Productive or reproductive exercises – what is appropriate for low attainers?

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1 introduction

In recent years there is clear evidence of a changing view of learning and teaching, which emphasizes the constructive activity of the learner. Corresponding consequences for primary schools have been developed in the meantime. The question arises whether those conceptual results can be valid for special education with learning disabled children too.

2 low attainers learning mathematics

One will find low attainers not only in special schools but in primary schools too. The learning processes are comparable with those of non-handicapped children, however, the processes are characterized by a temporal extension and a higher error-rate. For all subjects and especially for mathematics a high degree of differentiation and practice is needed. Still the generally shared opinion is that low-attaining pupils cannot cope with more demanding and complex problems. Accordingly, the isolation of difficulties and step-by-step learning are guiding principles and the students' 'activities' are confined to bare reproduction. Too often mechanistic drill and practice replace insightful learning. In contrast to this, several studies and experiences in general have confirmed that investigative learning combined with productive exercises is appropriate especially for children with special needs (cf. Ahmed 1987; Scherer 1995; Van den Heuvel-Panhuizen 1991). Productive work is to be understood in contrast to bare reproduction of knowledge. It should enable pupils to think, to construct and to extend their knowledge (cf. Wittmann 1990).
What are the specific demands of low attainers?

- *differentiation*
  Different offers made by the teacher bear the risk that some pupils are overtaxed, misjudged or fixed at a specific level. The following teaching

units contain a kind of natural differentiation which means that the pupils can choose their level of working by themselves, and that it isn't fixed at the beginning. At the same time a natural differentiation makes it easier for the teacher to organize the learning processes: all students are working on the same task and there is no need for the teacher to present plenty of different work sheets;

- *complex learning environments*
  Investigative learning and productive exercises are usually connected with holistic approaches, and especially for low attainers it is important to get an overview and see relationships (Ahmed1987). With regard to low attainers holistic approaches are usually avoided in favor of splitting up the subject matter into fragments (cf. Donaldson 1978). But that does not contribute to solving certain difficulties like understanding the structure of our number system;
- *support of self-confidence*
  Another important problem is the lack of self-confidence: It is necessary to give the pupils the opportunity to show what they are capable of, for example by using more open problems.

3 examples

In the following several examples of productive exercises and their advantage concerning the demands of low attainers are illustrated.

games

A first form of activity are games that are often used for low attainers. But too often the one and only reason for their use is the lack of motivation and you often find pseudo-games. However, there are strategy games to improve thinking and especially anticipating actions.

NIM-game

The following version of the so called NIM-game is limited to the space up to 20 (cf. Scherer 1996) and can be used for school beginners.

Rules of the game:

1 Two children play with red and blue counters on a game-board with 20 fields arranged in a line (numbered if needed; fig.1).

2 Alternately the players place one or two counters serially on the board starting at field 1.

3 The player who has reached field 20 wins the game.

Children can play this game without knowing the winning strategy in general and in this way different levels of working/playing are possible. It is not necessary to know all the numbers, although this makes it easier to discover the strategy, but a lot of children who have difficulties with numbers are able to discover a winning strategy.

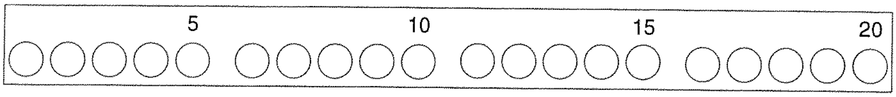

figure 1: game-board for the NIM-game (number-line up to 20)

At first the children realize the first step of the strategy. If the partner has placed his counter on field 18, I have to place two counters to win the game. In the next step they identify field 17 as a winning position (fig.2), usually arguing for the own position. If I have placed my counter on field 17, the other player has just two possibilities which do not lead to success. If he places one counter, I will place two. If he places two counters, I will place one. In both cases I will win the game.

