Principles and practices in arithmetic teaching
Innovative approaches for the primary classroom

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individual teacher. Over time these balances are continually shifting according to the degree of power held by different interests, and in particular depending on the changing degree of centralization.

The adjustments needed in what would be a complex task, even in stable conditions, can be potentially exciting but can also be potentially exhausting and undermining of confidence and self-esteem. Getting the degree of centralization and the pace of change right is something which we in England have not yet achieved.

Realistic mathematics education in the Netherlands

Marja van den Heuvel-Panhuizen

Introduction

This chapter can be seen as a guided tour through some main aspects of the Dutch approach to mathematics education and will focus on the number strand in primary school mathematics. The main questions that will be considered are:

- How do we teach arithmetic in primary schools in the Netherlands?
- What does our arithmetic curriculum contain?

This guided tour will not take you into classrooms nor will it provide you with a representative sample of Dutch classroom practice (this can be seen in the chapter by Julie Menne in this volume). Instead, it will introduce you to a theoretical framework of teaching mathematics and the teaching activities that are in tune with the ideas this involves. What it will do is show you attainment targets identified in the Netherlands regarding mathematics, and give you an idea of the position that we want to reach in the end. Do not expect to get complete answers and a thorough overview of the Dutch approach to mathematics education from this trip as our approach to mathematics education is too complex. Moreover, the difficulty is that there is not a unified Dutch approach. Instead, there are some shared basic ideas about the what-and-how of teaching mathematics.

These ideas have been developed over 30 years and the accumulation and repeated revision of these ideas has resulted in what is now called Realistic Mathematics Education (RME). Inherent in RME, with its founding idea of mathematics as a human activity, is the idea that it can never be considered as a fixed and finished theory of mathematics education. We see RME as 'work under construction' (van den Heuvel-Panhuizen 1998). The different accents are the impetus for this continuing development. (For a more detailed discussion about the RME approach by means of developmental research, see Chapter 11, this volume.)
This chapter will outline key characteristics in the teaching approaches now widely adopted in the Netherlands and will identify the thinking and research that has led to these ideas. It will show some of the work that is done to implement the RME approach in classroom practice.

The Dutch approach to mathematics education

History and founding philosophy

The development of Realistic Mathematics Education started almost 30 years ago with the foundations laid by Freudenthal and his colleagues at the former IOWQ, the oldest predecessor of the Freudenthal Institute. The impulse for the reform movement was the inception, in 1968, of the Whiskobas project, initiated by Wijdeveld and Goffree. The present form of RME was mostly determined by Freudenthal’s (1977) view that mathematics must be connected to reality, stay close to children and be relevant to society. Instead of seeing mathematics as subject matter that has to be transmitted, Freudenthal stressed the idea of mathematics as a human activity. Education should give students the ‘guided’ opportunity to ‘reinvent’ mathematics by doing it within a process of progressive mathematization (Freudenthal 1968).

Later on, Treffers (1978, 1987b) explicitly formulated the idea of two types of process, distinguished as ‘horizontal’ and ‘vertical’ mathematization. In horizontal mathematization the students come up with mathematical tools that can help to organize and solve a problem located in a real-life situation. Vertical mathematization is the process of reorganization within the mathematical system itself, for instance, finding shortcuts and discovering connections between concepts and strategies, and then applying these discoveries. In short, horizontal mathematization involves going from the world of life into the world of symbols, while vertical mathematization means moving within the world of symbols (Freudenthal 1991: 41–2).

Misunderstanding of ‘realistic’

Despite this clear statement about horizontal and vertical mathematization, RME became known as ‘real-world mathematics education’. The reason, however, why the Dutch reform of mathematics education was called ‘realistic’ was not only because of the connection with the real world, but was related to the emphasis that RME puts on offering the students problem situations that they can imagine. The Dutch translation of ‘to imagine’ is ‘zich REALISEREN.’ It is this emphasis on making something real in your mind, that gave RME its name. For the problems to be presented to the students this means that the context can be a real-world context but this is not always necessary. The fantasy world of fairy tales and even the formal world of mathematics can be very suitable contexts for a problem, as long as they are real in the student’s mind.

