CHANGING REPRESENTATIONAL INFRASTRUCTURES CHANGES MOST EVERYTHING: THE CASE OF SIMCALC, ALGEBRA AND CALCULUS

Jim Kaput *University of Massachusetts-Dartmouth* jkaput@umassd.edu

Roberta Schorr *Rutgers University-Newark* schorr@rci.rutgers.edu

This paper addresses the question of how research processes interacted with design and development processes throughout the SimCalc Project. The long-term goal of the SimCalc Project is to democratize access to the mathematics of change and variation, including the ideas underlying calculus. The core means of achieving this goal involve building and testing webs of representational and simulation tools and teaching and learning activities that exploit broadly available kinesthetic, visualization, linguistic and other human resources. From the outset, it became clear that we were unwrapping a series of fundamental representational innovations whose implications would require the better part of a generation to unfold. While empirical research played a role at several junctures, the bulk of the research was and continues to be intimately tied into iterative design processes centered on trying various representational options, answering questions such as: a) How to make a smooth, comfortable interface with a flexible authoring capability within the constraints of next-generation software and hardware? b) Does it make sense to users, and how does it change their capabilities over time, for both students and then teachers? c) How to control and scale complexity of the user experience? and, d) What are the curricular implications and how do these change as authoring capability changes? These questions give rise to implementation questions as the curriculum and technological platforms stabilize and mature. These questions include: a) How can our curricular option spaces be integrated with what standards-writers and others regard as important? and, b) How can these approaches be made implementable within the near-term capacities of teachers and their support infrastructures, including pre- and in-service teacher education systems? In order to make our research story comprehensible, we will begin with a series of vignettes taken from studies involving middle-grade students from an extremely low SES population in an unusually challenged middle school in central Newark (NJ). With these student-performance images of progress towards our democratization goal, we will then reflect upon the historical context for the SimCalc representational approaches with a view to contextualizing the research issues that have arisen. To understand the story, it is important to keep in mind: (a) the profound role that representational infrastructures such as writing systems and algebra play in determining what and how people think, and what they are capable of doing (Cole, 1997; Donald, 2001—A mind so rare); (b) how their learnability defines not only their associated communities of practice and the social support systems for their learning such as our education system but also societal norms regarding who is educable; (c) how today's communities of mathematical practice/use and our education system historically co-evolved with our historically received representational infrastructures (e.g., reading, writing, arithmetic, algebra) under the semiotic constraints of static, inert media to serve the needs of a narrow knowledge-producing/using elite; and

(d) that the greatly relaxed semiotic constraints of the dynamic and interactive computational medium enable the rapid evolution as well as deliberate design of new representational infrastructures, such as spreadsheets, dynamic geometry, interactive diagrams, visually editable graphical systems, state-space visualizers of dynamical systems (Stewart, 1990), as well as add new interactivity to traditional representational infrastructures as is reflected in CAS's. Historically, changes in representational infrastructures and changes in associated communities of practice (and literacy) as well as social institutions were slow, and, more importantly, on the same time-scale. This consonance in time-scales no longer prevails, which creates new tensions and new opportunities, as changes in representation and its learnability are occurring on much shorter time-scales than changes in the surrounding social systems. Research of the sort described in this volume is necessarily done in relation to expectations defined by pre-existing social systems—whether or not those expectations are appropriate to changed representational infrastructure. We will describe how research in the SimCalc Project has come to terms with this fundamental disjuncture of expectations.

Part I: An Overview of the SimCalc Project

Goals and Strategies of the SimCalc Project

The SimCalc Project began in 1993 as an NSF research project intending to work towards democratizing access to the Mathematics of Change and Variation (MCV), including the ideas underlying calculus, by developing and classroom testing new software and curriculum materials for Grades 6–13, with a first focus on middle school and early high school mathematics, gradually spreading upward in traditional mathematical level and outward to the more contemporary iterative and visual mathematics of dynamical systems. Access to the MCV had for centuries been limited to a small intellectual elite comprising at most 5–10% of the population in recent years because the mathematics was assumed to require a long series of algebraic prerequisites that effectively filtered out most of the population, especially those from less advantaged families, neighborhoods and schools. To democratize access the project needed to confront deeply institutionalized patterns of belief and practice regarding the nature of mathematics in general and the MCV in particular, who might learn it, how it may be learned, and it needed to confront the long-standing curriculum structures built into American education, and the assessment practices separating different levels of schooling. Working through this deep change is a generation-long process that continues today.

Two key representational strategies are intended to provide broad entry to this critically important domain of mathematics: (1) to use interactive simulations hot-linked to new forms of visually editable graphs and visualization tools, and (2) to build the fundamental relationships between rates and accumulations (embodying in computational form what is normally referred to as the "Fundamental Theorem of Calculus") into both the structure of the software and associated curriculum. The second strategy is akin to the way the extraordinarily powerful hierarchical structure of the number system is built into efficient representations that rapidly became the standard means by which numbers could be used and operated upon.

SimCalc implementation strategies range widely, including technological generalization across platform-types, curricular integration with standards-based and other curricula, developing and testing teacher education materials, scaffolding teacher practices within teachers' instructional materials, creating commercial partnerships, and supporting further research. The further research includes work by independent researchers as well as work by SimCalc staff, especially research involving curricular integration and the potential of wireless connectivity in combination with the representational strategies identified above (see below).

Three Stages of the SimCalc Project as of 2001

The initial project covered 7-years in two funding cycles beginning in 1993, but was based in prior and initially independent work by the second author and by Ricardo Nemirovsky at TERC in the late 1980's. The project continues today in several different forms as described in III below.

I. **1993–96: SimCalc I** amounted to a successful proof-of-concept, as software and instructional materials were successfully piloted by various project partners with populations of inner city and at-risk students in Massachusetts, New Jersey, New York, Michigan and California. SimCalc I showed that mainstream students, indeed, students who typically were not expected to enroll in algebra, let alone calculus, could learn the ideas and skills if given the opportunity. The vignettes below illustrate this work. By choice, we worked in some of the most challenging school circumstances in the country, e.g., in Newark, New Jersey (see Part II), and in high schools that had lost their accreditation, on the assumption that if we could succeed there (in terms of student learning of central topics), we could succeed in easier contexts. The Project also investigated detailed issues of learning, component software development, interface design, the interaction between physical and simulated motion, among many others.

II. **1996–2000: SimCalc II** addressed issues of scale-up: How to integrate these ideas and materials in Grades 6–13 curricula in ways that respect the accountability and other constraints that prevail in schools today? How to build functional software for school-affordable platforms? And especially, how to create an approach to the new uses of software and curriculum that builds upon and expands the <u>existing</u> capacities of teachers, teacher-educators, schools and districts? The project also spun off a technology-oriented investigation led by Jeremy Roschelle to examine the promise and challenges of component based design. This became part of the Education Object Economy Initiative and eventually became the NSF-funded Educational Software Components of Tomorrow (ESCOT) Project in 1998.

III. **2000–present: SimCalc III** spreads across several endeavors, funded independently and directed by pairs of colleagues from the original three SimCalc PI's (Jim Kaput, Jeremy Roschelle at SRI International, and Ricardo Nemirovsky at TERC):

1. Developing assessment systems that provide reliable data on student learning and on the surrounding conditions that promote sustainable implementation of SimCalc and SimCalc-like innovations (an IERI funded planning project directed by Roschelle & Kaput);

2. An investigation of the affordances and constraints of wireless networks of diverse device-types in mathematics classrooms (an NSF-funded Research on Learning and Education (ROLE) project with Texas Instruments and Palm as commercial partners—directed by Kaput & Roschelle);

3. An investigation, including neurophysiological aspects, of how experiences with physical devices can feed into and influence the learning of mathematical ideas, especially those involving the math of change & variation (an NSF-funded ROLE project—directed by Nemirovsky & Kaput).

4. The development of commercial software and curriculum products, initially in partnership with Texas Instruments, with particular focus on Flash software for TI's bread-and-butter lines of graphing calculators serving core curriculum in Grades 7–13 (independently funded using the vehicle of a corporation-partnership between SimCalc personnel and the University of Massachusetts, SimCalc Technologies, LLC).

5. Both research-oriented and commercial partnerships in development with publishers and developers of standards-based curricula to embed and extend SimCalc materials and technologies into existing or new instructional materials in mathematics and science.

We will now turn to a series of vignettes that are intended to illustrate the SimCalc *representational* strategies at work. The *implementation* strategies will be discussed later, as we examine the role of research in the project.

Part II: An Illustration of Students Exploiting the SimCalc Representational Strategies

Summary of SimCalc Representational Strategies

Before looking at students' work, we will summarize the core web of five representational innovations employed by the SimCalc Project, all of which require a computational medium for their realization. The fourth and fifth—not discussed below in detail—are mentioned for completeness. The fourth, especially through the importing and then re-animating of students' physical motions, plays an especially important role in SimCalc instructional materials to anchor the visual experience of the simulations in students' kinesthetic experience. Cross-platform software, Java MathWorlds for desktop computers can be viewed and downloaded at http://www.simcalc.umassd.edu and software for hand-helds can be examined and downloaded from http://www.simcalc.com.