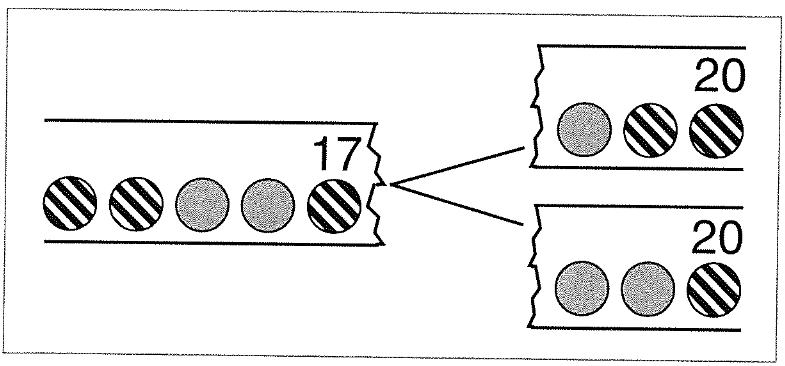

figure 2: winning position '17'

So the 'new' game is to reach field 17 which leads to field 14 as the next winning position. At last it becomes clear that field 2 is a winning position (fig.3) and that the beginning player has the possibility to win. Children may discover the strategy in a natural way by trying out, supposing a strategy and checking different hypotheses.

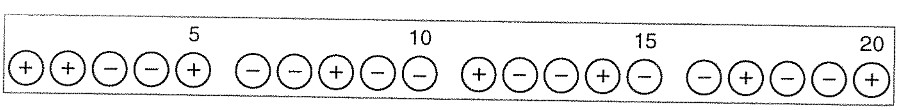

figure 3: winning positions in total

After having discovered the strategy in general this game can be varied:

1 Variation of the game-board (10, 12, 20, 100 fields etc.).
2 Variation of the second rule: You must place 1, 2 or 3 (or more) counters.
3 Variation of the third rule: the player who has reached field 20 has lost the game.

Pupils can play this game when dealing with the space up to 20, but moreover it is appropriate in higher classes to get the children to more flexible thinking. Another 'thinking-game' is 'triangle-memory', an African game that is played on a game board with numbers arranged in the form of a triangle (cf. Wittmann & Müller 1990, p.21f; Zaslavsky, 1973). The problem is to remember the numbers and reconstruct the field according to a special geometric pattern. This game is helpful in the case of children who have internalized the number line in a rather mechanistic way, who are fixed to the specific arrangement of a line and for instance have difficulties to handle with the telephone as there is a different arrangement of the numbers.

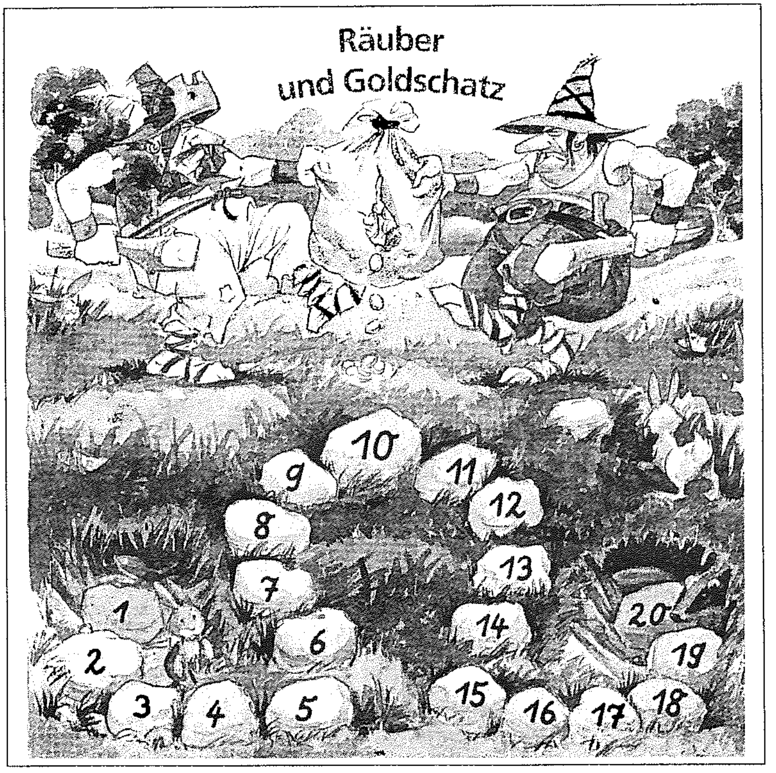

figure 4: game-board for 'the robbers and the treasure' (number-line up to 20)

the robbers and the treasure

The following game is an exercise for orientation but the transition to calculating is fluent. Presenting a game-board shown in figure 4 the following story is told (cf. Wittmann & Müller 1990, p.17f). Two robbers live in a deep forest, each of them in a big cave. The path between these two caves is covered with 20 stones (numbered). One day the robbers discovered a big sack. With great difficulty they were able to get it out. A gold nugget was glittering. Immediately a fight broke out and each of them claimed that he owns the treasure because he had discovered it first. After a time when they both fell to the ground exhaustedly they thought that the dice should decide who will get the treasure:

> The treasure (piece) is placed on field 10. We throw dice alternately and move the piece to our cave according to the number the dice shows. Who gets the treasure at first in his cave should own it.

