Doing mathematics while practising skills

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1 introduction

Primary school practices cannot be changed on a large scale, if no alternative perspectives on the practising of skills are offered to teachers. The German project 'mathe 2000', directed by G. Müller and E. Wittmann, has worked on bridging the gap between seeing learning as a constructive process on the one hand and the prevalent way of practising based on behaviourism on the other. The contribution at hand aims at giving an overview of the conception of the so-called 'productive practising' that allows to achieve higher order goals – like reasoning or communicating mathematically –, even while facts, skills or algorithms are practised.

2 two different ways of practising

Since the early eighties, German primary education has seen a reform that was – to a considerable amount – coming from 'the inside': besides a couple of researchers and teacher educators it was mainly the *teachers* themselves who realised that they had to change the way in which they taught. Thus, children were more and more seen as active participants in the teaching/learning process and given increasing responsibility to direct their own learning.
During the mathematics lessons they consequently were given the freedom to decide *when* to work on certain tasks and to choose between worksheets showing different degrees of difficulty. In addition, there was a strong movement to no longer deal with pages of bare sums, but to make the practising of skills more motivating. Let me give one example (Krampe & Mittelmann, 1992, p.193): here, the children are to work out the results first and then connect the crosses according to the sequence of the numbers belonging to them (fig.1). If a child succeeds, he/she will end up with a nice picture of a sailboat. If he/she makes mistakes, the picture will be

spoiled. This might lead to rework the problems which do not seem to have being solved correctly. Many children really prefer these kinds of tasks instead of doing boring pages of sums. But are activities like these really contributions to letting children learn actively and autonomously, as the reform movement intended?

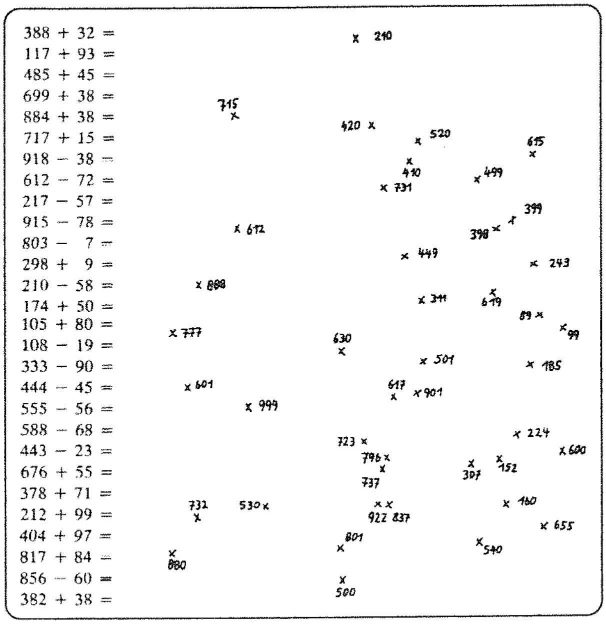

figure 1: connecting crosses

Before answering this question, let me give a second example (see Wittmann & Müller, 1992, pp.117-119): Children were asked to solve the following series of tasks and to put down what they had noticed.

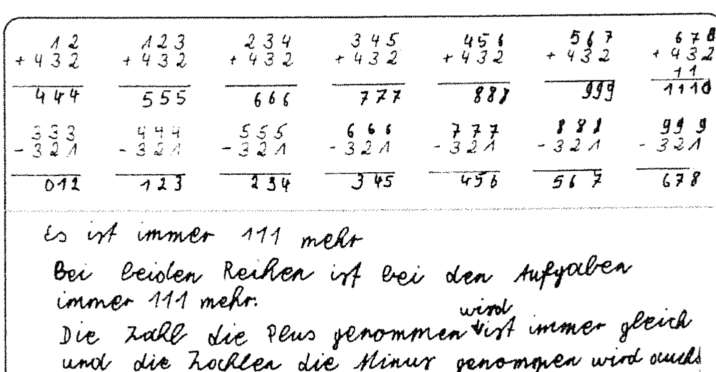

figure 2: nice numbers

They also had to find the first addend (minuend) in the three (four) last tasks each. Marc-André had worked out all the answers correctly and produced the following text (fig.2): 'It is always 111 more. In both series the answers are 111 more. The number to add is always the same and the numbers you have to minus as well.' I asked him what he meant by the second sentence, and he explained that the first numbers (the first addend and the minuend) showed a difference of 111, if being compared with each other.

