Summary

1 Research questions

The project ‘Learning algebra in a computer algebra environment’[1] reported upon here is a case of research on the integration of information technology (IT) into education, and on its influence on the learning of mathematics in particular. In this study we focus on two issues that are relevant to mathematics education today: the learning of algebra and the integration of technology – and particularly of computer algebra – into it.

Algebra has always been an important topic in school mathematics curricula, and often is a stumbling-block to students. The difficulties of learning algebra include its formal and algorithmic character, the abstract level at which problems are addressed, the object character of algebraic expressions and formulas, and the compact algebraic language with its specific conventions and symbols. Because of these difficulties, students often do not perceive algebra as a natural and meaningful means for solving problems (Bednarz et al., 1996; Chick et al., 2001).

The integration of IT is one of the ways in which solutions for these problems of learning algebra are sought. IT use is expected to contribute to the visualization of concepts, and can free students from carrying out operations by hand, thus directing their attention towards concept development and problem-solving strategies. In this way, IT use might lighten the traditional algebra curriculum for them. In the meantime, the integration of technology raises questions concerning the goals of algebra education and the relevance of paper-and-pencil techniques, now that they can be left to a technological device. For algebraic skills, the use of a computer algebra system (CAS) is of particular interest, as it provides a complete repertoire of algebraic procedures and operations (Heid, 1988; O’Callaghan, 1998).

The issue of integrating computer algebra use into the learning of algebra brings us to the main research question of this study:

> *How can the use of computer agebra promote the understanding of algebraic concepts and operations?*

This question needs further specification. As algebra as a whole is too big a topic to address in this study, we confine ourselves to the concept of parameter. Parameters may emerge quite naturally from concrete contexts, and may also be means of generalization and abstraction. Therefore, addressing the concept of parameter may stimulate students to enter the algebraic world of formulas, expressions and general solutions. Also, the concept of parameter allows for revisiting the different roles of ‘ordinary’ variables that the students have met before. The use of parameters may

1. Granted by the Netherlands Organization for Scientific Research, project number 575-36-003E.

improve the students' insight into the meaning and structure of algebraic formulas and expressions (Bills, 2000; Furinghetti & Paola, 1994). Therefore, the general research question is specificated in the first research subquestion:

> 1. *How can the use of computer algebra contribute to a higher level understanding of the concept of parameter?*

Previous research on the integration of computer algebra into mathematics education shows that the idea of technology carrying out the elementary operations, so that the students can concentrate on conceptual understanding, is too simplistic (Artigue, 1997; Drijvers, 2000; Guin & Trouche, 1999; Lagrange, 2000; Trouche, 2000). The technical skills that the students need to carry out procedures in the CAS environment both require some conceptual understanding and affect that understanding. As a second focus of this study, therefore, we consider the intertwined development of techniques in the computer algebra environment and of mathematical understanding in terms of the students' mental schemes. This instrumental approach to IT use – which concerns the instrumentation process of establishing the dual relation between conceptual understanding and techniques in the IT environment – is the topic of the second research subquestion:

> 2. *What is the relation between instrumented techniques in the computer algebra environment and mathematical concepts, as it is established during the instrumentation process?*

2 Theoretical framework

While phrasing the research question, we wondered what theoretical perspectives could help us to investigate these issues. A ready-to-use theoretical framework for the study of learning algebra in a computer algebra environment was not available. Therefore, we selected those theoretical elements from research on learning algebra and mathematics in general that seemed promising for application in and adaptation to the focus of this study. As a consequence, most of these elements were taken out of their usual scope and localized for the aim of this study. This eclectic approach to theory is called 'theory-guided bricolage' (Gravemeijer, 1994). The following elements were included in our theoretical framework.

- *The domain-specific instruction theory of Realistic Mathematics Education*
  Key elements in the domain-specific instruction theory of Realistic Mathematics Education (RME) are guided reinvention and progressive mathematization, horizontal and vertical mathematization, didactical phenomenology and emergent modelling (Freudenthal, 1983; Gravemeijer, 1994; de Lange, 1987; Treffers, 1987a, 1987b). Guided reinvention, progressive mathematization and didactical phenomenology were supposed to be useful for developing a hypothetical learn-

ing trajectory for the concept of parameter, and for designing instructional activities. The issue of horizontal and vertical mathematization might clarify the relation between concrete problem situations and the abstract 'microworld' of the computer algebra environment, which tends to have a top-down character (Drijvers, 2000). The notion of emergent modelling was supposed to be useful for distinguishing levels of activity. As is shown in Fig. 1, Gravemeijer (1994, 1999) distinguishes four levels of mathematical activity. The idea of emergent modelling is that models, which initially refer to a concrete context that is meaningful to the students, gradually develop into general models for mathematical reasoning within a mathematical framework.

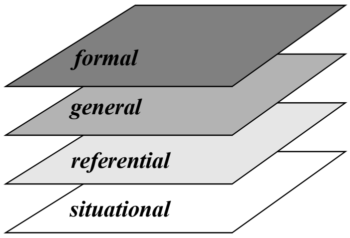

Figure 1 Four levels of mathematical activity (Gravemeijer, 1994, 1999)

- *Level theories*
  The first research subquestion speaks of 'a higher level understanding of the concept of parameter'. To make explicit what is meant by that, several perspectives were considered. First, Van Hiele's level theory distinguishes between a ground level, a second level and a third level of insight (Van Hiele, 1973, 1986). This distinction might be useful for defining levels of understanding of the concept of parameter. A second approach to the levels involves the concepts of emergent models and levels of activity (see Fig. 1). In fact, in this study we aim at the transition from referential to general level of activity.
- *Theories on symbol sense and symbolizing*
  The notion of symbol sense concerns the sense for algebraic entities in general and the insight into formulas in particular (Arcavi, 1994). In this study, we defined symbol sense as the insight into the meaning and the structure of algebraic expressions and formula. Working in a computer algebra environment with parametric formulas was supposed both to require this kind of symbol sense and to foster it.
  How does one acquire symbol sense? Theories of symbolizing stress the parallel development of symbols and meaning by means of a signification process (Gravemeijer et al., 2000). Because giving meaning to algebraic techniques, formulas and expressions as they appear in the computer algebra environment

seemed to be an important aspect of the instrumentation process, a symbolization perspective was supposed to be relevant in this study.