Principles underpinning teaching methods in the Netherlands

RME reflects a certain view of mathematics as a subject of how children learn mathematics and of how mathematics should be taught. These views can be characterized by the following six principles,1 some of which originate more from the perspective of learning and some of which are more connected to the teaching perspective.

1 Activity principle

The idea of mathematization clearly refers to the concept of mathematics as an activity which, according to Freudenthal (1971, 1973), can best be learned by doing (see also Treffers 1978, 1987b). The students, instead of being the receivers of ready-made mathematics, are treated as active participants in the educational process, in which they develop by themselves all sorts of mathematical tools and insights. According to Freudenthal (1973), using scientifically structured curricula, in which students are confronted with ready-made mathematics, is an ‘anti-didactic inversion’. It is based on the false assumption that the results of mathematical thinking, placed in a subject-matter framework, can be transferred directly to the students. The consequence of the activity principle is that the students are confronted with problem situations in which they can, for instance, produce fractions and can gradually develop an algorithmic way of multiplication and division, based on an informal way of working. Related to this principle, students’ own productions play an important role in RME.

2 Reality principle

In RME the overall goal is that the students must be able to bring into action mathematical understandings and tools when they have to solve problems. This implies that they must learn ‘mathematics so as to be useful’ (see Freudenthal 1968). The reality principle is, however, not only recognizable at the end of the learning process in the area of application; reality is also conceived as a source for learning mathematics. Just as mathematics arose from the mathematization of reality, so must learning mathematics also originate in mathematizing reality.

In the early days of RME it was already emphasized that if children learn mathematics in an isolated fashion, divorced from experienced reality, it will be quickly forgotten and the children will not be able to apply it (Freudenthal 1971, 1973, 1968). Rather than beginning with certain abstractions or definitions to be applied later, one must start with rich contexts demanding mathematical organization or, in other words, contexts that can be mathematized (Freudenthal 1968, 1979). While working on context problems the students can develop mathematical tools and understanding.
3 Level principle

Learning mathematics means that students pass various levels of understanding: from the ability to invent informal context-related solutions to the creation of various levels of short cuts and schematization, to the acquisition of insights into the underlying principles and the discernment of even broader relationships. The condition for arriving at the next level is the ability to reflect on the activities conducted. This reflection can be elicited by interaction.

Models serve as an important device for bridging this gap between informal, context-related mathematics and more formal mathematics. First, the students develop strategies closely connected to the context. Later on, certain aspects of the context situation can become more general which means that the context can assume, more or less, the character of a model, and as such can give support for solving other, but related, problems. Eventually, the models give the students access to more formal mathematical knowledge. In order to fulfil the bridging function between the informal and the formal level, models have to shift from a 'model of' a particular situation to a 'model for' all kinds of other, but equivalent, situations (see Streefland 1991; Treffers 1991a; Grave-meijer 1994b; van den Heuvel-Panhuizen 1995). The bus context (van den Brink 1989) is an example of a 'daily life' context which can evolve to a more general and formal level. In the beginning the situation is more or less pictured to describe the changes at the bus stop (see Figure 4.1). Later on the bus context has become a 'model for' understanding all kinds of number sentences. Then the students can go far beyond the real bus context. They can even use the model for backwards reasoning (see Figure 4.2).

An important requirement for having models functioning in this way is that they are rooted in concrete situations and that they are also flexible enough to be useful in higher levels of mathematical activities. This means that the models will provide the students with a foothold during the process of vertical mathematization, without obstructing the path back to the source.

