dot moves across the field in a number of moves, while reaching new positions in the field.
That could happen like this for example:



The numbers in the figure represent the successive positions you reach, and these positions are connected to the moves. You start in position 0, after the first move the dot reaches position 1, after the second move the dot reaches position 2, etcetera.
The small dots are the gridpoints, the big ones are the successive positions of the moving dot on the grid.

How do you determine the route the dot follows?

To do so, you have a number of buttons you can activate.
You control the game with five buttons (see illustration on the right).
In every move you press one of the buttons. Pressing a button influences the *speed* of the dot during that move, as well as the direction in which the dot moves along a straight line (O = east, N = north, W = west and Z = south).
If you don't want to change the speed during a turn, you press P (for Pass).



You get the route in the figure above like this:

At the start the dot is stationary in position 0.

Move 1: Press O. The speed of the dot is now: one unit per move to the right. So after move 1 the dot has moved one unit to the right and is in position 1.

Move 2: Press P. The speed doesn't change. The speed was: one unit per move to the right and that stays the same. So after move 2 the dot has moved one more unit and reaches position 2.

Move 3: Press O. You have to add one unit to the east to the speed. So now the speed becomes: two units to the right per move. So after the third move the dot has moved two units and has reached position 3.

Move 4: Press N (north). An upward component is added to the speed: two units to the right per move (as it already was) **and** one unit up per move. So now the dot is moving upwards diagonally and reaches position 4.

Move 5: Press W. The speed now becomes: per move one unit to the right, and one up. So again diagonally up to position 5.

Start 1

a. Which button has been pressed in move 6?

b. Which button should you press if you want to stop the dot in move 7?

c. Could you also have stopped the dot in move 5?

It's useful to describe a series of moves with 'button language'.
For instance, the button series

NNWZZPPZ

means 'Press N for the first move, press N for the second, press W for the third, ...'

We agree that you **always start from a stationary position**.

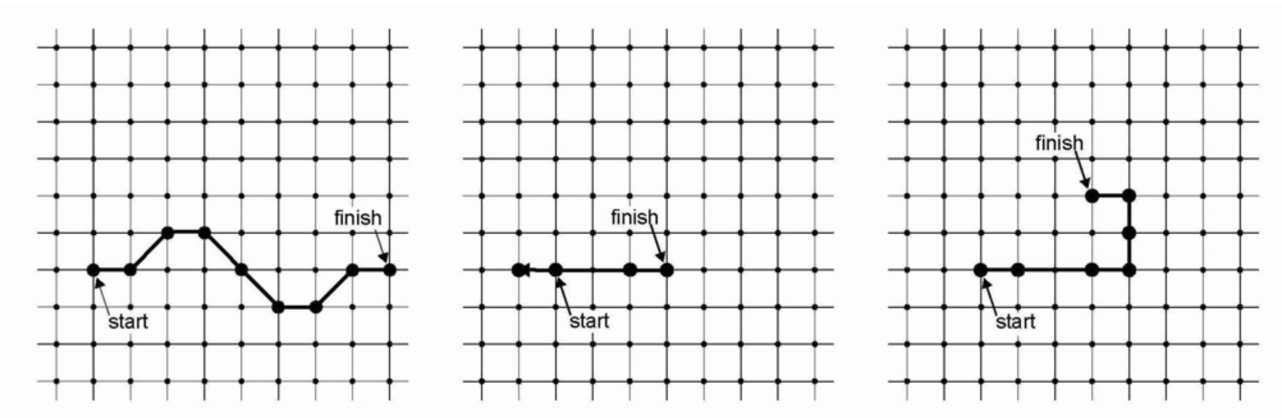
Start 2

a. Draw the route for the button series NNWZZPPZ.

b. You want to reach a gridpoint that is four units to the left and three up from the starting point. Describe a button series that will get you there.
You don't have to stop at the end, but you must arrive in exactly that point.

