Different students solving the same problems

- the same students solving different problems -

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1 introduction

The article focuses on the learning processes of low achievers in mathematics, especially on the topics multiplication and division. In Germany, teaching practice in special education as well as for low achievers in regular schools still proceeds in a small-step-way, without making use of central relations, which can ultimately represent a learning aid. The study presented here examines how and on what level the different students solve the same given problems. Moreover, the effects of a variation of the problems (e.g. context-free or context-embedded; initiating a mental image) were studied, on the one hand in the form of a paper-and-pencil test and on the other hand through interviews. The results show - beside other aspects - non-uniform profiles with individual children. It becomes obvious that a conception of small-step instruction cannot do justice to different students: it neither deals with existing difficulties nor makes the introduction of their individual abilities possible for the children. As a consequence, central aspects for teaching are pointed out, namely the support of own strategies, support of base knowledge, effective use of representations, relation-rich learning and practising and a more conscious selection of numerical data.

Recent large scale studies document that students in school show different competences and different levels of abilities. Those studies not only give information about correct and incorrect solutions, but one also gets information about typical difficulties and - depending on the type of problems - about the underlying strategies. However, those studies focus in general on a macro-level and usually it is not possible to go into details. For teachers at school it is also important to have a closer look at the specific classroom situation. What are the existing competencies, difficulties and processes in her/his class? For an adequate design of teaching and learning mathematics the teacher has to study the situation on a micro-level. Referring to this micro-level two foci should be distinguished: how and on what level do different students solve the same given tasks? At the same time it is necessary to identify the factors that influence the learning process of the individual child, namely the conditions under which one and the same student solves or does not solve a problem. It is helpful to know the effects of a variation of:

- the problem (e.g. context-free or context-embedded; initiating a mental image);
- the setting (e.g. mathematics lessons, support lessons or one-to-one situations);
- the design of the study (e.g. paper-and-pencil test, interview or regular classroom situation)

In the following these ideas will be illustrated by a small case study (low achievers solving multiplication and division problems). Consequences and perspectives for research as well as for teaching are presented.

In addition it should be mentioned that the kind of instruction is of great importance: investigative learning is one of the guiding principles for primary schools but not for special education. In Germany the usual teaching practice in special education can be characterized by learning step-by-step in a rather mechanistic and reproductive way.

From the point of view of mathematics education this kind of approach is inappropriate, and children's existing difficulties are often a consequence of this kind of instruction. Hardly ever do these pupils have opportunities to show what they are capable of. In this sense the following study can be understood as a plea for a change in teaching practice.

2 multiplication and division: typical difficulties

Multiplication and division are a central topic in primary school mathematics, and a solid understanding and automatization of certain base abilities is absolutely necessary for later topics. However, one often encounters deficits - especially with learning-disabled students - which can refer to the following domains: 'Automatization, 'Relations between single exercises', 'Extensions' and 'Change of the representation level'. I will elaborate these here.

automatization

Even in higher grades, not all problems of the multiplication table are automated. Many children have to calculate a problem like 6×8 anew every time, often by means of reciting the whole multiplication-row (8, 16, 24, 32, 40, 48; cf. also Lorenz/Radatz 1993, p.138). In many cases, this is connected with finger calculation, with this help being quite demanding and thus also unsafe for multiplication, as the part results on the one hand and the multiplier on the other hand must be kept in view (cf. to this also the example by Anghileri, 1997; also the interview excerpt by Karsten in paragraph 5). Two examples of this: in the frame of a mini project, in which fifth grade students of a school for learning disabled had measured and projected that they could walk 4 km in an hour, they themselves introduced the problem of how many kilometres they might cover in one day, in 24 hours. Problems such as 24 · 4 had not yet been treated in their mathematics instruction (for the detailed representation cf. Scherer, 1997).

Sandrina wrote down the problems from 1·4 until 24 · 4 without yet calculating them (fig.1). She started with the easiest one and calculated the following results one after another, where she interchanged the lines and finally abandoned this laborious method.