One child is going forward (adding) and therefore he is called the 'Plus-robber'; the other - going backwards (subtracting) - is called the 'Minus-robber'. This game includes a kind of natural differentiation too. The treasure can be placed step by step (counting), a child can get the next field simultaneously or a child can get the right field by calculating. Those levels depend on the space what means that a child calculates, e.g. in the space up to 10, but places the treasure one-by-one in the fields from 10 up to 20. Moreover it depends on the operation (addition is much easier than subtraction) and on the specific tasks (it's easier to calculate 7 + 2 than 7 + 6). But the transition from one level to another is fluent and not fixed by the teacher. Beyond this the teacher has the possibility to pose question during the game:

> The treasure is placed on field 17.
> What must the die show to win the game?

Moreover the transition to calculating is possible: you can take down the course of the game in form of addition tasks. In the space up to 100 you can continue to play it on the number line up to 100.

orientation

In order to extend a new number space it is necessary to orientate in that new area. A crucial point for teaching is the use of appropriate manipulatives and representations. It is important to give pupils enough time and suitable experiences to build up an internal image of numbers and operations. One has to choose those materials which represent the mathematical structure (e.g. field of hundred) and of course flexible representations which can be continued (cf. Wittmann, 1994).

representing numbers

One of the major problems with low attainers is counting one-by-one or calculating by counting without using the decimal structure. Counting one-by-one can be avoided by the 'flash exercise' (cf. Wittmann & Müller 1990, p.149) 'How many?': presenting numbers at the field of hundred with the 'number-angle' just for a short moment (fig.5a, 5b).

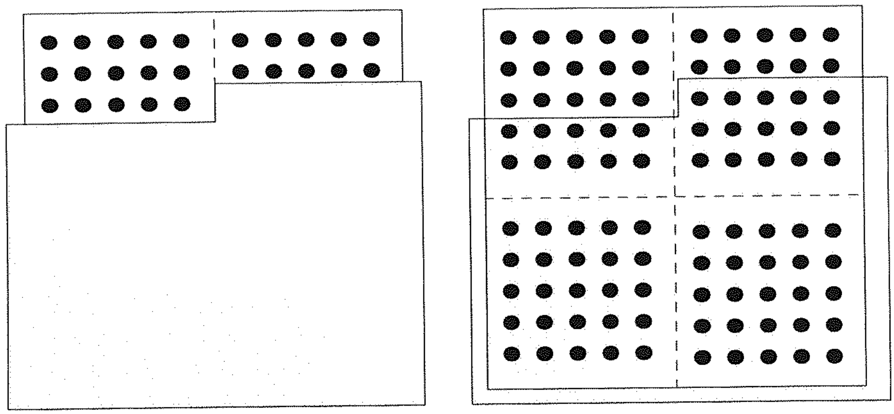

figure 5a and 5b: representing numbers in the field of hundred with the number-angle

The children are forced to use the given structure (5 or 10) in order to tell the given number.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

figure 6: patterns in the table of hundred

The earlier you start with these activities (e.g. when dealing with the numbers up to 20) the better the internal images will be built up. But also for higher domains it is an essential and fundamental activity but often neglected.
Referring to patterns when using manipulatives can help the children to do more than pure calculating. In the 'table of hundred', pupils have to fill in the missing numbers (fig.6) and to think of these numbers. What similarities are found? What are the differences? (also possible for the flash-exercises). In phases for orientation those numbers should be used that cause difficulties (inverse numbers, numbers with two equal digits; cf. Scherer 1995, 176ff). That means children should be confronted with the problems at an early time when using manipulatives and representations as this can contribute to understand the number system.