Start 3

What should you do to get the following three routes?
Take care: sometimes you stay in the same place for several moves!



(a) **(b)** **(c)**

Start 4

Why can you only make a right angle turn if you stand still for one turn in the corner.

The goal of this game is to go as fast as possible from the starting point to an end point.
'*As fast as possible*' means '*in as few moves as possible*', so with as little button pressing as you can.
The final move must finish exactly in the intended end point. The speed of the dot is not important.

Start 5

You want to move four units to the right from the starting point.

a. Why can't you do it in two moves?

b. How could you do it in three moves?

c. Find all ways to move four units to the right in three moves. Also include the final speed for each way you find.

Part 2 Racing along a straight horizontal line

An important question is: how can I reach a specific position in as few moves as possible?
For simplicity's sake we will first look only at *horizontal* moves. So the N and Z buttons do not come into play.

Problem 1

a. Compare the following series of buttons (5 times O and 1 P and the last one with just the 5 Os):

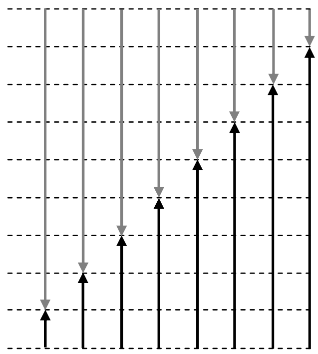
OOOOOP
OOOOPO
OOOPOO
OOPOOO
OPOOOO
POOOOO
OOOOO

Which series of buttons gets you furthest?

As you can see, the moment you use the P button is important for reaching the various positions.

b. Show that you can move every arbitrary positive whole number distance (so, 1, 2, 3, ...) eastward by only pressing O and *at most* once pressing P.
You have to select the necessary number of moves yourself.

Problem 2



a. From a stationary position, you press just O for 8 moves.
Show that the position you reach, can be found using the calculation: $\frac{1}{2} \cdot 8 \cdot (8 + 1)$

Hint. To the right you see an image showing the consecutive distances per move as black arrows. They have also been placed as grey arrows, rotated over 180 degrees, over the black arrows. How can you use this to calculate the sum of the lengths of the black arrows?

b. Now you make n moves from a stationary position pressing just O.
Show that the position you reach is given by $\frac{1}{2} \cdot n \cdot (n + 1)$

Problem 3

a. Take another look at the series of six moves at the start of problem 1. What is the final speed for all these cases?

b. Formulate a general statement about the final speed after a series of moves in which you use only the P and O buttons.

Opgave 4

From a stationary position you make n moves in which you only press W and O.
Which positions can you have reached after those n moves?

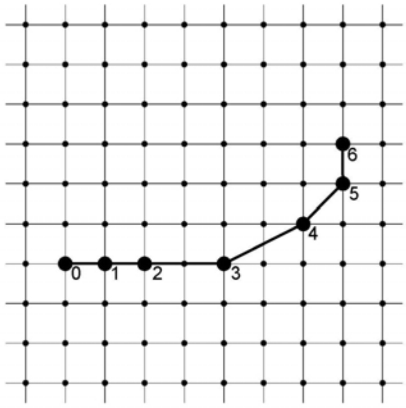
The next problem has an extra requirement. You not only have to arrive at a given position, but you must also stop there.

Problem 5

Describe, for a given positive integer x, a way to go **exactly** x units to the right in as few moves as possible. Make sure that your final speed is zero.

