Going one's own way

- a teaching experiment in grade 3 -

C. Selter
IDM, Universität Dortmund

1 introduction

Many reform documents throughout the whole world, such as the Dutch 'Proeve van een nationaal programma voor het reken-wiskundeonderwijs op de basisschool' (Treffers, De Moor & Feijs, 1989), the German 'Richtlinien und Lehrpläne Mathematik für die Grundschule in NRW' (Kultusminister, 1985), the English 'Cockcroft Report' (Cockcroft, 1982) or the American 'Curriculum and Evaluation Standards for School Mathematics' (National Council of Teachers of Mathematics, 1989), advocate important shifts (Becker & Selter, in press; Verschaffel & De Corte, in press):

- mathematics is no longer seen as a collection of unrelated concepts or skills, but as a coherent, living organism;
- the goal of mathematics teaching is no longer seen as transmitting ready-made mathematics, but as helping pupils to acquire a mathematical disposition;
- learning is no longer seen as absorbing elements of information, but as the active construction of knowledge to a structured entity;
- teaching is no longer seen as transmitting knowledge, but as encouraging learning. It is acknowledged that teaching cannot guarantee learning, but can raise the probability that learning occurs.

In this connection, it seems to be of essence to build on children's mathematical abilities, as they have lots of ideas to offer that can be used as the engine keeping the teaching/learning process running - provided they have not unlearned these.

In the contribution at hand I want to illustrate how a third grade teacher and I tried to do justice to children's thinking by giving as much initiative to them as possible (Sundermann & Selter, 1995). We encouraged them to go their own (sometimes narrow or long-winded) ways instead of directly forcing them to use the express motorways we - as adults - have learned to 'drive on'. This did not mean neglecting the fact that certain goals prescribed by the curriculum were to be achieved at the end of the course. In other words: I intend to illustrate how we understood the principle of *progressive mathematisation* (Treffers, 1987).

2 overview

The course I will be describing dealt with strategies for adding and subtracting in the domain of 1 through 1000. Twenty-seven eight- and nine-year-old children worked on this topic for fifteen lessons. Before I go into further detail, I briefly want to sketch the structure of the course, by enumerating its six parts:

1 'The size of the cinemas' - Understanding the context.
2 'That's the way I do it' - Children's own methods of adding.
3 'That's the way I do it' - Children's own methods of subtracting.
4 'Doing it like ...' - Becoming familiar with different strategies.
5 'Doing it skilfully' - Reflecting about different strategies.
6 'That's the way we did it' - Reflecting about the course.

As we were looking for a good context to serve as the starting point, we decided to deal with the so-called cinema problems. There was a modern cinema centre (called the UCI) near our school with the total of 4128 seats, the size of its eighteen cinemas ranging from 98 to 444 seats. The UCI-context can be used in different ways that might suggest:
- to sum up: x persons are sitting in cinema y. They are being joined by z persons;
- to take away: x persons are sitting in cinema y. During the break z persons leave;
- to add on: Cinema y has a total of x seats. Z persons are already sitting there.

Of course, critical remarks with regard to the quality of the context are absolutely justified. (1) It might not develop very naturally into the model of the empty numberline that was used throughout the whole course. (2) A person would hardly have to work out the number of seated persons, then find out how many of them are leaving during the break in order to determine the number of people remaining. But (1) the context was understandable and meaningful for our children and (2) its main purpose was not to understand 'reality better through mathematics', but to understand 'mathematics better through reality'. In varying detail I now want to describe the six parts of the course.

the size of the cinemas - understanding the context

In order to get accustomed to the UCI-context the whole class made an excursion to the cinema-centre. Here, a cinema-rally was organized where one of the activities consisted of estimating the sizes of the different cinemas.

that's the way I do it! - children's own methods of adding

During the second and the third part of the course the children were asked to solve a couple of addition and subtraction problems from the cinema context with their own methods and to put these down as entries in their maths diaries. In addition, they conducted *maths conferences* where they discussed their ways of working with their class-mates and in return learned about the strategies the other children had developed.

Here, the empty numberline was used - a horizontal stroke on which pupils can put down the numbers and the operations they need in an informal and approximate way (Treffers, 1991, pp.40-52; Gravemeijer, 1994, pp.120-129). The empty numberline is not just one representation of the fundamental idea that our numbers form in infinite series (Wittmann, 1995, p.25), but also represents a means for using informal notation. Some solutions of the problem: '128 persons are sitting in cinema 4. They are being joined by 96 persons', might illustrate this (fig.1).