- *The process-object duality*
  The process-object duality concerns the idea that a mathematical concept can be considered both as a process and as an object. Often, students first experience the process aspect; on the basis of this they may develop the object aspect, which is a prerequisite for conceptual progress. This development is called reification (Sfard, 1991) or encapsulation (Dubinsky, 1991) and results in proceptual understanding (Tall & Thomas, 1991). These theoretical notions concern mathematics in general, but can also be applied specifically to the learning of algebra.
  In this study we considered the reification of expressions and formulas, because perceiving formulas and expressions as objects is important for developing the concept of parameter. Furthermore, computer algebra use might affect the relation process-object. Reifying formulas and expressions involves overcoming the lack of closure obstacle, viz. the students' inability to see expressions and formulas as results if these contain operators; in $a+b$ or $x+3$ for example, the students would prefer to carry out the addition (Collis, 1975; Küchemann, 1981; Tall & Thomas, 1991). The reification of formulas and expressions does not imply the reification of function, which involves more aspects.

An important theoretical element in our study is the instrumental approach to IT use. We address that issue separately in Section 4.

3 Analysis of the concept of parameter

This study included a conceptual analysis of the concept of parameter.

We started this conceptual analysis with an investigation into its historical development. Essential in this development was the transition from syncopated to symbolic algebra that is characterized by the work of Diophantus and Viète (Boyer, 1968; Harper, 1987). Whereas Diophantus (ca. 250 A.D.) used literal symbols to denote unknown variables but not parameters, Viète (1540 - 1603) provided general, parametric solutions. He distinguished unknowns from parameters by using vowels and consonants. Apparently, he accepted expressions as solutions to general, parametric equations and thus considered them as objects. In our study, we aimed at this jump 'from Diophantus to Viète', or from the referential to the general level (Fig. 1). The time that this step took in history evinces its complexity.

The conceptual analysis of the notion of parameter led to the idea that a parametric formula or expression represents a second-order function. For example, the short arrow in Fig. 2 indicates that $x \to a \cdot x + 5$ for a fixed value of the parameter a represents a – linear – function in x. As soon as the value of a changes, the long arrow in Fig. 2 indicates a second-order function, with the parameter as argument and linear expressions as function values (Bloedy-Vinner, 2001). As the main difficulties of the concept of parameter we identified this hierarchical relation with ordinary variables, represented in Fig. 2 and in the expression ‘variable constant’, and the distinction of the different roles of the parameter, which may change during the problem-solving process.

$$a \longrightarrow (x \to a \cdot x + 5)$$

Figure 2 The parameter as argument of a second-order function

The conceptual analysis led to the identification of four parameter roles, which were similar to the roles of ordinary variables:

- The parameter as *placeholder* represents a position, an empty place, where a numerical value can be filled in or retrieved from. The value in the ‘empty box’ is a fixed value, known or unknown, that does not change. This is the ground level of understanding the concept of parameter.
- The parameter as *changing quantity* concerns the systematic variation of the parameter value. The parameter acquires the dynamic character of a ‘sliding parameter’ that smoothly runs through a reference set (van de Giessen, 2002). This variation affects the complete situation – that is, the formula as an object, and the global graph – whereas variation of an ordinary variable only acts locally.
- The parameter as *generalizer* generalizes over a class of situations. By doing so, this ‘family parameter’ (van de Giessen, 2002) unifies such a class, and represents it. This generic representation allows for ‘seeing the general in the particular’, for the generic algebraic solution of categories of problems, and for formulations and solutions at a general level. This general solution of all concrete cases at once by means of a parametric general solution requires the reification of the expressions and formulas that are involved in the generic problem-solving process.
- The parameter plays the role of *unknown* when the task is to select particular cases from the general parametric representation on the basis of an extra condition or criterion. This often requires a shift of roles and of hierarchy (Bills, 2001).

The conceptual analysis and the theoretical framework provided means to define the higher level understanding of the concept of parameter. Fig. 3 visualizes this level structure. In terms of the Van Hiele levels, we considered the placeholder level as the ground level of understanding the concept of parameter that is the basis for the second level. The three roles of changing quantity, generalizer and unknown share the property that formulas that contains them are considered as objects. Therefore, they are part of the second level understanding. The most important of the three was supposed to be the generalizer, as generalising is a key activity in algebra. The targeted higher level understanding, therefore, involved the jump from placeholder to the other parameter roles, ‘from Diophantus to Viète’, or in terms of the four-level structure (Fig. 1), the transition from the referential level to the general level.

