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On Reform Movement and the Limits of Mathematical Discourse

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In this article, I take a critical look at some popular ideas about teaching mathematics, which are forcefully promoted worldwide by the reform movement. The issue in focus is the nature and limits of mathematical discourse. Whereas knowing mathematics is conceptualized as an ability to participate in this discourse, special attention is given to meta-discursive rules that regulate participation and are therefore a central, if only implicit, object of learning. Following the theoretical analysis illustrated with empirical examples, the question arises of how far one may go in renegotiating and relaxing the rules of mathematical discourse before seriously affecting its learnability.

SOME QUESTIONS ABOUT TEACHING AND LEARNING MATHEMATICS ANY REFORMER MUST ASK

One does not have to be an educational researcher to agree with this: Mathematics is one of the most difficult school subjects. It is therefore understandable why teaching mathematics has always been, and will probably always remain, subject to change and improvement. Improving the teaching of mathematics is the principal aim of the reform movement, whose influence can be felt around the world. The exact shape of the required changes may vary from one country to another, but their multiple manifestations obviously have a strong common core. Indeed, they all seem to be rooted in the same educational philosophy generated by a number of generally shared basic principles. This is why the words *reform movement* may be

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expected to evoke similar connotations in most educators, regardless of their geographical placement.

The apparent consensus is a good thing, provided the assumptions underlying the agreed-upon position are clear and sound. This, however, may be not the case with the present reform. Although the intentions behind the word *reform* seem relatively clear today, the basis for what is actually being done appears vague at times. Despite decades of intensive research in mathematics education, many questions about students' learning, quite vital to any pedagogic decision,[1] still wait to be answered. To illustrate this claim, I will now look at one specific case of a school mathematical subject and will then discuss practical issues raised by its pedagogical aspects. In this way, I will exemplify the dilemmas that, according to my understanding, should be tackled before any decisions about teaching are made.

Let us talk about *negative numbers*—a subject introduced in most middle school mathematical curricula. Although obligatory, the topic is by no means easy to learn, and it may be problematic even in the eyes of the brightest students. Nobody managed to describe the learner's exasperation with the intricacies of the notion of negative number in a more convincing way than the French 19th-century writer, Stendhal (the pseudonym of Marie-Henri Beyle, 1783–1842):

> I thought that mathematics ruled out all hypocrisy, and, in my youthful ingenuousness, I believed that this must be true also of all sciences which, I was told, used it. … Imagine how I felt when I realized that no one could explain to me why minus times minus yields plus. … That this difficulty was not explained to me was bad enough (it leads to truth, and so must, undoubtedly, be explainable). What was worse was that it was explained to me by means of reasons that were obviously unclear to those who employed them.
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> M. Chabert, whom I pressed hard, was embarrassed. He repeated the very lesson that I objected to and I read in his face what he thought: "It is but a ritual, everybody swallows this explanation. Euler and Lagrange, who certainly knew as much as you do, let it stand. We know you are a smart fellow. … It is clear that you want to play the role of an awkward person …"
>
> It took me a long time to conclude that my objections to the theorem: *minus times minus is plus* simply did not enter M. Chabert's head, that M. Dupuy will always answer with a superior smile, and that mathematical luminaries that I approached with my question would always poke fun at me. I finally told myself what I tell myself to this day: It *must* be that minus times minus must be plus. After all, this rule is used in computing all the time and apparently leads to true and unassailable outcomes. (quoted in Hefendehl-Hebeker, 1991, p. 27)

[1]Let me be clear: I am not implying that knowledge of learning processes is a sufficient basis for clear-cut didactic decisions. However, it can certainly be beneficial to them, if not necessary. Although there is no direct route from research on learning to instructional design, understanding students' difficulties is an important factor that should inform choices of teaching approaches.

The techniques for adding, subtracting, multiplying, and dividing signed numbers may not be very difficult to master, but there are serious conceptual dilemmas that students invariably encounter if they have an urge to understand what the notion of negative number is all about. The question of why a product of two negative numbers should be positive is probably the most famous of these dilemmas. It is now time to pause and ask the following question:

> Provided we agree that negative numbers should be included in school curriculum, how should educators teach the topic to address Stendhal's complaints?

I have no doubt that, despite familiarity with the subject, or perhaps because of it, this question is surprisingly difficult to answer. It may well be because of this difficulty that some mathematics teachers resort to the strategy epitomized in the unforgettable rhyme: "Minus times minus is plus, the reason for this we need not discuss" (W. H. Auden, quoted in Kline, 1980, p. 115).

The unexpectedly unyielding question warrants some serious thought. To begin, let us listen more carefully to what Stendhal has to say. It is notable and thought provoking that the French writer's complaint was not so much about the inaccessibility of the mathematical definitions he was trying to digest as it was about the nature of his teachers' explanations. Indeed, it is rather unlikely that the "mathematical luminaries" had nothing to say on the subject or that they tried to withhold "the truth" from the young man in need. Despite Stendhal's accusations, the odds are that his teachers did try to explain and that they did it the best they could. Alas, their best was evidently not good enough for the young Stendhal. Whatever the teachers' explanations, the boy evidently found them unconvincing. Thus, the difficulty was an ineffective communication rather than an absence of any serious attempt to communicate. A preliminary issue must thus be tackled before answering the question on the ways to teach:

> Why are certain mathematical concepts inadmissible and certain mathematical arguments unconvincing to many students?

By stressing the issue of admissibility, this question brings to the fore the similarity of the difficulties experienced by mathematics students to the difficulties noted by the mathematicians of the past (see Fauvel & van Maanen, 1997; Sfard, 1992, 1994a; Sfard & Linchevski, 1994). According to historical sources, the conceptual obstacles tackled by the early mathematicians who spoke of negative numbers were, indeed, surprisingly close to those reported by Stendhal. For at least three centuries, mathematicians did know the rules of operating on the signed numbers, they did acknowledge the inevitability of these rules, and they still felt that whatever explanation they would be able to produce could not count as a fully fledged *justification* of the idea of negative number.[2] The phenomenon is truly mind-boggling: How can people have a clear sense of an inevitability of certain definitions and rules and, at the same time, claim their inadmissibility? How can a person believe in two opposite things without being able to decide between them? If the facts contradict each other, where does the confidence in their veracity come from?

A good understanding of this and similar phenomena is indispensable to those who wish to build their pedagogical decisions on a basis more solid than their unschooled intuitions. In the reminder of this article, I will try to meet this challenge by conceptualizing learning as gaining an access to a certain discourse. As I will show, the discursive perspective brings the promise of new insights into phenomena such as those described previously, and it is thus likely to provide effective tools with which to tackle some nagging questions about reform.

SETTING THE SCENE FOR ANSWERING REFORMERS' QUESTIONS: SPEAKING OF LEARNING IN TERMS OF DISCOURSE

Today, rather than speaking about "acquisition of knowledge," many people prefer to view learning as *becoming a participant in a certain discourse.* In the current context, the word *discourse* has a very broad meaning and refers to the totality of communicative activities, as practiced by a given community (to avoid confusion with the everyday narrow sense of the term, some authors, such as Gee, 1997, proposed capitalizing it: Discourse).[3] Within the discursive research framework, it is understood that different communities, with the mathematical community just one of many, may be characterized by the distinctive discourses they create. Of course, it must also be understood that discourses are dynamic and ever-changing entities, and thus, determining their exact identities and mapping their boundaries is not as

[2]For example, late Medieval and early Renaissance mathematicians Chuquet, Stifel, and Cardan called these numbers such unflattering names as "absurd," "false," "imaginary," or "empty symbols" even as they were using them. Descartes regarded negative numbers as "false, because they represent numbers smaller than nothing," whereas Pascal left no doubt as to what he was thinking about negative-number users in an exclamation as arrogant as it must now sound naive: "I know people who don't understand that if we subtract 4 from zero, nothing will be left" (cf. Kline, 1980).

[3]Just how important and encompassing this term is can be seen from Gee's (1997) functional description of Discourses:

> Discourses are sociohistorical coordinations of people, objects (props), ways of talking, acting, interacting, thinking, valuing, and (sometimes) writing and reading that allow for the display and recognition of socially significant identities, like being a (certain sort of) African American, boardroom executive, feminist, lawyer, street-gang member, theoretical physicist, 18th-century midwife, 19th-century modernist, Soviet or Russian, schoolchild, teacher, and so on through innumerable possibilities. If you destroy a Discourse (and they do die), you also destroy its cultural models, situated meanings, and its concomitant identities. (pp. 255–256)

straightforward a task as a researcher would hope. Moreover, discourses of different communities are constantly overlapping, and this results in their incessant crossbreeding. All these difficulties notwithstanding, the notion of discourse proves clear enough to spur a steady flow of highly informative research that has the power of eliciting hitherto unnoticed aspects of learning.

