# On German primary children's solutions to addition and subtraction

*- Problems with Three-Digit Numbers -*

C. Selter
Pädagogische Hochschule, Heidelberg

## 1 introduction

The present paper describes the methods (mental, informal written, standard), the success and the strategies almost three hundred primary students used while working on six addition and six subtraction problems with three-digit numbers.

| **problem** | **example for a skillful strategy** |
|---|---|
| 527 + 399 | auxiliary task (+ 400; – 1) |
| 199 + 198 | auxiliary task (for example 199 + 200; – 2) |
| 250 + 279 + 250 | combining (250 and 250) |
| 119 + 120 + 121 | combining (119 + 121) or balancing (3·120) |
| 286 + 437 | no specific one |
| 345 + 634 | no specific one |
| 845 – 399 | auxiliary task (– 400; + 1) |
| 649 – 347 | auxiliary task (49 – 47; or adding up: 47 + --- = 49 |
| 701 – 698 | adding up (698 + --- = 701) or auxiliary task (– 700; + 2) |
| 610 – 590 | adding up or auxiliary task |
| 836 – 567 | no specific one |
| 758 – 515 | no specific one |

figure 1: the twelve problems

These twelve problems were administered repeatedly by means of a class test: in February (grade 3; nine-year-olds) before the standard methods were introduced, in June after they had been dealt with and in October at the beginning of grade 4 (see Selter 2001 for an extended paper).
Six addition and six subtraction problems were selected (fig.1). Four problems each were constructed in order to offer the possibility to make use of a strategy of skilful number arithmetic, two additional problems were control problems which did not lend themselves to being solved the same way. One of these was constructed as an easy one requiring no 'carrying', the other one required 'double carrying'

# 2 methods

The solutions were categorized according to the main methods used: standard algorithm, mental (no writing) and informal written. A fourth category was called mixtures and was meant for those cases where children used several methods, for example first to add two numbers mentally and then to add the third one by means of the standard algorithm.

### different dates

Figure 2 shows that the standard methods were given clear priority (between 53 and 60 percent) by the children after their introduction following the first test in February.

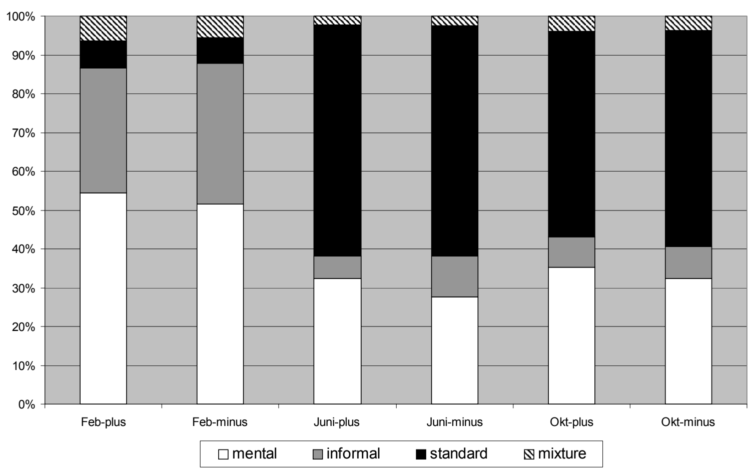


figure 2: different methods used in February, June and October

The informal written methods almost completely disappeared (as a rule not more than 10 percent), whereas mental arithmetic was still used by approximately one third of the children.
Taking into account the great amount of time given to the standard methods in comparison to informal written and especially to mental arithmetic it is surprising that so many children used mental arithmetic. It is remarkable that there were no big differences in this respect if addition and subtraction are compared. However, it should be noted that, as a rule, subtraction translated into slightly less mental and slightly more informal (February and June) and then standard written arithmetic (October) repectively.

### different problems

The data in figure 2 are average scores for all addition and all subtraction problems. Thus, it is necessary to examine how the various methods are distributed if one looks at single problems. The analysis of the data suggests that the children showed a stable decision pattern in all three tests. The October test serves as an example in the following (fig.3).

