Shortest pathways

- interaction and semiotic apprenticeship in a sixth grade mathematics classroom -

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1 introduction

This paper explores the notion of classroom interaction in relation to teaching and learning mathematics. We address this theme in the context of a one-year study of an heterogeneous sixth grade class attended by 26 students of African-American, Latin-American, and Asian-American backgrounds (ages 11-12), many of who are recent immigrants struggling with English as a second language. Within the year-long curriculum enacted in this classroom, this paper focuses on an one-month long instructional sequence, within the Probability Strand, that involved finding the number of shortest routes from one point to another on various kinds of graphs. The models and strategies developed throughout this sequence were then applied to problems involving road maps and traffic flow diagrams (see Appendix).[1]
Our data source included field notes from once-a-week classroom observations; transcribed videotapes of four lessons; student written work in scrap paper, chart paper, and notebooks; and transcripts of interviews with six students in this class.[2]
Preliminary analysis of this material shade light on the centrality of speech and writing in this setting. Furthermore, it made us aware that the function of discourse was not limited merely to that of a vehicle for communicating finished ideas. Rather, spoken and written linguistic exchanges served the function of ‘mediating participation in activity and simultaneously providing a medium in which activity is represented and thus made available to be reflected on’ (Wells, 1999, p.164).
Relying on Glaser and Strauss’ (1967) constant comparative method, we looked for patterns and regularities in the interactions occurring within and across the activity structures making up the lessons[3] (Lemke 1990). Lemke argues that, in most classrooms the interactions between students

and teachers take on an IRE form made up of teacher initiation (I), student response (R) and teacher evaluation (E). Initial analysis of transcripts found the pattern occurring most frequently in this 6th grade class not fitting neatly into the IRE form, but instead taking on an IRF form where the third term is a non-evaluative ‘follow up’ (F) move (Sinclair & Coulthard, 1975, Wells, 1999). We further classified the follow up moves using Wells’ (1999) coding scheme, to which we added a few categories that emerged from our data. This coding scheme, designed for inquiry-based classrooms, allows for analyzing the various socio-semiotic functions of the moves made by students and teacher in oral as well as written interactions.

This paper initially focuses on four whole class episodes within the unit above outlined. While in the first two of these episodes teacher and students jointly frame problematic situations, in the last two the teacher frames writing activities around students’ mathematical contributions. Following the analysis and discussion of these fragments, we highlight and further elaborate upon two central themes weaving throughout the pathways instructional sequence. In the ensuing analysis and interpretation of our data, we make explicit our theoretical and analytical tools and illustrate and discuss our findings providing all along examples of students’ comments and mathematical work.

2 connecting ideas

> One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else. (Polya, 1957, p.15).
>
> Melika: A past idea is something that you did before and you realize that it can connect to something that you have just done. That’s why Ms. H. would tell us to remember things. Because things that happened in the past can relate to things in the present or in the future. It can all tie together in one way and when it connects, you don’t realize it until you think about it.
>
> Kaylan: Ms. H. knows that we can do the problems with our minds, if we work hard enough and think about what we did in the past, about the different kinds of solutions that we had, by starting out with one small thing and building upon it into more solutions.

Over the previous two weeks, students had been working on counting pathways in square lattice diagrams. This new situation entailed a shift to a different kind of graph.

Teacher: I'm gonna give you something that looks a little different than what we have done so far, but it's actually very, very, very similar. It's gonna be really quick. You don't have to copy this down.

The teacher draws the following graph on the board (fig.1):

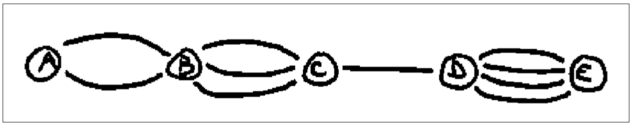


figure 1

Carlos: What's that? That's so big!
Teacher: Well... each of the circles represents a little city. And I'm just labeling them A, B, C, D, E. And these lines represent roads or bridges that connect the cities together. So, guess what my question is? Nyree?
Nyree: Mm... mm... how many… How many bridges are there to get to each ... city?
Teacher: Anyone wants to modify that? Arlene?
Arlene: I think... I think the question is gonna be how many miles does it take to get to E. And then you gonna show us the key of one mile looks like, and then... we have to figure it out.
Teacher: That sounds like an excellent idea. But, since I don't have the key, maybe we will need to modify it a little bit. But, I will save that idea for later. I think that's really good. Sonia?
Sonia: Oh, mine's like… I think that... the lines, like the roads.
Teacher: Yeah.
Sonia: We're gonna see how many lines it takes to get to E.
Teacher: From where?
Sonia: From there, from A.
Teacher: Ok. So, how many pathways are there from A to E. Three minutes. Let's go.
Arlene: I don't get it! Can you go like this? (makes a gesture in the air with her hand indicating a trajectory)
Teacher: You can go like this (gesture pointing to one possible pathway) or like this (gesture indicating another pathway), or this way, and I get to E. So, how many different ways for me to get from A to E? I'm gonna...
Beatriz: I wanna add something. I recognize something.
Teacher: What do you recognize?
Beatriz: This is like the bus problem that we did ... from the bus station to school. It's like that!
Teacher: Oh, really? So, I wonder if anything from that you can borrow that we can use to solve this problem. Ok? Three minutes. Think about that.

