What does it mean to understand some mathematics?[1]

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1 introduction

To understand mathematics *as a whole* would require a discussion of the roles mathematics plays in everyday personal affairs, in schooling, in occupations, in other fields such as physics, and in its existence as a discipline studied for its own sake. In contrast, this chapter is primarily concerned with what it means to understand *some* mathematics, which generally means to begin with a bit of mathematics and to subject it to detailed analysis, usually from the perspective of the learning of that bit.

My perspective is that of a curriculum developer, from decades of writing materials for students that attempt to lead them to understand the mathematics they are being asked to learn. My perspective falls somewhere between Freudenthal's 'Didactical Phenomenology of Mathematical Structures' (1983) and the American mathematics education researchers Hiebert and Carpenter's chapter on learning and teaching with understanding in the 'Handbook of Research on Mathematics Learning and Teaching' (1992).

2 understanding school mathematics

There is a common saying attributed to Confucius, which in English is sometimes translated as: 'I hear and I forget. I see and I remember. I do and I understand.' Yet we often hear it said that students can 'do' certain mathematics but do not understand what they are doing.
This roughly parallels the difference between what in psychology are sometimes called *behaviorism* and *cognitivism*. We in education act both as behaviorists and cognitivists. We view 'understanding' as something that goes on in the brain without external actions yet we want students to *exhibit* their understanding by responding to tasks we present before

them. Specifically, as behaviorists, we want students to answer questions correctly and sometimes do not care how they got their answers. As cognitivists, we want to know what students are thinking as they work with mathematics and we ask students to show their work.

In 1976, the British mathematics educator Richard Skemp (1919-1995) wrote on this subject with the phrases *instrumental understanding* and *relational understanding* essentially meaning *procedural understanding* and *conceptual understanding*. He wrote: 'I now believe that *there are two effectively different subjects being taught under the same name, "mathematics"*.' Skemp's dichotomy is, I believe, now the most common broad delineation of what is meant by mathematical understanding.

I agree with Skemp that instrumental and relational understanding are different but I do not think that they are different subjects. I view them as different aspects of understanding the same subject. In this paper, I argue that there are more than two aspects or types of understanding, as different from each other as Skemp's two types. For reasons I explain later, I call these aspects *dimensions of understanding*.

What is mathematics? Mathematics is an activity involving objects and the relations among them; these objects may be abstract, real, or abstractions from real objects. The activity consists of *concepts* and *problems* or *questions*: we employ and invent concepts to answer questions and solve problems; we pose questions and problems to delineate concepts. So a full understanding of mathematics requires an understanding both of concepts and of problems and what it means to invent mathematics.

It is natural for mathematics teachers to view the understanding of mathematics from the standpoint of the *learning* of mathematics. But the *full* or *complete* understanding of mathematics in schools requires more than the learner's perspective. It includes the understanding of mathematics also from the standpoints of educational *policy-making*, from the standpoint of the *teaching* of mathematics, and from the standpoint of those who *invent* or *discover* new mathematics. In this paper, I briefly deal with educational policy, and to mathematical understanding from the point of view of mathematicians, then elaborate on understanding from the standpoint of the learner, and close with understanding from the standpoint of the teacher.

Educational policy towards mathematics includes the selection of content to be covered in school, who should encounter that content, and when. In the selection of content, a fundamental question concerns what constitutes mathematics appropriate for schools.

Here are some answers as reflected in the Common Core State Standards (CCSS) that have recently been adopted by 45 of the 50 states in the United

States. Is statistics mathematics? (Yes, statistics is in the standards from grade 5 on.) Is physics mathematics? (No. There is no physics.) Is formal logic a part of mathematics? (No, not a part of school mathematics according to these standards.) Is applied mathematics part of school mathematics? (Yes.) Should telling time be a part of the mathematics curriculum? (Yes.) What about reading tables of data (Yes.) or locating one's home town on a map of a country? (No.) What about doing a logic puzzle such as a Sudoku puzzle? (No.) Is computing using a calculator *doing* mathematics or *avoiding* it? (The CCSS are inconsistent on this.) These questions bring out differences among us in what we think mathematics is, and differences in what we think is *real* or *good* mathematics. An entire paper could be devoted to these questions but policy is not the focus of this paper.
The understanding of the invention or discovery of mathematics from the mathematician's perspective has been the subject of many books, of which the classics by Hadamard (1945), Hardy (1940) and Polya (1962) are probably the most well-known. The understanding of mathematical invention would not be complete without consideration also of the inventors, mathematicians themselves, through the many biographies that are available. The recent book by Hersh and John-Steiner (2010) falls into this broad category.

Let me move now to understanding from the standpoint of the learner. I begin with the observation I stated earlier, that mathematics consists of concepts and problems.
On problem solving, Polya's 'How To Solve It' (1957) has long been the most cited work in our field. He identifies four major steps in solving a problem: understanding the problem, devising a plan, carry out the plan, and looking back. Here is what Polya has to say about understanding the problem.

