What works?

Research in the primary mathematics classroom[1]

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> It was in the Thirtieth already, when in Germany still we had many village schools with just one class, where all age groups from 6 through 14 were sitting together (grade 1 through 8). The teacher entered the classroom and before he could say a word a little girl raised her hand and said: '56!' The teacher, surprised, asked her 'Why that?', and she responded decidedly: 'We do have a math lesson now, don't we?' - 'Sure' - 'Yeah, and you always open the math lesson with mental arithmetic.' - 'Yes' - 'And you start with the most difficult task of the multiplication table, and that is 8 times 7.'[2]

What we can learn from the story: Children always have expectations of what is coming to happen now. In particular in school and in a mathematics lesson they have quite developed images and expectations according to their actual interpretation of the situation in the classroom. What teachers usually are *not* sufficiently aware of is, that such expectations are the outcomes of the experiences which teacher and students have together. In other words: children from a special *mathematical habitus* across their related school experiences, subconsciously, but functioning very reliable - especially where we, the teachers don't like it - and very powerful. The whole range of relevant aspects of the child's mathematical activities from 'what is taken as a math task' to 'how does it have to be said' (or 'represented with material', or 'written with symbols') and 'what is accepted as an answer', 'which commentaries (or critique, or associations, or supplements) are allowed etcetera is formed this way. Teachers usually do not know about the covert functioning of the birth of all these regulations. They are an outcome of the specific *classroom culture.*

The specific 'culture' of the classroom

I prefer to use the word 'culture' for the social processes, because it transports the connotation of *indirect* learning, of getting introduced through *participation,* of 'knowing how to do things' but not necessarily why. Let me give another example from my own classroom experience.[3]

Early in first grade, the children knew the numbers from 1 to 6 and some simple addition within that section, the teacher intended to introduce the related ordinal number symbols. She asked six students to come out of the room with her and to prepare something for the others. So I was left with the rest of the students who asked me curiously, but I also did not know what was to come. The door opened and one after the other the children came in, holding over their heads cardboard squares with '1' through '6' written on them. They lined up in front of the class and then the issue was discussed intensively. The question 'What is special with the numbers?', directed the children's attention to the points next to the figures. It was cleared that these notions are to be read differently, 'the first', 'the second' etcetera, that the '4' comes after the '3' and ahead of the '5'. The children then handed their card each to another classmate and the new group lined up in the adequate order, supported by the class. At a sudden Matthias' voice broke into the game: 'The zero is missing!' Everybody seemed to be struck. Did we miss something of importance? The teacher and I we were affected too and looking into each others eyes we decided not to intervene and let things go.

Matthias insisted: 'There must be another child with a card and a zero on it!' The debate among the students quickly developed: 'Yes, one must stand at the beginning with a zero!' 'Why that?' 'That's the zeroest.' 'And the zero must have a dot!' 'But', said Katrin, a bright girl, thoughtful, 'that won't work. Think of the door, who is coming in first then?!' Disappointment comes up. No word, everybody thinks of a way out of the troubles.

Then a Turkish boy offered: 'Perhaps we can give the zero to a girl and place her after 6?' (There are many other stories to be told about the specialities of little Turkish boys.) 'Rubbish', replied an experienced boy, a repeater, 'that makes sixty!' Once again, silence and reflection. Now Matthias had his great moment: 'Perhaps we simply put a card with a zero on the bottom ahead of the first one, and no child with it?' Isn't it a marvelous solution? Without knowing about the difficulties of the number line - counting the marks (strokes or points) versus counting the sections (distance or length) in between - he has made a special issue out of the zero, a starting point.

Clearly, it won't work, if you *want* to have it this way. It just happens. But it is a case of the culture of the classroom, whether it *becomes possible* at all or not, whether it is appreciated and encouraged by the teacher or not, whether the related and necessary social organisation of the class has become a habitual part of every student or not. Sometimes, I come to think, that we, the teachers, are by far to rigid with our insisting on precise mathematical presentations compared with our wide tolerating of social or communicative disorder, whereas the other way round might be a more promising way for developing mathematical reasoning: Being very strict with communicative regulations but very tolerant with every mathematoid contribution of the children. It is not the teacher alone who forms the culture. Both teacher *and* students constitute the culture of their classroom through their interaction. It is their way of living together, of jointly experiencing and developing mathematical activities, and of forming related conventions.

