Natural Differentiation in Mathematics - the NaDiMa project[1]

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1 heterogeneity and differentiation

Differentiation in mathematics education is not a new problem teachers have to cope with. Since the early seventies of the last century, many publications can be found, and those references are still cited in actual publications (Bönsch, 1976; Geppert, 1981; Klafki & Stöcker, 1976; Winkeler, 1976). Though it is an old problem, it can be stated that the extent of heterogeneity, the range between slower and faster learning students, low achievers and bright children, has been expanded (Beutel & Ruberg, 2010). In one classroom there may be students whose range of proficiency can spread over three grades.

You will find in a first grade classroom children whose mathematical knowledge is restricted to numbers up to 10 and others who calculate with numbers up to 1000.

Besides this, 'differentiation' has become a modern term in pedagogy and educational policy, if not to say a label which is expected to serve as an all-purpose weapon for optimizing the output of achievement at school. Differentiation and individualization are two of those pedagogical terms which are so universal and by this so ambiguous that it makes sense to have a closer look at opportunities and forms of differentiation, because it remains a convincing leitmotiv to create the most favourable learning conditions for each individual student (Wielpütz, 1998, p.42).

classical ways for coping with heterogeneity

Analyzing the literature and the common practice of teacher education in Germany, it is obvious that till today nearly the same solutions as 20-30 years ago are offered for coping with heterogeneity (Bönsch, 2004; Paradies & Linser, 2005; Vollstädt, 1997). Heterogeneity is mostly seen as a *problem* because of didactical traditions in teachers' heads which still may be oriented to an ideal of a homogenous learning group (Reh, 2005; Tillmann, 2004). This habit in itself already can be questioned, which is

part of our project as well, but will be addressed later. In our view - and our project experiences confirm this view - heterogeneity, under certain conditions, can be a source of productivity and a chance and great *advantage* for teaching and learning.
But what are the suggested (classical) solutions in literature:

- social differentiation (individual work, partner work, group work): The teacher puts together a heterogeneous group and the faster learning pupils should help the slower learning ones;
- differentiation via teaching methods (e.g. projects, courses): In project work that touches different subject matters the pupils could choose an activity according to their strengths;
- differentiation via media (e.g. textbooks, worksheets, manipulatives): Low achievers may use counters or a calculation frame for a task whereas the bright children solve the same task mentally;
- quantitative differentiation (same amount of time for different amount of content, or different amount of time for same amount of content): In a phase of individual work pupils are expected to finish one or two worksheets;
- qualitative differentiation (different aims respectively levels of difficulty): For a specific topic, like calculations up to 20, the teachers prepares three different worksheets on three different levels.

This list could have been taken from a current catalogue of pre-service and in-service courses. But it dates back to a publication from Winkeler (1976).
As our concrete examples illustrate, most of the decisions are usually made by the teacher and they are made in advance. That's not to say that those mentioned solutions *per se* are to be valuated in a negative way, nor do we say they are ineffective. But as we think, there are several indications:

- that they are not always implemented in the intended way;
- that they are necessary, but not at all sufficient.

what about mathematics?

One major problem is the fact that the discussion concerning heterogeneity and differentiation is mostly guided by organizational and methodical questions (e.g. see above social differentiation). Secondly, if one looks at the literature, the discussion about conceptual forms of differentiation is dominated by pedagogy. This neglects the essential importance of the subject matter, here mathematics, and its specifics. Some suggestions in literature even can be seen as counter examples, related to essentials of mathematics education.

Although by all means there *are* several proposals for learning environments in mathematics education where desirable forms of differentiation can take effect – because, in a sense, it is implemented in the topic itself (Hengartner et al., 2006; Hirt & Wälti, 2009). But what is missing, is a more comprehensive contribution which deals with a concept of natural differentiation from a theoretical point of view and with the perspective of mathematics education. It is merely mentioned on a half page in a teachers' manual of the German textbook 'Zahlenbuch' (Wittmann & Müller, 2004, p.15).

natural differentiation

The following attributes (Wittmann, 2001a; Krauthausen & Scherer, 2007, p.228 f.) are constituent for natural differentiation (ND):

- All students get the same material. So there is no need for a vast number of additional worksheets or materials.
- This offer must be holistic (referring to the content, and, here, not meant in the sense of 'head, heart & hand'), and it may not fall below a specific amount of complexity and mathematical substance (complexity is not necessarily the same as complicated). It could be an open task, a problem to be solved or an activity that looks for structures and patterns. These kind of challenging and complex learning environments (in contrast to common isolated tasks) are absolutely not just an advantage for better learning students (Scherer, 1999; Van den Heuvel-Panhuizen, 1996).
- Holistic contexts in that sense by nature contain various levels of demands which must not be determined in advance. The level of that spectrum which is actually worked on is no longer assigned by the teacher, but by the student him- or herself. This not only subserves the support of increasingly realistic self-assessment, but the student him- or herself can rate his or her specific abilities better than the teacher who is not able to look inside the student's head. Though, students have to learn that kind of self-assessment, and that does not happen by simply tossing them again and again into such situations. Rather they need situations, consciously planned and organized by the teacher, where they can talk and reflect about demands and their criteria on a meta-cognitive level (Treffers, 1991, p.25).
- In addition to the level the students decide to work on, they can freely make their own decisions concerning: ways to solve, use of manipulatives and facilities, kinds of notation, and even the problems they decide to solve (problem solving also includes problem posing).