Part 3 Racing in two dimensions

Now we go back to the whole board: the N and Z buttons come back into play. It makes sense to use *x*- and *y*-coordinates for the points on the board. You always start at (0,0).
Below you can see the route and moves described in Part 1 again:



This route, with starting point (0,0), can be represented compactly in a table as follows:

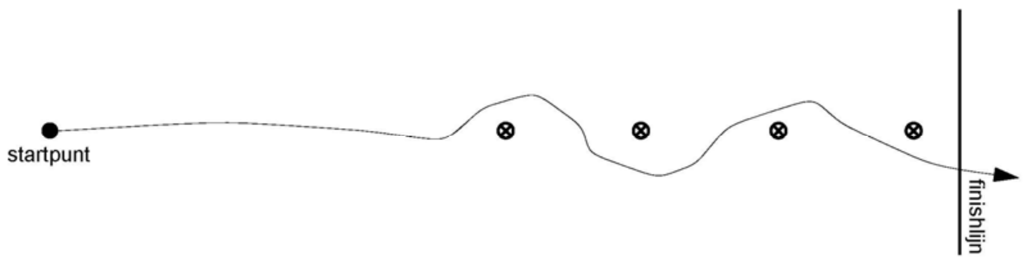
move	action	speed during move	Position after move
1	O	(1,0)	(1,0)
2	P	(1,0)	(2,0)
3	O	(2,0)	(4,0)
4	N	(2,1)	(6,1)
5	W	(1,1)	(7,2)

The fourth move, starting from position (4,0), is the action "press N". As a result the speed is determined by two steps to the right and one up, represented as (2,1), with move 4 ending in position (6,1).

There is an Excel-file which lets you draw routes quickly, which may be useful. Of course you are free not to use the file. In the appendices a short description of how to use the file is given.

Problem 6

One of the games you can play, is slalom. You are standing still in (0,0). Your goal is to cross the finishing line $x = 200$ as quickly as possible. On your way to the finish you must slalom around the points (100,0), (130,0), (160,0) and (190,0), for example like this:

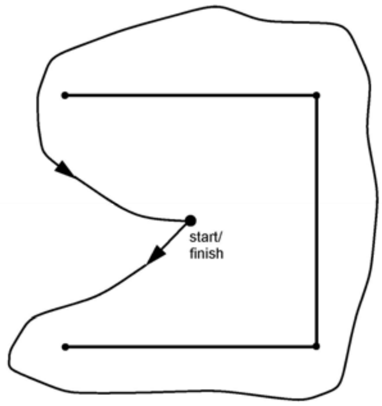


Try to cross the finishing line in as few moves as possible.
You do not have to end up exactly *on* the finishing line, but you must cross it completely.
Of course your route isn't allowed to go through any of the four given points.

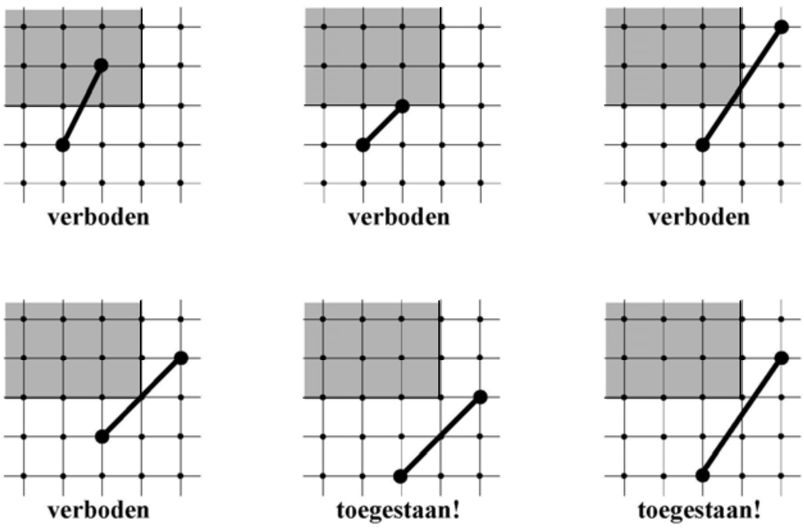
Problem 7

On the next playing field you must go *around* the four points (–10, –10), (10, –10), (10,10) and (–10,10). You start at the origin and you have to end in (0,0) as well, but you don not have to come to a stop there. The route cannot cross the lines that have been drawn to connect the points.

