The role of realistic situations in developing tools for solving systems of equations

Martin van Reeuwijk
Freudenthal institute
Utrecht University, The Netherlands

paper presented at the 1995 annual AERA conference in San Francisco,
at the symposium:

The learning and teaching of Algebra: Exploring and Contrasting Two Cyclic approaches to Research and Development

The title of this paper contains many words; They just fit on one line. To address all these issues – “realistic situations”, “developing”, “tools”, “solving”, and “systems of equations” – in depth, we would need a whole day of discussing, working, and doing mathematics to experience the point we want to make. Since this is not feasible, we restrict ourselves to one example of developmental research of realistic mathematics education; an example about solving systems of equations using informal and pre-formal methods.

Introduction

The place and role of algebra in school mathematics are under intense review on many fronts. The reasons for reform have been many and varied. In the traditional algebra curriculum in the United States, algebra is presented as a symbol language and a fixed structure. Students learn to copy the rules and tricks of algebra without a real understanding of the matter. The pedagogy is top-down, and very little attention is paid to the generalizing aspect of algebra, and to the dynamic aspects of variables, that is algebraic reasoning. The jump to the formal level is made too quickly, and there is almost no time for students to develop their own schemes. The traditional algebra course is seen as sterile, disconnected from other mathematics and the “real world” (Romberg & Spence 1993).

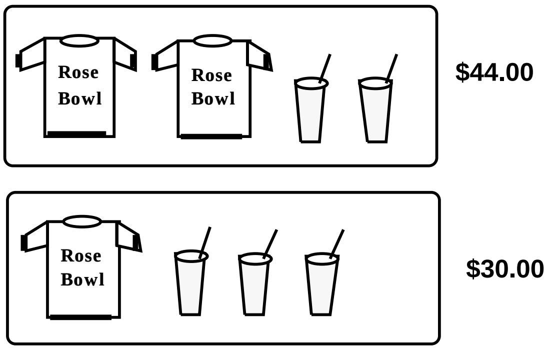
A large and growing body of research began to elaborate the cognitive underpinnings of algebraic understanding (see e.g. Booth 1984, 1988, Kieran 1992, Schoenfeld 1987, Wagner & Kieran 1989). Algebraic reasoning in its many forms, and the use of algebraic representations – including graphs, tables, and formulas – are very powerful intellectual tools that should be made available to all students (see e.g. Janvier 1978).
Research has shown that students have much difficulty with algebra topics as “making equivalent expressions, substituting numbers and variables, solving (systems of) linear equations with two or more unknowns” (see Booth 1988, and chapters in Wagner & Kieran 1989).

In this paper we report on the development process of instructional materials in which the topic of "solving (systems) of equation" is addressed. Our research does not take the traditional curriculum as a given as was done in much of the past algebra research. As a consequence, the nature of algebra learning as we will discuss will necessarily be deviating from such research outcomes. This was also concluded by Kieran (1994) who started rethinking the teaching of algebra based on exploiting the potential of technology.

The unit *Comparing Quantities*

In this paper we use the instructional unit *Comparing Quantities* (Kindt et al, 1998)[1] to illustrate the cyclic developmental process that is so characteristic for the research methodology at the Freudenthal Institute (Gravemeijer, 1994).

Spending Money at the Game



1 How much does a T-shirt cost?
And how much is a soda? Explain how you got your answers.

Figure 1: Sample problem from *Comparing Quantities* assessment.

The problem in figure 1 is a typical problem from the unit *Comparing Quantities*. It may remind you of the infamous algebra word problems from Algebra 1: First you have to rewrite the problem in mathematics -- write equations in this case -- and then solve it. However, the sixth grade students for whom this unit is designed do not know any formal algebra to solve equations. They do bring a lot of other knowledge and common sense, and they are already able to solve this kind of problems. Figure 1 presents the problem by means of a picture. In the unit, many other representations are used: stories, diagrams, symbols, and at the end also bare equations.