– The postulate of social learning from and with each other is fulfilled in a natural way as well, because it makes sense from the content itself: it is obvious to exchange various approaches, adaptations and solutions. In doing so, insight and understanding can grow or be deepened.

> All students will be confronted with alternative ways of thinking, different techniques, variable conceptions, independent from their individual cognitive level. Rigid inner differentiation more likely will just complicate this opportunity. (...) So, the various, individually organized ways of solution also have an impact on affective, emotional areas. They leave a cognitive margin to students what can facilitate their identification with the learning demands. In this way, the direct experience of autonomy can lead to motivation and interest.
> (Neubrand & Neubrand, 1999, p.155, translated PS/GKr; also cf. Freudenthal, 1974, p.66 ff.).

These attributes have to be described in more detail (e. g. what does the same offer mean) and have to be put in concrete terms for teaching and learning processes. This is the aim of the project that will be described in the following. The presented concept can be seen in accordance with 'realistic mathematics' (Treffers, 1991, p.24 ff.): Learning mathematics is a constructive activity, moves at various levels of abstraction, is promoted through reflection and interaction and leads to a structured entity of mathematics. The different types of problems and tasks can be seen as 'free productions' or 'own productions' in the sense of developing own strategies (Streefland, 1990).

2 the project *NaDiMa* (natural differentiation in mathematics)

project group

The European project NaDiMa 'Motivation via Natural Motivation in Mathematics' is supported by the Comenius-program 'LifeLongLearning' (LLL; duration: 01.10.08-30.09.10). The official scientific project partners are from Poland (Eva Swoboda), Czech Republic (Alena Hospesova, Filip Roubicek & Marie Ticha), the Netherlands (Maarten Dolk) and Germany (Günter Krauthausen & Petra Scherer). Moreover, this international cooperational project includes partner schools and associated schools.[2]

Aims and objectives

The project NaDiMa stresses the necessity of a differentiation which amplifies the nature of the subject matter. That is not the same as to make the formalism of the mathematics science absolute. Differentiation which lies within the subject, can be given complete expression under the premise that the topic is not step by step anatomized and supplied to the students bit by bit, but as a holistic, sufficiently complex learning environment (see par. 'Natural Differentiation' on page 35). We explicitly see the teacher as responsible for the sound identification, selection and framing of the problems to work on in classroom that means: the first two of the constituent attributes of ND mentioned above. Very helpful for that can be the four characteristics which Wittmann (2001a) has established as a definition of a so called 'substantial learning environment' (SLE) (Krauthausen & Scherer, 2007, p.197 ff.):

1 It represents central objectives, contents and principles of teaching mathematics at a certain level.
2 It is related to significant mathematical contents, processes and procedures beyond this level, and is a rich source of mathematical activities.
3 It is flexible and can be adapted to the special conditions of a classroom.
4 It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research (Wittmann 2001a, p. 2).

Criteria like these, as well as others which are postulated for 'good problems', can be stated as necessary prerequisites, but do not guarantee effective teaching and learning by themselves (cf. Griffin 2009). With his criteria, Wittmann gives priority to arithmetical topics but the criteria are also applicable with geometrical or context-embedded topics.
In NaDiMa the concept of 'natural differentiation' will be investigated. Our hypothesis is that for the learners this concept will contribute:

– to a deeper mathematical understanding, as well as
– to the development of general learning strategies, and
– to high (intrinsic) motivation.

Although the relations between motivation (respectively its various components) and achievement in mathematics are not consistent in diverse studies (for an overview see Moser Opitz, 2009), we see intrinsic motivation as a general aim in mathematics education. In addition to this, the theoretical concept and the term of natural differentiation should be sharpened.

Design and methods

Together with selected teachers, different learning environments were designed and tried out in primary school (in Germany grade 1-4). The lessons (for pilot study and field test 1 completely and for field test 2 partly video-taped) are analysed and evaluated with respect to the realization of natural differentiation. By a cyclic process of evaluation (pilot study, field test 1 & 2) with reflections of the participating teachers and an extension of classes and teachers the tested material should be tried out, evaluated and optimized (fig.1).

pilot study (grade 2 & 4; one class)	march/april 2009	– 3 lessons SLE 'number chains' – individual interviews about motivation (5 to 6 per class)
field test 1 (grade 2 & 4; one class)	may/june 2009	– pre-test SELLMO-S (grade 4) – 8 lessons SLE 'number triangles' – post-test SELLMO-S (grade 4) – adapted TIMSS-items on motivation in mathematics respectively specific SLE – individual interviews about motivation (5 to 6 per class) – teacher interview
field test 2 (12 classes; different grades from 3 schools)	october 2009 – january 2010	– implementation of different SLEs – adapted TIMSS-items on motivation in mathematics respectively specific SLE – if applicable, individual interviews about motivation (5 to 6 per class) – if applicable, teacher interview or questionnaire

figure 1: overview on content, time schedule and methods for NaDiMa (Germany

In addition to this, the materials should be prepared for pre-service as well as for in-service courses.
For measuring motivation, different instruments and methods have been chosen dependent on the children's age and on the phase of the project (fig.1). As standardized methods the TIMSS-items (Walther et al., 2008) as well as the SELLMO-S (Spinath et al., 2002) were chosen. For a qualitative analysis guided interviews took place (in all project phases video-taped) with selected children in each class. In addition to other aspects, the children were asked about their favorite subjects and their attitude towards mathematics, about the level of difficulty of mathematics and the specific learning environment they had experienced in the study.
Whereas the other participating countries tested context-embedded or geometrical learning environments, in Germany we focused on arithmetical SLEs for different grades. For the SLE number chains used in the pilot study, the structure of the series of lessons was the following:

– 1st lesson: introduction/revision of the format; open problem: own number chains (see par. 'Natural differentiation ...' on this page);
– 2nd lesson: problem structured activity: reaching a specific target num-

ber (see par. 'Natural differentiation ...' on page 43);
- 3rd lesson: different investigations (e.g. effects of operative variations; investigation of specific starting numbers). For the investigation of the third lesson the teachers were free to choose which and how many they tried out.

Field test 1 was extended to the following teaching unit, using the SLE number triangles:

- 1st lesson: introduction / revision of the format; own number triangles;
- 2nd-4th lesson: investigating operative variations;
- 5th-8th lesson: diverse investigations, e.g. number triangles with numbers from the times-table, reaching even / odd exterior fields (see par. 'Natural differentiating on page 46), number triangles with three given exterior fields etc.

Teachers and researchers met several times for discussing the learning environments and the teachers were equipped with didactical literature, suggested lesson plans and worksheets. These materials should serve as a framework, and the teachers were free in selecting specific topics of the series and modify the given plans.
For field test 2, besides number chains and number triangles, we offered the learning environments 'time-plus-houses' (Verboom, 2002; Valls-Busch, 2004), 'minus walls' (Krauthausen, 2006; 2009) and 'bars and angles on the field of 20' (Hirt & Wälti, 2009). For these SLEs again various types of problems (operative and problem structured as well as open tasks) were offered.

3 selected results

pilot study and field test 1

In the following, three different types of problems will be illustrated showing the opportunities for natural differentiation and discussing the different outcomes. The first two examples refer to findings from the pilot study (SLE: number chains), whereas the third example is taken from field test 1 (SLE: number triangles). At first, an open task in the sense of a free production will be presented. The second and the third example deal with problem solving activities including the discussion of structures and mathematical connections.

natural differentiation with open problems: number chains

For our first SLE we focused on 'number chains' and chose the length of four numbers (cf. Scherer, 2007a). The underlying rule is the Fibonacci rule: Choose two (arbitrary) numbers as starting number, and write them side-by-side. Work out the sum and write it to the right. Finally, add the second and the third number, and write the result as a target number to the right. Examples:

2 4 6 10 or 12 3 15 18

For both classes (2nd and 4th grade) that format was new, and after a short introduction on the blackboard the students completed a worksheet with four given examples to ensure the correct application of the rule. The worksheet also contained empty formats to fill in own number chains and to become familiar with the format. Moreover, the students had a second worksheet with more empty formats at their disposal. Especially with own productions and without any specific demands, the students have the opportunity to work on their own level (Treffers, 1991, p.25). The question arose if and to what extent the students make use of this openness. In both classes the students strongly concentrated on their own worksheet. The results differed with respect to the quantity of tasks and worksheets as well as with respect to the concrete number chains and the chosen numbers. Anyhow, the quantity does not say anything about the chosen level of difficulty. The concrete tasks indicated a wide spectrum and the following categories are quite representative.

dimensions of starting numbers

In grade 4, the material for the starting numbers varied from numbers less than ten up to numbers with millions. The biggest numbers were used by Julian (fig.2). The individual spectrum of one and the same child differed: just two students restricted their number chains to target numbers close to 100.

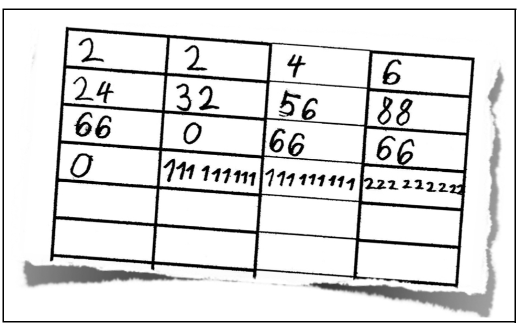

figure 2: Julian's (grade 4) own number chains

As expected, the second graders used smaller numbers. Many of the children stayed in the official number space up to 100. But we could also detect excursions into 'unknown' number spaces up to 1 000, 10 000, 100 000, and even up to one million (fig. 3a and 3b).

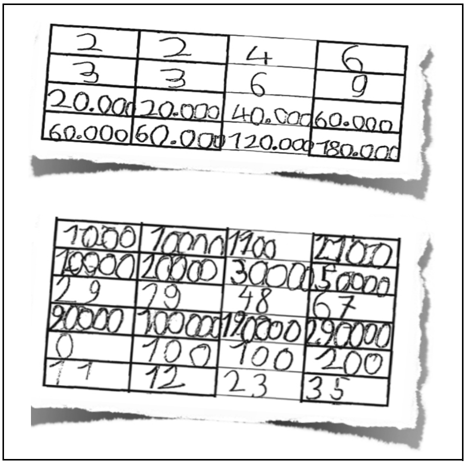

figure 3a and 3b: Konrad's and Ben's (both grade 2)
own number chains

special starting numbers

A lot of students chose own productions as starting numbers, e. g. number chains with zero, pure tens/hundreds/thousands, numbers with the unit 5 or numbers with equal digits (multiples of 11, 111, 1111, ...). Two second graders made number chains with zeroes in the following way (fig.4a and 4b). Here, the possible individual (mis)understanding of zero or a different meaning (counters?) has to be clarified through the classroom discussion.

