What is Arithmetic?

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1 introduction

What arithmetic is or is about, depends on who asks and why they are asking!

What do students think arithmetic is?

This is an empirical question to be answered by asking them, though they may try to guess what you expect or want them to say, rather than revealing their idiosyncratic concept images (Tall & Vinner, 1981).

What do teachers think arithmetic is?

This is also empirical, but teachers are more likely to respond with the answer you think they are looking for, and not necessarily with their concept images or what impressions their teaching makes available to students.

What do researchers think arithmetic is?

Many psychologists treat arithmetic as a relatively unproblematic domain of right and wrong answers, factual knowledge retained or not, in which to test theories.

What do politicians think arithmetic is?

Again this is an empirical question, which can be addressed by looking at their public statements. They are not always as well informed as one would hope, considering the power that they wield.

What do mathematicians think arithmetic is?

You might get multiple responses under different conditions.
A plausible conjecture is that answers to these questions would differ widely, both within, and especially between groups. The lecture at the 30th Panama-conference forming the basis of this article was about how one mathematician-mathematics educator sees arithmetic.

2 method of enquiry

My method of enquiry is to examine closely my own experience in order to sensitise myself to what learners may be experiencing. The 'data' being offered here are two-fold: what the reader notices when engaging in the tasks presented, and what comes-to-mind, resonated from the past with triggered associations. What actually matters is not what you have known, what has come-to-mind in the past, but what comes-to-mind in the future, in new situations as well as in familiar one. Thus effectiveness resides in the freshness and enrichment of awareness when engaging with learners in the future, rather than statistical evidence about learners in the past. This approach follows the 'Discipline of Noticing' (Mason, 2002).

3 claims

My aim in the lecture was to try to justify experientially two claims:

> that in order to be sensitive to students' mathematical development it is essential to engage in mathematical thinking yourself, in some manner that parallels the sorts of challenges that students are meeting;
>
> that arithmetic is the study of actions, usually on numbers, with numbers. It is about the structural relations amongst numbers, seen as instantiated properties, and on the basis of which one can reason about numbers. Calculations are a by-product, and not part of *arithmetic* in its fullest form.

4 a selection of tasks

What follows is a selection of tasks, most of which were in the talk, but without the luxury of time to get audience comment and reflection. I see tasks given to learners as intended to generate activity (Christiansen & Walther, 1986). Activity provides experience (of the use of mathematical powers, of encounters with mathematical themes, of getting stuck and getting unstuck, of meeting challenges, exercising skills, etc.). Rarely is experience sufficient. Indeed one thing we don't seem to learn from experience is that we don't often learn from experience alone. Some sort of reflexive stance is also required, withdrawing from the action constituting the activity, and considering what was effective and what could be improved in the future, and what connections with past experience emerged, whether at the time or in retrospect. Teaching takes place (as acts) *in* time, while learning, takes place *over* time (during sleep!) as a maturation process, like bread, beer and wine making. With each task, pause before reading on and consider what you noticed happening inside you, how you used yourself, and perhaps how you might want to improve 'next time'.

pre-counting and counting

These initial tasks were intended for the audience, not for children, and aimed to provide some insight into what children might be experiencing when they struggle to count or to skip count.
Imagine an animation in which various objects are moving around the screen. Counting the objects can be quite difficult because they keep moving. It is possible that for some children, although the objects are stationary, their attention is darting about somewhat analogously to the animation. One-to-one correspondence, pointing and arranging the objects to be counted are ways to steady attention and make counting possible.

task 1: counting

How many rectangles make up figure 1?

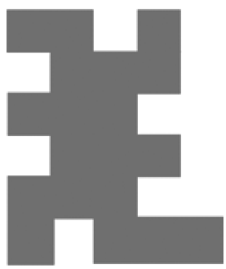

figure 1

Of course it is not possible to count until you have discerned the things you are to count ... and in this case there are different answers according to how the shape is decomposed (all vertical slices, all horizontal slices, or a mixture of both). It is well known that secondary students struggle with such tasks. One way to lay the groundwork is to get them to construct shapes from a specified number of rectangles. Once they become familiar with construction, they can be challenged, or challenge themselves to find shapes that can be decomposed in different ways with different numbers of rectangles. One thing to emerge will be the notion of 'efficient' decompositions, expressed in some way or other, capturing the idea that none of the decomposition rectangles form a larger rectangle within the figure.
This illustrates an important pedagogical strategy: if students are