Comparing Quantities is a unit for sixth grade (students of 10 to 11 years), and for most students it is the first time that they are doing algebra. Students are encouraged to use informal and pre-formal methods and strategies to solve the problems. By discussing and reflecting on the students' use of strategies, they start realizing that formal strategies

1 *Comparing Quantities* is one instructional unit of the curriculum *Mathematics in Context, A Connected Curriculum for Grades 5*-8. The curriculum development project *Mathematics in Context* (MiC) is sponsored by the National Science Foundation, and is a collaboration between the National Center for Research in Mathematical Sciences Education (NCRMSE) at the University of Wisconsin, Madison, and the Freudenthal Institute (FI) at the University of Utrecht, The Netherlands.

of a more general character exist. The topic “solving systems of equations” is revisited in MiC units in grades 7 and 8, and only in grade 8 the process is completely formalized. However, students are encouraged to make the step to a formal level of solving systems of equations in the unit *Comparing Quantities*, but they are not forced; it is not the goal of the unit.

Ancient societies (Greek, Chinese) used different strategies for solving systems of (linear) equations than what is usually taught in algebra classes. Some of these old strategies were used to design this unit, combined with the knowledge that students bring in themselves. The strategies as developed in the unit go from informal through pre-formal to more formal.

guess and check

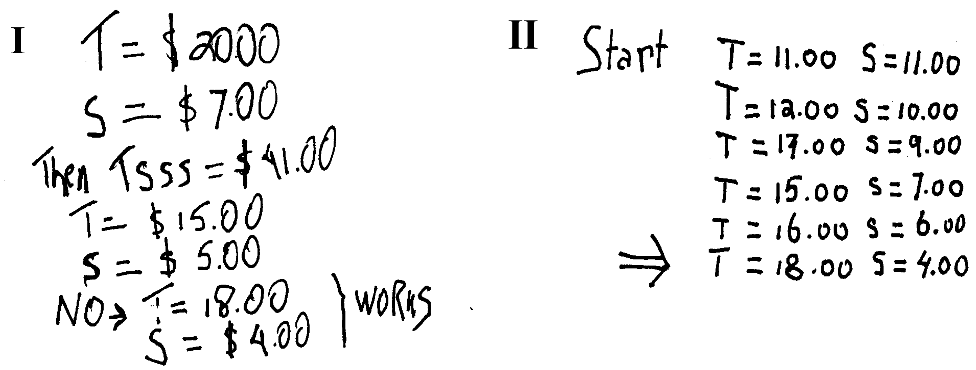


Figure 2: “Guess and check” strategy.

A common strategy on a low cognitive level is ‘guess & check’. Example of ‘guess and check are presented in figure 2. Guess and check is often recognized as a random unorganized method to solve a problem (student I). This strategy however, can also be used in a more sophisticated way: choosing a good starting price for the two items, and improving the first guess in a systematic way (student II). Only looking at student work, it is sometimes hard to distinguish between ordinary ‘guess and check’, and smart ‘guess and improve’.

reasoning through exchanging

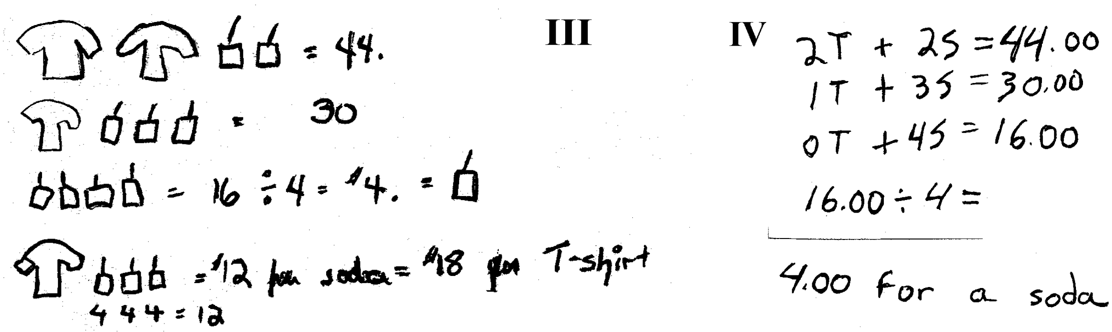


Figure 3: Reasoning by exchanging items.