In substituting the word *discourse* for *knowledge*, the philosophers made salient the central role of speech in human intellectual endeavor. For many researchers, studying mathematical communication became a task almost tantamount to studying the development of mathematical thinking itself, even if they do not say so explicitly (e.g., see Bauersfeld, 1995; Forman, 1996; Lampert & Cobb, in press; Morgan, 1996; O'Connor, 1998; Pimm, 1987, 1995). However, the shift of focus, which is evident in the renaming, goes further than the change in emphasis. First, it may count as an act of "putting body back" into the process of knowledge construction. Knowledge viewed as an aspect of a discursive activity is no longer a disembodied, impersonal set of propositions, the exact nature of which is a matter of "the true shape" of the real world; rather, it is now a human construction. Furthermore, because the notion of discourse only has sense in the context of social interaction, speaking of discourse rather than knowledge precludes the possibility of viewing learning as a purely individual endeavor. Moreover, because thinking may be conceptualized as an instance of discursive activity (Bakhtin, 1981; Sfard, 2000), putting discourse in the place of disembodied knowledge brings down the conceptual barriers that separated the "individual" from the "social" for centuries. Indeed, the discursive rendition of the issue of knowing and learning makes it clear that the demands of communication are the principal force behind all human intellectual activities, and thus, all these activities are inherently social in nature, whether performed individually or in a team. Finally, the word *discourse* seems more comprehensive than *knowledge*. Researchers who speak about discourse are concerned not only with those propositions and rules that constitute the immediate content of the specific discourse but also with much less explicit rules of human communicative actions, which count as the proper way of conducting this particular type of discourse. One may therefore speak about *object-level rules* of mathematical discourse, that is, rules that govern the content of the exchange, and about *meta-discursive rules* (or simply meta-rules), which regulate the flow of the exchange and are thus somehow superior to the former type of rules, even if only implicitly. (The prefix *meta-* signals that the rules in question are a part of discourse about discourse; that is, they have the discourse and its part as their object.[4])

[4]The rules such as "If you are to solve the equation $3x + 2 = 5$, the actual physical shape of the letter used is unimportant," or "If you want to be sure of the veracity of a mathematical statement on numbers, you have to derive it from axioms on numbers with the help of deductive inferences," or "If it is true that statement A entails B and statement B entails statement C, then statement A entails C" are all clearly meta-discursive because their objects are mathematical statements (and not the entities that are objects

There Is More to Discourse Than Meets the Ears: An Example

Let me elaborate on the issue of meta-discursive rules because this is the aspect of learning that, although ubiquitous, has not been given any direct attention by teachers and researchers until quite recently. The long-lasting neglect is hardly surprising. Meta-rules are only implicitly present in discourses, and their learning occurs spontaneously, without being deliberately planned by teachers, without being intended by the students, and without being consciously considered by anybody. Yet, these invisible rules are responsible not only for the ways we do things but also for the very fact that we are able to do them at all. Their behind-the-scene influence on our spontaneous discursive decisions may sometimes be dramatic. To make this point clear, let me use an example.

The following observations were made rather accidentally, when I was teaching two university courses in parallel. One of the courses, called "Didactic Seminar in Mathematics,"[5] was a compulsory class for undergraduate students majoring in mathematics and preparing themselves to be mathematics teachers in high schools. The other was a seminar, "Discursive Approach to Research on Mathematical Thinking," intended for graduate students. Although no special knowledge of mathematics was required from the participants in the latter seminar, all the students in the group did have at least a high-school mathematics background, and some of them held a university degree in mathematics. A few of them were working as mathematics teachers in high schools.

Both groups were supposed to discuss, among others, the popular idea of learning mathematics through writing. To provide the participants with material for later reflection, I engaged them in the activity shown in Figure 1: The students were asked to "write a letter to a young friend," in which they would try to convince this friend that a certain equality they believed true must hold for any natural number. Unexpectedly, although not altogether surprisingly, the results obtained in the two classes turned out quite differently. Two samples of students' responses are displayed in Table 1. It is important to stress that the samples are truly representative, in that each one of them displays certain critical characteristics typical of

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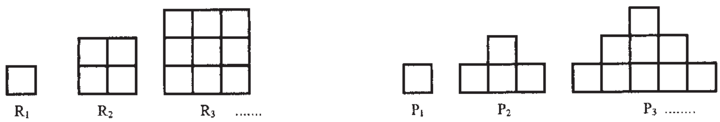
of mathematical statements). As an example of object-level rules in mathematical discourse, one can offer the laws of numerical field that govern the relationships between real numbers or, in everyday discourse, the rules according to which one decides whether there is a possibility of rain on the basis of what is presently known about the weather. Knowledge of object-level rules is indispensable in evaluating the veracity of utterances. Of course, so are the laws of logic, which, themselves, belong to the meta-discursive category. I must stress that, within a radical discursive approach, the seemingly straightforward distinction between object-level and meta-level becomes somehow blurred (for further elaboration, see footnote 9).

[5]What in Israel we use to call a "didactic seminar" is close to what is known in other parts of the world as a "methods course."

Look at the following two sequences:

the sequence of **squares**: and the sequence of **"triangles"**:



1. What can you say about the relation between the area of R_{200} and P_{200}? Write your claim down.
2. Write a letter to a young friend trying to convince him or her that your claim is correct.

FIGURE 1 The task given to students in two different classes.

TABLE 1
Representative Samples of Students' Responses in the Two Classes

1. Undergraduate mathematics student in the didactic seminar

$R_{200} = P_{200}$.

This is true also for $R_1, \ldots, R_n$, and $P_1, \ldots, P_n$, the first squares and triangles. Moreover,

$$R_n = n^2$$
$$P_n = n^2$$
$$P_n = R_n.$$

Let's assume this is true for n and let's prove for $n + 1$:

$$\overset{?}{R_{n+1} = (n+1)^2}$$
$$R_{n+1} = \underset{\displaystyle R_n}{\underset{\downarrow}{n^2}} + 2n + 1 = R_n + 2n + 1$$

$$P_{(n+1)} = (n+1)^2 = n^2 + 2n + 1$$
$$P_n = R_n \text{ (according to the assumption)}$$
$$\Rightarrow P_{n+1} = R_{n+1}$$
$$\Rightarrow P_{200} = R_{200}$$

2. Graduate student in the discourse seminar

Dear Friend!

The area of R_{200} will be equal to P_{200}. Let me explain.

First, the area of R_1 is equal to the area of P_1. This means that they are exactly the same size. Now, if we add the same area to P and R, the relation between them will stay the same. Let's take an example from a different domain (I just want to explain why the relation remains constant when the basic magnitudes are the same and we enlarge them [by the same amount] respectively).

Let's look at scales. On one side we put an apple, on the other side an orange of equal weight. Now, let us add to each [side] a fruit so that both added fruits are of the same weight. It's clear that the weights on the two sides remain equal to each other.

The same is true about our squares and triangles. The first small squares, R_1 and P_1, are equal, and then this equality is preserved when we add the same amount to both.

all the other texts produced in the same class. For me, the difference in the performance of the two classes was thought provoking; in the remainder of this section, I will explain why.

The difference between the two sample texts is striking and is obvious at first sight: Whereas one piece makes extensive use of mathematical symbols, the other one can hardly be recognized as belonging to mathematical discourse (this difference, by the way, is not less visible in the original Hebrew versions of students' responses; one does not have to understand the language to notice the dissimilarity of the two texts). However, there are many other differences as well. The general styles of the two "letters" are incompatible—one may have difficulty believing that the two texts try to answer the same question. The didactic seminar student put his argument in the well-known form of proof by mathematical induction, whereas the discourse seminar student used a less specialized, more commonsense form of convincing.

On the surface, the remarkable difference may seem surprising, as the two classes are almost identical in three important respects: their mathematical background, their teacher, and the task at hand. As long as we believe these three aspects to be the major factors responsible for the manner in which students act in the classroom, we have no explanation whatsoever to the observed incompatibilities in the performance of the two groups. The situation changes once we use the metaphor of learning by becoming a participant in the discourse and start paying attention to the elusive issue of meta-discursive rules. We can now say that the two groups *participated in different discourses* and that this difference was a matter of invisible meta-level aspects rather than of the explicit object-level factors. The didactic seminar students acted, or at least tried to act,[6] as *mathematical discourse* participants, whereas those in the discourse seminar class were obviously geared toward *everyday discourse*—a discourse that has meta-rules quite unlike the forms of communication commonplace in an ordinary mathematics classroom.