expected to 'work on' or transform expressions (for example, a sequence of arithmetic calculations, equations, or an algebraic expression; factoring numbers or algebraic expressions, etc.), then providing experience of constructing the objects first, offers learners a sense of where these objects may have come from as well as insight into structural relationships. Another important pedagogical strategy is to ponder different approaches in order to seek a maximally efficient approach that works in as many varied situations as possible. Finding the areas of rectilinear figures is a case in point.

The next task really requires a group of people all chanting together.

task 2: skip counting

Try counting forwards in blocks of 4, thus, 1, 2, 3, 4 then 2, 3, 4, 5 then 3, 4, 5, 6 etc.

Now do it to a rhythm of threes (strong, weak, weak; strong, weak, weak; ... perhaps by someone clapping the rhythm).

Starting at 101, count down in steps of $1\frac{1}{10}$.

If done in a group, you may find that you can lose the thread but pick it up again from the on-going chant. Children almost certainly move 'in and out' of lessons mentally in a similar manner. The important thing is to trap what you do mentally in order to do this counting, and what obstructs your facility ... there may be analogues for children who struggle with skip counting. For the third one, counting down in steps of $1\frac{1}{10}$, everyone prepares the first number, and can 'see' or 'feel' or 'sense' how the next ones will go. The rhythm develops nicely until you get to $91\frac{1}{10}$ and you realise that there is going to be trouble ahead! Constructing and experiencing your own skip counting task deepens appreciation of what makes such a task valuable or trivial.

tasks exploring arithmetic as the study of actions

The remaining tasks can be and have been used with children of various ages. My aim is to draw your attention to tasks that expose arithmetical structure.

task 3: what's the difference?

You are about to subtract one number from another, but before you do, someone adds one to both of them. How has the difference changed (what is the difference in the difference)?

Instead, add one to the first number and subtract one from the second number ... now what is the difference in the differences? Extend and generalise.

Of course there is no change in the first case, something that quite young children appreciate intuitively. It is the basis for subtracting through equal addition. The point is that you don't need to know the actual numbers in order to check what happens.

Drawing on 'variation theory' (Marton & Booth, 1997), an important question to ask is: 'What can be changed so that still the task is done in much the same way?' For example, the two numbers themselves can be any numbers at all; the one and the two can be any number, and the add and subtract can be either add or subtract. What happens if instead the first number is doubled and so is the second? The first number doubled but the second left alone ...?
These variations illustrate the point that in order to learn *arithmetic* in its full and proper sense, it is actually necessary to think algebraically (Hewitt, 1998); that thinking arithmetically actually involves generality, which can be expressed in algebraic symbols (but does not need to be). If the original 'subtraction' is changed to multiplication, then analogues to the original task emerge, suggesting analogues between adding and multiplying.

task 4: think of a number (THOANs)

I am thinking of a number; I add 8 and the answer is 13; what is my number?

I am thinking of a number; I add 8, then multiply by 2 and the answer is 26; what is my number?

I am thinking of a number; I add 8, then multiply by 2 then subtract 5 and the answer is 21; what is my number?

I am thinking of a number; I add 8, then multiply by 2 then subtract 5, then divide by 3, and the answer is 7; what is my number?

What do you do to the sequence + 8, x 2, – 5, :3, 7 in order to recover my number?

What role does 7 play? What role do the actions play?

This is a reversal of the usual 'Think Of A Number' puzzles, popular since Fibonacci introduced them from unknown sources in the 12^{th} century. You can ask people to think of a number, then lead them through some computations and end either by eliminating their original number and telling them the answer, or, given their answer, you can tell them what they started with. Young children are amazed, and want to know how it is done. The answer of course is by algebra, which can be accessed by young children initially through 'tracking arithmetic', a didactic tactic which will be illustrated in task 7.

