Young Children's
Spatial Structuring Ability
and Emerging Number Sense

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Spatial Structuring Ability
and Emerging Number Sense

Ruimtelijk Structureervermogen
en Ontluikend Getalbegrip van Jonge Kinderen

(met een samenvatting in het Nederlands)

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For Koen

Preface

In the final stages of completing this thesis, I began to realize that the four-year period of working on the MENS research project was very quickly coming to an end. That realization spurred me on to reflect on this period and its "developmental trajectory", particularly regarding all the people who supported me along the way.

After graduating from university and just before I traveled to Canada, a vacancy at the Freudenthal Institute caught my attention. It described a Ph.D. position in an interdisciplinary project that combines mathematics education and neuroscientific research. I became excited about the prospect of contributing to such a unique project; it would relate to my background in Cognitive Psychology, my interests in mathematics, and my experiences with working with young children. I looked forward to exploring ways of helping others learn through better (mathematics) education. I was also attracted to the opportunity of supplementing my background in traditional, quantitative research with insights into qualitative research. As such, the project would offer me learning opportunities for becoming a competent and a more all-round researcher. It all came together when I heard that I could start the project as soon as I returned from Canada, in September 2005. Exactly four years and a few hours later, I completed my work on the project with this thesis.

In light of this background, I would first like to express my gratitude to Prof. Jan de Lange and Prof. Edward de Haan who were head-supervisors of the project. I greatly appreciate how, from the very first contacts, Jan de Lange seemed to have an unconditional trust in my competencies. He warned me that I had to be able to work independently, and in retrospect I am very grateful for his way of throwing me in at the deep end: it resulted in a thesis that in its entirety is the product of my development as a researcher.

This development would not have come as far, however, without the dedicated support that I felt I could always count on from Michiel Doorman, Dolly van Eerde, and Jo Nelissen. Their patience, expertise, and committed role as supervisors guided me along the ups and downs of the research trajectory. Their encouragement and moral support was an enormous driving force. I thank each of my supervisors for all their help and hope they can catch up on their free time.

All along the data collection phase of the research, I realized how fortunate I was to be working with the staff, children and parents of the Jenaplanschool Cleophas in Utrecht. From the very beginning, the staff and parents acknowledged the importance of the research and went out of their way to help me schedule and collect the data. I greatly admire the staff's energy and passion for teaching. The repeated warm welcomes at school made me even more motivated to contribute to ways of helping the children learn and supporting the teachers in improving their teaching practices.

Mathematics education research at the Freudenthal Institute was a new field of research to me when I started working on this project. I thank my colleagues at the institute who, each in their own way, taught me about Realistic Mathematics Education and procedures for performing design research. Many thanks also to all those people who made it possible for me to present and discuss my work both at and outside of the institute. Coming into contact with people from different backgrounds in different settings gave an extra dimension to the issues that I encountered in mathematics education research.

I had a lot of help finishing the manuscript. In particular, I thank Han Hermsen for his help in learning to work with Adobe FrameMaker, Nathalie Kuijpers for editing the book, Bas Holleman for his advice on the screenshots and pictures, and Reyndert Guiljam for his illustrations. Betty Heijman was a great support in finalizing the lay-out and organizing the publication of the book.

I would like to thank the other "young" researchers (*Klasje008*) at the institute for their companionship in working on our projects. During countless lunchtime walks and tea talks, they encouraged me and brightened up the more demanding times. I wish them all good luck and enjoyment in completing their research. Thank you, Sylvia van den Boogaard and Iris Verbruggen, for also reminding me that there can be more to a day than working on a Ph.D. thesis.

This brings me to the people whom I hold most dear to me, my family and friends. I put them in the spotlight for their endless "behind the scenes" encouragement and enthusiasm. Koen, I dedicate this book to you for your unconditional support, your enlightening perspectives, and, above all, your patience.

Table of Contents

1 Introduction

This thesis documents the development and outcomes of a mathematics education research that was carried out between 2005 and 2009 at the Freudenthal Institute for Science and Mathematics Education, with the support of a grant from the Netherlands Organization for Scientific Research (NWO; project number 051.04.050). It is part of the Mathematics Education and Neurosciences (MENS) research project which is concerned with young children's emerging mathematical abilities from a combined mathematics education and neuropsychological perspective (see Appendix 1). The purpose of our research is to contribute to knowledge about young children's mathematical development with an investigation into the role of young children's spatial structuring ability in fostering their insight into numerical relations. In this introductory chapter, we sketch the background of the research and its setting within the MENS research project.

1.1 Young children, young scientists

What repeatedly stands out from studies on early childhood development is how young children (three to six years old) may be characterized by their natural drive to go out and explore the world. In fact, children's early competencies have been compared to the behavior of scientists (e.g., Gopnik, 2004; Gopnik, Meltzoff, & Kuhl, 1999). This suggests that children are born with certain conjectures about the world, which they continuously test and revise through new insights from daily experiences. Parallels are also drawn between children, scientists and poets who share a sense of wonder and an intense way in which they experience the world (Gopnik et al., 1999). As Dijkgraaf (2007) observes: "It is often said that young children are ideal scientists. They are curious about the world around them. They ask questions, make up theories, and carry out experiments". This is what is said to give both scientists as well as children their drive to learn (Gopnik, 2004).

Research has shown what mathematical understanding children possess before they start formal schooling, and that much of this knowledge is derived from their everyday settings (Anghileri, 1989; Ginsburg, Inoue, & Seo, 1999; Ness & Farenga, 2007). Recent studies, however, have expressed concern about how the cognitive capacities of young children are not sufficiently valued. A report from the National Research Council (NRC, 2005) concluded that:

> ... early childhood education, in both formal and informal settings, may not be helping all children maximize their cognitive capacities. (p. 3)

De Lange summarizes that the "curious minds of young children" have to be stimulated (De Lange et al., 2008). It is therefore disconcerting that a gap exists at the start of formal schooling between children's informal, intuitive knowledge and interests, and formal learning opportunities in school (Griffin & Case, 1997; Hughes, 1986; Murphy, 2006). Indeed, many early elementary mathematics curricula focus mainly on teaching number sense (Casey, 2004; Clements & Battista, 1992; Ness & Farenga, 2007). Gopnik (2004) put the problem as follows:

> If we could put children in touch with their inner scientists, we might be able to bridge the divide between everyday knowledge and the apparently intimidating and elite apparatus of formal science. We might be able to convince them that there is a deep link between the realism of everyday life and scientific realism. (p. 28)

In answer to these issues, a national research program called Curious Minds (Talenten-Kracht) was initiated (Van Benthem, Dijkgraaf, & De Lange, 2005). Several universities and institutions in the Netherlands collaborate in this program to gain a better understanding of what talents young children exhibit as they perform scientific activities, how these talents may be enhanced, and how they may be intertwined and possibly connected to language development. Talent in this project concerns qualitative characteristics that relate to high learning potential, environmental characteristics that relate to talent supporting surroundings, and individual characteristics. What makes the definition unique for this program is that it is prospective, looking ahead towards stimulating and supporting the expression and development of talent (De Lange et al., 2008). Hence the goal of the program is to bring scientists from various research perspectives together with parents and teachers, to chart the talents of young children and to scientifically underpin how these talents may be optimally cultivated.

The essence of the Curious Minds Program is foundational to the MENS research project. Through acknowledging young children's early competencies in spatial and geometric activities and through concentrating on what the children already can do and understand, we expect to contribute to an understanding of the development of mathematical thinking and learning. This research perspective may help to foster children's innate curiosity and eagerness to learn mathematics.

1.2 Young children's spatial structuring ability

Much research is concerned with how children's early numerical and spatial abilities may influence the development of their mathematical thinking. Most of these studies focus on the learning of numbers and operations in early childhood, making it perhaps the best-developed area in mathematics education research (e.g., Clements, 2004). Yet, early research has demonstrated a significant relationship between spatial thinking skills and mathematics achievement (Bishop, 1980; Guay & McDaniel, 1977; Tartre, 1990a, 1990b). This generates questions about how the development of children's spatial and numerical abilities may be related (cf. Clements & Battista, 1992). Indeed, the National Council of Teachers of Mathematics (NCTM) standards (1989, 2000) have strongly recommended increasing the emphasis on the development of spatial thinking in education.

Several more recent studies have specifically related elementary students' spatial structuring abilities to their counting skills (Battista & Clements, 1996; Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998) and their early school mathematical performance (Mulligan, Prescott, Papic, & Mitchelmore, 2006b; Mulligan, Mitchelmore, Martson, Highfield, & Kemp, 2008). Our research contributes to this body of research in two ways. First, we investigate young children's spatial structuring ability to (a) gain greater insight into the role of spatial structuring in the development of early number sense, particularly in terms of numerical relations, and to (b) design a local instruction theory that outlines an instructional setting which may stimulate the development of young children's spatial structuring ability. Second, it involves kindergartners because Kindergarten children (aged four to six years) are in the process of building solid foundations for number sense (Griffin, 2004b). At the same time, they are challenged to bridge their relatively intuitive and informal mathematical knowledge with the more complex mathematics that they encounter in formal schooling (Clements & Sarama, 2007).

On a scientific level, the importance of the research is that it should contribute to understanding how spatial structuring ability may support the development of number sense, particularly with regard to insight into numerical relations. On a more practical level, the research may offer teachers a sequence of interview activities for gauging young children's level of spatial structuring ability and number sense at a very early stage in the children's development. In addition, teachers may find inspiration in the classroom instruction activities that are designed to help children learn to make use of spatial structures for abbreviating numerical procedures.

This acknowledges the importance of a focus on spatial structuring in mathematics education (Clements, 1999a). In effect, the earlier children can be supported in learning to recognize and make use of spatial structures, the better they are prepared for the higher-order mathematical procedures that they learn as soon as they enter formal schooling.

1.3 Structure of the book

This thesis is about the development and outcomes of the mathematics education research in the MENS research project. In **Chapter 2**, we discuss relevant literature that describes the constructs of number sense, spatial sense, and spatial structure, and we present our conjectures about how these may be related. This *theoretical framework* is then embedded into the domain-specific learning instruction theory of Realistic Mathematics Education and into the socio-constructivist theory on learning. This leads to a definition of the research questions at the end of the chapter.

Chapter 3 sets out the *methodology* of the research. Design research is explained and the considerations for applying this methodology are outlined. After presenting the setting of the research, including the preparations, the participants, the type of school that was involved, and what role the teachers played in performing the research, we turn specifically to the procedures for conducting the interviews and for performing the instruction experiment. Next, we discuss how the software program ATLAS.ti was used for retrospective analyses on the data from the interviews and from the instruction experiment. Finally, the measures that were taken to improve the internal and external validity and reliability of the data analysis for both the interviews and the instruction experiment, are described.

Chapter 4 begins with a description of how the first versions of the *conceptual schema* for relating early spatial sense, emerging number sense, insight into numerical relations and spatial structuring abilities came about. This is followed by an explanation of how the *interview tasks* were designed, and what finally became the set of interview tasks. Analogously, in the second part of this chapter, we discuss how the *strategy inventory* was created and how it was used both as an instrument and as an interpretative framework for investigating children's performance on the interview tasks. The chapter concludes with an explanation of the final version of the conceptual schema which is founded in part on this interpretative framework.

In anticipation of the instruction experiment, we turn to the strategy inventory in **Chapter 5** and set out the method that was developed to condense children's interview scores. This method led to the identification of *four phases* that describe a general trajectory in children's development of spatial structuring ability within this interview setting. We refer to the Overlapping Waves Theory (Siegler, 2002, 2005) to illustrate how these phases are interpreted in terms of children's development. The chapter concludes with several quantitative outcomes of the *pre-interviews* that give insight into the children's level of spatial structuring ability at the start of the instruction experiment, as well as the reliability and validity of the interviews as compared to standardized tests.

In **Chapter 6** we describe how a conjectured local instruction theory with its corresponding classroom instruction activities emerged during our exploratory and literature studies. This is followed by a documentation of how the *hypothetical learning trajectory* and the *instructional sequence* of activities were developed and revised before they were tried out in the instruction experiment.

Chapter 7 offers the main outcomes of each instruction activity after they were performed in *Round 1 of the instruction experiment*. Five patterns are distilled from these observations that describe what characterizes an effective instructional sequence and what revisions were necessary in preparation for Round 2 of the instruction experiment. *An overarching context* that unites each instruction activity is presented as one of the most significant changes that was made to the instructional sequence. The chapter concludes with a description of the conjectured local instruction theory in terms of the final sequence of instruction activities that was designed for Round 2.

The retrospective analyses of Round 2 of the instruction experiment are discussed in **Chapter 8**. We begin with observations that support and supplement the conjectured learning moments that were presented in Chapter 6. Then observations that demonstrate the pro-active role of the teacher are discussed. Finally, a *general retrospective analysis* is conducted in which we define nine learning insights for the development of children's spatial structuring ability.

In **Chapter 9** we conclude the research by presenting the qualitative and quantitative outcomes of the *post-interviews*. These outcomes are interpreted in light of the pre-interviews to gain insight into how the children reflect on the instruction experiment and whether their spatial structuring ability may have improved as a result of their participation in the instruction activities. The chapter closes with a section on how the teachers evaluated the instruction experiment and what they thought may have changed in the children's mathematical insight as a result of the instruction experiment.

The *conclusions* of the study are presented in **Chapter 10**. The answers to the research questions culminate in a local instruction theory for teaching and learning about spatial structure. We conclude with a discussion on spatial and number sense, what constitutes an effective instructional setting, limitations to the research, and implications for educational practice. This leads to suggestions for future research.

2 Theoretical Background and Research Questions[1]

In Chapter 1 we indicated that the purpose of this research is to gain insight into how young children's spatial structuring ability can support the development of their number sense, particularly regarding insight into numerical relations. This chapter begins with a discussion about number sense and then turns to early spatial sense and how studies have put spatial structuring forward as an essential factor in early numerical development. Next, we present the domain-specific instruction theory of Realistic Mathematics Education (RME) and a socio-constructivist instruction theory on learning as the underlying context for the research. This takes us to the last section of the chapter in which the research questions are formulated.

2.1 Number sense

The concept of number sense can broadly be defined as the ease and flexibility with which children operate with numbers and quantities (Gersten & Chard, 1999). Berch (2005) states that:

> Possessing number sense ostensibly permits one to achieve everything from understanding the meaning of numbers to developing strategies for solving complex math problems; from making simple magnitude comparisons to inventing procedures for conducting numerical operations; and from recognizing gross numerical errors to using quantitative methods for communicating, processing, and interpreting information. (p. 334)

Early quantitative abilities include children's ability to *subitize* (defined as an automated perceptual process that all people can apply only to small collections up to around four objects; see also *perceptual subitizing*, Clements 1999a) and to compare quantities by laying correspondences (Clements & Sarama, 2007; Van den Heuvel-Panhuizen, 2001). Cognitive research has extended findings about children's early quantitative abilities to mathematics operations (Gelman & Gallistel, 1978; Hughes, 1986). Recently, Berger, Tzur and Posner (2006) found that six-month old infants can recognize simple addition errors (i.e., a doll added to another doll, followed by an incorrect solution of one doll versus a correct solution of two dolls) and that the corresponding brain activity can be compared to that of adults detecting an arithmetic error. These early abilities, however, may not involve explicit number concepts.

1. This chapter is based on a previous publication in TMME (Van Nes & De Lange, 2007)

The development of counting abilities is supported by the knowledge of five counting principles (Gelman & Gallistel, 1978; see also Clements & Sarama, 2007 for a review): the stable order principle (always assign numbers in the same order), the one-to-one correspondence principle (assign only one number word to each object; synchronous counting), the cardinal principle (the last count indicates the number of objects in the set; resultative counting), the order irrelevance principle (the order in which objects are counted is irrelevant), and the abstraction principle (all the other principles can apply to any collection of objects). As children progress in their ability to count, they explore easier ways to manipulate quantities and they come to understand that quantities can be represented by numbers in various ways which can also function as different points of reference (Berch, 2005; Gravemeijer, 1998; Griffin & Case, 1997; Freudenthal, 1973; Van den Heuvel-Panhuizen, 2001).

At first, numbers only play a role in the counting sequence, while counting is still a meaningless procedure. Then, in the process of mastering cardinality, children come to perceive numbers as adjectives that describe quantities. Finally, children learn to generalize number meanings for counting quantities and for relating quantities to each other. Because numbers are perceived as independent objects, a number has meaning in terms of its relationship to other numbers. This requires insight into *numerical relations* which can be achieved through the structuring (e.g., splitting or decomposing and composing) of quantities (Hunting, 2003; Steffe, Cobb, & Von Glasersfeld, 1988).

To better understand the importance of a well-founded number sense for children's learning and understanding of higher-order mathematical skills and concepts, we turn to the Central Conceptual Theory (Griffin & Case, 1997; Griffin, 2004a). Griffin and Case (1997) consider young children's ability to compare quantities and to count as two initially separate cognitive schemas. At the age of four, children have difficulty integrating these competencies, as if "the two sets of knowledge were stored in different 'files' on a computer, which cannot yet be 'merged'" (p. 8). A "revolutionary" developmental step is said to occur by the age of five or six, in which these two schemas merge into "a single, superordinate conceptual structure for number" (Griffin, 2004b, p. 40). This structure interrelates children's knowledge of the counting words, their one-to-one synchronous counting ability, their familiarity with finger patterns, and their ability to translate these insights to objects (Griffin, 2005). Such a conceptual structure is important because it builds on "the intuitive knowledge that appears to underlie successful learning of arithmetic in the early years of formal schooling" (Griffin & Case, 1997, p. 8).

The Central Conceptual Theory is an example of a theory that underlines the interrelatedness of four aspects in the development of number sense: recognizing and naming how many items are in a small configuration, learning names and eventually ordered lists of number words, enumerating objects, and establishing cardinality (Clements & Sarama, 2007). The progress of the first aspect (recognizing and naming the number of objects in a small configuration) can be summarized as developing from the nonverbal recognition of one to two objects, to the quick recognition of three to four objects (subitizing), to the decomposition and composition of larger groups (Clements & Sarama, 2007). In our research we specifically focus on this development because of its importance in developing numerical insight for higher-order arithmetic abilities. The ability to *(de)compose quantities* is essential for learning to differentiate, for example, greater and lesser magnitudes (Hunting, 2003; Steffe et al., 1988). Knowing that six, for example, is composed of four and two, helps to understand that four is a lesser quantity than six. This, in turn, establishes insight into numerical relations which underlie arithmetic skills that children learn in formal schooling (Van Eerde, 1996; Van den Heuvel-Panhuizen, 2001). The strategy of counting on, for example, where children continue adding on two from four to obtain six, depends on children's ability to conceptually embed the four inside the total to abbreviate the addition procedure (Steffe et al., 1988).

Early quantification has also been referred to as "spatial quantification" (Mix, Huttenlocher, & Levine, 2002, p. 139) to highlight the importance of recognizing specific configurations, and of (de)composing quantities in the development of insight into numerical relations. In the next section we turn to early spatial sense to investigate its role in "spatial quantification".

2.2 Spatial sense

Spatial sense can be defined as the ability to "grasp the external world" (Freudenthal, in National Council of Teachers of Mathematics [NCTM], 1989, p. 48). It is one of the many terms associated with spatial thinking, that has been the topic of much discussion among researchers, including speculations about its relationship to stimulating mathematical performance (see Chapter 1). Clements (1999b) proposes spatial abilities and spatial sense as the fundamental components of spatial thinking. Spatial sense, in turn, is said to presuppose the development of spatial orientation, spatial visualization and imagery. Alternatively, Van den Heuvel-Panhuizen and Buijs (2005) distinguish geometry, orientation and navigation, and constructing with shapes and figures as central factors for early spatial education.

In reflecting on the myriad of terms and definitions for spatial sense and how they could embody the term “spatial quantification” (Mix, Huttenlocher, & Levine, 2002), we focus our research on spatial visualization (including transformations), spatial orientation (including navigation), and shape (including constructing with shapes and figures) as the three main components of spatial sense that enable young children to “grasp the world” (cf. Clements & Sarama, 2007). In what follows, we describe each of the components to explain how specifically these components may play a role in young children’s number sense. It is important to keep in mind that in practice these components are far less dissociable and that their developments are intricately related to each other and often manifested even before these children begin their formal schooling. A cognitive study with four- and five-year old children, for example, provided evidence that at this age children can already compare proportions and figures (Sophian, 2000). The children in this study were able to match the correctly shrunken picture to the original picture without being distracted by pictures that not only were smaller, but also disproportionate to the original picture. Studies such as this one illustrate the remarkably developed spatial sense of many children prior to the start of formal schooling.

Spatial visualization. Spatial visualization involves the ability to mentally picture the movements of two- and three-dimensional spatial objects. In spatial visualization tasks, all or part of a representation may be mentally moved or altered (Bishop, 1980; Clements, 2004; Tartre, 1990a), requiring object-based transformations where the frame of reference of the observer stays fixed (Zacks, Mires, Tversky, & Hazeltine, 2000). Cognitive research suggests, for example, an early competency for judging distances that is manifested regardless of the presence or absence of references in the direct surroundings of the child (Huttenlocher, Newcombe, & Sandberg, 1994). This requires spatial visualization skills for creating a mental image of the location of the object.

Mental images are “internal, holistic representations of objects that are isomorphic to their referents” (Battista & Clements, 1992, p. 446). Owens and Clements (1998) modified Presmeg’s (1986) categories for different types of imagery to differentiate the following types of representations in spatial problem solving in the elementary classroom: concrete imagery (recognize a concrete, holistic picture), dynamic imagery (change shapes into new related shapes), pattern imagery (recognition of properties of concrete images and their relationships), action imagery (use of mental ideas to search for new shapes without explicitly identifying them), and procedural imagery (able to repeat a procedure of putting together pieces that form a composite shape). In developing a framework of imagery for assessing children’s early spatial mathematics learning, Owens (1999) concluded that visualization strategies are a key to the development of spatial sense.

Spatial orientation. The second component of spatial sense that we focus on is spatial orientation. This is what Clements (2004, p. 284) refers to in describing how we “make our way” in space. In spatial orientation, the self-to-object representational system is at work because the viewer reorients the imagined self (Hegarty & Waller, 2004). This is dissociable from the object-to-object representational system that is involved in mentally rotating and manipulating a mental image in spatial visualization tasks. An example is when one examines two photographs to compare the position of the camera. This involves changing the egocentric frame of reference with respect to the environment while the relation between object-based and environmental frames of reference stays fixed (Hegarty & Waller, 2004). One can navigate through space by operating on relationships between different positions in space with respect to one’s own position (Clements, 2004; Tartre, 1990a, 1990b).

Spatial orientation is considered to be a “core domain” in children’s development. It is an early cognitive strength that is necessary for localization, for navigating one’s environment, for spatial thought (i.e., reflecting about the spatial world), for operating with models and maps, for using a coordinate system and for spatial structuring (Clements & Sarama, 2007; Tartre, 1990a, 1990b; Van den Heuvel-Panhuizen & Buijs, 2005). The term spatial structuring in this context, is reserved for organizing two- and three-dimensional concepts (Battista & Clements, 1996; Battista et al., 1998; Outhred & Mitchelmore, 1992). It has to do with selecting, coordinating, unifying, and registering in memory a set of mental objects and actions (Clements & Sarama, 2007).

Shape. We refer to the third component of spatial sense as shape. Similar to spatial visualization, it has to do with mentally manipulating spatial forms from a fixed perspective (McGee, 1979; Owens & Clements, 1998). Young children can separate these forms from the figure in which they are embedded using gestalt principles. Several studies have related the gestalt laws to early numerical development. Spelke, Breinlinger, Jacobsen, and Phillips (1993), for example, found that adults’ perceptions of simple, unfamiliar visual displays, were strongly influenced by the gestalt relations of color, texture similarity, good continuation, and repetition and structural symmetry. The perceptions of five- and nine-month old babies, however, were only weakly affected by these particular gestalt relations, and the perceptions of three-month old babies were not at all affected. These differences suggest a development in these particular gestalt relations (Quinn, Burke, & Rush, 1993; Quinn, Bhatt, Brush, Grimes, & Sharpnack, 2002), showing that even infants as young as three months old are capable of distinguishing particular elements and of establishing crude perceptual coherence (i.e., a spatial structure).

Insight into shapes and their relations enables children to make reference to familiar figures such as one’s own body, to geometrical figures such as mosaics, and to geometric patterns such as dot configurations on dice or dominoes. As such, patterning involves recognizing similarity, symmetry, and regularity and using characteristics of spatial shapes for building and analyzing spatial constructions (Papic & Mulligan, 2005). School geometry can help young children learn about shape through position, ordering, and comparing spatial objects (Clements & Battista, 1992). Children come to improve their ability to relate objects in space, they extend the size of that space, and they link primary meanings (treating the spatial relations as one with their environment) to secondary meanings (taking the perspective of an abstract frame of reference) and uses of spatial information (Clements, 2004a, p. 281). Moreover, the communication that is involved in learning and teaching spatial relations helps to increase children's vocabulary, enrich their imagination, and sharpen their perceptions (Casey, 2004; Newcombe & Huttenlocher, 2000; Van den Heuvel-Panhuizen & Buijs, 2005). This explains why NCTM (1989, p. 48) described spatial sense as necessary for interpreting, understanding, and appreciating our inherently geometric world.

In the first section on number sense (2.1), we described how the ability to (de)compose quantities is essential for the development of insight into numerical relations. In this section on spatial sense, we zoom into a relationship that we note between the three components of spatial sense, and part-whole relations and the (de)composition of spatial objects (cf. Clements & Sarama, 2007). First, in spatial visualization, the ability to manipulate mental images can support children in rearranging objects to explore their composition. Second, the spatial structuring factor in spatial orientation involves integrating previously abstracted items to form new structures. Third, insight into shapes helps children perceive parts and wholes of geometric patterns, congruence, symmetry, and transformations. In the next section we suggest that the three components share a spatial structuring ability. As such, the focus of our research is on spatial structuring to see how it may influence young children's ability to (de)compose quantities for gaining insight into numerical relations.

2.3 Spatial structures

Several studies have emphasized children's spatial structuring ability as an essential factor in their mathematical development (e.g., Battista & Clements, 1996; Battista et al., 1998; Mulligan, Prescott & Mitchelmore, 2004; Mulligan et al., 2006b, 2008). Considering the importance of young children's ability to (de)compose quantities for insight into numerical relations, we suggest in our research that spatial structuring is a factor of spatial visualization, spatial orientation, and shape that appears to support this ability and that should therefore be fostered.

Defining spatial structuring. The act of spatial structuring as introduced in the section on spatial sense, can be defined as:

> ... the mental operation of constructing an organization or form for an object or set of objects. Spatially structuring an object determines its nature or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. (Battista & Clements, 1996, p. 503)

Spatial structuring in enumeration. In order to organize and make sense out of visual information, the process of perceiving structure requires a child to spatially visualize and flexibly recognize the structure both as a composite of parts and as a whole. As Smith (1964, as cited in Tartre, 1990a) put it:

> The process of perceiving and assimilating a gestalt...[is] a process of abstraction (abstracting form or structure)... It is possible that any process of abstraction may involve in some degree the perception, retention in memory, recognition and perhaps reproduction of a pattern or structure. (p. 218)

The mental extraction of structures from spatial configurations (i.e., identifying a *gestalt*) is also what Arcavi (2003) found to aid older students' counting processes. For Arcavi's students, the gestalt could involve "breaking and rearranging the original whole" or "imposing an 'auxiliary construction' whose role consisted of providing visual 'crutches', which in themselves were not counted, but which supported and facilitated the visualization of a pattern that suggested a counting strategy" (Arcavi, 2003, p. 229). Analogously, young children could use gestalts to, for example, rearrange a set of objects that is to be counted. The spatial structure that subsequently arises can help the child recognize (part of) the quantity and consequently abbreviate the counting procedure (Van Eerde, 1996).

Battista and Clements (1996; see also Leake, 1995; Outhred & Mitchelmore, 1992) put forward that students do not "read off" structure but rather construct structure according to how an object is perceived. Clements (1999a) refers to this as *conceptual subitizing,* since such spatial structures can support the advanced organization of a quantity. While *perceptual subitizing* refers to the natural ability to recognize a number without using other mathematical processes, conceptual subitizing requires the child to view number and patterns (spatial but also temporal, rhythmic or auditory) in terms of units of units, or part-whole relationships. As such, the ability to read off a small quantity develops from preattentive but quantitative, to attentive (perceptual) subitizing, to imagery-based subitizing, to conceptual subitizing (Clements 1999a; Steffe, Von Glasersfeld, Richards, & Cobb, 1983).

In studying how third, fourth, and fifth grade students count squares in two- and three-dimensional rectangular arrays, Battista and Clements (1996) and Battista et al. (1998) found students' spatial structuring abilities to provide the necessary input and organization for the numerical procedures that the students use to count an array of squares. The development of spatial structuring may be based on how students' perceptual and physical actions during counting become input for the structuring process (Battista & Clements, 1996). This underlines how attempts at enumeration sometimes engender spatial structuring and restructuring, which in turn provides the input and organization for enumeration. Hence, Battista and Clements suggest that spatial structuring precedes meaningful enumeration and that spatial structuring is therefore "an essential mental process underlying students' quantitative dealings with spatial situations" (Battista et al., 1998, p. 503). Their follow-up research highlighted how structuring two- and three-dimensional space is essential for geometric and visual thinking, which underlies an understanding of early algebra such as multiplication and area formulas (Battista et al., 1998). As such, spatial structuring as a form of organization through composition is considered to contribute to insight into important mathematical procedures and concepts such as patterning, algebra, and the recognition of geometric shapes and figures (see also Carraher, Schliemann, Brizuela, & Earnest, 2006; Clements & Sarama, 2007; Mulligan, Mitchelmore, & Prescott, 2006a; Papic & Mulligan, 2007; Waters, 2004).

Additional support for associating spatial visualization, spatial orientation, and space with young children's emerging number sense, comes from research by Mulligan et al. (2006a). They based their research on Goldin's (2002) model of cognitive representational systems and found that children with a more sophisticated awareness of patterns and structures excelled in mathematical thinking and reasoning compared to their peers and vice versa. Although the correlations could not reveal causal effects, their conclusion was that young children are capable of understanding more than unitary counting and additive structures alone. They suggest that instruction in mathematical patterns and structures could stimulate children's learning and understanding of mathematical concepts and procedures. This coincides with Battista et al.'s (1998) views about how students must learn to construct a meaningful structure themselves, and that students could improve their own use of structure if they recognize errors in their counting that result from inadequate spatial structuring.

Early advantages of spatial structuring. The process of determining a quantity includes a situation that is recognized (a collection of countable objects), a goal (find out how many), an activity (counting), and a result (unitary whole of counted items; Steffe et al., 1988). Hence, when asked to determine the quantity of a randomly arranged set of objects, young children initially tend to count each object. As the set grows, however, children are confronted with the time-consuming process and the difficulties of keeping track of which objects have already been counted. Therefore, the advantage of applying spatial structure to abbreviate numerical procedures becomes evident, for example, when reading off a quantity (i.e., recognizing the quantity of six as three and three, Steffe et al., 1988; Van den Heuvel-Panhuizen, 2001; Van Eerde, 1996; Van Nes & De Lange, 2007), when comparing a number of objects (i.e., one dot in every one of four corners is less than the same configuration with a dot in the centre, Clements, 1999a), when extending a pattern (i.e., repeating the structure, Papic & Mulligan, 2005, 2007) and when building a construction of blocks (i.e., relating the characteristics and orientation of the constituent shapes and figures, Battista et al., 1998; Van den Heuvel-Panhuizen & Buijs, 2005). For this reason, children's ability to grasp spatial structure appears essential for gaining insight into numerical relations and for using this insight to develop mathematical abilities such as ordering, comparing, and generalizing (NCTM, 2000; Papic & Mulligan, 2005; Steffe et al., 1988; Waters, 2004). In fact, Mulligan, Mitchelmore and Prescott (2006a) state that "the development of pattern and structure is generic to a well-connected conceptual framework in early mathematics" (p. 214).

More formal mathematical abilities that depend on insight into numerical relations require even further understanding and use of spatial structures. This is particularly the case for arithmetic abilities such as counting and grouping, for part-whole knowledge in addition, multiplication and division (e.g., 8 + 6 = 14 because 5 + 5 = 10 and 3 + 1 = 4 so 10 + 4 = 14), for using variables in algebra, for proving, predicting and generalizing, and for determining the structure of a shape in order to subsequently mentally rotate or manipulate it (Anghileri, 1989; Buijs, 2008; Carraher et al., 2006; Papic & Mulligan, 2007; Steffe et al., 1988). In fact, studies have shown that children with difficulties in learning mathematics tend to focus on non-mathematical features and prefer to count objects unitarily without using any form of structure (Butterworth, 1999; Mulligan, Mitchelmore, & Prescott, 2005; Pitta-Pantazi, Gray, & Christou, 2004). Clements and Sarama (2007, p. 473) noted that children who cannot conceptually subitize, are handicapped in counting on (e.g., temporal patterning) or addition (e.g., finger patterning).

Spatial structuring in practice. An anecdote of Richardson (2004) about the children in her preschool classroom illustrates how the abstraction of spatial structures may occur in practice. Richardson's children worked with cards showing dot configurations like those on dice so that they could learn to recognize amounts in such arrangements. When she asked the children to count out a certain number of counters, instead of correctly counting out the counters, the children made an "X" shape to match what they recognized to be the shape of five dots on a card, and they made a square shape to match what they recognized to be the arrangement of nine dots. These children apparently abstracted a shape from the individual dots and used the shape as a representation of a particular quantity.

Richardson (2004) concludes from this experience that teachers must always interact with the children to check whether what they are doing makes sense to them, because performing without understanding interferes with their mathematical development. Moreover, it is a practical example of how children are able to abstract a shape from individual elements. It also adds to the finding that infants make use of gestalt principles to make sense of the real world and to establish perceptual coherence. As such, the ability to process the gestalt, the whole, from its component parts is an important requirement for mathematical skills; it can support children in gaining insight into numerical relations which, in turn, could help to abbreviate children's strategies for determining, comparing and operating with (e.g., adding, subtracting, and multiplying) quantities (Van Eerde, 1996). These conclusions about children's tendencies to organize the world through the use of spatial structures, should encourage researchers and instructors to take care to attend to children's early spatial structuring abilities in mathematics curricula. To date, many teachers are unaware of spatial structuring and many textbooks still present quantities that discourage subitizing (Clements, 1999a).