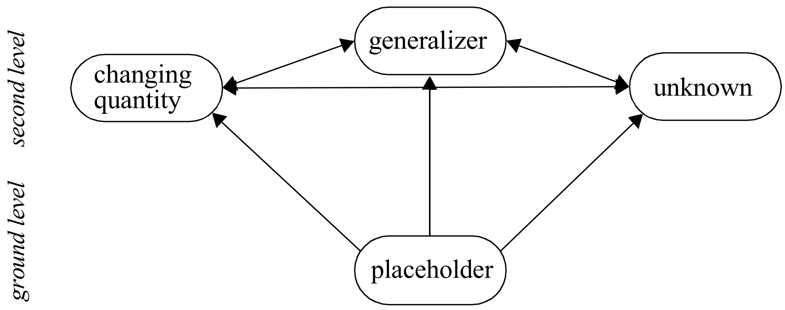

Figure 3 Levels of understanding the concept of parameter

The understanding of the higher parameter roles was supposed to require the reification of parametric expressions and formulas (Gravemeijer et al., 2000).

4 The instrumental approach to using computer algebra

The question now is how computer algebra use can contribute to the higher level understanding of the concept of parameter.
Generally speaking, computer algebra environments were supposed to offer opportunities for algebra education. Unlike other IT tools, the CAS has a full repertoire of algebraic procedures and representations. By freeing the students from the algebraic calculations, computer algebra would offer opportunities to concentrate on concept development and on problem-solving strategies. CAS use might help students to distinguish between concepts and skills (Monaghan, 1993) and to readjust the balance between them (Heid, 1988; O’Callaghan, 1998).
The integration of CAS into algebra education also might introduce some pitfalls. Because the CAS already ‘contains all the algebra’, the computer algebra tool might have a somewhat abstract and formal top-down character, and might turn out to be inflexible with respect to informal notation and syntax. Furthermore, the CAS might be a black box for students, as it carries out complex procedures in a way that is not transparent to them. Finally, the CAS might seem to be a microworld to the students that is not connected to the world of real-life problems or that of paper-and-pencil and mental mathematics (Drijvers, 2000).
For the purpose of understanding the concept of parameter, we identified the following CAS affordances. First, getting algebraic expressions as solutions to parametric equations and processing algebraic expressions further by means of substitution was expected to support the reification of algebraic expressions and formulas. Second, algebraic explorations that generated examples were supposed to foster generaliza-

tion over these situations, thus opening the way for the parameter as generalizer. Third, the flexibility concerning literal symbols and their roles that the CAS offers could be used to change the parameter role into that of unknown. Finally, although not exclusive to the computer algebra environment, the availability of a slider tool was supposed to support the view of parameter as changing quantity.
For the teaching experiments, we had to choose a specific computer algebra tool. For practical reasons we decided to use the handheld TI-89 symbolic calculator: the students would be able to use it permanently at school as well as at home, and it would not require a reorganization of the lessons. The limitations of the screen resolution were mitigated by having some lessons in the computer lab, using the TI-Interactive software package. We hoped that the individual character of the handheld calculator would not hinder collaboration between students, and decided to have them work in pairs in order to obviate that problem.
As we announced in the description of the theoretical framework, we used the instrumental approach to computer algebra use as a framework for understanding and interpreting the student-machine interactions. The central idea of the instrumental approach to using IT tools is that a 'bare' tool or artifact is not automatically a useful instrument. While using such a tool, the user has to develop mental 'schemes of instrumented action', or – to put it somewhat shorter – instrumentation schemes (Artigue, 1997, 2002; Guin & Trouche, 1999, 2002; Lagrange, 2000; Trouche, 2000). An instrument, according to this view, consists of the artifact (or a part of it), the mental scheme and the type of tasks for which it is appropriate (see Fig. 4).

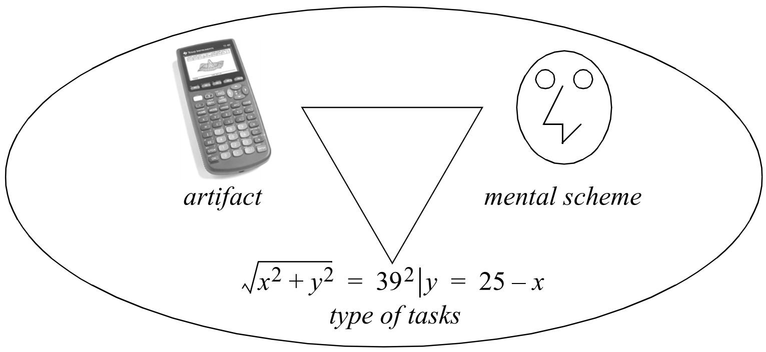

Figure 4 The instrument: a triad of artifact - mental scheme - task

The instrumentation schemes integrate technical skills and conceptual insights; difficulties in the development of the schemes – the instrumental genesis – often involve both aspects. As we cannot look insight the heads of the students to observe the mental schemes, we focused on the techniques, which can be considered as the observable parts of the instrumentation scheme, viz. the set of procedures in the en-

vironment that the students use for solving a specific type of problems.
In this study, the instrumental approach offered a framework to investigate the interaction between student and the computer algebra environment. The combination of technical and conceptual aspects within the instrumentation scheme made the theory promising for this purpose, because it both goes beyond the somewhat naive idea of reducing skills and reinforcing concepts, and takes seriously the difficulties of the instrumental genesis.