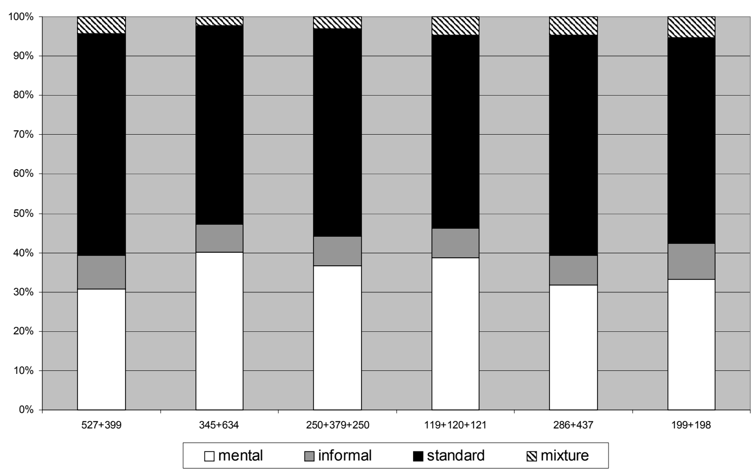


figure 3: distribution of methods in October, addition problems

The existence of a general trend does of course not translate into the nonexistence of performance differences. For example, 345 + 634 (no carrying required) was solved by 40 percent mentally, whereas for 537 + 399 (‘double carrying’ necessary) the percentage was 31,7 percent. Slightly bigger differences could be observed for the subtraction problems, the most extreme between 845 – 399 (26,7 percent mentally) and 610 – 590 (40,5 percent mentally). Nevertheless, as a whole the data for the problems did not spread that much around the average as expected. It seems as if the children did not make a task-specific decision as to which method to take. This means for example that problems like 701 – 698, 845 – 399, 527 + 399 or 250 + 379 + 250 were tackled by more than 50 percent of the children by means of the standard algorithm. None of the four addition or subtraction problems bearing the possibility of applying a skilful strategy was solved by more than 40 percent mentally and by more than 10 percent by means of an informal written strategy.

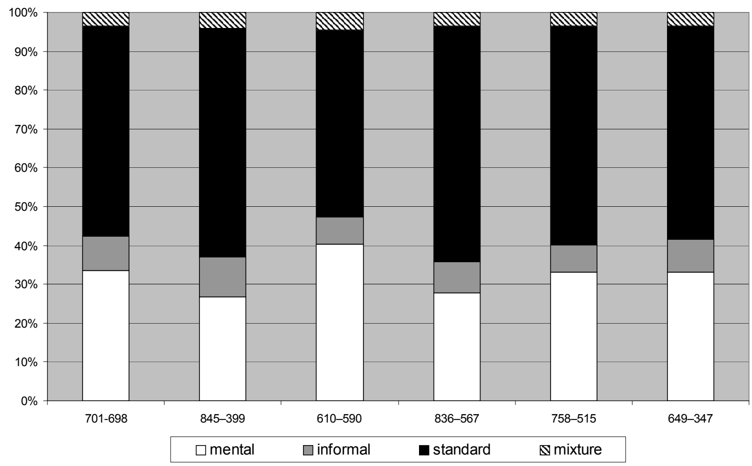


figure 4: distribution of methods in October, subtraction problems

# 3 success

## different dates

Figure 5 shows the percentages of correct solutions in February, June and October. For addition as well as for subtraction problems an increase of about 20 percent can be established.

| | February | June | October |
|---|---|---|---|
| addition | 61,6 % | 78,3 % | 82,8 % |
| subtraction | 41,2 % | 57,5 % | 64,4 % |

figure 5: percentages of correct solutions

A relatively stable difference of 20 percent can also be observed if one compares addition and subtraction. In addition, it should be noted that generally speaking about three third of the addition and subtraction problems were solved correctly at the beginning of grade 4.
It is surely not satisfactory that only roughly 60 percent of the addition and 40 percent of the subtraction problems were solved correctly in February, although mental and informal written number arithmetic had been dealt with in the lessons at that time.

### different problems

This deficit becomes even more obvious in figures 6 and 7, as the fact that some problems caused big difficulties in February cannot be overlooked. Only nearly or just over half of the children solved addition problems requiring a 'double carrying' correctly.