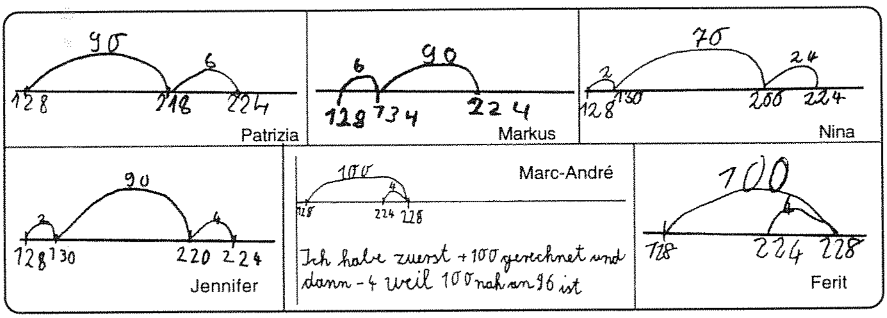


figure 1: different ways to solve an addition problem

For example, Patrizia (128 + 90 + 6) and Markus (128 + 6 + 90) added tens and ones of the second number successively, whereas Nina (128 + 2 + 70 + 24) and Jennifer (128 + 2 + 90 + 4) split them up in order to made use of convenient preliminary results. Ferit and Marc-André used an auxiliary task (128 + 100) and adjusted this by subtracting 4 at the end (for a more detailed analysis of children's strategies for addition and subtraction, see Beishuizen et al., this volume).

that's the way I do it! - children' s own methods of subtracting

Many different ways of working could be observed not only with regard to addition, but particularly to subtraction. Let me illustrate this by presenting some solutions of the following problem: 'cinema 2 has a total of 216 seats; 148 persons are already sitting there' (fig.2).

For example: Kristina took away tens and ones one after the other (216 – 100 – 40 – 8), whereas Patrizia (216 – 100 – 20 – 20 – 4 – 4) and Manuela (216 – 100 – 20 – 20 – 8) split them up in order to obtain steps that they were able to handle better.

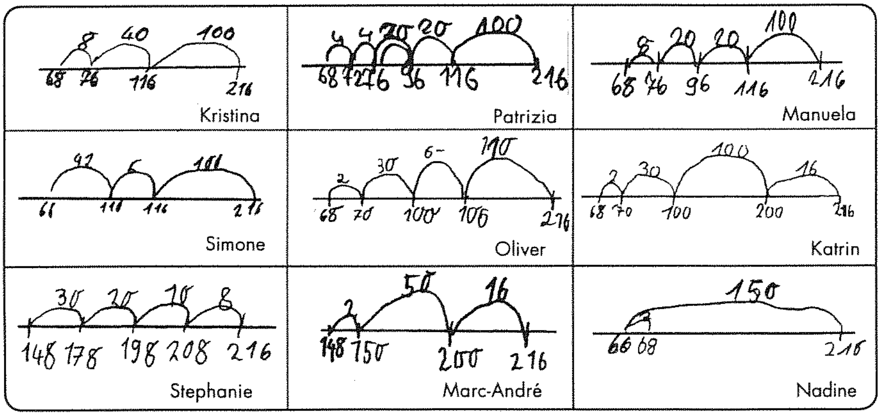


figure 2: different ways to solve a subtraction problem

Another strategy was to split up 148 in order to reach convenient preliminary results (Simone: 216 – 100 – 6 – 42; Oliver: 216 – 110 – 6 – 30 – 2; Katrin: 216 – 16 – 100 – 30 – 2). As well, some pupils used an adding up-strategy, like Stephanie who filled up the tens first and then the ones (148 + 30 + 20 + 10 + 8). Marc-André's method who made use of convenient numbers (148 + 2 + 50 + 16) shall serve as a second example here. Moreover, some pupils - like Nadine - used an auxiliary task (216 – 150 + 2).

The variety of solution strategies that could be observed was a central element of the classroom culture. But it should not mean that single children kept using their own strategies, even if they appeared to be long-winded and susceptible to mistakes. Thus, it was the core principle of the course to encourage children to develop more efficient and elegant ways of working. Here, the maths diaries and the maths conferences were of importance.