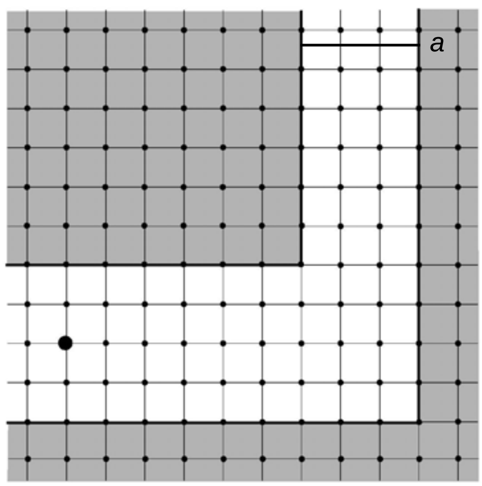
Construct a route with as few moves as possible.



The racing is more exciting if there are obstacles you have to avoid.
In the following images you can see what you are allowed to do and what not when you approach an obstacle (the grey areas).



The grey areas in the figure below are forbidden for the moving dot.



Problem 8

You start from a stationary position at the black dot, and you have to pass finishing line *a*.
Find the fastest route.

The Decomposition Theorem

The button series OPNOZO takes you from (0,0) to (10,2).

You can turn this series into a one-dimensional, horizontal movement by replacing every N or Z with P:

OP**N**O**Z**O → OP**P**O**P**O

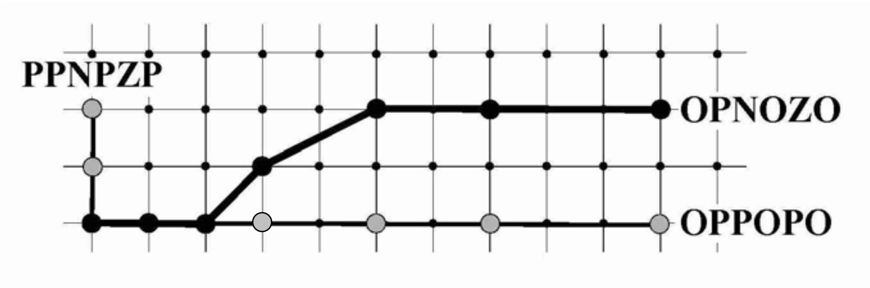
This one-dimensional series takes you to (10,0).

There is a similar method to turn this into vertical movement:

OPN**O**Z**O** → **P**PN**P**Z**P**

This takes you to (0,2).

Illustrated, this decomposition looks like this:



This idea can be expressed in a theorem.

Theorem

A button series consists of n moves. In each move, one of the buttons N, O, Z, W or P is pressed. Supposing that this series takes you from the starting position (0,0) to (x,y), then the rule is:

- If you replace every N and every Z in this button series by P, the new series of n moves will take you along the x-axis from (0,0) to (x,0);
- If you replace every O and every W in this button series by P, the new series of n moves will take you along the y-axis from (0,0) to (0, y).

Problem 9

A two-dimensional route of fourteen moves, starting in (0,0), is decomposed in a horizontal and a vertical component.
The horizontal component is: OPPWWPPPPOOPWP
The vertical component is: PNNPPZZZPPPPPZ

a. Reconstruct the two-dimensional route from these two components.

For each of the fourteen moves, consider the three speeds, in the horizontal component, the vertical component and in the two-dimensional route

b. What relationship you see between those three speeds?

c. Under what conditions does a button series of n moves have a final speed equal to zero?

Problem 10

Given a point (x,y) with x and y integer. Suppose that you can move in n moves from the origin to (x,0) and you can also move from the origin to (0,y) in n (different) moves.

Find out under which conditions it is possible to also reach (x,y) in n moves.

Palindrome routes

A *palindrome* is a word (or sentence) that looks the same whether you read it from left to right or from right to left. Some examples are: ‘eye’, ‘racecar’ and ‘Too bad, I hid a boot’. You can also make palindromes in button language. One example is NONZZWPWZZNON.
We call it a *palindrome route*.
If you make these moves, you will end up where you started from, with end speed zero.

