Tasks, teaching sequences, longitudinal trajectories: About micro didactics and macro didactics

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Anne Teppo
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Bibliographic Reference:
The aim of this short oral is to reflect on characteristics of instructional materials which have a different scope in teaching-time. We focus on tasks, lesson series or teaching sequences, and longitudinal teaching trajectories. By exploring how these instructional materials of different sizes differ with respect to their planning nature, their focus and their research methodology, our presentation sheds light on differences between micro and macro didactical approaches within mathematics education research.

First, we have to make clear how we see mathematics education as a discipline. We consider the “didactics of mathematics” as a scientific discipline that guides our research in mathematics education and to which our research in mathematics education contributes.

This approach to the science of mathematics education has a strong European tradition.
Science of mathematics education

**Didactics of mathematics**
*as a scientific discipline*

Didaktik der Mathematik (German)
Didactique des mathématiques (French)
Didáctica de las matemáticas (Spanish)
Matematikdidaktik (Scandinavian languages)
Wiskundedidactiek (Dutch)

- Freudenthal, 1978
- Wittmann, 1984
- Biehler et al., 1994
- Bengtsson, 1997
- Brousseau et al., 1997
- Niss, 1994, 1999


Science of mathematics education

Didactics of mathematics
as a scientific discipline

- Niss (1999)

**descriptive/explanatory dimension**
identifying, characterising, and understanding phenomena and processes actually or potentially involved in the teaching and learning of mathematics at any educational level

**normative dimension**
constructions of curricula, teaching approaches, instructional sequences, learning environments, materials for teaching and learning
Science of mathematics education

Didactics of mathematics

as a scientific discipline


focused on the design of substantial learning environments

“specific orientation from the requirements of the core”

“work in the core must start from mathematical activity”


Science of mathematics education

Didactics of mathematics
*as a scientific discipline*

**Design Science**

**Methodologies?**

What is designed can largely differ

- Tasks
- Teaching sequences
- Trajectories
<table>
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<td>Dominant Research methodology</td>
<td>Piloting and revising tasks through interviews/observations</td>
</tr>
</tbody>
</table>
Can lessons be replicated?
Angelika Kullberg
Göteborg University, Sweden

Task:
Are there numbers between 0.97 and 0.98?

This design was not successful

Revision:
Parts in between instead of numbers in between
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<td>cycle model</td>
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An example of such a teaching cycle is described by Simon (1995).* The goal of his teaching was to explore the multiplicative relationship involved in evaluating the area of rectangles.

The following problem was given to prospective elementary teachers:

“Rectangles problem 1. Determine how many rectangles, of the size and shape of the rectangle that you were given, could fit on the top surface of your table. Rectangles cannot be overlapped, cannot be cut, nor can they overlap the edges of the table. Be prepared to describe to the class how you solved this problem.“

Instead of covering the table as follows ...  

... Some of the teacher-students came up with:

Simon’s comment was:

„As we proceeded to explore the multiplicative relationship involved in evaluating the area of a rectangle, I came to believe that the context in which we were working (area) was not well understood by many students. They seemed to think about area as generated by multiplying length times width.“
This reflection led the teacher to adapt the problem.

Among other problems he presented the teacher-students the following problem.

The *blob problem*: How can you find the area of this figure?

In other words, the teachers’ hypothetical learning trajectory (HLT) needed to be adapted.
• An HLT is a vehicle for planning learning of particular mathematical concepts.
• Because of the hypothetical and inherently uncertain nature of this process, the teacher is regularly involved in modifying every aspect of the HLT.
• Generation of an HLT is based on understanding of the current knowledge of the students involved.

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| Focus | How can mathematical thinking be prompted? | How can this thinking be strengthened in subsequent lessons? | What mathematical content and competences should be taught? |

| Dominant Research methodology | Piloting and revising tasks through interviews/observations | Classroom experiments following a teaching cycle model | }
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We like to highlight that the concept of a “hypothetical learning trajectory” cannot be used in the context of a longitudinal learning pathway. A trajectory that covers several grades or even a complete school level cannot be tested by means of a classroom experiment.