The maths diaries (see similar ideas in Gallin & Ruf, 1993) are little logs in which the pupils document their learning processes and products in their own language, including success, difficulties and open questions. As the use of maths diaries can improve children's abilities to communicate mathematically, they can provide at least a little counterbalance to the speechlessness of mathematics in our schools. Besides, maths diaries are an important source of insight for the teacher who gets more information from each single child. This may help to reflect on the teaching/learning process and to plan it further.

On the first pages the children were asked to express their own methods for the addition and subtraction problems. As they were not used to doing this, they were offered some guidelines, such as putting down their method in a symbolic expression in the left half of the page and documenting their strategy by using the empty numberline and by putting down a brief description of what they had done in the right half (fig.3).

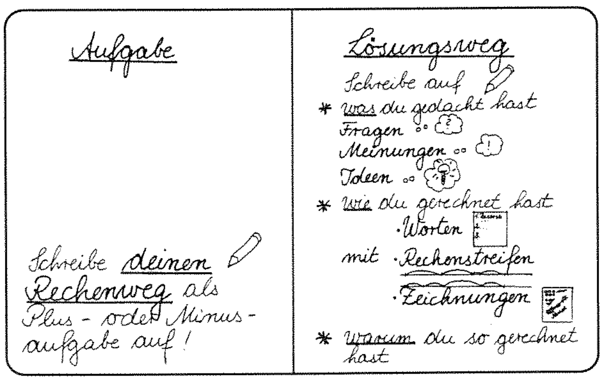


figure 3: guidelines for the first pages of the maths diary

Nina, for example, solved the problem '274 persons are sitting in cinema 15. They are being joined by 167 persons' by adding hundreds first, then ones and finally tens (fig.4).

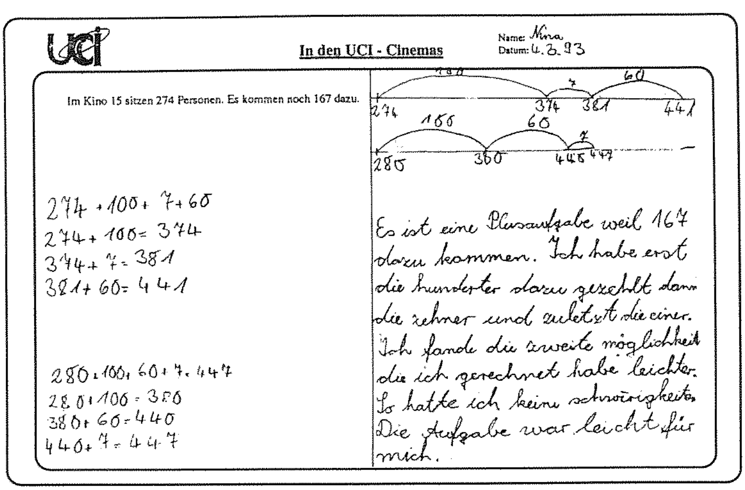


figure 4: a page from Nina's diary

A second method mentioned by her made reference to the auxiliary task 280 + 167, though not coming back to the initial numbers. She commented as follows - and we encounter a slightly different strategy in her verbal description:

> 'It must be an addition, because there are 167 joining. At first I added the hundreds, then the tens and in the end the ones. In my opinion, the second way of doing it was the easier one. I did not have any difficulties. The problem was very easy for me to solve.' (fig.4)

Two more examples shall give an idea of the methods the children developed and especially their ability to reflect on what they did (fig.5).