The next task offers an introduction to solving equations with one instance of an as-yet-unknown number to be 'de-coded' through appreciation of the doing and undoing relationships between adding and subtracting and between multiplying and dividing. Use is being made of the 'construct before de-constructing' pedagogic strategy mentioned earlier. This task exploits the same principles as task 4.

task 5a: doing & undoing

What operation (or action) undoes 'adding 5'?

What action undoes 'subtracting 3'?

What action undoes 'adding 5 then subtracting 3'?

There are two answers to the last part: 'adding 3 and then subtracting 5', or 'subtracting 2'. This dual approach supports the notion of flexibility ... there are different ways to carry out some sequences of actions, and some ways are more efficient that others.

Now, as an extra, what action undoes 'subtracting from 7'?

This is a bit of a mind-stopper at first. Suppose someone has a number, and they tell you the result of subtracting it from 7. How are you to reconstruct their number?
Try an example: if they (secretly) chose 5, then they would announce '2' as the result of the action. Your job is to reconstruct their original number knowing only the action ('subtract from 7') and the result (2).
Now, what are the analogues for multiplication and division?

task 5b: doing & undoing

What action undoes 'multiplying by 3'?

What action undoes 'dividing by 4'?

What action undoes 'multiplying by $\frac{3}{4}$'?

Again there are two answers to the last: 'dividing by $\frac{3}{4}$', and 'multiplying by 4 and then dividing by 3'. The playground 'rule of thumb' to 'invert and multiply' arises from exploiting awareness of undoing arithmetical actions. Corresponding to the surprising 'undoing subtracting from 7', there is the slightly ambiguous action of 'dividing into 12'. Here the intended meaning is not to divide something into 12 equal parts, but rather to divide 12 by the specified number.

What action undoes 'dividing into 12'?

As with subtracting from 7, suppose someone (secretly) chooses 6. They

then announce ‘2’ as the result of the action of dividing their number into 12 (that is, 12 : 6). Your job is to reconstruct their starting number knowing only the action (‘dividing into 12’) and the result (2).
Arithmetical operations as actions are usefully presented on a number line.

task 6a: actions on the number line

Imagine a number line with the integers marked on it (fig.2).

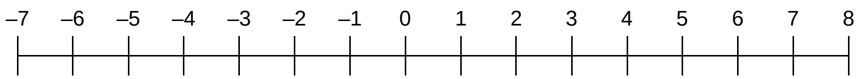

figure 2

Imagine an transparent copy of the number line, sitting exactly on top of it.

There are several actions we can perform. We can for example, shift or translate the number line a specified number of steps to the right or left. If you shift the (copy of) the number line 3 places to the right, then shift the result 4 places to the left, what is the overall effect? Denote the shift by the amount a by T_a. What then is the effect on the point x? What happens if you perform first T_a and then T_b?

We can scale the number line. If you scale the number line by a factor of 3, where does 2 end up? I am thinking of a number; where does it end up? Now scale that by a further factor of $\frac{2}{3}$; where does 2 end up now? I am thinking of a number; where does it end up under the composite action?\

We can also rotate the (copy of) the number line through 180° about the 0. Where does 3 end up? Where does -2 end up? I am thinking of a number; where does it end up?

Notice that translations can be composed without ambiguity, and scalings can as well, as long as it is understood that they are all carried out relative to the fixed original origin, not with respect to where the origin might be now!
There are several descriptions of where some as-yet-unknown number ends up under the rotation, using words like ‘same distance the other side’, or ‘minus your number’ or ‘change the sign of your number’, and it is learners’ flexibility in discourse that is of interest, so as to be confident that the language actually has meaning.
In each case, asking what can vary opens up possibilities for generalisation, underlining the role of algebraic thinking in the learning of arithmetic. In particular the next task is perhaps the most intriguing of this type, and challenges most teachers who have not previously thought about it.

task 6b: rotations of the number line

Again imagine a number line with an exact copy on top of it. Let R_a denote the action 'rotate the number line through 180° about the point a. Where does 5 end up if a is 3? Where does -2 end up? I am thinking of a number: where does it end up under this action? Generalise to $R_a(b)$.
Start again, and this time the action is 'first rotate the number line through 180° about the original point 3; then rotate it through 180° about the original point 4'. Where does 5 end up now? What about -2? What about x?