5 Methodology

We used a design research methodology in this study. Design research – which is also known as developmental research or development research – aims at developing theories and an empirically grounded understanding of 'how learning works' (Research Advisory Committee, 1996). Its main objective is to understand the students' learning process. This connects with the character of our research questions, which start with 'How can ... ' and not with 'Can ... '. One characteristic of design research is the importance attributed to the design of instructional activities, which is seen as a meaningful part of the research methodology as it forces the researcher to be explicit about choices, hypotheses and expectations (Edelson, 2002). Another important feature of design research is the adaptation of the learning trajectory throughout the research: instructional sequences and teaching experiment conditions are adjusted according to previous experience. Therefore, design research was particularly suitable for this study, because a full theoretical framework was not yet available and hypotheses were to be developed. Adapting the experimental situation is possible by means of the cyclic design that is used in design research. A macro research cycle consists of a preliminary phase (which in our case included the development of a hypothetical learning trajectory and the design of instructional activities), a teaching experiment phase and a retrospective phase, which includes data analysis and leads to feed-forward for the next research cycle (Gravemeijer, 1994).This study consisted of three main research cycles – G9-I, G9-II and G10-II – and one intermediate cycle, G10-I. The 9 and 10 stand for the grades in which the teaching experiments were conducted. The I and II refer to the first and second cohort of students involved. The tenth-grade population was a subset of the ninth-grade population, with the exception of a few students who entered the teaching experiment in tenth grade (see Fig. 5).
We will now briefly explain each of the phases within one research cycle. In the *preliminary phase*, a hypothetical learning trajectory (HLT) was developed (Simon, 1995). This involved assessing the starting level of understanding, formulating the end goal and developing a chain of mental steps towards that goal, as well as instructional activities that were expected to bring about this mental development. This was accompanied by the designer's description of why the activities were supposed to work and what kind of mental developments were expected to be elicited. Because

of its stress on the mental activities of the students and on the designer's motivation of the expected results, the HLT concept was a useful research instrument for monitoring the development of the hypotheses and for capturing the researcher's thinking.

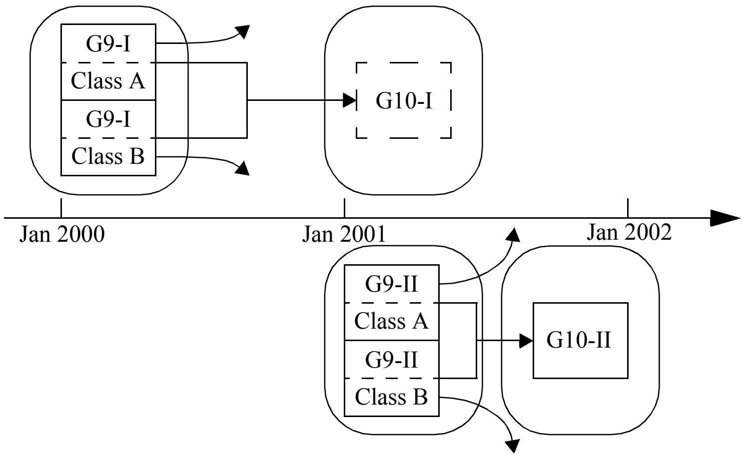

Figure 5 Arrangement of research cycles and teaching experiments

The development of the HLT was closely related to the design of instructional activities, which was done by using design heuristics such as guided reinvention, didactical phenomenology and mediating models. Key items in these instructional activities that would serve to monitor students' mental development in relation to the HLT were identified; also, questions for interviewing students on these key items were formulated and a prior global coding system for observations was developed.

The second phase of the design research cycle is the *teaching experiment phase* (Steffe, 1983; Steffe & Thompson, 2000). During the teaching experiments, we focused on data that reflected the learning process and provided insight into the students' thinking. The main sources of data, therefore, were observations of student behaviour and interviews with students. Most lessons were observed by two observers, who made field notes. The key instructional activities were assessed by means of mini-interviews with a selection of the students; video registrations were made of classroom discussions and of a selected pair of students. Written materials (notebooks, pretests and posttests) were gathered from all students. Altogether, 110 students were involved, and over 100 lessons were observed.

The final phase of the research cycle is the phase of *retrospective analysis*. A first step in this phase was the selection and analysis of data. The initial method of analysis was inspired by the constant comparative method (Glaser & Strauss, 1967; Strauss, 1987; Strauss & Corbin, 1998). The first findings led to adaptation of the prior coding system. Then the data were coded. The coding process was partially done by two researchers, in order to achieve an intersubjective agreement. The conclusions of the data analysis were translated into feed-forward for the next research cycle. The feed-forward comprised changes in the HLT and in the instructional activities, and even slight changes in the focus of the subsequent teaching experiment.

6 Through the research cycles

Fig. 5 shows the arrangement of the design research cycles in this study. We will now briefly review each of these cycles, and the teaching experiments in particular. These teaching experiments were carried out at a school in Bilthoven, a town not far from the city of Utrecht.

The first research cycle (G9-I) included a teaching experiment that was carried out in two ninth-grade classes of the pre-university stream (in Dutch: vwo, 14- to 15-year-old students). The experiments ran for five weeks with four mathematics lessons each week. The HLT for these experiments followed the line parameter as placeholder-generalizer-changing quantity-unknown. The instructional activities were presented in two booklets: 'Introduction TI-89', which was intended to introduce the students to the use of the TI-89 symbolic calculator, and 'Changing algebra', which was intended to help them to develop an insight into the concept of parameter.

The results showed that generalization was hindered by instrumentation problems, in particular with a scheme for solving systems of equations. Fig. 6 shows this scheme of isolate-substitute-solve (ISS) for the system $x + y = 31$, $x^2 + y^2 = 25^2$.