1. **Definition and direct manipulation of *graphically defined* functions, especially piecewise-defined functions**, with or without algebraic descriptions. Included is "Snap-to-Grid" control, whereby the allowed values can be constrained as needed—to integers, for example, allowing a new balance between complexity and computational tractability whereby key relationships traditionally requiring difficult computational and

conceptual prerequisites can be explored using whole number arithmetic and simple geometry. This allows sufficient variation to model interesting situations, avoid the degeneracy of constant rates of change, while postponing (but not ignoring!) the messiness and conceptual challenges of continuous change.

2. **Direct connections between the above representational innovations and simulations**—especially motion simulations—to allow immediate construction and execution of a wide variety of variation phenomena, which puts phenomena at the center of the representation experience, reflecting the purposes for which traditional representations were designed initially, and, most importantly, enabling orders of magnitude tightening of the feedback loop between model and phenomenon.

3. **Direct, hot-linked connections between graphically editable functions and their derivatives or integrals**. Traditionally, connections between descriptions of rates of change (e.g., velocities) and accumulations (positions) are mediated through the algebraic symbol system as sequential procedures employing derivative and integral formulas—but they need not be. In this way, the fundamental idea, expressed in the Fundamental Theorem of Calculus, is built into the representational infrastructure from the start, in a way analogous to how, for example, the hierarchical structure of the number system is built into the placeholder representational system for numbers.

4. **Importing physical motion-data via MBL/CBL and re-enacting it in simulations, and exporting function-generated data to drive physical phenomena** LBM (Line Becomes Motion), which involves driving physical phenomena, including cars on tracks, using functions defined via the above methods as well as algebraically. Hence there is a two-way connection between physical phenomena and varieties of mathematical notations.

5. **Use of hybrid physical/cybernetic devices embodying dynamical systems,** whose inner workings are visible and open to examination and control with rich feedback, and whose quantitative behavior is symbolized with real-time graphs generated on a computer screen.

The result of using this array of functionality, particularly in combination and over an extended period of time, is a qualitative transformation in the mathematical experience of change and variation. However, short term, in less than a minute, using either rate or totals descriptions of the quantities involved, or even a mix of them, a student as early as 6th–8th grade can construct and examine a variety of interesting change phenomena that relate to direct experience of daily phenomena. And in more extended investigations, newly intimate connections among physical, linguistic, kinesthetic, cognitive, and symbolic experience become possible.

Illustrating the SimCalc Representational Strategies Exploited By Students

Context for the Vignettes

Our work in schools, particularly in schools where academic achievement has traditionally been very low, has shown that young children, in grades as early as six, can build powerful understandings of the MCV when provided with appropriate learning environments, pedagogical practices, and curriculum materials. For example, middle school children in one of Newark's lowest performing middle schools as measured by State mathematics and reading tests, where in the previous year, only 2.9% of the 8th graders passed the state-mandated mathematics test,[1] have developed and are able to articulate meaningful insights into the velocity-position

[1] This school under-performed the Newark average for 8th graders, and was the second lowest scoring school in the district. It was the fourth lowest in the state. It should be noted that in many districts, particularly suburban districts, passing rates reach close to 100%.

relationships embodied in the Fundamental Theorems of Calculus. In fact, one student, upon developing and then confirming for himself (via generic reasoning from well-chosen special cases) a hypothesis about generating velocity graphs by using the slope of a corresponding position graph, and then generating position graphs by calculating the area under the curve in a corresponding velocity graph, noted that "this should be in books." He had come up with an idea that does indeed appear in books—books written by the greatest geniuses of western civilization and then appearing in Calculus textbooks over the ages that few, if any, students from his school community would study until reaching college—if and when that might ever occur.

The instructional context for the following vignettes is an after-school program for 15 students meeting 1.5 hours per week for 18 weeks taught by the second author with the cooperation and assistance of the school's science teacher. The excerpts below are taken from the fourth lesson. In the prior lesson, the students had explored mean values as they built graphical representations for an elevator that had variable velocities throughout a trip, compared with an elevator that traveled at a constant velocity throughout the trip, where both elevators traveled for the same amount of time, and had the same starting and ending positions. They used a velocity-graph only version of the configuration in Figure 1.

Description of the Representational Strategies Employed

Here the (red) staircase-shaped velocity graph drives the (red) elevator on the left side of the building appearing to the left of the graphs. The (blue) constant velocity graph drives the (blue) elevator to the right of the staircase-driven elevator. These embody Representational Strategies 1 and 2. By pointing, clicking and dragging, the user can create or visually edit any of the segments appearing. In particular, for the step-wise varying velocities initially used by the students, only vertical or horizontal stretches are allowed to segments. This ensures that area computations are sums of products, and by turning on "Snap-to-Grid" the values appearing in these sums of products can be made to fit those appearing in the axes tic marks—whole numbers in this case. The two corresponding position graphs are shown on the right side of the figure simply to illustrate how Representational Strategy 3 can appear in certain activities (note that a change to either velocity or position graph yields a change to the appropriate segment of its counterpart). Importantly, the contents of the tool-bar on the right, e.g., function-types, coordinate axes, editability of functions, can be configured by the teacher or curriculum author (via drag and drop) to fit the students or activities at hand.

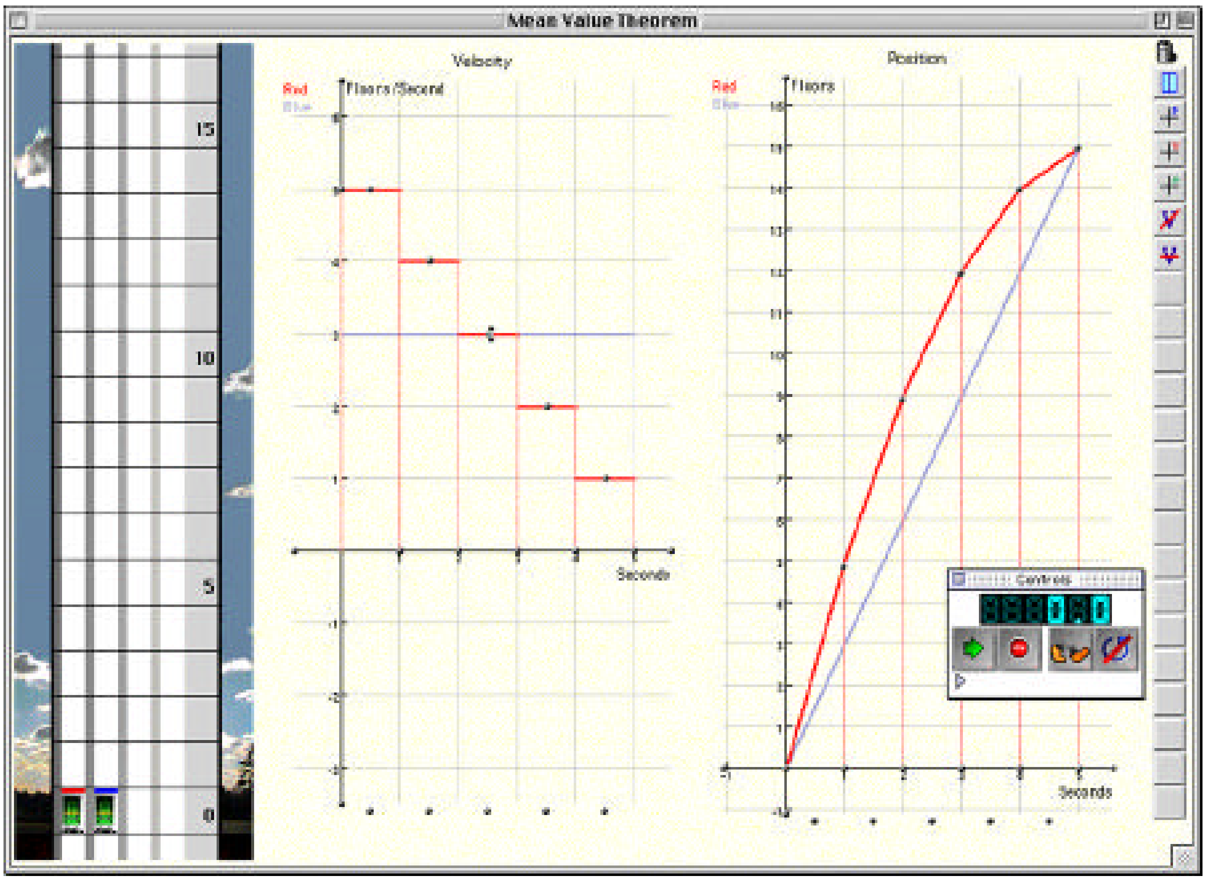


Figure 1. Averages from Both Velocity and Position Perspectives

Vignette 1: Area-Based Reasoning With Piecewise Constant Velocity Graphs

During this particular session, the students were given a piecewise constant velocity function (as in Figure 2 below) and asked to determine where the character ("frog") would be at the end of several different time periods, for example, after 8 seconds. They were also challenged to talk about the motion of the character over intervals during the trip, i.e. was he going faster or slower than in another interval, and they were asked to determine his change in position over the requested interval. During this particular episode, 8 students were gathered around the front of the room (while several others were working in pairs in the computer lab), and were viewing the graph on the overhead projection system. The researcher was in control of the computer for most of the interactions, although occasionally used the software in accordance with student requests. A series of transcript excerpts is provided below:

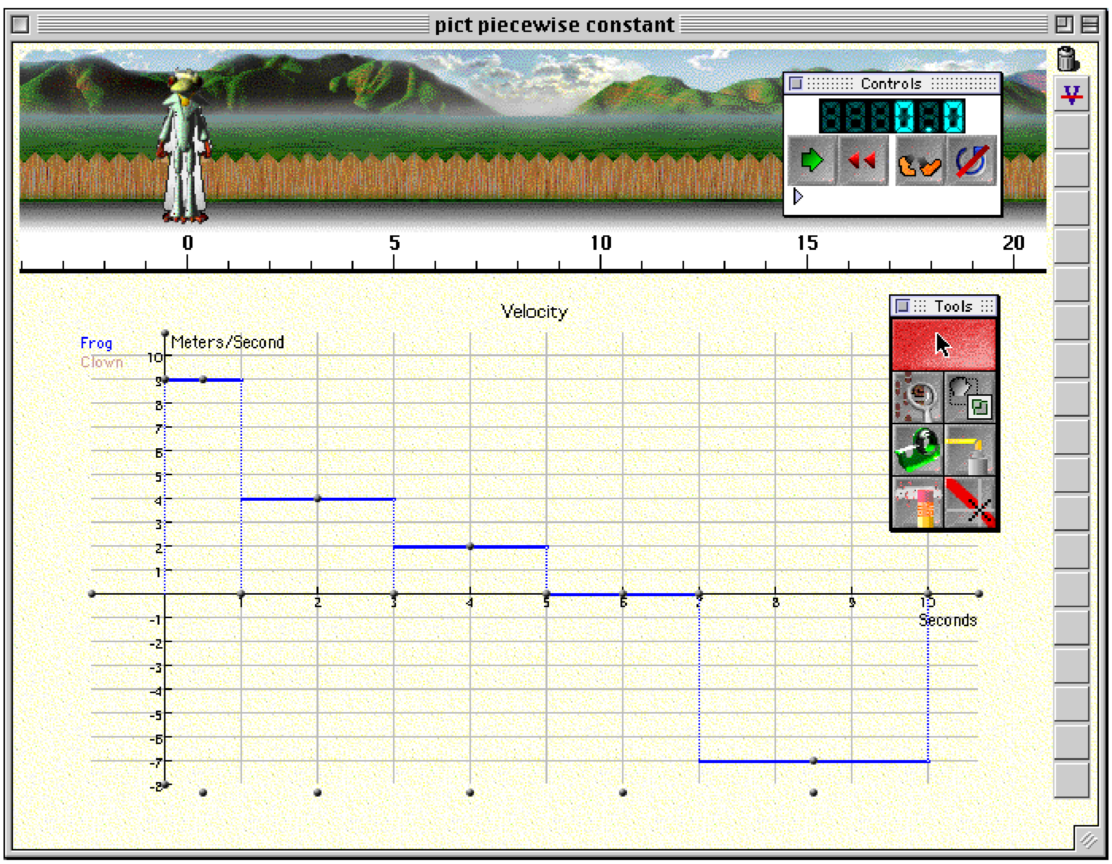


Figure 2. Piecewise Constant Velocity Function for Frog

Student I say it's huh, 14. Because right here on the 1st second…on the middle of the 2nd…right here he's on the 9th floor (at one second)[2]…right here he's gonna go on the 13 (at 2 seconds)…17 (at 3 seconds)…19 (at 4 seconds)…21 (at 5 seconds)…he's gonna stop here for 2 seconds (from 5 to 7 seconds)…and then you said take 21 from 7…it leaves you 14.

Researcher Why are you guys subtracting?

Students Because negative…because it's a negative (velocity).

In the above excerpt, one student calculated the area below the curve, if it the curve was above the horizontal axis. She, informally representing the consensus of the group, noted that when the velocity was 0, the character would stop. (The common understanding across the group is reflected in the chorus nature of the response to the "subtracting" question.) She then subtracted the area below the curve where it was below the horizontal axis. The students were able to use this strategy to calculate exactly where the frog would be at any given point in time. In addition, they were also able to determine when and why the frog was going faster or slower. For example, another student offered the following explanation to justify why he felt that the character would be going faster during the second segment than during the third:

[2] The reference, as quite often occurs, is in the language of the first motion system that the student has experienced—which was the Elevator system appearing in Figure 1. This is an indication that the student has already abstracted away certain particulars, such as the physical context for the motion, and is focusing on the displacement in terms of areas under the graph.

Researcher	So you are telling me that he actually slowed down over here (referring to the third segment). Then how come?
Student	Look, look I can tell you.
Researcher	Tell me why.
Student	He slowed down cause he is going 4 meters, right?
Researcher	4 meters per second?
Student	For 2 seconds. He goes 8, 8 meters. But here it goes 2 meters…yeah 2 meters per second for 2 seconds…4 meters. It will just be four and he will go faster here cause he will go 4 meters in just 1 second.
Researcher	Oh, you're saying he goes 4 meters in just 1 second here and here…
Student	It is just two in 1 second

In this particular exchange, the student compared the distance that the frog would travel in one second for two different segments to determine the relative speed over each.

Vignette 2: Generalizing to Linearly Varying Velocity

The researcher then presented the students, for the first time, with a single *linearly increasing* velocity function and asked them to describe the motion of the character, and the final position of the character. At this point, the students saw the first 5 seconds of the velocity graph in Figure 3. Several students suggested that they should find the area under the curve, to figure out where the frog would end his trip, even though the curve was not horizontal. They began to hypothesize about the final position of the frog as they used the concept of area under the curve and tied the motion of the frog to the graphical representation. They had used this area-method with constant velocity segments and sequences of such segments, and were spontaneously using it again in this instance, despite the increase in computational complexity. Representational Strategy 1 was at work. One student commented that "I think it is going to go from slow to fast because it is a slant…so it is going to change, it is going to change throughout its trip." In effect, this student is generalizing the step-wise variation he had experienced previously to continuous variation. The researcher then extended the velocity graph with two additional segments (up to 12 seconds) and repeated the questions, and the students extended their area-based reasoning to this situation.

The final episode of the session, in which the researcher added the last segment of Figure 3, from 12 to 17 seconds, helps reveal the level and depth of the mathematical thinking that often emerged with this class of students. In this excerpt, the students were challenged to figure out what would be happening during this last segment, which extended below the x-axis was added to the linearly varying velocity graph. From the outset, many of the students shouted out that the frog would be going backward because the velocity was negative. The researcher then challenged them to talk about where the frog would end up, and why.

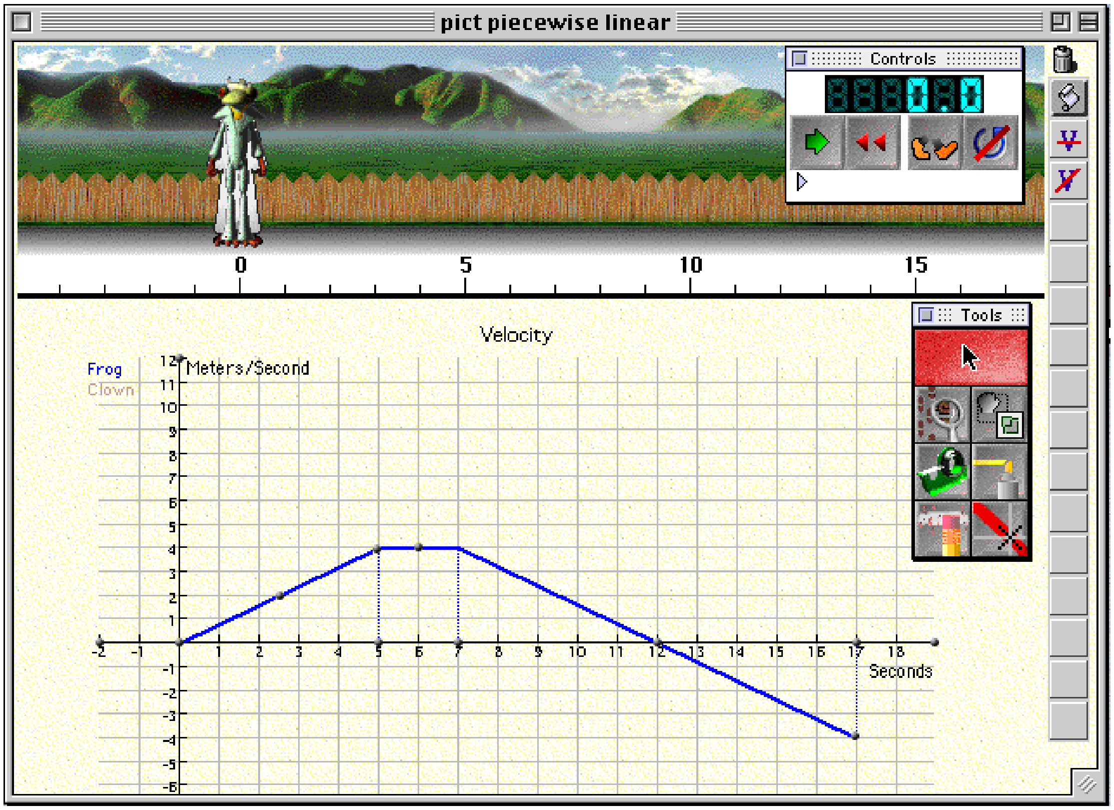


Figure 3. Linearly varying velocity

Student He's going back to 18. He's going back to 18 because if you look at it, there is going to be an invisible line right there to make that 10 (referring to a line perpendicular to the horizontal axis, extending upward from the end of the last segment to the horizontal axis). And then 28 (28 had been calculated as the position before the last segment had been added) minus 10 is 18.