> What does the problem say – can it be stated in your own words? What is the unknown? What are the data (givens)? What is the condition (or conditions)? Is it possible to satisfy the condition(s)? Will a figure help? Is it necessary or helpful to introduce suitable notation?

These questions are nice for a pure mathematics problem but for a real-world problem, one wants to add more questions.

> Could the situation described in the problem actually occur? Do we need to simplify it to treat it with mathematics? What kind of answer is expected? Is the mathematical answer feasible in the real world?

The process of solving a real-world problem with mathematics is illustrated in figure 1.
Modeling starts with a real problem. The first step in the process is sim-

plifying the problem, usually by ignoring difficult aspects of the real situation. The next step is translating the simplified problem into mathematics (that is, finding a mathematical model). The third step is to work with the mathematics. Then, once a solution has been found to the mathematical problem, there is the step of translating the solution back to the original situation, and then checking whether the solution is feasible.

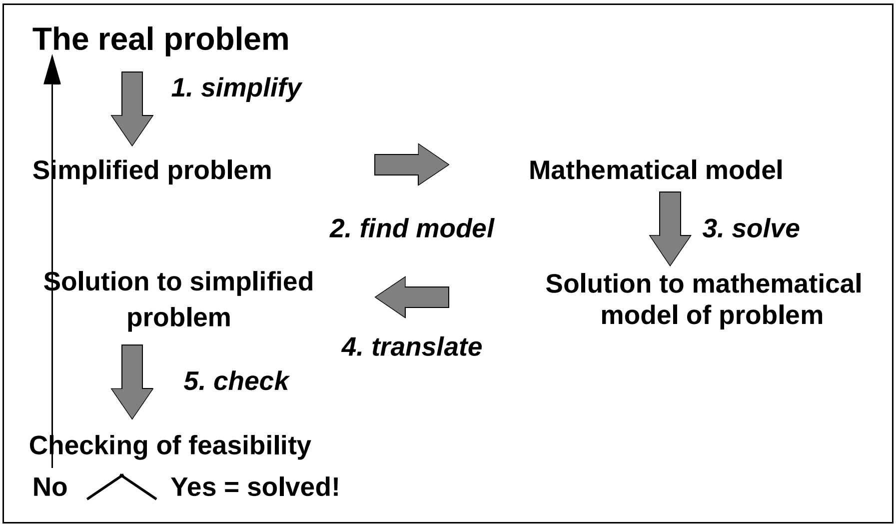

figure 1: the modeling process

The steps in the modeling process are quite similar to the steps in Polya's problem-solving scheme. We identify the problem, we look for a method of solution, and we carry out that method. Even the last step in the modeling process, the feasibility of the solution, is similar to deciding whether our solution is appropriate.

A point I would like to stress is that modeling begins in the primary school when we introduce numbers and the operations of addition and subtraction to answer questions such as How many? How much more? What is left? What is the total?

As an example I offer the following problem, a question that can be asked with a large number of contexts at a variety of grade levels. Consider three countries: the Netherlands, Belgium, and Germany. How can we compare the sizes of these countries?

We could compare by their areas, or by their populations, by their gross domestic product and by many other variables. Each of these is a measure of size.

Suppose we choose to compare by population. I looked in the World Almanac and found the following estimates for their populations in the middle of 2012:

Netherlands: 16,730,632
Belgium: 10,438,353.

Can you believe that these estimates are to the nearest person? Because people move into a country, move out, get born, and die, already we know that the data are not feasible. If we chose to compare by area, we have a choice of land area or total area, and we could feel more confident that the areas do not change much.

To mathematically compare the populations, we have a choice. In first grade, we can use an ordering model. Of two numbers, one is either larger, smaller, or equal to the other. In the early primary grades, we subtract. From the data, we might conclude that Netherlands has 6,292,279 more people than Belgium, but this is not a feasible answer since the population is always changing. So we might say that the Netherlands has about 6,300,000 more people than Belgium, or Belgium's population is about 6.3 million less than the Netherlands. Here are other statements of comparison.

N's population is 6,292,279 larger than B's.
N's population is about 6,300,000 larger than B's.
N's population is about 6.3 million larger than B's.
B's population is about 6.3 million less than N's.

These comparisons use additive models, that is, models based on addition or subtraction. But we can also use multiplicative models. Multiplicative models typically involve decimals, fractions, and percents, and lead us to reciprocals, so the mathematics is quite rich.

Netherland's population is 1.6028...times that of Belgium.
N's population is about 1.6 times that of B.
B's population is about 0.624 times that of N. (Rare!)
N's population is about 160% that of B.
N's population is about 60% more than B.
B's population is about 62% that of N.
N's population is about $\frac{8}{5}$ that of B.
B's population is about $\frac{5}{8}$ that of N.

I dwell on this simple idea of comparison because in the primary school we need to give attention to literacy, and the understanding of mathematics has to include the ability to deal with descriptions of the world around us.

Multiplicative models of comparison are particularly common when the numbers being compared are not close to one another, as with the populations of the Netherlands and Germany.

Netherlands, 16,730,632.
Germany: 81,305,856.