Another strategy to solve the ‘T-shirts’ problem, is the idea of exchange (figure 3). By exchanging a T-shirt for a soda, the price decreases with $14.00. One can continue this exchanging process till there are only sodas left, and then it is easy to find the price of only one soda. It is also possible to go up, and to exchange a soda for a T-shirt and ending up with only T-shirts. There are different levels of using this exchanging strategy. Students can draw pictures for the next situation, but they can also use letters as labels for the (prices of the) T-shirts and sodas to describe the relation between the items and the amount of money. They can then extend the pattern using this same notation.

combination chart

One more pre-formal strategy is the combination chart. Guess and check and exchanging are strategies that are quite close to students' reality. The combination chart is a new method for most students. It is introduced in the unit as a new way of representing the information and solving the problem. Moves in the chart represent an exchange of items. Students can analyze patterns in the chart – e.g. going one up and one left means $14.00 less – and make more of these diagonal moves to get on the sides of the chart where there are combinations with only one item in it.

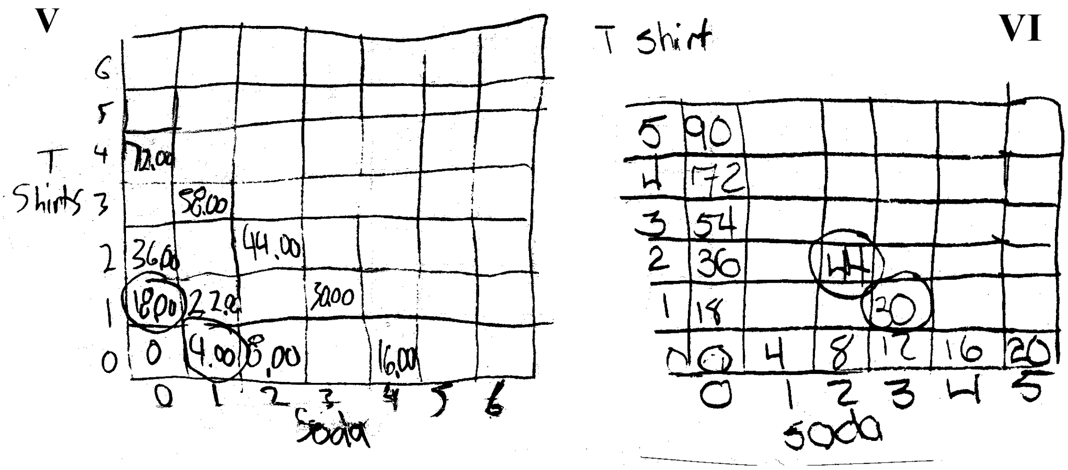


Figure 4. Combination Chart.

notebook notation

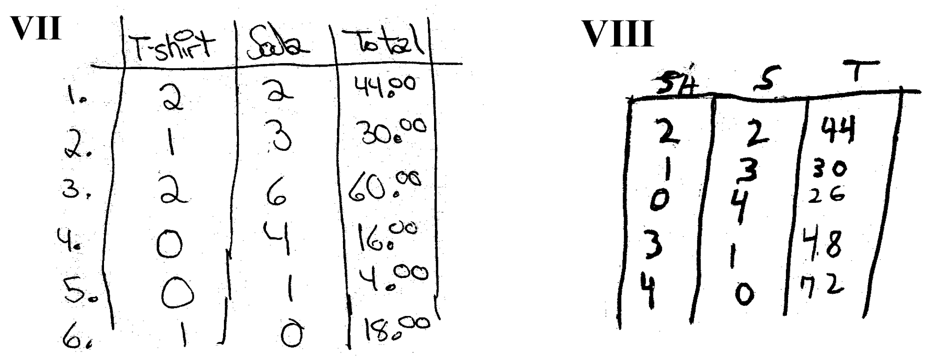


Figure 5. Notebook.

The notebook notation – or matrix – is another strategy that is introduced in the unit. This is again a different way method to represent the information, and to solve the problem. One can best compare this strategy with a notebook used by waiters in a restaurant; every row is a new order. The combination chart can only be used for combinations with two items. Reasoning by exchanging is also doable for combinations with two items, but becomes harder if there are more items involved. This notebook strategy can be used for any number of items. It is a nice and efficient way to record the combinations, and to keep track of new combinations that one can create by combining the known orders.