A succinct commonsense comparison of the rules that were evidently at work in the two environments is presented in Table 2. Due to the substantial differences between these two sets, the request to *convince* (or *prove*) was bound to evoke a completely different reaction in the two classes. What led the classes in such dramatically different directions was neither the task, nor the teacher, nor anything else that happened in the school on the particular day when the task was presented. Rather, the different meta-level choices were dictated by different connotations

[6]Note that, although it appears to be a regular inductive argument, the first piece of text in Table 1 does not present a truly meaningful statement. In fact, what the student claims to be proving is an evident tautology: $P_n = R_n$, while $P_n = n^2$, $R_n = n^2$. For this obviously true statement, which requires no proof, the student produced a lengthy, as-if inductive justification. A quasi-mathematical text such as the first example in Table 1 is a natural response for students whose view of themselves as teachers is shaped by typical mathematical textbooks.

TABLE 2
Comparative Description of the Activity of Convincing in the Two Classes

	Undergraduate Mathematics Students in the Didactic Seminar	*Graduate Students in the Discourse Seminar*
What is the *purpose* of the activity?	To make a mathematically convincing argument (an argument that can count as a mathematical proof)	To make a commonsensically convincing argument
What counts as a convincing argument?	An argument made according to the principles of mathematical proving (mathematical induction, in this case)	Any commonsense argument that has a chance to appeal to the interlocutor's reason and beliefs
What is the required *form* of the argument?	Formal, symbolic, and clearly distinct from everyday speech	No rigid rules: Any form of argument goes, provided it is comprehensible and does its job
To what kind of discourse does it belong?	Mathematical classroom discourse	Everyday discourse

and expectations with respect to the task, grounded in the different classroom traditions that had developed in the two classes throughout their past activities.

A Closer Look at Meta-Discursive Rules

The foregoing example has shown that two pieces of discourse substantially overlapping in their object-level aspects, that is, in their content, can be almost unrecognizable as such due to the fact that they have been constructed according to different meta-discursive rules. It is now time to clarify this notion in a more explicit way.

As mentioned previously, researchers' attention to discursive regularities, whether in mathematics or in any other kind of discourse, is a relatively new phenomenon. Yet, the study of repetitive discursive patterns has already spawned an impressive bulk of publications. To appreciate the range of the current interest in the issue of discourse and its modes, one has to be aware that the term *rule* is not the only one used in this context. Many publications in philosophy, sociology, anthropology, and related areas have been devoted to striking regularities in human discursive actions, regularities that go well beyond those that can be presented as purely linguistic, grammatical canons of behavior. When people engage in the activity of communication—speak or write to each other, read texts, or even lead an inner dialogue with themselves—they do not seem to act in accidental ways, and the resulting discourses do not seem to be arbitrary formations. This basic observation was made by many writers and has been presented and explained with the help of numerous theoretical constructs. Thus, Wittgenstein (1953) spoke extensively of human communication as an instance of rule-following activity. The rules are what allow a person to take part in the complex language games all of us play while talking to each other. This is what makes meaning possible, or rather, this is what meaning *is*. Indeed, the set of rules that govern the use of a specific word is offered by Wittgenstein as a definition of the concept of meaning. Similarly, in the heart of Foucault's (1972) theory of discursive formations lies the assumption on the existence of rules that regulate the discourse both "from outside" and "from inside" and without which the different discourses would neither be possible nor would have distinct identities: "The rules of formation operate not only in the mind or consciousness of individuals, but in discourse itself; they operate therefore, according to a sort of uniform anonymity, on all individuals who undertake to speak in this discursive field" (p. 63).

The motif of activity-regulating rules, often hiding under different names and referring to a wide range of related phenomena, recurs in the seminal work of the French sociologist Bourdieu (1999). Without making an explicit reference to communicative activities, Bourdieu contributes to our present topic when speaking of *habitus,* "systems of durable, transposable dispositions, structured structures predisposed to function as structuring structures, that is as principles which generate and organize practices and representations [thus, discourses]" (p. 108). Closer to home, one finds much attention to the regularities in mathematics classroom discourse in the work of Bauersfeld (1995), Forman (1996), Krummheuer (1995), O'Connor and Michaels (1996), and Voigt (1985, 1994, 1995, 1996), to name but a few. Notions such as "routines," "patterns of interaction," "obligations" (Voigt, 1985),[7] "participation structures" (O'Connor & Michaels, 1996), and "discursive practices" (O'Connor, 1998), although not tantamount to the idea of meta-discursive rule, clearly refer to the same phenomena. The related notions "social norms" and "sociomathematical norms,"[8] introduced by Cobb, Yackel, and their colleagues (Cobb, 1996; Cobb, Wood, & Yackel, 1993; Yackel & Cobb, 1996), have been picked up by many other researchers as a useful tool not only for analyzing mathematical learning in a classroom but also for thinking about practical matters, such as instructional design and improvement of learning.[9] Certain

[7]Voigt's (1985) work was inspired by, among others, Schutz's (1967) idea of "natural attitude," which expresses itself in our propensity for certain kinds of conduct.

[8]One should not be tempted to interpret the prefix *socio-* as suggesting that there are also norms that are not social in their origin.

[9]The question remains open whether one can also say that what Cobb (1996) called "mathematical practices" implies its own set of meta-discursive rules or not. After all, the practices in question are discursive practices. Are those rules that govern, for example, the operations on numbers meta-discursive or object level? The answer depends on the ontological status ascribed to the objects of these rules, such

subtle differences between the concepts of "rule" and "norm" will be discussed later, as the role and properties of meta-rules are presented. It should be clear that the term *meta-discursive rule* used in this article does not come as an entirely new construct but rather as an almost self-explanatory term to encompass all the phenomena signaled by the aforementioned notions. The present notion seems to cover roughly the same terrain as all the aforementioned notions taken together.

The regulating impact of meta-rules is felt in every discursive action. They tell us "*when* to do what and *how* to do it" (Bauersfeld, 1993, p. 4; cf. Cazden, 1988). It begins with such seemingly trivial issues as our routine decisions to respond to a given type of utterance (e.g., greeting) with a certain well-defined type of answer (usually another greeting) and continues with our use of logic in the construction of discursive segments, with the particular ways in which we shape interlocutors' mutual positioning, in the means we choose to convince our partners, and so on. In mathematics, discourse-specific meta-rules manifest their presence in our instinctive choice to attend to particular aspects of symbolic displays (e.g., the degree of a variable in algebraic expressions) and ignore others (e.g., the shape of the letters in which the expressions are written) and in our ability to decide whether a given description can count as a proper mathematical definition, whether a given solution can be regarded as complete and satisfactory from mathematical point of view, and whether the given argument can count as a final and definite confirmation of what is being claimed. To give one last example, until quite recently, unwritten meta-rules of classroom mathematical discourse allowed the student to ignore the actual "real life" contents of word problems and to remain oblivious to the issue of plausibility of the "givens." As was widely documented (e.g., see Even, 1999), this rule has often been seen by students as one that exempts them from worry about the plausibility of results.

The long list of examples makes it clear that the current attempt to speak of meta-discursive rules as self-sustained principles notwithstanding, these rules are

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as numbers, functions, and so forth. If these objects are viewed as external to the discourse itself—as having a discourse-independent existence—then the operations on them must also be seem as externally determined. In this case, the rules that govern these operations are clearly of a different kind than those that shape the discourse itself: The former are object-level rules, whereas the latter are meta-level rules. If, however, one rejects this Platonic position and views the objects of mathematics and the operations that can be performed on these objects as purely discursive phenomena, then the rules that govern the objects and operations on objects should also (probably) count as meta-discursive. To put it another way, the distinction between these two sets of rules becomes blurred, if not altogether untenable. However, I will keep this distinction, assuming that it is possible to distinguish between the object-level and meta-level statements without falling prey to Platonic implications. The distinction can be made sharper by saying, for example, that meta-level rules are those that speak of mathematical utterances, of their structures, and of relations between them, as well as those that deal with producers of the utterances, that is, interlocutors. In contrast, object-level rules refer to entities that the given discourse speaks about. As such, these rules, if formulated, may count as part of the discourse itself (e.g., "2 + 3 equals 5"; "to divide *a* by *b*, find the number *c* such that *b* __ *c* = *a*").

tightly connected to the discursive object-level and have a major impact on interlocutors' interpretation of the content. Let me now make a number of additional remarks on the notion of meta-rules, as it should be understood in this article.