Jan wanted to split 24 · 4 into 10 · 4 + 10 · 4 + 4 · 4, and he also started with calculating the results of the multiplication table up to 10 (fig.1).

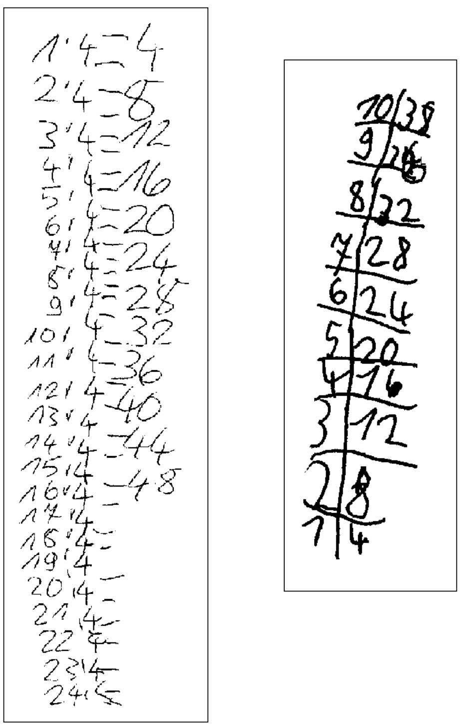

figure 1: Sandrina’s and Jan’s solution strategies

Unfortunately, he made a mistake with the table, which then continued throughout his calculation (9|34, later corrected, then 10|38).
It shows that even the 'simple' tasks like 10·4 are not directly written down as a matter of course. It could be that both children have this knowledge, but they do not apply it in this complex situation.

relations between single problems

Problems like 6·8 or 9·8 are not derived from easier ones (6·8 from 5·8 or 9·8 from 10·8; cf. Ter Heege, 1983, p.12f.). Learning-disabled students often do not use any reference point conceptions to so-called core or key problems.

calculation laws

Calculation laws, such as for instance the commutative law, are not used either, because of memory problems or of a lack of understanding: if children are given the problem 6 · 8 and right after the successful calculation the problem 8 · 6, quite a lot of children will start calculating it completely anew.

extensions

Extensions to the so-called step multiplication table are executed by means of a rather mechanistic use of rules: for the problem 3 · 70, for instance, the problem 3 · 7 out of the multiplication table is taken and a rule is derived: 'For the new result, one zero has to be appended'. Accordingly, two zeroes are appended in the task 30 · 70 ('Add the number of the zeroes in the factors and append just as many zeroes to the result'). Such a rule, memorized in a rather meaningless way, however, can confuse students when the result of the original multiplication table problem already has a zero at the end: 50 · 80 is to be calculated; the reference is 5 · 8 = 40 and many children write down 400 as the result. With further calculations beyond 1000 and with a higher number of zeroes, these uncertainties can increase.

change of the representation level

Retranslations to the iconic level do not succeed anymore. For many children, symbolic and iconic level have become different, completely separated worlds.
At this point, it is to be emphasized that these difficulties are not necessarily to be understood as features of the students themselves, but that they can also be consequences of the kind of instruction they have experienced (cf. also Van den Heuvel-Panhuizen (ed.), 2001): with a small-step instruction conception, which is currently still encountered in most Ger-

man schools for children with learning disabilities, the multiplication-rows are usually introduced and worked on isolated from each other; thus the children learn the problem 6 · 8 in the 8-row and at another point in time 8 · 6 in the 6-row. The fact that the students then do not use the relation of the commutative law is not a surprise.
Moreover the central aim is memorization of the multiplication table, ignoring how the process of memorization takes place. A rather mechanistic kind of instruction (here: learning the multiplication-rows isolated from each other) does not support the process of automatization:

> Children who can recite the multiplication tables do not (…) know any isolated multiplication fact by heart.
> (Ter Heege, 1985, p.378)

In order to gain findings for a suitable instructional treatment, understanding of the way learning-disabled students preceed with multiplication and division problems must be examined more exactly. As well as for other content domains, the inquiry into existing knowledge or foreknowledge is suitable for this.