![Figure 4.1 At the bus stop](Source: Streefland (1996: 15, 16))

![Figure 4.2 Two number sentences](Source: Streefland (1996: 17))

The strength of the level principle is that it both guides the growth in mathematical understanding and it gives the curriculum a longitudinal coherency. This long-term perspective is very characteristic of KME. There is a strong focus on the connection between what is learned earlier and what will be learned later. A powerful example of such a 'longitudinal' model is the number line (see Figure 4.3). It begins in first grade as (a) a beaded necklace on which the students can practice all kinds of counting activities. In higher grades this chain of beads subsequently becomes (b) an empty number line.

![Figure 4.3 Different appearance of the number line](a, b, c)
for supporting additions and subtractions (see the chapter by Julie Menne about her Jumping Ahead programme for underachievers in the early grades and Ruthven, Chapter 12) (c) a double number line for supporting problems on ratios, and (d) a fraction/percentage bar for supporting working with fractions and percentages.

4 Intertwine principle

Characteristic of RME is the fact that mathematics as a school subject is not split up into distinctive learning strands. From a deeper mathematical view the chapters within mathematics cannot be separated. Moreover, solving rich context problems often means that you have to apply a broad range of mathematical tools and understandings. For instance, the mirror activity in Figure 4.4 clearly shows how geometry and early arithmetic can go together. The strength of the intertwine principle is that it gives coherence across the curriculum.

5 Interaction principle

Within RME, the learning of mathematics is considered as a social activity. By listening to what is found by others and discussing these findings, the students can get ideas for improving their own strategies. Moreover, the interaction can evoke reflection which enables the students to reach a higher level of understanding. The significance of the interaction principle implies that whole-class teaching plays a very important role in the RME approach to mathematics education. This, however, does not mean that the whole class is proceeding together and that every student is following the same track and reaching the same level of development at the same moment. On the contrary, within RME, children are considered as individuals, each following an individual learning path. This view of learning often results in pleas for splitting classes up into small groups of students each following their own learning trajectories. In RME, however, there is a strong preference for keeping the class together as a unit of organization. Instead of adapting lessons to the different ability levels of the students by means of ability grouping, differences between students are catered for by providing them with problems that can be solved with different levels of understanding.

6 Guidance principle

One of Freudenthal’s (1991) key principles for mathematics education is that it should give students the ‘guided’ opportunity to ‘re-invent’ mathematics. In RME both the teachers and the curriculum have a crucial role in steering the learning process, but not in a fixed way by demonstrating what the students have to learn. This would be in conflict with the activity principle and would lead to pseudo-understanding. Instead, the students need room to construct mathematical insights and tools by themselves. In order to reach this position the teachers have to provide the students with a learning environment in which this constructing process can emerge. A requirement for this is that teachers must be able to foresee where and how they can anticipate the students’ understandings and skills that are just coming into view in the distance (see also Streetland 1985). Educational programs should contain scenarios which have the potential to work as a lever to reach shifts in the students’ understanding. Crucial for these scenarios is that they always keep in view the long-term teaching/learning trajectory based on the goals one wants to attain. Without this perspective it is not possible to guide the students’ learning.

What are the determinants of our mathematics curriculum?

Unlike many other countries, at primary school level the Netherlands does not have centralized decision making regarding curriculum syllabuses, textbooks and examinations (see Mullis et al. 1997). Teachers have flexibility with respect to their teaching and can make many educational decisions either by themselves, or as a school team, including choice of textbooks and even what curriculum to teach. To give some examples, teachers are allowed to make changes in their timetable without asking the director of the school (who often teaches a class too), and the teacher’s advice at the end of primary school, and not a test score, is the most important criterion for allocating a student to a particular level of secondary education. Despite this freedom in educational decision making, the mathematical topics addressed in primary schools do not differ a lot between schools. In general, all the schools follow the same curriculum. This takes me to the question: what determines this curriculum?

Until recently, there were three important determinants for mathematics education in primary school:

- the mathematics textbook series;
- the ‘Proeve’ – a document that describes the mathematical content to be taught in primary school; and
• the key goals to be reached by the end of primary school as described by the government.