The children were also encouraged to put down similar series of tasks; here, Marc-André produced four different series with multiples of 111 as results, using different rules of construction (fig. 3).

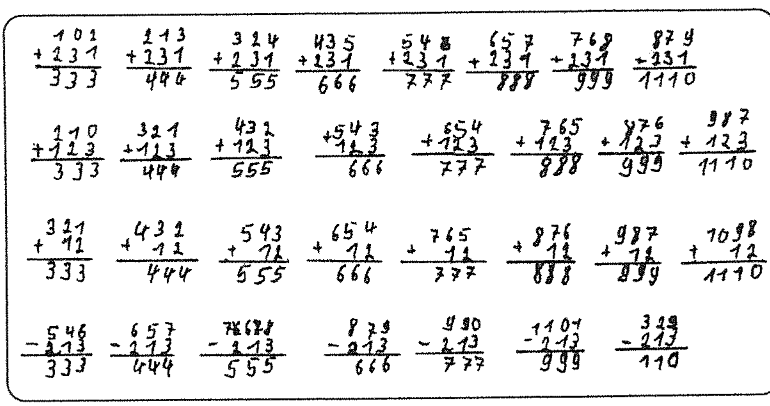

figure 3: Marc-André's own productions

Let us compare both ways of practising. At first both tasks aim at practising addition and subtraction in the domain of 1 through 1000. But these are almost all the similarities. 'Connecting crosses' is close to a behaviouristic understanding of practising as suggested by Thorndike (1922): A stimulus is given (the task to be solved), the pupil gives a response (he/she writes down his/her solution) and a positive (negative) reinforcement is given (the results can(not) be detected among the given numbers and finally a nice picture is (not) emerging).

This chain of 'stimulus-response-reinforcement' has to be repeated as long as the bond between giving a task (stimulus) and giving the correct answer (response) is not stable (Brownell, 1944). As activities like these were very prominent a couple of years ago - and to a certain extend still are nowadays - it can be concluded that despite all the positive developments, German primary school has seen, the reform in *mathematics* teaching practice often only happened on the surface, especially when skills were to be practised.

In comparison to 'connecting crosses', the single tasks chosen in the 'nice numbers' problem show a coherent structure; they have not been chosen at random, but are connected with each other. This underlying pattern allows children to discover, to describe, to reason, to be creative, to communicate, in brief: to really work mathematically. For example, they can work on the construction principles of the first series, being engaged in questions like: 'What do single tasks and their results have in common, in how far do they differ?' 'Is there an explanation why the result (in general) is made up of three identical digits?' 'What happens, if the numbers get too big?' Here, the practising of skills is not seen as primarily drumming in of bonds, but as the child's constructive activity, through which – whenever possible – the higher order goals are also trained. Teaching units like this were developed in the project 'mathe 2000' and embedded in the conception of productive practising (Wittmann, 1992) that I want to describe briefly in the following section.

3 the conception of productive practising

The theory of 'productive practising' embraces the practising of skills as a crucial part of the constructive learning process. Practising always contains an element of learning and learning always contains an element of practising (see also, Winter 1984, p.10). The project's theory emphasises the necessity to offer children (1) material-based and (2) coherent tasks.