Making issues 'ondubbelzinnig' or 'eenduidig' too early

Since I can discuss my main concerns related to research outcomes through examples and through selecting only, let me point another crucial issue. Teachers often complain about the terrible ways their students use when solving text problems. During the reading already many students 'intuitively' - better: habitually - form their ideas about numbers and adequate operations. And once they have done their calculations, only the right or wrong of the result is of interest but neither meaning nor dimension of it: '170 comes out', but whether meters, centimeters, Grandmothers, cents, or square miles, who cares?

If habits develop in our mutually generated classroom culture, then it might be helpful to look back at where we have introduced procedures, which might give support to the emergence of such unwanted subconscious strategies. We will have to go back as far as into the first grade. Here, all over the world, pictures are used to support the formation of 'number sentences' (the first equations with natural numbers). Nearly everybody having passed schooling will identify a picture with three birds sitting on a fence and another two birds flying in the air with number sentences like $3 + 2 = 5$ or $5 - 2 = 3$. The sad point in this issue is, that the picture allows a nearly unlimited heap of referable number sentences, according to the selected focus (on birds, eyes, wings, legs, toes, fence pales, clouds, etcetera).

The picture (fig.1) is taken from a common first grade mathematics textbook. If you asked first grade students ahead of any related convention they will produce a great variety of interpretations (among them surely the expected one: 'The wall has three bricks, the bricklayers bring another three bricks): 'Three thieves steeling big stones from a pile'; 'They check: the left one has the heaviest stone'; 'Craftsmen steeplechasing'; 'They hear the sign for break and will throw down their load yet'; 'They wonder about who has to come first'; etcetera. This is to say: the unequivocal mathematical interpretation the teachers wants to have can be brought up *through convention* only. It is the teacher's insisting on a one-to-one 'translation' into a number sentence, 3 + 3 = 6, which narrows the students' production. While in 'realistic' cases our actual interests and purposes do form the criteria of relevance for the selection of the focus on a situation or a picture. The pictures *never speak for themselves.*

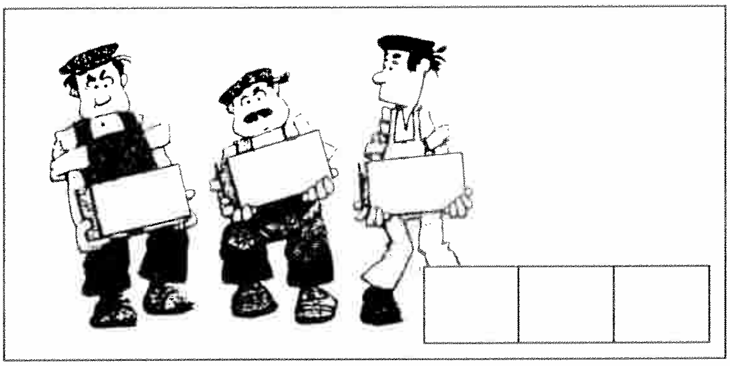


Figure 1: visualization of number sentences in first grade textbooks

Moreover, the same is true with presentations which teachers mostly would like to identify directly as 'truly mathematical' visualizations. Radatz has published a surprising piece of research work. At the opening of a mathematics lessons and at the opening of a religion lesson he had given a set of pictures to third and fourth graders (age 9-10) and he asked them for an interpretation, without any further commentary. The results are presented in figure 2.

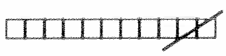
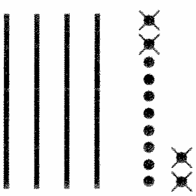
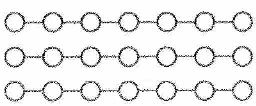
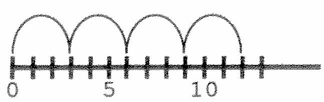
Presented 'visualization' 'What does it mean?'	**Percentage of mathematical acceptable interpretations at the start of**	
	mathematics	**religion**
	50%	27%
	26%	6%
	27%	14%
	8%	3%

Figure 2: third and fourth graders' interpretations of mathematical visualizations at the opening of a mathematical lecture and of a religion lecture

As interesting as the interpretations in detail are, in particular under the expectation of an religious instruction, it is sufficient for our discussion here to realize the striking differences. Obviously there is no unequivocal meaning of these representations.
The ascribed meaning, the interpretations, are highly specific to the child's framing of the actual situation.