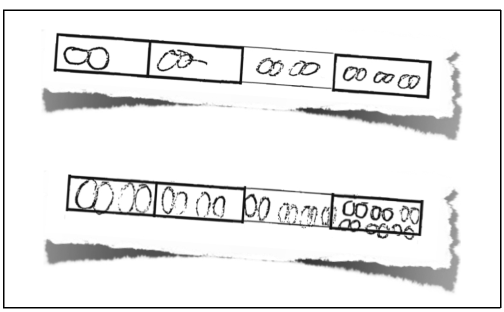

figure 4a and 4b: Louis' and Ben's (both grade 2)
use of zero

relations between starting numbers

The students explored the relation between the two starting numbers (e.g. starting numbers of nearly the same size, combination of a rather big number with a rather small one). Moreover, they chose starting numbers such a way that they could reach pure tens or hundreds with the third number.

relations between chains

Relations became obvious not only between the starting numbers but also between the chains. For example, many second and fourth graders tried out equal starting numbers or successive starting numbers. In that phase, some students (more or less consciously) already explored operative variations like constant sums or increasing both starting numbers by 1 or by the same amount (see also fig.5b).

For us, a sound phase of plenary reflection at the end of the lesson seems very important. In such phases students should present and commonly reflect their individual learning products (see also Treffers, 1991, p.25). The individual reasons can be quite different: size of the target number, easy or difficult number chains, appearance of patterns like 123 and 123 as starting numbers etc. (fig.5a). Two children also presented their operative variations (fig.5b).

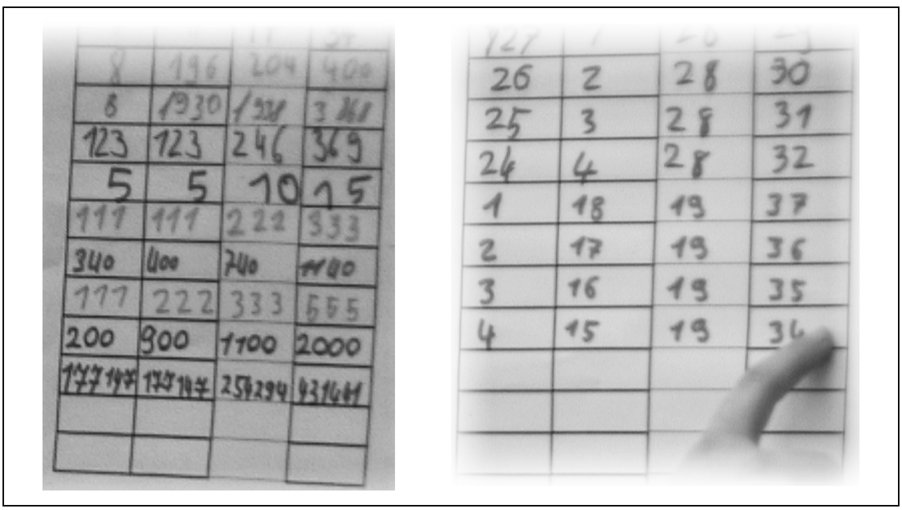

figure 5a and 5b: collection of number chains at the blackboard (grade 4)

Leon und Nina transformed the law of constant sums (in a different way for the first and second starting number). The change by 1 for the target number could be discovered, and moreover, the constant result of the third number. The difference between these two variations was not discussed in that phase. But this may serve as a starting point for further investigations in one of the following lessons: What happens if you increase the first starting number by 1 or both starting numbers? (Scherer 2007a). Discoveries like those are more likely the more the substance of the learning environment can foster that - a theoretical assumption which could also be affirmed practically in our project.

Summarizing the results for this type of problems, the self-chosen tasks represented a wide spectrum according to different criteria and with a wide-spread size of numbers. At first sight, one could object, that the students did not choose the size of numbers appropriate to their achievement level (e.g. the fourth graders actually calculated with numbers up to one million). But one has to take into account that this is allowed when dealing with an open problem. Moreover, the teacher could get helpful information with respect to the achievement level or more general for the work with open problems.
On the other hand, the above mentioned categories for choosing examples could show that special numbers, e.g. in the sense of operative variations, must not inevitably be done with bigger numbers. Perhaps they consciously shouldn't even be done with bigger numbers because then the calculation demands absorb too much cognitive energy. This energy should have priority for describing, explaining and justifying the patterns. As a consequence for field test 1, we wanted to ensure that the students make their choices with a greater (also meta-cognitive) awareness and we changed the corresponding worksheet for open problems for the SLE 'number triangles' in the way that children should differentiate between what they find 'easy', 'difficult' and 'special' number triangles (see also Van den Heuvel-Panhuizen 1996, p.145 f.).

natural differentiation with a problem structured task: number chains

In the second lesson of the pilot study, the problem of reaching a specific target number was posed (Scherer, 1999, p.283 ff.; 2007a; Van den Heuvel-Panhuizen, 2001, p.72 f.). On purpose, we chose the small number 50 for the target number. On the one hand, all second graders should have the opportunity to find solutions. On the other hand, the development of problem solving strategies and the discovery of patterns should have priority and not the training of calculation. Insofar, the target number 50 is also appropriate for the 4th grade and can be discussed with the students in a meta-cognitive way. Principally, the selection of numbers should be done according to the dominant aims and objectives of an activity.
For the target number 50, there are several solutions in IN_0 (fig.6) as well

as solution strategies to solve it. Deliberately, we formulated the instruction that approximate solutions are also allowed (or desired) which later could lead to exact solutions.