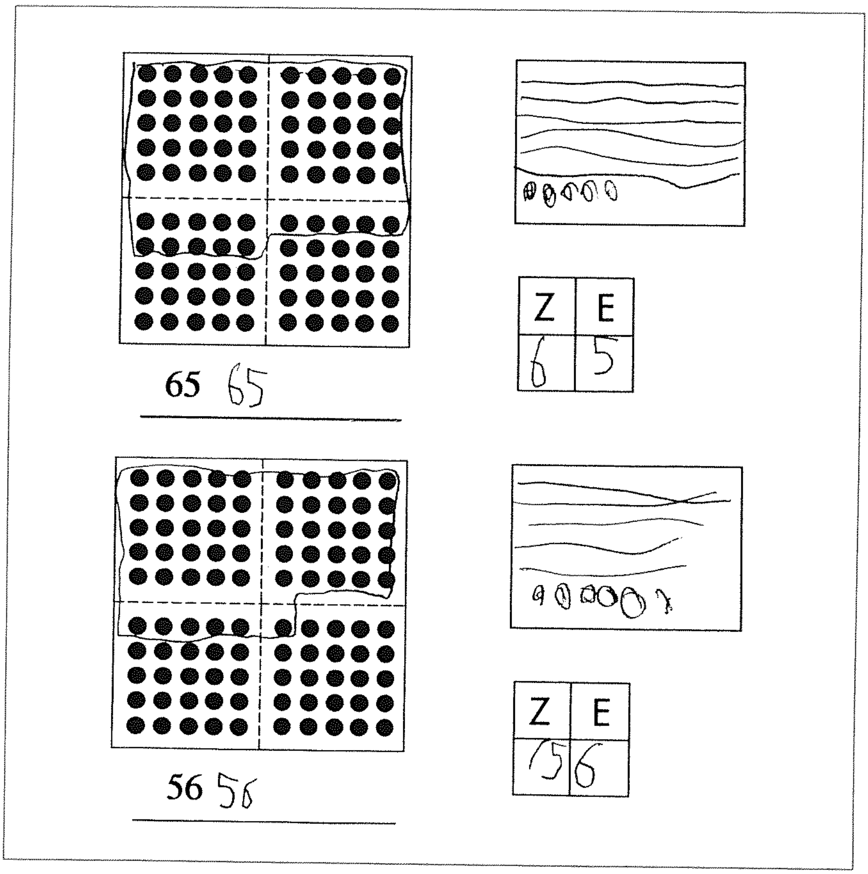

figure 7: coordination between the field of hundred, the place value table and the representation with bars and dots

Another crucial point is the coordination between several representations. As it is difficult for the children to see a similar or even the same structure (cf. Lorenz 1992) the transfer from one representation to another should be an important activity (cf. fig.7). During the phases of orientation the mental calculations already should be prepared e.g. by moving on the number-line or representing a number given in a sum (Show three tens and six ones in the field of hundred!).

calculating

One category of exercises are those showing relationships illustrated before at the table of hundred where the students had to identify patterns.

Operative exercises

In special education textbooks you find tasks like these that should support the understanding of place value (fig.8a). But these analogies can be solved in a rather mechanistic way by adding a zero. It's better to use sets where it is necessary to look exactly at the different place values (fig.8b).

8a	8b
6 + 2 =	40 + 5 = [45]
60 + 20 =	50 + 4 = [54]
4 + 5 =	40 + 50 = [90]
40 + 50 =	5 + 4 = [9]

figure 8a and 8b: two types of analogue sets

Dealing with addition and at the same time with subtraction it is reported that pupils mix up the different symbols and a lack of discrimination is presumed.

46 + 3 = [49]	46 + 3 = [49]
49 - 3 = [46]	49 - 46 = [43]

figure 9: operative sets for addition and subtraction

In my opinion in many cases the only reason for mixing up the symbols is that pupils are used to solve similar sets. Therefore they should be forced to think and not to solve tasks in a mechanistic way that cannot be avoided in any case (last set in fig.9).

Before starting to calculate children should reflect on the given relationships and think about e.g. how to minimize the calculations. Another form are operative series (for instance decreasing the second addend by 1) (fig.10) where several levels for differentiation exist.