analysis

Ms. H. drew a graph, indicated that its vertices represented cities and lines represented roads or bridges, and qualified the new situation as different and yet similar to what had been done so far. While valuing Nyree' and Arlene's contributions, both of whom failed to solve her semiotic puzzle, the teacher elicited further suggestions. Eventually, Sonia and Ms. H. jointly framed the question. Responding to Arlene's request for further information, the teacher traced two acceptable pathways on the graph while reformulating her own previous contribution. Whereas the initial formulation may lead to counting the graph's edges, the second one defines it as a task involving ways to go. As opposed to roads, pathways are not already there but appear only as one travels from point to point. Next, Beatriz claimed to have seen a connection between this and the situation of walking from the bus station to the school, the first problem in the ongoing instructional sequence. Acknowledging her recognition, Ms. H. suggested that students consider whether 'anything from that' could be borrowed for the present problem.

Ms. H. invited her students to formulate a question about the given graph that would fit within the ongoing instructional sequence, one that involved organizing 'ways to go' situations while satisfying the 'need, and skill to proceed systematically when counting' (Freudenthal 1978, p.207). The teacher did not tell the students what to do, nor did she show them where to start. Instead, she presented an indeterminate situation in the form of a drawing, a few verbal clues, and a request for them to guess the problem she had in mind. The question 'How many pathways are there from A to E?', co-formulated by Sonia and Ms. H. did not suffice to orient Arlene. The teacher then reworded it as an utterance marked by people engaged in actions in the world: 'How many different ways for me to go from A to E?' Aside from yielding a more imaginable situation, this reformulation invited students to connect the present problem with previous ones also involving the need for someone to travel between two points. While counting pathways from the bus station to school, students had reinvented the general rule for finding the number of shortest paths on a square lattice diagram of any given dimension and, later on, recognized in it a piece cut out of the larger Pascal triangle (Polya, 1962). The teacher's follow up to Beatriz's recognition invited students to compare and contrast present and past situations with an eye to considering whether or not an already appropriated tool was applicable.

discussion

In the episode above, the teacher presented an artifact and engaged students in a conversation regarding its possible meanings so as to orient them into the problem. She did so through a series of follow up moves, eventually echoing and slightly transforming student responses into the

terms of an imaginable situation, one populated by persons with needs and goals. The collective effort to make sense of what at first appeared as an indeterminate situation led to the formulation of a worthwhile mathematical question.[4]
Whole group activities aimed at framing problems and generating questions were a central component of mathematical inquiry in Ms. H.'S class. Whole group discussions did not occur only at the endpoint of the problem solving process in the form of the presentation of solution strategies, but instead began early in an inquiry, and continued intermittently throughout the various phases of the process. These conversations, wherein entry points were provided and imaginable situations shared, guided students towards making connections between past and present problems. Framing problems through whole group interaction contributed to making public likely threads of continuity within the instructional sequence at the level of prosaic imaginable situations.

3 details and models

> Consider your problem from various sides. Emphasize different parts, examine different details, examine the same details repeatedly but in different ways, combine the details differently, approach them from different sides. Try to see some new meaning in each detail, some new interpretation of the whole. (Polya 1957, p.34).
>
> Kaylan: A model is just an example, a preview of what you're gonna do, a preview of what your final answer is going to be.

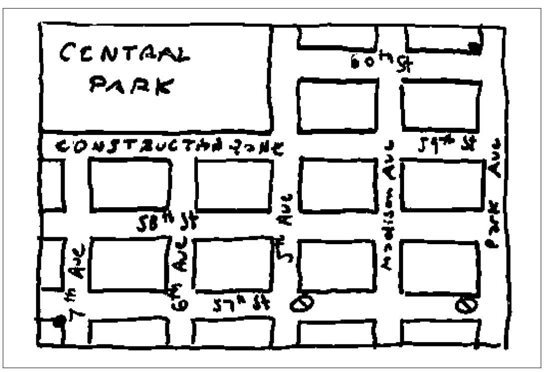


figure 2

Thus far students had worked on pathway problems of two types: lattice diagrams and graphs. Due to the constraints to walking caused by the (red) cross and the construction zone, this situation did not fit neatly into either the lattice diagram or graph, but instead combined elements of both models (fig.2). Below is the beginning of this lesson:

Teacher: I drew this map of New York City, near Central Park. This is 58th street, this is 59 street, Fifth Avenue, Sixth Avenue, Seventh Avenue, and Madison Avenue. Now, guess what? This is what happened: I went shopping at Bloomingdale's. My husband called me on the cell phone and said: I'm here with the kids at the corner of 57th Street and 7th Avenue. Can you come over and meet with us?
Lok Kei: I have a question!
Teacher: Hold on. I want to know how many ways. I need the shortest way, how many ways are there for me to walk from East, from 60th and Park Avenue to 57th and 7th Avenue?
Arlene: Take a cab!
Teacher: I wanna walk.
Ming: You are on the black dot, right? And where do you wanna go?
Teacher: I am over here in Park and 60th and I want to go... I need to go to 57th and 7th Avenue. I want you to ask me questions right now, questions about this map only.
Lea: So, how many ways to go, right?
Teacher: And I need the shortest way. I need more questions about this map. Nobody looked at these symbols yet.
Beatriz: Can you go through the construction zone?
Teacher: Can I go through the construction zone? What do you think?
Chorus: No!
Teacher: No, we cannot. Arlene?
Arlene: Can you go through the middle of the street, like can you cut like this? (gesture of diagonal movement).
Teacher: Do you mean like this? (gesture indicating an imaginary 'as the crow flies' line on the map).
Arlene: Yes.
Teacher: No, you cannot.
Ming: What does the red-cross mean?
Teacher: What do you think it means?
Chorus: Stop.
Teacher: There was a fire earlier that day and they closed this street. So, I cannot go through there, unless I live there. Unless you are a resident, they don't let you go through this part.
Lok Kei: Ms L., what direction can you go?
Teacher: I won't answer that question. Because I said I want the shortest way to go meet my husband and kids. I'm not gonna answer your direction problem. You need to figure that out. And, as a matter of fact, could there be only one shortest way?
Ming: What does it mean? What do you mean by shortest way?
Teacher: That means I don't want to walk all the way down and then up again and then have to walk down again ...
Ming: Can you go down and across? About how many times is short?
Teacher: I don't know. Think about that. And that has everything to do

> with Lok Kei's question, which direction you are supposed to go. Common sense, use your common sense. You are trying to get to a place quickly, what direction are you going to go? And I need you to create… this map is very complicated. If I ask you to draw it, it will take you another fifteen minutes. I don't want you to draw this map. I didn't give you a Xerox copy because I didn't want you to count like that. I want you to use symbols to make a modified map in your notebook to help you solve this problem. You have about fifteen minutes.

analysis

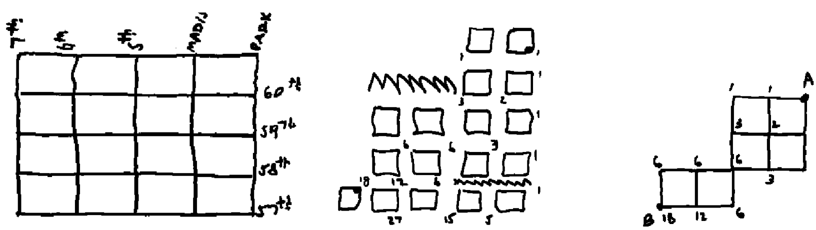
The teacher presented a map and told a story. For some students, this narrative transformed the map from a seemingly random section of the city into a model of an imaginable experience she has undergone. Ignoring Lok Kei's bid to talk, Ms. H. formulated herself the question that framed this as a problematic situation: How many shortest ways are possible for her to reach the destination? While the map was real and the story true, counting the number of shortest paths may have seemed as a rather contrived endeavor. Yet, as noted in the words below, such situations or stories were nonetheless easily imaginable and thus functioned as a scaffold orienting students to imagine or realize the situation at hand:

> Nyree: When she does the math, Ms. H. stops and talks about what it is. She gives us a clear picture of exactly what she's talking about. Let's say she's talking about… walking somewhere, she talks about details. She details what everything looks like. Like if she is trying to get somewhere, she would say she'll pass through a store, something like that, but Ms. D. (previous year's teacher) she would just say she passes something and you wouldn't really understand what she was talking about.

> Melika: The path problems give you a different idea... like you never really noticed when you're walking. You just know where you are going. But once you realize that there are so many ways to get from one point to another, you realize that you've been doing it all along but you really didn't see it... the math in it.

As she opened the floor for student contributions, Ms. H. guided the conversation by requesting questions about the map and then restricting these to questions concerning details on the map. Her call for questions set up the students as the initiators, while she took on the role of respondent. Initially, the teacher used 'telling' to frame the activity in terms of needs and wants: 'I need the shortest way, how many ways are there for me to walk from East, from 60th and Park Avenue to 57th and 7th Avenue? I'm over here in Park and 60th and I want to go... I need to go to 57th and 7th Avenue.' Yet, once the framing was attempted, including her implicit foregrounding of the modeling aspect of the activity, Ms. H.'S discursive moves shifted accordingly from telling to requesting and answering questions.

How did the constraints of the walking situation, symbolized with icons on the map, affect the number of shortest routes Ms. H. could to reach her destination? Which of the maps' features could be eliminated so as to construct a schematized version that would simplify the counting task without altering the referential situation? The teacher guided the asking and answering of the first question in a whole group conversation while the second one was left for the students to struggle with by themselves. As they continued to work on the problem - individually or in small groups - it was intended that the students would appropriate the questions asked by Arlene, Beatriz, Lok Kei, and Ming, using these as guides for the modeling task. Below is the work of three students in this class (fig.3).