0	25	25	50
2	24	26	50
4	23	27	50
6	22	28	50
8	21	29	50
10	20	30	50
12	19	31	50
14	18	32	50
16	17	33	50
18	16	34	50
20	15	35	50
22	14	36	50
24	13	37	50
26	12	38	50
28	11	39	50
30	10	40	50
32	9	41	50
34	8	42	50
36	7	43	50
38	6	44	50
40	5	45	50
42	4	46	50
44	3	47	50
46	2	48	50
48	1	49	50
50	0	50	50

figure 6: solutions for the target number 50, notated in a systematic way

In addition, the teacher emphasized that the students should try out things and they were requested not to erase. This request, however, was more and more ignored when working on the problem because finding an exact solution dominated.

The problem can be solved with different strategies and on different levels: Children can do it by trial and error (more or less systematically), by discovering and using operative variations ('If I increase the first starting number by 1, the target will increase by 1, too. I found a chain with the starting numbers 17 and 15 and the target number 47, so I increased the first starting number to 20 and got it!') or by calculating backwards (the sum of the third and second number equals 50; the first number can be found by subtracting the second from the third).

For our empirical study, in both classes the students at first should work individually. Besides the empty formats to try out, the students had so-called 'hit-chains', one big empty format per sheet, to fill in the exact solutions for the target number 50. Those papers were just collected individually and not presented yet. Compared with the open problem presented before (see par. 'Natural differentiation ...' on page 39), we could also identify that the children worked individually. But we found a stronger orientation on other students ('How many solutions do you already have?') This - seen positively - could be a motivation to find more solutions. On the other hand, it can put pressure on students, if one has not found as many solutions as the others. The role of the teacher is important here, to do justice to the individuality of all students (pointing out that this is not a competition against each other, but efforts to a common objective) and to give all students the opportunity to present results, their discoveries and, more general, their insights.

While collecting strategies and results, our students were mainly interested in collecting the solutions to make sure that they had found all. They were less interested in reflecting on their strategies, so this reflection was postponed to the following lesson.

The collection of solutions has to be organized methodically in a way that all students can contribute to the overall solution of the problem: In grade 4 every volunteering student should tape one solution (hit-chain) on the blackboard (at first unsystematically). At the same time, the others were asked to check that the solutions are correct and that there are no doubles. By and by, the blackboard arrangement became confusing and the students looked for a better arrangement, for a structure to get a better overview. At once, the students suggested to sort the solutions according to the size of the first starting number starting with the biggest one. With this strategy, they were able to find missing solutions and, lately, to find all solutions.

This activity was more demanding and time-consuming for the second graders, but they also found nearly all solutions. Their spontaneous suggestions for sorting were the following: Sorting the number chains according to the first starting number (starting numbers beginning with 40s on the left side of the blackboard, beginning with 30s in the middle and so on). Another suggestion was the distinction if the first starting number is bigger than the second or vice versa. A third proposal brought up by a second grader was to change the places of the both starting numbers to get all solutions. This could be disproved by a concrete counter example. With the discussion of different proposals, the students found out that the first starting number has to be an even number. In the end, also the pupils in grade 2 found all solutions for the target number 50. Nevertheless, if applicable, for grade 2 one could decide to reduce the size of the target number (for example 20), so that *all* students have the opportunity to develop problem solving strategies and are not hindered by calculations demands. That was realized in field test 2.

natural differentiation with a problem structured task: number triangles

For field test 1 we chose the SLE number triangles (Scherer, 1997; Wittmann, 2001b). The rule for this format is the following (fig.7). The sum of two adjacent interior fields is written down in the exterior field.