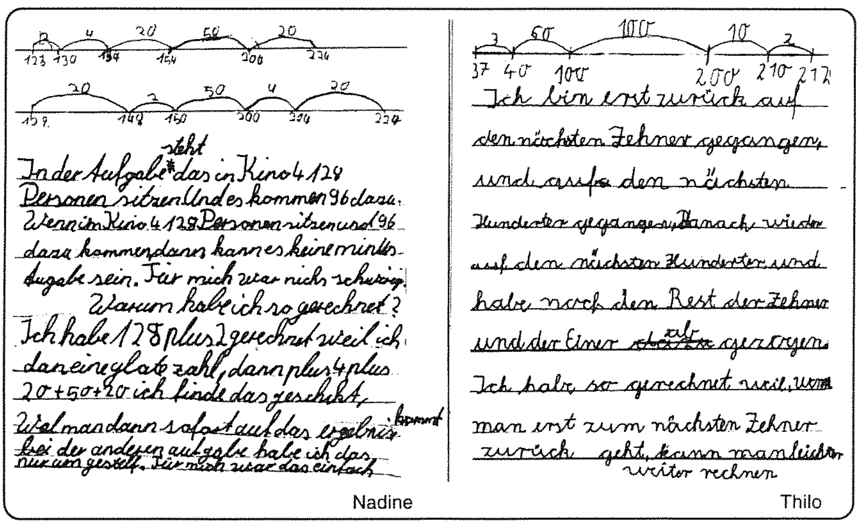


figure 5: pages from Nadine's and from Thilo's diaries

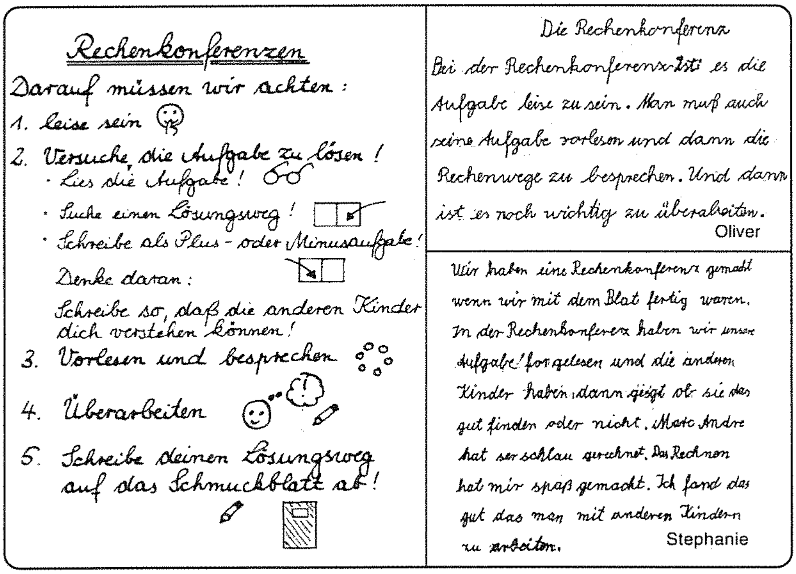
Nadine wrote:

> 'The text indicates that there are 128 persons sitting in cinema 4. And it says that 96 persons are coming. If 128 persons are already seated and 96 are coming, it cannot be minus. It was absolutely easy for me. Why did I do it in that way? I added 128 and 2, because I was able to reach a convenient number, then I added 4 plus 20 + 50 + 20. I think that this is clever, because you end up with the correct result immediately. I also did it in another way. I just changed the sequence of steps. It was really easy.'

Thilo put down:

> 'At first I went back to the next ten and then I went back to the next hundred. Then back to the next hundred, and then I just took away the remaining tens and ones. I did it in that way, because, if you go back to the next ten, it is very easy.'

Without doubt using maths diaries offers a lot of advantages; on the other hand it can result in an excessive individualisation, if the teacher fails to encourage children to learn cooperatively. Thus, they met regularly for so-called maths conferences in groups of two, three, four or five.



Rechenkonferenzen

Darauf müssen wir achten:

1. leise sein
2. Versuche die Aufgabe zu lösen!
   - Lies die Aufgabe!
   - Suche einen Lösungsweg!
   - Schreibe als Plus- oder Minusaufgabe!

   Denke daran:

   Schreibe so, daß die anderen Kinder dich verstehen können!
3. Vorlesen und besprechen
4. Überarbeiten
5. Schreibe deinen Lösungsweg auf das Schmuckblatt ab!

Die Rechenkonferenz

Bei der Rechenkonferenz ist es die Aufgabe leise zu sein. Man muß auch seine Aufgabe vorlesen und dann die Rechenwege zu besprechen. Und dann ist es noch wichtig zu überarbeiten.

Oliver

Wir haben eine Rechenkonferenz gemacht wenn wir mit dem Blat fertig waren. In der Rechenkonferenz haben wir unsere Aufgabe forgelesen und die anderen Kinder haben dann gesagt ob sie das gut finden oder nicht. Marc André hat ser schlau gerechnet. Das Rechnen hat mir spaß gemacht. Ich fand das gut das man mit anderen Kindern zu arbeiten.