Nyree's work **Emily's work** **May's work**

figure 3

discussion

Among the semiotic tools for this modeling activity were requested questions regarding details on the map. The teacher explicitly restricted the discursive moves she and her students were allowed to make, provided relevant information in response to their initiation moves, and often followed up these initiations with additional requests for questions. Ms. H.'s tight control of classroom discourse was not accomplished by means of an increasingly narrow, funnel-like set of questions aimed at eliciting specific answers from the students (Bauersfeld, 1988; Voigt, 1995). Instead, she delegated to her students the responsibility for framing the problem by eliciting in a public forum the kinds of questions she believed would guide their mathematical organization of the given situation (map).[5]
This episode reminds us also that the acquisition of a mathematical disposition requires consideration of all the details of a situation, and at the same time an inclination to ignore those mathematically insignificant in light of the problem at hand (Van den Heuvel-Panhuizen, 1996). This ability to distinguish between mathematical and 'real life' knowledge is far

from evenly distributed among students from different social classes and cultural backgrounds (Bernstein, 1990; Cooper & Dunne, 2000) and thus ought to be an explicit aim of mathematics instruction. A semiotic strategy Ms. H. employed to facilitate the development of this skill and disposition in her students was the creation of a semiosphere (Lotman, 1990) wherein they could practice, with her guidance, the asking of those questions that facilitate the pivotal discrimination aspect of the modeling task.

4 does it all make sense to you?

> If you release a marble at the edge of a bowl you know that it will end up at the bottom of the bowl, but you cannot predict, much less pre-determine, its trajectory - it might spiral downwards or rock back and forth… Thus a situation can be simultaneously closed for the teacher who knows where the marble will end up, and open for the student whose trajectory is all her own. (Brousseau, 1986, p.74)

When you see a lot of different ways that everybody else did, does that help you understand better than if you see only one?

Lok Kei: It gives you an idea of how you can solve it. Say a person did something different from mine, I can make a change in my solution, using what this person did and I can also use it to see if my solution was correct or wrong. That way I can change it and make it better so other people, when they read it, can understand what I did.

Even if your solution is right, would you change the way you did it sometimes, by using someone else's idea?

Lok Kei: Yes.

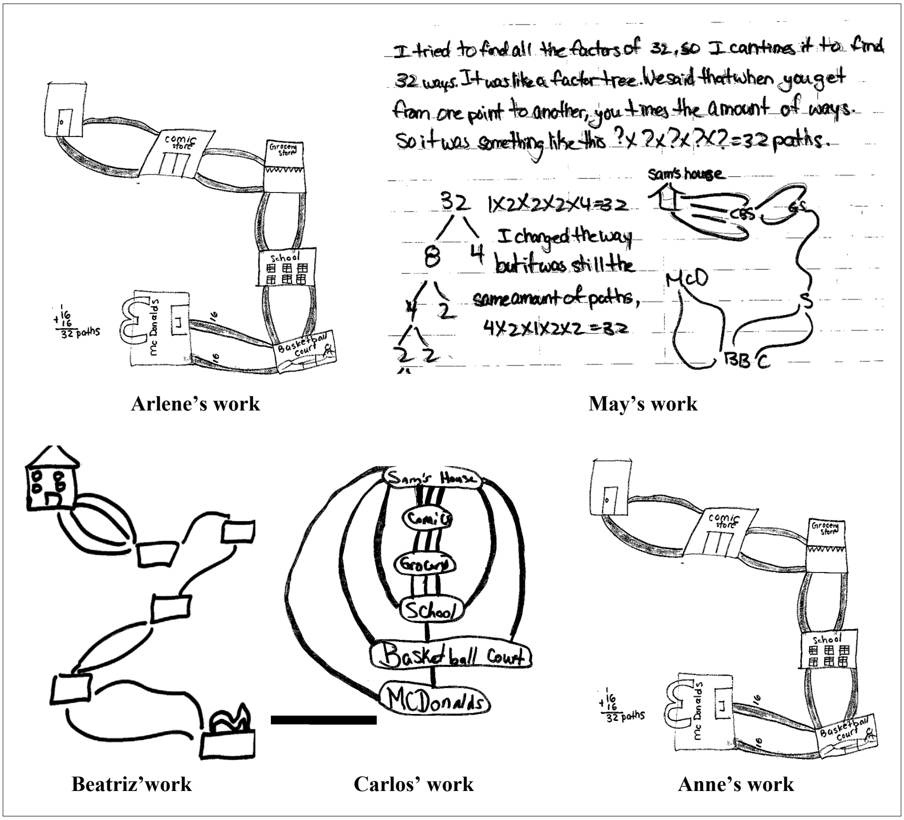
Why?

Lok Kei: Well, because it's making me understand what I'm doing more and gives me more knowledge to how I can change it.

So, it's your process ever fixed and final or is it always open and open to be changed?

Lok Kei: It is open.

The day before, students had been asked to 'make a map connecting Sam's house with the grocery store, the comic book store, the school, the baseball court, and McDonalds, such that he would have exactly 32 different ways to get from his house to McDonald'. This open production (Streefland, 1990) included the explicit constraint that the number of ways for Sam to go from his house to McDonalds had to be 32, as well as the implicit one that all pathways pass through the specified locations. At the start of the next day's lesson, the work of each and all the students was posted on the board. Below are the instructions for an in-class writing assignment followed by five samples of student work (fig.4).