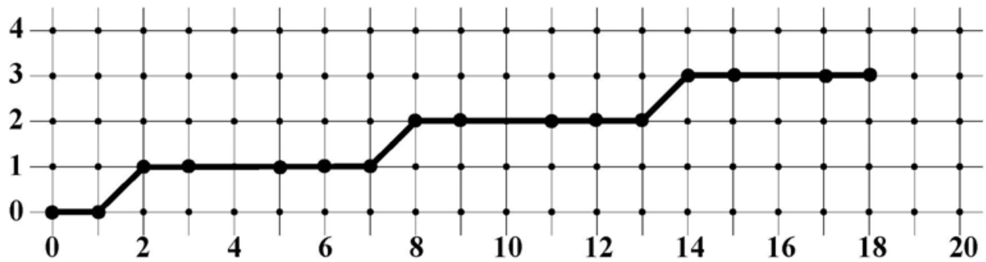
Problem 11

Find out whether it is always true that the end point is the same as the starting point with end speed zero for a palindrome route, or that there are additional requirements for the palindrome to achieve this. Give good arguments for your statements.

Periodicity

In this part you will look at repeats (periodicity). We assume a series of moves. The following phenomena can occur:

- The *button series is periodic.* This means that the entire series consists of two or more copies of a shorter series. For instance, the button series ONZOWWONZOWWONZOWW is periodic, since it is arrived at by repeating ONZOWW three times.
- The *route is periodic*: you can divide the image of lines and points on the playing field into two or more equal parts. For example:



- The button series has a *periodic displacement*. This happens when there is a repeat in the speed vectors. This may be clearer from the table below:
-

position	*speed*	*position*	*speed*	*position*	*speed*
(0,0)	**(0,0)**	(6,1)	**(0,0)**	(12,2)	**(0,0)**
(1,0)	**(1,0)**	(7,1)	**(1,0)**	(13,2)	**(1,0)**
(2,1)	**(1,1)**	(8,2)	**(1,1)**	(14,3)	**(1,1)**
(3,1)	**(1,0)**	(9,2)	**(1,0)**	(15,3)	**(1,0)**
(5,1)	**(2,0)**	(11,2)	**(2,0)**	(17,3)	**(2,0)**
(6,1)	**(1,0)**	(12,2)	**(1,0)**	(18,3)	**(1,0)**
				(18,3)	X

Problem 12

Examine the relationships between the phenomena that are described above. For example, does a periodic series of buttons always result in a periodic route? And if no, are there conditions under which this does happen?

Part 4 The final assignments

You can now make use of the insights you gained from the previous parts in the three circuits that are offered here.
Of course you may decide to do all three, but keep in mind that the assessment will look especially at the thoroughness with which you describe one, two or even all three circuits and justify your choices!

You will find the circuits on the next three pages. Here are the assignments to go with them.

Circuit 1 A constant width track

Start from a stationary position on the starting point. Cross the finishing line as quickly as you can.

Circuit 2 Racing around Oval Lake

Start from a stationary position in A.
Race clockwise around the lake and finish in B.

Circuit 3 An endless slalom

A circuit where you may not hit obstacles 1, 2, 3, 4, 1, 2, ..., but you must go between them. The position of the obstacles repeats without end.