Stephanie

figure 6: guidelines for and texts on maths conferences

These meetings gave good opportunities to present and to discuss the different ways of working. In the beginning a guideline offered some orientation (fig.6): try to solve the problem, put down your strategy in your diary in a way that can easily be understood by others, meet in small groups, present and discuss your productions and (if necessary) revise it. Some of the pages were copied (Schmuckblatt) and collected in a big diary to which all the children of the class contributed. This was handed over to the children in grade 2 at the end of the course. After having conducted several maths conferences the children were asked to describe them (fig.6). Oliver, for example, wrote:

> 'If you are coming to a maths conference it is very important not to be too loud. You have to read out the problem you have been working on and to discuss the solution strategies. And it is important to revise them in the end.'

Stephanie explained:

> 'We did a maths conference, after we had finished the entry in our diary. During the maths conference we showed our way of working and the other children made their comments whether they liked it or not. Marc-André

developed very skilful ways. I really liked to work on these problems, and especially to work with the other children.'

Thus, the purpose of maths conferences was twofold: on the one hand, the children got to know the methods of the others, reflected on them and made them part of their repertoire; on the other hand the ability of the author as well as of the critics to communicate mathematically might be extended.

In order to do justice to the heterogeneity of children a couple of different offers were made to differentiate. For example, they were asked to pose problems themselves with different degrees of difficulty, to solve them and to give them to their class-mates. Here, several pupils left the domain of 1 to 1000 which is usually dealt with in grade 3 (fig.7).

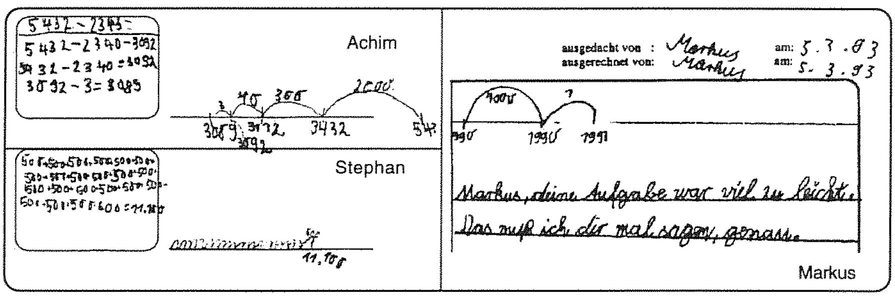


figure 7: self-invented problems

Achim posed the problem 5432 – 2345 (context-free) which he solved by taking away stepwise on the empty numberline (5432 – 2000 – 300 – 40 – 3); he put down the symbolic equivalent in a slightly different way. Stephan worked on adding 500 twenty-one times and 600 once on top of that. He coped with this challenge by drawing two jumps of 500 and simultaneously counting in steps of 1000. Markus put down the sum 990 + 1001 for his class-mates which he solved by adding stepwise (990 + 1000 + 1). Finally, he wrote down the following comment, addressed to himself:

'Markus, the problem you posed was much too easy. I really had to tell you that, indeed.'

doing it like - becoming familiar with different strategies

The guiding principle of the course was to make use of the different methods and abilities of the children - a plea for the heterogeneous learning group Freudenthal has often advocated.

Doing justice to the variety of children's thinking is by no means in contradiction to the goals that are prescribed by the curriculum. Thus, one consequence was to make elegant and efficient ways of working a topic of the lessons. This was realized in the maths conferences, but it was also an important part of whole-class discussion and pupils' work.

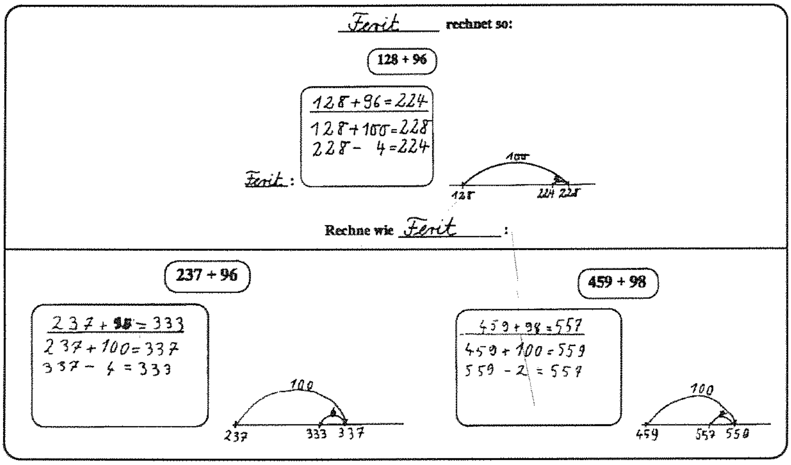


figure 8: Jennifer does it like Ferit

'Doing it like ...' was one type of problem that was dealt with during these lessons. The teacher had selected several methods that were used in the learning group and that seemed skilful to her.
The goal was that the children understood these methods and applied them to a couple of other problems. For example, they were asked to do it like Ferit who had displayed the auxiliary task strategy several times beforehand, like solving 128 + 96 by adding 128 + 100 first and then taking away 4. Jennifer's work shall serve as an illustration here. She was asked to apply Ferit's strategy to 237 + 96 and to 439 + 98 (fig.8).