Exploring compound actions on the number line reinforces the sense of arithmetic as the study of actions with and on numbers, as well as laying the foundations for geometric transformations (and indeed linear algebra!). Composing rotations through 180° about different points is easiest to think about if they are always specified with respect to the original point, rather than the place that that point currently occupies. Exploring a different set of actions, where U_a denotes the action 'rotate the number line through 180° about the point where a currently is' leads to a different 'algebra'. But algebraic notation is not necessary for experiencing the transformations, using transparent copies on an actual number line.
Another task that illustrates a general approach to arithmetic, what I call a 'didactic tactic' is called 'tracking arithmetic' (Mason, Johnston-Wilder & Graham, 2005).

task 7a: doing differing products

Write down four numbers in the cells of a two by two array. An example is shown below (fig.3).

Now for each row, multiply the numbers in that row. Then add the results. Now for each column, multiply the numbers in each column, and then add the results.

Now subtract the second from the first. That is the 'measure' of your starting array.

Sum of row products: 28 + 15 = 43
Sum of columns products: 20 + 21 = 41
Difference in sums: 43 – 41 = 2

4	**7**
5	**3**

figure 3

The measure of my array is 2, but there is nothing particularly interesting so far. It is simply a calculation, typical of many situations such as *arith-*

mogons (MacIntosh & Quadling, 1975, Mason & Houssart, 2000), *pyramids (*Russell, 1997). However, treating that as a 'doing', what is the 'undoing'?

task 7b: undoing differing products

> The measure of my grid was 2. Can you construct a grid for which the measure is 11? Or any other specified number?

The usual approach is to start trying numbers, hoping to alight on a solution. This 'guess and check' is a useful strategy if you take Pólya's advice (1945) and treat it as specialising. But the point of specialising (Mason, Burton & Stacey, 1982/2010) is to get-a-sense-of underlying structure, to detect possible structural patterns. 'Trial and improvement', is a better strategy than simply 'guess and check' because it implies trying to learn from the examples rather than simply accumulating them. But in this case it may not be clear how to adjust one example so as to force the final 'measure' to change the way you want it to. 'Tracking Arithmetic' comes to your aid:

> Repeat the calculations but don't actually do them!

For my initial grid, the calculations come out as

$$\begin{array}{r} 4 \times 7 + 5 \times 3 \\ \underline{4 \times 5 + 7 \times 3} \\ 4 \times (7 - 5) + (5 - 7) \times 3 \\ = 4 \times (7 - 5) - (7 - 5) \times 3 \\ = (4 - 3) \times (7 - 5) \end{array}$$

Now you can see exactly how the 'measure' of a grid relates to the entries, and you can construct lots of examples with any specified answer, but only if you treat the 'answer' as an indicator of structure, of providing an instance of a general property of these calculations. What you are doing is treating the initial grid numbers as place holders, so that you can see how they influence or involved in the result. This same strategy applies to many different situations where using arithmetic to encounter algebraic thinking can precede the use of letters.

My own preference is for using a cloud (signifying an as-yet-unknown number that someone is thinking of), and making use of insights of Mary Boole (Tahta, 1972). Here there are four numbers to track, but in ThOANs (see task 4) there is only one number to track … all the others can be calculated with as long as you leave the initial number alone (and don't calculate with that number). Replacing a particular starting number with a cloud is a clerical task but makes the thinking clearer, and provides learn-

ers with encounters with as-yet-unknown numbers, the very heart of algebra (and arithmetic).
The only difference between calculations in arithmetic and in algebra is that in arithmetic you are always proceeding from the known (data) towards the as-yet-unknown answer, whereas in algebra you begin with the as-yet-unknown number (as Mary Boole said, “acknowledge your ignorance”), denote it in some way, and then proceed to treat it as a number, expressing the various calculations in which it is involved. You end up with one or more constraints in your initial as-yet-unknown number, which you then proceed to try to satisfy.
The theme of arithmetic as the study of actions on and with numbers, has been described slightly differently as studying the relations among and between numbers by a wide range of authors. See for example Nunes, Bryant & Watson (2009). One way to expose children’s thinking is with tasks along the following lines.

task 8: relational or structural thinking

What is 37 + 48 – 37?

True or false: 57 + 93 = 93 + 57?