3 problems and method of the study

For the compilation of the test items, homogeneous groups were built, and different types of problems were chosen (Scherer, 1995; also Van den Heuvel-Panhuizen, 1990): countable as well as uncountable representations, for the latter a distinction between context-related and context-free problems was made. In the following several examples will be presented.

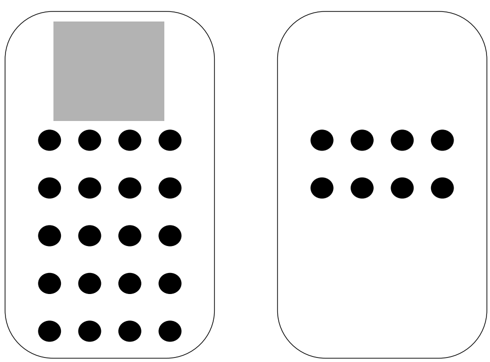

figure 2a and 2b : countable multiplication and countable division problems

countable problems (structural representations)

For the countable problems - group a; fig.2a: countable multiplication: 'How many counters does the girl have?' Fig.2b: countable division (distributing): '4 children equally share these counters.
How many counters does each child get?' or countable division (partitioning): 'These counters are to be divided equally. Each child is to receive 4 counters. For how many children is it sufficient?', cuttings out of the field of hundred are used, but without the segmentation into 5s, as with factors bigger than 5, the whole field is to be conceived. The question is whether children structure a bigger field themselves, possibly into a part field of 5s.

Furthermore, a countable multiplication with mental (if necessary also real) addition was recorded (group 1a*): A given field structure is partly covered and must be completed mentally. Here, a house front with windows, which are partly covered by a bush, is chosen (fig.3).

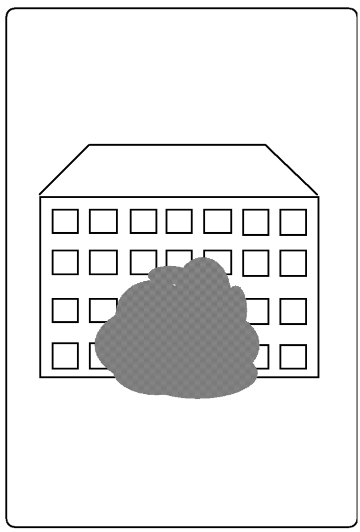

figure 3: countable multiplication problems with mental completion

The children are given the following information and question: 'With this house, you cannot see all the windows. Some are hidden by a bush. How many windows does the house have altogether?' Further realistic examples are well known in the Netherlands, such as tile patterns, lighted and/ or non-lighted windows, an incomplete puzzle, a curtain with a regular pattern etc. (cf. Hengartner & Röthlisberger, 1999; Van den Heuvel-Panhuizen (ed.), 2001, p.76; Wittmann et al., 2000). This kind of problem should also be taken up later in instruction proposals.

context-related problems

With the context-related problems to multiplication (group 1b), a representation of a dart game is given, in which the scored points are marked by means of thrown darts on the board (fig.4: Context-related multiplication ‘How many points has the boy scored in the dart game?’). For context-related division (group 2b and 2b*), word problems have been chosen, again on the one hand in the context of distributing, on the other hand in the context of partitioning. Context-related division (distributing): ‘There are 20 children in the gym. They are to form 4 groups. How many children are in each group?’ or context-related division (partitioning): ‘20 children are waiting at a cable railway. 4 children fit into a gondola. How many gondolas are needed?’

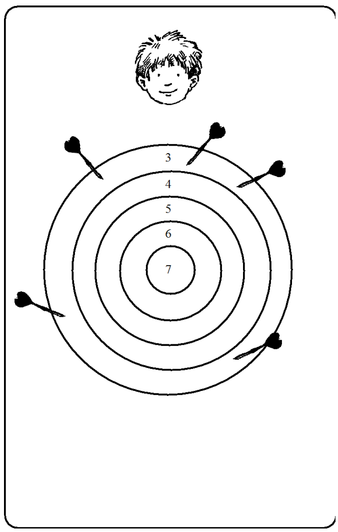

figure 4: context-related multiplication problem

context-free problems

The problems of type c are context-free. As the multiplication symbol might not be familiar, it was substituted by the word ‘times’. Like this, it can be ensured that the children can produce the connection to everyday multiplication situations. As the division symbol is probably unfamiliar as well, and as no meaningful context-free verbal representation for this is possible, context-free division was completely renounced (to the division symbol cf. also Spiegel, 1992 in Selter, 1994, p.82).