The interpretive nature of the concept of meta-discursive rules. Contrary to what seems to be implied by the notion of activity-regulating rules, most of these rules are not anything real for the discourse participant. To put it another way, except for some rare special instances,[10] one should not look upon these rules as anything that is being applied by interlocutors in an intentional manner. Meta-discursive rules are not any more in interlocutors' heads than the law of gravitation is in the falling stone. To use Bourdieu's (1999) formulation, although we deduce the existence of regulating principles from visible regularities in human activities, the patterned structures we see are "objectively 'regulated' and 'regular' without being in any way the product of [intentional] obedience to rules, they can be collectively orchestrated without being the product of the organizing action of a conductor" (p. 108). A similar idea has been conveyed by Wittgenstein in his attack on the "mythology of rules" as principles that govern human behavior in any real sense (as cited in Bouveresse, 1999, p. 45). In this article, I refer to meta-rules as observer's constructs, retroactively written into interlocutors' past activities and expected to reappear, possibly in a slightly modified version, in these interlocutors' future activities. Meta-discursive rule can thus be described as "an explanatory hypothesis constructed by the theorist in order to explain what he sees" (Bouveresse, 1999, p. 46). How this "hypothesis" should be constructed, that is, what methods should be used by a discourse analyst looking for implicit meta-discursive rules is a separate question, which still requires much conceptual investment.

The implicit nature of meta-discursive rules and their learning. Although meta-rules are an observer's construct rather than anything that governs human actions in the common sense of the word *govern*, there are certain patterns of action that have to be learned by those who wish to become skillful in a given kind of discourse. It is in this sense that we can go on speaking about the learning of meta-discursive rules. The question arises of how such learning occurs. Inasmuch as our rule following (or rather our compliance with rules) is unconscious and unintentional, so is our learning of patterned ways of interaction. The way we speak and communicate with others conveys the unwritten regulations. For example, students usually learn the rules of the mathematical game without a conscious effort, by a

[10]For example, see Cobb and his colleagues' (Cobb, Boufi, McClain, & Whitenack, 1997; Yackel & Cobb, 1996) attempt to make social and sociomathematical norms of classroom discourse an object of explicit negotiation and learning.

mere participation in the mathematical discourse. Of course, this does not exclude the possibility of elucidating rules and making them into an explicit object of reflection and change. Still, as Lampert (1990) put it, students would not learn the rules "simply by being told what to do anymore than one learns how to dance by being told what to do" (p. 58). Moreover, if the rules conveyed in an explicit way turned out to be in conflict with those implied in the participants' actual actions, the odds are that the latter rather than the former rules would be more readily picked up by beginners. In some cases, thus created "double bind" (Bateson, 1973) would probably impede any learning at all. All this is in tune with Wittgenstein's claim that "the person who follows a rule has been *trained* to react in a given way. Through this training the person learns to respond in conventional ways and thus enters into *practice*" (as cited in Fogelin, 1995, p. 175).

Unjustifiability of meta-rules. Being only implicitly present in human interactions, meta-discursive rules are an unlikely object for a rational justification. Existing discursive patterns and the meta-discursive rules that transpire from them developed spontaneously throughout the ages, and they are a matter of custom rather than of logical necessity. This message is conveyed in a powerful way by Wittgenstein (1956) who, to make this point, chose perhaps the least expected type of discursive activity, the activity of mathematical proving: "For it is a peculiar procedure: I *go through* the proof and then accept its results.—I mean: this is simply what I *do*. This is use and custom among us, or a fact of our natural history" (p. 61). He is even more explicit in his claim on inherent unjustifiability of meta-rules in the context of mathematical calculations: "The danger here, I believe, is one of giving a justification of our procedure where there is no such thing as justification and we ought simply to have said: *That's how we do it*" (p. 199).

On the face of it, this is a rather surprising statement. After all, nothing could be more rational than mathematical proof. Yet, the claim of unjustifiability does not regard the proof as such but rather the meta-discursive rules that govern the activity of constructing it. It is the justifiability of the meta-discursive conventions that is questioned and not the inner consistency of object-level inferences. Moreover, saying that the meta-discursive rules cannot be justified does not mean there are no reasons for their existence. It only means that, contrary to the Platonic view of mathematics, reasons that can be given are nondeterministic and have to do with human judgments and choices rather than an "objective necessity."[11]

[11]The strict meta-rules of modern mathematical discourse are the result of unprecedented efforts of 19th- and 20th-century mathematicians to reach maximally effective communication. Evidently, their undeclared hope was to create a discourse that would leave no room for personal idiosyncrasies and would therefore lead to unquestionable consensus. Such consensus would imply certainty of mathematical knowledge.

The impossibility of accounting for the meta-discursive rules in a fully rational way explains why their learning is usually a matter of practice. This may also be the reason why the well-known mathematician von Neumann reportedly declared that “one does not *understand* mathematics—one *gets used* to it.”

The nondeterministic nature of the rules. A certain danger in the use of the term *rule* is that it seems to be hinting at something deterministic in the way discourses enfold. Yet, nothing would be farther from the intentions of those who choose to speak of rules to describe and explain the phenomenon of discursive patterns. The rules of discourse do not tell us what to say any more than the rules of traffic tell us where to go. If anything, they make us aware of what would not be a proper thing to say in a given situation. By doing that, they make communication possible, just like traffic regulations make possible collision-free traffic. They are therefore not deterministic but enabling, and there is nothing causal in the way they regulate people’s participation in symbolic exchange.

The normative nature of the rules. As was repetitively stressed, meta-discursive rules do not usually take the form of explicit prescriptions for action, and they are not knowingly appropriated or followed by discourse participants. Yet, by incessantly repeating themselves, the unwritten and mostly unintended rules shape people’s conceptions of “normal conduct” and, as such, have a normative impact. In other words, many of these rules function as norms rather than mere neutral rules: They are value laden and count as preferred ways of behavior. Thus, many of the meta-discursive rules are conceived by discourse participants, even if only tacitly, as normative principles. As Voigt (1985) put it: “The interaction, especially in the classroom, acquires *normative* features” (p. 85). Cobb and his colleagues’ (e.g., Cobb, 1996; Cobb et al., 1993; Yackel & Cobb, 1996) notions of social norms and sociomathematical norms are normative counterparts of *general* meta-rules and *mathematical-discourse-specific* meta-rules (a more detailed treatment of this latter distinction will be presented later).

It is a simple, well-known truth that the prevailing forms of action tend to be regarded, after a while, as the preferred forms of conduct. The usual, the ordinary, and the dominant acquire the quality of the desirable and the privileged. The adjective *normal* turns into an assertion of merit, whereas anything that deviates from the normal is described as pathology, as wrong, and sometimes even as unethical. This fact expresses itself, for example, in the well-documented resistance common among teachers to change their professional habits. It also manifests itself in students’ common use of the words *not fair* to describe teachers’ requirements that cannot be implemented in habitual ways.

Due to the tacit nature of meta-discursive rules, the norms themselves are seldom made explicit either in mathematics classroom or by research mathematicians. Yet, sometimes they do.[12] In fact, making them explicit was the declared goal of the foundational movement in mathematics in the late 19th and early 20th centuries. Cobb and his colleagues (e.g., Cobb, 1996; Cobb et al., 1993; Yackel & Cobb, 1996) directed their research not merely toward exposing existing classroom norms but toward the possibility of shaping these norms in well-defined ways. It is worth mentioning that, as long as discursive norms are tacit, their grip on our thinking is particularly strong, and they are particularly difficult to change.

Dynamic nature of meta-discursive rules. The existence of repetitive discursive patterns is the reason for our present debate on meta-discursive rules. The stress on repetitiveness implies a relative stability of the discursive conduct and of the underlying principles. Although the repetitiveness and stability are our point of departure, it is time now to mention the *dynamic nature* of the discussed phenomena. Indeed, discursive patterns, incessantly created and re-created as interactions go on, do not ever reappear exactly as they were before, and as time goes by, they undergo substantial transformations. In fact, such transformations are the main purpose of learning. The student who arrives in a mathematics classroom is supposed to learn participation in a discourse that, so far, was inaccessible to him or her, and this means, among others, getting used to acting according to a new set of meta-discursive rules. The new discursive behaviors of the learner develop gradually as a result of classroom interactions. The way this happens deserves attention.