The sad point is: the early setting of such conventions makes teaching and classroom organization easier (for the teacher). But it cheats the students out of their developing of mathematical reasoning. Instead of eliciting different mathematical interpretations, relating different number sentences to the situation, and giving reasons for each choice, this early conventionalizing produces the illusion of clearness, of a one-to-one translation of reality into mathematics.
At the surface these conventions make classroom processes smooth and less conflicting, but at an unacceptable high price: poor meaning and no flexibility. These conventions replace insight by routine.

The alternative clearly is *not* deliberate interpretation and arriving at any nonsense in the end. Alternative classroom cultures may on the contrary accentuate the *negotiating* of useful meanings, of an adequate selecting of quantitative descriptions and operations, of producing reasons for each decision, and of a collecting and comparing of such ideas. These, I think, are helpful for the solving of text problems.

There are many more covert strategies which text-book writers implement in order to make teaching smooth, as Jörg Voigt has pointed at (publication in preparation). One of these undercover tricks is the exploiting of our habitual reading direction from left to right. Compare your own actual interpretations of the two pictures shown here! In commentaries of the above picture the keeper appears as to be the active part. Children say for example: 'The keeper hands bananas over to the chimp; he has three left.' Whereas interpretations of the reflected picture mostly see the ape as the subject, for example: 'The ape has taken two bananas from the keeper, now he laughs at him!' Here the keeper appears as the object, as the one who is the target of the subject's action.

Figure 3 and 4: picture and reflected picture 'read' under the same reading routine

Textbook writers apply this reading routine in order to promote the 'reading' of visualizations in the wanted way. The following textbook page (fig.5) is an illustration for this depriving technique of 'introducing' the students into pseudo-unequivocal translation habits. It becomes possible - but forcing through convention only - to 'translate' each picture into a number sentence by grouping the objects from left to right. Two cyclists are waiting, three are driving away, and so on.