In order to be able to parallelize the single competences regarding types a, b and c, the same number values were chosen. In order to examine mainly understanding and the solving of given situations and not arithmetical competences, only small number values were chosen. The tasks were worked out in the form of an individual written test and in individual interviews. All test items were given verbally; furthermore, the test sheet makes an individual understanding possible. Figure 5 shows a general view of the different task groups.

group		number values	number of tasks
1a	Countable multiplication (CM)	2·4 5·5 8·3 4·7 5·3	5
1a*	Countable multiplication with mental completion (CM/MENT)		5
1b	Context-related multiplication (CRM)		5
1c	Context-free multiplication (CFM)		5
2a	Countable division (distributing) (CD/DIS)	20·4 12·3 30·5 16·2 8·4	5
2a*	Countable division (partitioning) (CD/PART)		5
2b	Context-related division (distributing) (CRD/DIS)		5
2b*	Context-related division (partitioning) (CRD/PART)		5
			$\sum = 40$

figure 5: general view of the different task groups

During the work on the test, the children did not have access to additional material. However, the possibility of using drawings or other notes aids was explicitly pointed out to the children; but notes were rarely used. The tests (and the following interviews) were carried out with students of a 4th grade of a school for children with learning disabilities (12 children; 5 girls and 7 boys). 7 of these students had gone to primary school before, in one case also attended second grade. The topic for the 4th grade in special education is multiplication and division with numbers up to 100, whereas this topic is dealt with in 2nd grade in regular school.

4 results of the written test

notes

Altogether, the children very rarely used notes; these were restricted to structuring aids, such as for instance the separation of point fields with division or the marking of counted points or windows. With the word problems as well, there were no sketches or part results, which could have been helpful for the solution of the problems. This trend also shows analogously in the interviews.

general overview: differences within one class

Within the three test sessions, the children worked on 40 items (20 multiplications; 20 divisions). The overview of the number of correctly solved items (fig.6) shows a great heterogeneity within one class (max: 34; min: 1). The average value is 15.08, the standard deviation $\sigma = 9.24$. A gender-specific analysis shows an advantage for the boys with an average value of 16.71, while the girls only solve 12.8 tasks correctly on average. Altogether this shows a rather low achievement level in combination with a big standard deviation - an instruction situation which belongs to the most difficult ones (cf. Lienert & Von Eye, 1994, p.31).