All four strategies shown above are developed in the unit. Students realize that the notebook is more powerful than guess and check, but that a good reasoning strategy can sometimes be more efficient to solve a problem. At the end of the unit, all these strategies are related to each other, and students learn that they are isomorphic. The next step is to write the information of the problem in equations, and solve the problem using these equations. Since students have seen different representations, equations have a meaning now, and they can relate them to any of the other strategies.
In this way students develop an understanding of the equations and the role and meaning of the variables, and they can relate the meaning always to the context of the problem situation. As said before, students are not forced to quit using the more concrete pre-formal strategies. The goal of the unit is that students start using variables and equations, and that they realize where they come from. Students can always use any of the strategies mentioned to solve a problem. A strategy that they feel comfortable with, and that is appropriate for the problem situation.

We can summarize the structure of the unit in the following diagram shown in figure 6. This diagram is a kind of 'map' of the unit. It shows the progression of the development of the mathematical concepts. The problems in the unit are presented in many different ways -- pictures, stories, diagrams, symbols; students get in touch with and learn several informal and pre-formal strategies to solve these problems -- guess and check, reasoning (including discovering patterns), combination chart, notebook. These strategies and representations are conceptual mathematizations of the problem. By interaction and discussion, students (with the teacher) reflect these representation, and they are formalized by using variables and equations. At the end of the unit, students then apply the concepts and formal representations to solve problems. Realistic situations play an important role in the development of the mathematical concepts. First, they are the world of problems that need to be solved. The realistic problems are the source from which students develop the mathematics. Second, the students apply their mathematical knowledge to solve problems in realistic situations (Gravemeijer 1994).