The determining role of textbooks

Many reform movements around the world appear to be aimed at getting rid of textbooks. In the Netherlands, however, the contrary is the case. Here, the improvement of mathematics education depends largely on textbooks, which have a determining role in mathematics education and are the most important tools that guide the teachers’ teaching. This is true of both the content and the teaching methods, although regarding the latter the guidance is not sufficient to reach all teachers. Many studies have, for example, provided indications that the implementation of RME in classroom practice is not yet fully achieved (Gravemeijer et al. 1993; van den Heuvel-Panhuizen and Vermeeren 1999).

Currently, about 80 per cent of Dutch primary schools use a mathematics textbook series which was inspired, to a greater or lesser degree, by RME. Compared to even 10 or 15 years ago, this percentage has changed considerably; at that time, only half of the schools worked with such a textbook series (De Jong 1986). The development of textbook series is done by commercial publishers (chapter by Kees Buys in this volume). The textbook authors are independent developers of mathematics education, but they can also make use of the ideas for teaching activities resulting from developmental research at the Freudenthal Institute (and its predecessors) and at the Dutch Institute for Curriculum Development (the SLO).

The ‘Proeve’: a domain description of primary school mathematics

Looking back at our reform movement in mathematics education, it is clear that the reforms proceeded in a very interactive and informal way. There was no interference from the government. Instead, developers and researchers, in collaboration with teacher educators, school advisors and teachers, worked out teaching activities and learning strands. Later on, these were included in textbooks. An important aid to the development of textbooks has been the guidance which, since the mid-80s, comes from a series of publications called the ‘Proeve van een Nationaal Programma voor het reken-wiskundeonderwijs op de basisschool’ (Design of a national programme for mathematics education at primary school) (Trefers et al. 1989). It is of note that the title refers to a ‘national programme’ while, in fact, there was no interference from the government.

Trefers is the main author of the ‘Proeve’ and work on this series is still going on. The documents contain descriptions of the various domains within mathematics as a school subject and, although it is written in a very accessible style with a lot of examples, it is not written as a series for teachers. Instead, it is meant to be a support for textbook authors, teacher trainers and school advisors, many of whom are significant contributors to the series.

The key goals for mathematics education

Until recently there was no real interference from the Dutch government regarding the content of the educational programmes. A few years ago, however, the policy of the government changed. In 1993, the Dutch Ministry of Education published a list of attainment targets, called ‘key goals’. For each subject these goals describe what has to be learned by the end of primary school. For mathematics the list consists of 23 goals, split up into six domains (see Table 4.1).

Compared to goal descriptions and programmes from other countries it is notable that some widespread mathematical topics are not mentioned in this list, such as, for instance, problem solving, probability, combinatorics and logic. Another striking feature of the list is that it is so limited. This means that the teachers have a lot of freedom in interpreting the goals. At the same time, however, such a list does not give much support to teachers. As a result the list actually is a ‘dead’ document, mostly put away in a drawer when it arrives at school.

Nevertheless, this first list of key goals was of importance for Dutch mathematics education. The publication of the list by the government confirmed, and, in a way, validated the recent changes in our curriculum. The main changes have been that:

• more attention is paid to mental arithmetic and estimation;
• formal operations with fractions are no longer in the core curriculum;
• geometry is officially included in the curriculum;
• insightful use of calculators is incorporated.

However, not all these changes have yet been reflected in the textbooks or implemented in our present classroom practice. This is especially true for geometry and the use of calculators.

In the years after 1993 discussions emerged about these 23 key goals (see De Wit 1997). Almost everybody agreed that these could never be sufficient to give support for improving classroom practice, nor to assess the outcome of education. The latter is conceived by the government as a powerful tool for safeguarding the quality of education. For both purposes, the key goals were judged to fail; simply stating goals is not enough to improve practice, and the key goals are not formulated precisely enough to provide yardsticks for testing.