Here are some of the many descriptions.

1. N's population is 64,575,224 less than that of G.
2. N's population is about 65,000,000 less than that of G.
3. G's population is about 65,000,000 more than that of N.
4. N's population is about $\frac{1}{5}$ that of G.
5. G's population is about 5 times that of N.
6. N's population is about 20% that of G.
7. G's population is about 500% that of N.
8. G's population is about 400% more than that of N.

Ask students whether they prefer description (1) or description (2). I think many if not most will prefer description (1) because it seems to be more accurate and because students are often trained to give exact answers, thinking exact answers are always better than estimates. Yet description (1) is very likely to be false while description (2) is true. So we see that understanding mathematics in the real world can be quite subtle and quite different than understanding the same mathematics in an abstract problem in the classroom.

The topic of comparison is mathematically quite rich. When we wish to compare quantities that differ by huge amounts, as in comparing distances on earth with distances in the universe, we may want to use scientific notation. And to compare the intensity of sound or the brightness of stars, we use logarithmic scales measured in decibels or star magnitudes. The message is that understanding a real problem requires attention to mathematical literacy and to the feasibility of any solution that is offered.

Finally, let me indicate what does *not* constitute understanding. We say that someone *does not understand* a bit of mathematics when that person acts *blindly* or *incorrectly* to the prompts in the situation.

To exemplify the understanding of concepts, in the next two sections we consider two dissimilar concepts: *multiplication of fractions* and *rectangles*.

3 understanding the multiplication of fractions

skill-algorithm understanding of multiplication of fractions

If a random person on the street is asked: 'Do you understand the multiplication of fractions?', a typical response might be: 'Yes, you multiply the numerators and denominators to get the answer.' To the world outside school, understanding is often equated with getting the right answer.

Knowing how to get an answer is the essence of the procedural understanding of the multiplication of fractions or any other concept. Because algorithms are often done (and supposed to be done) *automatically*, we often contrast applying a procedure to understanding what is going on. However, there is much more to procedural understanding than merely applying an algorithm. With regard to the multiplication of fractions, the procedure seems very simple. If we are confronted with:

$$\frac{2}{3} \times \frac{4}{5} \qquad (1)$$

we merely multiply numerators and denominators to obtain the product $\frac{8}{15}$. However, the values of the numerators and denominators can alter what we do. Change the 4 in problem (*1*) to a 6:

$$\frac{2}{3} \times \frac{6}{5} \qquad (2)$$

and we may divide the 3 and 6 by 3 and thus get $\frac{2}{1} \times \frac{2}{5}$ and now multiply numerators and denominators to obtain the product $\frac{4}{5}$. Or we may divide the 3 into the 6 and write 1 and 2. Some people cross out the 3 and 6 in the process. These variants of the algorithm used in the first case are different enough to require days of instruction in a typical classroom.
Change the 4 in (*1*) to a 3, and we think about it even another way:

$$\frac{2}{3} \times \frac{3}{5} \qquad (3)$$

We ignore the 3s (some people cross them out) and just write down $\frac{2}{5}$.
Change the $\frac{4}{5}$ in (1) to 4 and there is another algorithm:

$$\frac{2}{3} \times 4 \qquad (4)$$

I multiply the 2 by 4 and write down $\frac{8}{3}$. Some students feel the necessity to replace the 4 with $\frac{4}{1}$ and then they treat the problem as if it were of the first type and multiply numerators and denominators to obtain $\frac{8}{3}$.
Change the $\frac{4}{5}$ to 60 and there is still another algorithm:

$$\frac{2}{3} \times 60 \qquad (5)$$

Now we may divide 3 into 60 and then multiply the quotient 20 by 2. Or, since the numbers are so simple, we might multiply 2 by 60 and then divide by 3.
Change the $\frac{4}{5}$ to $\frac{3}{2}$ and there is still another algorithm:

$$\frac{2}{3} \times \frac{3}{2} \qquad (6)$$

We recognize that these numbers are reciprocals and immediately write down the product 1.
Change the $\frac{4}{5}$ to $1\frac{4}{5}$ and there is still another algorithm:

$$\frac{2}{3} \times 1\frac{4}{5} \qquad (7)$$

Some children think this calculation cannot be done. One way is to rewrite $1\frac{4}{5}$ as $\frac{9}{5}$ and then apply the algorithm we used in (*1*).
If there are more than two fractions to be multiplied, combinations of these strategies are applied.
Multiplication of fractions is perhaps the simplest algorithm in all of primary school arithmetic. And yet the skillful arithmetician has at least seven different ways of multiplying two fractions, depending on the numbers involved in the situation.

Skill is sometimes thought of as a lower order form of thinking. I disagree. Those who are skillful make all sorts of decisions while performing the skills. They possess *skill-algorithm understanding*. You and I exhibit this type of understanding of the multiplication of fractions when we do it and get the right answer. We exhibit a higher form of this same type of understanding when we know many ways of getting the right answer (that is, we know different algorithms), choose a particular algorithm because it is more efficient than others, and can check our answers using a different method than we employed to get the answer. Many people possess this type of understanding because we spend so much time working on the skill.

property-proof understanding of multiplication of fractions

For many people, understanding has a completely different meaning than obtaining the correct answer in an efficient manner. You don't *really* understand something unless you can identify the mathematical properties that underlie *why* your way of obtaining the answer work.