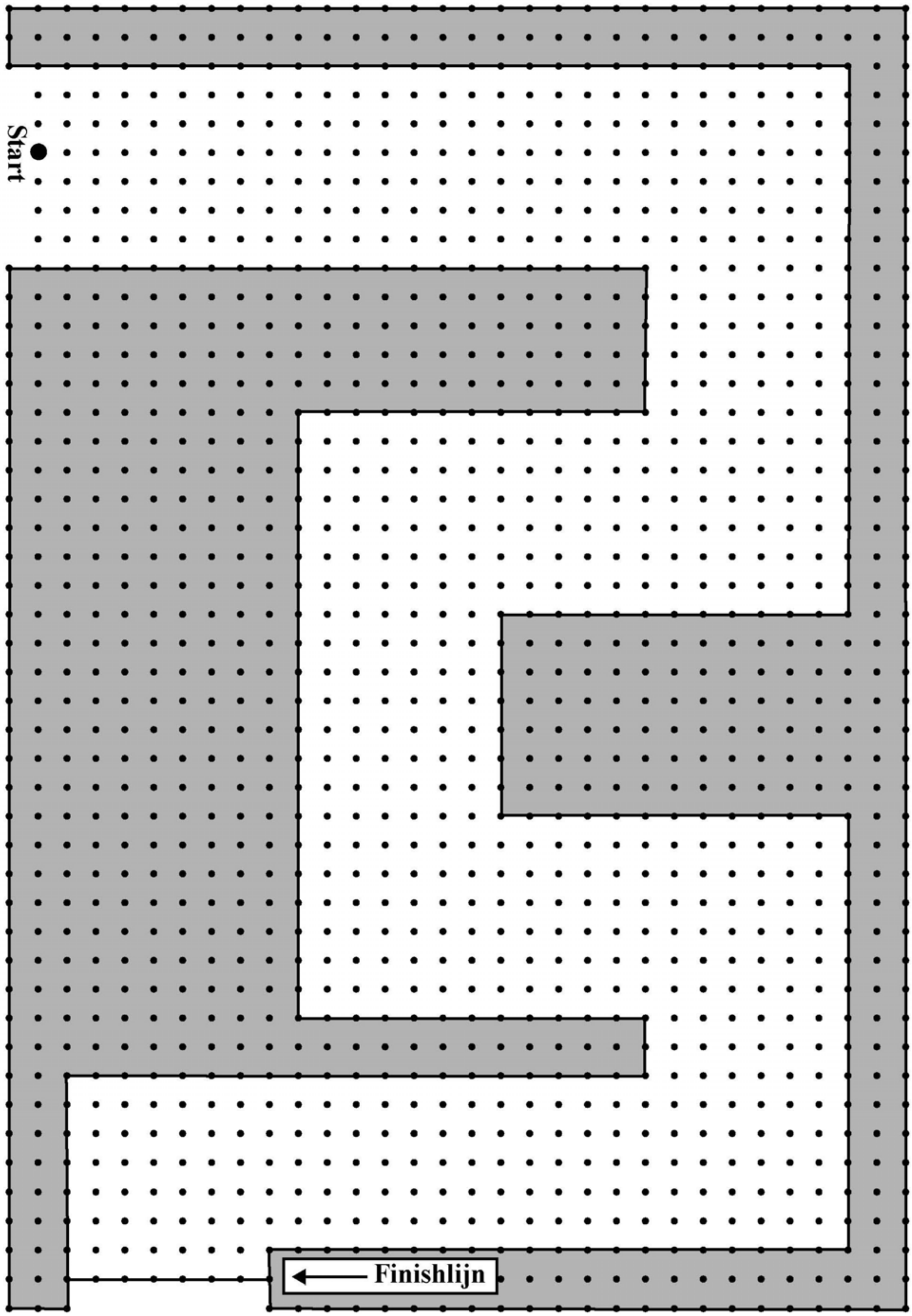
You make a route in the order:
- first between 1 and 2 in the direction indicated by the arrow
- then between 2 and 3 in the indicated direction
- then between 3 and 4
- then between 4 and 1

From here on repeat the previous steps.