Researcher So you're prediction is what?

Student It's an invisible line right there where the 12 is just goes down and you equal it up with the line over there by the…

Researcher	Can you show me? I don't see what you mean. Yeah, go ahead you had a good idea that you were about to share.
Student	Right here, if you put an invisible line and equal it up with that, it will be the same as that and when it goes down, it is a negative. So subtract it.

This student was pointing to the positive-area section of the graph that was congruent to the new section (time interval from 7 to12 seconds) that had just been formed (time interval from 12 to17 seconds). She correctly identified the two as having the same (numerical) area, and therefore came up with an answer of 18 by subtracting off the negative area. Several other students echoed her solution. The researcher continued the questioning, in effect applying Representational Strategy 2.

Researcher	And what does that me in terms of the motion of the frog?
Student	It's gonna go backwards.
Researcher	When does the frog turn around and go… you tell me he is going backwards…I hear that from you guys. When does the frog turn around
Student	…28 (meters)…

The researcher asked the students to clarify by talking about when, in terms of seconds, he turned around. To that, the students said "As soon as he gets to 12 (seconds), he is gonna turn around, but he is gonna be walking slower." Not only did the student notice when the Frog would turn around, she also predicted that he would be walking slower. This prompted the researcher to ask the students to talk more about the motion of the frog, continuing to apply

Representational Strategy 2. An interaction between three students (two boys and a girl) shows the interplay between the geometry and the MCV resulting from the joint application of Strategies 1 and 2.

Student 1	… when it go to the end it's gonna be fast. Because if you look upside down, it's going up, it's going fast.
Researcher	So over here, is he going faster or slower than over here (first pointing to the graph at 13-14 seconds, and then 15-16 seconds)?
Student 1	Over there he is going slower than over there because…
Researcher	Wait, wait, wait… I don't know what you meant. Do you want to just point to where…
Student 1	Wait, look upside down…if you look upside down it goes from downward to upward. So that means it's going faster.
Researcher	So he is going faster…is he going faster here (at 13 seconds)…
Student 1	Yes
Researcher	Than here (at 15 seconds)?
Student	Yes.
Student 2	No, I think it's the opposite.
Researcher	He thinks it's the opposite. That's a really interesting thing. You think he's going slower here than here?

Student 2	Yeah, because look at the top of the other one, uh look.
Researcher	Show me.
Student 1	Right here it goes fast right, then right here it goes slow so this to the top right here it's gonna go fast and then from this right here he gonna go slow.
Researcher	What do you think?
Student 3	I think…I think I know.
Researcher	You think, you think…that's important to think about what you're thinking.
Student 3	I say, I say it's gonna go fast when he get like down here.
Researcher	Why?
Student	Because when it's…because when this is up there, at the bottom…it would go…and it would go up…it would go slow. So I say it would go slow.
Researcher	Agree? Disagree? Guys, agree? Disagree?
Students	Agree
Researcher	Are you ready to huh…does anyone else have a speculation?
Student	No.

Researcher	I like your upside down approach. Want to tell me about that some more?
Student	Yeah, because it's going downward.
Researcher	Tell me what you mean.
Student	Since it's going down, it's almost the same thing but it's down. But it you tilt it up, it will be going up.

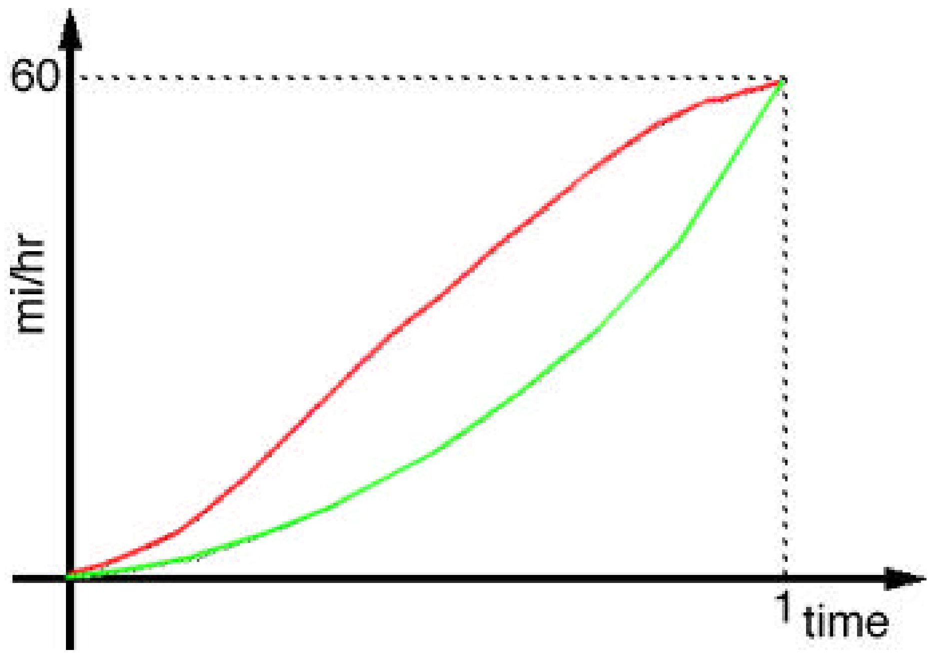
Reflections on the Vignettes

Not only could these middle school students—students coming from extremely low socio-economic backgrounds, discuss complex mathematical ideas after a relatively brief intervention of 4 sessions, in this case, the Mathematics of Motion, they could also fluently move between and among different representational systems—including those involving symbols. Representational Strategies 1 and 2 appear to work under these circumstances in the sense of providing the means by which students could begin with very simple functions and motions that tapped into their simplest intuitions about motion and leveraged their limited arithmetic and geometric competencies. Indeed, we have repeatedly seen that this approach feeds back into and intensely engages their prior mathematical skills and understandings, in effect, becoming a vehicle for teaching core mathematics such as signed number arithmetic, signed areas, dimension analysis (especially when non-motion rates are considered), rate and ratio, among others. These mathematical experiences also lead directly into algebraic formalizations, especially linear and quadratic functions as the position functions associated with constant and linearly changing velocities, respectively. This should not be a surprise, and reflects a contextualization-oriented curricular strategy widely used in the US in standards-based curricula and elsewhere in the world, e.g., **[Realistic Mathematics references].**

Students in this group routinely solved problems without using the technology. One young girl summed it up by saying that she was so sure of her method and solution that she didn't need to see the simulation. This level of understanding-based conviction was not unique to this group of children. For example, after using SimCalc-based instructional materials for a 5-week period (4 sessions per week with 1.5 hours/session) in a summer course designed for a group of seventeen pre-college economically disadvantaged students, not a single student used any technological resources to solve problems similar to those described in previous sections on their post-test, despite the fact that they took the test at their computer with the software loaded and running. This is a frequently reported phenomenon among users and helps answer the frequently voiced concern that students in such technological environments would become so dependent upon the technology that they simply would not, or could not solve the problems without it. The opposite appears to be the case.

The young students whose work we sketched above articulated their mathematical thinking with fluency and fluidity, despite test scores suggesting that they were not very good in mathematics. When fully engaged with challenging ideas, they were able to marshal linguistic resources to match the task. And when they were confronted with new concepts, they were able to use what they had already learned in innovative ways. Recent work deliberately mixing writing with these kinds of mathematical activities suggests that the same positive feedback and engagement that occurs with other mathematical ideas may apply to the development of reading, writing and speaking capacities.

Our work has consistently shown that middle, high school, and college level students are able to solve problems that have been documented as being problematic—even for college students enrolled in traditional calculus courses. For example, consider the following problem (Monk & Nemirovsky, 1992; Thompson, 1994):



Both the red car and the green car reach 60 mi/hr after one minute as shown on the two velocity graphs. Are they side by side after one minute or is one ahead?