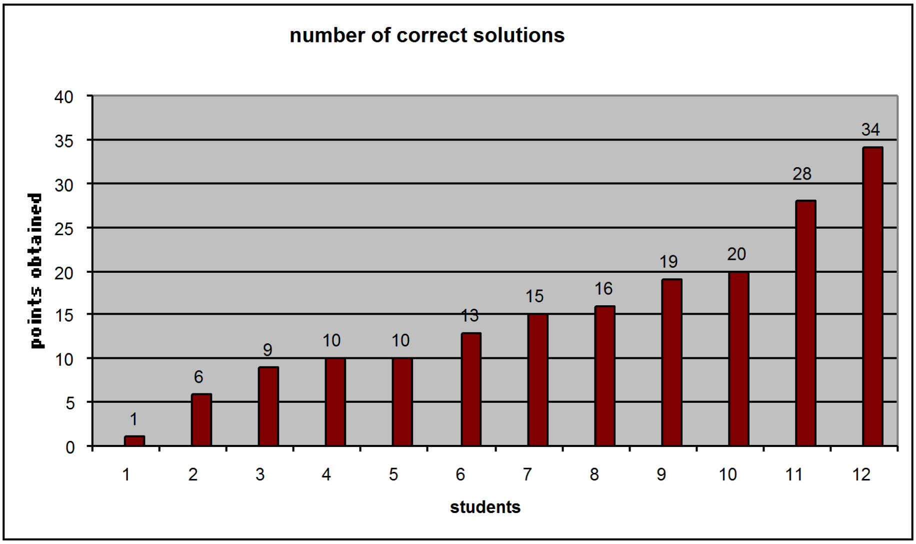

figure 6: general overview

However, according to the achievement level, one should consider that for almost all children the tested contents have either been treated long ago or not yet systematically and that this is not a learning objective control, but an inquiry of foreknowledge. On the other hand, a task such as countable multiplication requires merely a quantity determination, which should be sufficiently familiar.

different task groups

If an 80%-level is established for the classification as master in one group, this means solving at least 4 out of the 5 tasks in one group correctly. Having established this 80%-level, non-uniform profiles for single children are revealed (fig.7), so that one cannot proceed on the assumption of a hierarchical development of the different competences. Also, a rising number of correctly mastered groups does not automatically follow out of the rising order of precedence of the obtained number of points. Regarding among others the low achieving children, for instance those who could be classified as masters in two of the eight groups (four children), a different profile emerges. One child is a master for countable and context-free multiplication, another one in the groups countable multiplication and countable multiplication with mental completion. The two other children could be rated as masters in the groups countable multiplication and countable division/partitioning. A similar picture showed the combinations of children who were masters in three groups.

	CM	CM (MENT)	CRM	CFM	CD (DIS)	CD (PART)	CRD (DIS)	CRD (PART)	Σ
S 1	–	–	–	–	–	–	–	–	0
S 2	–	–	–	–	–	–	–	–	0
S 3	+	–	–	–	–	–	–	–	1
S 4	+	–	–	+	–	–	–	–	2
S 5	+	+	–	–	–	–	–	–	2
S 6	+	–	–	–	–	+	–	–	2
S 7	+	–	–	–	–	–	–	–	1
S 8	+	–	–	+	–	+	–	–	3
S 9	+	–	–	–	–	+	–	–	2
S10	–	+	–	–	+	+	–	–	3
S11	–	+	+	–	+	+	–	–	4
S12	+	+	–	+	+	+	+	+	7
Σ	8	4	1	3	3	6	1	1	27

figure 7: group-specific overview of the masters on the 80%-level

However, it can be stated that some groups are significantly easier than others, for instance the countable multiplication (group 1a) as well as countable division/partitioning (group 2a*). In group 1a, eight out of twelve children could be counted as masters, in group 2a* at least six out of twelve. Other groups proved to be much more difficult.

These results also show that the same operation (division) on the same representation level (countable representation) has a significantly different degree of difficulty, caused by the context of distributing on the one hand and that of partitioning on the other hand.

Even if countable multiplication on the whole now seems easier than for instance, countable multiplication with mental completion, there are individual cases in which the opposite is true (S10 and S11). It is possible that the incomplete representation in group 1a* encourages single children to multiplication and thus to use more effective strategies than mere counting.

different number values within one group

With the countable multiplication (group 1a) the first item (2·4) was most frequently solved correctly, followed by the fifth item (5·3). This was the effect of the small result values with both of the tasks (8 and 15, both within 20). Altogether, with multiplication, the influence of the size of the result becomes evident: As a simplified trend, one could formulate for multiplication 'The higher the result value, the higher the mistake rate'. Exception here is the task '5·5', which is easier to solve for many children.
In opposite to multiplication, the situation with division was less clear. Spiegel & Fromm (1996, p.360) were able to show that the size of dividend or divisor or their numerical relation in dependence from the chosen strategy (distributing or partitioning) plays a decisive role. Regarding this study one has to comment that the numerical data for the dividends was much bigger than in the present study. But here as well, one could formulate in a simplified way: 'The bigger the dividend, the higher the mistake rate'. Exception is again a task with 5, here 30 : 5.
Even though the number space up to 100 had already been introduced to the children in 3rd grade and even though rather small number values had deliberately been chosen for the problems, there partly are significant problems and the influence of the numbers is quite high.
Altogether, the typical mistakes show that the operation with its arithmetical challenge or the given context, or the chosen strategy as well are the central factors for mistakes. Here, the existence of task groups has advantages as compared to single items: hypotheses regarding the underlying wrong strategies or misinterpretations can be examined for all tasks in a group. However, strategies can also change within one group.
The limits make it clear that for more detailed analyses qualitative methods are necessary. Last but not least, the children's lacking notes (no part results, etc.) do not allow much information about present competences. Thus additional qualitative examinations should also be carried out for the competence-oriented diagnostic.