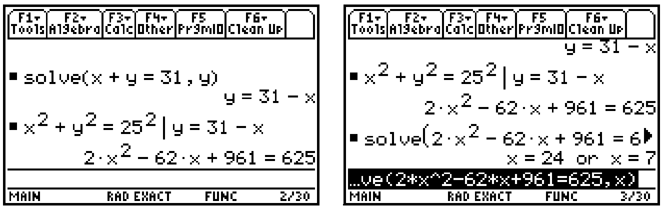

Figure 6 The isolate-substitute-solve scheme on the TI-89

In some cases students felt no need for generalization, or the generalization had a superficial character of pattern recognition without intrinsic understanding. The parameter as changing quantity was understood better, although we missed features for visualizing the dynamics on the TI-89. The parameter as unknown did not receive much attention. The shift of roles of the parameter that was involved presented the students with difficulties. The substitution of expressions and the appearance of expressions as solutions of parametric equations supported reifying expressions and overcoming the lack of closure obstacle. The results of the posttest confirmed that both the generalization and the application of the ISS scheme were bottle-necks for many students. As feed-forward from the G9-I research cycle, we decided to reorder the global HLT line into placeholder-changing quantity-generalizer-unknown, in order to delay generalization until a more dynamic and hierarchic view of parameter was established. For this purpose, we aimed at finding a better means to represent the sliding parameter on the TI-89. To overcome instrumentation problems, we included practising simple instrumentation schemes for solving and substituting more extensively before integrating them into the composed ISS scheme.

The teaching experiment of the second research cycle (G9-II) was carried out in a second cohort of two ninth-grade classes of the pre-university stream (14- to 15-year-old students). The experiment was similar to the G9-I teaching experiment and ran for five weeks with four mathematics lessons each week. The HLT followed the line that emerged from the feed-forward from G9-I. The concept of parameter as changing quantity was supported by two TI-89 programmes called SHOOT and SLIDE. The student text included a part that introduced the students to the use of the TI-89 and a part that focused on developing an insight into the concept of parameter.

The results showed that the concept of parameter as changing quantity was supported by the SHOOT and SLIDE applications, although some students tended to see the parameter as running only through integer values. Starting the HLT with this parameter role was an improvement compared to the G9-I HLT. For generalization, we encountered similar difficulties as in G9-I, though to a lesser extent. Despite the attention to simple instrumentation schemes for solving and substituting, the composed ISS scheme still caused difficulties. Also, the students often did not feel the need for generalization, perhaps because of the abstract and complex problem situations. The parameter as unknown received more attention than it did in G9-I, but the complexity of the problem situations prevented the students from keeping track of the problem-solving strategy and the changing roles of the parameter therein.

The results of the posttest confirmed these findings, although the performance of the ISS instrumentation scheme and the generalizations with parameters slightly improved compared to the G9-I results. After the posttest, final interviews were held with 19 students. During these interviews the students did not have a symbolic calculator at their disposal. The results of these interviews suggested a transfer of substitution as a computer algebra technique to substitution as a paper-and-pencil technique, and an improved reification of expressions. For example, the students were able to substitute $y = a - x$ into $x^2 + y^2 = 10$ by hand, and one of them used the TI-89 notation for substitution while doing so (Fig. 7).

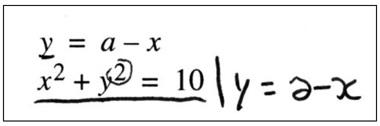

Figure 7 Transfer of notation

As feed-forward from the G9-II research cycle, we first noted that the global HLT along the line placeholder-changing quantity-generalizer-unknown was adequate and needed no further change. For the parameter as changing quantity, the use of a continuous slider tool might be used to avoid the notion of a discrete reference set. The findings suggested that the parameter as generalizer might be better addressed without the complication of the instrumentation of the ISS scheme. This required looking for problem situations that could be solved with one equation rather than with a system of two equations. Furthermore, the problem situations were targeting the general level too directly. The use of realistic contexts in which the parameter would have a meaning to the students might lead to easier and more meaningful generalizations, and would address the referential rather than the general level. Graphs might be used as contexts as well, and could be useful for the transition from the parameter as changing quantity to the parameter as generalizer. The use of realistic problem situations might also address the parameter as unknown in a more natural way.

The G10-I intermediate cycle included a five-lesson teaching experiment in tenth grade. These students had opted for the exact stream and all but one had participated in G9-I as well. Because of a lack of time and the full regular curriculum in tenth grade, this intermediate cycle rather was more of a short 'finger exercise' for G10-II than a full research cycle. The results showed that the students easily recalled the TI-89 skills they had learned a year earlier. Once more, using a slider tool for the parameter as changing quantity was preferred to replacing parameter values by substitution.

The third and last complete research cycle (G10-II) included a 15-lesson teaching experiment in one tenth-grade class of the exact stream of pre-university stream. All but two students had participated in the G9-II teaching experiment. In accordance with the feed-forward from G9-II, we stuck to the global trajectory of placeholder-changing quantity-generalizer-unknown. For the parameter as changing quantity, the continuous slider tool of the TI-Interactive software package was used. To address meaningful parameters and formulas and natural generalizations, realistic problem situations were more prominent than in the instructional activities of the previous teaching experiments. We expected this would also improve the understanding of the parameter as unknown.

The results showed that the use of the continuous slider tool improved the perception of the changing parameter running through a continuum. Contexts in which the parameter had an evident meaning in the graph allowed for meaningful generalizations. Realistic contexts also improved insight into the meaning and structure of formulas and expressions. Meanwhile, entering complex expressions with fractions and powers and recognizing equivalent expressions provided difficulties. The fact that the problem situations led to one rather than to two equations averted the difficulties of the ISS instrumentation scheme. The parameter as unknown was addressed with more success than in previous experiments, probably also due to the more meaningful problem contexts. However, the complexity of the formulas involved seemed to be a crucial factor.