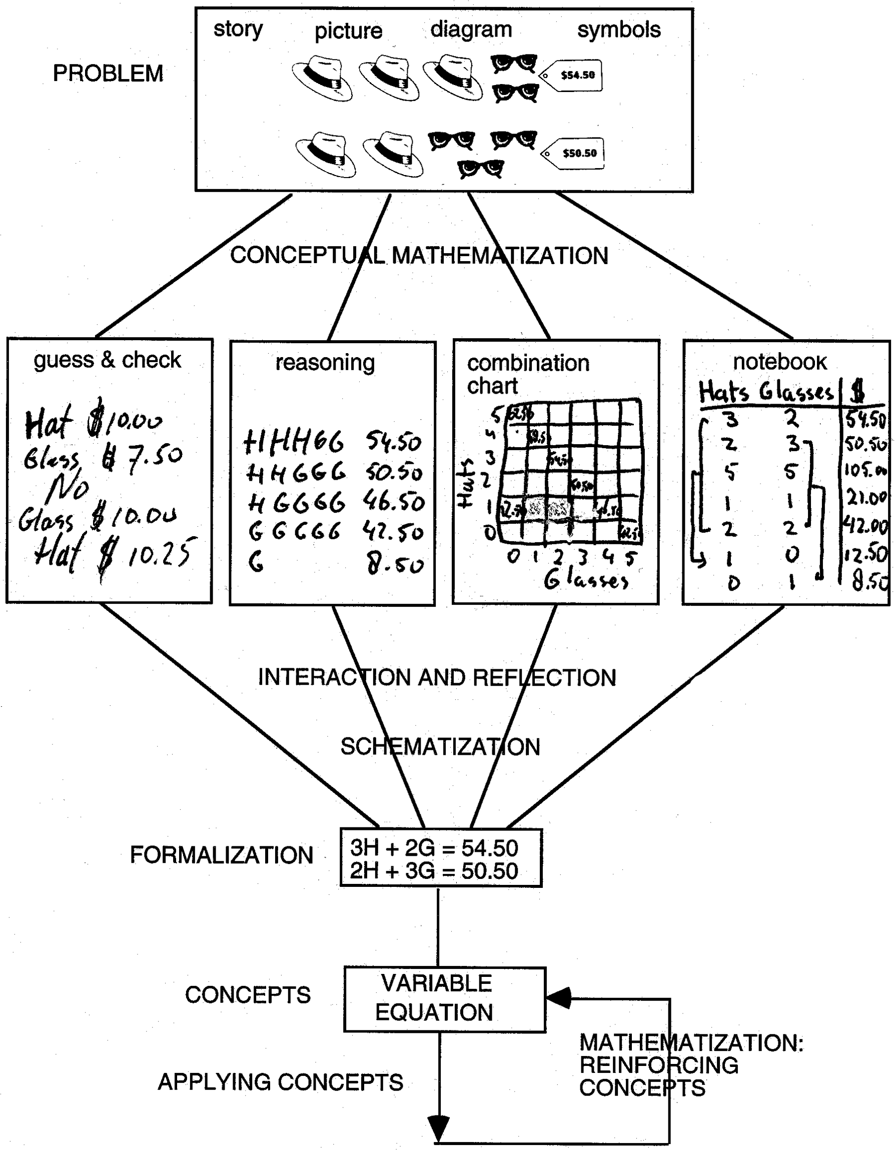


Figure 6. Structure of *Comparing Quantities.*

We have now described the content and the structure of the unit are, and the underlying ideas of this unit. These ideas were not invented behind a desk, and then written up to result in the instructional unit that then went into the classroom. At the moment we have a third version of the unit, and this version will also be revised based on results of the field test. It should be clear that not only the instructional materials evolve during this cyclic revision process, also the ideas underlying this unit – the theory – evolve. In the following section we will focus on the development process that resulted in the unit as shown in figure 6. We will describe some of the decisions that were made during this process to illustrate the considerations that a developer has to deal with.

The development of the unit

Comparing Quantities is a unit in the algebra strand of MiC. It is part of the series of units that deal with equations, constraints, and in grade 8 leading to simple linear programming problems. It is the introductory unit in this series. Based on the professional experience of the designers, was chosen to use realistic situations to develop students' conceptual understanding of solving systems of equations. The problems would be represented in pictures and stories to stay close to the world of the students. Reasoning would play an important role as an alternative for the structural approach as often done in the traditional algebra course (see also figure 6).

Very first versions of parts of the unit were tried out in a Dutch classroom to find out if the ideas of the designers were feasible. It appeared that students were more creative and could perform on a higher cognitive level than was anticipated. The step to more sophisticated strategies and models to solve systems of equations was not as hard as we at first thought. Students brought in much more knowledge of solving these kind of problems than we anticipated. As long as the problem made sense to the students -- that they could realize or imagine what the problem was about -- they made a start to solve it., and in most cases could solve the problem. We learned that students could do more than we might have thought at first. This resulted in the pilot version for the American MiC project. The unit as described in the preceding section is the result of the first 'Dutch' tryout, and how it was piloted in American classrooms.

During the pilot we found out that the unit was too open. **S**tudents needed to be challenged more and sometimes they needed more direction to make the step to a higher level of abstraction. From observations of the lessons, and from analyzing student work, we learned that the unit needed somewhat more structure to help students making the step to a higher level of mathematical sophistication.

An analysis of students' use of strategies on the problems in the end-of-the-unit assessment shows that quite a number of students used the 'guess and check' strategy. The T-shirt problem shown in figure 1 was one of the four problems on this assessment. The distribution of strategies that students used to solve this problem is representative for their strategy use on the whole assessment. In table 2, the results of 97 students -- 4 classes -- that took this assessment after they had finished the unit are presented.

Table 1. Results on the T-shirt and sodas problem, version **1994**

Total 4 classes, number of students: 97

strategy used	correct answer	partially correct	wrong/ no answer	TOTAL
guess & check	7	3	2	12 (12%)
reasoning (exchanging)	35	3		38 (39%)
combination chart	23	4		27 (28%)
notebook	10	7		17 (18%)
other/ unclear	2		1	3 (3%)
TOTAL	77 (79%)	17 (18%)	3 (3%)	**97**

The results show that 12% of the students used guess and check, and 3% used a strategy that we could not identify. From a closer inspection of the student work we learned that about a third of the number of students who used the reasoning strategy did not show anything they had learned in the unit. Based on interviews with teachers and students, observation reports, the results on the assessment, we concluded that quite a number of students had not learned much during the unit. At least they did no show some mathematical growth. They might be solving the problem in the same way as that they would have if the problem was presented before they started working on the unit.