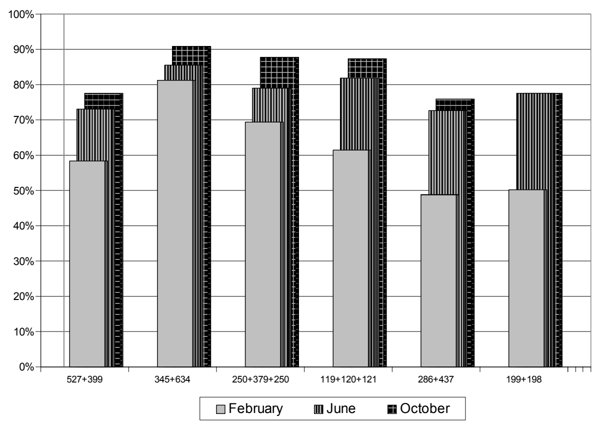


figure 6: percentage of correct solutions for addition problems

Something similar applies even more to subtraction: only 16,1 percent (26,2 percent) correct solutions for 845 – 399 (836 – 567) simply is not an acceptable result. In June, an overall increase of correct solutions can be observed. For 119 + 120 + 121 we can state almost 20 percent and for 199 + 198 not less than 27,5 percent more correct results. For subtraction an increase is visible as well, ranging from 35,1 percent (845 – 399) down to 4,6 percent (610 – 590). An analysis of the October results reveals that there was no regression, but also no big progress after the holidays. Besides, it should be noted that the percentage of correct solutions for some problems which held the possibility of skilful arithmetic was considerably: only 48 percent for 701 – 698, just 54 percent for 845 – 399 and not more than 60 percent for 610 – 590.

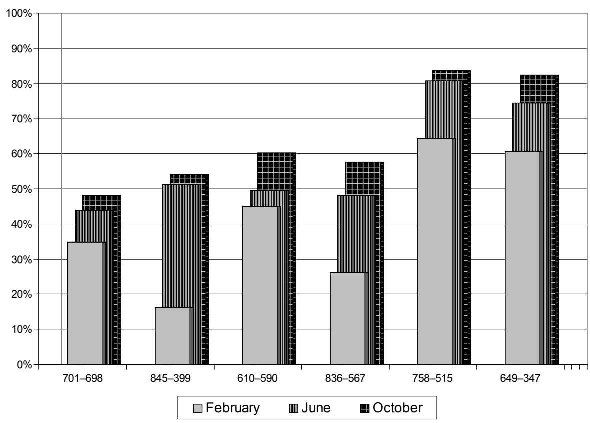


figure 7: percentage of correct solutions for subtraction problems

Something similar can also be observed for the respective addition problems. Finally, it is worth mentioning that the percentages did not differ that much for addition (between 76 percent and 91 percent) whereas bigger differences could be observed for subtraction (from 48 percent to 84 percent).

# 4 strategies

The number arithmetic strategies to be observed were coded according to the guidelines listed in figure 8. It should be kept in mind that the data presented in the following are, in most cases, related to those solutions where informal written arithmetic was shown. It is obvious that we cannot say anything about the mental strategies that were used by almost 30 percent of the children. In consequence, this means that we are speaking about one third (February) or not more than 10 percent (June and October) respectively, of the solutions.

The strategies ‘stepwise’ and ‘htu’ (hundreds, tens, units) were classified according to the two versions listed above. It is of course possible that students indicated the use of the shorter version 2 and solved the problem by using version 1. Here, we encounter the general and well-known problem that an analysis of written notes cannot translate into more than a justified hypothesis about the student’s thinking.

| strategy | comment |
|---|---|
| htu, version 1 | unit per unit, e. g. 345 + 634 → 300 + 600; 40 + 30; 5 + 4; 900 + 70; + 9 |
| htu, version 2 | tens and units or hundreds and tens simultaneously, e. g. 300+600; 45 + 34; 900 + 79 |
| stepwise, version 1 | units separately, e. g. 345 + 600; +30; +4 |
| stepwise, version 2 | tens and units or hundreds and tens simultaneously, e. g. 345+600; +34 |
| stepwise and htu | e. g. 300+600; + 45; + 34 |
| auxiliary task | e. g. 845 – 399 → 845 – 400; + 1 |
| adding up | e. g. 845 – 399 → 399 + ? = 845 |
| simplifying | e. g. 845 – 399 → 846 – 400 |
| combining | e. g. 250 + 379 + 250 → 500 + 379 |

figure 8: coding guidelines for the strategies

**different dates**

Figure 9 displays the information about the occurrence of the different strategies in percentages (for the three main strategies) and in absolute numbers. With respect to 'stepwise' and 'htu' it should be noted that the first and the second term of the sum represent version 1 respectively version 2.