Of course the ‘difficulty’ of the numbers can be adjusted to the experience of the children involved. Some children start calculating and (if they do it correctly) get an answer; some children start calculating and then say ‘Oh’, and get the result more quickly than those who calculate everything; some children quickly get the result.
The latter children are processing before doing the first action that comes to mind. They can be said to be thinking relationally. Others are at various stages along the way to relational thinking. The first are liable to develop a habit of ‘doing the first thing that comes to mind’, whereas once one possible action comes to mind it is useful to ‘park’ it and look for something more efficient.
There are all sorts of variants of this task, exploiting children’s awareness of properties of particular numbers such as 0 and 1, 9, 99 and 999, and so on, as well as exploiting general properties (rules of arithmetic) such as commutativity, associativity and distributivity. You can also explore awareness of place value with something like:

True or false: 37 + 48 = 38 + 47

True or false: 14 + 26 = 7 + 26 + 7

Again some students may start calculating. They are among the ‘eager beavers’ who carry out the first possible action that comes to mind, whereas the effective thinker is pleased that something possible has come to mind, but parks that and pauses in order to see if there is an alternative, perhaps more subtle or efficient action to undertake.

task 9: fractional actions

In the first diagram, can you see something that is $\frac{2}{5}$ of something else? $\frac{3}{5}$ of something else? $\frac{2}{3}$ of something else? $\frac{5}{2}$ of something else? $\frac{1}{3}$ of something else? What other fractional actions can you 'see'?

In the second diagram, can you see something that is $\frac{1}{4}$ of something else? Something that is $\frac{1}{5}$ of something else? Something that is $\frac{1}{4} - \frac{1}{5}$ of something else (fig.4)?

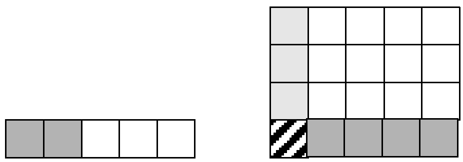

figure 4

Pat Thompson, from whom this idea comes (Thompson, 2002) offers the pedagogic strategy in a session by asking people to 'put your hand up when you can see ...' That way he can choose whether to get people to describe or point to their choices, or whether virtually everyone is confident that they can indeed see what is being sought. An extended version of this strategy is the use of small whiteboards or slates on which children make a response to hold up for the teacher to see.

Many teachers and many more children find it unusual to shift from one perceived whole to another. They are used to assuming that the whole of what they discern is 'the whole'. But it is essential to develop flexibility as to what constitutes the whole (actually, the unit) and what constitutes 'part' of that unit. The language of 'whole' is misleading because it presents an obstacle to seeing something as $\frac{5}{3}$ of something else, for example.

The second part of the task generalises. For example, you can 'see' a rectangle divided into $\frac{1}{n}$ and $\frac{1}{(n+1)}$ parts horizontally and vertically, respectively, and the difference has to be $\frac{1}{n(n+1)}$ of the whole rectangle taken as the unit.

5 reflection

It is not the task that is rich; it is not the activity that is rich; it is the way that the task is used that is rich, and this is what influences whether the experience is rich, setting up the possibility to learn from that experience. But even a rich experience may not guarantee learning, as teachers who engage students in physical activities and games discover: children may participate fully, but may not appear to influence their future behaviour as a result. The same applies to novice teachers and in-service teachers encountering some professional development activities.

When learners are given the chance to make significant (mathematical and behaviours) choices, when they are stimulated to make use of (and develop) their natural powers of sense-making, such as imagining and expressing, specialising and generalising, conjecturing and convincing, they experience a taste of the pleasure that can come from thinking mathematically, and that is what will encourage them to continue thinking mathematically.

In order to be sensitive to what learners are experiencing, it is at least helpful and perhaps even necessary to have recent parallel experiences oneself. Consequently any group of teachers in a school will benefit from jointly working on mathematics themselves, in a mathematical atmosphere of mutual respect, in which everything said is treated as a conjecture that needs to be tested in experience (if it is pedagogical in nature) or tested structurally (if it is mathematical in nature). Only when people are moved to 'modify my conjecture' can it be said that mathematics, or at least mathematical thinking, is being supported and stimulated.

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