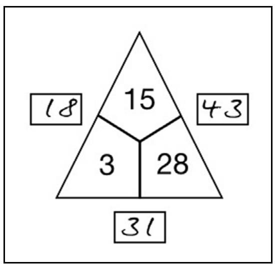

figure 7: number triangles

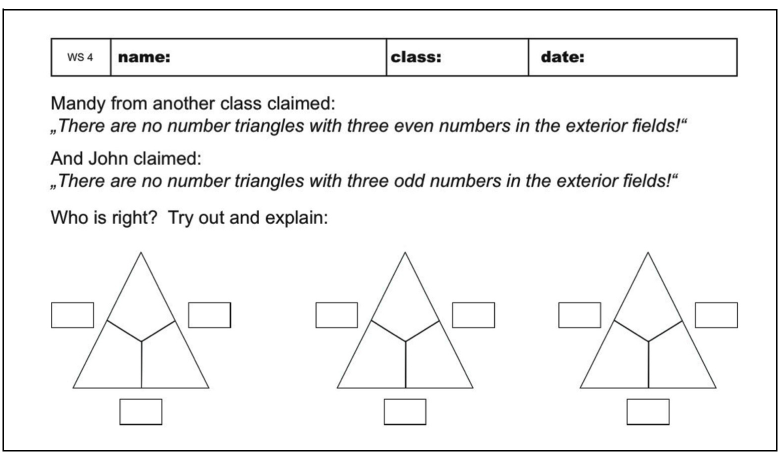

figure 8: worksheet of pupils

For primary grades we (among others) designed the following problem for investigation (fig.8) which was dealt with in the 6th lesson in the series of number triangles (see par. 'Design and methods' on page 37).
This format can be challenging for pupils and they might 'happily play the role of sceptic, looking for falsifying examples' (Burton, 1987, p.11).
Checking whether the statements are true or false, can be done more or less extensive and on different levels. Obviously, Mandy's statement is false and can be disproved by *one* counter example (some children did so). Moreover, the situation can be investigated in a systematic way with systematic reasoning (three even or three odd exterior fields).
This could be found with several students: They had a look at several examples and different cases (three even interior fields with three equal or different numbers; or analogous for odd interior fields). Others used case discriminations in order to explore what will happen with a combination of even and odd interior fields. On the one hand, the chosen level could be 'based on examples (arithmetical)' (fig.9a and 9b).

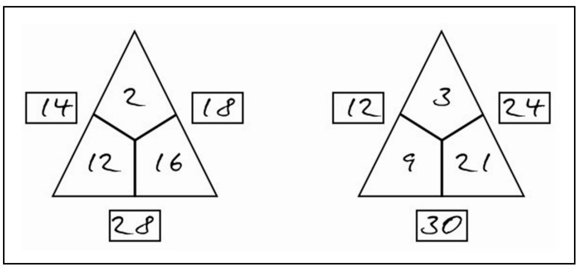

figure 9a and 9b: even exterior fields with three even or three odd interior fields

Moreover, this problem aims at developing algebraic thinking - not on a formalistic level, but pre-algebraically: The children argued with even and odd numbers and used the complete words (fig.10a) or the abbreviations (fig.10b).

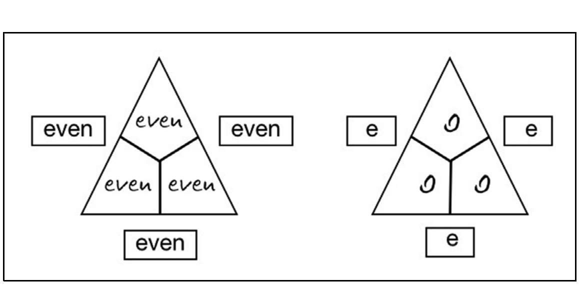

figure 10a and 10b: even exterior fields on a pre-algebraic level

This was realized in grade 2, with a distinction by different colors. The formal 'algebraic' representation (as *2n* for even numbers and *2n + 1* for odd numbers) is not intended for primary level, but for secondary students (and of course for teacher students and teachers).

The exploration of John's statement on the mentioned worksheet is more demanding and can be illustrated on the pre-algebraic level: If you want to reach three odd exterior fields, you have to choose for two adjacent interior field one odd and one even number (fig.11a, right side). To get an odd exterior field also in the bottom field, you have to add an odd number to the already given even interior field. With this, the numbers for the interior field on the left side are fixed and, by force, make the exterior field an even one (fig.11b). Therefore, you will never get a number triangle with three odd exterior fields if natural numbers are used.

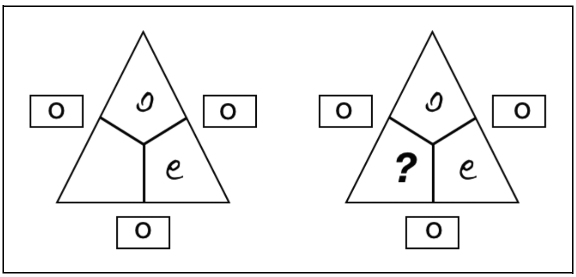

figure 11a and 11b: trial to get three odd exterior fields

According to the intention of this investigation, we chose a true and a false statement (for primary students) on purpose, to analyse the potentially different argumentations. Unsolvable tasks may challenge arguments and reasoning in a special way, but at the same time demand adequate mathematical competences. Beyond content-related objectives, here, process-related objectives are required: the development of problem solving strategies, argumentations and reasoning (Scherer, 2007b).

As shown before, the students dealt with number triangles during six lessons (introduction, open problems, operative-structured activities (see par. 'Design and methods' on page 37). For this specific problem solving activity, the students should not only calculate but be encouraged to verbalize and write down their discoveries. Those verbalizations and notations could and should be done with the help of number examples and drawings/sketches, so that different representation levels can be integrated in a natural way.

For primary students, the second statement (John) is an unsolvable situation when using natural numbers; that means John's statement is true. In our empirical study we found a variety of students' discoveries:

Carl (2nd grade) showed a typical solution, he tried out a lot (exclusively with even numbers) and came to the conclusion:

> John is right. Because I tried out very often if there are only odd numbers.

Some of the students ascribed the fact that they could not solve the problem to their own incapability or to the fact that they did not try enough. This is a crucial point for the plenary reflection, to work out the reasons for not finding a solution (see also Scherer 2007b).