On the surface, the rules of the classroom mathematical game are established exclusively by the teacher, the person whose expertise in the discourse renders him or her a position of relative power and authority. Indeed, the teacher's discursive ways are privileged by all the participants, and attainment of mastery in this privileged type of discourse is the general goal of learning. Yet, this unidirectional vision of learning is an oversimplification. When speaking of social and sociomathematical norms, Cobb (1996) stressed the *reflexivity* of the relationship between students' mathematical activity and communal classroom practices: "This is an extremely strong relationship and does not merely mean that individual activity and communal practices are interdependent. Instead, it implies that one literally does not exist without the other" (p. 97).

This statement has many entailments, one of them being the reflexivity inherent in the very process of practice building: Discursive norms, rather than being implicitly dictated by the teacher through discursive behaviors, are seen as an evolving product of the teacher's and the students' collaborative efforts. To put it in Bauersfeld's (1988) words: "Teacher and student(s) *constitute the reality* of class-

[12]Of course, no analysis would ever exhaust all the meta-discursive rules, and no amount of explicit rules would ever suffice to reconstruct the whole of the discourse. Therefore, whatever number of norms is formulated, it will only cover a restricted discursive area.

room *interactively*" (p. 37), and this means, in our context, a never-ending negotiation of rules through which the discourse of a given class is being shaped and reshaped. Yackel and Cobb (1996) devoted much of their research to the study of the ways in which activity-regulating principles are *interactively constituted* by all participants.

The observation on the reflexivity of the constitutive processes is very important, but lest it be taken too far, a word of caution is necessary. Whether applied to norms or to rules, the claims just made should not be understood as saying that the discursive principles in question are created in the classroom from scratch, and they are not implying that the class and the teacher are autonomous in the choice thereof. The mathematical discourse that is being learned in the classroom is a well-established part of the cultural heritage, and the students are supposed to become its participants exactly for this reason. The rules of this discourse are modified each time a new teacher and a new class start their work together. After all, children come to the classroom with their own discursive habits, and these habits are bound to leave their mark on the discursive habits of the whole class. Yet, the teacher will make the decisive contribution to classroom discursive practices. Being the "carrier of the tradition," the teacher is obliged to ensure that the students would eventually be able to participate in the mathematical practices of the broader community. This is probably why Cobb and his colleagues required the teacher to play a *proactive* role in shaping the rules of classroom discourse (see Cobb & McClain, 1999).

Meta-rules as a double-edged sword. The propensity for habitual conduct is related to the general human need for meaningful interactions with others. Behaving according to rules is a necessary condition of effective communication. The invisible meta-rules have an enabling effect in that they eliminate the infinite possibilities of discursive moves and leave the interlocutors with only a small number of reasonable choices. Without this preselection, we might be deprived of the ability to participate in any discourse. Just imagine that you are required to "investigate the function $f(x) = 3x^3 - 2x + 5$" and you are not sure whether you are supposed to list the properties of the graph (yet to be drawn!) or to admire its aesthetics, to count the characters with which the function has been recorded on the paper or to express your opinion about them, to make an investigation of the effects of real-life applications of the formula $3x^3 - 2x + 5$ or to check possibilities of transforming it, and so forth.

Thus, the enabling impact of the meta-rules seems all-important. Yet, it is noteworthy that the gain has its price: The constraining influence of the meta-rules may go beyond what is helpful. Every so often, they may close problem-solvers' eyes to promising possibilities or nonstandard routes that sometimes must be taken if a problem at hand is to be solved at all. Mathematics educators have identified a whole assortment of problems that they used to call "insight" problems, which are particularly difficult to solve not because of the intricate mathematical techniques required but because of the special nonstandard "Aha effect" necessary to launch a successful solution process. Insight problems are those problems that simply cannot be solved within the confines of the accepted rules of classroom mathematical discourse. Indeed, rules—any kind of rules—would often create mindsets. To give just one anecdotal example, a person asked during a mathematics lesson about the "odd number out" in the set {2, 3, 8, 10} is unlikely to give the answer that I have in mind: "It is 8, because 8 is the only number in the set whose English name does not begin with the letter *t*." More often than not, a mathematics student would not think about this kind of answer simply because considering the letters composing the number names does not belong to the repertoire of activities that count as mathematical. In sum, discursive meta-rules are both confining and indispensable. Although too much rigor is paralyzing, so is a complete lack thereof.

A Closer Look at the Concept of Mathematical Discourse

In the rest of this article, while trying to come to grips with the reformer's question, I will focus on a special type of meta-discursive rules that, as I will show, may be among the most influential factors in the processes of learning and understanding mathematics. The rules I will be dealing with are those that render mathematical discourse its unique identity. The specific rules of traditional mathematical discourse impose, among other things, extremely rigorous, precisely defined ways in which to argue about mathematical claims. The belief in proof as a formal derivation to form axioms, and the mathematicians' prerogative to establish axiomatic systems in any way they wish, provided the systems are free of contradiction, belong to this category.[13] With discourse-specific meta-discursive rules, people also decide, usually in an instinctive way, what kind of action would count as proper in a given context and what behavior would look rather out of place. To be more specific, the meta-rules in focus are those that determine the uniquely mathematical ways of communicating. These are the principles that regulate such discursive activities as delineating the meaning of concepts (defining), validating assertions on these concepts (proving), preparing written records, and so forth. Clearly, this subset of meta-discursive rules also determines the epistemological–ontological infrastructure of mathematical discourse.

At this point, an alert reader is likely to remark that this description is not clear because the central notion, mathematical discourse, is not unequivocal. For one thing, there is something vague about the term *discourse* itself, and then, even if

[13]It is noteworthy that this rule is a relatively new one; until at least the 18th century, only those propositions that were believed to be universally and objectively true were regarded as axioms.

the term is accepted, discourses as different as those led by professional mathematicians and those conducted by school students cannot be put in the same category. Let me elaborate on these two problems while making the case for the tenability and usefulness of the term *mathematical discourse.*

It follows from the former remarks on the dynamic nature of meta-rules that discourses cannot be seen as invariant entities with immutable properties. Rather, they should be compared to rivers that flow, change, and never stop, but which preserve their identities if only because of the continuity of this process. Moreover, any group of people engaged in an ongoing conversation, and any school class in particular, is bound to create its own idiosyncratic breed of discourse, with a somehow unique set of meta-rules.

All this said, I still sustain that there is something distinctive and relatively invariant about those discourses that we identify as mathematical. Moreover, there are grounds to believe that the similarities that make people say “this class is learning mathematics” whenever children are engaged in certain kinds of activity cross the boundaries of particular classes, schools, languages, and countries. What makes mathematical discourse distinct and easily recognizable is not just its content. The feature looked for instinctively is a special subset of meta-discursive rules that are distinct from anything known from other discourses.

The many discourses identified as mathematical may still differ considerably in certain subsets of their meta-rules. For instance, it is important to distinguish between academic (or research) mathematical discourse,[14] school mathematical discourse, and everyday mathematical discourse (Rittenhouse, 1998). Each of these discourses has its own unique set of meta-rules, and as anyone who has tried to compare a university lecture in mathematics and a school lesson in mathematics can testify, these sets differ in many different ways.[15]

Of course, the common descriptor “mathematical” signals an object-level similarity as well: It says that utterances coming from different discourses may still speak of the same objects. Thus, whether rendered in a formal scholarly language or in everyday careless parlance, an utterance dealing with operations on numbers or with transformations of geometrical forms may qualify as mathematical. Many mathematical concepts can be treated with reasonable precision within the flexible boundaries of everyday discourse, and thus, many mathematical facts can be presented and discussed in conversations that do not display many typical traits of ac-

[14]For insightful essays on different aspects of professional mathematical discourse, see Davis and Hersh (1981).

[15]Compare the notion of *didactic transposition* introduced by Chevallard (1985, 1990; see also Sierpinska & Lerman, 1996) to denote the change that inevitably occurs in discourses in the course of their transition from academia to school. In its original version, the term referred to the fact that professional knowledge must change in accord with the needs of the institution in which this knowledge is being practiced; in the language of discourse, we may say that we are concerned here with the transformation of discourse in accord with the needs and requirements of different communities.

ademic discourse. This is certainly true about almost any mathematical notion learned in elementary school. The focus in this article is on mathematical discourse that goes beyond everyday discourse in an essential way, that is, discourse dealing with mathematical concepts that the student is unable to incorporate into everyday discourse.[16]

USING THE DISCURSIVE LENS TO ANSWER THE ORIGINAL QUESTIONS

Let us return to the question asked after listening to Stendhal's complaint about his teachers' unconvincing explanations of the rule "minus times minus is plus": Why are certain mathematical concepts inadmissible and certain mathematical arguments unconvincing to many students?