Different from a hypothetical learning trajectory that covers only a small number of lessons, a longitudinal learning-teaching trajectory describes how the understanding of students can develop over several grades. A longitudinal learning-teaching trajectory describes the benchmarks in their development and the learning environment and learning activities that stimulate this development. To give this overview and to make clear how the students’ learning develops, a longitudinal learning-teaching trajectory describes how the different stages of the development are connected. It offers a didactical structure of key concepts (see Wittmann’s (1999) “fundamental ideas”). Therefore a longitudinal learning-teaching trajectory is called conceptual.

But, what should be the methodology when classroom experiments will not do?
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<td>What mathematical content and competences should be taught?</td>
<td>Collecting achievement scores and determining ability scales?</td>
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</table>
The following ability scale for ratio (see next slide) resulted from the PPON study* in the Netherlands.

According to this scale estimation (or approximate calculation) in multiplication problems (see the problems 2 and 3) turned out to be easy for students in grade 6.

<table>
<thead>
<tr>
<th>Items</th>
<th>Order and degree of mastery</th>
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<tr>
<td>10. Estimating the result of a multiplication of two composed numbers; e.g. 5 1/49 x 7 19/20 is approximately ...</td>
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<tr>
<td>9. Estimating the result of a multiplication in which a whole number is multiplied by a decimal that is close to 0.5; e.g. 48 x 0.497</td>
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<tr>
<td>6. Estimating the answer of a division in which the dividend is close to a number that easily can be divided by the divider; ƒ6327,75 † 8 → ƒ6400,- † 8</td>
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<td>5. Estimating the answer of additions and subtractions (including whole numbers and decimal numbers); e.g. 17000 – 2997 – 2999 – 2996 ≈</td>
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<tr>
<td>4. Indicating about what percentage is 397 of 809</td>
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<tr>
<td>3. Figuring out whether the answer is more or less than the estimated answer; e.g. 51 x 41□2000</td>
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<tr>
<td>2. Indicating by what calculation the answer can be estimated; e.g. 203 x 496 can be estimated by ...</td>
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<td>1. Estimating the result of simple additions</td>
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</table>

**Percentiles**

<table>
<thead>
<tr>
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<th>10</th>
<th>25</th>
<th>50</th>
<th>75</th>
<th>90</th>
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<tr>
<td><strong>MASTERY</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Good</td>
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<td>(≥ 80%)</td>
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<td>Moderate</td>
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<td>(50%~80%)</td>
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<tr>
<td>Insufficient</td>
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<tr>
<td>(&lt;50%)</td>
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Percentiles: 10 25 50 75 90
As a result of this finding, teachers might think that estimating in $\times$ and $\div$ is easy for primary school students.

The following problem will make clear that estimating in $\times$ and $\div$ actually is beyond the scope of primary school.

In the hall, there are 18 rows of 32 seats each. Approximately how many seats are there all together?

Is your estimate ($20 \times 30 = 600$) lower or higher than the exact calculation of $18 \times 32$?
The estimate \((20 \times 30)\) is more ...
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**Focus**

- **Micro planning**
  - How can mathematical thinking be prompted?  
- **Meso planning**
  - How can this thinking be strengthened in subsequent lessons?  
- **Macro planning**
  - What mathematical content and competences should be taught?  

**Dominant Research methodology**

- Piloting and revising tasks through interviews/observations  
- Classroom experiments following a teaching cycle model  
- Empirical data will not do. Needed are ...
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This is also true for teaching sequences.
Two examples to make clear that classroom experiments on their own are not enough to design teaching sequences, but that mathe-didactical analyses are needed to design teaching sequences.

**Example 1**: Two different approaches to teaching equivalent fractions

**Example 2**: Two different approaches to teaching calculations up to 100

Example 1  

Two different approaches to teaching equivalent fractions

Simon & Tzur (2004)*

HLT was tested and revised in a classroom experiment

“The tasks [for the teaching sequence] were designed to encourage the setting of a goal of finding the new numerator and the specific activity of partitioning fractional parts.”