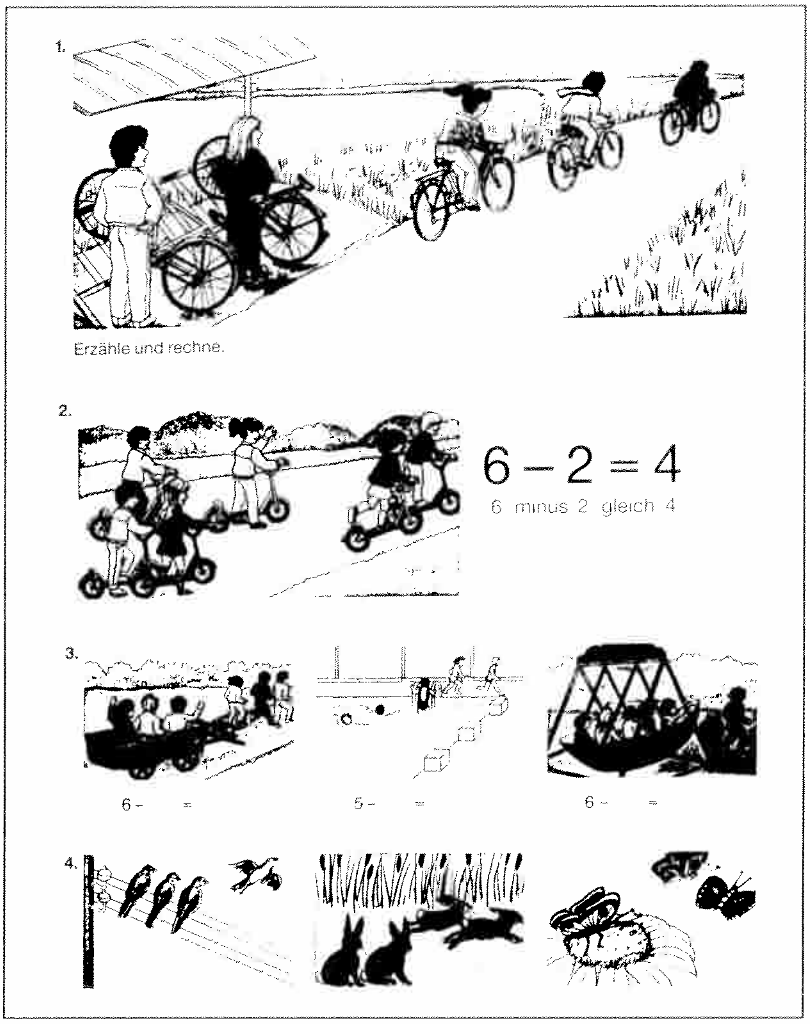


Figure 5: page from first grade textbook, representing visualizations for subtraction in the left-to-right reading direction

An example from an alternative culture

Instead of making pictures and representations as 'clear' and unequivocal under the focus of the wanted meaning it can become helpful to raise their complexity. At the flannel board the teacher had arranged five nice green apples (fig.6), (cardboard with a piece of sandpaper on the backside), one apple with two worms, three with one worm each, and one apple without a worm. (Biologists may forgive us, that we put two worms into one apple, which in nature does not come to happen!)

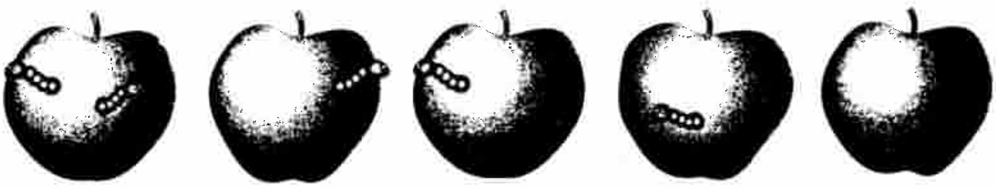


Figure 6: the wormly apple presentation

The teacher wrote '4 =1 + 3' on the blackboard and waited for commentaries of the students. But the children 'read' or saw different things there: 'There are two worms in one, and then three with one, and one has none.' (One has to listen carefully to catch this excellent presentation!) Our zero-specialist Matthias got nervous already. Another boy said: 'There are four apples with a worm and one without one.' Now Matthias burst in: 'There are two in this apple (points), and in the other three is one only, and there is none. There (points) we must write a zero on top, because there is no worm in ... *zero worm*!'

Now an intensive struggle arose. The class split up into an apple party and a worm party. 'There must be a *1*!' (Sure, it is *one* apple.) 'Non, no - *zero*!' 'A zero stands, if there is nothing at all.' 'Write 1!' 'Ha-ha, but there is no worm in it!', insisted Matthias. Astrid, a long-suffering repeater and well experienced with teacher's tricks, suspected: 'There is very well a worm in it! She (the teacher) has turned over the apple!' Obviously and inevitably, our realities are our formed expectations. The teacher quickly turned over the cardboard apple. There was sandpaper only, no worm.

'Aha, there *has* to be a zero!', shouted a triumphant Matthias. But Katrin, with insight and mediating power, injected: 'This is not a matter of worms, it is a matter of apples' and she offered the number sentence '4 + 1 = 5'. With this the apple party was satisfied. And after some additional discussion (without teacher's invention) the worm party arrived at the number sentence '2 + 1 + 1 + 1 + 0 = 5'. Surprisingly and as a new experience for me, the children all applauded in the end. An indicator for how intensive they themselves had experienced tension, effort and final achievement. By the way, since that event Matthias' position as *the* specialist for every zero question is proven.