5 individual cases

In addition to the written test, clinical interviews on one item of each group have been carried out as well. In the following, the results of the written tests are referred to, completed at some points by findings from the interviews, which can partly explain strategies but also false solutions. The combination of these two methods leads to further findings (cf. to this also Scherer, 1996).

In the following, three individual cases are presented to obtain an impression of the learning profile of individual children. All three students could be classified as masters in two groups, but obtained different numbers of points.

Karsten: dependence on number values

Student S 9 (Karsten; 9 years) has solved all in all 19 of the 40 tasks correctly and is above-average (fig.8).

	CM	CM (MENT)	CRM	CFM	CD (DIS)	CD (PART)	CRD (DIS)	CRD (PART)
S4 Julia	5	0	0	4	1	0	0	0
S6 Vladimir	5	0	0	2	1	5	0	0
S9 Karsten	5	3	2	2	2	4	0	1

figure 8

Group 1a, countable multiplication, is solved completely right, while in group 1a* (countable multiplication with mental completion) three of the tasks are solved correctly. The results of the other two tasks diverged by 1 from the correct result. For group 1b, Karsten worked out two right solutions ($2 \cdot 8$ and $5 \cdot 5$). For the other tasks, it seemed that he guessed the solution, giving pure tens as the result. How did he solve $5 \cdot 3$ (cf. fig. 4) in the interview situation?

I: This boy has played a dart game. He has thrown several times, and then hit the board. And you should tell me how many points he scored in total.

Karsten: (tries to work out the task with the help of his fingers; extends three fingers of each hand] I don't know it at the moment. Don't know.

I: Mhm. Then we will do it together. What did you want to do?

Karsten: Calculate these here together (points to the arrows).

I: Mhm. What's the meaning of one arrow, how many points does it mean?

Karsten: Three.

I: Mhm. Then you can start there below, can't you.

Karsten: (starts writing; fig.9).

I: Yes, that's a good idea.

Karsten: (always notes three points for three scored points and separates these by a short vertical mark; counting softy).

I: You can count out loud.
Karsten: One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen.
I: Mhm.

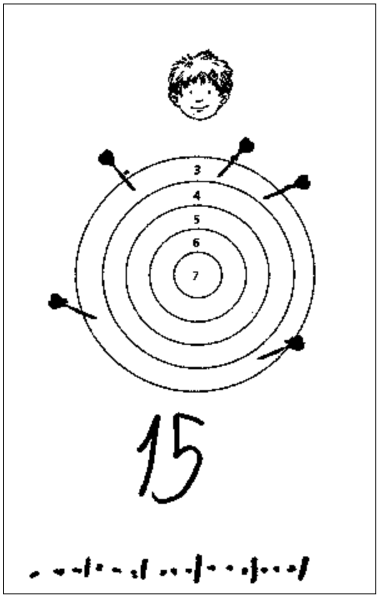

figure 9: Karsten's notation for the context-related multiplication problem

Karsten doensn't have a sufficient number of fingers to work out this task. The interviewer indicated in a very vague manner to start at the bottom, and Karsten develops a suitable notation (fig.9). With context-free multiplication (1c) Karsten again obtains two correct results. Surprisingly, now the tasks 8 · 3 and 4 · 7 are correct. The problems of countable division are solved with different outcomes. While the context of partitioning is managed well, there are only two correct tasks in the context of distributing (20 : 4 and 8 : 4). As wrong solutions emerge:12 : 3 = 3, 30 : 5 = 8, 16 : 2 = 2.
Context-related divisions dominate neither in the context of distributing nor in the context of partitioning. With the exception of the right solution 8:4 = 2, Karsten always gives the divisor as the result. Summarizing his results, one could see that there is just one group in which he did not obtain any right solution. Within several groups, the number values are the influencing factors for solving or not solving a task. He understood nearly all types of problems and could at least solve one or two tasks of a group. His dominant strategy was calculating with the help of his fingers.