For example, for the multiplication problem $\frac{2}{3} \times \frac{4}{5}$, we wish to justify the rule: For any numbers *a, b, c*, and *d* with $b \neq 0$ and $d \neq 0$:

$$\frac{a}{b} \bullet \frac{c}{d} = \frac{ac}{bd}$$

by showing how it follows logically from a set of simpler properties.

$\frac{a}{b} \bullet \frac{c}{d} = (a \bullet \frac{1}{b}) \bullet (c. \frac{1}{d})$	(definition of division)
$= a \bullet (\frac{1}{b} \bullet c) \bullet \frac{1}{d}$	(associative property of multiplication)
$= a \bullet (c \bullet \frac{1}{b}) \bullet \frac{1}{d}$	(commutative property of multiplication)
$= (a \bullet c) \bullet (\frac{1}{b} \bullet \frac{1}{d})$	(associative property of multiplication)
$= (ac) \bullet (\frac{1}{bd})$	(uniqueness of multiplicative inverse; each is the inverse of *bd*)
$= \frac{ac}{bd}$	(definitation of division)

This aspect of understanding, *property-proof understanding*, is obviously quite different from skill-algorithm understanding. From this mathematical derivation, a student learns that the rule for multiplication of fractions is not arbitrary but a property that can be deduced from more basic properties of multiplication and division. The derivation also shows the importance of thinking of the fraction $\frac{a}{b}$ as being equal to $a \bullet \frac{1}{b}$, the relevance of reciprocals, and so on.

Technically, the proof that $\frac{a}{b} \bullet \frac{c}{d}$ only accounts for the algorithm in the first of the seven cases described earlier, $\frac{2}{3} \times \frac{4}{5}$, a multiplication involving two fractions with no common factors in numerators and denominators. To justify the case in which the denominator of one fraction is a factor of the numerator of the other fraction, as in $\frac{2}{3} \times \frac{6}{5}$, we must show that $\frac{a}{b} \bullet \frac{kb}{d} = \frac{ak}{d}$. A full justification of the ways in which we multiply fractions requires proofs also for the other five cases mentioned earlier. Thus the complexity of the algorithms is matched by a complexity of the mathematical underpinnings.

Some people believe that if you understand these sorts of mathematical derivations and use language correctly, then you will be more skillful. But in the 'new math', that era from about 1960 to 1974, it was found that the transfer from understanding properties to understanding skill was not automatic. Skill requires practice, and also requires flexibility to choose among various possible algorithms, a quality that the mathematical derivations do not convey. When people contrast *procedural understanding* with *conceptual understanding*, they are often contrasting skill-algorithm understanding with property-proof understanding. However, there is more to understanding than these two facets.

use-application understanding of multiplication of fractions

A person may know *how* to do something and may know *why* his or her method works, but - particularly to people who use mathematics in their daily lives and on the job - a person does not fully understand the multiplication of fractions unless he or she understands *when* to multiply fractions. I call this kind of understanding, which is a hallmark of the work of the Freudenthal Institute, *use-application understanding*. The understanding of applications does not come automatically. There are students who can multiply fractions but who cannot use them. Uses must be taught in order for most students to learn them. Here are some examples of situations that might lead to that multiplication. That these five situations represent five different *types* of applications, not merely five different application contexts, can be seen by examining the units of measure (or lack of units) involved:

1 A rectangular region on a farm is $\frac{2}{3}$ km by $\frac{4}{5}$ km. What is its area? (A measure is multiplied by a measure.)
2 If an animal travels at a rate of 2 km in 3 hours (i.e., at $\frac{2}{3}$ km per hour), how many kilometers will it travel in 48 minutes (i.e., $\frac{4}{5}$ hour)? (A rate, a measure with a derived unit, is multiplied by a measure.)
3 If two independent events have probabilities $\frac{2}{3}$ and $\frac{4}{5}$, what is the probability both will occur? (Two scalars, numbers with no units, are multiplied.)
4 If a segment on a sheet of paper is $\frac{4}{5}$ unit long and is put in a copy machine at $\frac{2}{3}$ its original length, what will be its final length? (A scalar is multiplied by a measure.)
5 If something is on sale at $\frac{1}{3}$ off (i.e., at $\frac{2}{3}$ its original price) and you get a 20% discount (to $\frac{4}{5}$ of the sale price) for opening a charge account, your cost is what part of the original price? (A scalar is multiplied by an unknown measure; then the product is multiplied by a scalar.)