Development of spatial structuring ability. In an analysis of structure present in 103 first graders' representations for a variety of mathematical tasks, Mulligan and colleagues (2004, 2006a) found that mathematical structure in children's representations transfers across various mathematical domains. In addition, the researchers were able to code the individual profiles of these children as one of the following four broad stages of structural development:

(1) *Pre-structural stage*: representations lack any evidence of mathematical or spatial structure; most examples show idiosyncratic features.
(2) *Emergent (inventive-semiotic) stage*: representations show some elements of structure such as use of units; characters or configurations are first given meaning in relation to previously constructed representations.
(3) *Partial structural stage*: some aspects of mathematical notation or symbolism and/or spatial features such as grids or arrays are found.
(4) *Stage of structural development:* representations clearly integrate mathematical and spatial features. (Mulligan et al., 2004, p. 395-396)

This developmental trajectory for spatial structuring inspired the researchers to develop the Pattern and Structure Awareness Program (PASMAP). The pattern-eliciting tasks were designed to improve students' visual memory, the ability to identify and apply patterns, and to seek structure in mathematical ideas and representations. It was found that young, low-achieving students (aged 5 to 12 years) can be taught to seek and recognize mathematical structure and that this can lead to an improvement in their overall mathematics achievement (Mulligan et al., 2006b). In a subsequent study, Mulligan and colleagues (2008) implemented PASMAP with ten Kindergarten children and found that "explicit assessment and teaching of structure, using instructional approaches suitable for low-achieving students, has the potential to effectively improve students' abstraction of mathematical processes within a relatively short time frame" (p. 135). These findings are extended by a case study in which four-year-olds excelled in patterning and structuring after a six-month intervention (Papic & Mulligan, 2007).

In short, the research above gives reason to suggest that spatial structuring ability should be fostered as a key factor in the development of number sense, particularly regarding insight into numerical relations. Hence, the assumption of our research is that children can improve their numerical understanding by helping them associate numerical procedures with spatial structuring using spatial structures such as finger patterns, dice configurations and double-structures.

2.4 Motivation for and purpose of the research

Considering the role of spatial structuring in (de)composing quantities, we explain the motivation for the research and its purpose.

Focus on Kindergarten children. Most research on patterning and structuring with respect to early algebraic thinking has been concerned with children who are at least six years old. Relatively little is known about the developmental trajectories of younger children. Yet, studying and stimulating the mathematical development of particularly kindergartners is important because kindergartners' progress in mathematics throughout their formal schooling strongly depends on the extent to which school instruction succeeds at relating to the child's level of understanding, and at bridging the child's initially informal learning methods with relatively formal teaching methods (Aubrey, Dahl, & Godfrey, 2006; Baroody, 1987; Clements & Sarama, 2007). This is crucial for fostering learning progression as well as for preventing learning difficulties that may arise at a later stage in formal schooling (Allsopp et al., 2003; Freudenthal, 1991; Henry & Brown, 2008; Jordan et al., 2007; Starkey, Klein, & Wakeley, 2004).

More specifically, the importance of studying the development of kindergartners' number sense has to do with the "revolutionary changes" that children at this age are said to experience in their understanding of numbers and quantities (Griffin, 2004a; Griffin & Case, 1997; see also section 2.1). The conceptual structure that develops by the age of five or six integrates children's intuitive understanding of quantity with number, and provides them with the conceptual foundation for number sense which is believed to underlie all higher-level mathematics (Griffin, 2004b). This alludes to a crucial period in the development of mathematical thinking. As such, our focus on young children and their emerging number sense can help to support children in attaining and maintaining “central conceptual structures”. A greater understanding of the development of kindergartners’ spatial and number sense may help to design instruction that can support the development from very early onwards.

In analyzing structure in children’s representations as they solved pattern-eliciting tasks, Mulligan et al. (2006b, 2008) were particularly concerned with the progress of low-achieving children. Our focus is less on low-achievers and more on the numerical procedures of kindergartners in general as they solve tasks that cohere with RME principle in a regular classroom setting. This should provide an understanding of developmental trajectories as they are manifested in a relatively ecologically valid setting. By exploring the strategies that children use to solve the specially designed interview tasks (Chapter 3), we may find support for the developmental trajectory for spatial structuring ability that Mulligan and colleagues defined. This may contribute to a greater understanding of what characterizes the development of young children’s spatial structuring ability.

A classroom instruction experiment. As Mulligan et al. (2006b) noted, “there is a real need to develop classroom-based studies that seek to identify teaching and learning influences that promote the development of structure and generalization in children’s mathematics learning” (p. 376). Therefore, to not only contribute to an understanding of the developmental trajectory of spatial structuring, but to also gain insight into factors that may influence this trajectory, we used design research and performed a sequence of instruction activities in a classroom during an instruction experiment (Chapter 3). Our instruction experiment included instruction activities that mainly involve *conceptual subitizing,* and it specifically focused on understanding *how* a particular instructional setting could stimulate greater awareness and use of spatial structures. The classroom implementation of the activities created a concentrated and therefore powerful instructional setting in which the children could interact with each other and with the teacher (Cobb & Yackel, 1996). This setting helped to understand the influence of a particular intervention on children’s awareness of spatial structuring, and on their ability to make use of spatial structures for abbreviating numerical procedures such as determining and comparing quantities.

Purpose of the present research. Research on the development of young children’s structuring and patterning ability has fuelled many questions that require further investigation. After identifying four stages in structural development, Mulligan et al. (2004, 2005), for example, ask why some children do and why others do not develop structure in their representations of mathematical concepts. This requires more insight into the characterization of the developmental trajectory, as well as into the influences that the instructional setting may have on children's development of spatial structuring ability. Indeed, Papic and Mulligan (2007) state that more research is needed to understand the effects of the curriculum and teacher pedagogy that is more focused on encouraging representation, abstraction, and generalization of repeating and growing patterns, on young children's mathematical development. Furthermore, Henry and Brown (2008) identified a need to "design, implement, and then study curricula that help students move beyond counting to memorization and derived-fact strategies during first grade" (p. 179). Translated to the Kindergarten setting, this means that research must concentrate on designing interventions that stimulate the development of children's understanding of quantities, numbers and numerical procedures.

Taken together, the present study contributes to the body of research above by:
(a) outlining the development of kindergartners' spatial structuring ability;
(b) designing a local instruction theory that may foster their spatial structuring ability.

2.5 Context of the research

The research is predominantly based on the domain-specific learning instruction theory of Realistic Mathematics Education (RME) and the socio-constructivist perspective on learning. We discuss these frameworks in the following paragraphs.

2.5.1 Realistic Mathematics Education

Realistic Mathematics Education inspires the development of mathematics education by offering a pedagogical and didactical perspective on mathematical teaching and learning. It was devised at the Freudenthal Institute in the 1970s and 1980s and has been further developed and propagated internationally since then (cf. Freudenthal, 1991; Gates & Vistro-Yu, 2003; Gravemeijer, 1994; Mason & Waywood, 1996; Streefland, 1991; Treffers, 1987). The term *realistic* implies that the problem situation is set in a context that gives a problem meaning and that brings forward the mathematics that "begs to be organized". At an initial level of learning, "realistic" does not have to be true in real life (e.g., it may be a context with fairy tale characters or a context in a mathematical setting), as long as it is "experientially real" to the student, so that it gives meaning to the student's mathematical activity. Such a context can be motivating, but it is especially important that it acts as a model for stimulating personal strategies that can be used as building blocks for the mathematics that is the focus of the discussion.

As such, mathematics in RME is seen as a "human activity" that is driven by the act of mathematizing (Freudenthal, 1973, 1991). *Mathematizing* means more than gaining mathematical knowledge and becoming adept at performing mathematical operations. Rather, it involves understanding underlying mathematical abilities such as ordering, classifying, generalizing and formalizing (Treffers, 1975). Freudenthal (1973, 1991) posits the importance of *guided reinvention* for stimulating mathematization. This principle underlies the construction of knowledge as if the student "reinvented" it. It differs from the discovery perspective, which states that symbolic systems exist and that children must try to understand them with help from adults. In the (re)invention perspective, the children really (re)invent the mathematics by being placed in situations that require mathematizing (Gravemeijer, 1998). Hence, the designer's role in this "active learning process" is to encourage the children's spontaneous strategies without imposing the mathematical knowledge on them. It is the task of the designer to offer children tools with which to build upon earlier knowledge for constructing and internalizing new insights. Together, this is implied by the term "guided".

In practice, guided reinvention means that the teacher can anticipate the types of mathematical thinking that children may develop at particular moments, and provide the concepts, models and symbols when the children may find them necessary. At a classroom level, the designer must help the children converge their mathematical ideas with each other and with the teacher so that a shared mathematical standard can be achieved. For this, concepts, models and symbols are introduced in the (re)invention perspective that are directly related to what is necessary for the children's learning and understanding at that particular moment, and ideally builds upon the children's present knowledge. Hence, for learning to occur, mathematizing is said to require a series of steps where:

> ... the activity of the lower level, that is the organizing activity by means of this level, becomes an object of analyses on the higher level: the operational matter of the lower level becomes a subject matter on the next level. (Freudenthal,1973, p. 125)

Importantly, Freudenthal emphasizes that the key to learning is reflection, where students mathematize their own mathematical activities. This highlights the iterative character of mathematics; concrete experience helps to validate and test abstract concepts, while feedback contributes to a social learning and problem-solving process (De Lange, 1987). Hence, RME is aimed at supporting conceptual mathematization (development of mathematical concepts and ideas starting from the "real" world) at the level of initial learning, as well as at the general and formal level of reinforcing mathematical concepts and ideas through application.

The RME conception of learning is based on the idea that reality is not only a source for applying new insights after a transition in learning levels has taken place, but that reality forms part of the input for students' understanding that is necessary to facilitate such a transition. *Reality* in this sense, is defined as "what common sense experiences as real at a certain stage" (Freudenthal, 1991, p. 17), and the goal of RME is to support students in creating a new mathematical reality. The following five design principles outline the fundamental characteristics of RME (Treffers, 1987):

(1) *Didactical phenomenology*: Mathematical ideas are described in terms of what is to be mathematized. This implies that proper instruction should derive from an in-depth analysis of what phenomena in reality "beg to be organized" through mathematics (Freudenthal, 1984). It is the designer's task to determine how the subject matter should be presented to optimally support children's learning.
(2) *Bridging learning levels using vertical supporting materials*: Where horizontal mathematizing involves the transition from a "realistic" context problem to the definitions of a problem in mathematical terms and the interpretation of the solution in light of a "realistic" setting, vertical mathematizing supports progressive mathematization (Freudenthal, 1991; Streefland, 1985; Treffers, 1978). Progressive mathematization describes every step that underlies the transition from one learning level to another as part of a general progression towards mathematical understanding (Streefland, 1983). As such, vertical materials such as models, diagrams, and conceptual schema support the transition from an initial naive, informal and context-dependent level, towards a level that requires more formal, systematic and reflective mathematical behavior.
(3) *Learning is a constructive activity*: In light of the constructive view on learning, this principle highlights the important role of children's own solutions as a means for designers to gauge their levels of learning and to subsequently develop appropriate instruction that can stimulate the child towards the next learning level.
(4) *Learning through interaction*: Considering the complex classroom situations in which learning typically takes place, this principle acknowledges the value of investigating the different types of strategies that children use in one instructional setting. This enables children to set examples for each other and to learn from each other, and it provides input for the teacher for discussing efficient and generalizable strategies.
(5) *Learning strands are intertwined*: This principle refers to the intricate relationship between different mathematical concepts and learning strands (e.g., fractions, decimals and percentages). It highlights how an instructional sequence is not isolated. Instead, the theoretical conceptualizations must take account of other knowledge that children acquire both within and outside a particular domain.

To stimulate the transition from informal knowledge to formal knowledge, children are encouraged to construct models such as schemes, notations, or descriptions. Such a model is at first context-specific and inspired by students' informal strategies (Gravemeijer, 1994, 1999). Throughout progressive mathematization, students encounter many contexts that can be represented by a particular model. This supports the generalization of the model across situations. Hence, rather than being a model "of" a particular mathematical situation, the model becomes a model "for" the coinciding mathematical conception (Gravemeijer, 1994, 1999). This transition underlines the bottom-up character of the reinvention principle. Emergent modeling identifies the following four levels of learning:

(1) *Situational level*: interpretations and solutions depend on the student's understanding of how to act in the setting of the activity
(2) *Referential level*: the model derives its meaning from the reference to the activity in the task setting
(3) *General level*: the focus is on mathematical relations and strategies and the model begins to derive its meaning from these
(4) *Formal level*: formal arithmetic no longer depends on the support of a model and the student can work with conventional procedures and notations

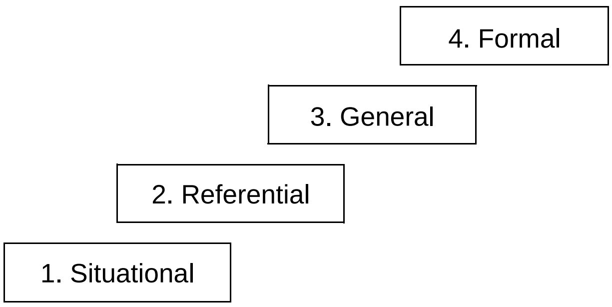


Fig. 2.1 Four levels of learning in emergent modeling (Gravemeijer, 1994)

The emergent modeling heuristic has been used in various design research projects in mathematics education (Doorman & Gravemeijer, 2009; Gravemeijer, 1999; Rasmussen & Blumenfeld, 2007). In the next paragraph the principles of RME are supplemented with insights from the socio-constructivist perspective that inspired the analyses of the influence of the instruction experiment on children's learning in a social context.

2.5.2 A socio-constructivist approach to learning and the emergent perspective

The compatibility of socio-constructivism and the emergent perspective with RME lies in the shared focus on learning as an iterative and constructive process. Socio-constructivism is derived from (radical) constructivism and social-interaction theories (Bauersfeld, 1980).

Radical constructivism characterizes knowledge as a model that is made based on experiences. This implies that knowledge is adaptive and that it cannot merely be a reflection of the objective outside world. For this reason, knowledge cannot be passed on passively from one person to another, but rather has to be actively constructed by the learner. Yet, if everyone's interpretation of the outside world is one that is acceptable, then teachers have no foundation for defining learning goals and for guiding students in their learning. This makes constructivism as a theory difficult to apply to education. Socio-constructivism provides a pragmatic solution to this problem by taking general agreement as the criterion for knowledge (Cobb & Yackel, 1996; Yackel & Cobb, 1996). The "socio" in socio-constructivism refers to the stimulation of discussions in class for sharing and reflecting on mathematical ideas as a means to progress towards a higher level of learning. As such, (mathematics) education can be founded upon:

> ... the taken-as-shared mathematical interpretations, meanings, and practices institutionalized by wider society. (Cobb, Yackel, & Wood, 1992, p. 16)

The *taken-as-shared* aspect of mathematical meaning-making, learning and practice, highlights how socio-constructivists interpret the type of communication that occurs between student and teacher in order for learning to take place. The teacher and student must reach a point in their discussion at which their conceptualizations are synchronized and subsequently "taken-as-shared". In this sense, learning occurs through the interactive process of readjusting conceptualizations that come to light when teacher and student encounter a miscommunication.

This socio-constructivist framework puts the social learning environment of a classroom setting forward as an alternative to analyzing learning processes across a group of students or for each student individually (Gravemeijer & Cobb, 2006). This framework suits our instruction experiment where students' learning processes are regarded as evolving from interactions and discourse in a social and constructive instructional setting. To gain insight into the complexity of such an instructional setting, we use design research to study the intricate relationships between elements that make up the *learning ecology* and children's learning. A learning ecology includes elements such as:

- the tasks or problems that students are asked to solve;
- the kinds of discourse that are encouraged;
- the norms of participation that are established;
- the tools and related material means provided;
- the practical means by which classroom teachers can orchestrate relations among these elements (Cobb et al., 2003, p. 9).

A learning ecology can be interpreted in light of the “emergent perspective” (Cobb & Yackel, 1996; Yackel & Cobb, 1996), a framework that helps to “account for students’ mathematical development as it occurred in the social context of the classroom” (p. 178). The social perspective in this framework consists of three components: the classroom social norms, the socio-mathematical norms, and the classroom mathematical practices.

Classroom social norms. The classroom social norms concern “the expected ways of acting and explaining that become instantiated through a process of mutual negotiation between the teacher and students” (Gravemeijer & Cobb, 2006, p. 31), and is sometimes referred to as a “didactic contract” (Brousseau, 1990). Such classroom social norms differentiate different types of classrooms on the basis of, for instance, how the students are expected to respond to questions or how the teacher evaluates and responds to students’ work and behavior. The classroom social norms that were identified in the instruction experiment are discussed in section 7.2.

Socio-mathematical norms. The socio-mathematical norms refer to norms of explaining and acting in class that specifically relate to mathematics. This may involve what students and teachers consider to be correct, efficient, or different mathematical solutions and explanations. These norms emerge through the process of negotiation in the instruction experiment. Awareness of spatial structure as an alternative to unitary counting, is an example of a socio-mathematical norm that is established in our instruction experiment. Socio-mathematical norms are valuable because they support students’ development of mathematical beliefs and values and therefore “foster the development of intellectual autonomy” (Cobb & Yackel, 1996, p. 179) of the class.

Classroom mathematical practices. The classroom mathematical practices are “normative ways of acting, communicating and symbolizing mathematically at a given moment in time” (Gravemeijer & Cobb, 2006, p. 32). These practices are specific to a particular mathematical idea or concept and they evolve during the course of an instruction experiment to become a new mathematical truth. For example, if a particular strategy, such as organizing objects into groups to read off their quantity, required an explanation at the beginning of the instruction experiment, this strategy no longer needs justification at the end (see Chapters 7 and 8). Qualitative differences in children’s mathematical interpretations may exist as they participate in the same mathematical practices (Cobb & Yackel, 1996). This contrasts with the socially accepted mathematical practices and illustrates the association between the development of individual minds and the social setting in which the children learn.

The socio-constructivist perspective on learning contributes to design research a way of analyzing the instructional sequence and students’ learning in a complex classroom setting. We use the framework for our instruction experiment to organize analyses of children's learning and to inspire the types of revisions of the instructional setting that are necessary to understand the role of the learning ecology in children's learning.

2.6 The research questions

Based on the literature review described above, we have come to suggest the following. Young children's emerging number sense includes insight into numerical relations, which requires the ability to (de)compose quantities. This ability involves insight into part-whole relationships, which is also an important factor in three components of spatial sense, namely spatial visualization, spatial orientation and shape. The act of spatial structuring, then, can help to mentally or physically rearrange spatial objects to elucidate part-whole relationships. Such insight into part-whole relationships may help to clarify numerical relations and find ways to abbreviate numerical procedures such as determining, comparing and operating with small (up to 10) quantities. Hence, we propose that children's insight into numerical relations may be supported by learning to recognize and make use of spatial structures for abbreviating numerical procedures. These conjectures generate the following research questions:

1. *What strategies for solving spatial and numerical problems characterize young children's spatial structuring abilities?*

To gain insight into the development of young children's spatial structuring ability, we chart and analyze the variety of approaches that children take to perform specially designed spatial and numerical tasks during an interview (Chapters 4 and 5). This operationalizes numerical relations in the research. Further, insight into the developmental trajectory of spatial structuring ability provides an interpretative framework that inspires our approach to the second set of research questions. The outcomes of the interviews and the strategy inventory are interpreted in light of the child's school assessment scores to assess their reliability and validity. This is to validate the use of these instruments as tools for gauging the development of young children's spatial structuring ability and number sense (Chapter 9).

2a. *How can young children be supported in learning to recognize and make use of spatial structures for abbreviating numerical procedures?*

After charting and analyzing the strategies that young children apply to solving numerical tasks, the next focus of the research is to see whether and how the development of young children's spatial structuring ability can be stimulated. For this we develop a hypothetical learning trajectory according to design research principles. This inspires the design of a sequence of instruction activities for a classroom instruction experiment (Chapters 5 and 6). The analyses of the classroom instruction experiment help to refine the learning trajectory with revised instruction activities (Chapters 7 and 8). The final instructional sequence operationalizes spatial structuring in the research. Further, we trace influences of children's participation in the instruction experiment on their spatial structuring strategies, by comparing children's spatial structuring ability before and after the instruction experiment (i.e., the strategies that they used on the pre- and post-interviews, Chapter 9). The analysis of converging and challenging observations regarding influences of the instructional sequence on children's learning, culminate in the design of a local instruction theory about how the development of young children's spatial structuring ability may be supported (Chapters 8 and 10).

2b. What characterizes a learning ecology that can facilitate the development of children's spatial structuring ability?

This research question is emphasized separately from research question (2*a*) to focus on the role of the learning ecology in supporting children's learning. Yet, considering the interrelatedness of (2*a*) and (2*b*), we first analyze them together and elaborate on question (2*b*) separately. Through investigating the influences of the instructional sequence on the development of children's spatial structuring ability, we abstract components of the instructional setting that contribute to a supportive learning ecology (Chapters 7-10). We look, for example, at the role of the teacher, at whether and how the class develops a shared vocabulary, and at the extent to which socio-mathematical norms of spatial structuring are established. This analysis is to contribute to a greater understanding of the complexity of such an instructional setting and its influence on children's learning.

In the next chapter, we describe the methodological approach that was taken to answer these research questions.

3 Methodology

Building on the context of the research that was presented in Chapter 2, we first explain what characterizes design research and why this type of research was appropriate for answering the research questions that are stated in section 2.6. We present the setting in which the research took place, including the children and the school that was involved, and what role the teachers and researchers played in performing the instruction experiment. In section 3.3, the procedures for conducting the interviews and the instruction experiment and for analyzing the data are described. Finally, we discuss the measures that were taken to improve the validity and reliability of the study.

3.1 Design research

As introduced in Chapter 2, our research is aimed at answering two research questions. The first question is concerned with charting and analyzing young children's spatial structuring strategies. For this, we developed tasks that were administered during one-on-one interview sessions with the children (see Chapter 4). The second question involves the development of a learning trajectory for supporting spatial structuring ability. In this section, we describe design research and explain why it was used to answer the second research question.

3.1.1 Theory development in design research

Design research is based on the procedure for generating empirically based theory that was first described by Glaser and Strauss (1967). It has further been developed into a methodology that is characterized by the cumulative, cyclical process of theory generation and validation (cf. Freudenthal, 1991; Gravemeijer, 1994; Gravemeijer & Cobb, 2006). This process involves an iterative procedure of theory-driven adjustments to the intervention and revisions of the hypotheses which led to an improved theory about learning (Cobb, Confrey, diSessa, Lehrer, & Schable, 2003; Gravemeijer, 1994).

Design research in mathematics education has both a theoretical and an applied purpose: theoretical in terms of coming to an understanding of mathematical thinking, teaching and learning, and practical in terms of using this understanding to improve mathematics instruction (Schoenfeld, 2000). Freudenthal (1991) used the term "instruction experiment" to refer to a type of research design in which an instructional sequence is created to broaden the children's insight into a particular mathematical construct. At the same time the instructional sequence should provide the researchers with a greater understanding of the children's learning processes. As such, researchers do not settle for answers that merely illustrate what works best. Instead, they use methods that both aim to understand *why* as well as describe *how* mathematics instruction works in practice (cf. Freudenthal, 1983; Gravemeijer, Bowers, & Stephan, 2003; Gravemeijer & Cobb, 2006).

Theory development in design research involves the development of a local instruction theory (Gravemeijer, 1994; Gravemeijer & Cobb, 2006). This theory includes a learning trajectory that is based on mathematical, psychological, and didactical insights about how researchers expect the children to progress towards an aspired level of reasoning. In line with the emergent perspective on learning (paragraph 2.5.2), such a progression should take into account both the cognitive development of the individual students, as well as the social context (i.e., people, classroom culture and type of instruction) in which the instruction experiment is to take place (Cobb & Yackel, 1996; Gravemeijer & Cobb, 2006). In practice, a local instruction theory encompasses an instructional sequence, as well as a description of the coinciding learning processes, the classroom culture, and the proactive role of the teacher.

The local instruction theory may inspire other researchers and teachers to implement the instruction experiment in their own settings as a way to evaluate and contribute to the development of a more encompassing theory (Gravemeijer, 2004). Since complete replicability is not necessarily desirable or possible, it is the trajectory towards revising and improving the instructional sequence, that is key to generating knowledge about children's learning (Simon, 1995). In fact, differences in learning ecologies (e.g., teachers, tasks, and materials) and their role in influencing the impact of the instructional sequence, may ultimately contribute to the ecological validity of the research (Gravemeijer & Cobb, 2006).

The cyclical process that characterizes design research (Gravemeijer, 2004) is illustrated in Fig. 3.1. To come to a local instruction theory, researchers conduct thought experiments to define a hypothetical learning trajectory (HLT, Simon, 1995) that shapes the design of each activity in the instructional sequence and that determines the type of data that is to be collected. Then an instructional sequence is tried out during an instruction experiment, retrospective analyses are performed on the transcripts from these sessions, their hypothetical learning trajectories are adjusted accordingly in a thought experiment, and the instructional sequence is improved to cohere with the revised hypotheses. Finally, the procedure is repeated by trying out the new instructional sequence in a subsequent cycle, and by learning from the class-experiences to, once again, begin the next thought experiment.

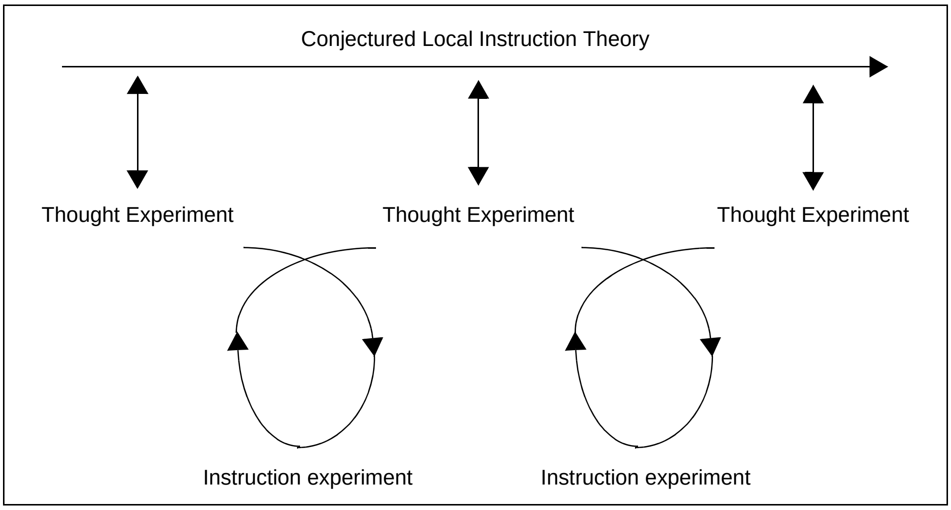


Fig. 3.1 The cyclical procedure of design research (Gravemeijer, 2004)

Design research is characterized by the interaction between data analysis and theory development. At the level of analyzing the raw data, Jacobs, Kawanaka and Stigler's (1999) model of the cyclical analytical process for analyzing video data illustrates the (re)organizing, (re)examining, and (re)coding of data that occurs in one cycle of analyses. In Fig. 3.2, we embed Jacobs et al.'s (1999) model into Gravemeijer's (2004) relatively theoretical model to come to a more encompassing schema that describes the intricate interactive processes of data analysis that are essential for theory development in design research.

In design research, the RME design principles (see section 2.5.1) interplay to help the researcher find ways of supporting students' progressive mathematization (Gravemeijer, Bowers, & Stephan, 2003). Regarding guided reinvention, the researcher anticipates the thinking trajectory that children may experience during a particular activity. Inspired by practical classroom experiences or by prior knowledge from literature, the researcher defines a learning trajectory, and develops instruction activities that can guide the students towards the learning goals.

In the initial stages of conceptual mathematization, the context of the instruction activities must at least be "realistic". This implies that the context is meaningful to the children although it may not be set in a real world. The setting may even be a mathematical one. Yet, as higher levels of conceptual mathematizing are approached, the context must be experientially real so that students can validate and test the mathematical concepts and ideas in more real life settings.

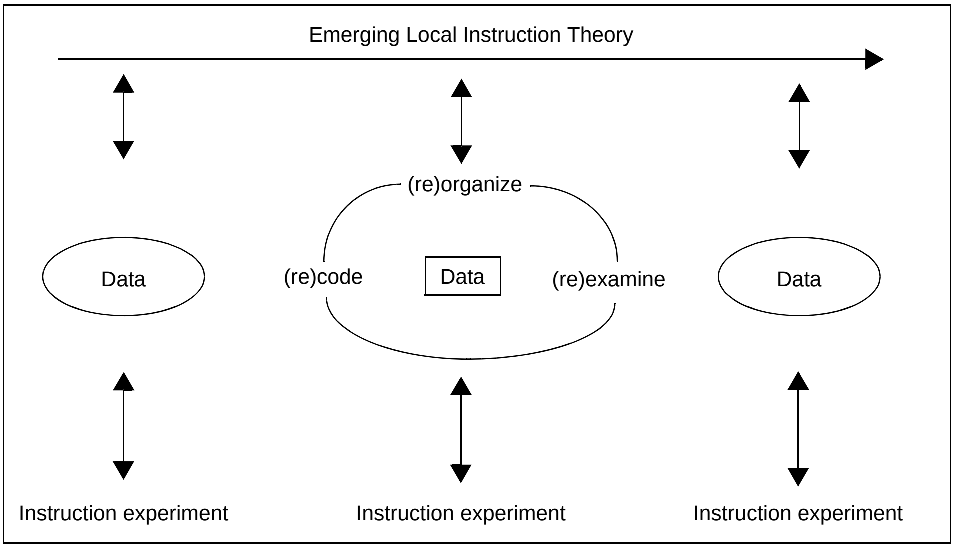


Fig. 3.2 Figure of the model that integrates Jacobs et al.'s (1999) model of the cyclical analytical process for analyzing video data with Gravemeijer's (2004) model for describing the cumulative cyclic process of design research

Hence, rather than searching for a context that suits a particular mathematical problem that must be covered in class as dictated by the mathematics curriculum, the researcher should first look for phenomena that "beg to be organized" and that trigger mathematical questioning (Freudenthal, 1991). For this, the researcher must engage in a didactical phenomenological study to investigate what and how phenomena are organized by mathematical concepts or procedures (Gravemeijer et al., 2003). Such knowledge can then contribute to defining a hypothetical learning trajectory.

Taken together, the researcher's assumptions about the students' learning trajectories, coupled with the search for experientially real contexts, should help find ways to model student's informal strategies. In turn, these mediating models should support the formalization of children's mathematical conceptions and operations along the four levels of learning (Situational, Referential, General and Formal, see section 2.5). These design research guidelines underlie the design of the interview tasks and instruction activities in our research.

3.1.2 Differences with traditional experimental designs

It is important to understand the differences between the design research methodology and traditional experimental design so that the choice for design research is grounded from the start of the research (Collins, Joseph, & Bielaczyc, 2004; Shavelson, Phillips, Towne, & Feuer, 2003). In comparing qualitative and quantitative methodologies, it becomes apparent that the two research traditions have strong reasons for adhering to their particular methodologies. The quantitative methodology in a traditional experimental setting, for example, results in quantitative findings that have the advantage of allowing the aggregation and generalization of data (Strauss & Corbin, 1998). Factors that are essential in a controlled experiment are the random allocation of the participants to experimental conditions, identifying and controlling key variables and excluding others, providing a particular treatment to the experimental group while holding other variables constant for all groups of participants, comparing control and experimental groups after the treatment, and generalizing the results (Cohen, Manion, & Morrison, 2007).

Purely quantitative methodologies cannot adequately accommodate design research outcomes for several reasons (Van Nes & Doorman, 2006; Van Nes, in press). One reason is that the complex qualitative data has to be reduced and transcribed to a more manageable amount before any analysis can take place (Strauss & Corbin, 1998). Strauss and Corbin (1998, p. 13) highlighted the need to organize the raw data in qualitative data analysis, since the process of organizing itself contributes: (a) to building rather than testing theory, (b) to providing researchers with analytic tools for handling masses of raw data, (c) to helping analysts consider alternative meanings of phenomena, (d) to being systematic and creative simultaneously, and (e) to identifying, developing, and relating the concepts that are the building blocks of theory. In this way, the type of data that is to be collected in design research should help the researcher gain insight into behavioral patterns that may confirm or challenge the hypotheses and contribute to answering the research questions (Gravemeijer & Cobb, 2006; Powell, Francisco, & Maher, 2003).

Another reason why a quantitative methodology can be inadequate for design research, is that, to keep the complex qualitative data manageable, a qualitative study is limited in size, with not enough data to perform generalizable statistical procedures (Strauss & Corbin, 1998). Given the small-scale and often exploratory type of studies in qualitative design research, it is also challenging to guarantee external validity of emerging (theoretical) conjectures (Jacobs et al., 1999). This is because qualitative analysis “draws on both critical and creative thinking” and relies on “the interplay between researchers and data” (Strauss & Corbin, 1998, p. 13). In theory, qualitative researchers may enlarge their pool of data, but in practice, the time-consuming data collection and analyses often place limits on the size that the study can grow to.

A question is whether it may be desirable to study mathematics education from a traditional experimental perspective (Freudenthal, 1991; Schoenfeld, 2000). Statistical analyses indicate whether significant difference exist in performance between the two groups. This methodology depends on prior theories and hypotheses as well as on previously developed and validated measures (Jacobs et al., 1999), which contradicts the exploratory theory-building that is inherent to trying to understand the complex processes that arise during instructional interventions. As a result, such outcomes have limited value in design research where the object of the research extends beyond establishing merely whether or not an intervention works. The value of small-scaled design research is that it acknowledges the complexity of a learning ecology and therefore allows for an in-depth analysis of an instructional setting to generate a theory about why the instructional design works and how it may be adapted to new settings (Cobb et al., 2003).

After interpreting the qualitative design research methodology against traditional experimental methodologies, we conclude that the formative (rather than confirmative) characteristics of design research are most appropriate for answering the research questions (Collins, Joseph & Bielaczyc, 2004).

3.2 Setting

3.2.1 The participating children

The instruction experiment was conducted in one of the four Kindergarten classes of a local Jenaplan elementary school. The class consisted of 21 children ranging in age from four to six years. The children came from mixed social and cultural backgrounds, several of whom spoke Dutch as a second language. The children's school standardized test scores were very heterogeneous. The children mostly scored in the highest and lowest percentiles with very few scores in between. This concerned the school director and he therefore welcomed research that could improve the level of mathematics education at his school.