The results of the final task that ended the G10-II teaching experiment were positive. Many appropriate generalizations were found and the students dealt adequately with complex formulas. The realistic problem situation on the speed of a traffic flow helped them to give meaning to the results.

7 Results on computer algebra use and the concept of parameter

The first research subquestion of this study concerns the contribution of computer algebra use to a higher level understanding of the concept of parameter. This higher level understanding was defined as insight into the higher parameter roles of changing quantity, generalizer and unknown. We will now summarize the results on this issue.

As expected from our conceptual analysis, the placeholder view of the parameter was the starting level for almost all students. To extend this concept to the role of *changing quantity*, the computer algebra environment allowed for changing the parameter values one by one. Investigating the dynamic effects of the 'sliding parameter' by means of a slider tool – which is not an exclusive CAS feature – proved to be more efficient. This led to the perception that parameter change affected the complete situation and the graph as a whole, and was a second-order change. If the parameter had a meaning to the student in the context or as a graphical property, examining the sliding graph invited algebraic verification and generalization. However, there was a risk of only superficial examination without reflection or verification. Furthermore, the parameter as changing quantity tended to dominate the students' view of parameter, so that they sometimes seemed to forget the other parameter roles. Altogether, a higher level understanding of the parameter as changing quantity was achieved.

The work in the computer algebra environment supported the transition to the parameter as *generalizer* by allowing the repetition of similar procedures for different parameter values. This generated examples that were the basis for generalization and for solving parametric equations. The fact that parametric equations led to expressions rather than numerical solutions fostered the reification of expressions and formulas, and the overcoming of the lack of closure obstacle. The dynamic graphs that appeared from the work with the sliding parameter in some cases elicited generalization, for example in the case of the graph of $y = (x-a)^2 + a$. One pair of students noticed that the vertex of the graph had coordinates (a, a) for all values of a. One of them commented that 'the vertex is at a' and that 'the y- and x-coordinates are the same as a'. Later, during the whole-class discussion, the other student explained this generalization by referring to an exemplary parameter value.

Maria: *Yes, I think that vertex, that were the x and y coordinates, they were, so to say, equal to the a. The vertex was (1, 1), then a was 1 too.*

Many students achieved a higher level understanding of the parameter as generalizer only to a limited extent. In some cases, the students did not feel the need for generalization. In other cases, generalization came down to phenomenological pattern recognition without intrinsic understanding. The instructional activities were aimed too directly at the general level, and the formulas in some cases were too complex. Finally, instrumentation problems hindered generalization. The essential step of generalising relations – in which parameters are used to unify a class of situations – seemed to be primarily a mental one, upon which computer algebra had hardly any influence.

The possibility to solve equations in the computer algebra environment with respect to any unknown improved the students' flexibility concerning the roles of the literal symbols. The results on parameter as *unknown* were mixed: although this parameter role was understood in simple cases, in more complex ones the students failed to distinguish the changing roles and meanings of the variables and parameters. The change in the hierarchy between variable and parameter in complex cases was more difficult to keep track of.

Two issues seemed to influence the development of the higher level understanding of the concept of parameter: the use of realistic problem situations and the insight into the meaning and structure of expressions and formulas. We have already mentioned the importance of *realistic problem situations*. Despite our initial ideas, using the CAS as a mathematical environment that would make references to reality redundant, did not work out for the students of this age and level. Addressing the general level directly did not work out either, because the students had not yet developed an appropriate mathematical framework. Instead, they needed meaning for parameters, expressions and formulas from the realistic problem situation at a referential level. Meanwhile, realistic problem situations did not hinder generalization and abstraction beyond these specific contexts.

The *expressions and formulas* that the students encountered were more complex than usual because of the presence of parameters. The CAS output, whether algebraic or graphical, invited a closer inspection of the formulas and expressions involved, particularly when this was suggested by the task or by the teacher. Entering complex formulas required insight into the algebraic structure of the formulas and expressions. Therefore, although insight into the meaning and structure of expressions and formulas was an obstacle during the CAS work, such work seemed to improve the insight. The work in the computer algebra environment also supported the development of symbol sense, the reification of formulas and expressions, and the overcoming of the lack of closure obstacle. As an overall conclusion on the contribution of computer algebra use to a higher level understanding of the concept of parameter, we did indeed find that the CAS activities contributed to the development of insight into the different parameter roles. This contribution was more evident for the parameter as changing quantity than for the generalizer and the unknown. Furthermore, the reification of formulas and expressions was supported. In the meantime, we found that realistic starting points were indispensable for meaningful work in the computer algebra environment.

Several factors might optimise the targeted learning process. First, the data from the final teaching experiment suggest that it might be fruitful to integrate rather than separate the different parameter roles, so that the students develop a more integrated view of the parameter. Second, we noticed that starting with models that refer to concrete problem situations leads to better results with students of this age and level. Third, we conjecture that the contribution of computer algebra to the conceptual development would improve if it were used for a longer period, so that instrumentation difficulties were less dominant. Finally, we think that whole-class discussions, guided by the teacher, might have stimulated the collective conceptual development to a greater extent. We notice that the teachers considered the student cohorts involved in this study as weak cohorts. They might have needed more guidance.

The difficulties the students encountered while working with the CAS suggest that the instrumentation process is important. We will discuss this issue in the next section.

8 Results concerning the instrumentation of computer algebra

The second research subquestion concerned the instrumentation of computer algebra and the relation between computer algebra techniques and conceptual development. We will now summarize the results of this study on this issue.