Figure 4. The two cars problem

Three fourths of all students could not only determine which car would be ahead, they could describe the motion of the cars, the approximate position of the car at any given time, and make predictions about where the cars would be, given various scenarios that extended the time the cars would be traveling. Interestingly, they had never previously dealt with non-linear continuous change as depicted in the graphs of Figure 4. They also successfully dealt with isomorphic scenarios involving changing rates and accumulation (such as those involving debt, salary, etc.). Their understanding of these ideas appears to be robust. In fact, as evidenced by follow-up interviews with 7 of the students[3] approximately 1–2 years after their initial exposure and without having taken an intervening calculus course, students in the summer course mentioned previously were still able to solve problems such as the one in Figure 4. In fact, one of the 17 students who had been identified as “at risk” due to poor overall performance during the two years that followed the summer class was able to draw upon her experience to convince a group of high school teachers of her solution to a problem. This student had been invited by the second author to attend a workshop for high school mathematics teachers. Much to her surprise, she was the only participant able to correctly answer a problem[4] presented to the teachers as part of a warm-up activity. Her explanation was based upon her experience with SimCalc two years before, and entailed an imagined description of what she envisioned would happen in a particular simulation involving two actors and a graphical representation of their motion as they walked in opposite directions. She was able to explain and defend her solution despite the fact that *all* of

---

[3] Additional students were either unavailable or had withdrawn or transferred from the University and therefore could not be interviewed. Students that were interviewed were representative of the group in terms of overall academic performance.

[4] The problem was as follows: There was a young monk who sought spiritual clarification by climbing to the top of a steep mountain. She embarked at sunrise, following a winding path wide enough for only one person, stopping occasionally along the way to rest or eat. By sundown she reached the mountain-top. She meditated there that night, and the entire next day. The following morning, at sunrise, she began the journey down along the same narrow path she had used for her ascent, again stopping intermittently for food and rest.

Assuming that she left at precisely the same time on both treks and returned to the same spot, can you prove (or disprove) that there is some place along the path that she passed at exactly the same time of day on both journeys?

the teachers present, at least initially, disagreed with her.

The fact that students seem able to generalize from highly constrained initial functions and contexts to less constrained ones, to articulate their understandings with fluency beyond what they normally display, and recall and use their understandings on problems well-documented as challenging after a more than two years suggest that the representational strategies being employed tap into well-established and stable systems of interpretation, both visual and linguistic. Unlike the learning of techniques based in formal systems of representation, where fragility and quick erosion of competence are the norm, the learning that we have repeatedly seen **[refs]** seems to be robust and long-lasting.

What the Vignettes Do Not Address: Implementation Strategies

The fact that the vignettes were taken from classes taught by a researcher outside the students daily mathematics class means that, while the students were unusually under-prepared in mathematics, the instructor was specially prepared to teach them the mathematics in question, in terms of both the content itself and the pedagogical approaches that fit the approach. And the context was well-designed to ensure success—a relatively small number of students, the lack of curricular and other constraints on the instructor and the students, and good technological support. In effect, we factored out the kinds of school and teacher factors that, inappropriately dealt with, prevent most innovations from succeeding. Similar, indeed startlingly positive results as measured formally by pre- and post-tests, have been documented (Kaput & Cabral, in preparation), and in somewhat less "hot-house" conditions by Nickerson, Nydam, & Bowers (in press). These kinds of results indicate the power of the representational strategies *if appropriately implemented.* Current work, as indicated above, is addressing issues of implementation on several fronts (Bowers & Doerr, 2001; Kaput, 1999).

However, our focus here is on representational issues, and an understanding of these, especially from a historical perspective, will help contextualize work on implementation. In particular, it will help us understand the depth of the change that is involved here and, in turn, help us understand the role of research in the kinds of major changes that technology makes possible and necessary—which from our point of view are rooted in the representational changes that technology brings

Part III: Stepping Back for the Long View: The Evolution of Representational Infrastructures and Their Material Technologies

Kaput (2000) traced a deep and dual analogy, one level involving changes in learnability and the other involving changes in availability, of representational infrastructures. One analogy relates the evolution of western writing systems and the impact of these changes on expressivity and especially *learnability* on one hand, to the corresponding changes in the representational infrastructures of mathematics as they have become increasingly visual and more expressive and learnable on the other. Below, because of space limitations, we will give the “punch-lines” of the story.

Learnability of Writing Systems

Over a period of several thousand years in the Mid East, early writing (based in the need to create external records of quantitative information (Schmandt-Besserat, 1978, 1992)) evolved from ideographic to phonetic/acoustic and eventually alphabetic. In its ideographic forms, writing was a specialists’ tool, requiring a lengthy apprenticeship to learn the subtleties of 600 or more signs and highly context-dependent, nonlinear interpretation processes. Such writing systems made large demands on human memory and interpretive skill, and hence were laboriously learned and used only by specialists—scribes. For example, approximately 15% of all the 100,000 existing cuneiform tablets were used to train scribes (Davies, 1987; Walker, 1987). Note also that the complex non-phonetic system and the lexical lists used to train scribes during the 3rd millennium B.C.E. remained essentially unchanged for more than 600 years—a hint that the conservative nature of education is not something new! It was driven and constrained by many factors beyond the semiotic creativity of the scribes.

As writing gradually became more phonetic, it tapped into highly expressive sound-based systems of meaning-making and communicating that had been evolving for the previous several hundred thousands of years and well supported physically and neurophysiologically (Deacon, 1997; Donald, 1991). Writing tapped into a powerful pre-existing system. However, while the writing systems became more expressive, they remained hard to learn and hence remained a specialists' endeavor.

In its next stages, however, in the two millennia BCE, phonetic syllabaries evolved to alphabetic writing systems in several languages (Arabic, Hebrew, Aramaic, and Phoenician), and the Phonecian alphabet was adopted and modified by the Greeks to become the system that is used in all western languages today (Woodard, 1996). This extraordinary achievement allows an extremely efficient and learnable bidirectional mapping between the sound stream and sequences of 2 dimensional characters that amounts to one of the supreme achievements of the species (Ong, 1982). Roughly two dozen characters are all that's needed, an optimization that has never been improved upon. In Greece, anyone who wanted to learn these mappings (reading and writing) and was allowed to by virtue of social class, could do so at a relatively young age (Havelock, 1982).

Writing, as a fundamental representational infrastructure, changed the means by which humans constructed their world individually (Nelson, 1996) and culturally (e.g., Cole, 1997; Donald, 1991; Havelock, 1982; Olson, 1985; Shaffer & Kaput, 1999). Humans became able to communicate, build, and accumulate knowledge (and all that comes with knowledge—including power and control) across time and space.

Availability of Writing Systems

The second level of the analogy involves the *availability* of representational infrastructures, as reflected in the availability-changes associated with the printing press on one hand, and now the diverse forms of computing devices in which the new visual representational infrastructures are becoming available. Hence, another two millennia passed before reading and writing became widely distributed across the populations of the west, when written documents became widely available. As we know, this availability of representational infrastructure required yet another technological innovation, the printing press (Eisenstein, 1979). As argued by Haas (1996), however, this technology was slow to develop into a broadly useable form and hence its impacts required some two to three centuries to be realized. See Kaput (2000) for more detail.

It is important to keep in mind the distinction—between a change in representational infrastructure, such as alphabetic writing, and a change in the material means by which that infrastructure can be embodied, such as the printing press and inexpensive paper—which participates in a different kind of infrastructure, a combined technological, physical and social infrastructure. We now turn to the other side of the dual analogy, the side that involves mathematics and mathematics learning.

The Representational Infrastructures—Operative Systems—of Arithmetic and Algebra

I will not recount the histories of these systems here. See (Kaput, Noss, & Hoyles, 2001) for a more detailed account. However, the evolution of each was a lengthy process, covering thousands of years before the achievement of an efficient symbol system upon which a human could operate. Unlike written language, which supported the creation of *fixed* records in static, inert media, the placeholder system of arithmetic that stabilized in the 13th–14th centuries supported rule-based *actions* by an appropriately trained human upon the physical symbols that constitute quantitative operations on the numbers taken to be represented by those symbols. This system and the algorithms built on it, seems to be optimal in an evolutionary sense similar to the way the alphabetic phonetic writing systems seem to be optimal. Each has remained relatively stable for many centuries and has spread widely across the world. The arithmetic system, although initially a specialist's tool—for accounting purposes—came to be part of the general cultural tool-set as needs for numerical computation arose in Western societies. Interestingly, the early algorithms developed for accounting in the 14^{th}–15^{th} centuries and that appeared in the first arithmetic training books at that time have remained essentially unchanged to this day, and continue to dominate elementary school mathematics (Swetz, 1987).

Algebra, of course, began in the times of the Egyptians in the second millennium BC as evidenced in the famous Ahmes Papyrus by using available writing systems to express quantitative relationships, especially to "solve equations"—to determine unknown quantities based on given quantitative relationships—the so-called "rhetorical algebra" that continued to Diophantus' time in the 4^{th} century of the Christian era, when the process of abbreviation of natural language statements and the introduction of special symbols began to accelerate. Algebra written in this way is normally referred to as "syncopated algebra." Achievement to that point was primitive, with little generalization of methods across cases and little theory to support generalization.

But, in a slow, millennium-long struggle involving the co-evolution of underlying concepts of number, algebraic symbolism gradually freed itself from written language in order to support techniques that increasingly depended on working with the symbols themselves according to systematic rules of substitution and transformation—rather than the quantitative relations for which they stood. Just as the symbolism for numbers evolved to yield support for rule-based operations on symbols taken to denote numbers, where attention and mental operations guide actions on the *notations* rather than what they are assumed to refer to, the symbolism for quantitative relations likewise developed. Bruner (1973) refers to this as an "opaque" use of the notations rather than "transparent" use, where the actions are guided by reasoning about the entities to which the notations are assumed to refer. In effect, algebraic symbolism gradually freed itself from the (highly functional) ambiguities and general expressiveness of natural language in order that very general statements of quantitative relations could be very efficiently expressed.