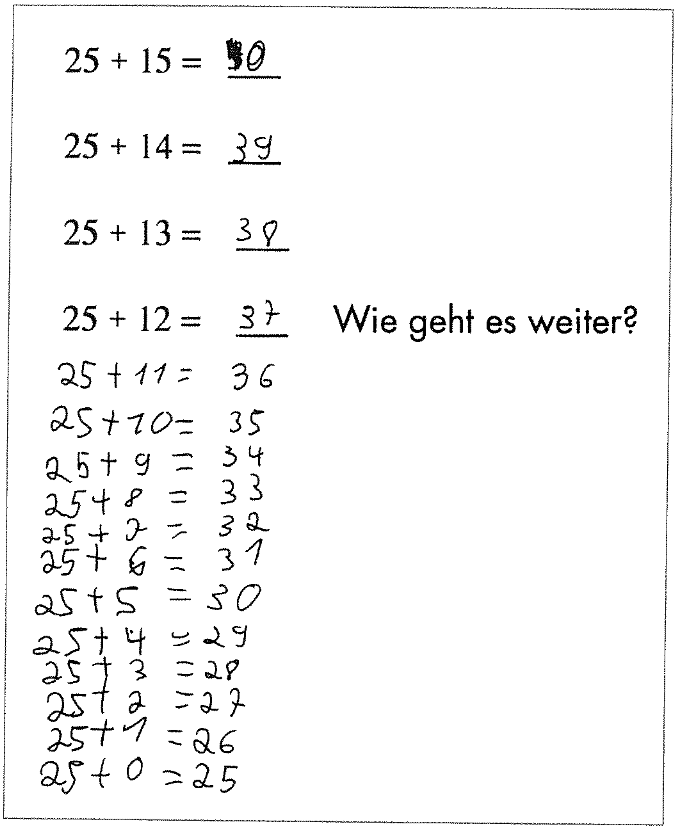

figure 10: Benni's solutions

- children could work out the given tasks by mental calculation or with the help of manipulatives and representations;
- children could identify the patterns and continue the series (the number of tasks isn't fixed), here they have the possibility to represent the addends at the table of hundred. Children have the opportunity to continue the series with the help of the visual pattern and may reflect afterwards about the numerical relationships;

– children could describe the relationships between the tasks or even between the results. Benni continued that series up to the task 25 + 0 without working them out, a rather sly solution. Then he began with the easiest task 25 + 0 and went up (fig.10). It is important to vary such relationships for the danger of solving these tasks in a rather mechanistic or schematic way: Andreas increased the addend by 1 not by 2 (fig.11).

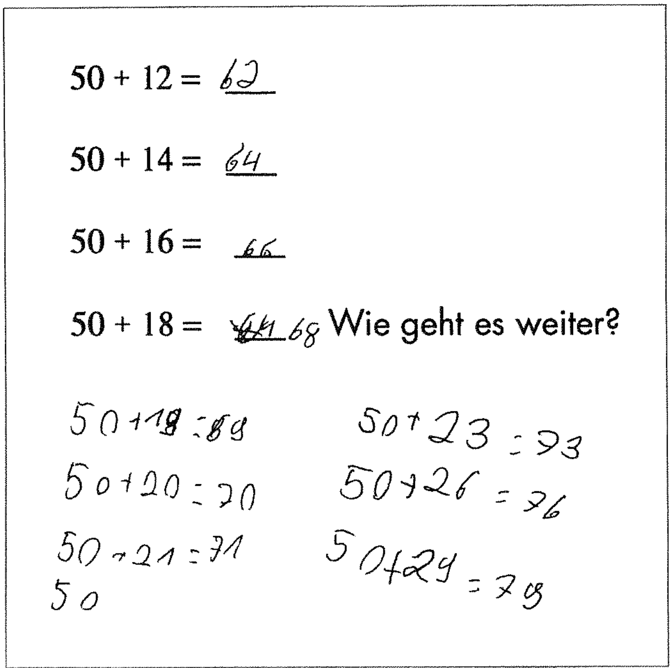

figure 11: Andreas' solutions

Perhaps he remembered a similar set he had done before. He asked if his solutions are correct. In those cases pupils should be encouraged to think once more on their own and he replied 'Oh, there must be two more'. He was advised to write three more tasks and increased the second addend by three. The ambiguity of our language is clearly seen.

activities with number cards

The following number cards are given: [1] [2] [4] [5] and the children should build two-digit numbers and add the two numbers. Find as many tasks as possible. What do you remark?

For some children it is necessary to have real cards as they cannot operate with the cards mentally. Pupils like Marc solved the tasks by mental calculation (fig. 12a). For the children several equal results were surprising. There were different tasks (commutative property) but also different numbers which led to the same result. When reflecting about those phenomenons a representation on the blackboard used by other pupils was helpful (fig. 12b). The children could identify that just the tens and the ones had changed.