Let me explain what is meant by *material-based* tasks first: Lorenz (1992, p.2) has shown that a shortcoming of mathematics teaching is that making the jump from ('concrete') representations to ('abstract') mathematical concepts and operations is almost completely left to children. The missing links are the development of mental images as well as the increasing ability to mentally operate with material that represents the concepts (like e.g. the arithmetic rack or the empty number line). Thus, enough tasks are needed where the children use material or mental images of material in order to practice. Of course, children should learn to work without 'concrete' material at hand in the long term, but they should use it just as long as they need it: so using material should on the one hand by no means be regarded as not needed in the classroom, but on the other hand its use should not be over-exaggerated.

The second aspect I would like to mention are the so-called *coherent* tasks. These are not selected at random, but show a coherent structure, having either a (pure) mathematical core or being rooted in real life. The ways of solving the problems and their results can support and correct each other.

Coherent tasks cannot only further the understanding of concepts and operations, but can also make a contribution to achieving higher order goals, such as those mentioned above. Not to be misunderstood: not every problem solving activity can be regarded as providing coherent tasks for practising. The precondition for the latter always is that the same skill or the same set of facts has to be used in order to work on a *series* of similar tasks.

The conception of productive practising embraces all types of practising: material-based tasks as well as formal; coherent series as well as those where the single tasks do not show any relationship. Nevertheless, the project clearly emphasises that material-based and coherent tasks should be given priority in order to replace the *premature* mechanisation that unfortunately seems so prevalent in mathematics teaching practice almost all over the world - despite the fact that many mathematics educators have convincingly pointed out the danger of too early and too much 'drill and practice' (Baroody, 1985; Brownell, 1954; Madell, 1985; Rathmell, 1979; Ter Heege, 1985).

4 examples

Let me now sketch some teaching units. Due to page constrains, I will neither deal with material-based tasks nor with real life contexts. I do not want to carry too many owls to Athens. The reader can find many of our examples in Wittmann and Müller (1990 and 1992).
Instead I will be discussing four teaching units representing the beauty of numbers and their relationships. Each of them is introduced by an activity for one of the four grades of German primary school (6 to 10 year-olds), but it is also shown, how variations of the problem context make it possible to use it in the other grades of primary school as well as in teacher education courses. This is meant to illustrate our ideas about a *conception* of rich and meaningful contexts that offer substantial activities on different levels - a counterbalance to (just) nice, unrelated activities.

arithmetic triangles

The arithmetic triangle teaching unit (Wittmann, Müller et al., 1994) is a slight variation of the idea posed by McIntosh & Quadling (1975). An equilateral triangle is divided into three congruent kites, each of which contains several counters (fig. 4). There is a simple rule: determine the number of counters in adjacent kites and write their sums on the corresponding sides, as to be seen in examples 1 and 2. It is also possible to give the number of counters in two kites and one sum (3) in order to work out the missing information. Counters can be used by the pupils to determine the solutions, which can be approached on a 'guess and check' basis, as well as by systematic variation based on the operative principle (Wittmann, 1985).

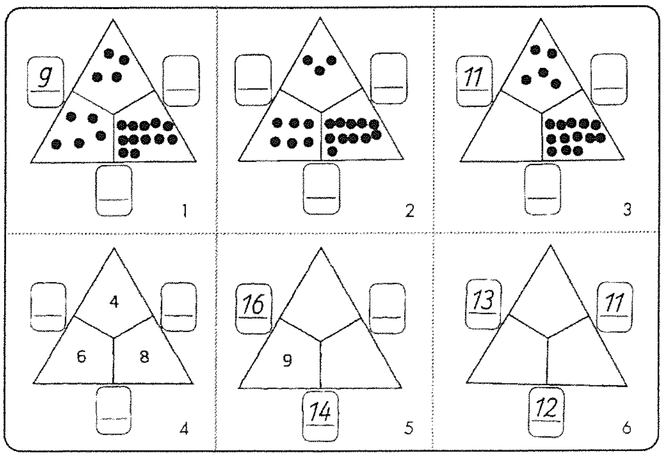

figure 4: arithmetic triangles – activities for first graders

Some further variations shall be mentioned:
- number symbols are used instead of counters (4);
- two sums and the number (of counters) in one kite are given (5);
- all three sums are given (6).