Don (2nd grade) expressed the effects of addition of even and odd numbers, directly addressed to John:

> You were right because the numbers are mixed, at least one is even, because you get an even when you add an odd to an odd.

He concluded, that at least one even number is necessary.
The same direction could be seen with Carsten's argumentation (4th grade), expressed in a rather elaborated way:

> Even exterior numbers exist. But there aren't any odd exterior numbers because one number is always even, namely the number you need to get the first two odd numbers.

Also Lena (4th grade) reasoned, that one even number is necessary:

> There are number triangles with even numbers but no number triangles with odd exterior numbers. Because 3 odd numbers always have 1 even number. If you would have a two-sided figure or a number quadrangle, it could happen that you have all results odd.

She went one step further, looking for alternatives to make the problem solvable (here turning the figure into a 'symmetric' one; in field test 2 it happened that students explored this idea more systematically by investigating polygons with even and odd numbers of edges like pentagons, hexagons, ...).

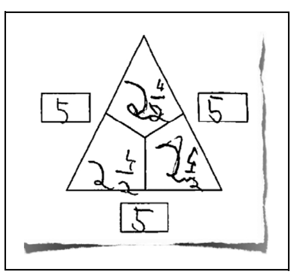

figure 12: Damian's (grade 2) number triangles with fractions

Damian (2^{nd} grade, fig.12) found a number triangle with fractions. Understandably, problems with this new symbolic expression occur.
Verbally he explained: 'Two and two fourth', so he mixed up numerator and denominator.

4 interim conclusions and perspectives

SLE and opportunities for natural differentiation

Our first results could show, that the students really made use of the containing substance that allows working on different levels. Moreover, it could be stated that the different types of tasks, such as open problems, free explorations and productions, more purposeful problems, problems which contain the construction, discovery and explanation of patterns and structures have different effects. Beyond this, it became clear that the size of the numbers is an important and influencing factor and has to be reconsidered with respect to the underlying aims and objectives.
Absolutely essential for the use of substantial learning environments like these is the opportunity to both foster and demand the general, process-related competencies, besides the content-related ones. The implementation and the fulfilment of just these objectives has become more and more valued and stressed by education policy (NCTM, 2000; KMK, 2005).

The concrete opportunities of natural differentiation became obvious with respect to the:

– *selection of numbers*
  The widespread selection was presented with the open tasks in the format of number chains (see par. 'Natural differentiation ...' on page 39). This became true also for other formats and free productions like creating patterns for others. With the second problem, reaching a specific target number (see par. 'Natural differentiation on page 39), this freedom was not given, but we could state that some students first tried out 'easy' numbers (pure tens or multiples of 5). The target number 50 consciously was chosen as an easy one and the importance of such a selection became clear. With the presented activity for number triangles, reasoning about the given statements (see par. 'Natural differentiation ...' on page 46), the children were completely free to choose the numbers. Many children chose numbers up to 10 or 20 as a maximum, which is quite sensible for such an investigation, as you may concentrate on the pattern. This is a crucial point for the common reflection with respect to the development of more general

problem solving strategies. If the focus is on investigation and explanation of structural relationships, then it is clever to reduce the own calculation demands because the size of the numbers does not play any role for the structure.

The children also used own productions like equal numbers or successive numbers or multiples of 11. During the plenary reflection on John's statement, looking for three odd exterior fields, a fourth grader commented: 'I tried it out with a zero' and another: 'I guessed, perhaps it will work with prime numbers'. This makes clear that the children do not choose the numbers accidentally but on the basis of mathematical experiences and knowledge.

– *number of examples*

  The problems also differentiate according to the numbers of examples (which is not independent from the former point; see also above: quantitative differentiation).

– *problem solving strategies respectively general (solution) strategies*

  The second and the third problem-structured activities focused on problem solving. As problem solving strategies we could find argumentations with respect to concrete numbers or in a more general manner. The students should gain experiences with these different levels and we found a wide spectrum. We could also find students who considered particular cases, distinction of cases, up to completeness of a proof. Such types of investigation could challenge a general requirement of reasoning and proofs (Winter, 1983). For example, some of our children thought you just have to examine 'enough' examples to have a proof.

  Also with other types of tasks (e.g. inquiry of structures or self-designed patterns) we could find different ways and levels of working which could provide important aspects for reflection and discussion.

– *searching for alternatives to turn a problem into a solvable one*

  For our second statement (John), it turned out that this was difficult for many students, mostly because they had only few experiences with such an activity (Scherer, 2007b). But this type nevertheless lead some children to further investigations. The proposals could be amusing, like: 'If 6 and 6 would make 13, then it could work'. Other ideas showed complex mathematical thinking processes, lead to fractions or variations of the format as shown before: Johanna (4^{th} grade) suggested to change the operation of number triangles: 'If you would multiply, then it would work'.