In the United States, we spend large amounts of time teaching arithmetic paper-and-pencil skills, including weeks on multiplication and division of fractions alone. We spend relatively little time teaching students how to apply these operations with fractions. As a result, performance on application is lower than performance on skill and many people believe that application is more difficult than skill. I believe that some application is harder than some skill, but some skills are harder than some applications. Applications involve a different kind of understanding, not necessary one that is at a higher level.

representation-metaphor understanding of multiplication of fractions

Even these three types of understanding do not encompass the entire scope of what it means to understand a mathematical concept. To cognitive psychologists with whom I have discussed this topic, the three types of understanding discussed so far do not convey the real true understanding of mathematics. From psychology we obtain the notion that a person does not really understand a concept unless he or she can represent the concept in some way. For some, that way must be with concrete objects; for others, a pictoral representation or metaphor will do. A well-known way to represent $\frac{2}{3} \times \frac{4}{5}$ stems from the basic application of multiplication to area of rectangles. In the area representation $\frac{2}{3}$ and $\frac{4}{5}$ are side lengths of a rectangle and the product is the area. We represent $\frac{2}{3}$ by splitting a unit square horizontally into three parts and shading the top two parts. We represent $\frac{4}{5}$ by splitting the same square vertically into five parts and shading the four parts on the left with a different shading than used for $\frac{2}{3}$. This splits the square into fifteen rectangles, eight of which have both shadings, a picture of $\frac{8}{15}$ (fig.2).

A discrete version of the area representation is with an array of dots. With an array, $\frac{2}{3}$ is represented by putting two of three dots above a horizontal line; $\frac{4}{5}$ is represented by putting four of five dots to the left of a vertical line; and the product consists of the eight of fifteen dots that are both above the horizontal and to the left of the vertical representation.

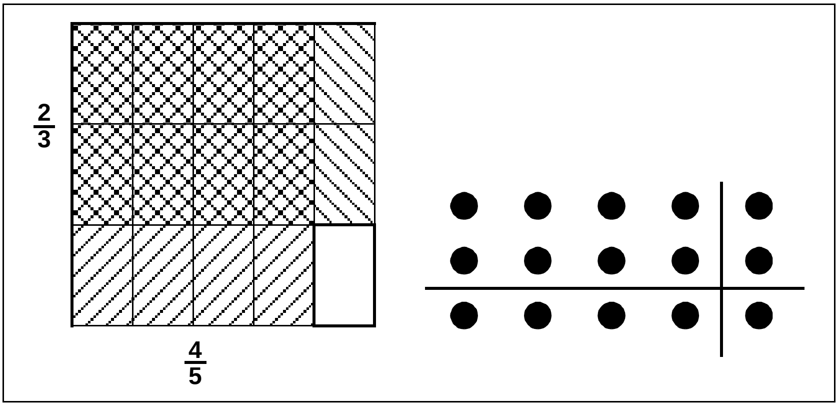

figure 2: two representations of $\frac{2}{3} \times \frac{4}{5}$

A third representation is quite different and views the $\frac{2}{3}$ and $\frac{4}{5}$ not as equal partners but the $\frac{2}{3}$ as operating on the $\frac{4}{5}$. In this representation, we begin with any geometric figure (here, the larger trapezoid) on which there is a segment of length $\frac{4}{5}$. Then we draw segments from some point (here, point C) to the vertices of the trapezoid. Then, on each segment from C, we pick points $\frac{2}{3}$ of the way to the vertices of the trapezoid. Connecting those segments results in an image trapezoid whose sides have $\frac{2}{3}$ the length of the corresponding sides of the original trapezoid (fig.3).
Although this representation seems like a lot of work for such a simple arithmetic operation, it has wide applicability.

A person can have a rather deep knowledge of multiplication of fractions even if the person has never seen these representations. Thus concrete or pictoral representations do not have to precede acquisition of the other types of understandings, even though some people think about representations in the same way that others think about knowing properties; if students are brought carefully to understand (in the representational sense) what they are doing, then they will ultimately be better at skill.

It is not difficult to apply these four dimensions of understanding to other arithmetic topics and to the understanding of topics from algebra and analysis. For instance, with linear equations in algebra, it is easy to distinguish skills, properties, and uses, and the common representation is with graphs. So, for the second example, I have picked a concept that is quite different, rectangles in geometry.

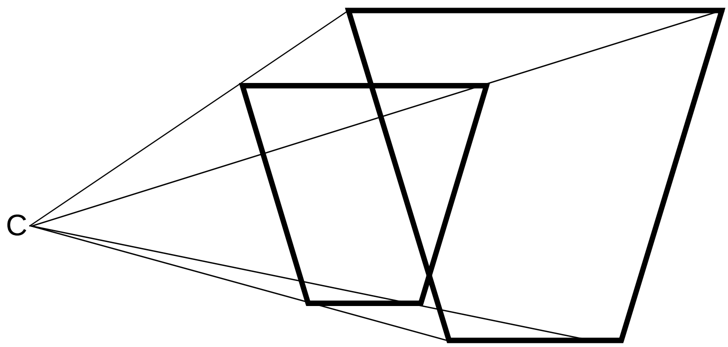

figure 3: a third representation of $\frac{2}{3} \times \frac{4}{5}$

4 understanding rectangles

skill-algorithm understanding of rectangles

The skills of geometry are drawing and visualization or recognition. A rectangle may be drawn with a ruler or constructed with ruler and compass, or shown on a computer screen with the help of drawing software.