However, the more important aspects of the new representational infrastructure are those that involve the rules, the syntax, for guiding operations on these expressions of generality. These emerged in the 17th century as the symbolism became more compact and standardized in the intense attempts to mathematize the natural world that reached such triumphant fruition in the "calculus" of Newton and Leibniz. In the words of Bochner (1966, pp. 38–39):

> Not only was this algebra a characteristic of the century, but a certain feature of it, namely the "symbolization" inherent to it, became a profoundly distinguishing mark of all mathematics to follow. ... (T)his feature of algebra has become an attribute of the essence of mathematics, of its foundations, and of the nature of its abstractness on the uppermost level of the "ideation" a la Plato.

Beyond this first aspect of algebra, its role in the expression of abstraction and generalization, he also pointed out the critical new ingredient:

> ... that various types of 'equalities,' 'equivalences,' 'congruences,' 'homeomorphisms,' etc. between objects of mathematics must be discerned, and strictly adhered to. However this is not enough. In mathematics there is the second requirement that one must know how to 'operate' with mathematical objects, that is, to produce new objects out of given ones (p. 313).

Indeed, Mahoney (1980, p. 142) points out that this development made possible an entirely new mode of thought "characterized by the use of an operant symbolism, that is, a symbolism that not only abbreviates words but represents the workings of the combinatory operations, or, in other words, a symbolism with which one operates." This second aspect of algebra, the syntactically guided transformation of symbols while holding in abeyance their potential interpretation, flowered in the 18th century, particularly in the hands of such masters as Euler, to generate powerful systems of understanding the world.

But this operative aspect of algebra is both a source of mathematics' power, and a source of difficulty for learners. However, another learning-difficulty factor is the separation from natural language writing and hence the separation from the phonetic aspects of writing that support tapping into the many powerful narrative and acoustic memory features of natural language. Indeed, as well known via the error patterns seen in the "Student-Professors Problem", the algebraic system is in partial *conflict* with features of natural language (Clement, 1982; Kaput & Sims-Knight, 1983). For many good reasons, traditional character string-based algebra is *not* easy to learn. It is perhaps as learnable as ideographic writing. And, it should be noted, algebra *is* an ideographic system, despite its roots in phonetic writing. And, till relatively late in the 20th century, it was regarded as a specialists' tool.

Indeed, in contrast with the arithmetic system, *the algebra system was built by and for a small and specialized intellectual elite* at whose hands, quite literally, it extended the power of human understanding far beyond what was imaginable without it. Importantly, it was designed and used by specialists without regard for its learnability by the population at large. The effect of these learnability factors did not really become felt until the latter part of the 20th century when education systems around the world began to attempt to teach algebra to the general population, but especially in the United States, where it is treated as a subject isolated from other mathematics. Prior to the middle of the 20th century, the algebra literacy community was quite small, quite analogous to the small literacy communities of specialists associated with early writing.

By the end of the 20th century, with the growth of the knowledge economies, the need for quantitative insight spread across the population of industrialized countries in a way analogous to the way it spread for arithmetic skill earlier. This general need combined with the politically driven need to democratize *opportunity* to learn higher mathematics, typically assumed to require knowledge and skill in algebra, has produced considerable tension in many democracies, especially the United States, where access to algebra learning has come to be seen as political right (Moses, 1995). The attempts to democratize access to traditional algebra in the Unites States have not been especially successful despite much work and energy.

Deep Changes in the Nature of the Representational Infrastructure—Writing and Algebra Linking Into Existing Powerful Visual Systems of Acting and Seeing

Just as writing gradually tapped into another, previously established human system of meaning making and communicating, and changed radically in the process as it became phonetic, algebra may likewise be in the process of doing so. In this case, instead of the auditory-narrative system, it is the visuo-graphic system. Although it may not have been Descartes' (or Fermat's) intent, anticipated by Oresme (Clagett, 1968), he (and Fermat) laid the base for tapping into humans' visual perceptual and cognitive capacities previously employed only by geometry. Recall Joseph Lagrange's comment

> As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited. But when these sciences joined company, they drew from each other fresh vitality and thenceforward marched on a rapid pace towards perfection (cited in Kline, 1953, p. 159).

From 350 years ago to our contemporary graphs of quantitative relationships, we see an analogy to the gradual transition that occurred as writing became more phonetic—the newer ways coexist with the old in various combinations as we graph algebraic functions which are defined and input into our graphing systems via character strings. And more recently, we have been able to hot-link these in the computer medium as reflected in most CAS.

Can we make the *next* transition that might make the representation and manipulation of quantitative relationships broadly learnable analogous to the invention of the alphabet? Below, we shall propose an analogous step for algebra.

The Development of Material Means By Which Access to Learnable Representational Infrastructures Might Be Democratized—Analogy With the Printing Press.

With the emergence of computers, we are involved in an extended process analogous to the evolution associated with the development of the printing press. The first stages involved expensive and hence rare central computers, what we have known as "mainframes" and "mini computers." Then came the microcomputer and networks—connectivity, local and global. More recently, we have seen the emergence of hand-helds and, even more recently, connectivity across device-types. We are moving towards the kind of ubiquitous access and full integration into life and work that was achieved by printed writing materials that eventually occurred by the time of the Industrial Revolution. But, as our cursory examination of the printing press evolution suggests, the process will take time and will depend on many other changes taking place along the way.

The Shift from Static, Inert Media to Dynamic, Interactive Media

The systems of knowledge that form the core of what was taught in schools and universities in the 20th century were built using some representational infrastructures that evolved (e.g., alphabetic and phonetic writing) and others that were somewhat more deliberately designed, mainly by and for a narrow intellectual elite (e.g., arithmetic and, to a greater extent, operative algebra). In all cases, they were instantiated in and hence subject to the constraints of the static, inert media of the previous several millennia. But, the *computational* medium is neither static nor inert, but rather, is dynamic and interactive, exploiting the great new advance of the 20th century, autonomously executable symbolic processes—that is, operations on symbol systems not requiring a human partner (Kaput & Shaffer, in press). The physical computational medium is based on a major physical innovation, the transistor, that, in turn, was the product of the prior knowledge system (Riorden & Hoddeson 1997). The longer term development of the computational medium is reviewed in Shaffer & Kaput (1999).

Reflecting the fact that we are in the midst of a huge historical transition, we see three profound types of consequences of the development of the new medium for carrying new representational infrastructures:

Type 1: The knowledge produced in static, inert media can become knowable and learnable in new ways by changing the medium in which the traditional notation systems in which it is carried are instantiated—for example, creating hot-links among dynamically changeable graphs equations and tables in mathematics. Most traditional uses of technology in mathematics education, especially graphing calculators and computers using Computer Algebra Systems, are of Type 1. The Carnegie Learning *Algebra Tutor* is a concrete example of a complete Type I system, and most uses of CAS in this volume are likewise of Type I.

Type 2: New representational infrastructures become possible that enable the reconstitution of previously constructed knowledge through, for example, the new types of visually editable graphs and immediate connections between functions and simulations and/or physical data of the type developed and studied in the SimCalc Project—to be described briefly below in Part III. Cabri Geometry and Geometers Sketchpad embodiments of Euclidean

Geometry are also of Type II, as are certain broadly expressive computer languages such as Logo and its derivatives (Abelson & diSessa, 1981; diSessa, 1986; Noss & Hoyles, 1996, in press; Papert, 1980).

Type 3: The construction of new systems of knowledge employing new representational infrastructures—for example, dynamical systems modeling or multi-agent modeling of Complex Systems with emergent behavior, each of which has multiple forms of notations and relationships with phenomena. This is a shift in the nature of mathematics and science towards the use of computationally intensive iterative and visual methods that enable entirely new forms of dynamical modeling of nonlinear and complex systems previously beyond the reach of classical analytic methods—a dramatic enlargement of the MCV that will continue in the new century (Kaput & Roschelle, 1998; Stewart, 1990). The AgentSheets (Repenning, 2000) approaches and materials to agent-based modeling, as well as the Participatory Simulation Project and materials (Wilensky & Stroup, 2000) provide concrete Type III examples (both involve Logo-like derivatives).

Tracing any of these complex consequences is a challenging endeavor, particularly since they overlap in substantive ways due to the fact that knowledge is co-constituted by the means through which it is represented and used—it does not exist independently of its representation (Cobb, Yackel, & McClain, 2000). And its deployment depends on many factors outside the realm of its immediate use—as was the case for all the representational infrastructures reviewed above. This aspect of the SimCalc story is the focus of Part IV.

This sets the stage for our case study based in our recent work in the SimCalc Project involving the Mathematics of Change & Variation (MCV), of which a subset concerns the ideas underlying Calculus. Thus we are focusing on a Type 2 change.