One advantage of SLEs is the fact that the levels of demand are not determined in advance, but develop quite naturally within a substantial research problem with floating transitions. This facilitates natural differentiation for the students, combined with an economical expenditure for the teacher who therewith wins free zones a) for diagnostic work, and b) for framing the plenary phase at the end of a lesson. We consciously used the benefit of the mixture between individual approaches, partner or small group work, but also with common phases of reflection and integration. In doing so, the important social learning is strongly taken into consideration, and real deepening mathematical knowledge can take place (cf. Treffers, 1991).

findings with respect to motivation

The first results based on the interviews as well as on the TIMSS-items have shown a rather high motivation of the students with respect to mathematics education and mathematics learning in general. These results are in accordance with the findings of large scale studies for this age (cf. Walther et al. 2008, p.78). In our study this positive attitude is not only true for the bright students, but for nearly all students. Exemplary statements in the interviews:

- It's interesting.
- It's tricky.
- Because it is easy.
- I like difficult problems.
- Because there exist easy and difficult problems.
- You have to do it with your brain and are not required to write so much stuff like in German language.

Some of the students commented on the given SLE, that they liked those activities more than the regular mathematics lessons, because they were more successful than usually. Exemplary statements:

- I like this format very much.
- Because every time you could do something new with it.
- I was successful with the number chains;
- When I made sense of it, I liked it very much.

In the ongoing project we will scan and analyze different aspects of motivation in more detail (Krauthausen & Scherer, in preparation). Our examples certify that intrinsic motivation is not imperatively bound to a realistic, everyday context situation: Pure inner-mathematical problems and contents can be motivating for all children as well. More important than the question of contexts versus inner-mathematical structures is the enclosed 'substance'.

consequences for teacher education

The crucial demands for the teacher and the teacher's role are the following (see also Scherer & Steinbring, 2006; Speer & Wagner, 2009):

– own mathematical exploration of the format or problem in order to be able to class and value the different strategies and solutions,
– an anticipated reflection of possible strategies and levels for pupils, as well as
– an analysis of real student documents,
– the reflection on integration of different strategies, solutions and argumentations, and
– the sound capability to moderate plenary discussions, which includes e.g. posing questions, initiations, even irritations if needed.

> The responsibility for ensuring that these mathematical conversations are productive 'ultimately lies with the teacher'. To organize, manage, and support such classroom dialogue, teachers must model appropriate discursive practices for students, present meaningful tasks, and actively monitor students' interactions. Teachers should listen to and help guide the conversations by requiring explanations and clarification of ideas as well as provide feedback about students' ways of thinking about the problem and their solutions. Students should also have sufficient time to work on tasks, grapple with mathematical ideas, consider alternative approaches, and formulate reasonable explanations to support their solutions (Kilic et al., 2010, p.352; *emphasis* GKr/PS).

In our view, a subserving classroom culture is of major importance (see also Griffin, 2009). It is a prominent task for teachers to provide their students with a sound image of mathematics as a science. Inherent to this is to consciously value the central attributes of doing mathematics. And that cannot be successful by just giving isolated information, but with the teacher him- or herself offering a continuous, explicit model, and habitually initiating and maintaining of meta-cognition.

For example referring to the level of difficulty of a specific task. Questions like: 'What is an easy or a difficult task?', will usually make the students yield just statements - with a reduced meaning and mostly without any consequences. Referring to this, not only the sensibility to questions like these or the consciousness and diversity of the answers must be fostered. Rather often, deeply internalized underlying pre-experiences have to be discussed. As traditional classroom culture more likely emphasized the content-related competencies and neglected the process-related ones, students internalized implicit valuations and units of value - unconsciously or caused by that kind of teaching and learning.

One is rather proud if one can deal successfully with difficult operations, unusual number systems, like fractions or negative numbers, or big numbers. But students still do have less role models for cherishing process-related competencies. And that is also true for attributes of desirable attitudes of sound working, like endurance, persistence, patience. Students must have criteria in order to be able to rate levels of difficulty because 'difficulty' can be related to:

- the complexity of calculation (kind and size of numbers);
- the operations involved (addition / subtraction is often said to be easier than multiplication / division);
- the demand of thinking and reflection, strategic understanding, process-related competencies;
- the understanding of the given tasks (language understanding);
- the amount of necessary (written) text production.

From this, for the implementation of a sound culture of teaching and learning, the following consequences can be drawn.

The teacher has to make it plausible for the students that the amount of reflections, the flexibility of strategic ways of solution, an attitude of questioning, and the individual request for proving (Winter, 1983) at least are good reasons to feel proud of one's own capability. The easier that goes, the more a steady culture of meta-cognition is implemented in the classroom, so that everybody can talk about the problem in a natural and habitual way.
Students by themselves will offer reasoning and proofs for solutions quite naturally if the teacher has asked 'why' before (in the beginning: in substitution for the child).
In the same way, with the role model of the teacher, the students' sensibility, diversity, and valuations of the aspects mentioned above will increase and be established as an attitude. This points to an educational assignment of mathematics education which leads far beyond the subject and the subject matter.

5 final remarks

The project NaDiMa has reached more than half time, and we have already gained important experiences. In the second part of the project an extension and evaluation of the first project activities will take place (at the moment the second field test is running, partly finished). In the center of the project are the teachers as well as the concrete design of mathematics lessons.
We also suggest to keep in mind that the described concept of natural differentiation within substantial learning environments is not meant as an exclusive one for the whole range of mathematical learning and teaching, though rather likely for its major part. Training the basic facts e.g. or intro-

ducing a specific procedure, may require other practices. Above all, successful and motivating learning processes should be possible and are desirable for as many students as possible.

notes

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2 See for more details: www.nadima.eu

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