As simple instrumentation schemes, the solve scheme and the substitute scheme were investigated. For both schemes, which are related to the concept of parameter, the instrumental genesis involved the extension of conceptual understanding. For the *solve* scheme, the syntax and the application to parametric equations led the students to realize that an equation is always solved with respect to an unknown, that solving a parametric equation leads to expressing one variable in term of others, and that the solution in that case is an expression. For *substitution*, the substitution of expressions within the computer algebra environment led the students to extend their conceptual understanding with the notion that only isolated forms can be substituted, and that expressions can be considered as ‘things’ that can be ‘pasted’ into a variable. In line with Sfard’s ideas, the operation on the expression enhances its object character (Sfard, 1991).

Although the instrumental genesis of the solve and substitute schemes seemed to progress smoothly, combining them into the composed *isolate-substitute-solve* (ISS) scheme caused persistent difficulties. This indicated that the integration of simple schemes into a more comprehensive scheme requires a high level mastering of the component schemes. The instrumental genesis apparently needed more time than was given to the students.

We identified a number of obstacles that hindered the instrumental genesis, such as the way in which the CAS deals with the difference between numerical approximations and exact algebraic results, the difficulties with entering expressions containing parentheses, square root signs and powers, and interpreting results. Recognizing the equivalence of CAS output and expected results was a difficult issue. Although many of these obstacles were related to a lack of symbol sense, they also fostered its development. And although the students' lack of insight into the structure of expressions and formulas led to errors in the CAS work, these problems stimulated a closer look at these structures. In that sense, obstacles also provided opportunities for learning, when managed appropriately by the teacher.

Some of the observed difficulties with entering expressions and interpreting algebraic results were related to the paper-and-pencil work that the students were used to. Incongruence between computer algebra technique and paper-and-pencil technique explained some of the instrumentation problems. When the techniques in the two media were congruent, the transfer of notation, strategy and technique between the paper-and-pencil and computer algebra environments was observed. An explicit comparison between computer algebra technique and paper-and-pencil method was effective for making students record the degree of congruence. A second condition for transfer was the transparency of the computer algebra technique: when the students were able to understand the way in which the CAS arrived at its results rather than perceiving it as a black box, transfer to the paper-and-pencil work took place. The data indicate that the students developed different preferences concerning paper-and-pencil work versus work in the computer algebra environment.

The teacher played an important role in the instrumental genesis. Different teacher behaviour led to different instrumented techniques. Whole-class demonstrations and discussions proved to be important for collective instrumentation; this aspect of instrumentation should have received more attention in the teaching experiments. Furthermore, a new didactical contract had to be established concerning the relation between by-hand work and machine work, and between numerical-graphical methods and algebraic methods. The fact that for the teachers the integration of computer algebra technology into teaching was new and lasted for a short period made it hard to establish a new didactical contract and to foster collective instrumental genesis.

Overall, the results of this study suggest a close and reciprocal relation between CAS techniques and conceptual understanding. Students had to build up instrumentation schemes that combined technical and conceptual aspects. This instrumental genesis required time and effort, and obstacles had to be overcome. Seemingly technical difficulties were often related to a lack of conceptual understanding. Paper-and-pencil routines were also involved in this relationship. As consequences for teaching, we recommend developing a new didactical contract, fostering collective instrumentation by means of whole-class discussions and demonstrations, and discussing the congruence or incongruence between paper-and-pencil and CAS techniques.

9 CAS use and the learning of algebra in general

In this section we first discuss the conclusions concerning the main research question. Then follows a reflection on the study. We conclude with some recommendations.

Conclusions

The main research question concerns the contribution of computer algebra use to understanding algebraic concepts and operations in general. By means of extrapolating the answers to the two research subquestions to this more general level, we conclude the following.

First, some of the findings not only concern the understanding of the concept of parameter, but also show that computer algebra use can contribute to the understanding of algebraic concepts and operations in general. Symbol sense could be developed, as could insight into the meaning and structure of formulas and expressions. Algebraic notions were extended. Computer algebra offered affordances for combining different representations, repeating procedures as a preparation for generalization, and dealing with expressions as objects. Critical for capitalising on these affordances were the didactic embedding and the use of meaningful formulas that referred to realistic problem situations, rather than to the general level directly.

Second, the observed instrumental genesis of CAS instrumentation schemes, provided that it was orchestrated adequately, included conceptual development concerning algebra in general. The development of the solve scheme and the substitute scheme led to new insights into solving and substituting. Although incomplete instrumentation hindered learning, instrumentation obstacles provided learning opportunities. Conditions that fostered the instrumentation of computer algebra were the congruence between CAS technique, mental conception and paper-and-pencil technique, and the transparency of CAS procedures. These turned out to be hard issues, as some CAS procedures were not transparent to the students, and CAS notations and syntax differed from what the students were used to from paper-and-pencil experience.

When we compare these finding with our initial expectations, we notice that the instrumentation difficulties were more persistent than we had expected, and that the role of the teacher was more important than we had foreseen.

Furthermore, our ideas on focusing on the general rather than on the referential level did not come true, and realistic problems could not be missed as starting points. What is realistic to students depends on their age and level: we conjecture that students in advanced mathematics can more easily experience the CAS as a realistic environment on its own.

Reflection

When we look back at this study, what can we say about the role of theory in it, about the methodology that we used and about the generalizability of the results? We will first look back at the elements of the *theoretical framework*.