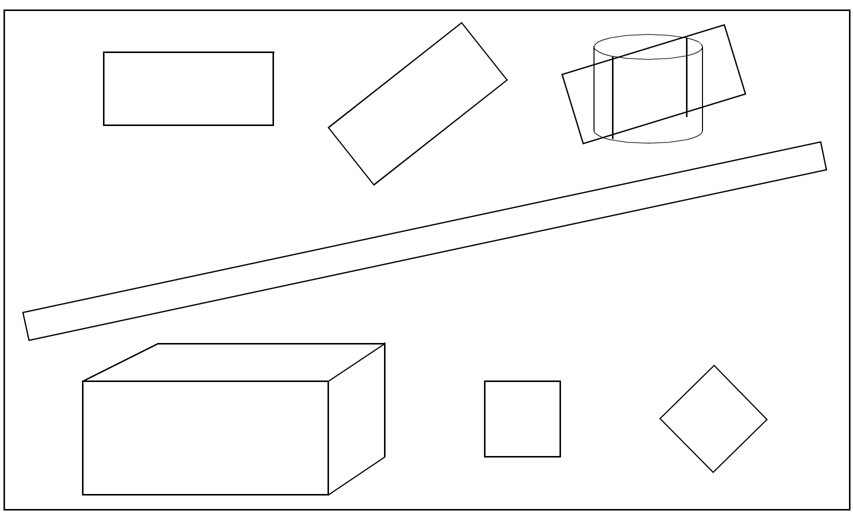

figure 4: rectangles

Recognition is also not obvious. Very young students have difficulty seeing that every square is a rectangle, or that a thin or tilted rectangle is still a rectangle. In the Van Hiele theory of development (Van Hiele, 1986), recognition is at a low level but it is also related to definition which in that theory is at a higher level. Visualization is not always obvious either. Young students may not realize that all six faces of a box are rectangles. And, if a cylinder is sliced by a plane that goes through its axis, it is not obvious that the intersection is a rectangle. These are some of the skills associated with rectangles (fig.4).

property-proof understanding of rectangles

The property-proof dimension of understanding is the aspect of rectangles that is often given most priority in secondary schools. It includes working from the definition to show that adjacent sides are perpendicular and opposite sides are parallel, and that the diagonals are of equal length. There are properties inherited from parallelograms, such as that the diagonals bisect each other, and that each diagonal splits the rectangle into two congruent triangles. Then there are the properties that rectangles inherit from trapezoids, and the properties that are true of all quadrilaterals.

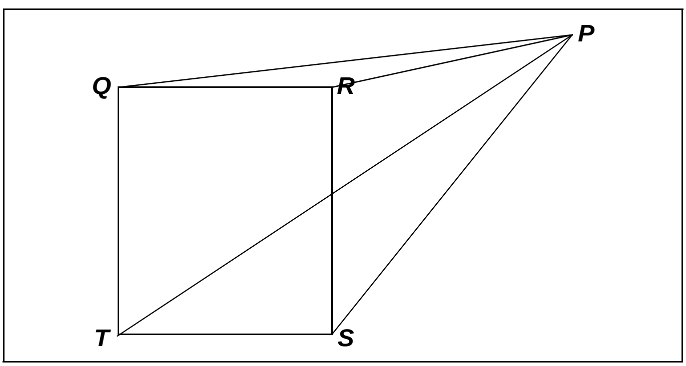

figure 5: a property of rectangles not obvious by sight.
For any rectangle $QRST$ and any point P (in or not in the plane of $QRST$), $PQ^2 + PS^2 = PR^2 + PT^2$

Advanced aspects of this understanding involve such things as the fact that there are no rectangles on the surface of earth and some interesting theorems such as the fact that if $QRST$ is a rectangle and P is any point in the plane, then $PQ^2 + PS^2 = PR^2 + PT^2$ This theorem holds even if P is a point not in the plane of $QRST$, so the optical illusion in the drawing, in which P may or may not be in the plane $QRST$ makes no difference.

use-application understanding of rectangles

Rectangles are everywhere in our world, so finding examples of their use is easy. The question that this raises is why there are so many instances of rectangles. Why are the tops of most beds shaped like rectangles? Why are pages in books almost always rectangles? Why is a football field a rectangle? Why are some tables rectangles and why are others not? The answers are important for inventors, designers, and manufacturers because they need to know the range of shapes that are possible for a product.

Beds tend to be reflection-symmetric because our bodies are reflection-symmetric. A mattress needs to be turned around every so often so that it does not sag, so it has to have point symmetry. If a quadrilateral is reflection-symmetric and it has point symmetry, then it is a rectangle or a rhombus. But a rhombus-shaped mattres would have a great deal of unused surface.