The natural thing to do now is to see whether the just presented discursive conceptualization of the notion of learning helps in finding an answer. We do not have any information about the explanations offered by Stendhal's teachers, and even though it appears they were somehow faulty, we have no reason to dismiss the possibility that Stendhal's problem was to be found elsewhere. It may well be that the difficulty he experienced had to do with the *implicit meta-rules* responsible for the type of argument he was given, rather than with the explicit contents of the argument. The kind of argument that counted as proper and final in the eyes of the teachers might have seemed inadmissible or insufficient in the eyes of the student.

Here is one possibility of what Stendhal's teachers' justification could look like.[17] Taking as a point of departure the request that the basic laws of numbers, as have been known so far, should not be violated, and assuming that the law "plus times minus is minus" and the rule $-(-x) = x$ have already been derived from these laws (Stendhal seemed to have had no problem with these!), the explainer may now argue that, for any two positive numbers, a and b, the following must hold:

On the one hand,

$$0 = 0 \cdot (-b) = [a + (-a)]\,(-b), \tag{1}$$

and on the other hand, because of the distributive law that is supposed to hold,

$$[a + (-a)]\,(-b) = a(-b) + (-a)(-b). \tag{2}$$

[16]As will be explained, this difficulty stems from a certain incompatibility between advanced mathematical concepts, such as negative numbers, and those meta-rules of everyday discourse that generate its epistemological infrastructure.

[17]It is important to keep in mind that what follows is speculation and that the justification presented is an explanation of the reasons for defining a product of two negatives as positive.

Because it was already agreed that $a(-b) = -ab$, we get from Equations 1 and 2:

$$-ab + (-a)(-b) = 0.$$

From here, and from the law $-(-x) = x$, one now gets:

$$(-a)(-b) = -(-ab) = ab.$$

One may be outraged by the degree of formality of this justification. A different kind of explanation, more convincing in the eyes of the student, could only come from everyday discourse. Indeed, secondary school students' classroom conversations, not yet a case of a full-fledged mathematical discourse, are typically a result of crossbreeding between everyday discourse and modern mathematical discourse. In everyday discourse, claims about objects count as acceptable (true) if they seem necessary and inevitable and if they are conceived as stating a property of a mind-independent "external world." This applies not only to material objects but also to numbers, geometrical forms, and all other mathematical entities found in colloquial uses.[18] It is this "external reality" that is a touchstone of inevitability and certainty. In mathematics, as in everyday discourse, the student expects to be guided by something that can count as being beyond the discourse itself and independent of human decisions. This is what transpires from the words of Dan, one of the students who responded to my questionnaire, as he was trying to account for his difficulty with negative numbers:

01	Dan:	Minus is something that people invented. I mean … we don't have anything in the environment to show it. I can't think about anything like that.
02	Anna:	Is everything that regards numbers invented by people?

[18]Of course, this sense of externality and mind independence can be seen, by itself, as a by-product of discursive activities, a feature that we witness whenever objects of the discourse have a long history and constitute an inextricable part of one's everyday reality. Because this reality is in the never-ending process of discursive construction almost since the day we are born, it becomes for us ultimate and objective, and we accept it unreflectively as external to ourselves, as necessary, and as the only possible. The objects that populate this reality, although discursive constructions in themselves, function in our everyday discourse as if they had a life of their own. This sense of independence is strengthened by the fact that our everyday discursive use of objects is usually massively supported with visual experience and is mediated by operations we perform on images associated with the objects. Availability of such visually manipulable means, either actual or only imagined, underlies our ability to communicate about the objects and operate them discursively and is, therefore, a condition of our intuitive acceptance of these objects; the more so that we are "discursively conditioned" to regard as external whatever comes to us in the form of a perceptual experience (the folk models implicit in the ways we speak present the perceptually accessible objects as having an independent, external existence). This is one of the tacit meta-rules of everyday discourse that is evidently missing when negative numbers are considered.

03 Dan: No, not everything …
04 Anna: For instance?
05 Dan: For example, the basic operation of addition, 1 plus 1 [is 2], and according to the logic of the world, this cannot be otherwise.
06 Anna: And half plus $\frac{1}{3}$ equals $\frac{5}{6}$. Does it depend on us, humans or …
07 Dan: Not on us. You can show it in the world.
08 Anna: I see … and 5 minus 8 equals minus 3. It's us or not us?
09 Dan: It's us.
10 Anna: Why?
11 Dan: Because in our world there is no example for such a thing.[19]

Thus, the safest way for the student to understand and accept negative numbers and the operations on these numbers would be to make them a part of everyday discourse. Alas, in the present case this does not seem possible. Although people usually can incorporate negative numbers into sentences speaking of everyday matters, these discursive appearances are incomplete, in that they rarely include operations on numbers and thus, in fact, refer to such entities as –2 or –10.5 as labels rather than full-fledged numbers. This is evidenced by the results of my experiment in which 18 students were asked to construct sentences with the number –3 as well as questions that admit –2 as an answer. In both cases, they were encouraged to look for utterances with "everyday content." As can be seen from the results presented in Table 3, not all the students were up to the task. The few everyday uses of negative numbers were made solely in the context of temperature, latitude, and bank overdraft. In all these cases, the negative numbers were applied as a label rather than as a measure of quantity.[20]

The supposition that the formal derivation shown previously might have been the one that was offered to the young Stendhal by his teachers is thus more than

[19]If the object lacking "real world" support has somehow been admitted into discourse (and this may happen just because this is what is requested by the teacher!), it will always remain in its own separate category of "human inventions." This means a status of a second-rate citizen within mathematical discourse. This belief in the inferiority of entities that cannot fully fit into everyday discourse is sometimes expressed with the adjective *imaginary,* which implies that other, full-fledged objects are "real," that is, mind independent (cf. Sfard, 2000).

[20]It is also noteworthy that many of the "everyday" questions to which the answer was supposed to be –2 suffered from out-of-focus syndrome; that is, although the negative quantity was somehow involved in the situation presented in the question, the actual answer to the question should be 2 rather than –2 (see the last example in Table 3).

This result is easily explicable because of the former remarks on the crucial role of perceptual mediation in discourse. Although we can visually support some of the operations involving negative numbers with specially constructed models, no such model would enable making clear-cut discursive decisions about the way multiplication between two negative numbers should be performed (any such decision must be supported by assumptions on the model that, in the eyes of the learner, would often look arbitrary; see also footnote 21).

TABLE 3
Examples of "Everyday" Utterances Involving Negative Numbers Given by Students

Example	*%*
Sentence with –3	
"The temperature went down to –3."	42
Questions in which the answer may be –2	
"Temperature went down 12 degrees from 10 degrees. What is the temperature now?"	42
"How much money do you owe to John?"	25

plausible simply because the learned explainers did not seem to have had much choice: The just presented formal argument does not have a genuine alternative.[21] Yet, for a student who looks for objects and for ways to operate upon them in the external world, any idea brought into existence on the sheer strength of a logical argument must seem difficult to accept. Modern mathematical justification can only sound convincing if one admits the primacy of axioms and acknowledges the convention that consistency with a possibly arbitrary set of axioms is the ultimate condition of acceptability. This meta-discursive rule departs considerably from the epistemological infrastructure of everyday discourse. It is therefore rather obvious that formal derivation did not, and could not, have much appeal in the eyes of the boy who had yet a long way to go to become a fluent participant in the formal discourse of modern mathematics.

In summation, while speaking of the same thing, Stendhal and his teachers might, in fact, be participating in different discourses, regulated by different meta-mathematical rules. No wonder, then, that the teachers' argumentation did not convince the puzzled student: One cannot lose or a win in a game he is not playing! So, how can we proceed if we insist on teaching negative numbers and, at the same time, wish to honor the student's need for understanding? Mathematicians themselves overcame their difficulties with negative numbers by adjusting the rules of the mathematical game to the new needs. This, however, was a lengthy and painful process. Yet, throughout three centuries of persistent discursive application, mathematicians eventually got used to the new meta-discursive regulations to such an extent that, over time, the new rules became for them as natural, self-evident, and seemingly inescapable as the former set of meta-rules was for

[21]On the surface, this claim may be contested because many ideas have been proposed to explain and to model negative numbers (e.g., there is the model of movement in which time, velocity, and distance can be measured in negative as well as positive numbers; numbers may be represented as vectors, etc.). Yet, at closer look, all of these explanations and justification turn out to be derivatives of the same basic decisions about preserving certain former rules of numbers while giving up some others; these fundamental choices are exactly the same as the ones that find their expression in the acceptance of axioms of numerical field as a basis for any further decision, and they must be (tacitly) accepted prior to any justification.

their predecessors. A similar revolution must probably take place in classroom mathematical discourse if the operation of multiplying negative numbers is to be accepted. Whether and how it can be attained in today's classrooms is a separate question, which I revisit in the closing section, while reflecting on the current reform movement.