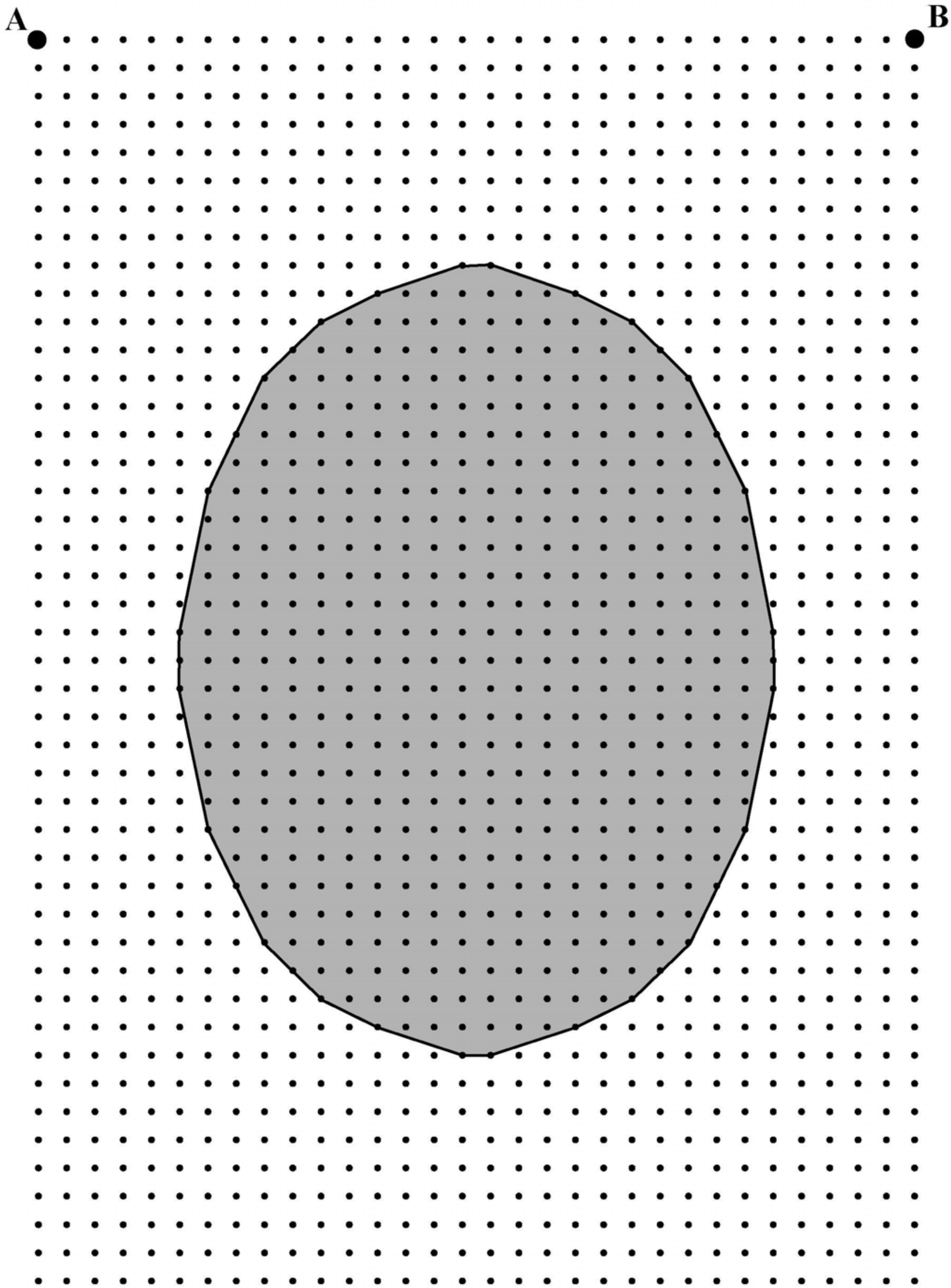
Think of a pattern for your route that will repeat endlessly.
Pick a sensible starting position and starting speed for your dot.
So this is the only case where you don't have to start from a stationary position!

Good luck!

Circuit 1 A constant width track



Circuit 2 Racing around Oval Lake



Circuit 3 An endless slalom