The domain-specific instruction theory of *Realistic Mathematics Education* helped us to understand why the students needed realistic problem situations to start with: the algebraic objects and procedures were not yet part of an appropriate network of mathematical relations. The distinction of the referential level and the general level (see Fig. 1) clarified our goals as well as the difficulties that the students encountered. The notions of horizontal and vertical mathematization made it clear why generalization and abstraction did not work out on many occasions. Finally, the RME view was helpful in understanding the problem of the black box character of using the CAS.

The *level theories* acted as background for defining the higher level understanding of the concept of parameter and helped in developing the learning trajectory and for designing instructional activities.

We defined *symbol sense* as insight into the structure and meaning of formulas and expressions. Looked at in this way, it was a means of interpreting the students' behaviour with formulas and expressions in the computer algebra environment: symbol sense was a prerequisite to productive CAS use, but could also develop from it. As feedback to the theoretical notion of symbol sense, we would suggest that it needs a more precise definition.

Theories on symbolizing made it clear that the character of the computer algebra environment and the initial choices of this study did not offer much opportunity for a bottom-up signification process. However, the instrumentation of computer algebra fostered the development of meaningful mathematical objects, and of a framework of mathematical relations, which was connected with symbols and expressions that the students encountered while working in the computer algebra environment. Therefore, the relation between the instrumentation approach and theories on symbolizing deserves further investigation.

The *process-object duality* was useful for identifying the influence that computer algebra use had on the development of an object view of formulas and expressions by the students. Together with the notion of symbol sense, reification was a frame of reference for monitoring the progress of algebraic insight into formulas and expressions.

The *theory of instrumentation* helped us to interpret student behaviour in the computer algebra environment and to observe the relation between technical and conceptual components within instrumentation schemes. Sometimes the instrumental genesis was hindered by conceptual barriers, but at other times it fostered further conceptual development, as was suggested by the theory. This study worked out some concrete schemes and stressed the relevance of the congruence and transparency criteria. We suggest paying more attention to the link between the computer algebra technique and the corresponding paper-and-pencil technique.

Altogether, we see that most of the elements of the theoretical framework contributed to this study, despite the fact that the elements of the framework – except for the instrumental approach – were not dedicated for use in the context of this study. One of the main results of this study for the theoretical framework, therefore, is the theories' perceived contribution in a setting for which they were not developed, which suggests a wide applicability.

How do we look back at the *methodology*? Reflecting on this, we feel that the design research paradigm was appropriate for the purpose of this study. The cyclic character allowed for adapting experimental settings, and the design of the learning trajectory and instructional activities served for making explicit our intentions. In the preliminary phase, the hypothetical learning trajectory was a productive way of monitoring the development of the hypotheses and expectations throughout the research. Prior identification of key items and their expected outcomes in the design phases guided classroom observations during the teaching experiments and data analysis in the retrospective phases. The mini-interviews on these key items provided relevant data, although this data gathering technique required a good attunement with the teacher and refrainment from playing the role of assistant teacher. The whole-class videos did not contribute much to the data, as the whole-class discussions in most cases did not address the key items or concepts that we wanted to observe. In the retrospective phase, the data analysis method – which combined a prior coding system with an approach similar to the constant comparative method – was effective. Working together with a second researcher who coded the data independently improved both the quality of the data analysis and the coding system. The method of formulating feed-forward for the next research cycle was helpful for capturing research progress.

How about the *generalizability* of the findings? Do the findings of this study depend on the specific choices that were made, or can they be generalized to other situations? Many issues that showed up in this study – such as the reification of formulas and expressions, and the development of symbol sense – are not specific to the concept of parameter. Therefore, we think that the scope of these findings exceeds the understanding of the concept of parameter and can be generalized to learning algebra in general. We also think that the conclusions can be generalized as regards instrumentation. Despite differences between computer algebra environments at a detailed level, these systems are based on common principles and similar instrumentation issues play a role. The question whether the instrumental approach can be applied to other IT environments deserves further research.

Generalization to other Dutch schools can be made as well, as long as the specific educational setting is taken into account. Generalization to CAS use in other grades should be considered with care. We expect more instrumentation problems to appear in lower grades, and we recommend longer periods of CAS use in ninth and tenth grades. For higher grades, we expect similar instrumentation issues and affordances to play a role, and we conjecture that in those grades the students will have more mathematical means for overcoming obstacles and benefiting from opportunities.

Recommendations

The conclusions of this study led to recommendations for teaching, for software design and for further research. As regards *teaching algebra using computer algebra*, the results suggest that it is important to anticipate on computer algebra use, to be explicit about the changing didactical contract, to orchestrate individual and collective instrumentation, to have students compare CAS techniques with paper-and-pencil techniques and to have students reflect on the way CAS works.

As for *designing algebra software* for educational purposes, we recommend taking care of the criteria of the transparency and congruence of the technological environment. The black box character and idiosyncratic features inhibit instrumentation. Flexibility with regard to notations, syntax and strategy is a third criterion that should be met.

As regards *further research* on learning mathematics in a technological environment, we recommend taking into account the complete pedagogic situation rather than investigating the use of the technological environment as an isolated phenomenon, so that the social and the psychological perspective can be coordinated (Yackel & Cobb, 1996). Here, the cultural-historical activity theory – which considers tool use as embedded in a social practice – could provide a fruitful additional perspective. Furthermore, the relation between the instrumental approach and theories on symbolizing would be an interesting starting point for further theoretical development. A longitudinal study on symbolizing in a computer algebra environment, including adequate teaching that prepares for computer algebra use, is recommended. The extension of the scope of this study to other mathematical topics – such as the concept of function – and to other IT tools – such as software for dynamic geometry or Java applets – is also recommended. Finally, the role of technology in assessment requires further study.

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