These are just some types of activities that seem to be suitable for first graders. This context can also be expanded by posing problems that deal with systematic variations, for example:
- what happens, if one additional counter is placed in each kite?
- or if one counter is moved from the top to the bottom right?
- if a counter is taken away from both the top and bottom left?
- if one is taken away in the top and one is moved from the bottom left to the right?

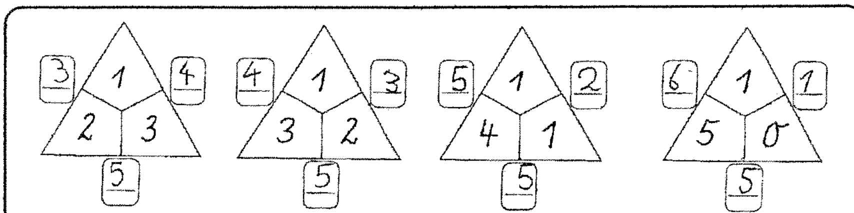

figure 5: Mathias' solutions to an arithmetic triangle-problem

An illustration for problems of this kind follows. Mathias has worked on the following one: 'What can be observed, if one counter is moved from bottom right to left?' Reflecting on the series he had completed, he said that the bottom sum does not change, whereas the other two are decreasing respectively increasing by one (fig.5).
Activities like these can easily be modified by taking larger integers, fractions or even algebraic expressions. Some of these and other related problems – for example involving arithmetic squares or hexagons – can also be investigated in teacher education courses. The arithmetic triangle is thus an example of a teaching unit that can be used at different levels of a long-term learning process and as such represents a shining example of the substantial problem contexts the project advocates.

number walls

The rule for the number walls problem context is as follows: you start with an almost triangular wall, consisting of stones, on which numbers are written. The number in one stone is the sum of both stones underneath; for communication purposes the stones in the bottom row are called bottom stones, the one on the top is called top stone (Wittmann & Müller, 1990, pp.103-106). Figure 6 illustrates one type of activities for second graders: the children are asked to fill in the missing numbers.

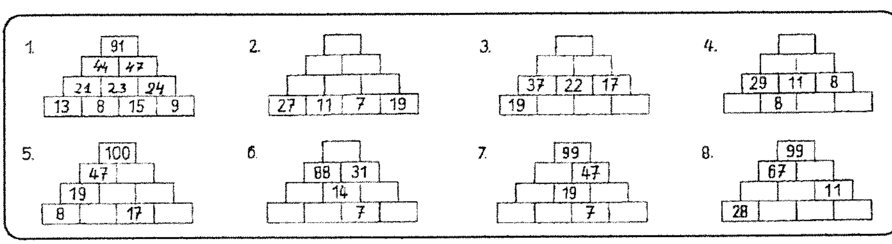

figure 6: number walls – activities for second graders

Another activity consists of investigating how the top stone changes, if the numbers in the bottom stones are made bigger or smaller. One example: how does the top stone change, if you take a wall with three rows and make the left base stone bigger? Second graders were asked to complete number walls where the numbers in the bottom stones were given (13, 20, 9; 14, 20, 9; and so forth) and to write down what they have noticed. In addition they should apply their findings to a fifth wall where two of the base stones (20, 9) and the top stone (66) were given (fig.7).
Here, Heinz put down that the left base stone was always changing: 'At first it is 13, then 14, 15, 16.' Subsequently he stated that the left stone in each row was getting bigger: 'The left side is always changing.' Bernd

referred to the first two examples, writing: 'At first it was 13, then 14. At first it was 33, then 34. At first it was 62, then 63.' Without using any example, Jörg noted that he had found out that both the left stone in the bottom row and the top stone were getting one bigger: 'The numbers in the left bottom corner get one number more. The top stone also gets one number more.' Finally, Helga's solution shall be presented who made mention of the left stones in each of the three rows: 'I always go one more in the left stone in the bottom, in the left one in the middle and in the left one on top.'