The children were not just encouraged to use these strategies, but also to reflect on them. In the maths conferences and during whole class discussion it was pointed out, for example, that certain children preferred specific methods, independent of the problem structure and context: some always took away, even if the two numbers were very close to each other which might suggest to add up; others always made use of convenient numbers as preliminary results.
Besides such individual preferences it was also discussed that certain types of problems might suggest a specific strategy. If, for example, one of the numbers to add was close to a multiple of 100, it was clever to use Ferit's strategy, but on the other hand there seem to be many problems where this method appears not to be very appropriate.

doing it skilfully - reflecting about different strategies

During the fifth part of the course the reflection about advantages and disadvantages of different strategies was put into the foreground. The children were posed several problems that suggested to add (subtract) step by step (examples 2 and 3 in fig.9), to add up (when both numbers were very close to each other, example 4) or to use an auxiliary task (examples 1, 5 and 6). Nevertheless it was not prescribed that these strategies had to be used.

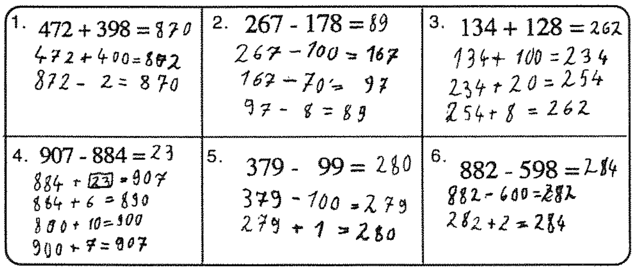


figure 9: Angela does it skilfully

Besides, the pupils developed other strategies that they thought to be skilful ones (fig.10). Simone, for example, took the subtrahend as the difference when she solved the (context-free) problem 312 – 278 so that she arrived at some sort of adding up backwards.

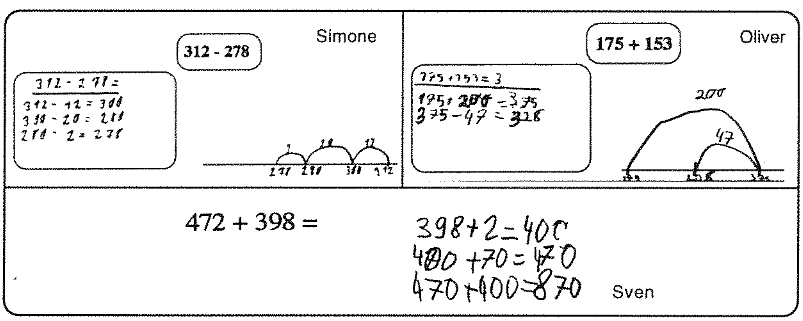


figure 10: Simone's, Oliver's and Sven's methods

A second example: Oliver always took an auxiliary task, even if both numbers were not close to a convenient one. When being asked why he said that this method was his favourite one and that he had no difficulties in using it. We cannot absolutely be sure whether this really was the case; nevertheless, this example shows that what seems to be a skilful way for us adults need not be clever for (individual) children and vice versa.

Finally, Sven's invention was to change both the numbers. He had found out that the ones could be combined to a (convenient) ten so that he could add up to 400 which was his first preliminary result. Subsequently he added tens and in the end hundreds of the number that initially was the first one (398 + 2 + 70 + 400).

that's the way we did it - reflecting about the course

Reflections about what has been done and learned should always be an integral component of mathematics learning. At the end of this course, reflection was given an extra amount of time: the children worked on a big maths diary for the second graders that was also shown to their parents, where they put down their experiences during the whole course.