The pre- and post-interviews were conducted with the children who participated in the instruction experiment (i.e., the intervention group, "IG") as well as with a comparable Kindergarten class (i.e., the non-intervention group, "NG") of 17 children who only participated in the pre- and post-interviews and not in the instruction experiment. Even though this group is not a strict control group, it was included in the research to increase the data set for at least developing and analyzing the interviews and the strategy inventory. In addition, the analyses on the IG children's post- compared to pre-interview performances could be enriched by seeing whether these outcomes showed any striking differences with the outcomes of the NG. The teachers of the NG were not informed about the instruction experiment that was conducted in the IG class. Instead, they continued teaching the standard curriculum, and the teachers of the IG group were asked to not talk about their experiences in the instruction experiment.

Five children in the IG were appointed as a focus group. The focus group sessions were intended to supplement the classroom discussion with more detailed observations of children's strategies and social interaction (Cobb, Stephan, McClain, & Gravemeijer, 2001). These sessions also provided the children with more opportunities for social interaction that could evoke a greater variation of strategies. Based on experiences in previous exploratory studies, the focus group consisted of five instead of seven children. A smaller focus group was expected to minimize distractions and improve the quality of the work. The five focus group children were selected on the basis of several criteria. Most importantly, the children had to have shown evidence of a spatial structuring ability that fit the description of the second (Recognition) and third (Usage) phases in the pre-interview (see Chapter 5). Children who fit the Recognition phase, recognized regular spatial structures, but did not make use of these structures to abbreviate numerical procedures such as determining, comparing and operating with small quantities. Children who fit the Usage phase not only recognized particular spatial structures, but also made use of available structures to circumvent unitary counting procedures. What differentiates these children from children in the fourth (Application) phase, is that they did not yet show their own initiative to spontaneously apply structure to unstructured arrangements of objects in a goal-directed effort to abbreviate numerical procedures.

The reason why Recognition phase and Usage phase children were selected for the focus group, is that these children were expected to benefit from the instructional setting for learning to make use of spatial structures within the time frame in which the instruction experiment was to take place (cf. Siegler, 2005). It would require much more time and instruction to support children in the first (Unitary) phase, who could not yet count resultatively and synchronously. Conversely, fourth (Application) phase children were apparently already at the level of strategy usage that the other children were expected to reach through participating in the activities in the instruction experiment. Other selection criteria for the focus group were that the children were between 4.5 and 5.5 years old, that they had responded openly during the pre-interview (e.g., no language barriers), that the group would consist of both boys and girls, and that the parents of the children gave permission for not only involving the children in the research, but for also having it recorded on video. Finally, the selection of children was based on the teachers' impression of which children they thought would be a good representation of the class.

3.2.2 The Jenaplan school

The school that participated adheres to the Jenaplan teaching approach. According to Jenaplan principles, children take part in several classroom activities and discussions throughout the day, but the emphasis is on letting the children work and learn at their own pace.

This implies that the children are used to working in small groups. The teaching is based on what is called "world orientation" which means that it focuses specifically on what the child needs to cope in society. As such, much of the work concerns projects that relate to real life situations and that emphasize interactions between the children and the position of the child in the world (Van Noorden, 2005).

This teaching approach at the school is important to take into account when interpreting the activities. At one point, for example, the teachers suggested that the classroom activity should not take too much time nor require too much effort on the part of the children because the children are not used to long centralized discussions. They preferred to work in small groups. Indeed, towards the end of the classroom discussions, several children were asking when they were allowed to "work", making reference to small projects that they wanted to get back to. This was spurred on by the fact that the children's age and developmental levels were sometimes so different that the classroom discussions could not always keep all the children involved at the same time anyway.

The children's experience with working on theme-related projects resembles how, in line with the principles of RME, the activities in the instruction experiment were set in a context. The context was important to support the children in their understanding of how spatial structuring can help them solve numerical problems. Moreover, in agreement with the Jenaplan teaching approach, a context was important to relate the activity to real life situations as a way to give meaning to the questions that the teacher was asking. Hence, the RME and Jenaplan perspectives converge in the way children are regarded as the constructors of their own knowledge in practical, real-life settings. However, the Jenaplan background of this school was not the reason for including this school in the research; the school was one of the few schools in the area that welcomed researchers and that was open to suggestions for improving their mathematics education.

3.2.3 The role of the teachers in the instruction experiment

Two teachers were in charge of the intervention group. Teacher Alice[1] taught the class for the first three days of each week and Teacher Tracy taught the class for the remaining two days. The teachers performed the same instruction activities in both rounds of the instruction experiment. To prepare for the instruction experiment, the teachers were provided with a manual that described in detail the aims, requirements and instructions for each of the instruction activities (see Appendix 5). This manual was also used to discuss each of the activities with the teachers before the instruction experiment. After introducing the

1. All names of participants in this thesis are anonymized

principles of RME, the manual outlines the instructions, the procedure and the questions that the teacher can ask the children during the activity. This procedure allowed the researcher time to prepare the teachers and to help them become aware of the important contextual and conceptual connections that exist between the instruction activities. This made the teachers responsible for guiding the classroom discussions.

The teachers took their role very seriously and asked critical questions about the theoretical background of the activities. The teachers made suggestions about how the activities could work better in their classroom setting. These suggestions also gave an impression of the classroom social norms which later helped to interpret the outcomes of this instructional experiment. One particularly important suggestion concerned the duration of the classroom discussions. Since the children were used to working in small groups, the teachers preferred to use the classroom discussions as an introduction to the activity before continuing in smaller groups for a more effective setting with in-depth discussions.

Based on this suggestion, each of the classroom discussions was conducted in a way that would introduce the children to the activity without requiring too much continuous attention and time. The implication for the research is that the classroom discussions could not be expected to result in as much in-depth analysis as was originally anticipated. That could occur more in the focus group. On the other hand, it strengthens the ecological validity of the research because this instructional setting is more true to typical classroom social norms such as the types of interactions and discourse that normally occur.

3.3 Procedure

3.3.1 The pre- and post-interviews

To answer the first research question, several interactive tasks were compiled that the children performed on two separate occasions (the pre- and the post-interviews) in a one-on-one interview setting with the researcher (see Appendix 2). The development of the interview tasks will be described in Chapter 4. The outcomes of the pre-interview were to contribute insight into a general developmental trajectory for children's ability to recognize and make use of spatial structures in numerical procedures (see Chapter 5). They also offered an important starting point for gauging the children's initial approaches to the tasks and the role of their prior instruction therein. The post-interviews were meant to gain insight into the influences of children's participation in the instruction experiment on their advancements in spatial structuring ability.

The interview technique was based on the clinical research method, proposed originally by Piaget and elaborated later for educational research purposes (Van Eerde, 1996). By asking the child to perform tasks and answer questions in a non-standardized setting, the researcher can gain insight into how the child approaches a particular problem (e.g., Battista et al., 1998). Next to observing the children's behavior, the researcher may ask additional questions to ascertain the observations and to encourage the children to elaborate on or to clarify their explanation (Van den Brink, 1989; Van Eerde, 1996). This offers children a chance to rephrase their explanations or to rethink and reflect on their answers. It also gives them the benefit of the doubt about whether, for example, they may have made an unsystematic or careless error that would otherwise give a wrong impression about their actual level of understanding. These qualities of the interactive setting contrast with a paper-and-pencil setting in which the questioning occurs less flexibly. As a result, much of (young) children's conceptual knowledge may remain obscure.

Each child was interviewed by the researcher for 20-30 minutes in a separate room where the video-camera was installed and the activities were set out on the table (Fig. 3.3). All the children performed every task, and both the pre- and the post-interviews spanned a two-week period. The understanding between the researcher and the child was that the interview would be conversational in style; the researcher told the child that she was curious to know how the child solved the questions, she encouraged the child to explain his/her thinking as he/she was working, and she casually responded to the child's comments and stories while manoeuvring the focus back to the interview task.

Fig. 3.3 The video set-up for the interviews

The researcher made use of a script of the opening questions for each task. These were asked identically in each interview (see Appendix 2). Our experiences in the exploratory studies had helped to formulate these questions in a way that would encourage the children to interpret them properly and to increase the chances of answering the question correctly. As soon as the researcher noticed that the child misunderstood the question, the question was clarified without providing any additional information. When it became apparent that the questions were too difficult for the child (usually the youngest children during the pre-interview), the task was simplified and the researcher made note of the changes in the script for the analyses. The researcher proceeded analogously when the tasks did not seem challenging for the child (i.e., for several bright and/or older students).

As the researcher was taking notes of what the children were saying and of how they were rearranging the objects, the children were given the opportunity to correct the researcher's interpretations and to explain why they approached the task in their particular way. Although children typically vary in the types of strategies that they use to solve a problem, and although studies have suggested that such reflection can further stimulate the children's learning (Siegler, 2002; Cheeseman & Clarke, 2007), our objective was to gain a general impression of the children's level of spatial structuring ability through an as valid as possible interview procedure.

Due to the circumstances, only one researcher was available to conduct the interviews. Still, the value of having only one person, is that it added a consistency in following the script and in asking additional questions. It also offered the opportunity to spend a considerable amount of time with each of the participating children which contributed to the validity of the data collection. The researcher took great care to follow the script and to keep the interviews separate from each other. That is why we only reflected on the interviews and started analyzing the videos after every child had participated.

Every additional question that was asked after the standardized question in the script, was meant to gain as much insight into the child's conceptual knowledge of the problem as possible. This supported the intricate process of analyzing the children's responses in light of the strategy inventory. This inventory was developed during the exploratory studies to chart the strategies that the children applied to solve the interview tasks (Chapter 4, see Appendix 4), giving an impression of the children's level of spatial structuring ability and number sense.

3.3.2 The instruction experiment

In parallel with studying the developmental trajectories of children's spatial structuring ability, we designed a hypothetical learning trajectory and a corresponding instructional sequence to answer the second research question. This sequence was tried out with the intervention group during two rounds of an instruction experiment. The first round included six activities while the second round was adjusted to five activities (see Chapters 6 and 7). These rounds spanned two weeks each, and they were conducted in between the pre- and post-interviews. The purpose of this instruction experiment is (a) to better understand how children learn to make use of spatial structures for abbreviating numerical procedures such as determining, comparing and operating with small (up to 10) quantities, and (b) to highlight ways to improve the instructional sequence so that the instruction activities better interweave with the children's mathematical reality, and promote spatial structuring to support children's numerical development.

Several meetings were held with the two teachers to discuss the general plan for the study. During the first meeting, the researcher asked the teachers of both the intervention and non-intervention groups to create mind maps of the constructs "number sense" and "spatial thinking". This was before the intervention group teachers were informed about the focus on spatial structuring in the instruction experiment. The idea was to get an impression of how the teachers interpreted these constructs, how their approaches to the constructs differed, and whether or not and in what way their approaches may be reflected in their teaching. The mind maps are discussed in section 9.4.

A two-week period was reserved for the instruction experiment to minimize disruptions by other school activities or holidays. The activities were also scheduled during schooltime (early in the morning or after playtime) to keep the children most concentrated. Before each session, the researcher met with the teachers to discuss and prepare the instruction activity. The total time for preparing, performing and discussing the activity took approximately one hour. Following the activity, the teacher was asked to complete a questionnaire about her impression of the classroom session and the activity itself. This debriefing was important to develop a shared interpretation between the researcher and the teacher about what was happening in the classroom (Gravemeijer & Cobb, 2006). The teacher included the activity in her notes to the other teacher so that the teachers could relate the activities to each other. The researcher reflected on the activity by working out the field notes into a report and by using this report to prepare for the next session.

Each activity in the instruction experiment started with a classroom discussion. The children were sitting on their chairs in the middle of the classroom in a U-shaped arrangement, facing the teacher who was sitting on her chair. The camera was set up just behind the teacher to record the expressions of the children as they responded to the activity. The teachers prepared the activity for themselves before the session and they made use of the manual to guide them through the activity during the session. The teacher agreed to end the classroom discussion as soon as she noticed that the children lost their interest and concentration. The researcher, in turn, was free to interrupt with suggestions for questions that could help to optimize the activity.

The role of the researcher during the classroom discussions was to observe and take notes of the class performing the activity, to coordinate the data collection and to provide suggestions for improving the discussion or the classroom interaction. This time, in contrast to the previous exploratory studies, the researcher guided the focus group while the teacher entertained the rest of the class. The researcher was fully in charge of the focus group, because teacher effects were sometimes noticed in the exploratory studies (e.g., “pulling” rather than “guiding” the children towards a particular solution, without reflecting on the meaning of the activity). The disadvantage of not having the teacher present during the focus group sessions, however, is that the researcher had to put in more of an effort to keep the children disciplined and focused on the tasks.

The researcher’s active involvement in standing by the teacher during the instruction experiments did not go unnoticed. Considering the impact of being a stranger and onlooker with a camera in a classroom, the researcher therefore decided to make use of her role and interact with the children whenever that was appropriate. When children turned to her to show her how they had solved a task, for example, the researcher responded encouragingly. Sometimes she also asked the children questions from behind the camera when an opportunity arose for stimulating the children’s thinking, which the teacher could not attend to at that time. The children became less intimidated by the camera, as they became increasingly comfortable with the researcher’s presence in class.

The focus group sessions immediately followed the classroom discussions. In two of the sessions, the researcher took the focus group into a different room so that the rest of the class could continue working in the classroom, and in the other sessions the focus group stayed in the classroom while the rest of the children went to play outside. In the focus group setting, the researcher asked the questions, described the context, and guided the children in discussing the role of spatial structure in each of the activities. One child was absent during the fourth activity and another child missed the classroom discussion about that activity. The researcher ended a focus group session as soon as the discussion was becoming fruitless with the children being too tired and distracted. Typically, this happened after about twenty minutes. After each focus group setting, the researcher recapitulated the session with the teacher, gave the teacher a questionnaire with questions to reflect on the session and the instruction activity, and wrote a report about the classroom and focus group session, with an analysis of the instruction activity itself.

The time between the first and the second round of the instruction experiment was necessary for conducting retrospective analyses. This involved revising the hypothetical learning trajectory (HLT) and the instruction activities based on analyses of the field notes, analyses of the videos using ATLAS.ti (see section 3.4), the teacher's suggestions, more literature, expert meetings, and a better formulated research perspective. This resulted in five improved activities that were tried out in the second round of the instruction experiment (see Chapter 7). The second round of the instruction experiment was performed in the same way as the first round. In preparing the teachers for the second round, the researcher highlighted the differences between the design of the activity in the first compared to the second round. In the debriefing talk, she also asked the teachers whether they found that the activity had improved and, if not, what further revisions they would suggest. Two months after the instruction experiment, the teachers were interviewed to determine what they retained from the intervention and whether they continued to perform similar activities with a focus on spatial structuring (see section 9.4).

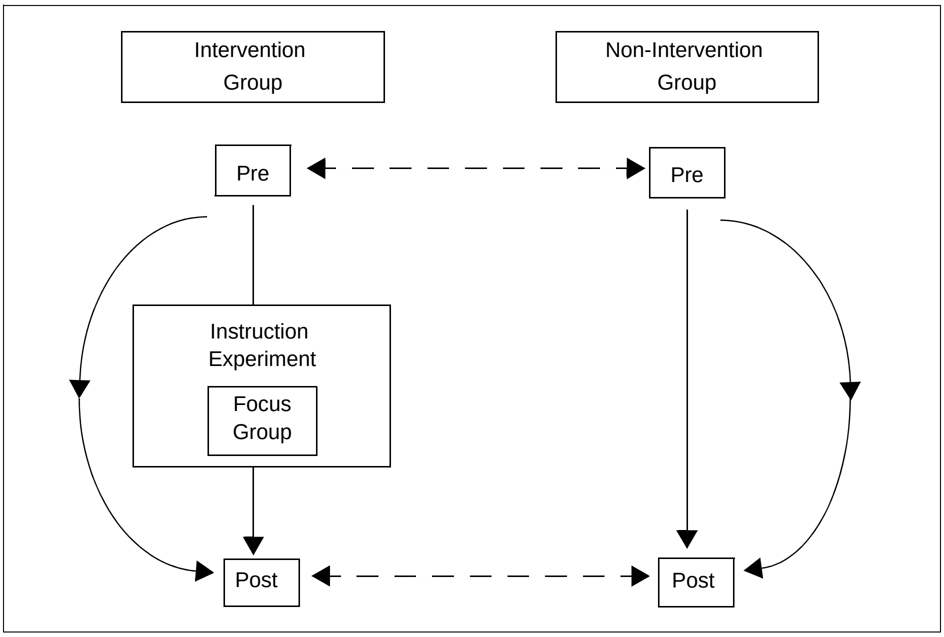


Fig. 3.4 The chronological procedure for performing the interviews and the instruction experiment

Taken together, the data gathered from the interviews with the teacher, the two rounds in the instruction experiment and the reflective analyses, all serve as input for more insight into how the instructional setting can support the development of children's spatial structuring ability. Fig. 3.4 summarizes the chronological procedure for the interviews and the instruction experiment in the study. The vertical arrows indicate the progression of the study from pre-interviews to post-interviews, with the addition of the instruction experiment and focus group for the intervention group. The curved arrows indicate the comparisons between the outcomes of the pre-interviews and the post-interviews between and within the IG and NG (see Chapters 5 and 9). The dotted horizontal arrows indicate the careful (i.e., not controlled) comparisons between the pre-interview outcomes and the post-interviews outcomes of the IG and NG.

3.4 Data analysis

The children were video-taped during the interviews and the classroom instruction activities. This resulted in more than fifty hours of video recordings. The multimedia data analysis program ATLAS.ti was an essential tool for organizing the raw video data. This visual qualitative data analysis software program allows for the interactive coding of rich text, image, audio and video materials. After importing raw data in the form of, for example, a video, screenshots or scans of written work into the program, the researcher can organize the data in ATLAS.ti by segmenting the data into “quotations” (i.e., video clips or “meaningful chunks”; Stigler, Gallimore & Hiebert, 2000). By adding comments to quotations, creating codes to label the quotations and linking the appropriate codes to specific quotations, we could make sense of how the children were solving the problems, how they were developing in their understanding, how the researcher, the teachers and the instruction activities had played a role in this development, and how proactive individual and classroom instruction could ultimately support the children’s learning. The insights were supplemented with data from the debriefings with the teachers and reflections on the interviews with the children and the classroom activities.

3.4.1 Multimedia video data analysis in the interviews

In analyzing the interviews, the focus was on the types of strategies that the children used to solve the tasks before compared to after participating in the instruction experiment. The children’s responses to each of the tasks in the interviews were classified using the strategy inventory (see also Chapter 4). The strategies are listed in the order of their complexity so that each strategy can be interpreted as a particular level of sophistication in the children’s spatial structuring and numerical ability.

The role of ATLAS.ti in this part of the research was to provide a mould for organizing the raw video data, as well as for helping to create a user-friendly and reliable method for coding the interview questions. We used the program as follows:

(1) We segmented each main question in the interviews into one quotation.
(2) We entered each category of the strategy inventory into the program as a preliminary list of codes.
(3) We analyzed the interviews by linking the relevant codes to each quotation.

The coding procedures supported the tracing of general trends in children's level of strategy use. This contributed to insight into variations in young children's spatial structuring abilities that outline a developmental trajectory for children's spatial structuring ability (see Chapter 5).

3.4.2 Multimedia video data analysis in the instruction experiment

Compared to the deductive analyses of the interviews, the analyses of the instruction experiment were inductive. ATLAS.ti helped to configure and organize the often large amounts of raw qualitative data into quotations that mark critical events. This occurred in the following steps.

Defining the unit of coding and clipping quotations. Several studies have suggested making quotations of each activity (Stigler et al., 2000), lesson event (Clarke, 2003), or episode (Andrews et al., 2004). The unit of data that is appropriate depends, however, on the nature and goals of the research. Stigler and colleagues (2000), for example, studied cross-cultural differences in teaching patterns and devised a coding scheme based on the activities in a classroom session. Clarke (2003) analysed the same data in terms of the "form" and "function" of various lesson events across more than just one lesson. In response to Clarke, Andrews and colleagues (2004) stated that such lesson events are too broad and that, instead, cross-cultural studies, for example, can suffice with coding episodes within one lesson. These studies aim to compare several lessons across several studies. Our research is more limited in scope because it concerns one classroom at one school. Therefore, we settled on using codes that illustrate whether, and if so, how each of the children that took part in the instruction experiments met the observation criteria.

Observation criteria are descriptions of how, based on literature and on the previous exploratory studies, the instruction activity is expected to proceed in terms of how the children respond to the instruction activities and what teachers can do to improve the learning ecology (see sections 6.2 and 6.4). These observation criteria are defined in terms of both verbal (e.g., the child may say "it looks like the five on dice") and nonverbal behaviors (e.g., the child arranges the eggs into two rows of three). Hence, the unit of coding the videos of the instruction activities was based on the observation criteria. This implies that the intervals of footage vary in length and can be part of an activity, an episode (defined as "a sequence of observed interactions, negotiations and activities with a single didactic or managerial purpose"; Andrews et al., 2004, p. 7), or a lesson event (defined as "a form of classroom interaction occurring within a lesson, but at a level of social complexity greater than just a statement or action taken by an individual"; Clarke, 2003, p. 10).

Defining codes and supercodes. The codes and supercodes (i.e., clusters of codes) that were defined during the analyses for Round 1, concerned the child, the teacher, or the researcher. Examples of the codes are "the child makes a connection to a preceding instruction activity", "the child expresses insight verbally", "the teacher makes a supportive remark", and "the researcher re-formulates the question". The supercode "creating links" refers to the collection of codes that relate to connecting insights between and within activities. Most of the codes and supercodes were already defined after the analyses of Round 1. This spurred the analyses of Round 2 because the coding could begin as soon as an episode was clipped.

Refining the codes. As the coding continued, it became apparent that more focus was put on particular codes and less on others. Some codes became less essential in the second than in the first round, while inspiration for other codes kept emerging from the second round. For example, in Round 2 we coded less thoroughly whether the children were distracted and we focused less on the difference between the teachers' teaching styles because the focus was more on instances that related to the activities and spatial structuring of the children. Apparently, during the first round of analyses, far more codes were differentiated than was necessary or relevant for answering the research questions.

Towards tracing behavioral patterns. Multimedia data analysis can relieve researchers from the cognitive strain that may be associated with trying to keep track of all the thoughts and observations that could be shaping the conjectures and contributing to the theory. This coincides with the method of constant comparison (Glaser & Strauss, 1967; Strauss & Corbin, 1998) which was applied to systematically reflect on the codes to assess their relevance to the research questions. In this method, incidents of participants' activity are documented and compared to one another to give rise to general themes or patterns. The process of constantly comparing incidents to current conjectures leads to ongoing refinements to the theoretical categories that are developed from the data. Indeed, as more behavioral patterns emerged in the data, so-called memos were defined which served as notes of our developing conjectures. These memos joined similar observations and ideas throughout the instruction experiment.

Refining, organizing and analyzing the memos. The defining of memos continued during the analyses of the second round of the instruction experiment. With an increasing number of quotations that fit particular codes and memos, several memos started to overlap (e.g., memo "type of question" and "questioning") so we combined some of them to elucidate the coding process. The memos themselves were linked to each other in the same way that the codes and supercodes had been linked. A diagram of the corresponding network helped to define and focus on the physical and theoretical connections between the clips and on the conjectures that emerge during the process of organizing and making sense of the data (Fig. 3.5). Importantly, such a network view is as close to a "picture" for answering the research questions as it can get using ATLAS.ti. Although working with the program took an excessive amount of time, the final outcome was not a presentation of the theory for answering and providing illustrations for all the research questions. Still, the long-term advantage of using this program outweighed its short-term tediousness because each time the data was analyzed or prepared to share with others, the quotations and descriptions made it significantly easier and less time consuming to find those particular instances that were referenced.

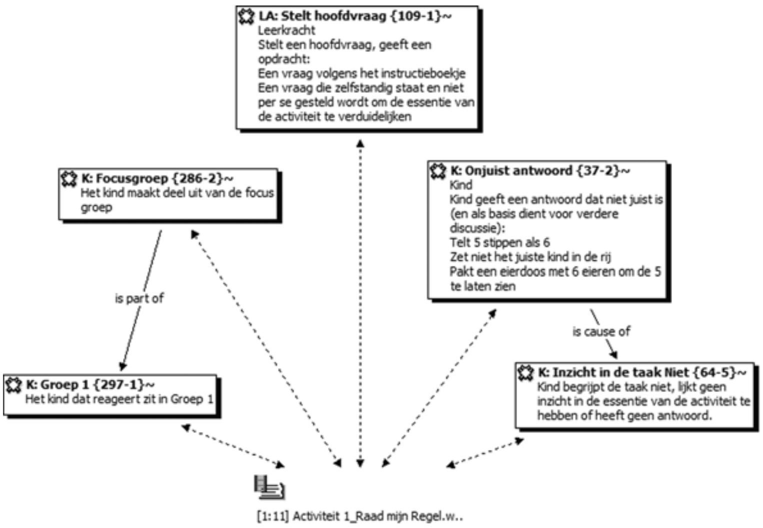


Fig. 3.5 An example of a network of nodes for the quotation 1:11 in Activity 1, "Guess my Rule". Each node signifies a code that is attached to this quotation and two sets of nodes are linked to each other ("is cause of" and "is part of") to illustrate their underlying relationship

Towards theory generation. The entire process of first organizing and coding, then re-coding, noting patterns in memos, condensing, expanding and redefining memos greatly contributed to the theory generating process that marks this design research. Through constantly comparing the rapidly growing list of memos, differences were noticed in the way that the various memos could be prioritized. It was decided to categorize the list, which up to now consisted only of memos, and change the memo type either to commentary, memo or theory (Fig. 3.6).

A commentary is a type that is broader than a code because it acts like a piece of notepaper for the researchers' thoughts. The commentary refers to one and possibly more instances that are interesting to keep in mind, but that are still difficult to make sense of or to place in the research perspective. A memo is type that refers to logistic issues and to issues that should be kept in mind while developing the theory. A theory is a type that directly contributes to the theory generating process of the research. It is supported mostly by memos and illustrated by quotations.

Commentary-type memo:
Dori repeatedly confuses the 6 and the 9. Is this because she is making counting errors?

Memo-type memo:
The teacher makes use of the children's confusion to explain the essence of the activity to the whole class

Theory-type memo:
The context generates surprise and enthusiasm, the teacher creates a mysterious and exciting setting, the children are keen to participate in the activity, they refer to the context spontaneously

Fig. 3.6 Examples of the three types of memos (commentary, memo and theory) as defined in ATLAS.ti for the instruction experiment

The more the clips were studied and the codes and various types of memos were shuffled, the more it put weight on the theory-type memos, and the more the theory-type memos converged towards a coherent understanding of how the instruction activities contributed to answering the research questions. This iterative process resulted in the articulation of learning moments (see section 6.2 and Chapter 8). Learning moments are a collection of observations (i.e., quotations contributing to a theory-type memo) that either contribute to the conjectured learning moments (section 8.1) or supplement them with perspective broadening insights (section 8.2). Finally, during reflective analyses, these learning moments were summarized into nine broad cumulative learning insights that underlie the hypothetical learning trajectory (section 8.4), and that contribute to a local instruction theory for supporting the development of young children's spatial structuring ability.

3.5 Research validity and reliability

A wealth of intricate data resulted from the interviews and the instruction experiment. In the following paragraphs we describe the measures that were taken to ensure the validity and reliability of the instruments and the analyses of this data. Validity may be defined as the absence of systematic bias, and reliability as the absence of unsystematic bias (Maso & Smaling, 1998). Each paragraph relates first to the analyses of the interviews and then to those of the instruction experiment.

3.5.1 Validity

The internal validity of the research concerns the quality of the data collection and the credibility of the outcomes. We first discuss the *interviews*. The reasoning behind the design of each interview task and the underlying theoretical framework is discussed in sections 4.2 and 4.3. Several measures were taken to prevent self-fulfilling prophecies from influencing the analyses. The analyses of the interviews began by noting observations that were of interest to the research incentives. Only later were the episodes reinterpreted with regard to the specific research questions. In addition, the children's individual analyses were constantly compared to each other to keep using the strategy inventory in the same way for each child. Expert researchers were also regularly consulted, and interpretations were compared to interpret the episodes from different perspectives. Since the purpose of the study was mainly to develop a theory (rather than to test a hypothesis), as much input as possible was gathered. This helped to attend to both converging and diverging observations.

To improve the quality of the data collection and the accuracy with which the conclusions were drawn for internal validity of the interviews, we discussed our experiences with trying out the tasks with different children of varying age levels, and we compared the sequence of tasks with similar studies. Children's general level of performance on the tasks also agreed with their performance on two standardized tests: the Leerling Volgsysteem Test (LVS) and the Utrecht Numeracy Test (UNT; Van Luit, Van de Rijt, & Pennings, 1994). The LVS test is a standardized school based assessment with tasks that cover shapes, classification, seriation, comparisons, counting and numbers. The UNT is a standardized test that covers topics such as comparing, classifying, correspondence, seriation, using counting words, synchronized and shortened counting, resultative counting, and applying number knowledge. During one of the exploratory studies, each child completed the UNT in an additional half-hour session. The children's LVS scores were provided by the teacher. Insights about some of the children could sometimes also be supplemented with the teachers' impressions.

We acknowledge that other factors outside of the pre- and the post-interview will influence children's development. Similarly, the children's development in age and experience in between the two interviews cannot be ignored. Yet, regarding threats to the internal validity of the interviews, the non-intervention group was included to highlight that outside factors will influence both the intervention and the non-intervention groups. Despite the non-controlled research setting, by qualitatively studying the interview outcomes of both groups, we looked for possible differences in the group's performance regarding their spatial structuring development in particular.

It was assumed that the children would not remember the details of the interview questions after a period of four months, and that this would result in minimal testing effects. Even if the children would recognize the contexts and the questions in the tasks, this would be equally the case for the intervention group as well as the non-intervention group. Some of the quantities that were involved in the tasks were also changed to certify that the children would have to give different answers from the pre-interview. This is in line with preventing statistical regression because no more difficult questions were structurally devised for the post-interviews. Rather, the same level of difficulty was kept, while ways to simplify the questions or to challenge the children were defined. We expected statistical regression because the aim of the interviews was to gauge children's spatial structuring ability up until the last phase in the development of spatial structuring ability (see section 5.2). Children whose repertoire of strategies fit this phase, appeared to be prepared to develop more formal mathematical procedures based on a strong foundation of spatial structuring ability. In this research we were particularly interested in children who had not reached this stage yet and who might benefit from instructional support to develop their spatial structuring ability.

Regarding the internal validity of the *instruction experiment*, the observation criteria were approached with a questioning attitude (see section 3.4). Since the unit of coding was based on observation criteria that were devised for the purposes of the instruction experiment, it would be tempting to focus only on these criteria and to neglect other behaviors that may add to the observation criteria or challenge the criteria and the conclusions that we would draw. Since the observation criteria are a part of our theory (i.e., what is expected to be observed as children learn), open-mindedness was necessary towards what kind of behavior in the videos may be indicative of children's learning processes. This corresponds to the developmental characteristics of design research in which the data itself can revise and improve our theory.

Considering the external validity of the interviews, we acknowledge that the interview setting will offer a moment's impression of a child's mathematical understanding. Nevertheless, the process of developing the tasks and constructing the strategy inventory occurred at three different schools, with children from six different classrooms who participated both alone and sometimes in pairs and who ranged in age from four to six years. Moreover, the interviews evolved on the basis of tasks that have been evaluated and reported in previous research, and many of the strategies that are listed in the inventory can be traced back to generally accepted developmental trajectories of particular mathematical abilities of young children (e.g., the counting stages, Gelman & Gallistel, 1978). Further, many of the issues that were encountered during the process of creating the inventory (e.g., identifying and distinguishing components of spatial structuring ability and number sense), are supported by other research (e.g., Van den Heuvel-Panhuizen & Buijs, 2005; Mulligan et al., 2005).

The interview contains questions that repeatedly tap a particular ability within spatial structuring and number sense, in various contexts, at different instances during the interview, and in different words. Together, the collection of codes for the strategies that the children used to solve these questions should provide a general impression of the children's spatial structuring ability and number sense. In this respect we agree with Bruce and Threlfall (2004), who in their research have suggested a developmental sequence in the cardinal and ordinal aspects of number, and who conclude that the developmental levels may raise:

> ... awareness of what may have preceded and what may follow the current approach used by the child, so that appropriate input and intervention can be shaped to enable the child to move forward. (p. 24)

Finally, the children's performance on the post-interview and their participation in the classroom instruction activities during the instruction experiment should shed more light on their performance during the pre-interview and contribute to the external validity of both interviews.

As only three schools (two pilot schools and one for the instruction experiment) took part in this research, we contend that the outcomes of the research cannot yet be generalized. Instead, the study is an exploratory (rather than confirmative) study into the learning trajectories of the children at this particular school. Still, the interaction between this specific instructional setting and the researcher strengthens the ecological validity of the study. The conclusions that are drawn deduce from the observations so other researchers can understand the reasoning that underlies the conclusions that are drawn. This should contribute to the trustworthiness of the research: "the reasonableness and justifiability of inferences and assertions" (Gravemeijer & Cobb, 2006). To approach generalizability, the next step in the research is to adapt the learning trajectory to other classrooms, by studying the effects in those settings and by revising the trajectory accordingly.

Ultimately, the integration of the outcomes of the trajectory as tried out in various settings, should contribute to a conjectured local instruction theory about young children's spatial structuring ability. In time, the final instruction activities could be tested in a larger scaled traditional experimental setting to determine differences in effects on students' learning due to their participation in the instruction experiment.

3.5.2 Reliability

To determine the internal reliability of analyzing children's performance on the interviews using the strategy inventory, several interrater reliability analyses (Cohen's Kappa) were conducted with expert colleagues. This occurred on various occasions throughout the development of the inventory (see section 4.3). Much time was spent discussing any discrepant scores so that remaining ambiguities in the strategy inventory could be removed. Further, the large data set that resulted from the combined intervention and non-intervention group's pre- and post-interview outcomes, contributed to the creation of an increasingly reliable and refined strategy inventory.

For the instruction experiment to come to internally reliable results, we combined several data sources through a method called triangulation; collecting data either from various sources (data triangulation), through various methods (method triangulation) or by various people (researcher triangulation), to approach the research situation from more than just one, relatively subjective or instantaneous, perspective. As such, the primary data sources in the instruction experiment consisted of video recordings of each of the instruction activities, the questionnaires that the teachers completed for debriefing, the log that was written about what happened during the activity, and additional notes from discussing the activity with the teacher before and after the session. The data from these sources was analyzed against the background of the research questions and the insights from the pre-interviews.