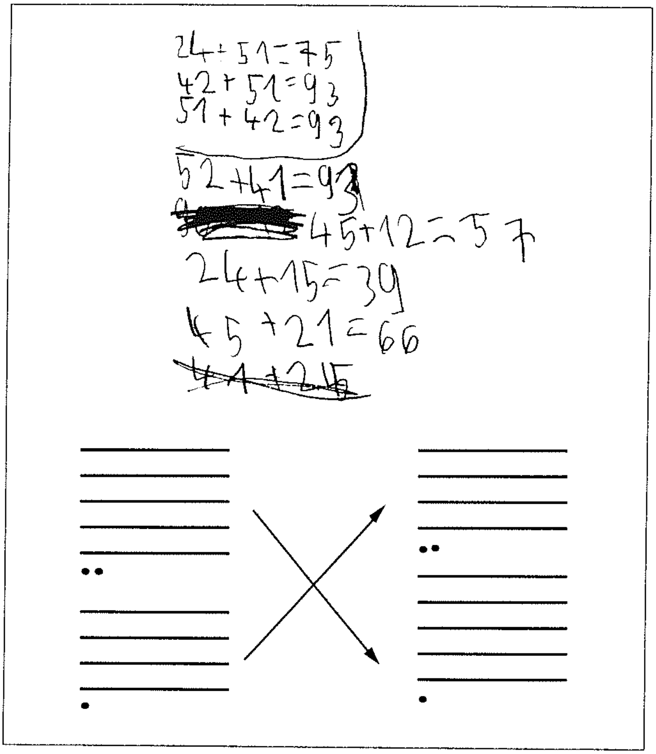

figure 12a and 12b: representation used as a proof

The representation is used as a proof: that means you can generalize this special example (Wittmann, 1988). This use is helpful in the case of a negative image of representations. In my view the process of becoming independent should not be a one-way street. One should be able to use and understand representations in every situation (cf. Scherer, 1995, 248f). Those activities fit for every number space for every grade (e.g. number cards from 1 to 9 and build three (four)-digit numbers etc.). What is important is that pupils can try out, as there does not exist the one and only correct answer. Moreover pupils do more than pure calculating, they think about the numbers, their relationships etc.

problem structured exercises

Another category are exercises within a problem which means that problem gives precedence over the exercise. These are illustrated through the example of magic squares.
Nine numbers are arranged in a 3 × 3-square. Adding a line, a column or a diagonal, in every case you get the same magic sum. One activity could be to fill in magic squares where some numbers are missing. The pupils have to work out addition or subtraction tasks, pay attention just to one dimension and solve the square directly (fig.13a). Or they have to overview several dimensions and their relationships. Some pupils just look for the right sum but do not look for other dimensions (fig.13b, 13c).

10	3	5	= 18
1	6	11	= 18
7	9	2	= 18

10	13	12
11	9	
6	5	8

10	4	12
11	9	
6	14	8

figure 13a, 13b and 13c: activities with magic squares

There should be the possibility to solve the problem by trial and error (by placing number cards), then perhaps try out systematically. Therefore it is helpful that this activity contains a kind of self-control which usually is not used automatically by the children at the beginning.
Further examples are for instance magic triangles (cf. Wittmann et al, 1995) or number chains' (Fibonacci 5'; cf. Scherer & Selter, 1996).

open problems

Open problems give freedom to the student in choosing tasks and methods of calculation. An example: find tasks with the result 100! This problem may help to clarify existing misconceptions, but at the same time it enables the child to show his or her capabilities. Moreover, this exercise covers a natural differentiation: it includes tasks on different levels, that are neither fixed nor determined by the teacher.
Some pupils just found a few tasks or just added tens (fig.14a), others like Mustafa found 24 tasks in total, including all additions with two addends which are possible for this problem, also the bigger addend in second position (fig.14b). He wrote down a lot of difficult tasks and also easy ones (pure tens for the addends) and made just two mistakes. The false task 73+29 may go back to a careless error, as well as the term 79 + 19. In this

case one has the possibility to represent the decomposition of 100 at the field of hundred with a transparent number-angle (cf. fig.5b). A lot of pupils found tasks officially provided several months later on, numbers they haven't dealt with yet (proximal zone of development) and one can avoid the danger of overtaxing because the pupils can choose tasks on their individual level. It is found that pupils often choose easy tasks perhaps in lack of self-confidence, but I think they often just want to play with easy numbers. Those open tasks can be realized in every form of exercise like for instance in the number walls (Scherer 1995, 274ff).