Arlene's work **May's work**

Beatriz'work **Carlos' work** **Anne's work**

figure 4

> I want you to look at each of these pictures carefully. And you should have your notebook in front of you. I need you to write interesting things you notice, questions that you have, and in particular, I want you to think about: Are any of these alike? Are any of these different? If they are, in what way are they alike, or, in what way are they different? (...) Make sure there are 32 ways. You have to check them. I haven't checked any answers. And if it makes sense to you (...) Please, be kind, and don't criticize people's writing style or drawing style, ok? Let's focus on what we need to look at (...) And, obviously, people worked differently. Some people used trial and error and some people thought of it, let's say, more mathematically. Does it all make sense to you? (fig.4)

analysis

Ms. H. asked students to carefully attend to all the graphs, requesting that they offer comments and questions in regard to each. She instructed them further to think 'in particular' about possible similarities and differences among the various pieces of work. This was followed by an invitation for them to evaluate whether or not each graph yields 32 ways. 'And if it makes sense to you' may be interpreted both as 'do you to understand how each person made his/her graph?' and 'do all these productions comply with the given specifications?' By requesting that each piece of work be evaluated in regard to these two demands, the teacher hoped to foreground Beatriz's graph, giving only 16 ways, as well as Arlene' and Carlos's divergent graphs. Attention to the latter two productions - respectively, 4×8 and $3 \times 3 \times 3 + 5$ - would serve as an occasion to discuss the relationship between the structure of a graph and the use of addition, multiplication, or a combination of these as tools for counting pathways on it. Finally, Ms. H. stated 'and, obviously people worked differently', qualifying such differences as being indicative of several levels of mathematical sophistication on a continuum from 'trial and error' to 'let's say, more mathematical[ly].' Among the latter was obviously May's framing of the situation as 'a factor tree.'

discussion

Everyone's work was presented to all the students at the same time. Ms. H. mediated the interaction between each student and the works of all her or his peers through requests for questions and comments as well as directions for evaluating and ordering the graphs. She wanted her students to interact critically with the work of their peers. By requesting that students make sense of the entire collection of graphs, the teacher guided them towards reconstructing the randomly posted set of works into an organized whole. Her guidance included a request for questions and comments, as well as instructions to evaluate, search for similarities and differences, and identify the mathematical strengths and weaknesses of each and all the productions.

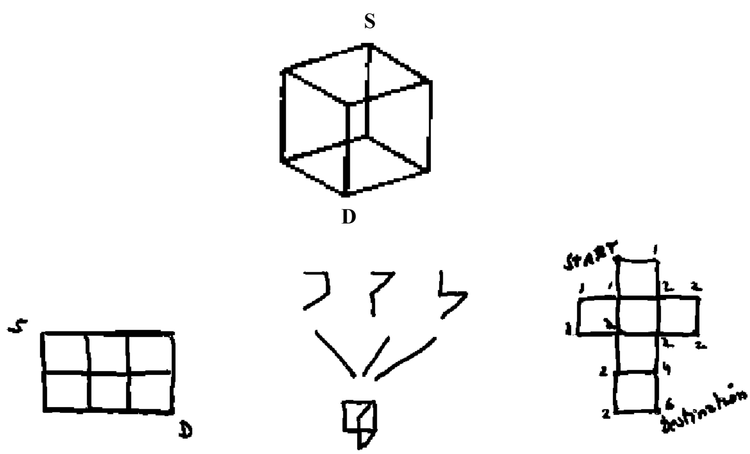
This assignment structured classroom interaction in an IRE/IRF modality. Within this pattern - where the initiation was the open ended task of making a graph that yields 32 pathways and the response was the myriad of student productions - evaluation and follow up moves were shared by the teacher and her students. Postponing for later any verbal explanations of each piece of work, Ms. H. asked students themselves to evaluate and organize the work in front of them and to do so through writing in the privacy of their notebooks in preparation for the ensuing whole group talk. As they interfaced with the exhibited graphs, mediated by the teacher's guidance, students entered into dialogue with their classmates.

5 can you do better?

As we approach the end of this one-month instructional sequence and in line with our ongoing discussion, we offer Nyree's reconstruction of this teaching and learning experience.

For the past couple of weeks we have been doing problems like trying to go from one distance to another. Last week we were trying to figure out how this boy named Denny could get from his house to the sports center. But it was different; she wrote the path different from what she would usually do. She gave us the challenge to try to figure out the different ways there were. Then one day Ms. H. was trying to get to her family from where she was at, but it was like in a grid situation... walking. And we were like… what we saw from the first problem, we tried to use in the second problem, to find the different ways for her to get there. And today it was kind of a little different because she had a box and she said that there was a spider trying to travel from one corner to another. But what kind of threw me off was that it was a spider. It wasn't like using intersections. Ms. H. said how come we didn't use words like intersections like we used in the other problem. But what threw me off today is that it was a spider, and spiders are in jungles so there probably aren't that many intersections.

For homework, students were asked to study the three diagrams representing the spider on the cube situation (fig.5).