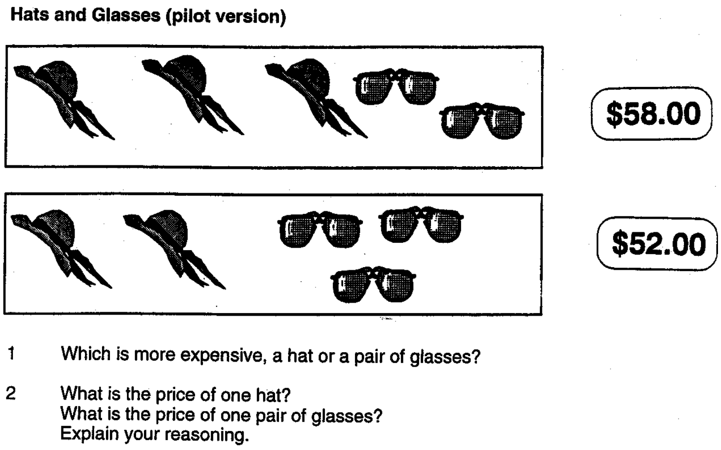
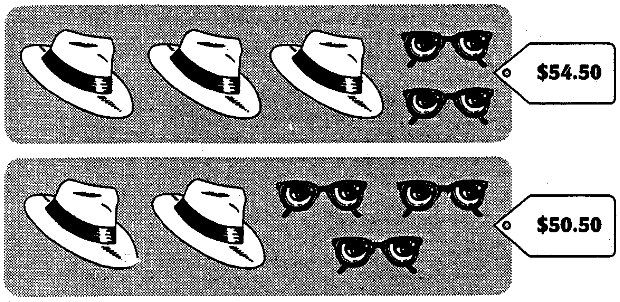


Figure 7. A composed problem from the pilot version of Comparing Quantities.

In the revision of this pilot version we tried to take care of this issue. Apparently the problems in the unit allowed too much for 'low level' strategies. There was not always a need for the students to use more sophisticated strategies -- combination chart, reasoning, notebook -- to solve the problems. The problem in figure 7 is illustrative for the openness of the unit that we are discussing. For students question 1 was not related to question 2. Asking for which of the two items is more expensive did not help the students discover a pattern, or helping them to use a specific strategy. During the pilot, many students answered question 1, and then answered question 2 by a guess and check strategy. The students were not moving from the informal to the (pre-)formal strategies.
So, in the revision process we inserted some more directed questions to have the students move from guess and check to more general and efficient strategies. By changing some numbers in the problems, so that guess and check would become a cumbersome strategy, and by inserting questions about interpretation and meaning of the problem before asking to solve the problem, we intended to overcome the problem of 'getting stuck with guess & check'. The revised version in figure 8 shows how the students are now guided to move away from guess and check, and to use a reasoning strategy to making new combinations of hats and glasses.

Hats and Glasses (field test version)



1 Without calculating the prices for a hat, or a pair of glasses, which is more expensive? How much more expensive?

2 Use the two pictures to make a new combination of glasses and hats, and write down the cost of this combination.

3 Make a combination of only hats or only glasses, and find its price.

4 What is the price of one hat? And of one pair of glasses?

Figure 8. A revised field test version of the composed problem from *Comparing Quantities*.

The new version of the unit was field tested a year later (1995) with the same teachers in the same school. Interviews with teachers and students, observation reports, and the results on the same assessment task lead to the conclusion that the changes were improvements. Almost all students now moved away from guess and check, and thanks to somewhat more structure the goals of the unit became much clearer, to the students and to the teachers. The results on the T-shirt problem in the assessment in 1995 (table 2) show that indeed there is a wider variation in strategies of a higher cognitive level.

Table 2. Results on the T-shirt and sodas problem, version **1995**
Total 4 classes, number of students: 73

strategy used	correct answer	partially correct	wrong/ no answer	TOTAL
guess & check		1		1 (1%)
reasoning (exchanging)	27	5		32 (44%)
combination chart	19	4	1	24 (33%)
notebook	14	2		16 (22%)
other/ unclear				
TOTAL	60 (82%)	12 (17%)	1 (1%)	**73**

Comparing the results of 1994 and 1995, there is not much difference in students answering the question correct or wrong. However, there are far less students who used guess and check, and there were no students who used an unclear strategy. A closer look at the student work shows that eight students solved the problem by using both a combination

chart and a notebook notation. Six students used both a reasoning strategy and a notebook notation. These students explained that the strategies they used are similar, and that they give the same result.
Although, table 2 provides only an overview of strategies used on one of the assessment problems, it is a good representative for the overall impression we got of student performance with the 1995 field test version. It seems as if the decisions made to improve the unit were the correct ones.