Regarding the external reliability of the research, it is necessary to establish the "trackability" of the conjectures (Gravemeijer, 1994; Gravemeijer & Cobb, 2006; Maso & Smaling, 1998); the research is documented in such a way that it could be retraced or virtually replicated (Maso & Smaling, 1998) by other researchers. This elucidates how the research was conducted, what choices were made and for what reasons, what conclusions were drawn from what kinds of observations, and how this affected the local instruction theory. Working with ATLAS.ti contributed to the reliability of the research because the quotations made it easier to mark and refer to specific events and instances during the interview and instruction activities. As described in section 3.4, the use of ATLAS.ti helped to organize the data and keep track of the theory development. Moreover, the progression of the design was regularly discussed with experts to reflect on experiences in the classroom and to evaluate the choices that were subsequently made. Such explicit and justifiable reasoning was necessary for such discussions to be fruitful.

In the next chapter, we discuss the development of the conceptual schema that is based on the literature review of Chapter 2. This underlies the design of tasks for an interview and an inventory for charting the types of strategies that the children used to solve the tasks. The development of the interviews and the inventory, together with the analysis as described in Chapter 5, contribute to answering the first research question about what spatial structuring strategies children use to solve spatial and numerical problems.

4 Developing a Conceptual Schema, the Interview Tasks, and the Strategy Inventory

This chapter marks the beginning of the development of a plan for the study. In the first section we describe how an outline of the conjectured relationship between spatial sense, number sense, and young children's spatial structuring ability was made based on the literature review as outlined in Chapter 2. This initial conceptual schema was fundamental to the design of the interview tasks and the strategy inventory, which are discussed in sections 4.2 and 4.3. The process of revising the interview tasks and refining the strategy inventory, shed more light on the role of spatial structuring ability in the development of numerical insight. This is reflected in the final conceptual schema that is presented in section 4.4.

4.1 Relating spatial sense and number sense through spatial structure

To organize the ideas about how young children's spatial structuring ability may be related to their emerging number sense (Chapter 2), we studied relevant research (e.g., Griffin & Case, 1997; Van den Heuvel-Panhuizen & Buijs, 2005), discussed our ideas with experts and observed Kindergarten children at a local elementary school as they performed spatial and numerical tasks during one-on-one interviews. Our general perspective on how spatial sense, number sense and spatial structure may be related is summarized in the first version of the conceptual schema as in Fig. 4.1:

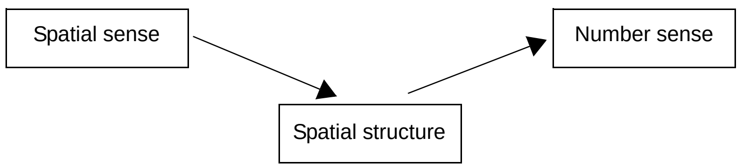


Fig. 4.1 The first version of a conceptual schema relating spatial sense to number sense through spatial structure

A second version of this conceptual schema was necessary to clarify that the supposed influence of early spatial sense on the development of number sense is (a) but one of many possible influences, and that (b) the influences may be reflexive. Another factor that plays an important role in the development of both spatial sense and number sense, is language. The words and symbols that children learn to use, offer them a tool to, for example, express their thinking and communicate their reasoning (Hughes, 1986). Language also bridges children's conceptual understanding and promotes the development of vocabulary; the more children know how to use language to verbalize their insights, the more they can evaluate their own insights in light of those of others, and the richer their understanding of mathematical ideas becomes (Freudenthal, 1984; Van Eerde, Hayer, & Prenger, 2008). This illustrates the mutual relationship between mathematical abilities and language.

Another factor that deserves attention in the conceptual schema, is the influence of, for example, early perceptual subitizing skills for recognizing parts of a whole spatial structure to facilitate conceptual subitizing (Clements, 1999a). In addition, children must have mastered at least a resultative level of counting if they are to learn to make use of spatial structures to move away from unitary counting procedures (Battista et al., 1998). To cover such external influences and to clarify that the focus of the study (i.e., fostering children's spatial structuring abilities) is *one* of many possible factors that can support the development of number sense, we proposed a second version of the conceptual schema (Fig. 4.2):

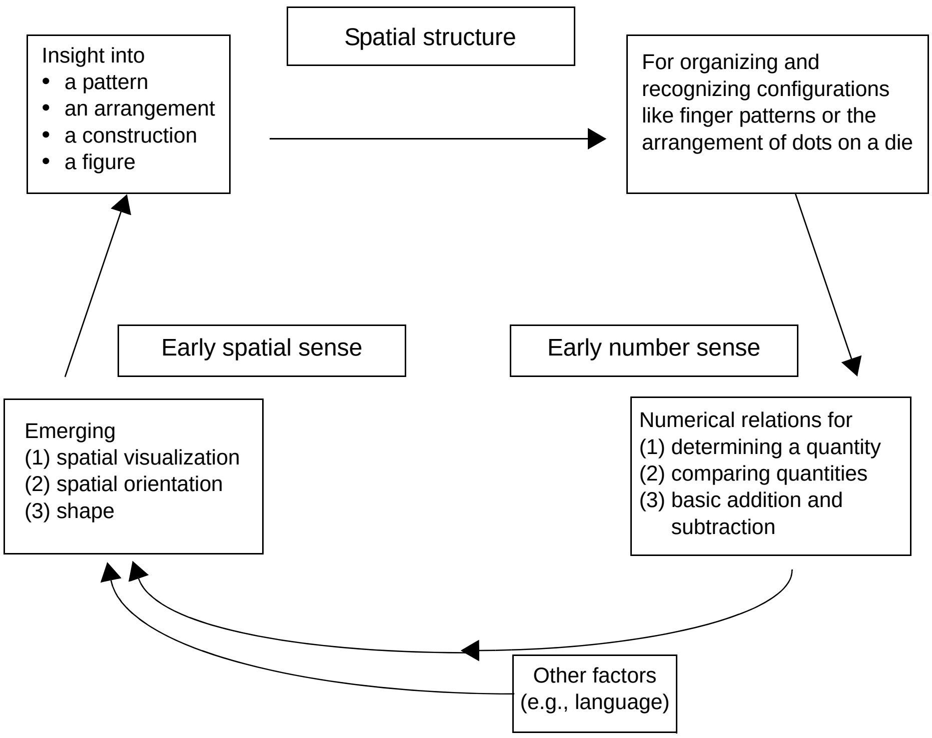


Fig. 4.2 The second version of a conceptual schema that specifically relates early spatial sense, early number sense, and spatial structure, while acknowledging influential effects of other factors

The ongoing process of revising this conceptual schema, contributed greatly to the tuning and refining of the research focus; it provided insight into how the three components of early spatial sense relate to spatial structuring, and, in turn, how spatial structuring could influence the development of number sense. It also inspired the development of an interview to gauge children's spatial structuring and numerical strategies.

4.2 The interview tasks

Based on the research literature, discussions with experts, and the conceptual schema, a sequence of tasks was designed for a one-on-one interview setting. The original purpose of conducting the interviews was to chart and gain insight into the strategies the children used to solve spatial structuring and numerical tasks. This was to indicate the extent to which a child made use of spatial structures for abbreviating numerical procedures in the interview tasks (cf. Battista et al., 1996, 1998; Mulligan et al., 2006a). The strategy inventory contributes to answering the first research question about what spatial structuring strategies children use to solve spatial and numerical problems (see section 5.3). Although the development of the interview tasks and the strategy inventory are indissolubly connected, we focus in this section on the explorations that were involved in designing the interview. The details of the strategy inventory are described in section 4.3.

4.2.1 Designing the interview tasks

The process of constructing the interview involved iterative cycles of designing the tasks, trying them with several children, reflecting on the observations in light of the conceptual schema, revising the schema if necessary, incorporating the children's strategies into the initial strategy inventory, and adjusting the tasks so that they would reflect more of children's conceptual knowledge about spatial structuring and number sense.

The first period of the development was particularly important for exploring the setting of the research in terms of questions such as what level of mathematical reasoning can be expected from 4-, 5-, and 6-year olds, what level and type of language do children of this age typically use, what level and type of language can they be expected to understand, and what kinds of differences exist in how kindergartners are approached within and across classrooms. It was also necessary to explore what contexts are experientially real for these children so that the interview tasks would be meaningful and inspiring to them (section 2.5).

With more experience in working with kindergartners, the focus shifted towards improving the quality of the tasks and constructing an inventory of the types of strategies that the children used to approach the tasks. The first set of tasks was divided into a sequence of spatial tasks and a sequence of number sense tasks that were to gauge the children's spatial sense and number sense. The tasks were administered in two separate half hour sessions. The number sense tasks involved comparing quantities, counting and dividing up quantities, sequencing numbers, adding quantities, and determining large quantities. The spatial tasks involved mosaics, constructing with blocks, counting blocks, determining a route, and classifying shapes. The choice for these mathematical concepts and processes was based on previous research (cf. Battista & Clements, 1996; Bruce & Threlfall, 2004; Clements et al., 1999b; Owens, 1999; Van den Heuvel-Panhuizen, 2001; Van den Heuvel-Panhuizen & Buijs, 2005) and discussions with experts. Each task was embedded in an appealing context that was meaningful to the children. A script was made of the questions (Appendix 2).

We discuss the first number sense task as an example of how the tasks were evaluated and revised with regard to the first try-outs. In the first task, two blue sheets of paper represented two ponds. A number of ducks were swimming in the ponds in either a structured or an unstructured arrangement. The children were to determine in which pond the greatest number of ducks were swimming (Fig. 4.3). Most children only used the area that was taken up by the ducks to compare the quantities. What was missing from the task was a standard counting question. That would guide the focus of the interview tasks more towards determining quantities rather than perceptual comparisons.

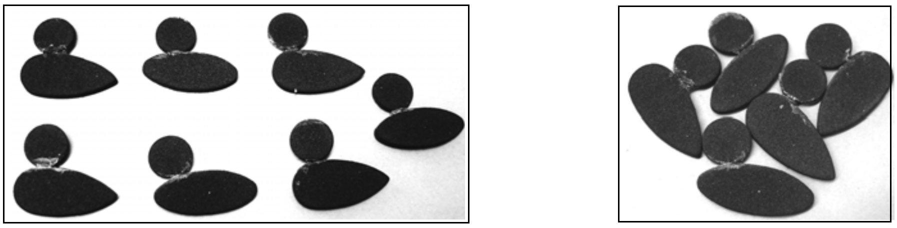


Fig. 4.3 A structured and an unstructured example of an arrangement of ducks in a pond. Children are asked to compare the number of ducks in each pond

In the revised activity, the children were asked to determine the number of ducks in the pond and to explain how they came to their answer. The 5, 6, 7, or 8 ducks were placed either in a structured or an unstructured arrangement in the pond. The reason for choosing these quantities is that the five and six can directly be related to familiar spatial structures such as dice configurations (cf. perceptual subitizing, Clements, 1999a), while the 7 and 8 challenged the children to search for and to construct their own structures (cf. conceptual subitizing, Clements 1999a). For determining smaller quantities, the children were expected to rely more on their perceptual subitizing (i.e., recognizing a quantity without using other mathematical processes) rather than on their conceptual subitizing skills (i.e., recognizing a pattern based on higher-order organization of a composite of parts; Clements, 1999a). Larger quantities were included to maximize the chance that all children within this age range would succeed at determining the quantities either by unitary counting or by spatial structuring strategies.

The activity about comparing quantities was revised to contrast the spatial structures more. In the previous version of the comparison task, children could easily see the difference between four and five ducks in the pond because one of the ducks was missing in the middle of one pond. This time the spatial structure for four ducks, for example, was changed to three ducks in a diagonal with one on the side. This configuration required more mental manipulation and quantity comparison than just a perceptual comparison between the two sets of ducks (cf. Clearfield & Mix, 2001; Wood & Spelke, 2004).

In summary, what became clear through analyzing this activity, is how strongly related specific characteristics of a task and children's resultant behavior are. Hence, in designing the tasks, instead of concentrating mainly on what kinds of tasks cover the important competencies that are inherent to early spatial and number sense, the focus had to lie more on what kind of behavior the task would invoke. Hence, the reciprocal question was what kind of behavior would provide sufficient insight into children's understanding of certain mathematical concepts and the strategies that they use to solve the tasks. As such, the next step in the developmental process was to study the behavior and language that the children used to solve the tasks. We then had to determine which types of behavior best reflect children's spatial and number sense, and examine whether this type of behavior was appropriately operationalized in the tasks.

To illustrate this developmental process of designing tasks and reflecting on their relevance for the research, we note how the set-up for the interviews was initially concentrated on the two general domains of space and number (see the upper part of the conceptual schema in Fig. 4.18). Literature studies inspired the conjecture that early spatial abilities could play a stimulating role in the development of number sense based on observations and literature about children's remarkable early spatial sense (e.g., Ness & Farenga, 2007). Yet, this research focus was still very broad, so we started looking for ways to narrow down the research scope.

After testing several practical adjustments to the tasks, we became increasingly intrigued by the role that spatial structures may play in determining and comparing quantities. Some children, for example, explained how they determined or compared quantities by making reference to how the objects were arranged. Moreover, some children described how they abstracted a structure to elucidate the process of determining a large quantity. At the same time, some children could "see" and conveniently make use of a particular structure in one context, while they became confused in another context. There also appeared to be a marked distinction between recognizing and actually making use of a particular spatial structure in a task. Some children recognized six if it was presented to them in an arrangement such as the dice configuration. Yet, when asked to arrange the objects in such a way that someone else could "easily see" that there are six, these children would not spontaneously make us of the dice configuration. Instead, sometimes they would leave the objects as they were, in a pile or in a line, and revert to unitary counting procedures, explaining that that was an "easy way to see how many there are". This may imply that the children did not understand the question. In general, however, children who not only seemed to be familiar with spatial structures, but who also made an effort to use the structures, tended to perform better on the interview tasks than children who were not familiar with the structures or who did not apply the structures.

Observations such as these motivated us to narrow down the research focus; they inspired the investigation into whether and how children's spatial structuring ability could be cultivated to foster early numerical abilities such as determining and comparing quantities. In practice, this meant that the operationalization of each task was evaluated in terms of its relevance to spatial structuring ability. A task about ordinality, for example, became relatively less relevant to the research than patterning tasks. From this perspective, a series of tasks was developed that involved the essential components of the number sense and spatial sense domains in terms of spatial structuring. Hence, the spatial and numerical tasks were no longer separated and the tasks were administered in one interview session.

Throughout the process of developing the interview tasks, we teased out the spatial structuring abilities that are necessary for recognizing, using, and extending spatial structures (i.e., patterning), and the number sense abilities that are necessary for determining, comparing, and operating with small quantities. These form the foundations of the strategy inventory, which was used to analyze the children's strategies on the interview tasks (see section 4.3).

4.2.2 The resultant sequence of interview tasks

In this paragraph we discuss the resultant sequence of interview tasks and describe their contribution to the research. To highlight the distinction between recognizing spatial structures, knowing how to use readily available spatial structures, and knowing how to spontaneously apply spatial structures to abbreviate numerical procedures such as determining, comparing and operating with small quantities, the interview started with several assessment tasks (see Fig. 4.4 for an outline of Part 1 of the interview). First, the children were asked to show “how well they could count to fifteen”. This task was to assess children’s counting ability while it usually also broke the ice and encouraged the children to share their reasoning with the researcher (see section 3.3 for details about the procedure).

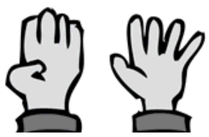
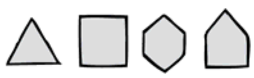
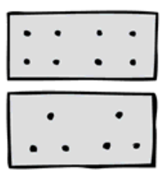
Flashcard Task	Underlying Question	General Evaluation
Quantities up to 15	Can the child count to 15?	Errorless and without guidance, but with room for corrections
Finger patterns 	Does the child recognize the finger patterns? Can the child produce finger patterns him/herself?	Recognizes and can produce the 1 to 6 and 10 finger patterns
Dice configurations 	Does the child recognize dot configurations on dice?	Recognizes all the dice configurations, but may have trouble with the 5-structure
Corners of shapes 	Does the child recognize the number of corners in a shape?	Recognizes at least the triangle and square. The pentagon and hexagon provide additional information about insight into symmetry for double-structures (e.g., “3 corners here and the same on the other side makes 6”)
Structures composed of two sets of 3 and two sets of 4 dots 	Does the child recognize double-structures of dots?	Provides additional information about the child’s insight into compositions of structures

Fig. 4.4 Part 1 of the interview: flashcards were used to evaluate children’s ability to recognize particular spatial structures

A set of flashcards was used to determine which children recognized what types of spatial configurations. This set included eight cards with three to ten raised fingers for recognizing finger patterns, four cards with either a triangle, a square, a pentagon or a hexagon on them, and four cards, each with three to six dots structured as dot configurations on dice. Since one and two are trivial quantities, they were omitted from the configurations. The cards were presented one-by-one for no longer than three seconds. This was enough to see whether the child recalled the spatial structure or whether the child had to count each of the fingers or dots unitarily to determine the quantity.

Assuming that the children are familiar with most finger patterns, these were presented first to them. Then the children were asked to raise their own fingers to show four and six. We took note of the spontaneity with which they showed a particular finger pattern. Next, the cards with the dot configurations were presented and the children were asked to determine the number of dots on the die. For each of the geometric shapes on the cards, the children were to determine how many corners the shape has. In this task, we were particularly interested in observing whether the children counted each corner, or whether they were familiar enough with the shape to either recall the number of corners (as in the case of the triangle and square) or to apply gestalt-like principles for analyzing the shape, recognizing symmetry, and for abstracting the quantity through conceptual subitizing (Clements, 1999a). One child, for example, explained how she saw three corners on one side and three on the other, and reasoned that that makes six corners in total. The last two tasks in the flashcard section were included to challenge the children in abstracting a particular spatial structure from a relatively unfamiliar arrangement. One card pictured six dots that were arranged in two groups of three, and the other card pictured eight dots that were arranged in two groups of four.

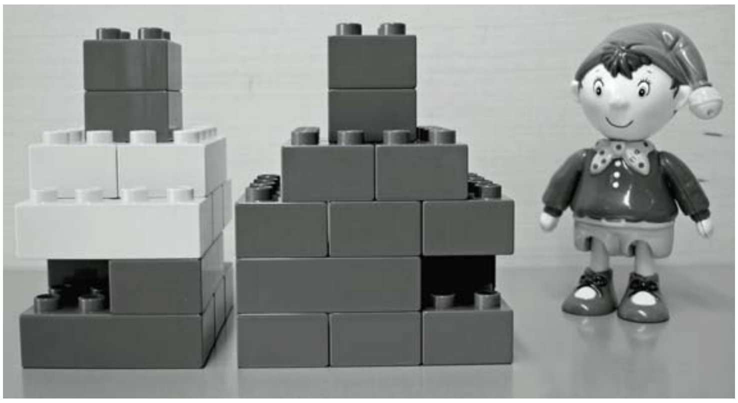


Fig. 4.5 Noddy the Dwarf ®, the main character of the interview tasks, with block houses

Part 2 of the interview involved the interactive tasks that were developed to gauge children's spatial structuring and numerical strategies. First, Noddy the Dwarf® was introduced to the children (Fig. 4.5). Noddy is a toy that accompanied the children as they worked on the tasks. The idea behind involving Noddy is that children tend to feel more comfortable when they can share their ideas with a toy rather than with a relatively intimidating researcher (c.f. Van den Heuvel-Panhuizen & Buijs, 2005).

The story was about Noddy the Dwarf® who went to pick flowers to decorate his home for his birthday party. The children were first asked to use a finger pattern to show that Noddy had turned eight years old. In the meantime, the researcher arranged four sets of plastic flowers on the table in front of the child (Fig. 4.6). The top-left set contained five flowers, the bottom-right set had twelve flowers and the flowers of both sets were placed in an unstructured group arrangement. The top-right set contained eight flowers, the bottom-left set had eleven flowers and the flowers of these sets were arranged in a structure: two rows of four and three rows of three with two on the side.

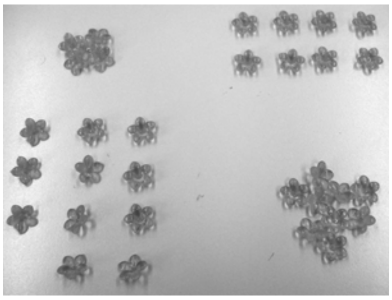


Fig. 4.6 The flowers in a task that required the children to determine, compare and operate with structured and unstructured quantities

The questions in this task originate from the task with ducks in a pond that was discussed in paragraph 4.2.1. The first question was about helping Noddy find the group that contained eight flowers, which is just as many as his age. Then the child was asked to find the group(s) that contained more than eight flowers. Both questions were aimed at observing how the child determined and compared quantities. The researcher asked as many additional questions as necessary to understand what strategy the child had used.

For the third question, the four groups of flowers were removed and replaced with a bunch of five flowers. After first asking the child to determine how many flowers were in this bunch, the next task was to determine how many flowers would be necessary so that the bunch would have twelve flowers. After recognizing and using structures in the first three questions, the child was expected to possibly be more aware of using spatial structures to simplify the procedures. This implied that the child would either physically or mentally add-on flowers to come to a total of twelve. The child may, for example, count using the fingers or physically or cognitively add flowers. Alternatively, the child could apply a doubling strategy with two sets of five flowers and add another two flowers to come to twelve.

Finally, the child was asked to arrange the group of eight flowers in a way for Noddy to also quickly and efficiently be able to see that there are eight flowers. For this question, the child was not only expected to *recognize* and make *use* of spatial structures, but also to spontaneously *apply* spatial structures to an unstructured arrangement. This would be a way for the child to communicate insight and understanding of the convenience of spatial structuring.

The questions for the second task revolved around Noddy's house. Two houses were constructed out of Duplo® blocks (Fig. 4.7). Both houses were made up of ten blocks but the blocks of one house were structured while the other house was unstructured. In this context, "structured" implies a degree of symmetry and a pattern in how the layers of blocks are placed on top of each other. The structured house in this task was made up of three layers of two rectangular blocks, topped with a layer of one rectangular block, two squared blocks and another squared block. The unstructured (i.e., asymmetrical) house was just as wide and tall as the structured house, but with the blocks piled up in a way that made it difficult to understand the construction.

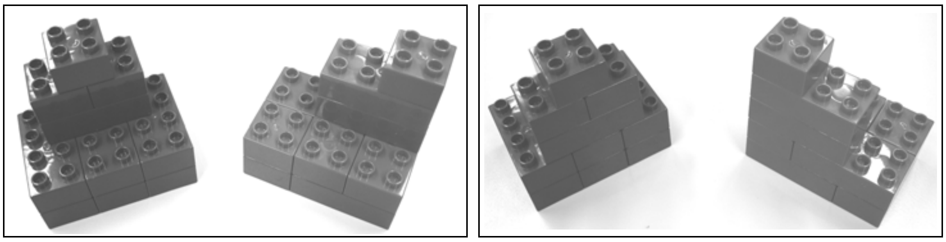


Fig. 4.7 Two side views of the structured (to each left) and unstructured (to each right) block constructions

This task illustrates the practical connection between spatial insight into a construction of blocks and ways to determine a quantity (Clements & Sarama, 2007; Ness & Farenga, 2007). It resembles the task in Battista and Clements' (1996) research in which the aim was to describe how students' spatial thinking is related to their enumeration strategies as they dealt with 3-D rectangular arrays of cubes. In the first question, the children were asked which of the two "houses" they thought was larger. The children's explanations could show

how they perceived the houses and whether or not they took note of the structures. Next, the children were asked how many blocks they thought would be necessary to build Noddy's (i.e., the structured) house, and then his neighbor's (i.e., the unstructured) house. The purpose was not for the children to count the blocks, but rather to see whether just by examining the structure of the structured house, they could find a way to determine the quantity.

Only after they attempted to determine the quantity without pointing to the blocks, were the children allowed to count the number of blocks. The children's counting procedures give an impression of the extent to which the children recognized and made use of the spatial structure of the structured house and whether they may have noted a difference between the two houses based on their structure. Finally, the children were asked to build Noddy's house using another set of blocks. The way the children approached this building task offered another indication of whether the children were aware of the structure of Noddy's house, and whether they were able to make convenient use of this structure. Finally, the children were asked to also rebuild the neighbor's (unstructured) house. This was expected to be a more challenging task for the children. Therefore, the last question was to compare the two houses to stimulate a discussion about the convenience of using structure for building the construction.

The third interview task was a patterning task. Following Mulligan and colleagues (2006a), we define a pattern to be a "numerical or spatial regularity", while "the relationship between the various components of a pattern constitutes its structure" (p. 209). Previous versions of this activity involved patterning with beads on necklaces, but the difficulty with such a context was that the children remained too attached to the specific length of the chain and to what they thought would look attractive, regardless of what the pattern looked like. Instead, to fit Noddy's context, the story in the present task was about his birthday party. Noddy's friends had lined up their dwarf hats outside the house (Fig. 4.8).

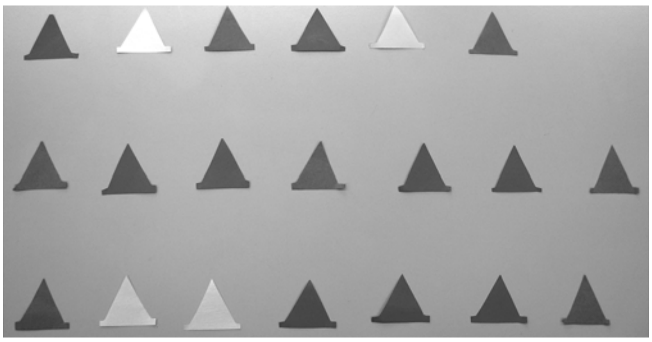


Fig. 4.8 The three patterns of red, blue and white colored dwarf hats that the children were asked to extend

The problem arose when a gust of wind blew part of the line of hats away, mixing up the order of the hats. The child was asked to help the dwarves reconstruct the line of hats so that the dwarves could find their hat again (A. Roodhart, personal communication, July 11, 2006).

For each of the three patterns, the child was asked to describe to Noddy what the sequence looked like (i.e., in terms of the colors of the hats) before the wind blew the hats away, so that Noddy could continue the pattern by himself. The first pattern (*a, b, a, b* ...) was relatively simple so that the children would quickly understand the task. The second one was more complex in that the child had to see the variation in the number of blue hats in between the single red hats. The last pattern offered the child several options in that it could be extended by increasing the number of the next elements (*a, bb, ccc, dddd* ...) or that the sequence could start again from the beginning (*a, bb, ccc, a, bb, ccc* ...). Through discussing the pattern with the children, the researcher tried to understand whether the child recognized the structure that is repeated to create a pattern. As such, by having the children describe the pattern, they were encouraged to think about and reflect on the characteristics of the pattern. This highlights the importance of part-whole relationships in understanding patterns and spatial structure (e.g., Mulligan et al., 2006a).

The third and final part of the interview (Part 3) involved a task on spatial orientation. The reason for including this component in the interview was to see how the children perform on the spatial structuring tasks relative to, not only their spatial visualization and shape abilities, but also to their localization and navigation abilities. This was to gain additional insight into other components of spatial orientation, apart from spatial structuring (Battista & Clements, 1996; Battista et al., 1998; Clements & Sarama, 2007) that could be influential in the development of number sense. In this spatial orientation task, the children were presented with a simple map of the Kindergarten classrooms (Fig. 4.9).

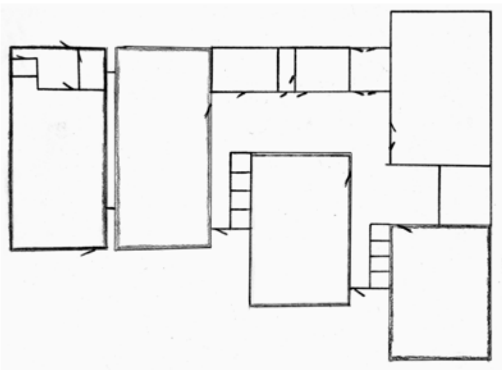


Fig. 4.9 The map of the Kindergarten classrooms that was used for the spatial orientation task

First, the map was discussed with the children so they could become familiar with its notations. The children were then asked what room they thought the researcher was pointing to on the map. Next, the children were asked to trace how they would walk from one particular room on the map to another. This requires the children to picture in their minds the route that they normally take, and then translate that image to the the map (Van den Heuvel-Panhuizen & Buijs, 2005).

Although this task assumes a high level of spatial orientation, it offered a valuable opportunity for some children to show their competence on this aspect of the spatial tasks that was not related to their performance on the enumeration tasks. If the questions for the tasks were too difficult, the children were asked to point in the air in the direction of a particular room in the school. This question was more accessible to most children because it eliminated the step of translating a mental picture to a map. To conclude the interview, the children were asked whether they had enjoyed the tasks, which task they liked best and whether or not they thought the questions were difficult.

4.3 The strategy inventory

As the tasks for gauging kindergartners' spatial structuring ability and number sense were (re)designed, an inventory was constructed of the types of strategies that the children used to solve the interview tasks. This inventory became the instrument that served to chart children's spatial and number sense in terms of their spatial structuring ability. By comparing the children's repertoire of strategies in the post-interview to that of the pre-interview, we also expected to gain additional insight into the development of children's spatial structuring ability in relation to their participation in the instruction experiment (see Chapter 9). In this section we present some of the developments that were necessary to optimize the reliability and validity of the inventory.

4.3.1 Creating the strategy inventory

The original list of strategies was compiled from a literature review (e.g., Battista & Clements, 1996; Bruce & Threlfall, 2004; Buijs, 2003; Gelman & Gallistel, 1978; Nunes & Bryant, 1996; Siegler & Araya, 2005; Van Eerde, 1996). The strategies that were observed during the exploratory studies contributed to elaborating this initial list so that strategies for each type of task (e.g., counting small structured amounts) could be ordered from relatively unsophisticated (e.g., pointing to each object and counting out loud) to more complex (e.g., perceptive counting).

First versions: Extensive, task related strategies within the two domains. Initially, the focus of the research was on the two broad domains of spatial sense and number sense. Therefore, the inventory first contained a cumulative list of strategies for each interview task within each of the two domains. With more experience in observing children's approaches to the tasks, more details were included in the strategy inventory. The strategies gradually became less specific to a particular task, and more oriented towards general aspects of children's mathematical development. Fig. 4.10 is an excerpt from one of the first versions of the strategy inventory.

Type of Task		Strategy
Counting and Comparing Quantities	A	Guessing
	B	Asynchronous 1-by-1 counting
	C	Synchronous 1-by-1 counting by i. pointing to the object ii. moving the object aside
	D	Resultative 1-by-1 counting by i. pointing to the object ii. moving the object aside
	E	Applying configurations to i. recognize a specific arrangement (e.g., dots on dice) ii. recognize 2 and 3 as 5 (subitizing)
	F	Arithmetic: i. recognizing part of the configuration and adding on the rest ii. addition (3 + 3 = 6) iii. comparing sums (5 + 5 = 10 but 5 + 4 = 9 so 5 > 4)
	G	Geometry: i. comparing surface areas in structured situations ii.comparing surface areas in unstructured situations

Fig. 4.10 Excerpt from one of the first versions of the strategy inventory: strategies relating to counting and comparing quantities within the domain of number sense

Second versions: Task-independent aspects of children's mathematical development. As described in section 4.2, following the literature review and our experiences in the exploratory studies, we became increasingly interested in children's spatial structuring ability. This resulted in a shift from studying children's behavior from the perspective of the spatial and numerical interview tasks (e.g., counting and comparing quantities of ducks in a pond), to studying children's behavior from the perspective of more task-independent aspects of their mathematical development (i.e., numerical procedures such as the ability to determine a quantity) and the role of spatial structuring ability therein (Fig. 4.11).

Type of task		Strategy
Determining a Quantity	A	i. guessing, no idea ii.unclear counting strategy
	B	Asynchronous 1-by-1 counting by: i. moving the object aside ii. pointing to each object iii. perceptive (by looking) How: i. out loud ii.mentally Order: i. randomly ii.systematically: rows or groups before single objects
	C	Synchronous 1-by-1 counting by: i. moving the object aside ii. pointing to each object iii. perceptive (by looking) How: i. out loud ii.mentally Order: i. randomly ii.systematically: rows or groups before single objects
	D	Resultative counting by: i. moving the object aside ii. pointing to each object iii. perceptive (by looking) How: i. out loud ii.mentally Order: i. randomly ii.systematically: rows or groups before single objects
	E	Applying spatial structures: i. subitizing (with ≤ 3 or combinations such as 2 and 3 is 5) ii. instantly recognizes the configuration (e.g., dice, “1 in the middle”)
	F	Combined strategies: i. abstracting the spatial structure and adding on the rest ii. (repeated) addition (3 and 3 and 3 is 9) of configurations or already counted rows; abbreviated counting of larger quantities

Fig. 4.11 Excerpt from the second version of the strategy inventory: the strategies that relate to determining a structured quantity are identical to the strategies that relate to determining an unstructured quantity

This change of perspective motivated revisions to the tasks to refine the operationalization of the research focus on spatial sense and the development of number sense in terms of children’s spatial structuring ability. Hence, each component of spatial sense and number sense in the second versions of the inventory, consisted of a list of task-independent mathematical concepts and procedures, and each of these topics was described by a cumulative and increasingly detailed list of strategies.

Third versions: From detailed and overlapping strategies to theoretical concepts. As the strategies became more and more specific, the strategies within one topic started to overlap with strategies within another topic. Fig. 4.11 shows part of a version of the inventory where, within the number sense domain, the strategies for determining structured quantities were identical to the strategies for determining unstructured quantities. In effect, these strategies all relate to the overarching concept of cardinality. Hence, in the third versions of the inventory, the topics were extrapolated further away from the specific tasks and more towards the theoretical conceptualizations (section 4.1). This resulted in the following mathematical concepts and procedures for the domain of number sense: cardinality (i.e., distinguishing structured and unstructured quantities), creating structure, comparing quantities, dividing quantities, and early arithmetic skills. For the domain of spatial sense, the components were patterning, constructing, counting elements of a construction, perspective taking, and recognizing figures and shapes. This revised organization of the strategies made it easier to trace the role of spatial structuring strategies in the number sense and spatial tasks. Excerpt from the second version of the strategy inventory: the strategies that relate to determining a structured quantity are identical to the strategies that relate to determining an unstructured quantity

Fourth versions: Tracing the role of spatial structuring in both domains. In the fourth version of the inventory, the spatial structuring strategies within each component were marked with “s” (Fig. 4.12). Two important conclusions could be drawn from this. First, the strategies that were marked, were typically categorized as the highest strategies within a particular topic. This implies that children who made use of the most complex strategies for a component, also knew how to make effective use of spatial structures. Moreover or alternatively, this could mean that children who knew how to make effective use of spatial structures, were successful at applying relatively complex strategies to solve a particular task. Hence, this version of the strategy inventory contributed to an understanding of the role of spatial structures in stimulating children’s performance on the interview tasks

The second conclusion that could be drawn from extrapolating the spatial structuring strategies from the inventory, is that the spatial tasks concerning figures and shapes, perspective taking and orientation, in contrast to the tasks about constructing with blocks and patterning, did not include strategies that were marked with an “s” and hence did not seem to involve spatial structuring strategies. Moreover, some of the number sense tasks included less spatial structuring strategies than others. Therefore, we reflected again on the conceptual schema and additional literature to improve the operationalization of spatial sense, number sense and spatial structuring.