Felicia's work Ming's work May's work

figure 5

Below is the teacher's framing of this assignment, followed by an excerpt from Lea's notebook in response to this task.

> I wonder whether these representations really represent this problem... Study these three diagrams. See which one of these relates to the original problem. Do you understand it? Do they make sense? Does it relate to the problem? If not, can you do better? In other words, can you design a better representation for us to share on Monday?

> I just thought of the cube as a map having six different places, but that got me confused. So I just went by the cube. When I found out the answer to this problem, I asked myself if the answer had to do with the number of sides on the cube. I noticed something that could have helped me figure out right away. I could have cut the cube in half diagonally and count the number of bars there were. Each half has three bars and since I cut it in half, I time the three bars by two… […] While some people were sharing, I had thought to myself if I could have really thought of the cube as six places, and I found out that it was true, I could have done that. I just didn't think hard and long about that idea, so I just moved on to a different strategy.

> I think that May's way was interesting, because she broke up the cube into a cross and found out all the intersection numbers. […] I think that what May did was a whole different stage that only her would have thought of. She took something that she knew and used it for this activity. If I had to do this activity again, I would try May's strategy…[…] I think that she probably thought that this was related to the path problem and I agree with her.

discussion

This episode also illustrates the centrality of writing as a modality of interaction in this classroom. Here too students were asked to interact in writing assignments with the ideas, explanations, and diagrams offered by their peers, in a writing activity mediated by the teacher's voice through her instructions and requests for questions. This homework assignment foreshadowed the need for students to be attentive and take notes in class that day, both important necessary conditions for later study (Lampert, 2001) and central elements of teaching and learning in this classroom.

Before interacting in writing with the work of her classmates, Lea reflected back on her own thinking. By reflection we mean here that she asked herself questions. This self-questioning attitude - in itself also a form of interaction - may be interpreted as evidence of internalization of her teacher's disposition. To illustrate Ms. H.'S wondering attitude towards mathematics, we offer below a follow up move she made in a conversation with Lea during small group work.

> If that is the case… what would be a reasonable move. Think about science, you have a question, and you're wondering. (...) You're going to do a

trial. Tell me, what would be a reasonable hypothesis… for this question that you have?

Lea's reflective comment in her notebook - 'I asked myself if the answer had to do with...' - reveals that appropriating the voice of a more knowledgeable and authoritative other is not merely internalizing his or her words, but involves also taking-over the attendant disposition. This notebook fragment points out to yet another modality of interaction in this classroom, one occurring between the voice of a student and that of her teacher.[6]

Lea announces her intention to take on May's way of looking at problems. When working on the map situation, May reconstructed the model of the city into a model for efficiently applying the 'adding intersections strategy'. Leaning back on her prior experiences, May 'broke up the cube into a cross' and 'found all the intersections'. Although May's transformation was eventually proven incorrect in that it yields a model of a cube rather than a model for counting paths along its edges, the study of her idea provided an opportunity for students to critically consider the match between situations and their mathematical representation. Lea appreciates her classmate's mathematical disposition, one marked by a search for ways to simplify situations such that one may apply an already acquired tool: 'She took something that she knew and used it for this activity'.

6 findings

We have come to view learning in Ms. H.'s sixth grade classroom as a semiotic apprenticeship. Building upon the ideas of Vygotsky, (1978, 1986), Rogoff, (1990), Lave & Wenger, (1991) and Wells (1999) defines semiotic apprenticeship as participation in discipline-based forms of inquiry whereby apprentices appropriate cultural tools and practices for making meaning under the guidance of more knowledgeable others. In what ways did the modalities of interaction occurring in Ms. H.'S class further this semiotic apprenticeship process? We found that two features of interaction significantly contributing to this apprenticeship were requests for questions on the teacher's part and the use of writing as a modality of discursive interaction that complements and furthers face to face exchanges.

We offer below two comments by Lea in the context of our interview with her. When asked to describe in what ways math with Ms. H. was different than math with other teachers, Lea offered the following characterization: 'She (Ms. H.) gets us thinking. She asks us questions. Why this is happening…she gets us thinking… she asks us why this is happening, whether we can do this in another way. Or, how are we going to prove that this is right.' Earlier in the interview, Lea commented that 'Ms. H. gives us problems that have a lot of patterns ... (T)here are a lot of observations that you can make.' When invited to further elaborate by giving an example, she referred to a situation where students were involved in searching for patterns on their own: 'Cause, in the pyramid problem, 24 always shows up. But then we…. some of us asked why that happens. Last year we didn't have to think of that.' How did it happen that students were able to appropriate such questions? At first the teacher acted as a model, performing the role of the asker of questions. Progressively, she handed over the responsibility for asking questions to the students, creating guided opportunities for them to imitate her, and to practice this discipline-specific skill in the social contexts of whole group talk and in reflective studying through writing. Below are a few examples of requests for questions.