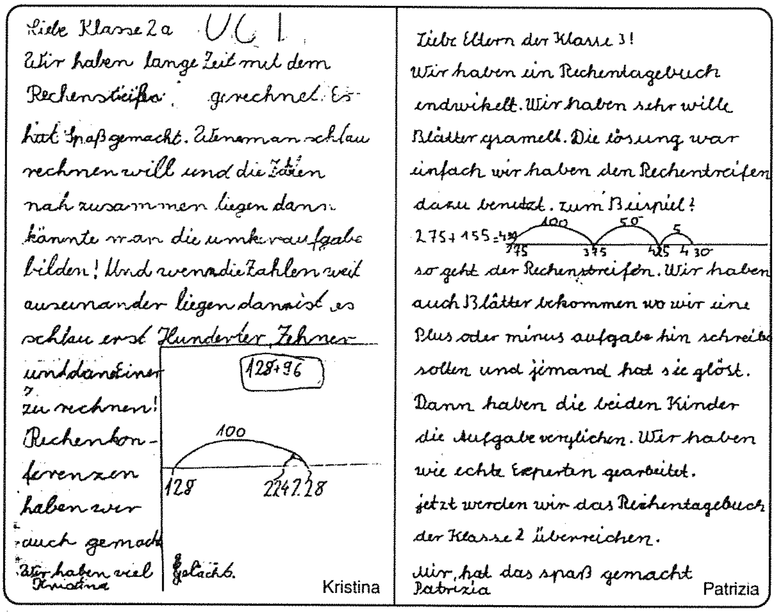


figure 11: Kristina's and Patrizia's introductions

This big diary (fig.11) contained not only selected sheets on which the children described their own methods (see 'that's the way I do it'), but also documents from the 'doing it like'- and the 'doing it skilfully'-parts of the course. Besides, the children included a couple of self-invented problems and wrote an introduction. Kristina for example put it this way:

> 'Dear second graders! We have been working with the empty numberline for quite a while. If you want to do it skilfully and the two numbers are close to each other, you can use the reverse task. And if the numbers are not close to each other, it is clever to start with hundreds, then tens and finally ones! We also did maths conferences. And we laughed a lot!'

She also gave an example where she drew an empty numberline on which she solved 128 + 96 by using an auxiliary task. Some children chose to write a second introduction for their parents, like Patrizia:

> 'Dear parents of grade 3! We have developed a maths diary. We have collected many, many sheets. The solution was easy; we have used the empty numberline for it. For example 275 + 155. (She then includes a solution.) That's the way the empty numberline goes. We also have received some sheets where we had to put down an addition or a subtraction problem and somebody else solved it. Subsequently both children compared their solutions. We have been working like real experts. Now we are going to hand over the big diary to the second graders. I have really enjoyed that.'

3 concluding remark

Encouraging children to find their own ways is anything else but a new idea: the writings of the German mathematics educator Johannes Kühnel (1869-1928), for example, are full of similar demands. So let me close by an abbreviated quotation from one of his books (Kühnel, 1930, p.91, translation by C.S.):

> 'I love to spend my holidays climbing in the mountains. Over the last years I have had more than twenty guides, all of them extremely conscientious. They went ahead, slowly and cautious, and the tourist had to follow them, slowly and cautious as well, trying to take exactly the same path as the guide took. Always staring to the ground, the tourist got ahead little by little. When reaching the ice region, the tourist was roped up and took even more care to step in his guide's footprints. Finally, they made it to the mountain peak, the tourist sometimes more being pulled up than walking himself. Beside these, I happened to know a couple of other guides, behaving in quite a different way. They let me go ahead and followed behind. That really was a different situation! It was neither possible to step in the guide´s footprints nor to stare to the ground. You had to be aware of everything that happened, all your muscles and nerves were working. Deciding which way to take claimed your full attention. You got to know the mountains in quite a different manner, you learned to decide for yourself and to take responsibility.
> Is that not a brilliant metaphor for our teaching? Were we not acting like conscientious guides, leading our children to what was regarded as important subject matter - guides who went ahead and insisted that the children had to step into our footprints? Should we not behave like real educators instead and let the children go ahead, let them decide for themselves which way to take - always being able to show them a more convenient one and giving them the security that nothing really serious can happen?'

There are a couple of reasons why Kühnel did not succeed with his visions in the long term. One of them was, that he was a loner - not very many colleagues shared his view on what mathematics education could be. I see a clear parallel between the learning processes of children and those of mathematics educators: going one's own way does also mean to share ideas with others and to learn from and with them. It is promising that nowadays we seem to be more than just sharing a few similar ideas about how the teaching of mathematics by adults can be replaced by the learning of mathematics by children.

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