Type of Task			Strategy
Determining structured quantities Cardinality	A		i. guessing, no idea ii.unclear counting strategy
	B		Asynchronous 1-by-1 counting by: i. moving the object aside ii. pointing to each object iii. perceptive (by looking) How: i. out loud ii.mentally Order: i. randomly ii.systematically e.g., Rows or groups before single objects Insight i. not resultative ii. resultative
	C		Synchronous 1-by-1 counting by: i. moving the object aside ii. pointing to each object iii. perceptive (by looking) How: i. out loud ii.mentally Order: i. randomly ii.systematically *e.g., Rows or groups before single objects* Insight i. not resultative ii. resultative
	D	S	Applying spatial structures: i. subitizing *e.g., With ≤ 3 or combinations such as 2 and 3 is 5* ii. instantly recognizes the configuration *e.g., "like on dice", "1 in the middle", "I've counted it with my fingers once"*
	E	S	Combined strategies (mostly with >6 quantities): i. abstracting the spatial structure and adding on the rest *e.g., "4 and 1 is 5" or "3 and 3 and 1 is 7"* ii. (repeated) addition of configurations or already counted rows; abbreviated counting of larger quantities *e.g., "3 and 3 and 3 is 9"*

Fig. 4.12 Excerpt from the fourth version of the strategy inventory: the strategies that pertain to spatial structuring for determining quantities within the domain of number sense are marked with "S"

Van den Heuvel-Panhuizen and Buijs (2005), for example, define orientation and navigation, construction, and insight into figures and shapes as the three important components of early geometry. The authors also differentiate measurement and geometry, stating that measurement is closely related to numerical aspects of space, while geometry is aimed at the development of skills in spatial visualization and spatial reasoning. This perspective relates to our study because these components of early geometry are analogous to the spatial visualization, spatial orientation and shape components of spatial sense that were defined in section 2.2. As a result, we organized and clustered the interview tasks in the spatial domain to cohere more with the components of spatial visualization, spatial orientation and shape so that more strategies would relate to spatial structuring. Regarding the number sense domain, the next version of the strategy inventory included components that were specifically associated with spatial structuring within the number sense domain: determining quantities (cardinality), comparing quantities, applying spatial structure, and basic arithmetic skills.

Final versions: Associating spatial structuring and number sense strategies. Considering the increasing focus on spatial structuring rather than spatial visualization, orientation and shape in general, in subsequent versions of the inventory the spatial structuring strategies were distilled from the number sense and spatial domains. This resulted in three clusters of strategies in the inventory, one for the domain of spatial sense, one for number sense, and the third for spatial structuring. The domain of spatial sense included strategies that pertain to spatial visualization, spatial orientation, and shape. The domain of number sense included strategies related to cardinality (e.g., determining unstructured quantities), comparing quantities, and basic numerical skills (e.g., adding). The domain of spatial structuring was defined according to the following general categories of strategies:

- Rest category.
- Uses no type of arrangement to simplify a procedure.
- Uses a type of arrangement to simplify unitary counting.
- Applies a type of organization for unitary counting.
- Applies a type of organization for unitary counting, but sometimes makes use of an already present spatial structure to determine a quantity.
- Makes goal-directed use of an already present spatial structure, and sometimes applies structure spontaneously to abbreviate a numerical procedure (e.g., determining, comparing and operating with small quantities).
- Spatially structures spontaneously and in a goal-directed way to abbreviate a counting procedure.
- Spatially structures spontaneously and in a goal-directed way to abbreviate an arithmetic procedure for relatively large (>6) quantities.

- Uses arithmetic independently and in a goal-directed way without having to use the organization or spatial structure of objects that are physically present.
- Uses abstract structures in a goal-directed way to represent a quantity.

These changes to the design of the inventory marked the beginning of a shift away from an inventory that focused on spatial sense and number sense strategies in general with implicit connections in terms of spatial structuring strategies. Rather, by comparing the cluster of spatial structuring strategies to the numerical strategies, we could gain more insight into the relationship between spatial structuring strategies and numerical performance. This design helped to narrow the scope of the research to: investigating the role of spatial structuring in the development of insight into numerical relations for number sense, as a means to abbreviate numerical procedures such as determining, comparing and operating with small quantities.

Along this process of (re)designing the strategy inventory, a point was reached where so much detail was included in the strategies, that it became extremely difficult to use the inventory as a practical and reliable instrument for scoring children's behavior on the tasks. Therefore, we tried to generalize the strategies by integrating the spatial strategies with the spatial structuring strategies. This resulted in two new clusters of strategies, one describing the domain of spatial structure and one describing the domain of number sense. After differentiating spatial structuring strategies and number sense strategies, an attempt was also made to combine the two lists of strategies into one with parallel codes as pictured in Fig. 4.13.

By limiting the number of strategies ("codes" for purposes of analyzing the data) to eight (rather than the more than fifty codes and sub-codes which occurred at one point), we could simplify the coding procedures to make it more accessible to other users and (therefore) contribute to the instrument's reliability. What this new version of the inventory suggests, however, is that the development of strategies concerning spatial structuring occurs in parallel with number sense strategies.

Hence, although this design helped to make the instrument more user-friendly, the initial interrater reliability analyses showed that it was not reliable enough. Apparently, too much detail was eliminated from the original inventory, which confused raters' understanding of what characterizes the development of spatial structuring ability and how it may be related to early number sense. The revisions for the final version of the strategy inventory are discussed in the next paragraph.

Code	Spatial Structuring	Number Sense
1	Too unclear to interpret	
2	Has no idea of the problem or the way to approach the problem	
3	Does not seem aware of organization as a way to clarify an action	Counts asynchronously and not resultative and compares quantities usually on the basis of perceptual factors
	• Usually does not recognize spatial structures. • Leaves the objects unstructured or moves them without an organizational purpose (e.g., places objects into a shape “because that’s fun”). • Has no attention for regularity in a pattern and can not verbalize the regularity.	• Does not count systematically and therefore has difficulty keeping count (e.g., counts ten butterflies one-by-one and concludes that there are eleven). • Compares quantities only on a perceptual level, without actively reorganizing the objects (e.g., compares the general surface areas and relative positioning of two groups). • Exaggerates a quantity in simple addition tasks (e.g., “there are some missing, so there must be ten missing!”).
4	Applies a type of organization (usually by trial and error) to act in a clear, but not yet an abbreviated, way and to come to more accurate conclusions	Counts objects verbally from one to ten usually synchronously and resultatively by numbering them. Starts to compare quantities on a numerical basis
	• Recognizes context-dependent spatial structures for small quantities (e.g., finger patterns representing a particular age; the four on dice). • Spreads out and organizes objects to clarify a procedure. • Sometimes recognizes characteristics of a pattern, but still has trouble verbalizing them (e.g., can name the correct colors of a pattern but disregards the order of the colors).	• Counts and does arithmetic context-dependently and more systematically to come to more accurate results. • Organizes and starts to estimate so that quantities can be compared not only on the basis of perceptual factors, but also numerical factors. Sometimes compares groups by physically relating the objects to each other (e.g., “that group has one more than that group, so that group is larger”). • Adds by counting all the objects and adding the rest on. Does this with the help of the objects and own fingers

Fig. 4.13 Excerpt of a final version of the strategy inventory in which the spatial structuring strategies are listed together with number sense strategies for each code

4.3.2 The final strategy inventory

In this paragraph, we discuss the version of the strategy inventory that was used to evaluate children's responses to the tasks in the pre- and post-interviews of the instruction experiment. The full strategy inventory can be found in Appendix 4. Although this version included less codes than other versions, it kept to the thoroughness of the earlier versions of the strategy inventory by accompanying each code with as many illustrative examples of the strategy as possible. This improved the reliability of the instrument; a final interrater reliability analysis resulted in a very high Cohen's Kappa value of 0.87.

The development of the strategy inventory is never completed. Even after conducting the numerous exploratory studies, every newly observed strategy was incorporated into the list during the instruction experiment. This is necessary for the list to become as conclusive as possible, and to be able to draw inferences that are accurate and illustrative of how children approached the tasks. As such, the strategy inventory is not only an instrument for gauging children's insight into numerical relations in terms of their spatial structuring ability. It also forms the basis of a theoretical model about the development of spatial structuring ability and its relationship to insight into numerical relations for arithmetic skills. Therefore, any adjustments to the content and organization of the inventory contribute to a more thorough and accurate theoretical description of this development.

In distinguishing spatial structuring strategies from number sense strategies, it became possible to trace general mathematical conceptions and processes in both domains. We refer to these as *components* of a domain. The components of the spatial structuring domain are the ability to recognize, use, and extend spatial structures. These three components agree with research on developmental trajectories in spatial structuring ability (e.g., Mulligan et al., 2005). Children must first become familiar with spatial structures before they can use or apply them to mathematical tasks. The flashcards in the interview helped to examine the extent to which the children spontaneously recognized particular spatial structures. The expectation is that, as soon as children recognize spatial structures, they may come to use the structures that are readily available. Several tasks in the interviews involved readily structured configurations of objects that could stimulate the child to make use of the structure for determining or comparing quantities. Apart from recognizing and using spatial structures, children should also learn to recognize structure in a pattern as a means to extend the pattern (Waters, 2004). This ability is covered by the patterning task in the interviews. The three components of the number sense domain are the ability to determine, compare and operate with quantities to cover the essence of early numerical procedures (e.g., Griffin & Case, 1997).

The strategies for each of the three components within both domains are listed in a cumulative order. This cumulativity requires that if a child's approach to a task is evaluated as a particular strategy, then the child should have also mastered the previous strategies in the list. In Chapter 5 we elaborate on the Overlapping Waves Theory (Siegler, 2005) to illustrate the learning progression that is implied in this cumulative compilation of strategies. Each horizontal level of the inventory is labeled by a number, which is the code that identifies a strategy. Hence, for a particular component within either the spatial structuring or the number sense domain, this code denotes the corresponding strategy that the child used to approach the relevant task.

Importantly, although the strategies for each component within both domains are in columns that are listed parallel to each other, they do not depend on one another. Hence, if a child scores a particular code (i.e., a strategy) for one of the components in a domain, it does not imply that the child will have used the same level of strategies on the other two components or on the components of the other domain. Although it is likely that the children will score in the same range of codes across the components, the key is to identify any discrepancies. For instance, a child may score the highest code for recognizing a particular structure, but at the same time score a lower code for how the structure is used to determine a quantity. This gives insight into the extent to which the child recognizes spatial structures compared to how the child's spatial structuring strategies contribute to abbreviating numerical strategies.

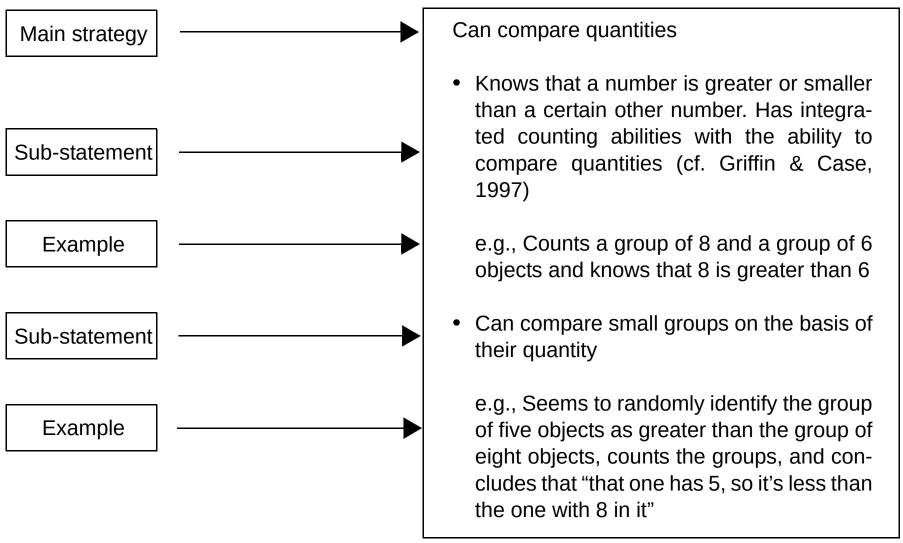


Fig. 4.14 An example of a cell in the strategy inventory

The main strategy is stated concisely in the top line of each cell (Fig. 4.14). The bullets in the cell elaborate on the strategy. These sub-statements are accompanied by practical examples of what a child may have said or done to give the impression that the child's approach may be defined by this particular code. As such, in evaluating children's behavior, the interpretation of the main strategy must be supported by at least one of the sub-statements and by at least one of the examples.

Although the main research incentive of these interviews is to understand how the children solved the tasks and to gauge children's spatial structuring ability and number sense, it is equally important to take note of how the strategy level coincides with the accuracy of the child's responses. We operationalized this type of mastery in terms of the accuracy of children's responses. A child may, for example, apply the most complex strategies to solve a task, but fail to answer the questions accurately. Conversely, a child may answer all the questions correctly while using relatively unsophisticated strategies. These behavioral patterns have consequences for how the children's level of strategy use should be interpreted. As such, we differentiated three levels of mastery of an interview task (Fig. 4.15).

Code	Mastery	Description
1	Not mastered	• The child understands what is being asked, but does not know how to approach the problem. e.g., To add flowers to a group, the child spreads out the flowers (i.e., an irrelevant strategy) and gives an incorrect answer. • The child uses an appropriate strategy, but does not apply it pro-perly to come to an accurate result. e.g., To determine the number of flowers in a group, the child starts counting the flowers, but asynchronously with an incorrect result.
2	Unclear	• The child does not understand the question. e.g., It is unclear whether the child has mastered the task because the child's answer or approach to the problem is unclear. • The instructor's guidance played too large a role to judge whether the child understood the task and whether or not the child came to a correct/incorrect answer independently. e.g., The child gets too much guidance with too many clues.
3	Mastered	• The child (mostly, but possibly with some guidance) performs the task correctly. e.g., The child's answer may first be incorrect, but the child may correct the answer after the instructor asks a question to check the child's understanding (e.g., a careless error). • The child understands the questions and uses an appropriate approach. e.g., The child used a goal-directed strategy and did not guess the answer

Fig. 4.15 Three levels of mastery for assessing the accuracy of children's responses to the interview tasks

Children's responses to the interviews were analyzed only after all the children had participated in the interviews. As explained in section 3.4, the interviews were first imported into the multimedia data-analysis software program ATLAS.ti so that quotations could be made for every question in every task of the interview. These clips greatly simplified the analyses because they could conveniently be retrieved without searching through the entire video.

A score sheet was created to analyze the children's response to every question. Due to the evolution of the tasks, not every question corresponded to every component in both the spatial structuring and the number sense domain. Extending a structure, for example, is only relevant to the patterning task. Likewise, the third part of the interview, the orientation task, could strictly only be assessed on accuracy because it did not involve spatial structuring or number sense strategies. Therefore, the cells that were not relevant for scoring were blocked out in the score sheet. Furthermore, the user-guidelines described the intention of each question and what strategies are considered most relevant to the task.

Using this score sheet to analyze every quotation and the field notes, the child's approach to every question was coded with respect to the relevant components of the two domains. This resulted in a score sheet that was filled with numbers (i.e., the codes/strategies) that give insight into a child's spatial structuring ability and number sense regarding the tasks of this particular interview. In Chapter 5 we describe how these score sheets were condensed to simplify the interpretation of the results.

4.4 The final conceptual schema

The process of designing the interview tasks and creating and refining the strategy inventory, was initially based on the conceptual schema in Fig. 4.2. Yet, this version of the conceptual schema was also continuously revised with respect to additional literature studies and observations of children performing the interview tasks. In fact, the strategy inventory served as an interpretative framework for redefining and refining ideas about the development of spatial structuring ability and about relationships between early spatial sense, number sense and spatial structuring. In this section we present the final conceptual schema that is the result of the development of the theoretical model that underlies the strategy inventory and the interviews.

To support our argument, we outline the conceptual schema in three parts, the first of which is presented in Fig. 4.16. This part (part A) of the conceptual schema acknowledges the domain of space and the domain of number as the foundations for young children's emerging spatial sense and number sense. In section 2.2, spatial sense was defined in terms of three spatial components, namely spatial visualization, spatial orientation and shape. These three components are interrelated, and each are considered fundamental to children's ability to "grasp the world" for making their way in space. Similarly, in section 2.1, children's ability to count and compare quantities was described as two main components of their early number sense. These components underlie children's ability to develop from an intuitive notion of quantity and number to a more formal understanding.

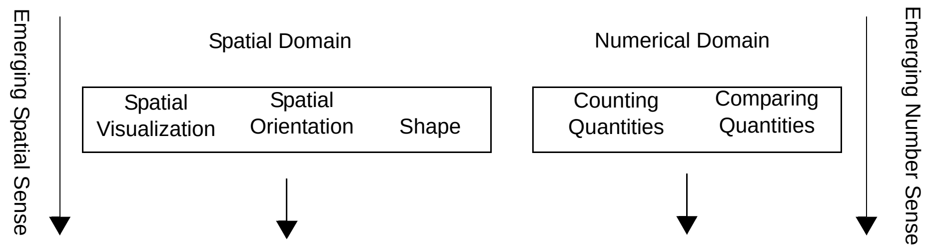


Fig. 4.16 Part A of the final conceptual schema: within the domain of space, spatial visualization, spatial orientation and shape are defined as major components in the development of spatial sense. Within the domain of number, counting quantities and comparing quantities are fundamental to the development of number sense.

Each of the three spatial components contribute to children's ability to (de)compose quantities through spatial structuring (Fig. 4.17). By spatially visualizing a set of unstructured objects, children construct a physical form or mental image that may be more structured or familiar to them (Markovits & Hershkowitz, 1997). For example, in determining how many buttons there are in a basket, at first sight the buttons may have no organization that could help to determine the quantity. Yet, through physically or mentally rearranging the buttons into a spatial structure, the children may refer to the image to discuss their strategies. Similarly, through recognizing a particular shape or figure in a set of objects, children can recognize a spatial structure. This may be the case when, for example, the child places one button in every corner of a sheet of paper, recognizes the rectangular shape and remembers that that is also what four dots on dice look like (cf. Clements, 1999a). Finally, children may apply their spatial orientation skills to spatially structure their environment to become aware of the different structures in their surroundings which may be applied to organize objects for abbreviating numerical procedures such as determining, comparing and operating with small quantities.

Spatial sense is sometimes associated with measurement (e.g., Van den Heuvel-Panhuizen & Buijs, 2005), but measurement is different from spatial structuring because it involves organizing a whole rather than abbreviating a numerical procedure through focusing both on the whole and on the unit (see also Freeman as cited in Clements, 1999a). This is why, for purposes of this study, measurement is differentiated from spatial structuring ability. Measurement can, however, become more relevant as a form of spatial structuring at a later stage in mathematical development. When children learn about repeated addition and multiplication, for example, they must first be able to recognize and identify structure, in order to be able to reason about its repetition (Anghileri, 1989; Steffe et al., 1988). This can be compared to defining a unit of measurement and repeating (or multiplying) it to find a distance or a weight.

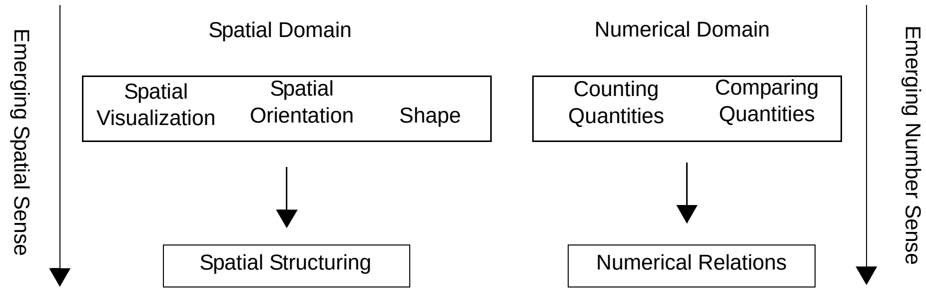


Fig. 4.17 Part B of the final conceptual schema: the three components of the spatial domain support the development of spatial structuring ability while the integration of the ability to count and compare quantities contributes to emerging number sense with insight into numerical relations.

In the numerical domain of the conceptual schema, the Central Conceptual Theory (Griffin & Case, 1997; Griffin, 2004a, 2004b) underlies the assumption that development of an overarching schema for counting and comparing quantities contributes to children's number sense. This new conceptual structure for number "connects number with quantity and enables children to use the counting numbers without needing the presence of physical objects to make a variety of quantity judgements" (Griffin, 2004a, p. 40). In this way, children are believed to acquire the conceptual foundation for number sense. As described in section 2.1, an important part of emerging number sense is children's insight into numerical relations. In light of the conjectured role of spatial structuring in the development of insight into numerical relations and the importance of such insight for mathematical understanding, this part of number sense is the focus of the present research.

Part C in Fig. 4.18 combines Part A and Part B for an outline of the complete conceptual schema that underlies the study. Children can gain insight into numerical relations by learning to associate numbers with specific quantities so that they may, for example, compare their magnitudes and consequently understand the meaning of numbers and operation signs (Griffin, 2004a). This ability to (de)compose quantities is shared with the spatial domain in terms of spatial structuring ability (Battista et al., 1998; Mulligan et al., 2006b).

Hence, as explained in Chapter 2, the central issue in this research is how spatial structuring ability may support the development of insight into numerical relations (highlighted by the darkest arrow in the figure). In Chapter 5 we explain how the development of insight into numerical relations is operationalized, while the development of the instructional sequence illustrates the operationalization of spatial structuring (Chapters 6 and 7).

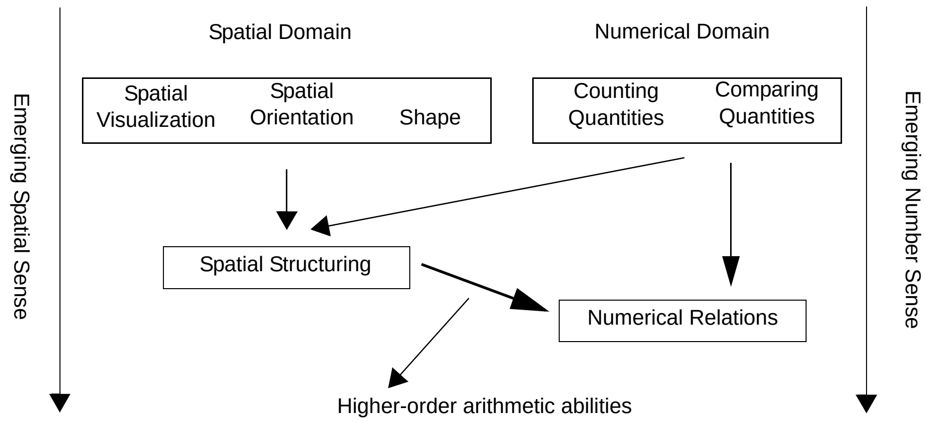


Fig. 4.18 Part C of the final conceptual schema: children's spatial structuring ability is expected to support their insight into numerical relations. In turn, spatial structuring ability and insight into numerical relations lay the foundations for the development of higher-order arithmetic abilities.

Insight into numerical relations is important for abbreviating and enhancing children's ability to determine and compare quantities, which is necessary for higher-order numerical procedures such as addition, subtraction, and multiplication (Anghileri, 1989; Buijs, 2008; Van Eerde, 1996). The structures should stimulate the child to "read off" a quantity rather than revert to unitary counting. The phrase "read off" in this thesis is used differently from Battista and Clements (1996; see section 2.3). Instead, we use the phrase to refer to what Clements (1999a) identifies as conceptual subitizing to denote how children may actively create a structure through mental or physical actions that concern the objects (Battista & Clements, 1996).

We add a connection between "counting quantities" and "spatial structuring" to acknowledge that children's spatial structuring ability itself may be influenced by their counting ability (Clements, 1999a). Children need a preliminary level of unitary counting ability before they should even concern themselves with trying to abbreviate counting procedures. This also relates to Battista and Clements' (1996) conclusion that although spatial structuring provides the input for enumeration, attempts at enumeration can engender spatial structuring just as well (see section 2.3). However, the difference between the present research and that of Battista and Clements (1996) is that the type of numerical insight that we are concerned with, precedes that of the children in grades 3, 4, and 5 who participated in Battista and Clements' (1996) study. The perceptual and conceptual subitizing that is implied by the spatial structures in our research involve small quantities (up to 10) that assist in the development of mental images of spatial structures (Owens & Clements, 1998). Once such images are established, they may, in turn, help to determine larger and more complex quantities such as the composition of structures. Hence, our research concerns a preliminary step to a two-way relationship between children's spatial structuring ability and their number sense. It may be conjectured that as children gain more experience with operating with small quantities, their insight into numerical relations improves, which, in turn, can help them (de)compose quantities to understand spatial structure. That appears to be the level at which Battista and Clements studied children's spatial structuring ability.

In the next chapter, we discuss how children's scores for the pre-interview were condensed into one of four phases describing spatial structuring ability. These phases are interpreted in light of the Overlapping Waves Theory (Siegler, 2002, 2005). Finally, we present the practical and theoretical outcomes of the pre-interview and explain what they contribute to the design of the instruction experiment.

5 The Strategy Inventory as a Foundation for the Hypothetical Learning Trajectory

In the previous chapter, we discussed the development of the interview tasks, the strategy inventory and the conceptual schema relating young children's spatial structuring ability to the development of their number sense. This chapter focuses on the strategy inventory. We explain how the coding procedures were simplified so that the inventory could function as an instrument for evaluating children's performance on the interview tasks. The process of condensing the strategy scores and relating them to children's performance on the interviews is summarized as follows (Fig. 5.1).

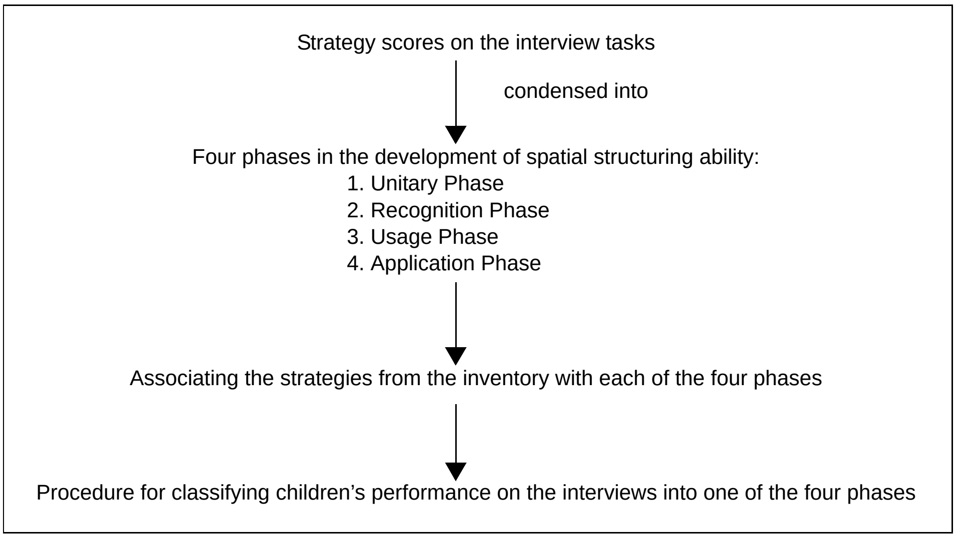


Fig. 5.1 Outline of the process of condensing children's interview scores (i.e., their strategies) to come to a general description of the child's performance on this particular interview in terms of one of four phases regarding the child's development of spatial structuring ability

In section 5.1, we describe how the scores were condensed. This leads to section 5.2 in which we explain how condensing the scores resulted in the identification of four phases that (a) outline a trajectory for the development of spatial structuring ability in this study, and that (b) offer an efficient and reliable way for comparing children's spatial structuring ability (cf. Mulligan et al., 2005). These phases can be interpreted in light of the Overlapping Waves Theory (Siegler, 2002, 2005) as described in section 5.3. These three sections operationalize insight into numerical relations in the research, and bridge the first research question (the development of children's spatial structuring ability) with the second research question (creating an instructional sequence in a learning ecology that can foster spatial structuring). The chapter concludes with several quantitative outcomes of the pre-interviews. This sets the stage for the comparison between the pre-interviews and the post-interviews which will be discussed in Chapter 9. In Chapter 6 the hypothetical learning trajectory (HLT) will be defined to underpin the development of a sequence of instruction activities.

5.1 Condensing the strategy scores

Children's approaches to a particular question in the interview were scored both in terms of the strategy as identified in the strategy inventory (i.e., the procedure) and the accuracy of the outcome (i.e., the product). This resulted in detailed scores of the strategies that called for a way to condense the scores, so that more general conclusions could be drawn about the children's spatial structuring ability and number sense. The main challenge was to balance the richness of the descriptions of the strategies as symbolized by the scores, with a more user-friendly and reliable instrument that can differentiate children's mathematical understanding on the basis of how they approached the tasks.

The first step to condensing the scores was inspired by patterns of observations of several children. These children recognized particular spatial structures, but did not tend to make use of those spatial structures to abbreviate numerical procedures such as determining, comparing and operating with small (up to 10) quantities. This led to the specification of three types of children:

(1) children who do not recognize spatial structures and hence do not make use of them in the task.
(2) children who do recognize particular spatial structures, but do not make use of them.
(3) children who recognize particular spatial structures and who spontaneously make use of them in a goal-directed way to abbreviate numerical procedures.

As described in paragraph 4.3.1, at one point in the development of the strategy inventory, the strategies that seemed most related to spatial structuring strategies were highlighted with "S". Figure 5.2 shows an excerpt from the number sense domain in the strategy inventory, depicting two kinds of strategies that were highlighted as spatial structuring strategies ("D" and "E") for determining quantities.

After abstracting the spatial structures from the domain of number sense and spatial sense, we specifically focused on the spatial structuring strategies and the number sense strategies (paragraph 4.3.1) to try associate the strategies with each of the three types of children outlined above. As such, in one of the earlier versions of the strategy inventory, the first two (relatively unsophisticated) strategies were associated with type (1), the next three strategies with type (2), and the last four (relatively complex) strategies with type (3). This created a first impression of children's level of spatial structuring ability with respect to the kinds of strategies that they used for the interview tasks.

D	S	Applying spatial structures: i. subitizing *e.g., with ≤ 3 or combinations such as 2 and 3 is 5* ii. instantly recognizes the configuration *e.g., "like on dice", "one in the middle", "I've counted it with my fingers before"*
E	S	Combined strategies (mostly with >6 quantities): i. abstracting the spatial structure and adding on the rest *e.g., "4 and 1 is 5" or "3 and 3 and 1 is 7"* ii. (repeated) addition of configurations or already counted rows; abbreviated counting of larger quantities *e.g., "3 and 3 and 3 is 9"*

Fig. 5.2 Strategies "D" and "E" are highlighted as spatial structuring strategies ("S") in one of the earlier versions of the strategy inventory

Such an explicit link between the conjectures about types of children's spatial structuring ability and the strategies that were defined in the strategy inventory, marked a first step towards condensing the strategy scores. The next step was to explore ways of aggregating the scores for each strategy in the first (the flashcards) and second (the interactive tasks) parts of the interview. Although it was tempting to devise a quantitative method to calculate a grand total, the key to this strategy inventory is that each score refers to a detailed strategy. Therefore, much information could be lost if the scores are manipulated in a formula. Instead, the scores were manipulated in a qualitative way (i.e., studying counts in terms of what strategy the scores referred to). It was decided that it would be most meaningful to aggregate the scores by taking the statistical mode (i.e., the statistical measure for the score that is most often observed) of the scores for each particular component (see also section 5.3) for Part 1 and Part 2 separately. Hence, for the first part of the interview this mode mainly indicates the children's ability to *recognize* particular spatial structures, and for the second part of the interview the mode reflects children's ability to *use* and *apply* spatial structures (i.e., the three components for spatial structuring) to abbreviate procedures for *determining*, *comparing* and *operating with* small quantities (i.e., the three components for number sense).

As more interviews were assessed, however, it became increasingly difficult to summarize all the scores with a mode that would accurately represent children's ability for that particular component (i.e., recognizing, using and applying spatial structures and determining, comparing and operating with small quantities). First, not all questions were equally relevant for assessing children's ability on a particular component. For example, it was decided that in Part 1 of the interview, the ability to recognize spatial structures should be prioritized by the finger patterning cards, the dice configurations on the cards, and the shapes on the cards because these tasks were more related to recognizing spatial structures than the relatively more complex tasks that involved differently structured dots on cards or the construction of finger patterns. The ability to use structures, on the other hand, was more present in the task with dotted cards because children were expected to have to look for (a combination of) spatial structures rather than merely recognize the structures. As such, a manual was created that indicates what interview tasks should be prioritized when analyzing children's total scores.

	Spatial structuring			Number sense			Mastery	Notes
Component / Task	Recognize	Use	Apply	Determine	Compare	Operate		
Counting 1-15				4			3	
Finger patterns	4			5c		4	3	counts 6, 8, 7
Own fingers		7		4b			3	sees all
Dots on dice	5			7			3	
Shapes	4			5c			3	
Dots (2 x 3)	3	5		5c		4	3	
Dots (2 x 4)	3	5		5c		4	3	
Total	**4**	**5**		**5c**		**4**	**3**	
Notes	Context-dependent recognition, synchronous resultative counting							

Fig. 5.3 Sample score-sheet for Part 1 of the interview in which the scores (i.e., strategies) for each task on each component of both domains (i.e., spatial structuring and number sense) and mastery are condensed into one total score. The notes reflect some of the researcher's thoughts about the child's performance. The scores in this table are taken from the pre-interview of a 5-year old girl.