Teacher: What would you like to share, Beatriz? Everybody should be paying attention. And if you have any questions about what they suggest, what do you do?
Chorus: Write it down.
Teacher: And then?
Melika: Ask.
Ming: It doesn't work because in a 3 by 3 grid, there are twenty ways. So, for a 4 by 4 grid, it has to be more than 20 ways.
Teacher: No questions? I can't believe none of you want to ask Ming something about what he just said!
Teacher: Hold on... Does anyone have any questions?
Emily: How did you come up with this way and how do you know that it would work for all of it?
Teacher: I need you to write interesting things that you notice, questions that you have, and in particular, I want you to think about: are any of these alike; are any of these different? If they are, in what way are they alike, or, in what way are they different…. If you see something, write it down…. It doesn't have to be complete sentences. Either a comment or a question.
Teacher: So, finish writing up. I don't want the answer only. I want you to find a way to explain what kind of problem did you have in trying to figure out this answer. And each of you has to think of one question: What does this problem remind you of? What are you wondering? What is the thing that you're not too sure about?

The teacher's requests did not set up an entirely open field. Rather the kinds of questions she sought are constitutive of a relatively stable and finite set, one that marks the speech genre (cf. Bakthin, 1986) of mathematical activity. When asked to reflect on the mathematics in the pathways problems, Kaylan advanced the following thoughts:

Kaylan: Ms. H. asks us to look for patterns every time we are working on a problem. I decided to use those patterns and think about the solution…. Mathematics is a subject where you're starting from scratch. This is what I did, I started out with something small and then I decided to add on to what I did and when I did that I saw that we are kind of multiplying… or you are adding these intersections. We are using addition to figure out these problems.

And where did you learn to do that?

Well, I just thought it out myself. I was looking at this and I was asking to myself: Do these numbers have a pattern? I kept on starting on with this and trying to get somewhere. But then I started with the small one, 1 plus 1 is 2, 1 plus 2 is 3, 3 plus 3 is 6, and so on... and that's how it is mathematics.

It seems worth emphasizing the use of 'ask' in Kaylan's first utterance. As opposed to 'telling,' 'asking' indicates a request on the teacher's part for students to adopt the habit of searching for patterns. 'Every time', is a marker of continuity and stability and illustrates the extent to which searching for patterns was an essential component of the mathematical attitude valued in this classroom.

In Ms. H.'S class, written texts functioned as thinking devices (Lotman 1988), allowing writers and readers to participate in generating new meanings as well as refining those meanings already represented (Wells 1999). As a guided practice extending classroom interaction beyond face-to-face encounters, writing occurred in various modalities, genres, and media. This included: scrap paper scribbling; group-made posters produced and used in whole class discussions; note-taking; copying down homework; brief write ups on solution processes, conjectures, and proofs; and full-fledged (and edited through multiple drafts) narrative accounts of work done on interrelated problems.

This writing-intensive instruction seems to have had a strong effect on students' ability to remember in great detail their common experiences of framing problems, asking questions, generating and testing ideas, and arguing for or against solutions. This is most revealing in the following excerpt from our conversation with Lok Kei.

Researcher: You are sitting here describing this problem that you did and I'm wondering how, why do you remember all the different solutions. Don't you think that's kind of strange?

Lok Kei: Well, people can really remember a lot of stuff.

Researcher: Yes, but last year in math, do you remember different kids' solutions to math problems?

Lok Kei: No.

Researcher: So, why do you remember those this year? You did this a while ago and you're able to tell me all these different solutions. Why do you think that is?

Lok Kei: We had to learn how to take notes. When a person did something, we had to take notes of what they did. And I can remem-

> ber what they did because we had to write essays, questions, and comments.

Yet, as Lok Kei sees it, putting down on paper 'what I had in my head' for others to 'see all the pieces of my work' is far from a simple task.

Writing the problem is like writing an essay on the problem. But it's hard to put your words down, when you have it in your head. I wrote what I had in my head, but I had to write it over to make it more understandable… so that people can see all the pieces of my work... For students to make their thinking understandable to others - not just to the teacher - requires that write-ups go through successive revisions. Lok Kei's words reveal his awareness of Ms. H.'s expectations in this regard.

We just started this more than two weeks ago, and some people just write one draft and say this is my finished product. But it's really not because you didn't really give it another thought. I had to write almost five... almost... it's my fourth draft by now!

7 epilogue

Mathematics is an activity, a practice. If one observes its participants, then it would be perverse not to infer that for large stretches of time they are engaged in a process of communicating with themselves and one another; an inference prompted by the constant presence of standardly formal written texts (notes, textbooks, blackboard lectures, articles, digests, reviews, and the like) being read, written, and exchanged, and of informal signifying activities that occur when they talk, gesticulate, expound, make guesses, disagree, draw pictures, and so on (Rotman, 2000, p.7-8).