Calculus—A Ridiculously Brief History of Its Place in the Curriculum

Although the Greeks, most notably Archimedes—whose extraordinary computational ability compensated for the weaknesses of the available representational infrastructure in supporting quantitative computation—developed certain mainly geometric ideas and techniques, the Mathematics of Change and Variation leading to what came to be called “Calculus” evolved historically beginning with the work of the Scholastics in the 1300’s through attempts to mathematize change in the world (this history, including the emergence of today’s calculus curriculum in the 18th century, is reviewed in Kaput, 1994).

The resulting body of theory and technique that emerged in the 17th and 18th centuries, cleaned up for logical hygiene in the 19th, is now institutionalized as a capstone course for secondary level students in many parts of the world, and especially in the United States. These ultimately successful mathematizing attempts were undertaken by the intellectual giants of Western civilization, who, in so doing, also developed the representational infrastructure of algebra, including extensions to infinite series and coordinate graphs, as part of the task. Their work led to profoundly powerful understandings of the different ways quantities can vary, how these differences in variation relate to the ways the quantities accumulate, and the fundamental connections between varying quantities and their accumulation. These efforts also gave rise to the eventual formalization of such basic mathematical ideas as function, series, limit, continuity, etc. (Boyer, 1959; Edwards, 1979).

Over the past two+ centuries this community’s intellectual tools, methods and products—the foundations of the science and technology that we utterly depend upon—were institutionalized as the structure and core content of school and university curricula in most industrialized countries and taken as the epistemological essence of mathematics (Bochner, 1966; Mahoney, 1980) as noted above. This content has also been taken as the subject of computerization. That is, it has been the target of Type 1 reformulation as reflected in several chapters in this volume. We now turn to a further description of the SimCalc approach, which involves an epistemic-level reformulation of the content relating to calculus based in the representational strategies outlined earlier.

The Representational Strategies from an Historical Perspective

Importantly, taken together, the representational strategies are not merely a series of software functionalities supporting some curriculum activities, but amount to a reconstitution of the key ideas. Hence we are not merely treating the underlying ideas of Calculus in a new way, treating them as the focus of school mathematics beginning in the early grades and rooting them in children's everyday experience, especially their kinesthetic experience, but *we are reformulating them in an epistemic way.* We continue to address such familiar fundamentals as variable rates of changing quantities, the accumulation of those quantities, the connections between rates and accumulations, and approximations, but they are experienced in profoundly different ways, and are related to each other in new ways.

These approaches are not intended to eliminate the need for eventual use of formal notations for some students, and perhaps some formal notations for all students. Rather, they are intended to provide a substantial mathematical experience for the 90% of students in the U.S. who do not have access to the Mathematics of Change & Variation (MCV), including the ideas underlying Calculus, and provide a conceptual foundation for the 5–10% of the population who need to learn more formal Calculus. Finally, these strategies are intended to lead into the mathematics of dynamical systems and its use in modeling nonlinear phenomena of the sort that is growing dramatically in importance in our new century (Cohen & Stewart, 1994; Hall, 1992; Kaput & Roschelle, 1998; Stewart, 1990).

In terms of our historical perspective, we see this current work as part of a large transition towards a much more broadly learnable mathematics of quantitative reasoning, where both the representational infrastructure is changing as well as the material means by which those more learnable infrastructures can be made widely available. Taken together, it may be that the kinds of representational strategies outlined and illustrated above constitute the development of a new "alphabet" for quantitative mathematics which might do for mathematical representation what the phonetic alphabet did for writing, particularly if coupled to the right kinds of physical implementations, which are examined below.

An Illustration of SimCalc Representational Strategies 1–3 As a Reconstitution of Content: Determining Mean Values

Figure 1 shows the velocity graphs of two functions, respectively controlling one of the two elevators on the left of the figure. The downward-stepping, but positive, velocity function, which controls the left elevator, typically leads to a conflict with expectations, because most students associate it with a downward motion. However, by constructing it and observing the associated motion (often with many deliberate repetitions and variations), the conflicts lead to new and deeper understandings of both graphs and motion. The second, flat, constant-velocity function in Figure 1 that controls the elevator on the right provides constant velocity. It is shown in the midst of being adjusted to satisfy the constraint of "getting to the same floor at exactly the same time." This amounts to constructing the average velocity of the left-hand elevator which has the (step-wise) variable velocity. This in turn reduces to finding a constant velocity segment with the same area under it as does the staircase graph. In this case the total area is 15 and the number of seconds of the "trip" is 5, so the mean value is a whole number, namely, 3. We have "snap-to-grid" turned on in this case so that, as dragging occurs, the pointer jumps from point to point in the discrete coordinate system. Note that if we had provided 6 steps for the left elevator instead of 5, the constraint of getting to the same floor at exactly the same time (from the same starting-floor) could not be satisfied with a whole number constant velocity, hence could not be reached with “snap-to-grid” turned on.

The standard Mean Value Theorem, of course, asserts that if a function is continuous over an interval, then its mean value will exist and will intersect that function in that interval. But, of course, the step-wise varying function is *not* continuous, and so the Mean value Theorem conclusion would fail—as it would if 6 steps were used. However, if we had used imported data from a student’s physical motion, as in Figure 3, then her velocity would necessarily equal her average velocity at one or more times in the interval. We have developed activities involving a second student walking in parallel whose responsibility is to walk at an estimated average speed of her partner. Then the differences between same-velocity and same-position begin to become apparent. Additional activities involve the two students in importing their motion data into the computer (or calculator) *serially* (discussed below) and replaying them *simultaneously,* where the velocity-position distinction becomes even more apparent due to the availability of the respective velocity and position graphs alongside the cybernetically replayed motion.

Note how the dual perspectives of the velocity and position functions, both illustrated in Figure 1, show two different views of the average value situation. In the left-hand graph, we see the connection as a matter of equal areas under respective velocity graphs. In the right-hand graph, we see it through position graphs as a matter of getting to the same place at the same time, one with variable velocity and the other with constant velocity. Depending on the activity, of course, one or the other of the graphs might not be viewable or, if viewable, not editable. For example, another version of this activity involves giving the step-wise varying position function on the right and asking the student to construct its velocity-function mean value on the left. This makes the slope the key issue. By reversing the given and requested function types area becomes the key issue. Importantly, by building in the connections between rate (velocity) and totals (position) quantities throughout, the underlying idea of the Fundamental Theorem of

Calculus is always at hand, built into the experience of the user at every stage.

The fourth SimCalc representational strategy in its most widely used form amounts to linking students' physical experience of motion with the cybernetic experience by re-animating imported students' motions in order to interact with and compare with synthetically defined motions (using graphs or formulas).

Moving Towards Availability, Not Merely Learnability

Now, in the left two pictures of Figure 5 below are partially analogous software configurations—two elevators controlled by two velocity graphs. Instead of the clicking and drag/drop interface of the desktop software, most user interaction is through the SoftKeys that appear across the bottom of the screen which are controlled by the HardKeys immediately beneath them. The left-most screen depicts the Animation Mode, with two elevators on the left controlled respectively by the staircase and constant velocity functions to their right. The middle screen depicts the Function-Edit Mode, which shows a "HotSpot" on the constant-velocity graph. The user adjusts the height and extent of a graph segment via the four calculator cursor keys (not shown), and can add or delete segments via the SoftKeys. Other features allow the user to scale the graph and animation views, display labels, enter functions in text-input mode, generate time-position output data, and so on—very much in parallel with Java MathWorlds, but without the benefits of a direct-manipulation interface. The right-most screen shows a horizontal motion world with both position and velocity functions displayed (hot-linkable if needed, as with the computer software).

We have developed a full, document-oriented Flash ROM software system for the TI-83+ and a core set of activities embodying a common set of curriculum materials that parallels the computer software to the extent possible given the processing and screen constraints (96 by 64 pixels!). The parallelism is evident in the Calculator MathWorlds screens shown. We have also developed a prototype version of MathWorlds for the PalmPilot Operating System (see http://www.simcalc.umassd.edu to download it to a Palm device).

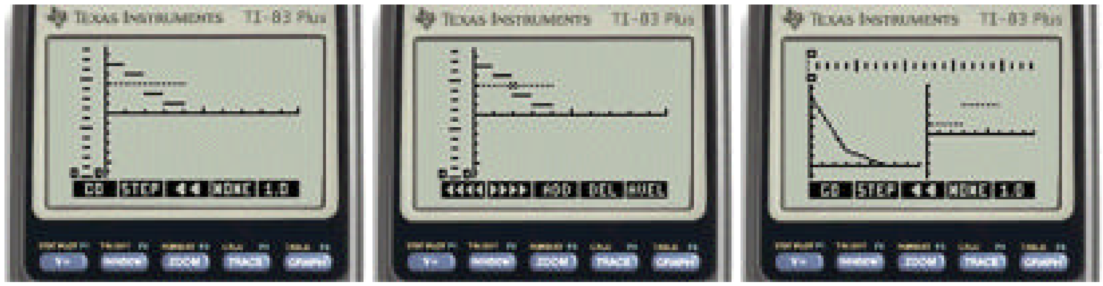


Figure 5. Calculator MathWorlds

As we saw above, it is one thing to have a powerful and learnable representational infrastructure, such as alphabetic writing, and it is entirely another matter to have broad access to that infrastructure. We pointed to the gradual impact of the printing press on the democratization of literacy. It is a major hypothesis of this chapter that a similar change in availability (coupled to changes in learnability) is now underway in the 21st century relative to the new representational infrastructures made possible by the computational medium.