The second difficulty with attending only to the statistical mode of a series of scores, was that it did not always reflect children's ability. Therefore, for each component, the meaning of the mode was interpreted qualitatively to determine what strategy in the inventory the mode was referring to and to evaluate whether that coincided with the teacher and researcher's impression of the child's ability. At the same time, the accuracy of the response was taken into consideration to check whether the strategy was in fact applied properly. For example, for extending a pattern and for operating with small quantities in the second part of the interview, it was decided that it would be more accurate to take the most sophisticated (i.e., highest) strategy that had been used to solve the related tasks, rather than the mode of the strategies. This is because the highest strategy that the children used, appeared to be a better reflection of their ability on these particular interview tasks. Fig. 5.3 shows a score-sheet that illustrates this process of condensing the scores for Part 1 of the interview.

With the scores on each of the tasks in the first two parts of the interview condensed to one strategy per component (i.e., the total), the next step was to develop a way to interpret this collection of total scores and come to a concise impression of a child's level of spatial structuring ability and number sense. For this, we turned again to the three types of children's spatial structuring ability that are defined above.

5.2 Four phases in spatial structuring ability

After condensing the scores for each task into one total score per component for Part 1, Part 2, and Part 3 (i.e., the spatial orientation task) of the interview, the next step was to find a way to give meaning to the collections of total scores for each component within the interview. Building on our initial conjectures about children's ability to recognize and make use of spatial structures for abbreviating numerical procedures, we continued to analyze the patterns of strategies in the inventory. At the same time, as the sequence of interview tasks began to take shape, and as the strategy inventory became more reliable and thorough, our conjectures about children's spatial structuring ability evolved from three to four type classifications in a developmental sequence.

Importantly, these classifications are not intended to confine a child to a particular level of spatial structuring ability. Rather, the types give a general indication of the kinds of strategies (interpreted in light of spatial structuring) that a child tends to apply most often during this particular interview. We therefore identify these type classifications as "phases" (rather than, for example, categories) to highlight their fading beginning and end points. (see section 5.3). They are listed cumulatively to reflect a progression in spatial structuring ability. Hence, these phases may be seen as four levels of sophistication in children's spatial structuring of the interview tasks. We conjecture that a child's repertoire of spatial structuring strategies may, in this particular interview situation, best be described as one of the following four phases:

(1) Unitary Phase: The child recognizes almost no spatial structures and consequently neither uses nor applies structures to abbreviate numerical procedures.

This phase is most applicable to the youngest children in the research who are typically still focused on improving their unitary counting skills. These children may already have encountered certain spatial structures such as dice or finger patterns, but without being familiar enough with these structures to associate them with counting procedures. Since these children barely recognize the spatial structures that are presented in the interview, it is not surprising that they neither recognize, nor use or apply them in the rest of the interview. It is especially interesting to see whether there are relatively older children who do not recognize the spatial structures and who may not use or apply them to abbreviate numerical procedures in the rest of the interview. This could shed light on the progress of their mathematical development. Similarly, it will be interesting to see whether there are children who do use and apply spatial structures in Part 2 of the interview, although they appeared to not recognize them in Part 1.

(2) Recognition Phase: The child recognizes several fundamental spatial structures, but rarely uses or applies spatial structures to abbreviate numerical procedures. Instead, the child may rationalize the use of spatial structures in hindsight.

Children whose general approach to the interview tasks fits this phase, will tend to use relatively sophisticated strategies in the way they recognize several types of spatial structures and use them to read off the flashcards (Part 1 of the interview). However, they will experience more difficulty in Part 2 of the interview where they are asked to not only recognize but also to use and apply the structures to solve the tasks. One subtle feature of this phase is that, although the children may neither make use of available structures nor apply their own structures, they may rationalize spatial structuring retrospectively (i.e., only after they counted). This points to their ability to recognize spatial structures when prompted, while they otherwise prefer unitary counting procedures over spatial structuring strategies.

(3) Usage Phase: The child recognizes and uses most available spatial structures, but rarely shows initiative in constructing and applying its own spatial structures as a means to abbreviate numerical procedures.

Like in the Recognition phase, children in the Usage phase tend to recognize the spatial structures in the first part of the interview. The difference with the previous phase, however, is that children in this phase do not hesitate to make use of the spatial structures that are readily available to them in Part 2 of the interview; a child will recognize the quantity six if the objects are arranged like the two rows of three that the child recognizes from, for example, dice. The challenge for the child, however, is to spontaneously apply structure to an unstructured group of objects. The child will tend to leave the objects the way they are or maybe organize the objects into a line because “that makes it easy to count them”. This suggests that although the child can recognize and make use of spatial structures, the child may not yet understand the convenience of spatial structuring as an alternative to unitary counting for abbreviating a numerical procedure.

(4) Application Phase: The child uses spatial structures in a goal-directed way and spontaneously constructs and applies spatial structures as a means to abbreviate numerical procedures.

Children in the Application phase are familiar with various types of spatial structures and tend to make use of the structures that are readily available to them (“there are six because they are in 2 rows of 3”). This means that they have come to understand the convenience of structure as more than the mere organization of the objects for determining, comparing and operating with small quantities. In addition, these children use this insight spontaneously to apply structure to objects that are initially unorganized. These children may spontaneously arrange the objects in the same configuration as dots on dice, or as eggs in an egg carton, for example, because the children understand that such configurations support numerical procedures. Hence, their spatial structuring is goal-directed. The children may even arrange objects in one or more organized lines, but their underlying intention is not to count the objects unitarily. Instead, they use a strategy that abbreviates the procedure for determining a quantity. This phase is expected to stimulate children’s insight into numerical relations and prepare them for high-order mathematical learning.

To give meaning to children’s repertoire of strategies for the interview, we first matched each strategy in the strategy inventory to a phase. For example, for the Unitary phase regarding the recognizing structures component in Part 1 of the interviews, it was expected that the child will have made use of the first three strategies for this component in the inventory (although the second strategy can apply to all four phases, see Fig. 5.4).

A table was constructed that associates each total score for each component of Part 1 and Part 2 with one of the four phases (see Appendix 3). The extent to which the strategy was applied properly (i.e., accuracy or mastery of the task, see paragraph 4.3.2) was also taken into account during the interpretation of the total scores. Further, the spatial orientation task (Part 3) was evaluated qualitatively and in terms of the accuracy of the response.

1	Does not seem to know how to approach the problem
2	The strategy is ambiguous, yet the result is acceptable
3	Does *not spontaneously recognize* spatial structures • (Usually) does not recognize spatial structures at first sight and must therefore count quantities e.g., Counts all the finger patterns; has trouble producing own finger patterns; counts the dots on all the flashcards; has no (grounded) preference for the structured construction and finds the structured house easier to count because (e.g.) "it's prettier" or because "it doesn't have to be turned around"

Fig. 5.4 These first three strategies for recognizing spatial structures are associated with the Unitary phase (phase 1) in Part 1 of the interviews. The second strategy can apply to each of the four phases.

Finally, the resultant six phase categorizations that corresponded to the three components of the spatial structuring domain and the three components of the number sense domain, were discussed and examined qualitatively to decide how these phases are best summarized into a single phase to describe the child's overall approach to the tasks in this interview. As such, the strategy inventory provided a qualitative means for gauging children's spatial structuring ability and number sense, with a particular focus on insight into numerical relations. It also provided a way to study differences in children's development of spatial structuring ability and insight into numerical relations.

Supported by Mulligan et al.'s conclusions that "children's perception and representation of mathematical structure generalizes across a range of mathematical content domains and contexts", and that "early school mathematics achievement was strongly linked with the child's development and perception of mathematical structure" (2004, p. 399), the commonalities between Mulligan et al.'s stages of structural development and our four phases contribute to the external validity and reliability of the interviews and the resultant conjectured phases (see also section 10.1).

Mulligan et al. (2005) also found a wide diversity in developmental stages for children of one particular age, including inconsistent developmental patterns for low-achieving children and improvements in structural representations for other children. Hence, before describing a means to support children in the development of their spatial structuring ability, we first explain how the four phases may be related to each other in terms of a developmental trajectory.

5.3 Relating four phases in a developmental trajectory

To illustrate how we interpret the development of children's spatial structuring ability in terms of the four phases defined above, we turn to the Overlapping Waves Theory (Siegler, 2002, 2005). The main assumption of this theory is that children typically use a variety of approaches to one problem at a given time. Therefore, development is seen as a qualitative shift in the types of strategies that children use to solve a particular problem, and as a quantitative shift in the frequency with which children use a particular strategy, the adaptiveness of the choice, and the efficiency of the use. Thus, learning and development can be characterized as processes of variability, choice and change. With age and experience, some strategies become less frequently used, others become more frequent, some become more and less frequent, while new strategies are discovered and some older strategies are eliminated.

The Overlapping Waves Theory inspired the image below (Fig. 5.5). The figure conveys our conjectures about how children develop throughout the four phases of spatial structuring ability.

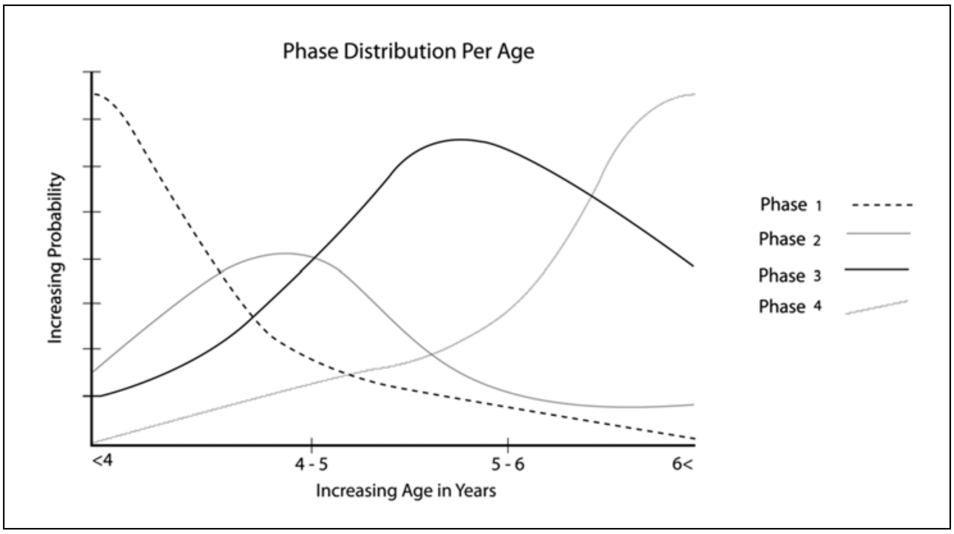


Fig. 5.5 An image of the conjectures about which phases appear most applicable to each age level. It is inspired by the Overlapping Waves Theory (Siegler, 2002, 2005).

Each wave in the figure corresponds to one of the four phases that are explained in section 5.2. As such, the dependent variable is not the percentage use of a particular strategy as in the original Overlapping Waves Theory, but rather the probability of using a strategy repertoire that belong to a particular developmental phase. The relative estimations for these probabilities evolved during several thought experiments throughout the exploratory studies. The ages are averages to acknowledge that some children may approach the problems in ways that either exceed or lag behind the level of children of the same age.

The essence of the Overlapping Waves Theory is reflected in the design of the strategy inventory and in how the phases are assumed to be related. First, in gauging children's strategy use on a particular task, we acknowledge that children use a variety of strategies to solve a particular task. Although the interview is a reflection of a particular moment in time, the key is that children use a variety of strategies that vary in levels of sophistication, but generally they tend to use more of one particular level of strategies during a particular point in their development. This agrees with Owen's (1999) findings that, although students did not necessarily apply the same strategies to all the questions in their spatial test, the *tendency* was that they did. We operationalized this assumption by using the statistical mode of children's scores to come to a general repertoire of strategies (see section 5.1).

Considering this variability in children's strategy use, our four phases are cumulative with faded beginning and end points. This means that children who appear to have used a particular repertoire of strategies that fit the Recognition phase (phase 2), will sometimes still make use of strategies that fit the Unitary phase (phase 1), and will sometimes make use of strategies that fit the Usage phase (phase 3). The main tendency, however, is for them to make use of Recognition phase strategies. In some cases, children's strategies fit a particular phase although they showed relatively strong tendencies towards the next phase, or they showed curious approaches to problems that strictly belonged to the Unitary phase. For these rare cases, a "+" or a "–" was added to indicate the child's tendency towards the next or the previous phase, respectively. These signs were used only as additional information for the quantitative and qualitative interpretations of children's performance on the interviews.

In designing the interview tasks, the level of the tasks had to be accessible to four-, five- and six-year olds, and the contexts (e.g., language use, story line) of the tasks had to appeal to the children's levels of understanding. Indeed, in Owen's (1999) research, it appeared that slight changes to tasks could make a task more difficult, with the result that the students did not use the same strategy as on other tasks. Hence, the greater the complexity of the task, the more the children are expected to turn to foregoing and more familiar approaches (i.e., less sophisticated strategies that belong to a lower phase) in an effort to answer the questions accurately. For example, a child may recognize a particular spatial structure in a small (e.g., 7) set of objects and know how to use that structure to determine how many there are. This involves a relatively complex strategy that is represented by a high score in the analyses. Yet, when the task involves a large (e.g., 12) set of objects, the child may revert to unitary counting procedures which, in terms of our research, is a relatively unsophisticated strategy. When these tendencies are generalized, the children's repertoire of strategy use may be such that it first fits best in the Application phase, but that the increased complexity of the task triggers the use of strategies that correspond better to the Usage phase, the Recognition phase, or the Unitary phase. In this way, the model gives insight into one possible reason for why children make use of a variety of strategies at a given point in their development.

Taken together, we assume that the phases in the developmental trajectory are not single, autonomous stages, but rather, that they correspond to how children apply strategies to solve particular tasks as illustrated by the Overlapping Waves Theory. This clarifies how the phases may be related to each other. In this we agree with Bruce and Threlfall (2004) who conclude the following from their research on developing cardinality and ordinality:

> The suggested detailed developmental sequences in both the cardinal and ordinal aspects of number (...) are not intended as a teaching model, in that they do not propose stages through which children should be progressively taught. Nevertheless, they may support the teaching of early number in each aspect, through raising awareness of what may have preceded and what may follow the current approach used by the child, so that appropriate input and intervention can be shaped to enable the child to move forward. (p. 24)

The identification of these four phases contributes to answering the first research question: while the strategy inventory provides the detailed description of the strategies that characterize young children's spatial structuring ability, the phases outline a developmental trajectory for children's spatial structuring ability and the relation between this development and children's emerging number sense, in terms of insight into numerical relations.

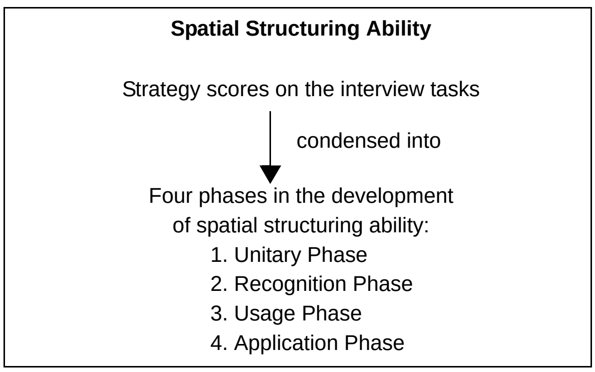


Fig. 5.6 Operationalization of the development of insight into numerical relations in terms of children's performance on the interview tasks, described as one of four phases in a developmental trajectory for spatial structuring ability

The outcomes of children's performance on the interview tasks (i.e., their strategy scores) and their corresponding strategy repertoire (i.e., phase classification) reflects children's spatial structuring ability. Regarding the proposed influence of children's spatial structuring ability on insight into numerical relations (Chapter 2), this operationalizes the development of insight into numerical relations in the research (Fig. 5.6).

In the next section we present some quantitative outcomes of the pre-interview that relate young children's spatial structuring ability to their mathematical development.

5.4 Quantitative outcomes of the pre-interviews

The figure below (Fig. 5.7) presents the number of children per classroom (Intervention or Non-intervention; IG or NG) and per grade (Kindergarten 1 or Kindergarten 2; K-1 or K-2) showing a repertoire of strategies in the pre-interview that coincides with one of the four phases. We note again that the non-intervention group was not intended to be a control group, but rather an additional source of data for developing and analyzing the interviews and the strategy inventory.

	Phase 1 (Unitary)	Phase 2 (Recognition)	Phase 3 (Usage)	Phase 4 (Application)	Total number
IG K-1	9	2	2	0	13
IG K-2	1	3	1	3	8
Total IG	10	5	3	3	21
NG K-1	6	1	1	0	8
NG K-2	1	1	1	6	9
Total NG	7	2	2	6	17

Fig. 5.7 The number of children per Intervention group (IG) and Non-intervention group (NG) and per Kindergarten 1 (K-1) and Kindergarten 2 (K-2) showing a repertoire of strategies in the pre-interviews that coincides with Phase 1 (Unitary), Phase 2 (Recognition), Phase 3 (Usage) or Phase 4 (Application)

This distribution of scores agrees with what was expected: the youngest children of both the IG and NG (i.e., Kindergarten 1) scored relatively more often in the lower phases than the eldest children (i.e., Kindergarten 2) and vice versa. In section 9.1 we elaborate on the significance of this distribution. This will explain how it contributes to the post-interview to provide insight into possible shifts in phase categorization for a particular child, and into possible differences between the development of the intervention group compared to the non-intervention group, as well as between the development of the youngest compared to the eldest children.

The children's phases were compared to their LVS scores (i.e., standardized test, see section 3.5) to relate their spatial structuring ability to their mathematical performance. Six K-1 children had not taken the LVS test and were therefore not included in the comparison. The distribution reflects a trend in the type of strategy use and the child's LVS categorization; LVS score *A* occurred most often in combination with the Application phase, LVS score *C* occurred most often in combination with the Unitary phase, LVS score *B* was evenly distributed across the four phases, and LVS score *D* only occurred once, and as expected, in combination with the Unitary phase. These LVS scores were not known to the researchers at the time of evaluating children's performance on the interviews and classifying children's approaches into one of the four phases. This trend supports the internal reliability and the internal validity of the interviews because the outcomes of the interviews converge with the children's standardized test scores.

Several children's strategy repertoires coincided less convincingly with their LVS scores than others. For example, two Unitary phase children and two Recognition phase children scored LVS *A*. Since these children were all Kindergarten 1 children, it raises the suspicion that the tasks were too difficult for them compared to the LVS tasks. Similarly, two Kindergarten 2 children scored LVS *B* while they showed a repertoire of Recognition phase strategies. The significance of this is that it could be interesting to monitor the mathematical development of these children and their response to the intervention to see whether the classification into the Recognition phase may be an early indication of relatively delayed mathematical development. This could show how the strategy inventory and the interviews can serve as a tool to trace potential delays at a very early stage in children's mathematical development.

In this chapter we described four phases in the development of spatial structuring ability that are founded on a detailed classification of spatial structuring strategies in the strategy inventory. These phases were related to each other in light of the Overlapping Waves Theory to explain their cumulativity and their faded beginning and end points. Finally, the children's phase categorizations were compared to their standardized math scores to conclude that the categorizations give an accurate impression of the children's spatial structuring ability.

Although the Overlapping Waves Theory may illustrate the relationships between the phases, it does not necessarily clarify how children may progress from one phase to another. This is where the socio-constructivist framework comes into play (paragraph 2.5.2): the child and the teacher can synchronize their conceptualizations and make them “taken-as-shared”. In other words, if children prefer to use a unitary counting strategy (i.e., pertaining to a relatively low phase) that differs from the teacher’s intended spatial structuring strategy, then the child and teacher must discuss the different strategies so that the child’s preference for a type of strategy will - eventually, taking developmental aspects into account - synchronize with the teacher’s intentions. We focus in our research on the development of a shared vocabulary to support “taken-as-shared” conceptualizations. Further, insight into differences in what strategies are preferred will help to create an instructional setting that can support children in this development. Analogously, insight into how children may be supported in this development, may help to understand how children progress from one developmental phase to another.

The construction of the strategy inventory and the resultant developmental phases feed forward into developing a HLT for answering the second research question. Therefore, the second research question may be interpreted as follows: if we may assume that children experience a sequence of developmental phases such as those outlined above, how could a learning ecology (i.e., instructional tasks and teacher input; Cobb et al., 2003) stimulate them into proceeding from one phase to the next and improve their ability to make use of spatial structure for abbreviating numerical procedures? In the next three chapters we elaborate on the HLT, and on the design of a sequence of instruction activities. This underlies the instruction experiment which was aimed at (a) supporting young children in the development of their spatial structuring abilities and at (b) contributing to a local instruction theory about the development of young children’s spatial structuring abilities for supporting the abbreviation of numerical procedures.

6 Refining the HLT and Developing Classroom Instruction Activities

This chapter begins with a description of how the principles of design research were applied to set up an instruction experiment for answering the second set of research questions. In section 6.2 we explain how learning moments were defined based on the hypothetical learning trajectory (HLT). This includes a description of the instruction activities that were designed to be implemented in the instruction experiment. In section 6.3 we elaborate on the revisions that were made to the hypothetical learning trajectory and to the instruction activities during several exploratory studies. In the last section we present the hypothetical learning trajectory and instructional sequence of six instruction activities for the first round of the instruction experiment.

6.1 The global HLT and learning goals

As described in section 3.1, design research involves an iterative procedure of empirically-based, theory-driven adjustments to the intervention and revisions of the hypotheses (Gravemeijer, 1994). First, a global hypothetical learning trajectory (HLT; Freudenthal, 1973, 1991; Gravemeijer, 2004) is defined that outlines the learning processes that are expected to occur on the basis of the instruction experiment. It is this envisioned learning route that summarizes expectations of the learning processes and the influences of the instructional setting (Simon, 1995). It guides the planning of the instruction activities and connects hypothesized observations to the conjectures. Hence, in terms of Simon's travel metaphor, the HLT is the "journey" that researchers and teachers prepare for the children to actually achieve a certain level of spatial structuring ability. The HLT is bidirectionally related to the local instruction theory which, according to this metaphor, describes the "travel plan" for children to learn to recognize and make use of spatial structures to abbreviate numerical procedures. By scrutinizing the HLT in retrospective analyses, the aim is to contribute empirical evidence for an overarching local instruction theory about the teaching and learning of mathematics in Kindergarten (Gravemeijer, 2004).

As described in Chapter 5, one of the main outcomes of working with the strategy inventory to evaluate children's spatial structuring ability and number sense, is the definition of four phases in the development of spatial structuring ability. These cumulative phases were particularly important for providing a way to generate conclusions about children's spatial structuring ability from the rich collection of scores, and for making the inventory more user-friendly and reliable. Turning to the second part of the research, it became clear that these phases also outline a global HLT that can inspire the development of a sequence of instruction activities. As such, the foundations for the HLT emerged from the cumulative range of strategies in the strategy inventory that was constructed on the basis of observations from the exploratory studies.

Child's predominant numerical strategy	Description of the strategy	Learning goal that an instruction activity may stimulate
Subitizing and counting asynchronously	Subitizing quantities ≤ 4 and counting by pointing to each object without attaching the counting word properly to one particular object	Through practice children learn to relate each counting word to an object (Gelman & Gallistel, 1978).
Synchronous, resultative 1-by-1 counting	Counting by pointing to each object and properly attaching the counting word to one particular object	Children can determine large quantities correctly, but experience that it becomes increasingly difficult to keep track of a larger number of objects.
Organizing before counting	Arranging objects in a way that makes it easier to count them	Children experience that organizing the objects can help to keep better track of which objects have already been counted. Yet, they also come to see that organization is not sufficient to quickly determine a quantity.
Counting more than one object at a time	Counting by twos for example	Children gain sufficient experience with the counting sequence so that they no longer have to count each object unitarily. This motivates them to structure objects using increments of two, for example.
Applying spatial structure	Applying spatial structure with familiar spatial structures (a) to read off small (≤ 6) quantities and (b) to be able to count large (≤ 10) quantities in an organized way	Children come to understand that spatial structure can help to keep track of which objects are already counted, but perhaps there is a quicker way to determine large quantities. The shape of the objects that are counted may confuse children at this preliminary stage of spatial structuring.
Spatial structuring	Applying spatial structure goal-directedly using familiar spatial structures to determine both small (≤ 6) and large (≤ 10) quantities in an abbreviated way	Children understand that spatial structures can abbreviate counting procedures because the quantity can easily be read off or calculated. The shape of the objects is less distracting because the children focus more on the overall structure of the set of objects that is to be counted.

Fig. 6.1 The initial learning goals that children were expected to achieve as a result of the instruction experiment

A global HLT consists of (a) learning goals for students, (b) a plan of the instruction activities and the tools that will be used, and (c) a specific conjectured learning process that is expected to occur as a result of the instructional sequence (Gravemeijer, 2004). The observations from the exploratory studies and the pre-interviews, together with complementary literature (e.g., Clements, 1999a; Gelman & Gallistel, 1978; Mulligan et al., 2005; Van den Heuvel-Panhuizen, 2001) and the four phases of a developmental trajectory (see Chapters 4 and 5), contributed to the global outline of learning goals for the instruction experiment, which is conveyed in Fig. 6.1.

Hence, the conjectured learning process that was expected to occur as a result of the instructional sequence, started from unitary counting, to learning to spatially organize objects, and, ultimately, to spatially structuring objects as a means to abbreviate numerical problems. The second and third components of the global HLT give the learning goals a more practical content. In the remainder of this chapter, we explain how a sequence of instruction activities with corresponding learning trajectories was developed to help children become more aware of spatial structure and the advantages of spatial structuring in numerical procedures such as determining, comparing and operating with small quantities.

6.2 Identifying learning moments and defining corresponding classroom instruction activities

In this section we present the crucial learning moments that the children were expected to encounter throughout the instruction experiment as they worked towards reaching the learning goals (Fig. 6.1). We continue with an explanation of how an instructional sequence and observation criteria were developed throughout two sets of exploratory studies. The outcomes of these studies contributed to refining the learning goals in a HLT to prepare for the instruction experiment.

6.2.1 Learning moments and a corresponding HLT

Based on observations from the exploratory studies in several Kindergarten classrooms at a local elementary school, the learning goals of the instruction experiment, and the four phases in children's spatial structuring ability, we hypothesized that the children would encounter the following interrelated learning moments as they progressed along the learning trajectory:

(1) organize objects as a step towards becoming aware of spatial structuring to simplify counting procedures;

(2) create a motivation for spatially structuring objects;
(3) use spatial structuring to elucidate numerical relations;
(4) develop abstract spatial structures that are less context- or task-dependent;
(5) use spatial structuring in a goal-directed way outside the instruction experiment.

Our task was to design corresponding instruction activities that would encourage children to overcome the challenges that are associated with these learning moments so that they may reach the instruction goals that are listed in Fig. 6.1. The first version of such a HLT with its sequence of instruction activities is outlined in Fig. 6.2. This version was tried out in an exploratory study. In paragraph 6.2.2, we elaborate on the concept contexts and essential features of these activities.

Activity		Concept Context		Essential Feature
1. Wooden 3-D Blocks	→	Reasoning about shapes and figures	→	Identifying part-whole relationships ↓
2. Patterns with Children	→	Extending patterns	→	Using insight into part-whole relationships to identify the structure of a pattern ↓
3. Counting Rooms	→	Determining the number of blocks in a structured and unstructured block house	→	Making a structure explicit for comparing structured and unstructured configurations. One configuration may be composed of several structures ↓
4. Bingo	→	Recognizing familiar configurations in structured arrangements	→	Applying insight into several structures for identifying a familiar structure that is part of a relatively larger structure arrangement (cf. gestalt) ↓
5. Counting Flowers	→	Keeping track of an originally (unstructured) quantity	→	Applying the ability to identify a familiar structure that is part of a larger structured arrangement to recognizing structures in larger unstructured arrangements ↓
6. Domino Train	→	Simplifying the representation of a number of objects	→	Generalizing the ability to identify familiar structures for representing an amount and comparing quantities

Fig. 6.2 Outline of the first version of a hypothetical learning trajectory with corresponding instruction activities. Each instruction activity in the instruction experiment draws on insights from the previous activity to guide children towards greater spatial structuring ability by the end of the sequence.

6.2.2 Developing corresponding instruction activities

In light of the important role of spatial structure in patterning (see Chapter 2), it was decided to start the learning trajectory with an activity about part-whole relationships (Cobb, 1987; Owens, 1999). The expectation is that insight into part-whole relationships can help children to recognize at a more abstract level the composition and decomposition of quantities. As such, the aim of the activity was to help children become more aware of the composition and decomposition of shapes. A square, for instance, can be combined with another square to make a rectangle, while a square itself can also be made up of two triangles. The children were asked to construct the silhouette of a house or a boat that was cut out of paper, using triangular, rectangular, and squared wooden blocks.

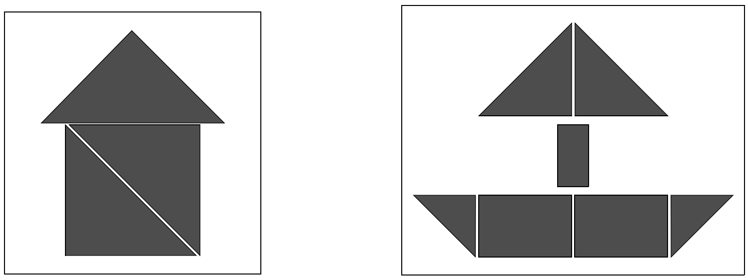


Fig. 6.3 Silhouettes of a house and a boat that the children were asked to build using squared, rectangular and triangular blocks to gain insight into part-whole relationships.

The second activity builds on the first activity in the sense that children were encouraged to use their insight into the composition of figures and shapes to elucidate the composition of a pattern.

Fig. 6.4 The class is discussing the difference between determining the number of blocks of a structured compared to an unstructured construction

This insight has been shown to be important for children to recognize the structure of a pattern and to support their understanding of how a pattern is characterized by the regularity of its structure (Papic & Mulligan, 2005, 2007). With more insight into recognizing structure in patterns, the next activity encouraged the children to look for structure in three-dimensional block constructions (Fig. 6.4). Presented with one structured (i.e., symmetrical and containing a regularity) block construction and one unstructured construction, the children were asked to determine the number of blocks in the construction. This activity was intended to stimulate a discussion about the difference between the way the blocks of the two constructions can be counted.

The fourth activity is the first activity that explicitly makes use of familiar spatial structures. In a type of Bingo game, first a large die was rolled; the children then had to find, on a card with a particular configuration of dots on it which the teacher held in front of them, the exact same configuration as shown on the die. This activity required that the children recognize dice configurations in larger dotted configurations. They were expected to make use of their experiences in the previous three activities with identifying part-whole relationships and structure in a relatively larger construction.

After four activities in which the children experienced the heuristic value of spatial structuring in various mathematical settings, the fifth activity was expected to encourage children to apply structure to unstructured configurations of objects as a means to simplify the process of determining and comparing quantities. The teacher placed twelve flowers (i.e., plastic chips) in a garden in the centre of the circle (Speciaal Rekenen, 2003). One child was asked to look away while the teacher came to pick flowers from the garden (Fig. 6.5).

Fig. 6.5 The children rearrange their flowers into a structure that can help them to easily determine the number of flowers in their garden

Then the child was asked to determine the number of flowers that the teacher removed. If the child's answer was incorrect, the teacher kept the flowers. If the child's answer was correct, the teacher replaced the flowers and added one more to the garden. The key to the game was that the children came to experience the value of applying structure to the flowers, especially as the number of flowers in their garden grew, making it increasingly difficult to keep track of the quantity by unitary counting alone.

The last activity of the instructional sequence is intended to let the children generalize their ability to identify familiar structures. The children were given a set of paper passengers. They were then asked to make a representation of this number of passengers in such a way that the conductor could conveniently see how many passengers were on the train. The children were expected to apply their insight into spatial structures for arranging the passengers into a spatial structure that would let someone else benefit from the arrangement and avoid unitary counting. The "wagons" (each child's representation of "their" passengers) could then be connected to each other so that the children could compare each other's structures and evaluate how the conductor could determine the number of passengers in each wagon.

In summary, these six instruction activities progress from a predominantly spatial focus, along a spatial structuring focus, towards a focus on number sense. This coincides with the underlying conjecture that children's insight into spatial structures can support them in recognizing, making use of, and applying spatial structures to abbreviate certain numerical procedures. As such, each activity is intended to draw on the insights that are the topic of discussion in the previous activity. Such an intertwinement of learning moments is what should give meaning to the instructional sequence and what should ultimately lead to so-called "Aha-moments" of insight (Freudenthal, 1984).

6.2.3 Defining observation criteria

To understand and to be able to draw conclusions from the observations in the exploratory studies, it was necessary to define observation criteria. These observation criteria operationalize the learning trajectories within and between each instruction activity. The preliminary set of observation criteria was outlined in terms of the following questions:

(1) The didactical effect of the instruction experiment:
- Are there indications of an improvement in the child's spatial structuring ability?
- Does it appear that the child has associated spatial structures with strategies for approaching the activities?

  - How do children of different age levels differ in the way they respond to the instruction experiment?
  - Would it be necessary to repeat the activities to improve their direct and retention effects?

(2) The effect of each of the instruction activities:
  - Do the children understand the objective and the essence of the activity?
  - Is the activity effective enough to be able to achieve its instructional intentions?

(3) The social effect of the instruction experiment:
  - The children's and teachers' responses to the activity: is the activity appealing?
  - Is the activity new and a valuable supplement to the regular classroom practices?

(4) The role of the teacher:
  - In what ways does the teacher influence how the instruction activity is implemented and its effects on children's learning?

6.2.4 Results of the first exploratory study

Two important changes were made to this sequence of instruction activities after one of the exploratory studies. First, it became clear that the last activity, "Domino Train", contributed very little to the instruction experiment. The children were so excited about drawing people in their wagon, that it distracted them from reasoning about spatial structures.

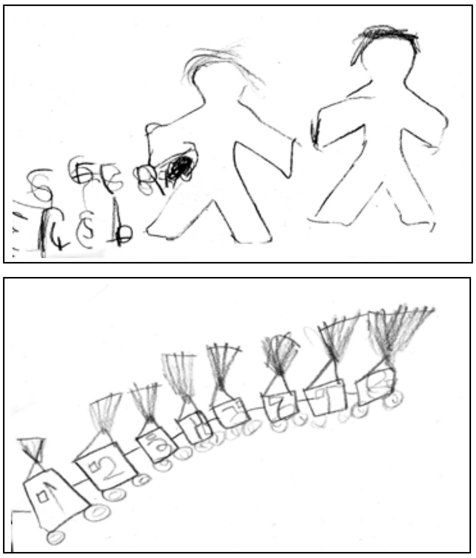


Fig. 6.6 Examples of children's drawings of passengers on the train. The top drawing shows two of the paper passengers and several number symbols (bottom left). The bottom drawing shows the wagons of a train, each marked with a number symbol.