Blueprints of longitudinal learning/teaching trajectories

For several years it was unclear which direction would be chosen to improve the key goals: whether for each grade a more detailed list of goals expressed in operational terms would be created, or whether a description that supports teaching rather than pure testing would be developed. In 1997, the government chose tentatively the latter and asked the Freudenthal Institute to work
The students have some basic concepts with which they can:

1. The students can count forward and backward with changing units.
2. The students can do addition tables and multiplication tables up to ten.
3. The students can do easy mental arithmetic problems in a quick way with insight in the operations.
4. The students can estimate by determining the answer globally, also with fractions and decimals.
5. The students have insight in the structure of whole numbers and the place-value system of decimals.
6. The students can use the calculator with insight.
7. The students can convert into a mathematical problem, simple problems which are not presented in a mathematical way.

The students can apply the standard algorithms, or variations of these, for the basic operations, addition, subtraction, multiplication and division, in easy context situations.

The students can compare ratios and percentages.

The students can do simple problems on ratio.

The students have understanding of the concept percentage and can carry out practical calculations with percentages presented in simple context situations.

The students understand the relation between ratios, fractions, and decimals.

The students know that fractions and decimals can stand for several meanings.

The students can locate fractions and decimals on a number line and can convert fractions into decimals; also with the help of a calculator.

The students can compare, add, subtract, divide, and multiply simple fractions in simple context situations by means of models.

The students can read the time and calculate time intervals; also with the help of a calendar.

The students can do calculations with money in daily-life context situations.

The students have insight in the relation between the most important quantities and the corresponding units of measurement.

The students know the current units of measurement for length, area, volume, time, speed, weight, and temperature, and can apply these in simple context situations.

The students can read simple tables and diagram and produce them based on own investigations of simple context situations.

The students have some basic concepts with which they can organize and describe a space in a geometrical way.

The students can reason geometrically for which they use buildings of blocks, ground plans, maps, pictures, and data about place, direction, distance, and scale.

The students can explain shadow images, can compound shapes, and can devise and identify cut-outs of regular objects.
The binding force of levels and the didactical use of them

It is this level characteristic of the learning processes that brings coherence to the learning/teaching trajectory. A crucial implication is that children can understand something on different levels and several can be working on the same problems without being at the same level of understanding. This distinction of levels in understanding, which can have different appearances for different subdomains within the whole number strand, is very fruitful for working on the progress of children’s understanding. It offers footholds for stimulating this progress.

As an example one might consider the levels in counting that we have distinguished for the early stage of the development of number concept in the early years (see TAL Team 1998):

- context-connected counting;
- object-connected counting;
- (towards) a more formal way of counting.

To explain this level distinction and to give an idea of how it can be used for making problems accessible to children and for eliciting shifts in levels of understanding, one could think of the ability to count up to ten. What do we have to do if a child does not make any sense of the ‘how many’ question (see Figure 4.5)? Does this mean that the child is simply not able to do the counting?

That this is not necessarily the case may become clear if we move to a context-connected question. This means that a plain ‘how many’ question is not asked, but that a context-connected question is used, such as:

- How old is she (while referring to the candles on a birthday cake)? (Figure 4.6)
- How far can you move (while referring to the dots on a die)?

![Figure 4.5](image)

Figure 4.5 How many . . . ?

![Figure 4.6](image)

Figure 4.6 How old . . . ?

Source: TAL team (1998: 26)

- How high is the tower (while referring to the blocks of which the tower is built)?