It is one thing to instantiate representational infrastructure innovations on expensive and hence scarce computational devices. It is entirely another to render them materially available on inexpensive, ubiquitous devices well-integrated into the flow of life, work, and education. A message from the history of the printing press is that the change needed to democratize access to the new infrastructures will be slow and will complexly involve many aspects of our educational culture. The message for implementation strategies is, at least at the general level, clear. **[Need elaborate on the many kinds of changes—teacher-education, assessment practices, curriculum integration, expectations regarding curriculum and standards, reshaping of epistemic views of what mathematics is, physical layouts of classrooms, activity structures, connectivity, etc.]**

Of course, “slow” in today’s terms is relative to rates of change that are at least an order of magnitude faster than in previous centuries, so that 300 years may shrink to 30 or less—but not likely to 3. The integration of automobiles, telephones, and television each took about a generation to reach wide penetration in industrialized societies. Penetration of the world wide web into everyday life and intellectual work seems to have taken only about half that time. E-time scales are remarkably short.

In today’s school climate, full-sized, desktop computers are a relatively costly and rare commodity, compared to, say, pencils or notepads. Therefore, most schools that have computers share their availability across many different uses and populations. Furthermore, despite falling costs for a given level of power and functionality, their maintenance cost, especially in network configurations, tends to be prohibitive for most schools to deploy them on a wide-scale basis except in the wealthiest communities. The school mathematics alternative has been the graphing calculator, which has been designed primarily to support Type 1 changes. It has typically taken the form of a full, open toolkit, isolated technologically from other computational devices, and independent of any particular curriculum, which has been supplied offline. This condition is now changing.

Part IV: Reflections On Where We Are from a Systemic Perspective [to be written]

We are early in an exciting new era for technology in mathematics education. Both the representational infrastructures are changing and the physical means for implementing them are changing. We are seeing new alphabets emerging, new visual modalities of human experience are being engaged, and new physical devices are emerging—all at the same time. Much work needs to be done.

The next draft of this paper will, in this last part, deal with how the SimCalc efforts play out in curriculum integration, teacher professional development, and assessment on a larger scale. The above has dealt primarily with issues of representational innovation, but not their consequences in real classrooms, schools, districts, and the larger educational system.

References

Abelson, H., & diSessa, A. (1981). *Turtle geometry: The computer as a medium for exploring mathematics*. Cambridge: MIT Press.

Bochner, S. (1966). *The role of mathematics in the rise of science*. Princeton, NJ: Princeton University Press.

Bowers, J., & Doerr, H. (2001). Modeling and mathematizing in a computer-based microworld: Pre-service teachers' insights when studying the mathematics of change. *Journal of Mathematics Teacher Education 4(2)*115-137.

Boyer, C. (1959). *The history of calculus and its historical development*. New York, NY: Dover Publications.

Bruner, J. (1973). *Beyond the information given.* New York: Norton & Co.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education, 13*(1), 16–30.

Clagett, M. (1968). *Nicole Oresme and the medieval geometry of qualities and motions*. Madison, WI: University of Wisconsin Press.

Cobb, P., Yackel, E., & McClain, K. (Eds). (2000). *Symbolizing and communicating in mathematics classrooms.* Mahwah, NJ: Erlbaum.

Cohen, J., & Stewart, I. (1994). *The collapse of chaos: Discovering simplicity in a complex world*. New York: Viking Books.

Cole, M. (1997). Cultural-historical psychology: A Meso-genetic approach. In L. Martin, K. Nelson & E. Toback (Eds.), *Socio-cultural psychology: Theory and practice of knowing and doing.* New York: Cambridge University Press.

Davies, W. V. (1987). *Egyptian hieroglyphs*. London: British Museum Publications.

Deacon, T. (1997). *The symbolic species: The co-evolution of language and the brain*. New York: W. W. Norton.

diSessa, A. A. (1986). Artificial worlds and real experience. *Instructional Science, 14,* 207–227.

Donald, L. (1991). *The origins of the modern mind.* Cambridge, MA: Harvard University Press.

Donald, M. (2001). A mind so rare: The evolution of human consciousness. New York: W.W. Norton.

Edwards, C. (1979). *The historical development of the calculus*. New York: Springer-Verlag, Inc.

Eisenstein, E. (1979). *The printing press as an agent of change.* (Two vols.) Cambridge: Cambridge University Press.

Haas, C. (1996). *Writing technology: Studies on the materiality of literacy*. Mahwah, NJ: Erlbaum.

Hall, N. (1992). *Exploring chaos: A guide to the new science of disorder*. New York: W. W. Norton.

Havelock, E. (1982). *The literate revolution in Greece and its cultural consequences.* Princeton: Princeton University Press.

Kaput, J. (1994). Democratizing access to calculus: New routes to old roots. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 77–156). Hillsdale, NJ: Erlbaum.

Kaput, J. (1999). Teaching and learning a new algebra. In E. Fennema, & T. Romberg (Eds.), *Mathematics classrooms that promote understanding (pp. 133-155).* Mahwah, NJ: Erlbaum.

Kaput, J. (2000) Implications of the shift from isolated, expensive technology to connected, inexpensive, ubiquitous, and diverse technologies. In M. O. Thomas (Ed.) *TIME 2000: An international conference in Mathematics Education*. Pp 1-25. University of Aukland, New Zealand.

Kaput, J., & Cabral, F. (in preparation). *Middle schoolers achievement in learning the math of motion: Context, belief, engagement.*

Kaput, J., Noss, R., & Hoyles, C. (2001). Developing new notations for a learnable mathematics in the computational era. In L. English (Ed.), *International handbook of technology in mathematics education.* London: Kluwer.

Kaput, J., & Roschelle, J. (1998). The mathematics of change and variation from a millennial perspective: New content, new context. In C. Hoyles, C. Morgan, & G. Woodhouse (Eds.), *Mathematics for a new millennium* (pp. 155–170). London: Springer-Verlag.

Kaput, J., & Sims-Knight, J. (1983). Errors in translations to algebraic equations: Roots and implications. *Focus on Learning Problems in Mathematics, 5*(3), 63–78.

Kline, M. (1953). *Mathematics in western culture.* New York: Oxford University Press.

Mahoney, M. (1980). The beginnings of algebraic thought in the seventeenth century. In S. Gankroger (Ed.), *Descartes: Philosophy, mathematics and physics*. Sussex, England: Harvester Press.

Monk, S., Nemirovsky, R., (1994). The case of Dan: student construction of a functional situation through visual attributes. *Research in Collegiate Mathematics Education.* 4, 139-168.

Moses, B. (1995). Algebra, the new civil right. In C. Lacampagne, W. Blair, & J. Kaput (Eds.), *The algebra initiative colloquium* (pp. 53–67). Washington, DC: U.S. Department of Education.

Nelson, K. (1996). *Language in cognitive development: Emergence of the mediated mind*. Cambridge, England: Cambridge University Press.

Nickerson, S., Nydam, C., & Bowers, J. (in press). Linking algebraic concepts and contexts:

Every picture tells a story.

Noss, R., & Hoyles, C. (in press).

Noss R. & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers.* Dordrecht: Kluwer.

Olson, D. (1985). Computers as tools of the intellect. *Educational Researcher, 14,* 5–8.

Ong, W. J. (1982). *Orality and literacy: The technologizing of the word.* London: Methuen.

Repenning, A. (2000). *AgentSheets®: An interactive simulation environment with end-user programmable agents.* Interaction 2000, Tokyo, Japan.

Riordan M., & Hoddeson, L. (1997). *Crystal fire: The birth of the information age*. New York: W. W. Norton.

Schmandt-Besserat, D. (1978). The earliest precursor of writing. *Scientific American,* 238.

Schmandt-Besserat, D. (1992). *Before writing: From counting to cuneiform* (Vol. 1). Houston, TX: University of Texas Press.

Shaffer, D., & Kaput, J. (1999). Mathematics and virtual culture: An evolutionary perspective on technology and mathematics education. *Educational Studies in Mathematics, 37,* 97–119.

Stewart, I. (1990). Change. In L. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 183–219). Washington, DC: National Academy Press.

Swetz, F. (1987). *Capitalism and arithmetic: The new math of the 15th century*. La Salle: Open Court.

Thompson, P.W., (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel and J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY:SUNY Press

Walker, C. B. F. (1987). *Cuneiform*. London: British Museum Publications.

Wilensky, U., & Stroup, W. (2000). Networked Gridlock: Students enacting complex dynamic phenomena with the HubNet architecture. In *the Proceedings of the Fourth Annual International Conference of the Learning Sciences*, Ann Arbor, MI.

Woodard, R. (1996). Writing systems. In B. Comrie, S. Matthews, & M. Polinksy, (Eds.), *The atlas of languages: The origin and development of languages throughout the world*. London: Quarto Publishing, Inc.