Many of the children also skipped spatial structuring procedures as they instead tried to use numerical symbols to represent a quantity (Fig. 6.6). The children seemed to not feel the need to use spatial structures if they could picture the quantities with numerical symbols anyway. Hence, children drew numerical symbols or irrelevant details such as hearts and suns. Most children drew the objects in a line, indicating a type of organization for keeping track of count rather than a spatial structure for abbreviating the counting procedure.

In reflecting on the intention of this activity, we came to the conclusion that the “Domino Train” activity was too ineffective and that it could be removed from the sequence. In fact, the “Picking Flowers” activity would be an appropriate final activity in the sequence because it also requires that the children use their insight into spatial structures to make sense out of larger unstructured configurations. It would be a large step for children to achieve this level of spatial structuring ability, since it involves a knowing how to recognize, use and apply spatial structures for abbreviating numerical procedures in a variety of contexts.

Several experiences in the exploratory study led to changes in some of the other activities to improve their effectiveness. The “Bingo” activity, for example, was chaotic because the children were too keen to be the first to call out the number of dots on their cards. As a result, the children neglected to take note of the structure of the quantity on the die, and scribbled a circle around the dots that they had counted as quickly as they could (Fig. 6.7).

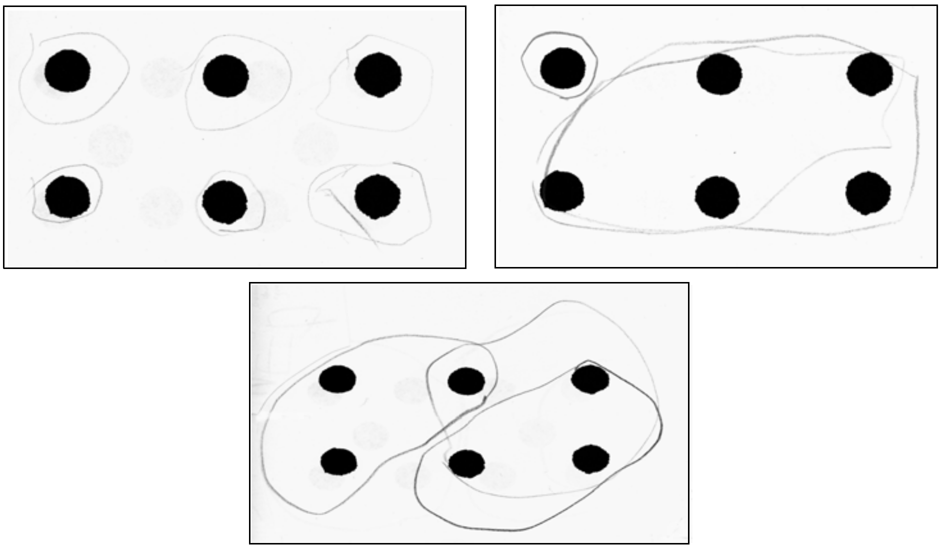


Fig. 6.7 Examples of children’s work in the “Bingo” activity. In the upper left card, the child circled each dot while counting unitarily. In the upper right card, another child first found 5 dots and then added the sixth dot. In the lower card, a child identified two sets of three dots and one set of four dots.

To organize the "Bingo" activity, we decided to centralize it by letting only the teacher roll the die. The teacher was to repeatedly encourage the children to look for the structure instead of just the quantity, and to have them raise their hands before showing the structure to the class. In this way, the teacher could stimulate the classroom mathematical practice of identifying a spatial structure, and the classroom social norms of raising hands, all for establishing the socio-mathematical norm of being aware of spatial structures (in large structured arrangements). This version of the activity was called "Got it!" and it was tried out in the follow-up exploratory study (see paragraph 6.2.6).

6.2.5 Revisions of the first instructional sequence and observation criteria

The second development in preparing for the instruction experiment, was to refine the observation criteria so that they reflect the HLT in each activity. Inspired by the strategy inventory (see Chapter 5), we listed strategies that the children were expected to use to approach the instructional activities. Fig. 6.8 conveys the HLT with the five revised activities and the corresponding observation criteria for the follow-up exploratory study.

At this point in the development of the instructional sequence, it was important to focus not only on the effects of the activities individually, but also on the relationship between the activities, because this interrelatedness composes the framework for the HLT. Therefore, the intended contribution of the instruction activities to the learning goals in the instruction experiment must be made explicit. The relationship between these five activities is outlined in Fig. 6.9. This figure illustrates how the first three activities are less interrelated than was originally anticipated. The children in the exploratory study did not explicitly make reference to part-whole relationships or patterning.

On the other hand, in reference to the conjectures about children's development of spatial structuring ability, it appeared that the first three activities can be analyzed alongside each other. Rather than considering the part-whole relationships as a prerequisite for recognizing structures in patterns, and recognizing structures in patterns as a prerequisite for recognizing structure in constructions, we decided to acknowledge the ability to recognize, use and apply spatial structures in the three activities independently of each other. The first ("Squared Triangles") activity supports the children in recognizing particular spatial structures. The second ("Guess my Rule") activity encourages the children to use structure to extend the pattern, and the third ("Huts and Castles") activity supports the children in applying their insight into spatial structures in a relatively larger structured or unstructured arrangement.

Activity	Goal of activity	Observation criteria
Squared Triangles	Analyzing and identifying the structure of a shape or figure in terms of its composition	• Shows no evidence of insight into the composition of a figure • Can independently compose a simple figure out of a minimal number of shapes • Can independently compose a simple figure out of numerous shapes of various sizes • Can independently compose a relatively complex figure out of numerous shapes of various sizes
Guess my Rule	Identifying and making use of the structure of a pattern to extend the pattern	• Names the elements of a pattern mechanically but cannot continue the sequence • Names the elements and continues the sequence on the basis of a rhythm • Understands the pattern but continues the pattern by explicitly naming each element without insight into the concise rule of the sequence (e.g., it's red, white, blue, red white, blue...) • Understands the pattern and explicitly summarizes the rule of the sequence (e.g., "it's red, white, blue every time")
Huts and Castles	Identifying the structure of a 3-D construction to understand and better be able to make use of its composition	• Counts the blocks unsystematically • Counts the blocks more systematically but has difficulty counting the blocks that are not visible • Counts both the visible and less visible blocks systematically • Can conceptualize the set of blocks to abbreviate the unitary counting procedure • Can conceptualize the set of blocks to read off the quantity
Got it!	Using insight into various types of structures to recognize a spatial structure in a larger arrangement	• Consistently counts the required number of objects unitarily • Is guided by examples to recognize familiar configurations in a set of objects • Can independently recognize various structures that are part of a larger set of objects
Picking Flowers	Using the ability to identify a familiar structure in a larger arrangement to recognize structures in randomly arranged configurations of objects and to learn to reason with larger numbers	• Places the objects in an arrangement either randomly or in the shape of a figure, but makes no explicit reference to structure • Places the objects in a figure that can make it easier to count the number of missing objects (e.g., in rows) • Places the objects in a structure that can make it easier to read off the number of missing objects (e.g., in groups of 5)

Fig. 6.8 A version of the hypothetical learning trajectory with the five revised instruction activities and corresponding observation criteria

As conveyed with a curly bracket in Fig. 6.9, the fourth activity is considered to build on to these three components of children's spatial structuring ability. We assume that the children may make use of their insight into recognizing, using and applying spatial structures to identify spatial structures in relatively larger structured configurations. Likewise, the next step in the developmental trajectory is covered by the fifth activity, in which the children are challenged to not only recognize spatial structure, but also apply it to unstructured arrangements.

Activity	Concept Context	Essential Feature
1. Squared Triangles	Reasoning about shapes and figures	Analyzing and identifying the structure of a shape or figure in terms of its composition
2. Guess my Rule	Extending patterns that consist of children or colors as the elements	Identifying and making use of the structure of a pattern in order to extend the pattern
3. Huts and Castles	Determining the number of blocks in a structured and unstructured block house	Identifying the structure of a 3-D construction in order to understand and better make use of its composition
4. Got it!	Finding familiar structures in larger structured arrangements of objects	Using the insight into various types of structures to perceive and identify a spatial structure in a relatively larger arrangement (cf. gestalt)
5. Picking Flowers	Keeping track of changes in the number of objects that are (originally) presented in a random configuration	Using the ability to identify a familiar structure in a relatively larger structured arrangement to recognize structures in randomly arranged objects and to learn to reason with larger numbers

Fig. 6.9 A revised version of the hypothetical learning trajectory with the five revised instruction activities

It is important to note that it is unlikely that all children start at the lowest level (learning to recognize spatial structures) and achieve the highest level (being able to apply spatial structures to unstructured configurations) by the end of this instruction experiment. The essence of this HLT, however, is that it is a general outline of the development of children's spatial structuring ability. This means that children of all levels of spatial structuring ability can begin at their level of understanding, participate in the classroom activity, and progress to a subsequent level based on their experiences in the classroom activity. Children who still count asynchronously, for example, can continue practicing their counting skills, while children who have already been applying spatial structures before the intervention, can be encouraged to find other and even more efficient spatial structures as a step up towards flexible spatial structuring abilities.

Children differ in the extent to which they progress in their mathematical development, so it will probably take more than five activities for a child to reach an "Aha-moment" and progress from one repertoire of strategies to another. Nevertheless, the importance of this instructional sequence is that it is based on a HLT that can guide teachers in improving their support for children's development of spatial structuring ability. The activities fit into this framework and can be repeated as often as necessary to improve children's spatial structuring ability. In Chapter 9, we will discuss how the teachers evaluated this instruction experiment.

6.2.6 Results of the follow-up exploratory study

In the follow-up exploratory study, the children were asked to construct houses with eight, nine, or ten blocks. This resulted in several interesting learning episodes. One girl had difficulty determining how to build a house with only three blocks. By turning the blocks, putting them together and with a little bit of luck, she eventually succeeded. Similarly, although with a little more help from the class and the teacher, two of the youngest children built a house with four blocks. These independent trial and error experiences offered a meaningful analysis of shapes and figures. This was clearly seen in the focus group, when two children were exploring how to construct a rectangle with two elongated triangles. One of the children first constructed the rectangle by chance and then spent time investigating how it was made and how he could remake it. His analyses motivated him to practice constructing the rectangle with triangles and to translate his experiences to construct a rectangle using two squares.

A more implicit learning process occurred in the "Huts and Castles" activity when the children picked the most structured and symmetrical houses as the houses that were "easiest to count". Only when the teacher asked them to explain, did they start to reflect on the differences between the structures of the houses. The discussion stimulated the children into verbalizing why they thought one house was easier to count than another. The teacher made use of the children's language by using their wording and by asking the class to comment on each other's remarks. In this way, the teacher led a classroom discussion that culminated in a reasonable verbalization of the children's implicit preference for structure. The focus group made this implicit preference more explicit when the children were asked to construct two houses that best show the difference between a structured and an unstructured house. The children all succeeded in building two distinctly different houses and in identifying which one of someone else's houses was easiest or most difficult to count (Fig. 6.10).

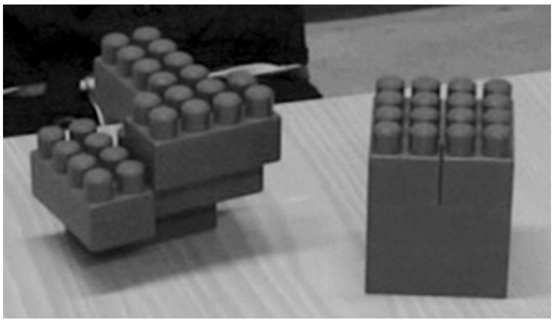


Fig. 6.10 Example of what a "difficult" (i.e., the unstructured construction on the left) and what an "easy" (i.e., the structured construction on the right) house means to one of the children in the exploratory study

A clear individual advancement in understanding occurred at the end of the "Bingo" activity. One girl had located five dots on her card, but the dots were not arranged like the dice configuration. When she was asked to find the structure like on a die, she looked at her card again and still located a different structure. In showing what the five looks like on a die, she drew what looked like a square in the air. When she was again asked to find that five on her card, she looked confused. The teacher then placed the die next to the card and asked her to search for the die in the card. As the girl (randomly) pointed to several dots on the card, the teacher guided her towards corresponding the dots on the card to the dots on the die (e.g., "yes, that's the middle one"; "those are the two bottom ones"). Quickly the girl then pointed to the correct configuration of five dots on the card. Apparently, the combination of seeing the die and connecting the positions of its dots to the dots on the paper, helped the girl recognize what dots on her card corresponded to the configurations of five on the die. She succeeded in recognizing the structure in a subsequent round of the activity.

A final example of a specific learning moment is one boy's advanced understanding in the "Picking Flowers" activity. The boy initially approached this activity in a numerical way; he used unitary counting procedures whenever he could, but these were not flawless and they often resulted in more incorrect answers as the tasks became more complex (up to 15 chips). In this activity, then, the more flowers were in his garden, the more difficulty he had keeping track of how many flowers were removed. As the activity continued, he started copying other children's configurations (e.g., placing the chips on the border of the paper, or in the shape of a flower, or in piles). This is most likely because he was more successful while using the other children's configurations to determine the missing number of flowers. When he repeated the activity in class the next day, he spontaneously arranged the fifteen flowers into five piles of three and succeeded in determining the missing number using this spatial structure. This suggests that he seemed to begin to understand the convenience of structure for determining a quantity.

Taken together, the individual learning moments illustrate the importance of opportunities for trial and error (constructing knowledge), of support from other children (social learning), of support from the teacher and task instructions that must carefully be connected to the children's points of reference (guided learning), to children's motivation to solve a meaningful problem (meaningful contexts) and to the difference between children's implicit and explicit explanations which can undermine children's understanding of a particular problem or situation. These are all points that are taken into account in designing the HLT and the sequence of instruction activities for the instruction experiment.

6.2.7 Revisions to the instruction activities after the exploratory studies

Despite these promising outcomes from trying out the instructional sequence in the exploratory studies, the instruction activities had to be revised for two main reasons. One reason was that some activities were not effective enough because of unclear instructions or too long durations. The second reason was because the children may have responded differently to the activity than expected. This suggests that the children needed other support than what was anticipated in the initial HLT. For example, at first the children had trouble disconnecting the characteristics of two different patterns. One child created a line of alternating blue and white chips, explaining that the blue chips are boys and the white chips are girls like in the patterns of children. Therefore, we introduced a patterning task about children holding colored papers which helped to translate the pattern that is made up of children to a pattern with only colored chips. This change to the activity was intended to improve children's understanding of how to pattern based on spatial structure, and therefore contribute to establishing the socio-mathematical norm of spatial structuring.

Further revisions to the "Got it!" activity were needed for it to become more meaningful to the children. The children did not experience a need to find the dice configurations since they could respond fast enough by just counting and subitizing the dots on the cards. In fact, some of the children preferred counting so that they could “show off” their numerical skills. Not all children counted accurately, however, and this is where insight into spatial structures could support their understanding of numerical relations. Hence, the activity should be such that the children prioritize looking for structures over counting strategies. In this way, the children and teachers would create a more taken-as-shared conception of spatial structuring. Hence, the last question in this activity was to ask the children to look for the exact configuration in the card as it appears on a die, and to circle the corresponding dots. This seemed effective because the children attended more to the structure of the number of dots that they were looking for. The activity was still set in a game-like context; the first child who located the die configuration won, while the other children could compare their configuration to the winner’s answer. In this way, the activity is exciting and is meaningful with its focus on dice configurations.

6.3 Optimizing the HLT and instructional sequence for Round 1 of the instruction experiment

The process of (re)designing classroom activities, (re)developing a HLT, (re)devising observation criteria, and (re)testing the sequence of activities in class, was the essential procedure for gaining experience in how children learn and respond to instructional support. The many specific learning moments that occurred during the activities, also contributed to more insight into how children may best be supported in their learning processes. According to Gravemeijer (2004), the learning issues that children are expected to encounter in performing the activities can become potential mathematical discourse topics that stimulate discussion about the different strategies that the children may use. The practical solution to the problem is called a “mathematical tool”, and it takes into account that the way that students use the tools builds on their experience with using other, more familiar tools.

To summarize the outcomes of the exploratory studies, we schematized the learning issues and related them to the instruction activities and the mathematical tools that were involved (Fig. 6.11). In constructing this schematization, it became clear that some of the learning issues were not represented well enough in the instruction activities. Therefore, more activities were developed to fill these gaps. For example, none of the activities had thus far specifically attended to double-structures, while these are equally as important as dice configurations in the development of young children’s spatial structuring ability (Speciaal Rekenen, 2003; Van Eerde, 1996). Therefore, for learning how to use spatial structures, an “Egg Carton” activity was designed to support children in learning to recognize and make use of double-structures in egg cartons.

General learning issue	Specific learning issue	Description of the corresponding activity	Mathematical tool
Familiarity with various spatial structures	Recognition and transfer of various 5- and 10-structures	“Hands and Feet”: Identifying different kinds of 5- and 10-structures in familiar settings and understanding how different structures can represent the same quantity	Fingers and toes, classroom attributes
	Part-whole relationships	“Squared Triangles”: Insight into the structure of a figure as the arrangement of its sub-components, and predicting the shape of a figure as the integration of smaller shapes	Small and large geometric blocks
	Structure of a pattern	“Guess my Rule”: Recognizing the structure in a pattern and generalizing the pattern with regard to its structure	The children and colored attributes
	Distinguishing structure of a 3-D construction	“Huts and Castles”: Insight into the difference between structured and unstructured 3-D constructions	Symmetrical and asymmetrical block constructions
Making use of spatial structures that are readily available in the task	Using spatial structure of a 3-D construction	“Huts and Castles”: Using spatial structure in a 3-D construction to abbreviate counting procedures	Symmetrical block constructions
	Translating (part of) a spatial structure to “double-structures”	“Egg Cartons”: Recognizing patterns in egg cartons and using them for insight into missing addends	Egg cartons for 4, 6, and 10 eggs
	Insight into structured dot configurations	“The Highest Card”: Reading off domino configurations up to 6 in relatively larger structured configurations	Sets of enlarged playing cards
Applying spatial structures goal-directedly to abbreviate numerical procedures	Insight into randomly arranged 3-D block constructions	“Huts and Castles”: Finding structure in a randomly arranged 3-D construction to help organize and abbreviate enumeration	Asymmetrical block constructions
	Insight into randomly arranged dot configurations	“Picking Flowers”: Arranging random configurations into spatial structures that help organize the elements and subsequently abbreviate numerical procedures	Own spatial structures
General: language issues	Vocabulary	Using appropriate words to describe spatial structure as a type of organization, and numerical procedures	Own constructions and social interaction

Fig. 6.11 Summary of the outcomes of the exploratory studies: the crucial learning issues, their corresponding instruction activities and the mathematical tools

It also appeared that the instructional sequence had no proper introduction that set a motivating context and invited the children to participate in the activities. We therefore adapted the “Search for the five and ten” activity from the Speciaal Rekenen Program (2003) to start the instruction experiment with an activity in which the children are stimulated to explore various five- and ten-structures in their surroundings. Further, given the complications with the “Bingo” and subsequent “Got it!” activities, the activity was changed to “Highest Card”, in which the children identified which of the two cards has the most dots. This version of the activity still required the children to recognize spatial structures in relatively larger spatial configurations, yet the competition between two children was easier to manage for the teacher than the excited classroom. Finally, we acknowledged the role of language as a learning moment that should continuously be taken into account when evaluating children’s performance on the activities (Fig. 6.11).

In design research, one of the researcher’s tasks is to observe any indication of children’s mental activities as they perform the instruction activities, and to compare these mental activities to what is expected in the HLT. In this way, the instruction experiment highlights observations that question the assumptions about learning that were originally set out in the global HLT (Gravemeijer, 2004). For this reason, a manual was constructed that the teachers could use to understand the aim of the activity, to know what materials each activity required, to follow the guidelines for what questions to ask and what wording to use, and to take note of what types of strategies the children were expected to use and what strategies the children may learn to use as they perform the activity (i.e., the observation criteria). The final version of the manual can be found Appendix 5.

The plan was to perform the instruction activities during the instruction experiment, with a period of analysis in between two rounds. In the first round, the children explored various two-, five-, and ten-structures in their surroundings (Speciaal Rekenen, 2003), studied the composition and decomposition of various shapes, practiced recognizing structure to extend a pattern, compared ways to determine the number of blocks of structured and unstructured constructions, and investigated double-structures in an egg carton context. These five activities were intended to offer children the opportunity to discover spatial structures in their surroundings and to see how these structures could conveniently be used to, for example, determine a quantity.

The second round of the instruction experiment was to begin with the “Picking Flowers” activity. This was to introduce the children to an unstructured situation that could be organized or, more specifically, spatially structured. Next, the children repeated the patterning activity to again practice abstracting a structure from a pattern. Then the children played the “Highest Card” game in which they practiced recognizing structure in larger structured settings. This was expected to prepare them for another attempt at the “Huts and Castles” activity in which they tried to use structure to determine a larger structured number of blocks. Finally, the children were encouraged to make use of all the insights that they gained from the previous activities to make another attempt at using and applying spatial structure in the “Picking Flowers” activity.

Similar to the analyses of the exploratory studies, the main issues that came up in planning and discussing this design of the instruction experiment with experts was (a) unclear or irrelevant intentions of several of the activities, and (b) lack of context in some of the activities. For example, through studying the role of the second activity (i.e., the composition and decomposition of shapes) in the instructional sequence, questions arose about what the activity could contribute to the conjectures, what the activity would mean to the children, and how the activity ultimately relates to the development of spatial and number sense. Although the original idea of attending to part-whole relationships is still very relevant to patterning abilities and ultimately to learning to identify spatial structures, it became doubtful whether starting the sequence of instruction activities at this fundamental point would stimulate the children enough to start to use and apply spatial structures to mathematical tasks. This activity seemed too tangential to fit the aims of the instruction experiment, while an activity such as counting blocks in a 3-D construction connects more strongly to our theoretical framework and still covers part-whole relationships. As such, we decided that the activity would best be eliminated from the sequence.

Regarding the role of meaningful contexts in RME, it was necessary to take another look at the activities in terms of their context and the perspective with which children approach the activity. The first activity about exploring various types of structures, for example, was not embedded in a context. This would make it difficult for the children to understand the aim of the activity, or even to be motivated to participate in the activity. Similarly, the RME construct of “guided reinvention” was not apparent in this activity because the children were only asked to collect and compare various spatial structures for particular quantities.

To improve the activity, we introduced the “Trick Box”, a box that was filled with various familiar objects such as finger patterns on cards, dice, and egg cartons. These objects represent spatial structures that have mathematical characteristics which make them useful for abbreviating numerical procedures such as determining, comparing and operating with small quantities (Van Eerde, 1996). The egg cartons, for example, illustrate double-structures and support the use of a doubling strategy. The objects were discussed with the teachers beforehand to ensure that the children would not be distracted by new materials, and that as much use of materials that are readily available in the classrooms or at home was made as possible.

The objects are called “tricks” because, as the children would discover throughout the instruction experiment, they can help children determine and compare quantities without having to count unitarily. Further, to foster a shared vocabulary, the teacher should take care to use consistent wording such as “tricks” and “convenient ways to determine how many there are” (see paragraph 2.5.2). Translated from Dutch to English, the word “easy” (i.e., makkelijk) may best be substituted with “convenient” (i.e., handig) to prevent children from “easily counting” a quantity rather than searching for a “convenient way to determine” a quantity.

6.4 The final instructional sequence

The final version of the sequence of instruction activities for the instruction experiment initially included two sequences of five activities. This is outlined as follows:

(1) “Guess my Rule”: Becoming aware of part-whole relationships through patterning.
(2) “The Trick Box”: Exploring types of spatial structures.
(3) “Giant Cards”: Making use of part-whole relationships (version 1).
(4) “Huts and Castles”: Spatial structure and three-dimensional constructions.
(5) “Picking Flowers”: Using and applying spatial structures.
(6) “Filling Egg Cartons”: Using part-whole relationships (version 2).

As described in Chapter 3, an instruction activity started with a classroom discussion during which the children were sitting on chairs in a circle facing the teacher. After about half an hour, the focus group was taken aside to perform additional tasks with the researcher. This provided additional opportunities for more detailed observations of how children approached a particular task. The classroom and focus group discussions and the children’s interaction with the teacher were later interpreted against a set of observation criteria that outline the HLT for each activity. These observation criteria are part of the manual describing the instruction activities (Appendix 5). In the following paragraphs we describe each of the instruction activities with their HLT within the instructional sequence.

6.4.1 “Guess my Rule”: Becoming aware of part-whole relationships through patterning

The Guess my Rule activity was developed to focus on the structure of a pattern as defined by the regularity of its elements. In light of this interpretation of structure and its fundamental effects on mathematical achievement (Papic & Mulligan, 2005; Waters, 2004), we decided to make this activity the first of the sequence. The conjecture was that insight into part-whole relations is necessary for children to abstract a fundamental element of an arrangement and notice its repeated occurrence (e.g., abstracting a row of three to enumerate three rows of three). As children become more aware of the components that make up a pattern, they may notice the order in which the components are arranged. Once the children become aware of its regularity, the structure of the pattern may make it easier for them to remember and extend it on their own. The process of abstracting a structure and noticing its repetition can also help children become aware of structure in less apparent arrangements.

The context of the activity was about a “fortune-teller” and the children were to construct a pattern that would be easy for someone else to extend. The children were to act as fortune-tellers and “predict” who belonged next in the line of children that the teacher arranged in front of the classroom in a pattern such as “boy, girl, boy, girl” or “tall, short, short, tall, short, short”. The class repeated this activity first with patterns of children holding colored papers, and then with patterns of plastic colored chips. In each case, the children were to identify the regularity in the pattern, find the repeated structure (e.g., color, size, gender) and use this structure to extend the pattern.

In accordance with the principles of RME, this idea of “predicting” the next element of a pattern is important for the meaning of the activity; the children were to come to understand the significance of finding and using structure and they were to relate structure in this activity with structure in the next activity. The envisioned connection between the first and second instruction activities was the focus on what makes it possible to predict the next element in a pattern. The second activity would continue this idea by having the children experience how the various spatial structures in the Trick Box contribute to “predicting” a quantity that is to be counted. For example, once the children become familiar with a structure of five such as that on a die, they may circumvent longer counting procedures by “predicting” a quantity on the basis of structures that they recognize.

6.4.2 “The Trick Box”: Exploring types of spatial structures

The Trick Box activity was based on an activity from the Speciaal Rekenen Program in which the children explore different structures representing 5 and 10, and later, double-structures as represented by egg cartons (Speciaal Rekenen, 2003). The idea behind including this activity in the instructional sequence was to create a strong context that can underlie the subsequent four activities; the structures that were represented by the objects in the box all related to structures that may be useful in solutions to the other activities. The objects in the box were referred to as “tricks” that could help to determine a quantity in “an easy way” in contrast to more tedious and error-prone unitary counting procedures. The teacher talked about these “tricks” as tools that everyone could command. In the upcoming activities, the children would be encouraged to refer to the box and see what structures could help them in performing the activity.

In line with the need in RME to appeal to children’s interests, the Trick Box itself was a colorful cardboard box that excited the children through how it was introduced in the classroom (i.e., the surprise of seeing such a box in the middle of the classroom). To associate this activity with the previous one, the teacher first asked the children whether they remembered how they could see how to extend a pattern. Such awareness of patterning was assumed to help the children in exploring the use of the contents of the Trick Box. As the children unpacked the box, the teacher could start a discussion about how they could conveniently see how many of each object there were. The box contained several objects and sets of objects that represented various structures and applications of structures. The one to six dots on two large dice were structured in ways that children are able to subitize at a very young age (Clements, 1999a) or may have familiarized themselves with earlier (cf. Teubal & Dockrell, 2005). A set of flashcards with finger patterns appealed to the children’s basic use of their fingers as a way of keeping track of count (Baroody, 1987; Butterworth, 1999).

The other objects in the box represented more applied spatial structures. A number of egg cartons containing either 6 or 10 plastic eggs were included to support the children’s doubling strategies (Speciaal Rekenen, 2003; Van Eerde, 1996). A set of enlarged playing cards encouraged the children to discover the dot structures on dice within the large structured dot configurations of the cards. Three towers of LEGO blocks were intended to appeal to the children’s interest in building with blocks and to challenge them to use the three-dimensionality of the construction to determine the number of blocks that it contains (Ness & Farenga, 2007). A set of little plastic butterflies was included as an example of unstructured objects that can be structured to be counted more easily. Finally, the three different bead patterns in the Trick Box linked this activity to the previous patterning activity.

The process of exploring the various types of spatial structures and their uses, catered to the different developmental levels of the children in the class. The objects that represented familiar structures (i.e., the dice and flashcards with finger patterns) appealed to most of the children because they had encountered them before, at least in class. The objects that represented more applied structuring (i.e., the egg cartons and butterflies), however, appealed mostly to children who were more aware of spatial structures and who were more experienced in making use of such structures in numerical procedures. In this way, the teacher could ask a child those questions that suited the child’s level of understanding.

The “guided reinvention” principle was manifested in this activity by how the children were solving the mystery of the contents of the box; the guiding role of the teacher was to ask appropriate questions that encouraged the children to think about the similarities and differences between the objects in the box and to explore what the objects could represent throughout the rest of the activities. The Trick Box offered children the experience of working with various structures that they may or may not already have been familiar with. The use of structures was not imposed on the children, but rather presented as objects in a box for the children to explore (see section 2.4). In this way they could familiarize themselves with objects that represent structures which would be useful in class discussions for solving the rest of the activities in the instructional sequence.

Hence, regarding the research questions, this activity was to contribute to insight into how children interpret various structures, how they relate their patterning experiences to other types of structures, and in what ways they tend to use structure to solve numerical problems. From a social perspective, the activity could illustrate instances where the instruction and content of the activity had to be negotiated with respect to the children’s responses and conceptual understanding. Such negotiations can provide insight into children’s understanding of spatial structures and how the instructional setting can influence the development of this understanding.

By this stage in the instructional sequence, the children had had an introduction to part-whole relationships and structure in the first activity, as well as opportunities to explore more types of spatial structures in the second activity. The third activity was the first activity that focused on one specific item from the Trick Box, namely the dice dot configurations. This was to highlight the important use of part-whole relationships in a numerical procedure such as determining a quantity.

6.4.3 “Giant Cards”: Making use of part-whole relationships

The third activity was a new version of the “Highest Card” game called “Giant Cards”. The intention of this activity was to support the children in gaining insight into part-whole relationships and, further, in learning to make use of such relations to recognize spatial structure for determining a quantity (cf. Arcavi, 2003; Henry & Brown, 2008). A pair of children was called forward to determine as fast as possible the number of dots on the enlarged playing card that the teacher had drawn from a pile of cards in the Trick Box.

The guidelines in the manual encouraged the teacher to repeatedly use phrases like “clever tricks” and “convenient ways of determining how many there are” as ways to encourage the children to make use of the contents of the Trick Box. The game-like context of this activity was inspired by the Speciaal Rekenen activities (2003) in which children were stimulated to explore different strategies that could help them win a game. Each pair of children consisted of a giant and a dwarf. The issue in this activity was that although giants are large and strong, this may not help to quickly determine a quantity. The dwarf could be more clever than the giant in finding a convenient way to see how many dots were on the card without counting. When one of the two children recognized a structure in a card faster than the other child, the class was expected to be motivated to improve their strategies. Such a motivation, along with the teacher's guidance, was to stimulate the child's strategy development and awareness of part-whole relationships (Ness & Farenga, 2007).

After concentrating on the dice dot configurations from the Trick Box, the next activity revolved around the LEGO blocks. This activity required the children to draw on their experiences with making use of part-whole relations in two-dimensional settings, in an effort to explore and make use of the spatial structure of a three-dimensional construction.

6.4.4 "Huts and Castles": Spatial structure and three-dimensional constructions

The design of the Huts and Castles activity was initially inspired by two observations that illustrate how children have a particular ability to spatially structure the world around them and how this ability could contribute to the development of the children's numerical insights (cf. Clements & Sarama, 2007). One observation concerns an eight-year old child who was determining the number of blocks in a construction drawn on paper. The child confidently explained that he could see ten blocks because there was a block of four blocks on top and three and three on the side. Apparently, this child had visualized the blocks and determined the quantity without having to count them unitarily. The second observation comes from a Kindergarten teacher who noted that one of the children pointed to the corners of a rectangle that was drawn on the blackboard while saying that it was "the same as four". Hence, this child seemed to associate a spatial structure for four with four as a quantity.

As explained in Chapter 2, several gestalt norms explain how structures can be detected and differentiated perceptually (e.g., Quinn et al., 1993, 2002). In this sense, a square resembles the dice dot configuration for four, whereas a rectangle could resemble two rows of three dots (Clements, 1999a). Ness and Farenga (2007) elaborated on children's early construction ability and its effects on mathematical performance. This motivated the design of an activity that would tap in to young children's natural interest in constructing and in symmetrical beauty (Freudenthal, 1984).

The activity was intended to make children more aware of the role of symmetry in structure and determining a quantity. In particular, the explicit comparison between a structured and an unstructured construction was expected to spark children's curiosity about why it seems easier to count the number of blocks in a structured (i.e., symmetrical) construction compared to the unstructured (i.e., asymmetrical) block construction. The challenge for the children was to estimate how many blocks are necessary to rebuild the sample symmetric and asymmetric block constructions. This required them to take note of the structure of the construction rather than to count each block unitarily. Hence, an important part of the activity involved connecting the children's methods of counting to the "convenient ways" of determining a quantity such as those discussed earlier in the Trick Box activity. An example is recognizing two rows of three blocks on the bottom of a construction such as, for instance, the two rows of three eggs in an egg carton, or two rows of three dots on the face of a die. As such, the children were expected to extend their insight into spatial structuring for numerical procedures to this 3-D setting.

Just like in the previous activities, this activity was accessible to different developmental levels because the children can also practice their counting skills in determining the number of blocks in the construction. It was expected that when the children experience difficulties in trying to keep track of the blocks that they already counted, they would be more motivated to explore and use the structure of the construction. From exploring patterns and various spatial structures, to making use of spatial structures in relatively more complex spatial structured settings, the four previous activities were intended to guide the children towards the fifth activity. This last activity in the instructional sequence required children to spatially structure objects that were set in relatively large, unstructured settings.