There is a different reason why pages in books are rectangles. They are cut from large pieces of paper. So that paper is not wasted, the pages need to tessellate. Since the large piece of paper is passed through a printing press, it is most efficient if the width of the piece is constant. These constraints define a rectangle.

In newer towns and cities, we see rectangular grids because if we want streets to intersect with equal angles of sight in both directions, then they must intersect at right angles. Then, if the streets are straight, the regions they bound will be rectangular regions.

The reader may wish to think about why football fields are rectangles and why most tables in restaurants are either rectangular or circular.

representation-metaphor understanding of rectangles

We represent arithmetic and algebraic ideas geometrically. The representation dimension of understanding of geometry goes the other way when we represent geometric figures with arithmetic or with algebraic equations.

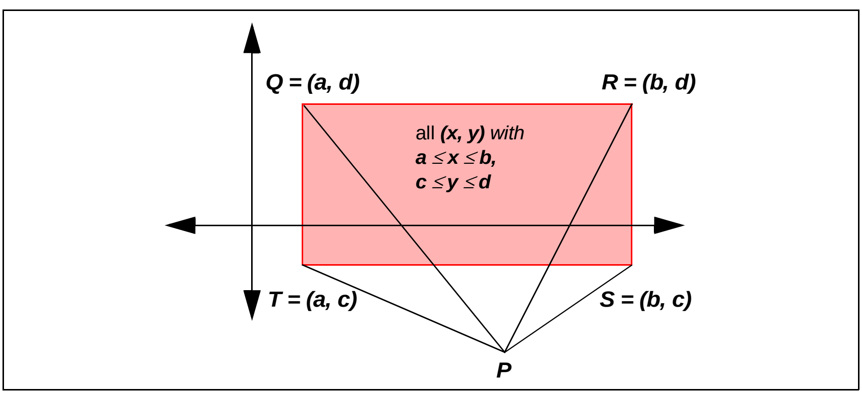

figure 6: rectangles on a coordinate grid

We can place two-dimensional geometric figures on a coordinate plane. For younger students, we may use a representation on a geoboard.
Rectangles are basic to coordinate grids. We can describe a rectangle numerically by four ordered pairs of numbers: (a, c), (b, c), (b, d), (a, d), as shown in figure 6. We can describe the rectangular region determined by this rectangle by two inequalities: $a \le x \le b$, $c \le y \le d$.
From this representation, using the distance formula between points on a coordinate plane, we can deduce rather easily the theorem mentioned earlier, that $PQ^2 + PS^2 = PR^2 + PT^2$.

5 a taxonomy of mathematical understanding

The four dimensions of understanding for multiplication of fractions and for rectangles have certain common qualities. Each dimension of understanding has supporters for whom that dimension is preeminent, and who believe that the other dimensions do not convey the real essence of the understanding of mathematics. Each dimension has aspects that can be memorized. Skills, names of properties, connections between mathematics and the real world, and even work with representations can be memorized.

Dimension	Description
Skill-algorithm	from the rote application of an algorithm through the selection and comparison of algorithms to the invention of new algorithms (calculators and computers included)
Property-Proof	from the rote justification of a property through the derivation of properties to the proofs of new properties
Use-Application	from the rote application of mathematics in the real world through the use of mathematical models to the invention of new models
Representation-Metaphor	from the rote representations of mathematical ideas through the analysis of such representations to the invention of new representations
History-Culture	from rote facts through the analysis and comparison of mathematics in cultures to the discovery of new connections or historical themes

figure 7: dimensions of mathematical understanding

They also have potential for highest level of creative thinking: the invention of algorithms, the proofs that things work, the discovery of new applications for old mathematics, the development of new representations or metaphors.

The four dimensions of understanding are relatively independent in the sense that they can be, and are often, learned in isolation from each other, and no particular dimension need precede any of the others. We like to begin with real-world situations whenever that is possible, but sometimes it is better to begin with another dimension of understanding.
It is because of the relative independence of skill-algorithm understanding, property-proof understanding, use-application understanding, and representation-metaphor understanding from each other, that I believe that the understanding of mathematics is a multi–dimensional entity, in the sense that there are relatively independent components that constitute what might be called ‘real true’, ‘complete’, or ‘full’ understanding. Figure 7 suggests the range of activities in each dimension.

history-culture understanding

As figure 7 shows, I believe there is at least one more dimension to a full understanding of a piece of mathematics, a dimension that is not usually found in school mathematics. It is the *history-culture dimension*. How and why did a certain bit of mathematics arise? How has it developed over time? How is it treated in different cultures? The understanding of the work and lives of mathematicians that I briefly mentioned earlier in this presentation falls into this dimension. Those who study the history of mathematics or cross-cultural mathematics obtain an understanding of mathematical concepts that is different from any of the understandings we have discussed so far. It is a fifth dimension.
Some important aspects of mathematics are primarily located in this dimension. Among these are: Mathematics is invented, or discovered. Mathematics has grown over time and so has the number and variety of its applications. Both formal and informal mathematics (ethnomathematics) are not necessarily the same in all cultures. The truths of mathematics are relative truths, deduced from definitions and a small number of postulates. There exists ‘recreational mathematics’, that is, mathematics done for fun.