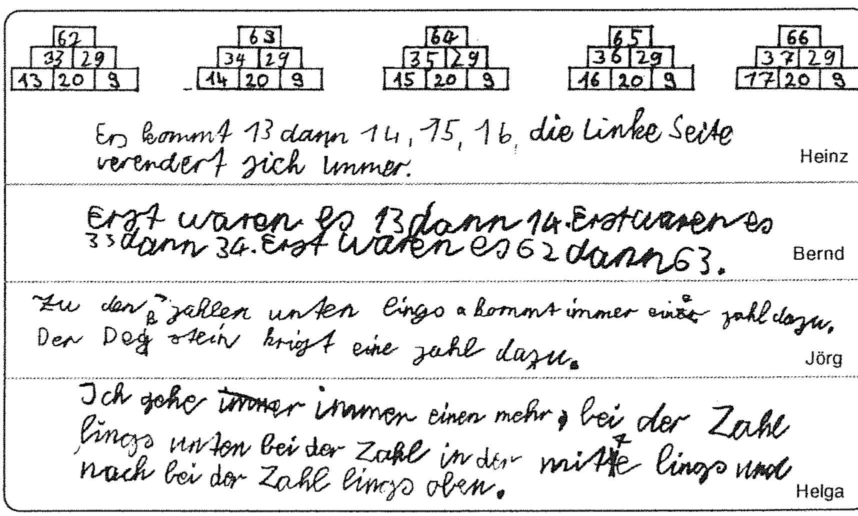

figure 7: second graders' investigations on number walls

There are lots of variations of this problem on different levels (see also Krauthausen 1995), like for example:

- What happens if you make the right stone in the bottom row bigger (smaller)?
- What happens if you make the middle stone in the bottom row bigger (smaller)?
- What happens if you enter the same integer in each of the bottom stones? If you use consecutive integers?
- Find different number walls with a certain top stone (for example, 20)! Is there a relationship between these?
- Three numbers (e.g. 4, 5, 9) are given. How do they have to be arranged so that the top stone is as big (small) as possible?
- You start from a given number wall (where e.g. the top stone is 43). You want to construct a number wall with the top stone 50. How can you use the initial wall?

- Is it possible, to reach a certain number (for example 100), if the bottom stones are identical (consecutive)?

All these problems can be investigated, if walls with more rows or other domains of numbers are taken.