What mathematizing and languaging have in common

That mathematics is learned like a language, one can find stated quite often, but mostly without drawing implications from it. Quite helpful, I think, the similarity can be referred to much more fundamental aspects[4] of all learning:

1. A person acquires and develops mathematizing as a special activity - like learning to speak - across *social interaction* only, through *participating* in (and also *actively contributing* to) a living culture. The richness or poverty of the culture therefore is crucial for the quality of the personal development.
2. The meaning of a word - in mathematics as well as in language in general - is *how it is used for* in a culture. Thus the meaning of a word or of a procedure are *social conventions*, and are therefore specific to the situation as well as to the (sub-)culture.
3. The larger part of these acquisitions (or better: interactively developed subjective constructions) come to happen covertly, emerge subconsciously. In a referable situation therefore the related knowledge is without delay *'at hand'* for the member of the culture. But the member can not give reason for it or reflect upon directly - 'you just do it that way'.
4. Likewise the regulations of social interaction - in the classroom as well as in society, with mathematizing as well as with languaging - are 'lived' by the members of the specific culture. Their regulations, their culture, *is* what they bring about in every moment.

The identification of such regulations[5] of actions through observers, in particular through research work, carries a misleading temptation for educators and teachers. They usually try to 'teach' these regulations as rules directly, expecting the outcomes of such teaching to be available to the students in all referable situations. But 'knowing' rules verbally is quite different from *being* a member. Having learned George Pólya's famous rules for problem solving verbally is by no means warranting good problem solving in real problem situations. Simpler, a person may 'know' about the rests of division by nine or eleven; but to control the own multiplications and divisions permanently and habitually by applying these techniques is part of a quite different *practice* (or culture).

Characteristics of an alternative classroom culture may be arrangements in which the students find themselves encouraged to work on challenging problems in different ways, developing and comparing their ideas and constructions, forming regulations through their own activities and engagement. It is the living culture then, which they bring about and which simultaneously supports the adaptation of subjective constructions and rules towards an acceptable practice.

Let me close with a story which I have found in Richard Skemp's recent book[6], who himself owes it to Edith Biggs. It is the sad report from a counter culture:

> 'A six-year-old said to me the other day: "give me a number and I'll double it for you." I gave him 37 (he was calculating in his head). He said "Two thirties that's sixty, two sevens that's fourteen, seventy-four." He continued, "Don't tell my teacher." I asked why not since his method was e good one. The boy answered: "She makes me write it down and I don't understand her method, so I do it in my head and then I write it down her way and I always get my mark, so don't tell her".'

Notes

1. What I feel has to be added now, is what I have not said opening my lecture, because I lose my balance all too easily: That Hans Freudenthal cannot be with us any longer makes us miss his orientation, his sharp critique, and his encouragement. Not only The Netherlands, we all have lost a great person. What he has left us, I think, is the obligation to carry on in his and the former IOWO's spirit for both the children and the teachers.
2. I owe this true story to my colleague Hans Kötter, professor of psychology at the university of Saarbrücken, who experienced it himself as a student at school in the Königsberg area.
3. In August 1988 I have started a little four years project, sharing the teaching of a mathematics class at school with an experienced woman teacher from the very beginning in grade one (age six plus) through the end of grade four. We are now in the third year. She is a specialist in arts and crafts, but not in mathematics, so we complement each other quite well. I have selected a school in an economically rather weak area. About a third of the students therefore are children from Turkish or Yugoslavian guest workers, another third are children from Russian remigrants, mostly Mennonites, and the last part comes from German low income strata, unskilled or unemployed workers o.s. Surely a difficult class with all problems from language to uncontrolled behavior and cases for special education, but forming an extremely rewarding encounter and with kids grateful for every challenge you provide for them. Most of what I am going to talk about here is related to this project. The main aim is developing both, the mathematics curriculum as well as our understanding of classroom processes.
4. See: Savigny, Eike von: *Die Philosophie der normalen Sprache,* Suhrkamp, Frankfurt/Main 1980, stw 29, 2.Auflage.
5. 'Regulation of action' refers to both here, to the regulations and pattern of social interaction as well as to the orientation and rules of personal actions.
   More details may be found in my paper 'Structuring the structures - Development and function of mathematizing as a social practice', *Occasional Paper no. 122,* IDM der Universität Bielefeld, 1990.
6. Skemp, R.R.: *Mathematics in the Primary School,* Routledge, London 1989, pag. 185.