Julia: dependence on context relations

Julia, a below average child (S 4; 9 years) solved 10 tasks correctly. Looking at the number of correct solutions within one group, an 'all or nothing' pattern becomes obvious (fig.8).

All tasks within countable multiplication (1a) are solved correctly. With group 1a* (countable multiplication with mental completion) all her results are close to the quantity of visible windows. With all problems of the context-related multiplication (1b), she names the multiplier, here the number of arrows, as the solution.

With the items in group 1c Julia makes one mistake ($5 \cdot 3 = 20$): Instead of $5 \cdot 3$, she probably calculated $5 \cdot 4$ or $4 \cdot 5$, using the commutativity. During the interview Julia worked out the task $8 \cdot 3$ with the help of her fingers without any problems. She calculated $3 \cdot 8$, again using the commutative law. In the groups of countable division only one result is correct. In most cases, Julia names the divisor as the result.

For nearly all tasks of the context-related division, she worked out the subtraction instead of the division. During the interview it became clear that the operation was not chosen in a mechanistic way, such as guessing the operation for the given numbers 12 and 3. She argued, referring to the given context that 3 children could take one gondola and that 9 children would remain at the cable railway and would have to wait. In summary, her results show a dependence on the kind of context: If she understands the given problem well, she is able to solve at least four out of five tasks of a group. The number of correct solutions does not depend on the given number values.

Vladimir: different dependences

Vladimir (10 years) is also one of the students below average (solving 13 out of 40 tasks correctly (fig.8)). Referring to different problems different dependences become obvious and the interplay of tests and interview is to be illustrated in an exemplary way.

All countable multiplication tasks (1a) are solved correctly. For countable multiplication with mental completion (1a*) none of his results is correct. Vladimir only counts the visible, sometimes partly covered windows.

If children do not take the hidden windows into consideration, they can still have thoughts about the context. In the interview, Vladimir also counted only the visible windows. The later thematisation of this problem reveals that Vladimir has indeed thought about the facts of the case and possibly recognized the field pattern. However, he argued that there cannot be any windows behind the bush, because one cannot see anything then.

I: Mhm. So can you imagine that there are also windows behind the bush?
Vladimir: (negatively) Mm … Doesn't work.
I: Why does that not work?
Vladimir: One doesn't see anything there.
I: Yes, but the bush can also … there can still be room in between.
Vladimir: Yes, there they can cut it.
I: Mhm. So could you draw the windows, which could be behind there?

Vladimir was able to do so and could solve the task. For the context-related multiplication (1b), during the written test, Vladimir always gives the answer 23. Probably he added all the numbers on the board (calculation error included). The interview gave more insight in his underlying thinking.

Firstly, Vladimir again added all the numbers on the board (3 + 4 + 5 + 6 + 7). He got the result 24, including again a calculation error. The interviewer asked him about the meaning of the arrows. Vladimir then added 5 to his first result and got 29. The interviewer started anew simulating the game: Imagine that we both play this dart game. One arrow means 3 points. Vladimir at once calculated the 'threes' together, again with a calculation mistake, and finally got the result 16. The interviewer reflected this new result:

I: Why did you do it another way?
Vladimir: Because … We have played now.
I: Yes. … And what's now the correct result? If you want to know how many points the boy scored in this dart game?
Vladimir: Twenty-nine.

Obviously, for Vladimir playing a game does mean a specific world, whereas the solution of a mathematical task takes place in another world, probably in a kind of 'mathematical world'.

For the context-free multiplication he obtained two correct results. For the others his results diverged by 1 or 2 from the right solution. Within the group of countable division/distributing he names the divisor in four of five cases. One task, 30 : 5 is solved correctly. Surprisingly, all countable division tasks in the context of partitioning are solved correctly. For the context-related division no right solution occurs. His strategy remains unclear.