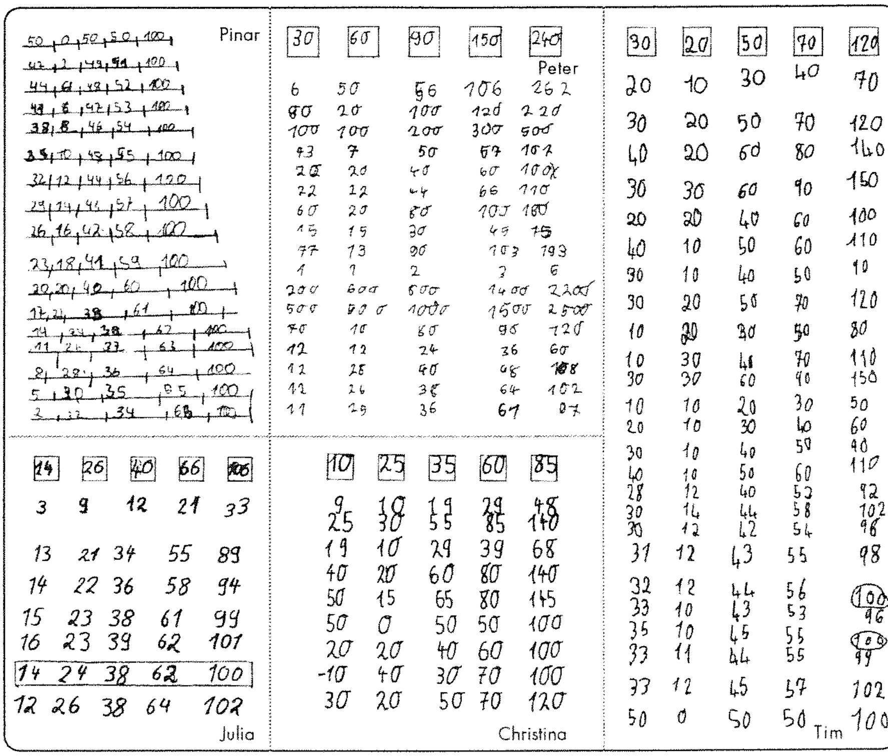

figure 8: children's work on a number chain problem

number chains

One possible number chain activity for grade 3 is as follows: take two integers (including 0), the (so-called) starting numbers, write them side by side, add them and write the result beside the second number. Now add the second and the third number! Write the result beside the third number. Finally add the third and the forth number. This sum is to be written as the fifth number and is called target number. Two examples:

3	7	10	17	27
64	8	72	80	152

In the beginning, children should explore this algorithm by playing around with the numbers. But when they have got acquainted with it, they should be given problems to work on. These could be of the following kind:

- Try to reach a high (a low) target number – without any precondition or within a certain domain of numbers!
- Reverse both starting numbers and observe the influence of this operation on the target number!
- Make one of the target numbers one (two, three) bigger (smaller). What do you notice? Make both bigger (smaller)! Or one bigger and the other one smaller!
- Find all (seventeen) pairs of starting numbers that add up to the target number 100.

Some examples of third graders' work on the last problem are given in figure 8. A didactical problem for the reader: try to find out in how far the children were working systematically!

Like all the other problem contexts I am discussing in this paper, the number chains can be used in grade one as well as in teacher education courses (Selter and Scherer, in press). Two activities shall illustrate this:

- First graders can be asked to find all pairs of starting numbers of chains (length 4) that reach the target number 20.
- Prospective teachers can work on the following problem: Which numbers cannot be target numbers, if one takes a chain with 4, 5, 6, ..., n numbers.

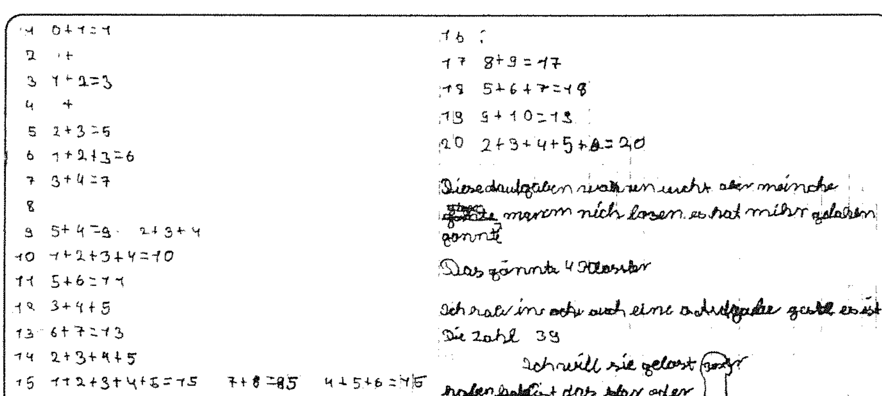

figure 9: Canan's writing about a consecutive integers problem

consecutive integers

The last problem context that I want to discuss is connected with one of the big problems of number theory: can you represent each natural number as a sum of consecutive integers? How many different solutions are there to a given number? Sylvester's theorem tells us that there are as many ways as the number has odd divisors other than 1. There are for example, three ways of decomposing the 15: 1 + 2 + 3 + 4 + 5, 4 + 5 + 6 and 7 + 8; the odd divisors of 15 different from 1 are: 3, 5 and 15.