| | February | | June | | October | |
|---|---|---|---|---|---|---|
| total | 1168 | | 324 | | 284 | |
| htu | 277 + 286 = 563 | 48,2% | 60 + 110 = 170 | 52,5% | 88 + 89 = 177 | 62,3% |
| stepwise | 129 + 281 = 410 | 35,1% | 16 + 65 = 81 | 25,0% | 25 + 44 = 69 | 24,3% |
| stepwise and htu | 147 | 12,6% | 38 | 11,7% | 11 | 3,9 % |
| auxiliary task | 23 | | 27 | | 22 | |
| adding up | 6 | | 2 | | 2 | |
| simplifying | 4 | | 2 | | 3 | |
| combining | 15 | | 4 | | 0 | |

figure 9: occurrence of strategies

This figure shows that 'stepwise' and 'htu' were used by the great majority of students. It should not be forgotten that for obvious reasons more than three times as much data were produced in February than in June or October. The increase of the occurrence of 'htu' and, in return, the decrease of 'stepwise' can possibly be explained by more children using 'stepwise' as a mental strategy (instead of informal written). However, at the present stage this is no more than a hypothesis to be confirmed or rejected by the interview study. It is interesting that both strategies were shown more often in version 2 than in version 1.

This means that the children used a strategy version they were not taught, as only version 1 was dealt with in class. The same was true for the mixed strategy 'stepwise' and 'htu' that did not occur in teaching. Once again, we witness the phenomenon that children develop their own methods. Very few students used the strategy 'auxiliary task', especially in June and October.

However, they did not do this in the expected way (845 – 399 → 845 – 400; + 1), as most of them solved a different task, which appeared to be easier, before they did the required balancing, as for example: 845 – 399 → 800–399; + 45.

The other three strategies listed in figure 9 were used by less than 1 percent of the students. In other words: the strategies of skilful arithmetic according to which four of the six problems each were constructed could barely be detected in the informal written methods. It is an interesting question whether this is also true for the mental arithmetic solutions.

### addition and subtraction problems

The general trends appearing in the preceding section can also be detected if one takes into account single problems. Some examples shall be outlined in order to illustrate this observation: '327 + 599' or '701 – 698' were almost without exception solved by means of the three main strategies of informal written arithmetic. It was not more than four times that the use of 'auxiliary task' could be observed in the expected way in all three tests.

| | Feb. plus | Feb. min. | June plus | June min. | Oct. plus | Oct. min |
|---|---|---|---|---|---|---|
| htu | 316<br>(c: 76%) | 247<br>(c: 34%) | 80<br>(c: 80%) | 90<br>(c: 32%) | 100<br>(c: 83%) | 77<br>(c: 22%) |
| stepwise | 129<br>(c: 58%) | 281<br>(c: 60%) | 20<br>(c: 60%) | 61<br>(c: 53%) | 25<br>(c: 64%) | 44<br>(c: 48%) |
| stepwise & htu | 82 | 65 | 26 | 12 | 9 | 2 |
| sum | 527 | 593 | 126 | 163 | 134 | 123 |

figure 10: occurrence of the main strategies for addition and subtraction problems

For obvious reasons not a single child documented that they used the 'adding up'-strategy for solving the problem '701 – 698' in their writings. It is striking that 'htu' was used quite often for subtraction problems although it had never been dealt with during lessons (figure 10).
The percentage in brackets represents the relative occurrence of correct answers. Of course, one has to take into account that the absolute numbers differ quite considerably in part. In addition, assumptions about all children cannot be derived from these data as these were produced by those children who indicated the use of an informal written method.
However, the discrepancies should be noted with respect to the success rates (1) between 'htu' for addition (roughly speaking 80 percent correct) and subtraction (about 30 percent correct) as well as (2) between 'htu' (about 80 percent correct) and 'stepwise' (60 percent correct).

# 5 implications for teaching

It should be borne in mind that a study conducted within the classroom - like this one - does no more but equally no less than to shed light on the procedural path taken by students inside school. It is well known that these can differ considerably from those which are applied in out-of-school-situations and that students often behave in school as they think it is expected from them. However, as we know how the students behaved in the school context, we can draw some conclusions for the teaching of arithmetic (in the number domain up to 1000) which in principle are typical for the German situation and indirectly relevant for the teaching of arithmetic in general.