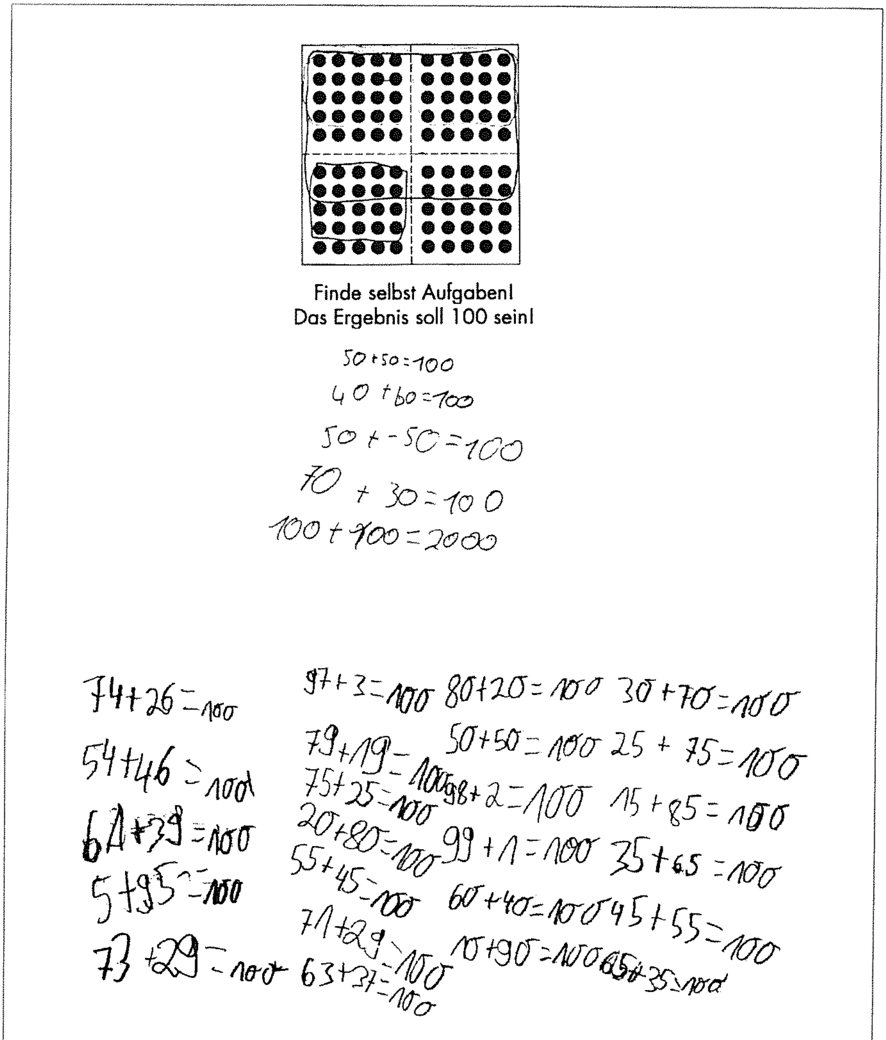

figure 14a and 14b: Ali's and Mustafa's tasks with the result 100

4 final remarks

Substantial or complex teaching units are not only appropriate for the 'good' students. Students who have difficulties in calculating do not necessarily have difficulties in discovering and using relations as well (cf. Scherer 1995). As there exist problems of different difficulty in a natural way, all students can contribute to the same (general) task. The danger of overtaxing the weaker pupils and undertaxing the better ones can be reduced extremely (cf. Scherer 1995; Wittmann 1990, 159). Giving freedom to the pupils to test their limits can support motivation and self-confidence. Supporting the children means making certain demands an aim at long-term learning processes and not only thinking of short-term success in learning.

The teacher must have patience as these processes will take more time in the beginning. For students who are used to work in a rather reproductive way for a long time it will take some time to change their attitude. But the time invested here will certainly pay and be more effective than bare quantitative extension of training.

The fact that the pupils have difficulties in learning mathematics and therefore a reduction of content is necessary may not lead to a 'light teaching' and to an insufficient mathematical education. Especially for those pupils the quality of teaching is of great importance. Good teaching that emphasizes the structure of a subject is probably even more valuable for the less able student than for the gifted one, for it is the former rather than the latter who is most easily thrown off the track by poor teaching (Bruner 1969, p.9).

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