I do not know much about the history of the *multiplication* of fractions, but most of us do have some history-culture understanding of fractions themselves. The first fractions were those for halves, thirds, and fourths. Over 3600 years ago, the Egyptians represented other fractions as sums of unit fractions. The first use of the bar to represent fractions seems to be among Arab mathematicians well over 1000 years ago but their common use did not appear until the 16^{th} century (Flegg 2002, pp. 74-75; Cajori 1928, p. 310). A sign very much like the slash for fractions first appeared in Mexico in the late 1700s (Cajori 1928, p. 313). Even today the symbols are not the same everywhere. In some places, the fraction a/b is represented by a : b, while in other places the symbol a : b represents a ratio that is mathematically not identical to a fraction. Simon Stevin, in his invention of decimals in the late 1500s called them *decimal fractions*, and some places still use that term. For mathematics education, the cultural history of fractions represents a dimension of understanding of the concept that is considered particularly important to those who believe in a genetic approach to learning, that is, a progression of learning activities that parallels the historical development of the subject. So, as with the other dimensions of understanding, there are those who believe that the history-culture understanding underlies the best learning sequence for a mathematical concept.

understanding mathematics from the teacher's perspective

As a last look at understanding, let us turn to the understanding of mathematics from the teacher's perspective. The teacher of mathematics is an applied mathematician whose field of application involves the classroom and the student. Like other applied mathematicians, in order to apply the mathematics, the teacher needs to have a good deal of knowledge about the field to which the mathematics is being applied – that is, about the educational process – as well as about mathematics. Thus, the understandings that a teacher needs involve more than mathematical understandings. The teacher also must take into account other student needs, classrooms, teaching materials, and the necessities of explaining, motivating, and reacting to students. Here are four realms of understanding mathematics from the teacher's perspective.

- Pedagogical Content Knowledge
- Concept Analysis
- Problem Analysis
- Connections to Other Mathematics.

I call these realms and not dimensions because operationally they are clearly very much interrelated. For the first realm, there is a substantial literature. The phrase that identifies it, *pedagogical content knowledge*, was introduced by Shulman (1986).

pedagogical content knowledge

- Designing and preparing for a lesson
- Analyzing student errors
- Explaining and representing ideas new to students
- Responding to questions that learners have about what they are learning.

The second realm deals with applying the understanding of mathematical concepts.

concept analysis

- Engaging students in proof and proving
- Choosing and comparing different representations for a specific mathematical procedure or concept
- Choosing and using mathematical definitions
- Explaining why concepts arose and how they have changed over time
- Dealing with the wide range of applications of the mathematical ideas being taught.

The third realm deals with the understanding of problems and problem-solving.

problem analysis

- Examining different student solution methods
- Engaging students in problem solving
- Discussing alternate ways of approaching problems with and without calculator and computer technology
- Offering extensions and generalizations of problems.

The fourth realm integrates the other three.

connections and generalizations to other mathematics

- Comparing different textbook treatments of a mathematical procedure or topic
- Extending and generalizing properties and mathematical arguments
- Explaining how ideas studied in school relate to ideas students may encounter or have encountered in other mathematics study
- Realizing the implications for student learning of spending too little or too much time on a given topic.

It is clear that teachers need understandings that go far beyond those of students, and that we could spend much more time discussing these understandings.

note

1 The multi-dimensional characterization of understanding described in this paper may be found in the textbooks for grades 6-12 of the University of Chicago School Mathematics Project. http://ucsmp.uchicago.edu/secondary/overview

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references

Cajori, F. (1928). *A history of mathematical notations. Vol I: Notations in elementary mathematics*. LaSalle, IL: Open Court.

Common Core State Standards (2010). *Mathematics standards*. http://www.corestandards.org/the-standards/mathematics

Flegg, G. (2002). *Numbers: their history and meaning*. New York: Dover Publications.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: D. Reidel Publishing Company.

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. Princeton, NJ: Princeton University Press.

Hardy, G.H. (1940). *A mathematician's apology*. London: Cambridge University Press.

Hersh, R. & V. John-Steiner (2010). *Loving and hating mathematics: Challenging the myths of mathematical life*. Princeton, NJ: Princeton University Press.

Hiebert, J. & T.P. Carpenter (1992). Learning and teaching with understanding. In: D. Grouws (Ed.). *Handbook of Research on Mathematics Learning and Teaching*, New York: Macmillan Publishing Company (Chapter 4, 65-97).

Hiele, P.M. van (1986). *Structure and insight: A theory of mathematics education*. Orlando, Fl.: Academic Press.

Polya, G. (1957). *How to solve it: A new aspect of mathematical method. Second edition*. Princeton, NJ: Princeton University Press.

Polya, G. (1962). *Mathematical discovery: On understanding, learning, and teaching problem solving*. Volumes I and II. New York: John Wiley & Sons.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 16*(2), 4-14.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*(1), 20-26. Reprinted (1978) in *Arithmetic Teacher, 26*(3), 9-15.