– *Strengthening the role of number arithmetic*
  The existing dominance of algorithms and the neglect of number arithmetic in mathematics instruction is questionable. The relatively low increase of success rates after the introduction of the standard methods - given the time invested in them - the difficulties the children still had with informal written and mental arithmetic - especially with subtraction - and the development of an attitude to use the algorithm for each problem that quite some children showed; these observations do not justify such an extensive treatment of standard written methods as was typical for the participating classes. More time should be devoted to mental and informal written methods.
– *Constantly encouraging reflection on method use*
  The inflexibility that could be observed calls for an approach to arithmetic where constant reflection on method and strategy choice is a nat-

ural element of the classroom culture. Quite some of them solved each problem mentally or each problem by means of the standard algorithm, regardless which number relations could have been used. Consequently, a dissolution of the sequence ‘first mental, then informal, finally formal written’ which the participating classes took seems to be necessary. Instead, all methods should be made topics in the lessons. They should be compared with respect to their suitability for certain problems (and certain children) from time to time.

– *Continuously developing a sense for number relations*
  Many children appeared to be somewhat blind to the relations between the given numbers. They more or less showed a stable decision pattern with regard to methods and strategies, regardless whether 836 – 567 or 701 – 698 had to be worked out. The results indicate that a feeling for number relations does not develop independently of instruction for most children, but has to be encouraged. It should be an essential component of the classroom climate that students (not only in bare number problems, but especially in context problems) wait a while before they perform a certain calculation and have a closer look at the given numbers.
– *Incorporating children's strategies*
  Several strategies of number arithmetic could be observed that have never been taught. On the one hand, their occurrence is an argument for a conception that gives children the opportunity to learn mathematics in their own ways instead of trying not to take children's mathematics into account. On the other hand, these untaught strategies, such as ‘htu’ for subtraction, the shorter versions of ‘htu’ and ‘stepwise’, the mixture of both main strategies or different ways of using an auxiliary task should become an element of textbooks or didactical handbooks and consequently of the teaching of arithmetic.
– *Establishing forms of short informal written arithmetic*
  Only a few children used informal written arithmetic and if so, they put down their calculations quite in detail. Short informal arithmetic where e.g. just preliminary results are put down could hardly be observed. In addition, almost 30 percent of the children worked out problems like 836 – 567 or 286 + 437 completely in their head - an enormous amount of information to be kept in mind while calculating. In order to be able to compare the ‘efficiency’ of the standard algorithms and mental methods respectively (no writing) short informal arithmetic has to be established as a normal method for solving problems.
– *Revaluing subtraction*
  The difficulties the students had with subtraction problems throughout the whole project were immense. For example, less than 50 percent of

the children were able to solve 701 – 698 correctly at the beginning of grade 4. As these difficulties already existed in February they are to a great extent caused by the neglect of subtraction in teaching practice, if compared with addition. Besides, subtraction is often dealt with as mainly taking away in Germany. The adding up to image (or adding down to respectively) of subtraction has to be taken into account from grade 1 on, especially if further stages in the mathematical learning process are anticipated (e.g. subtraction of negative numbers).

Finally: The results of the present study are in my opinion not in contradiction to results of other studies with most convincing results that clearly demonstrate children's capabilities of going their own mathematical ways creatively and productively, if they are continuously given the opportunity to find these. However, conceptions about mathematics teaching and learning - expressed in textbooks, teacher education or in didactic handbooks - can obviously contribute decisively to the development of a certain inflexibility in thinking or to an attitude that makes children believe that they are to show such a behaviour in school.

## acknowledgements

The present study was supported by the Pedagogical University Heidelberg (PUH) under grant No. 'FRG 1427/71#530201'. The opinions expressed do not necessarily reflect the views of the PUH. I thank my collaborators S. von Itzenplitz, V. Meseth, E. Neuhäusler and E. Ott for their cooperation.

## references

Selter, Christoph (2001). Addition and Subtraction of Three-digit Numbers: German Elementary Children's Success, Methods and Strategies. *Educational Studies in Mathematics*, 47(2), 145-173.