In the following I want to describe how fourth graders dealt with this problem (figs.9 and 10). They were not only asked to work out all the different solutions in the domain of 1 through 20, but also to describe what they noticed and to produce a problem for teacher students who were attending the lesson. Canan, for example, found out almost all decompositions into 2, 3, 4 respectively 5 addends, with exception of 18 = 3 + 4 + 5 + 6. Then she commented as follows: 'These problems were easy for me, but some problems can't be solved. Fourth graders are able to work on problems like these. I have also posed a problem, it is the number 39. And I want you to solve it, O.K.?'

Linda initially used the strategy to find all possible decompositions for 1, then for 2, for 3, and so forth. But she experienced some trouble here and changed her way of working, as she summed up consecutive numbers and related these to the corresponding results. Her text mirrors a little bit, how she felt: 'It was easy, but I got confused. But now it's your turn: Work out the 50!' Tim put down something remarkable: he had not only found out that there will be no solutions for 2, 4, 8 and 16, but he had also noticed the relationship between these numbers: '16 is a doubled 8; 8 is a doubled 4; 4 is a doubled 2.'

Indeed the powers of two are the only numbers having no solution, which we can also derive from Sylvester´s theorem, as they have no odd divisor apart from 1. Corinna wrote: 'These problems were fairly easy. Some of the numbers did not work. No. 2, 4, 8, 16. 15 had the most solutions. And now you shall do some work. Here is my problem: 1 + 2 + 3 + ... + 19 + 20.'

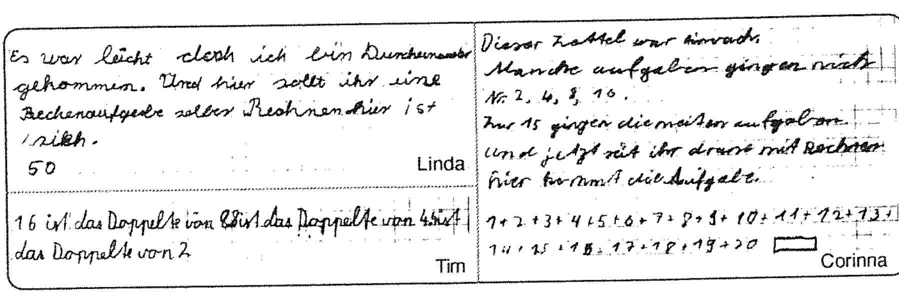

figure 10: fourth graders' work on a consecutive integers problem

Once again, there are many possibilities for variations, some examples:

- Children could be asked to continue the given sequence: 1, 3, 6, 10, and to describe what they noticed (Selter, 1996). Older pupils can be asked, if all numbers in the domain of 80 through 100 can be expressed as a sum of not more than three of these so-called triangular numbers (Kalthoff, 1995);

- They can be given the problem to find skilful ways to add consecutive integers, such as the sum given by Corinna (see above). This can be extended for older pupils to the famous Gauss problem to add the first 100 integers (Winter, 1985);
- Children can be encouraged to sum up multiples of consecutive integers, such as 5 + 10 + 15 + 20 + 25 as possible. They could also be asked, if there are five numbers with a constant difference that add up to 50, like 6 + 8 + 10 + 12 + 14 (Steinbring, 1995);
- Other geometric numbers offer excellent opportunities to practice skills, while doing mathematics really takes place: square numbers, rectangular numbers, pentagonal numbers, and so on.

5 concluding remark

In the present article I used – for purposes of easy reading, and thus maybe simplifying – the notion of practising of skills as an expression to represent the training of (1) basic facts, (2) flexible mental and non-algorithmic written calculation and (3) the standard algorithms. The project 'mathe 2000' has developed teaching units for all three domains, but under a certain background philosophy: it is our considered opinion that the basic facts are absolutely important and should be mental practised in a meaningful way. With respect to the other two domains we are advocating a shift from (3) to (2) (Krauthausen, 1993): For us, flexible and non-algorithmic written calculation is the crown of arithmetic in primary school.

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