In the context-connected questions, the context gives meaning to the concept of number. This context-connected counting precedes the level of the object-connected counting in which the children can handle the direct ‘how many’ question in relation to a collection of concrete objects without any reference to a meaningful context. Later on, the presence of the concrete objects is also not needed anymore to answer ‘how many’ questions. Through symbolizing, the children have reached a level of understanding in which they are capable of what might be called ‘formal counting’, which means that they can reflect upon number relations and that they can make use of this knowledge. Regarding the field of early calculating in Grade 1 (with numbers up to 20), the following levels have been identified (see Treffers et al. 1999):

- calculating by counting: calculating 7 + 6 by laying down seven 1-guilder coins and six 1-guilder coins and counting one by one;
- calculating by structuring: calculating 7 + 6 by laying down a 5-guilder coin and 1-guilder coins;
- formal calculating: calculating 7 + 6 without using coins and by making use of number knowledge about 6 + 6.

In the higher grades when students are doing calculations on a formal level the above levels can be recognized in the three different calculation strategies for addition and subtraction up to 100:

- the ‘jumping’ strategy, which is related to calculating by counting: it implies keeping the first number as a whole number, e.g. 87 - 39 = . . . 87 - 30 = 57 . . . 57 - 7 = 50 . . . 50 - 2 = 48;
- the strategy of splitting numbers in tens and ones, which is related to calculating by structuring: it implies making use of the decimal structure, e.g. 87 - 39 = . . . 80 - 30 = 50 . . . 7 - 7 = 0 . . . 50 - 2 = 48;
- flexible counting, which is related to formal calculating: it implies making use of knowledge of number relations and properties of operations, e.g. 87 - 39 = . . . 87 - 40 = 47 . . . 47 + 1 = 48

These ideas about counting on a number line as a base for counting up to 100 are further developed in the chapter by Julie Menne in this volume.

Didactical levels

Insight into such didactical levels provides teachers with a powerful mainstay for getting access to children’s understanding and for working on shifts in understanding. After starting, for instance, with context-connected questions (‘How old is she?’) the teacher can gradually push back the context and come to the object-connected questions (‘How many candles are on the birthday cake?’). The level categories for calculations up to 20 and 100 differ considerably from, for instance, levels based on problem types and levels based on the size of the numbers to be processed. They also deviate from the more general
concrete–abstract distinctions in levels of understanding and from level distinctions ranging from material-based operating with numbers to mental procedures; verbalizing is seen as an intermediate state.

As far as some of the main ideas behind the trajectory blueprints are regarded, we are just at the beginning of work on them. We do not yet know how they will function in school practice and whether they can really help teachers. Investigations to date (De Goeij et al. 1998; Groot 1999; Slavenburg and Krooneman 1999), however, have given us the feeling that the latter might indeed be the case and that we have triggered off something that can bring not only the children to a higher level but also our mathematics education. The interesting thing for us was to discover that making a trajectory blueprint was not only a matter of writing down in a popular and accessible way for teachers what was already known, but that the work on the trajectory also resulted in new ideas about teaching mathematics and involved revisiting our current thinking about it.

To conclude

In this chapter I have attempted to outline the main characteristics of an approach to arithmetic teaching that has been, and continues to be developed in the Netherlands. The metaphor of a guided tour through the Dutch landscape has a special meaning for Dutch mathematics educators as it was Freudenthal (1991) who called the last chapter in his last book, ‘The landscape of mathematics education’. This chapter probably inspired Treffers when he took a well-known poem of the famous Dutch poet Marsman to summarize Dutch mathematics education in primary school. Let me conclude with this poem.

Thinking of Holland

Thinking of Holland
I see wide rivers
winding lazily through
endless low countrysides
like rows of empty number lines
striping the horizon
I see multibase
arithmetic blocks
low and lost
in the immense open space
and throughout the land
mathematics of a realistic brand.

(after H. Marsman’s ‘Denkend aan Holland’; adapted by A. Treffers 1996)
Cambridge in March 1999; and the 23rd Annual Conference of the International Group for the Psychology of Mathematics Education, held at the Technion – Israel Institute of Technology – in July 1999 (Ruthven 1999b). I am grateful for the invitations to participate in these meetings, and for the many helpful ideas and suggestions gleaned.

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