HOW WELL DOES THE REFORM SERVE THE NEEDS OF THE BEGINNING PARTICIPANT IN MATHEMATICAL DISCOURSE?

Classroom Mathematical Discourse: Where Should Its Rules Come From?

A new question must now be answered: Where should the meta-rules that make a classroom discourse specifically mathematical come from? On the surface, the answer may be simple: Having to do with mathematics, this particular school discourse should be as close in its meta-rules as possible to the discourse led by mathematicians. However, the example discussed previously illustrated that the issue is not as simple as that. Forceful evidence also comes from experience with the New Math project, which in the late 1950s and 1960s tried to transport mathematicians' discourse directly from universities to school classrooms (cf. S. I. Brown, 1997). This attempt could not be fully successful simply because its conceivers did not take into account the adverse effects of the tension between the professional mathematical discourse and the student's former discursive experience.

Still, the need to preserve certain basic characteristics of professional discourse in school is unquestionable. After all, the decision to teach mathematics to everybody results from recognition of the importance of this discourse. In this vein, Lampert (1990) claimed that the proper goal of teaching should be "to bring the practice of knowing mathematics in school closer to what it means to know mathematics within the discipline" (p. 29). The words "the practice of knowing mathematics" signal that the emphasis is on the meta-discursive strata.[22]

What Do the NCTM Standards Say About Mathematical Discourse?

Since it is difficult to speak about the reform movement in general, I will take the National Council of Teachers of Mathematics (NCTM, 1989, 1991) Standards, is-

[22]This approach is germane to what Hicks called "deliberate 'genre instruction'" (as cited in Lampert & Cobb, in press) and to what Cobb et al. (1993) called "talking about talking about mathematics" (p. 96; see also Yackel & Cobb, 1996).

sued during the last decade, as a generic example, aptly representing the spirit and principles of the change pursued these days throughout the world. The language of discourse is present in the Standards, especially in the volume devoted to teaching. The authors of the Standards explained:

> Discourse refers to the ways of representing, thinking, talking, agreeing, and disagreeing that teachers and students use to engage. … The discourse embeds fundamental values about knowledge and authority. Its nature is reflected in what makes an answer right and what counts as legitimate mathematical activity, argument, and thinking. Teachers, through the ways they orchestrate discourse, convey messages about whose knowledge and ways of thinking and knowing are valued, who is considered able to contribute, and who has status in the group. (NCTM, 1991, p. 20)

This definition is an attempt to alert the implementers to the existence of implicit rules as well as to the indirect discursive ways in which these special contents are being communicated. The Standards, in their entirety, may count as a comprehensive attempt to make clear what, according to the authors, constitutes the proper classroom discourse. In this context, it is important to notice that, quite unlike more traditional documents, the Standards make space for certain concrete meta-level rules, such as the one that requires that students' own experience and reasoning, rather than the teacher and textbook, will be regarded as the main source of mathematical knowledge and certainty (e.g., see NCTM, 1989, p. 129).

How Is Reform-Inspired Classroom Discourse Different From Professional Mathematical Discourse?

"Mathematics presented with rigor is a systematic deductive science but mathematics in the making is an experimental inductive science," said Polya (1957, p. 11). This means, among other things, that the mathematical "discourse of doing" is much more natural and less constraining than the "discourse of reporting." Even the discourse of doing, however, is rather disciplined in comparison with other discourses, whether everyday or scientific. By greatly restricting the admissible ways of expression, mathematicians try to ensure that the exact shape and content of this discourse would count as independent of tastes, judgments, and preferences of interlocutors. One may rightly expect that classroom mathematical discourse would be a greatly relaxed, less rigorous, and more "popular" version of this discourse. Sometimes, however, the relaxation of rules may be so radical that it would start to count as "redefining what constitutes mathematics" (Wu, 1997, p. 954; cf. Sierpinska, 1995; Thomas, 1996). A careful analysis of the NCTM Standards' requirements shows that, contrary to the Lampert's (1990) recommendations quoted previously, the new school mathematics may indeed turn out very differently from "what it means to know mathematics within the discipline" (p. 29).

Let me give a very brief and incomplete account of the ways in which reform-inspired classroom discourse may be different from that of the professional (see also S. I. Brown, 1997; Love & Pimm, 1996). As I will show, certain values and norms professed by the Standards may be interpreted in a way that remains somewhat at odds with the norms that regulate the traditional mathematical discourse. This normative incompatibility can be seen mainly on the level of rules that speak of one's rights and obligations as a participant in mathematical discourse.

First, educators and mathematicians are often divided on the issue of what counts as a truly mathematical activity. Inspired by claims of the essential situatedness of learning (J. S. Brown, Collins, & Duguid, 1989; Lave, 1988; Lave & Wenger, 1991), the Standards promote embedding abstract mathematical ideas in familiar, concrete contexts (cf. Sfard, in press). The tendency to always look for real-life situations and to eschew dealing with "distilled" mathematical content is very much in the spirit of everyday discourses, but it contradicts what is often believed to be the very essence of mathematization. After all, mathematizing is almost synonymous with "flying high" above the concrete and about classifying things according to features that cut across contexts. Mathematicians would claim (e.g., see Wu, 1997) that the ability to strip the bones of abstract structures from the flesh of concrete embodiments is the main source of mathematics' unique beauty and strength. When we restrict ourselves to real-life based mathematics, we are tying mathematics back to the concrete and particular—and losing what for the mathematician is the gist of mathematical creation. In addition, mathematicians would argue, the great emphasis on putting mathematics into a real-life context creates an utilitarian atmosphere, foreign to the modern mathematical discourse: Wu deplored the disappearance of "the spirit of intellectual inquiry for its own sake" (p. 956).

In a similar way, educators and mathematicians would argue over the admissibility of nonanalytic arguments, such as those employing visual means. Although this latter type of argument is often recognized in the school as sufficient, it is still seen by mathematicians as far from decisive or final, even if helpful (Davis, 1993; Rotman, 1994; Sfard, 1998). More generally, the Standards put a great premium on heuristics, which is sometimes misinterpreted as a green light for acting without any restrictions. Indeed, every so often, the only instruction given to students to engage them in an activity of proving is, "Convince your partner." Yet, just like a mathematician's argument would often fail to convince schoolchildren, so would a child's argument fail to convince mathematicians. Because of the practically unlimited freedom in the choice of the ways of "convincing," what is unique to the mathematical discourse of proving may be lost. This is probably why critics speak of a "cavalier manner in which the reform treats logical argument" (Wu, 1997, p. 955), while deploring "suppression of precision" (p. 957).

Furthermore, school mathematical discourse engendered by the Standards turns out to be highly personal. Students are invited to speak and write about their mathematical experience in any way they choose, using first person language. The "subjectivization" of the discourse may be taken to an absurd—and, of course, unintended—extreme if the Standards' call for open-ended problems "with no right answers" (NCTM, 1989, p. 6) is misinterpreted as saying that "any solution goes." All this stands in stark contrast with classical mathematical discourse, the hallmark of which is an uncompromising impersonality. This latter style is what imbues the discourse with the air of objectivity and mind independence. A mathematician for whom Platonism is "a working state of mind," if not an outright article of faith (Sfard, 1994b), may find the personal note detrimental to the whole project.

Perhaps the most dramatic difference between school math and professional mathematics is to be found in the meta-rules that constitute the epistemological infrastructures of these discourses. The major change that occurs in the transition from school to academia expresses itself in the already mentioned transfer of the source of certainty from outside mathematics into the mathematical discourse itself. For mathematicians, the inner consistency and overall coherence of mathematics is the ultimate source of its justification. In modern mathematical discourse, the meaningfulness of a concept stems from its being an element of a harmonious system. For students, who have no means to appreciate this overall coherence, the little isolated pieces successively encountered in the course of learning may always remain somehow arbitrary.

What Can Go Wrong or How Can Certain Interpretations of the Reform Requirements Make Classroom Mathematical Discourse Unlearnable?