Streefland (1991)*

Draw rectangle with 1/2 shaded.
Draw lines on the rectangle so that it is divided into sixths.
Determine how many sixths are in 1/2.

2. Draw rectangle with 2/3 shaded.
Draw lines on the rectangle so that it is divided into twelves.
Determine 2/3 = ?/12

3. Draw diagrams to determine:
3/4 = ?/8, 4/5 = ?/15

4. 5/9 = ?/90, 7/9 = ?/72
Without drawing a diagram, but thinking of drawing a diagram

5. 16/49 = ?/147, 13/36 = ?/324
Similar to 4, but now with calculator

6. Write a calculator protocol for calculating a problem of the form a/b = ?/c

The tasks 4, 5, and 6 were particularly designed to foster reflection and give the students the opportunity to develop abstract understanding.

---

**Simon & Tzur (2004)**

The ‘equivalent fractions’ teaching sequence contains the following tasks

1. Draw rectangle with 1/2 shaded.
   Draw lines on the rectangle so that it is divided into sixths.
   Determine how many sixths are in 1/2.

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   Draw lines on the rectangle so that it is divided into twelves.
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**Streefland (1991)**

The ‘equivalent fractions’ teaching sequence has the following characteristics

- The teaching sequence starts with situations in which fractions emerge: “faire share” situations.
- The situations have been chosen in such a way that the unit is not the starting point (e.g., 3 divided by 4 instead of 1 divided by 4).
- While solving “fair share” problems, students can discover that the particular amount of pizza that one child gets can be named differently (3/4 is equivalent to 6/8).

The “equivalent fractions” teaching sequence contains the following tasks.

- Later, the students use the model in an abstract way.

Which fraction is larger?

\[
\frac{15}{12} \quad \text{or} \quad \frac{6}{8}
\]
Example 2

Two different approaches to teaching calculations up to 100

McClain (2002)*

TAL learning-teaching trajectory for whole number calculation*


Instructional sequence for adding and subtracting three-digit numbers based on a classroom-teaching experiment as described by McClain (2002)

- focus on algorithms
- focus on place value
- use of Unifix cubes

Sequence in TAL see Van den Heuvel-Panhuizen (2001)

- integrating written and mental calculation
- focus on whole numbers
- use of number line

\[
36 + 19 = 55
\]
Planning

Focus

Dominant Research methodology

Tasks

Mathematical thinking be prompted?

Piloting and revising tasks through interviews/observations

Micro planning

Teaching sequences

How can this thinking be strengthened in subsequent lessons?

Classroom experiments following a teaching cycle model

Meso planning

Trajectories

What mathematical content and competences should be taught?

Mathe-didactical analyses

This is also true for teaching sequences and tasks
I consider the design, analysis and empirical testing of mathematical tasks, whether for the purposes of research or teaching, as one of the most important responsibilities of mathematics education.

Behavior et al. (1994):
This responsibility carries great weight since the cognitive structures that children build reflect the nature of the tasks with which they engage.

Sierpinska (2003):
Research in Mathematics Education -- Through a Keyhole

Analysis of 55 research reports from PME26
Research question:

What mathematical tasks were used as tools in the research?

“I consider the design, analysis and empirical testing of mathematical tasks, whether for the purposes of research or teaching, as one of the most important responsibilities of mathematics education.

Doyle (1983):
Tasks influence learners by directing their attention to particular aspect of content … and how learners develop, use, and make sense of mathematics.


Sierpinska (2003): Research in Mathematics Education -- Through a Keyhole

“Too bad, though, that there is so little information about the design of the tasks in the report.”

“People are doing interesting teaching experiments but they choose to say very little about the reasons underlying the choice of the tasks and what difference would it make if they changed the tasks in this or that respect.”

“The reports are silent on the problem of the construction of systems of tasks which aim at consolidation of knowledge.”

Call for a Come-back of Task Analysis
Thank you!
Gomsa-hamnida!
m.vandenheuvel@fi.uu.nl
arteppo@theglobal.net

PME 31

Seoul
KOREA

www.fi.uu.nl/~marjah