In summary, there is no over-all pattern in his solving processes. It depends on the context and the given operation, but also on the given numbers.

6 conclusions for instruction

support of own strategies

For a successful solution, also of problems that are unfamiliar at first, it seems essential to encourage especially learning-disabled children in their own methods. These should be explicitly made the subject of discussion in class (cf. Ter Heege, 1983, p.12). Only in this way do these children learn that they themselves can solve given tasks or problems in general with their own ideas (cf. also Ter Heege, 1985, p.380).
Especially for the solution of word problems or context problems in general, the own notations and independently developed strategies play a central role. The knowledge gained in this way can be easier remembered and applied and it also contributes to the support of self-confidence and independence (cf. also Isenbarger & Baroody, 2001, p.468).
The lack of self-confidence in particular presents a big obstacle with learning-disabled children. Even though they are potentially able to solve the given tasks, lacking self-confidence or the fear of failure makes them fail.
The different achievements of the students regarding the division interpretations ‘distributing’ and ‘partitioning’ make obvious the necessity of the aware teacher’s role:

> The teacher must be aware of the differences in order to help those students for who the ability to apply the division is not yet situation-independent.
> (Hefendehl-Hebeker 1982, p.39)

Experience has shown that different strategy profiles emerge with children of one class, even though they had the same instruction (Spiegel & Fromm, 1996). With classes of a school for learning-disabled children with different school careers for individual children, one must assume even more difficulties, which also showed up with these exercise treatments. The concept of small steps and of the same steps, which is still common in this form of school, certainly does not represent a suitable instruction- and support-conception.

support of base knowledge

Certain base abilities are essential for solving multiplication and division problems if one does not want to be restricted to a mere learning by heart of the multiplication table. These are, for example, counting in steps, addition table or also subitizing. Tests and interviews have shown that this is where a big source of error is situated, especially when the children used their finger-counting methods.

effective use of representations

The representations that are central to multiplication and division should be made the subject of discussion in class and their advantages and disadvantages should be emphasized with the help of typical examples. An effective illustration is represented by structural fields, which however have to be used meaningfully. Here, teachers are also asked to identify and if necessary substitute less meaningful representations, and in any case to complete them with more suitable ones.

relation-rich learning and practising

This is about relations between single tasks (Ter Heege, 1999), between the single operations (multiplication and division, but also between multiplication and addition) as well as about relations between the different levels of representation (cf. Scherer, 2002). Only like this knowledge, which is not immediately accessible anymore, be reconstructed effectively.
This of course requires that there is importance attached to the understanding of the relations (Ter Heege, 1999; Van den Heuvel-Panhuizen (ed.), 2001, p.76ff.). Taking the children's non-uniform profiles into consideration, a comprehensive introduction, and learning and practising in many relations, are advisable already because of this.

more conscious selection of numerical data

In order to do justice to individual achievements on the one hand, a variety of easy and more demanding problems is recommended. This can, among other things be realized also by means of so-called open problems. The influence of the number values became obvious in the study. On the other hand, certain relations such as exchange tasks or derived tasks are to be explicitly practised by means of selecting the numerical data, as well as of course problems with 1 (as very easy) or problems with zero (as 'supposedly' demanding problems).
The frequent misconception $3 \cdot 0 = 3$ (in analogy to addition) can for example be talked about using the context of the dart game (cf. fig.4). It should be avoided that the children acquire a rule without meaning (for example 'Three times zero is equal to zero'). Thinking about number values and their more conscious use should not be underestimated.

7 closing remarks

Teaching practice in special education and also for low achievers in regular school still proceeds in a small-step-way, without making use of central relations, which can ultimately represent a learning help. With studies such as the one presented here, it becomes obvious that such an instruction conception cannot do justice to the different students: it neither deals with existing difficulties, nor makes the application of individual abilities possible for the children. It remains to be hoped that in the future differences between students will be taken more seriously, and that at the same time support will be given to improve their flexibility in solving different mathematical problems.

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