In summation, mathematical discourse that develops in the classrooms following the NCTM Standards may turn out quite different from the professional discourse of the working mathematician. This remains in contradiction with the declared goal of making the student "a legitimate peripheral participant" (Lave & Wenger, 1991) in true mathematical discourse. Nevertheless, one may deem this state of affairs fully justified. After all, the disparities mentioned previously are an inevitable result of an attempt to imbue the learning of mathematics with more progressive values and, above all, with respect for the student's ways of thinking. The relaxation of rules is further justified in view of the great diversity in students' needs and capacities. Many would claim that such a change does not have to be acceptable in the eyes of professional mathematicians to be sanctioned as pedagogically sound and necessary. In addition, the "expert practitioners," unhappy in the face of necessary concessions, must agree that a reasonable compromise is the only possible solution. Indeed, what is the use of trying to teach strict rules of professional mathematical discourse if almost nobody can learn them?

A compromise, however, can count as reasonable only as long as the discourse we are left with is well defined, intrinsically coherent, and generally convincing. If, on the other hand, we simply reject some of the basic conventions without replacing them with alternative rules, or if the changes we make are accidental and inconsistent, we may end up with a discourse so ill-defined and amorphous that it simply cannot be learned. As it turns out, such danger is real. Mathematics is a well-designed, highly organized system that cannot be arbitrarily modified in one place without creating problems in another. Just as the game of chess would become uninspiring and impossible to learn if we arbitrarily substituted or removed some of its rules, so may mathematics become somehow meaningless following too careless a "relaxation." As was shown, the idea of a negative number cannot be fully understood within a discourse that is regarded as describing the physical world because there is nothing in this world, as it is known to the student, that would dictate the rule "minus times minus is plus." Similarly, the request for rigorous definitions, which may count as "truly mathematical," cannot sound convincing without relation to the idea of mathematical proof; the mathematical rules of proving, in their turn, cannot be understood without the agreement that the ultimate criterion of a proper argumentation is the logical bond between propositions, not relations between these propositions and physical reality. None of these meta-rules can be arbitrarily removed or changed without affecting the congruity and cohesiveness of the discourse. On a closer look, therefore, because of the keen wish to respect students' need to understand, one may end up compromising the very feature of mathematical discourse that is the basic condition of its comprehensibility: its inner coherence.

WHAT CAN BE DONE TO SAVE BOTH CLASSROOM MATHEMATICAL DISCOURSE AND REFORM (AND TEACH NEGATIVE NUMBERS IN A MEANINGFUL WAY)?

The problem is intrinsically complex, and its solution is probably not just a matter of the good will of legislators and implementers. As educators, we are faced with a dilemma. On the one hand, the decision to give up an attempt to teach full-fledged discourse of modern mathematics is a result of the recognition that today's students, just like Stendhal before them, cannot possibly accept its formal and, seemingly arbitrary, meta-rules; on the other hand, without these meta-rules, the learner may not be able to regard certain more advanced concepts and techniques as fully justified. As if this was not enough, there is another didactic complication. I have already stressed the unjustifiability of meta-discursive rules or at least the impossibility to argue for their inevitability in a fully rational way. Yet, within mathematical discourse, rationality is the name of the game. While teaching mathematics, we create a belief that understanding can only be attained on the force of logical argument. Although this belief is intended to guide the student in judging the veracity of object-level statements, it may be trusted to "spill over" to the meta-level, creating expectations that cannot be satisfied: The student may demand rational justification of the meta-rules. And yet, no such justification can be provided.

The problem we are facing seems unsolvable. It is extremely difficult to establish appropriate measures of discipline and rigor in school mathematical discourse; it is even more difficult to decide about the ways to teach those advanced meta-rules we eventually deem as indispensable. Many solutions have been suggested, and each of them may be worth some thought (for a review of educational approaches to the issue of proof and proving, see Hanna & Jahnke, 1996). Aware of the basic unsolvability of the problem, some people suggest a radical change in the general approach to school mathematics or at least to high school mathematics. Thus, for example, some educators, evidently sensitive to the interdependence of the rules, tend to view the relaxation as a package deal and suggest giving up any kind of mathematical rigor. This clearly cannot be done, at least at the secondary level, if we wish the learning of mathematics to remain meaningful. Some other mathematics educators build on an analogy with poetry or music and propose that, beginning with a certain level, we teach students about mathematics rather than engage them in doing mathematics. After all, exactly like poetry and music, mathematical techniques do not have to be fully mastered to be appreciated as part of our culture (e.g., see Devlin, 1994, 1997). It is far from obvious, however, that this is a workable proposal: Although one can certainly appreciate and enjoy poetry and music even without being able to produce any, it is probably not the case with mathematics. Another radical solution would be to turn high school mathematics into an elective subject.[23]

If we insist on teaching mathematics that cannot be easily incorporated in, or derived from, everyday discourse, there seems to be no escape from introducing the students to the meta-rules of modern mathematics or at least to their selected subset. This must be done not in spite of, but because of, our respect for students' thinking, and out of our care for their understanding of the logic of the subject they are supposed to learn.

This goal, although extremely difficult to attain, may not be entirely beyond reach. Although the problems we are facing as teachers of mathematics may indeed seem intractable, much can be done to reduce their impact.[24] One can point to

[23]Some people would claim that the added value of such a move would be an overall improvement in teaching: We would have to try harder to make the subject attractive in students' eyes. The relationship between popularity and the quality of instruction is, however, far from obvious.

[24]One possible way to facilitate the transition from everyday to professional mathematical discourse has been offered by the designers of the Realistic Mathematics Education program in The Netherlands (e.g., see Gravemeijer, 1994; Gravemeijer, Cobb, Bowers, & Whitenack, 2000). According to

a number of principles that should guide us in teaching those parts of school mathematics that exceed the boundaries of everyday discourse. First, while deciding which mathematical meta-rules should be preserved and which can be given up, we need to be careful not to take out ingredients without which the whole construction might collapse. Second, being more explicit about the meta-level rules that are to be learned may also be helpful. Lampert (1990), Ball (1997), and Cobb and his colleagues (Cobb, Wood, & Yackel, 1991, 1993) have shown that this may be done even at the most elementary levels. In their studies, they investigate the ways in which teachers and children can negotiate the multifarious norms of their classroom discourse. Third, it is important to recognize the mutual dependence of certain sets of meta-rules and specific sets of concepts. It would be a mistake to think, for example, that the axiomatic method should be taught before any concept dependent on this method is introduced. In fact, it seems that the only way to bring about the recognition of the axiomatic method is to try to deal with concepts that depend on it for their justification. Thus, the acceptance of the concept of negative number and of the axiomatic principle can only come together as a result of a complex and lengthy dialectic process, in which one need stimulates the other. Needless to say, the principle of a disciplined discursive use of a new concept in conditions of persistent doubt is one of the meta-rules that should be turned into a norm in a classroom where students are supposed to proceed in this way.

While giving this advice, one needs to remember that the unique rules of mathematical discourse cannot be learned by simple articulation, and they cannot be reinvented by students engaged in discussing mathematical problems "in any way they regard as appropriate." Rules of language games can only be learned by actually playing the game with experienced players. The profound constructivist principles underlying the current reform movement are only too often misinterpreted as a call to teachers to refrain from any kind of intervention. Yet, the teacher who requires the learners to work on their own, who keeps from "telling," and who never demonstrates his or her ways of doing mathematics deprives the students of the only opportunity they have to be introduced to mathematical discourse and to its meta-rules. A mathematics teacher who abstains from displaying his or her own mathematical skills may be compared to a foreign language teacher who never addresses his or her students in the language they are supposed to learn. The historical reasons for the way mathematical discourse developed would not convince today's students. Thus, it is naive to think that either mathematical discursive habits or the ability to speak a foreign language could be developed by children left to themselves.

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their carefully planned scenario, everyday discourse is to be extended step by step, to include ever more advanced and abstract mathematical ideas. The guiding principle is that none of these successive extensions is done before a present version of the discourse, together with its objects, appears to the participant as realistic as the everyday discourse with which they began.

Finally, the point I was trying to make in this article may be illustrated by the story of a poor man who asked his wife to cook him a dish often served in a rich man's house and rumored to be truly delicious. The obedient woman did what she was asked to do, alas replacing or simply removing most of the luxurious ingredients that were beyond her means. Ever since the poor man tasted the result of his wife's efforts he could not stop wondering about the peculiarity of the rich man's taste: How could that unpalatable dish be an object of delight for the latter? Similarly, if we remove too many ingredients from the exquisitely structured system called mathematics, we may be left with a tasteless subject that is not conducive to effective learning.

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