From Freudenthal (1978, 1991) we have learned to think of mathematics as a human activity of organizing (mathematizing) subject matter and organizing mathematics and, correspondingly, we have come to view teaching and learning mathematics as the guided activity of mathematizing realistic situations via the reinvention of objects, symbols, and models. Yet, inasmuch as it is a human enterprise, mathematizing can be understood as a semiotic activity and, consequently, the teaching and learning of mathematics ought to be viewed as a semiotic activity (Van Oers & Wardekker, 1999; Van Oers, 2000). In the sixth grade classroom we have studied, teaching aimed at the creative appropriation of a discipline-specific set of reflective skills and dispositions. Acquiring this reflective competency involved mastery of the art of questioning oneself through conversations in speech and written texts. This learning process was akin to undergoing an apprenticeship wherein students first imitated, next practiced among others, and then appropriated these mathematical dispositions and tools.

notes

1 This instructional sequence was designed by the teacher, in collaboration with one of the researchers relying on Connected Mathematics, Mathematics in Context, and other sources (Freudenthal, 1978; De Lange, 1987).
2 With the teacher's help, we selected those students who seemed most likely to have something meaningful to say and were willing to share it with us.
3 The latter include whole group problem framing; students working on these individually, in pairs, or in small groups; exchanging ideas in whole group talk; taking notes and copying down homework done during lessons; doing the homework; and producing full-fledged narratives on mathematical problems, topics, or strands.
4 We use 'voice' in the sense of Bakhtin (1981).
5 As Christiansen (1997) suggests, in light of his study of students' interpretation of a modeling task in a high school classroom, it is imperative 'from the students' point of view that the activity has a well-defined goal, i.e. that they know the 'kind' of activity in which they should engage' (p.19). As he notes, when this does not occur, students try to extract directions from the teacher's implicit statements.
6 Here we think in line with Vygotsky for whom 'imitation is not to be interpreted as indiscriminate copying of someone else's actions, but as a form of meaningful reenactment of some cultural activity, based on interactive reconstruction and on the reflexive exchange of meanings' (Van Oers, 2000, p.139).

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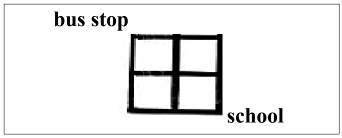
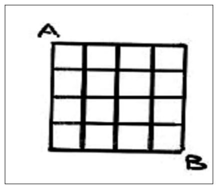
Appendix

1 How many shortest paths are there for the teacher to walk from the bus stop to the school on the street grid below?

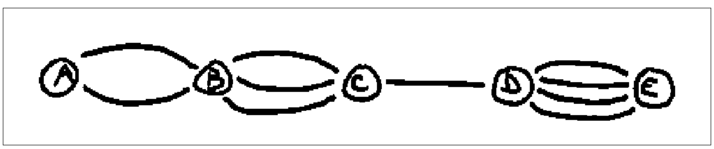


2 How many shortest paths are there to go from A to B?

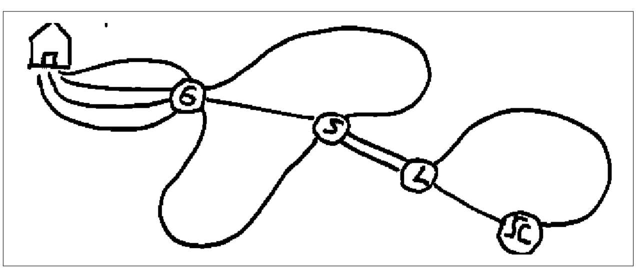


3 Homework: How did you start to solve these problems? What kind of system did you use to organize your thoughts? Share your discoveries. Did you find any patterns? How do your findings compare to your original prediction?

4 How many pathways are there from A to E?

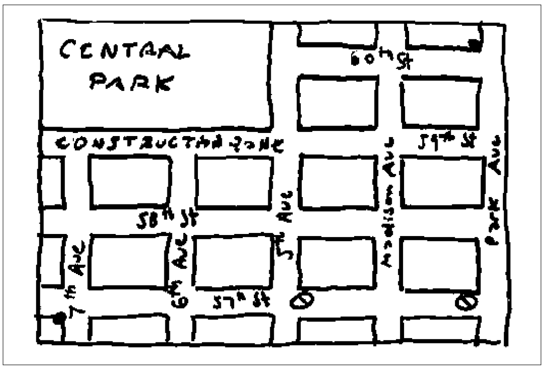


5 Every Saturday, Denny goes from his house to the Sports Center, passing through the grocery store, the school, and the library. He claims that, for an entire year, he can take a different route to get to the Sports Center passing through all those places. Is he right?



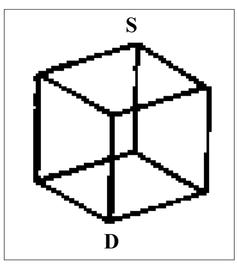
6 Homework: Make a map connecting Sam's house with:
the grocery store,
the comic books store,
the school, the baseball court,
and McDonalds, so that Sam would have exactly 32 different ways to get from his house to McDonalds.

7 Miss L. is on Park Avenue and 60th Street and she wants to meet her husband at 57th Street and 7th Avenue. How many different shortest pathways can she take?



8 Homework: Draw a map of a 'real neighborhood'. Invent a similar shortest paths problem. Make sure to provide a solution to your problem.

9 Imagine that you are a little spider moving from the start (S) to the destination (D) on the cube below. How many different shortest paths can you take?



10 Homework: Study each of the three diagrams presented today. Do you understand it? Does it make sense? Does it relate to the problem? If not, can you do better?