Conclusions and discussion

In this paper we have tried to describe the structure of the unit *Comparing Quantities* and the process of its development. We have used a couple of exemplary problems to illustrate what decisions have to be made, and what kind of issues play a role in curriculum development. Realistic situations are very critical to start the development of mathematical concepts with. Presenting students with problems in a natural context brings in that students use strategies that they may not have learned at school. The problem is solved in a way that makes most sense to them. However, since we want the students to move away from guess and check, to also mathematize vertically, to grow in their mathematical power, we need to use tools. These tools – or different strategies as we have called them in this paper – are used to bridge between the concrete and the abstract. The experiment learned that if we want students to use mathematical strategies to solve a problem, we need to provide problems that ask for that.
The alternatives for solving systems of equations as we have developed in this unit, seem to be worthwhile. More developmental research is being carried out at this moment (see also Streefland 1994, Streefland in press) to find out the possibilities of this new approach for learning algebra for younger students.

The cyclic approach of integrated research and development that we have given an example of in this paper, is a very powerful way to develop theory and curriculum that can be used almost immediately. There is a strong link with the daily practice of the teachers. One should realize however, that there is a difference between commercial curriculum development and curriculum development through developmental research. From our point of view we intend to develop ideas and theory on how we can improve the teaching and learning of mathematics.

note

I want to thank Martin Kindt and Mieke Abels – the original designers of the unit *Comparing Quantities* – for using “their” unit as a resource for study and research.

References

Booth, L.R. (1984). *Algebra: children’s strategies and errors*. Windsor, UK: NFER-Nelson.
Booth, L.R. (1988). Children’s Difficulties in Beginning Algebra. In: *NCTM 1988 yearbook, The ideas of Algebra, K-12*. Reston, VA: NCTM. pp. 20-33.
Freudenthal H. (1991). *Revisiting mathematics education*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
Gravemeijer, K.P.E. (1994). *Developing realistic mathematics education*. Utrecht, The Netherlands: CD-b press, Freudenthal Institute.
Janvier, Claude (1978). *The interpretation of Complex Cartesian Graphs Representing Situation Studies and the Acting Experiments*. Nottingham, GB: Dissertation, Shell Centre.
Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Hand-*

book of research on mathematics teaching and learning (pp. 390–419). New York: MacMillan Publishing Company.

Kieran, C, (1994). A Functional Approach to the Introduction of Algebra. Some Pros and Cons. In: *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education*, Vol I, 1994. Lisbon.

Kindt, M., M Abels, M.R. Meyer, and M. Pligge (1998), *Comparing Quantities,* In: National Center for Reserach in Mathematical Sciences Education & Freudenthal Institute (Eds.), *Mathematics in Context: A connected curriculum for grades 5-8.* Chicago: Encyclopedia Brittanica Educational Corporation.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Center for Reserach in Mathematical Sciences Education & Freudenthal Institute (Eds.) (in press), *Mathematics in Context: A connected curriculum for grades 5-8.* Chicago: Encyclopedia Brittanica Educational Corporation.

Romberg, T.A. & M. Spence (1993). *Some thoughts on algebra for the evolving work force*. Paper presented at a meeting of the MSEB, December 1993. NCRMSE: University of Wisconsin, Madison.

Schoenfeld A. (1987) ed. *Cognitive Science and Mathematics Education*. London: Lawrence Erlbaum Associates.

Streefland, L. (1994).*Pre-algebra from a different perspective*. In: Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education, Vol I, 1994. Lisbon.

Streefland, L. (in press). *Making Sweet Algebra*. Unpublished manuscript, Utrecht The Netherlands: Freudenthal Institute.

Usiskin, Zalman (1988). Conceptions of School Algebra and Uses of Variables. In: *NCTM 1988 yearbook, The ideas of Algebra*, K-12. Reston, VA: NCTM. pp. 8-20.

W12-16, team (1992). *Achtergronden van het nieuwe leerplan Wiskunde 12-16, band.* Utrecht / Enschede, The Netherlands: Freudenthal Institute / SLO.

Wagner, S. & C. Kieran (Eds.) (1989). *Research in the learning and teaching of algebra*. Hillsdale, NJ: Lawrence Erlbaum. pp. 11-24.