6.4.5 "Picking Flowers": Using and applying spatial structures

As described in section 6.2, the Picking Flowers activity was adapted from "Robbie de Rover" from the Speciaal Rekenen Program (2003). Presented with a number of flowers (plastic chips) in a garden (a sheet of paper), the children were asked to determine the number of flowers that the teacher picked from their garden. After determining the number of missing flowers correctly, the child won an additional flower. The teacher kept the missing flowers if the child answered incorrectly. Hence, the essence of this activity was for the children to use strategies that could abbreviate the numerical procedure. In this way they did not have to depend on unitary counting procedures that could become confusing as the number of flowers in their garden increased.

The main theoretical contribution for this activity came from children's spatial visualization ability to perceive, group and differentiate objects according to gestalt norms (see section 2.3). It is also based on research in special education that has shown how children typically have trouble letting go of unitary counting procedures and benefit from activities that challenge them to make more use of spatial structures (Boswinkel & Moerlands, 2002; Mulligan, Mitchelmore, & Prescott, 2005; Pitta-Pantazi, Gray, & Christou, 2004).

After studying the objectives of all the instruction activities, it became clear that this activity had the potential to stimulate children towards the fourth and final phase in the developmental trajectory for spatial structuring (see section 5.2). Children may keep practising their unitary counting skills, they may be inspired to use spatial structures as they start rearranging the flowers, or they may be challenged to spontaneously apply spatial structures to elucidate counting procedures. As such, they may at first approach the activity either by leaving the colored chips the way they are or by playing with the chips and arranging them into different shapes and figures. As the children experience the convenience of arranging the chips in ways that help to see how many chips are missing without unitary counting, they may explore more configurations that can help them read off the quantity. This can highlight which spatial structures are meaningful to young children and in what ways the children become familiar with such structures and learn to make use of them in a strategic way.

The design of the activity took into account all the structuring experiences that the children will have had after performing the first four activities. In addition, the context continued in line with the previous activities so that the children could recognize the socio-mathematical norm of spatial structuring that by this time should have started to be established. As such, the aim of the activity was to inspire the children to make use of structures from the Trick Box that they by now were more familiar with, or to create their own structures so that they could depend less on unitary counting when determining and comparing the number of flowers that were missing from their garden.

6.4.6 “Filling Egg Cartons”: Using part-whole relationships (version 2)

The second round of instruction activities is analogous to the first round, except that the “Giant Cards” activity was substituted by an egg carton activity called “Filling Egg Cartons”. Similar to the third and fourth activities, this activity guided the children in studying the egg carton structures in the Trick Box. It resembled the Giant Cards activity in how it gave the children an opportunity to explore spatial structures in larger structured arrangements. This activity was slightly more complex, however, in that the children were required to operate with quantities (i.e., determine totals and differences) rather than to only abstract the structures and read off the quantities.

The context of the egg carton activity was about helping a farmer who was having difficulty to quickly see how many eggs he had and how many egg cartons he would need to take the eggs to the market. While the teacher added and took away eggs from the carton, the children were asked to find “convenient ways” to determine how many eggs were still in the carton or how many have been removed.

This activity was included in the instructional sequence to discuss double-structures because double-structures are essential for gaining insight into numerical relations for basic addition problems (Van Eerde, 1996). When structure is defined as the rule that holds the elements of a pattern together (Papic & Mulligan, 2005), then mathematical operations such as multiplication (i.e., repeated addition) can be considered to draw on children’s patterning ability (Anghileri, 1989; Carraher et al., 2006; Waters, 2004). Another reason why double-structures may be useful to the development of kindergartners’ number sense, relates to the body of research about the effects of young children’s patterning ability on their later mathematical performance (cf. Papic & Mulligan, 2005; Waters, 2004). Consequently, along with for instance finger patterns and dice dot configurations, we consider it necessary for young children to become familiar with double-structures as one type of structure that can support their mathematical procedures.

The aim of this activity was very similar to that of the Giant Cards activity. It was originally inspired by activities from the Speciaal Rekenen Program (2003), and it was intended to stimulate the children towards recognizing double-structures in configurations and making use of such structures to abbreviate numerical procedures. The activity catered to different developmental levels in that children could practice their counting procedures, their ability to recognize different structures, and their ability to apply structures to other configurations. Hence there was a “zone of proximal development” in that the children were continuously stimulated by the varied levels of questions that the teacher asks, and their strategies were challenged by the strategies of other children.

Another way that this activity complemented the Giant Cards activity is that it involved egg cartons that, on the one hand, the children are more familiar with than Giant Cards, but that, on the other hand, are less straightforwardly connected to spatial structures in mathematics; while dice dot structures are usually already connected to game contexts and numbers, the children would have to consider egg cartons differently in this setting compared to their use in the household. The challenge then, was for the children to abstract spatial structures from arrangements that the child would not normally relate to mathematics. The significant contribution of this activity to the instructional sequence was to observe how children may be supported in learning to recognize familiar structures in relatively unfamiliar contexts.

Children were expected to be less familiar with recognizing double-structures compared to, for example, finger patterns. Hence, the challenge in designing an appropriate activity was to take care to relate the double-structures to structures that the children were already familiar with. It was also important to link the act of doubling to the act of patterning in the first activity, since both involved finding and then repeating the underlying structure. Therefore, in introducing the activity, the teacher had to connect the fundamental idea of structuring two rows of a certain number of eggs to structuring dots on a Giant Card, to structuring other objects in the Trick Box, to, finally, structuring a sequence into a pattern. The underlying motivation for the children was to discover why these structures were "tricks", how they were related to each other, and how they could be used to avoid unitary counting procedures.

This chapter described the process of developing the HLT and finalizing the instruction activities for the instruction experiment. In the next chapter, we explain what issues were encountered as the sequence of activities was performed in the first round of the instruction experiment. These outcomes have consequences for the design of the second round of the instruction experiment, as will be described in Chapter 8.

7 Analysis of Round 1 of the Instruction Experiment

In the previous chapter, we described the process of developing the HLT and corresponding instruction activities for the instruction experiment. This chapter begins with a comment about micro and macro cycles in design research and a description of the classroom social norms. Against this background, we continue with illustrations, analyses and evaluations of what happened during the first round of the instruction experiment. In section 7.4, we reflect on the outcomes to shed light on what determines the effect of an instructional sequence for particularly young children in an RME setting. One of the determining factors is the importance of a meaningful overarching context for supporting awareness of spatial structuring. Hence, section 7.5 elaborates on the role of “Ant” (“Miertje Maniertje” in Dutch) as an overarching context. Finally, in section 7.6 we describe how each instruction activity was revised to contribute to an optimized HLT that underlies the design of the second round of the instruction experiment, and that refined the operationalization of the construct of spatial structuring in the research.

7.1 Reinterpreting micro and macro cycles in design research

Before elaborating on the overall outcomes of the instruction experiment, we focus briefly on an important matter that affected the rest of the instruction experiment. What became clear as the first round of the instruction experiment progressed, is that the procedure in design research that helps researchers come closer to understanding how a particular intervention works, is not as pre-defined as our initial plan for conducting two consecutive sequences of instruction activities was. Our interpretation was that a pre-defined second round could offer the children a second opportunity to learn while performing the instruction activities. It would also give the researchers another opportunity to study the children’s behavior and the role of the instructional setting, and the learning ecology in general, in supporting the children’s learning. However, in design research the series of micro design cycles for constructing the initial HLT and instructional sequence (i.e., the exploratory studies described in Chapter 6) is followed by a macro design cycle to try out the resultant sequence of instruction activities and to adjust the initial HLT (Round 1), and is concluded by a macro design cycle to try out the revised HLT and sequence of instruction activities (Round 2). This process is illustrated in Fig. 3.1 in Chapter 3.

The micro design cycles within each macro cycle involve the anticipation of what will happen in a session (i.e., a thought experiment), the session itself (i.e., trying out the activity) and a critical reflection on the session. This reflection leads into a new thought experiment in which the HLT may be revised to inspire the development of another thought experiment about the next instruction activity in the subsequent micro design cycle. A macro design cycle spans the instructional sequence and inspires the design of a revised set of instruction activities. What this procedure clarifies is that the outcomes of the first round are a prerequisite for planning the second round. Therefore, in our instruction experiment we had to step away from planning the second round before having performed the first round. As such, we decided to perform the Egg Carton activity in the session following the Giant Cards activity, rather than in a separate instruction sequence where the Egg Carton activity was to substitute the Giant Cards activity (see section 7.6). In the next section, the classroom social norms are outlined that underlie the teacher's and children's behavior in the classroom discussions.

7.2 The classroom social norms

To prepare for an analysis of the outcomes of the first round of the instruction experiment, we present four classroom social norms that were observed throughout the classroom discussion. These norms paint the setting of the instruction experiment and are therefore important to take into account when interpreting the classroom observations.

(1) The children were to raise their hands and wait for the teacher to call on them before they made a comment or answered the teacher's question.
(2) The children were expected to listen to each other and to think about the teacher's questions even though it was not their turn.
(3) The teacher tried to involve each child in a classroom discussion by asking questions that fit the age and/or developmental level of the child.
(4) The teacher tried to make the children more involved in their own and each other's learning process by asking them to explain their own answer and someone else's answer, and to re-evaluate their answer and solution method.

The teachers especially referred to the first norm at the beginning of each activity, while the second norm became most apparent towards the end. The third and fourth norms occurred throughout the activity. These appear most important in coloring the analysis of the activity because they determine the nature and regularity of teacher's and children's interactions.

7.3 Key observations from Round 1

In this section we describe the micro design cycle that includes results of each instruction activity in Round 1 of the instruction experiment. Each paragraph begins with a short description of the hypothetical learning trajectory (HLT) for the instruction activity. The instruction activities are described in more detail in Chapter 6 and in Appendix 5. Next, we highlight observations that best illustrate how the children responded to the activity in the classroom and focus group settings, what effects the instructional setting had on children's performance, and how these results influence the HLT. These outcomes are clustered and discussed in several themes.

7.3.1 "Guess my Rule": Becoming aware of part-whole relationships through patterning

HLT for the instruction activity

The first activity of the instruction sequence was intended to offer children the opportunity to explore patterning. This activity was aimed at helping children to become aware of the part-whole composition of a pattern, which is a valuable tool for understanding part-whole relationships in spatial structures.

To start the activity with a meaningful context, the first sequences were made up of the children themselves as they were called up by the teacher to come stand in a particular order (e.g., boy, girl, boy, girl) in a line facing the class. During the classroom discussion, the children were asked to "predict" what element might be added to a sequence so that the sequence would continue to make sense. According to the HLT, we expected that some children would need more help than others to understand the objectives of the activity and to learn how to study the composition of a pattern. The children's types of reasoning could help the teacher gauge children's levels of patterning ability. Further, in the previous exploratory studies, a translation from patterning with children to patterning with the colored paper that each child in the line was holding, prepared them for patterning with relatively abstract sequences of colored chips. Hence, in line with the HLT, the teacher could create opportunities for guiding classroom discussions that focus on part-whole relationships. In this way, we expected children's different levels of patterning ability would become the shared topic of discussion for gaining insight into the part-whole compositions.

Analysis and illustrations

Creating awareness of the rule and regularity of a pattern. At the start of the activity, it was surprising to note that the children were very focused on the height of those who were standing in the line. As the discussion progressed, however, it became clear that the children were thinking of a previous classroom activity in which they learned to arrange themselves from short to tall. Hence, the teacher discussed the mathematical practice of looking at the beginning of a sequence to discover the characteristics of the pattern. She hinted to the children that each pattern that she constructed had a different “rule” to it, by asking the children whether the next child in line actually fit the pattern. She also let the children who were still seated explain what the new “rule” of the pattern was and why the two boys standing next to each other adhered to this rule. When she exposed the children to a line that had no apparent rule to it, she asked the children to explain what it was that made the last sequence more difficult to “predict” than the other two sequences. When one child answered that “it wasn’t a clear line”, the teacher took the opportunity to start a discussion about recognizing the components that make up a pattern (i.e., “a clear line”) by first decomposing the pattern into its parts and then recomposing the parts to extend the pattern (Fig. 7.1).

Fig. 7.1 Video frame of the teacher helping the children extend a pattern that alternates two girls and two boys

The second part of this activity supported the additional step of patterning with colored papers that was added to the activity in response to children’s confusion during the exploratory studies (see paragraph 6.2.7); the teacher succeeded in introducing the idea of extending a pattern that is made up of children to extending a pattern that is made up of colored origami papers or colored plastic chips. By this time the children also understood to notice the “rule” (i.e., a particular combination of elements that make up a unit of a pattern) of the sequence. This is illustrated in the following episode of the focus group:

Researcher:	(pointing to the last purple chip in an alternating blue and purple pattern) What color chip do you think should come after this one?
Focus group:	A blue one.
Researcher:	Yes, how do you know that?
Becky:	(points to one of the blue chips in the pattern) But there is also a blue one.
Researcher:	(pointing to one of the purple chips in the pattern) But there is also a purple one, so why don't you think the next chip should be a purple one?
Becky:	(first pointing to the blue chip at the beginning of the pattern and then to the purple chip at the end of the pattern) But here there already is a purple one.

In this episode, Becky shows that she understood the idea of a pattern that is made up of alternating (i.e., blue, purple, blue, purple...) colors. Yet, as illustrated in the next episode, she didn't show an understanding of a pattern that is made up of a variation in the number of elements for a particular color. After discussing three of the children's patterns that were made up of two alternating colors, the researcher turned to one of the children's sequence of chips that had no apparent pattern to it. She asked the focus group why this sequence had no "rule" to it and they remarked:

Becky:	It's not a good line because you have to do two of the same colors.
James:	And this one (pointing to his own sequence of two alternating colors) is much easier to make than that one (pointing to the sequence of randomly arranged colors). But you can also do three colors!

This episode illustrates how the children understood the "rule" of a particular pattern (i.e., alternating two colors). Like most children in the class, Becky still adhered to the classroom mathematical practice of patterning with alternating colors ("two of the same colors"). This is what the class had worked with so far in the instruction activity. At the same time, however, James showed a significant insight into the functionality of patterning by relating the absence of a pattern to the difficulty of constructing the line and extending it. This is a valuable insight for understanding the convenience of spatial structuring for use in mathematical procedures. Moreover, he generalized the two-colored patterns that had been discussed so far to more colors, suggesting that he understood more about patterning than what thus far had been the classroom mathematical practice.

What remains unclear, however, is to what extent these children understood and could identify the "regularity" of the pattern (i.e., the repetition of units of a pattern which consist of elements that are combined according to a certain rule). Most of the children seem to understand that a pattern is composed of a fixed number of colors (two or three); they showed no sign of being able to continue a pattern based on its "regularity". The children had not mentioned patterns with variations in the number of chips per color (e.g., *a-bb-a-bb*... or *a-bbb-a-bbb*...). Nevertheless, at the end of the focus group activity, Lisa created a pattern that illustrated a remarkably creative interpretation of patterning (Fig. 7.2).

Lisa:	Look, mine is easy!
Researcher:	Why?
Lisa:	Because... (waving one hand over the left side of her pattern and one hand over the right side) it's a little bit the same.
Researcher:	It's a little bit the same. What do you mean?
Lisa:	Yes, look, (pointing to the first chip and then to the last chip) here is yellow and there is yellow (continuing to point to each chip, moving to the middle of the pattern), here is blue, there is blue, here is white, there is white, here is yellow, there is yellow (now first pointing to the right side of the pattern and then to the left side), here is green, there is green (pointing to the middle left chip and then to the middle right chip), and here is purple and there is purple.
Researcher:	Great! I hadn't noticed that yet. So that is a "rule" isn't it, because you have to look at each side of the pattern and repeat it until you come to the middle.

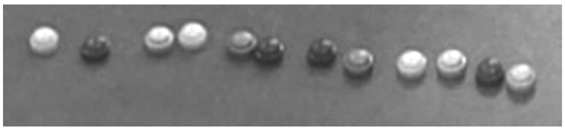


Fig. 7.2 Video frame of Lisa's pattern that mirrors the colors from both ends in towards the middle

The significance of Lisa's pattern is that it highlighted symmetry as a different type of "rule" than what the norm for patterning had been so far. This is interesting for the HLT because it is a different interpretation of the definition of a pattern and it eliminates the idea of "regularity" that the children were struggling with.

Creating a shared vocabulary for patterning. Lisa's unexpected insight into patterning and children's tendency to interpret a pattern in terms of unitarily alternating colors, suggests that children's general confusion about the "rule" versus "regularity" of a pattern could relate to the classroom mathematical practices and the weak context in which the activity was embedded. What is interesting for the HLT is that it shows how the teacher can send the children mixed messages; she was positively reinforcing the children as they correctly continued a pattern that was composed of two alternating colors. Yet, at the same time she was asking them to make their own pattern, meaning one with more colors and a "different rule" than the examples, and asking them to make a "good" sequence. Her choice of words evolved from the idea that a "good" sequence is one that has a "rule" that can easily be extended. Not surprisingly, children tended to make sequences that were similar to the previous ones (e.g., two alternating colors) because that fit their definition of a "good" sequence. This underlines the important role of language and a shared vocabulary in teaching and learning mathematical concepts (e.g., Van Eerde, Hayer, & Prenger, 2008).

From these observations, we deduce that the activity succeeded in evoking an awareness in children of the parts that compose a whole (i.e., the rule of a pattern) in the context of patterning. The children required more time and experience, however, to broaden their interpretation of a pattern to a sequence of any type of repeated element (e.g., color, quantity, gender) and units of combined elements (i.e., the regularity of a pattern). This underlined the role of regularity and symmetry in patterning as characteristics of a pattern that make up “a good line” that is “easy to make”. In terms of refining the HLT, these findings were expected to be sufficient for helping children interpret the contents of the Trick Box in terms of part-whole structures in various spatial contexts.

7.3.2 “The Trick Box”: Exploring types of spatial structures

HLT for the instruction activity

After exploring part-whole relationships in patterns, the Trick Box activity was designed to introduce the children to representations of several types of spatial structures (Fig. 7.3). According to the HLT, we expected awareness of part-whole relationships, which had been the topic of discussion in the previous activity, to help the children understand the composition of a structure (e.g., four dots on a die is composed of two and two dots) and how a structure may be embedded in relatively larger configurations (e.g., two finger patterns of five fingers make a total of ten fingers). To stimulate this connection, the teacher discussed the word “trick” to denote a “convenient or easy way of finding out how many of something there are”. She used these phrases to create a shared focus on spatial structure. According to the HLT, we assumed that such a shared focus and vocabulary would support the continuity of the activity throughout the rest of the instructional sequence. Hence, in discussing the different types of structures in the box, the teacher made use of phrases such as “easy ways” and “tricks” to encourage children to relate the structures to each other. It also anchored the idea of spatial structuring by broadening the scope of spatial structures that children were familiar with.

Analysis and illustrations

Developing a shared vocabulary. The main challenge for the teacher was to guide the children towards becoming aware of and making use of the spatial structures in the Trick Box. For this to succeed, it was first important that the children understood what it means to use an “easy trick” to determine a quantity. Hence, after the children counted the parts of several different objects (e.g., fingers on the flashcards, eggs in the egg carton, and blocks), the teacher explicitly asked them to show her an “easy way” of counting the dots on one of the enlarged playing cards. Several (older) children were already able to share “easy ways” of determining a quantity. One girl showed the class that a card with nine dots can easily be interpreted as two rows of four dots with one in the middle because “then you don’t lose count”. Another boy added that “otherwise you don’t remember which ones you’ve already done and (...) then (pointing to the card) you’ll go all the way across the middle”.

Most of the children, however, firmly held on to their (perceptual or pointing) counting strategies regardless of the teacher's suggestions and questions about finding "easier ways" to count. Given a card with two sets of five dots arranged like dots on dice, one girl, for example, pointed and counted each dot unitarily. When the teacher took the card away, the girl guessed eight instead of ten dots. This is curious because in the pre-interview she had repeatedly shown that she was familiar with the dice structure for five dots. The question is why she did not appear to recognize that structure in this card or, if she did recognize the structure, what prevented her from making use of this insight to circumvent the unitary counting procedure like the teacher asked her to. This supports the conjecture that recognizing structures does not automatically imply that children use spatial structures in various situations (see section 5.2).

Fig. 7.3 Video frame showing how excited the children are to see what is in the Trick Box

As the discussions about the different types of objects and spatial structures continued, some of the children were still confused about what the teacher considered to be "easy" or efficient methods for determining a quantity. It seems that the children kept to their counting procedures because, considering their familiarity with counting, to them that that was an "easy way" for determining a quantity. Some children even described their perceptual grouping strategy as "counting in their head", while these children seemed to have used spatial structuring strategies and, in the pre-interviews, they had explained their perceptual grouping strategy by saying that they "could just see it". Hence, unitary counting was still the predominant classroom mathematical practice while spatial structuring strategies were not self-evident. Apparently the setting was not successful yet in establishing a sociomathematical norm for considering spatial structure to be a valuable solution and an "easy" or "clever" way to determine a quantity.

Another reason why children continued to use a counting procedure could be that they were keen to respond exactly to what the teacher was asking (see paragraph 8.2.4). This could have been the result of some of the teacher's questions. Rather than consistently asking the children to use an "easy way of finding out how many there are", in her enthusiasm she sometimes asked the children to use an "easy way for *counting* how many there are". Early in the classroom discussion, for example, one girl told the teacher that she knew how many dots were on the die. According to the pre-interview, this child fit the Usage phase (phase 3) and should therefore be familiar with most structures and often use them to abbreviate counting procedures. Yet, when the teacher asked her to share her method with the class, the girl pointed to each dot as she unitarily counted six. This suggests that the girl was particularly focused on responding to the teacher's question by literally showing her how she counted the dots. Hence, the teacher's choice of words could have directed the children towards unitary counting procedures.

Comparing spatial structures. To stimulate the classroom mathematical practice of using the spatial structure, the teacher physically took apart or marked the structure, and explicitly compared the structures to each other. For example, as the children tried to explain how they perceived a structure in the dots on a card, the teacher pointed to the structure to highlight the relationship between the dots. Further, in the discussion following how one child had arranged the blocks into five rows of five, the teacher also grouped the rows in two and two and one so that the children could find a "faster way of counting" the blocks. Although several children continued counting the blocks one by one, one child counted the blocks "in her head" and another showed the class how the groups can be seen as "two rows of five which makes ten". At the end, the teacher pointed out to the children that they know that they also have five fingers, even though they don't have to count each finger every time either to know how many there are.

Judging from the children's excited responses, they seemed to have benefited from the teacher's efforts at explicating and comparing spatial structures. Becky, for example, tended to count objects unitarily at the beginning of the classroom discussion. Towards the end of the activity, she gave more analytical explanations. This is illustrated by how, in the focus group, she spontaneously placed the face with six dots on the die next to the playing card showing six dots. Hence, she overtly connected different structures for the same quantity. She experienced another important learning moment when the researcher guided the children in the focus group towards comparing how two objects (the egg carton and a die) represent one quantity. This insight seems to have spurred Becky's motivation, because towards the end of the activity she continued thinking about structure on her own initiative:

Researcher:	(pointing to the dice, the finger pattern cards, the giant cards, the blocks and the egg cartons on the table) So all these things are part of the Trick Box because they'll help us later on for counting things easily. Then we don't have to count everything one by one, but we can just see how many there are!
Becky:	(waking up after staring in the distance while the other children are partly listening and asking to go back to class) But I don't know, if you have a lot (motioning with her hands) of blocks, then I can't count them in my head!
Researcher:	That's right, but then we can arrange them into groups. Take these blocks (points to six blocks) for example, how can you arrange them so you can see how many there are?
Becky:	(arranges the blocks into two rows of three)
Researcher:	So this is a very easy way for you to count them.
Becky:	(pointing simultaneously to every two blocks) And it's also two, two, two.
Researcher:	That's right, then we count them by twos.
Becky:	(pointing first to four blocks simultaneously, then realizing that there were two left and not four, and pointing to the two blocks simultaneously) Or four, four (...) two (...) four and two.

This interaction between Becky and the researcher suggests that Becky was stimulated to explore how different objects can represent one particular quantity (six represented on a die compared to an egg carton) as well as how a quantity can be represented in different structures (six as two sets of three or a set of four and two). Regarding her hesitation in the four and two combination, it seems that she was also challenged into thinking beyond structures such as two sets of three or three sets of two which were relatively more familiar to her. Her enthusiasm appeared in the way she was not distracted by the other children and in how, on her own initiative, she was keen to show the researcher variations for seeing six, despite the researcher's use of the word "count".

Fig. 7.4 Video frame of the focus group discussing how the arrangement of dots on dice can help to "easily see" how many dots there are.

Taken together, the Trick Box context in which the various structures were presented appealed to the children. The children not only spontaneously referred to the Trick Box during the following activities, but, according to the teacher, also outside the classroom (e.g., recognizing egg cartons from the Trick Box). As such, in reference to the HLT for the instruction activity, it appears that the activity succeeded in creating an awareness of spatial structures. Still, the challenge is to disentangle children's understanding of "easy ways" and "counting"; the children must create a shared conception about how spatial structures offer "easy ways" to determine quantities. Such a shared conception is fundamental to the influence of the rest of the instructional sequence on establishing a socio-mathematical norm for spatial structuring.

7.3.3 "Giant Cards": Making use of part-whole compositions

HLT for the instruction activity

This activity built on the first instruction activity by focusing on one of the objects, the Giant Cards, in the Trick Box to investigate its structure in terms of part-whole relationships. This time the children were challenged to make use of part-whole relations to abbreviate procedures for determining the total number of dots on a card. According to the HLT, we assumed that most children would count all the dots on the cards unitarily. The game-like setting, however, was expected to motivate the children to search for more efficient methods of determining the number of dots on the card than unitary counting procedures. Children were called up in pairs and presented with a Giant Card that has structured configurations of six to ten dots (Fig. 7.5).

Fig. 7.5 Video frame of the children who are calling out as fast as they can the number of dots that are on the Giant Card that the teacher is holding up

They were to try to be the first to determine the number of dots by recognizing (part of) the structure. The role of the teacher was to encourage children to use “easy ways” such as those that were discussed in the Trick Box activity. This should have encouraged the children to use relatively more familiar structures such as dice or finger patterns.

Analysis and illustrations

Creating a shared vocabulary. As the class was recalling the previous activity, even the relatively low-achieving children remembered that they had explored egg cartons and were looking for “easy ways” to count the eggs. More significant, is that the children started to use the same phrases as those that the teacher introduced to create a shared vocabulary. Ali, for example, first mentioned “two lines”. Dori continued “yes, from the eggs”, saying that it made it “easy to count with (pointing with her finger in the air) one line here and one line there, together that makes six”. Other children corrected her by saying that it makes ten, but then Dori clarified that she meant “the other one, the one with three and three”. In spontaneously referring to the same type of spatial structure, namely the double-structures, these children showed how a classroom mathematical practice of spatial structuring and the corresponding shared vocabulary was evolving in the instruction experiment.

Lisa is an example of a child who showed progress in the activity while her responses sometimes suggested otherwise. In the classroom discussion during the Trick Box activity, Lisa repeatedly counted each of the dots, despite the teacher asking her to try to use a “trick” to easily see how many dots there are. In the focus group, however, Lisa did recognize the six as “three and three in a row” and she saw the ten as “five and five”. When presented with a card with nine dots, Lisa first counted each of the dots:

Researcher:	(while Lisa is counting) Can you think of an “easy way” to see how many dots are on the card? What do you see in the card?
Lisa:	Well (...) that here (pointing) there are two, here there are two and here there are (counting) five.
Researcher:	Look at that!
Lisa:	And together that makes nine.
Researcher:	That is a good “trick”! So now you can quickly see how many there are.

Although Lisa finally decomposed the structure, she referred to other structures that were discussed in the activity and she counted the five dots to show that there were indeed five. This shows how, although Lisa was aware of various spatial structures and sometimes also made use of them, she seemed to have difficulty describing the structures and she reverted to counting to simplify the question and to show why her answer was correct.

Taken together, the children showed how certain phrases (e.g., “easy ways”; “three and three”; “two lines”) were becoming shared, they appeared to have remembered the essence of the Trick Box activity, and they seemed to be progressing in their use of structure. At the same time, it must be taken into account that this progression is independent of the challenge for children to verbally explain their strategies (Hughes, 1986).

Comparing spatial structures. The effect of the Trick Box context already became apparent when the teacher asked several children to share with the class how they determined the number of dots in the card without counting each dot, and the children spontaneously started comparing the structure of the dots on the card to other structures. One boy compared the card to the corresponding face of the large die. Another boy related the card picturing four dots to his own age. Then someone showed the teacher that the four dots look like the four on the large die. This suggests that the children implicitly or explicitly remembered the comparisons that were made during the Trick Box activity. The teacher encouraged such comparisons by asking the children to think of other objects ("tricks") in the box that show the same quantity as on the card. One object that the children picked out of the box to compare the four dots on the card to, was a card picturing four fingers. The teacher subsequently encouraged the class to show each other what four looks like using their fingers. Some of the children spontaneously continued to explore counting using their fingers even when the teacher was already putting the objects back into the Trick Box. These comparisons within and between various representations of spatial structures extend the previous instruction activity in which the children started exploring the characteristics of the structures in the Trick Box, and contribute to creating an awareness of spatial structuring.

Taking the activity to yet another level, one girl picked the box with butterflies to show the class how it resembled the six dots on the card. She arranged the butterflies into two rows of three, with the first four green and the other two blue. What makes this interesting, is that she showed the class for the first time how objects that are initially unarranged, can be arranged into a familiar structure so that the quantity no longer needs to be unitized, but can rather be read off as a whole. The class could now see how the two rows of three butterflies is analogous to the two rows of three dots on the card, as well as to the two rows of three dots on the large die that was placed next to the card. Furthermore, the girl explained to the class how the six butterflies can also be seen as three sets of two, or four and two distinguished according to the colors. Again, the teacher's enthusiasm about these different strategies encouraged the children to talk about relationships between various compositions of quantities with each other.

This is valuable for children to learn to flexibly work with spatial structures in various settings. This is illustrated by Becky who, in the following episode, explains how she read off the number of dots on a card without having to count them unitarily:

Becky:	(turning the card vertically) But if you have the card like this, then (circling the collection of five dots in the middle) you have five here and (pointing to the two dots above the five dots) two here and (pointing to the bottom two dots) and two here. (circling the bottom four dots). Here are four, but there (pointing to the collection of five dots in the middle) can't be a five.
Researcher:	But you just showed me that there was a five in the middle.
Becky:	Yes, here is two and two and five make nine.

The way Becky decomposed this structure of nine dots, explored a different way of looking at it (in terms of four dots rather than five) and returned to her first explanation, is convincing of her flexible use of structure to gain insight into the number of dots on the card.

In comparing spatial structures, it appears that some children may be at a stage in which they recognize some, but not all structures. Yet, responses from children like Becky suggest that more familiarity with spatial structures can be stimulated through input from other children and through explicit comparisons between familiar and less familiar structures. This was encouraged by the game-like activity in which the children experienced the dilemmas of unitary counting. In this way, children may (a) first learn to recognize (compositions of) structures before (b) exploring how to make use of such structures in larger structured and unstructured configurations.

In summary, this activity has shown how the children came closer to a shared vocabulary for spatial structuring. In contrast to the previous Trick Box activity, the children seemed to understand the meaning of "easy ways" since they successfully spontaneously used these phrases themselves. Further, in building on the explorations in the Trick Box activity, the children compared different representations for spatial structures and spontaneously applied them to different settings. These are important abilities that the children continued to apply during the next instruction activity.

7.3.4 "Filling Egg Cartons": Using part-whole relationships

HLT for the activity

The Filling Egg Cartons activity was the second activity that revolved around one specific object from the Trick Box, namely the egg cartons. According to the HLT (similar to that of the Giant Cards activity), the children were to explore double-structures within relatively larger structured arrangements (i.e., large egg cartons). The added difficulty, however, was that the children would search for structure in an object that is less straightforward to read off than dots on a card. Continuing along the lines of the Giant Cards activity, the children were asked to arrange the eggs in the egg carton in such a way that would make it easy to see how many eggs there are. The children were expected to draw on their experiences with part-whole relations and to compare the double-structures to the structures from the Trick Box with which they were more familiar. Finally, the children were asked to help the farmer find a way to determine as fast as possible how many eggs were missing. Similar to the Giant Cards activity, this required that they tried to find and make use of a structure in the way that the eggs were arranged in the carton. This was expected to help them gain insight into double-structures and into the composition and decomposition of quantities in terms of eggs in an egg carton.

Analysis and illustrations

The convenience of spatial structuring. An important outcome of this activity is that the children tended to stay with their preferred ways for determining the number of eggs in the carton. Perhaps the duration of this activity was not long enough to result in explicit new insights, or the children were tired and not as motivated as during the previous activities. Except for several children who already made use of double-structures, many children preferred to count each egg unitarily and only retrospectively referred to particular structures as “two groups of four”, for example. Despite Mark’s structuring throughout the activity, when asked whether he had an “easy trick” to know how many eggs were left in the carton, he responded “by counting”. His short answer summarizes what was noted earlier in this activity and throughout the previous activities: most of the children were good and fast counters and they needed more stimulation and motivation to consider spatial structuring as a means to facilitate numerical procedures.

Nevertheless, several interesting learning moments occurred as the activity progressed. These moments may have at least sparked children’s awareness of double-structures in egg cartons. When, in the focus group, James was asked how many eggs were left in a carton that contained four eggs on the left, two eggs on the right and four empty spaces in the middle of the carton, he at first counted the eggs unitarily.

Researcher:	James, do you know how many eggs are left in the carton?
James:	(pointing) 1, 2, 3, 4, 5, 6.
Researcher:	Do you know a “clever trick” for seeing that there are 6?
James:	(examining the carton) Yes, because if these (i.e., the two eggs on the left) were there (i.e., in the empty spaces next to the other four eggs), then it is six.

This explanation suggests that he must have realized that the new configuration would again result in two rows of three that he knows make six and that “makes it easier to see”. It seems that this exercise stimulated James into thinking more flexibly about spatial structures; he eventually translated the spatial structures that he was familiar with (in this case the doubling of rows) to a relatively unstructured situation that could potentially be rearranged into rows. Similarly, at the end of the focus group activity, Becky observed that a quicker way would be to “move” the eggs to “see that there are two rows of three”. Her explanation was important for reiterating the intended classroom mathematical practice of making more use of spatial structuring rather than unitary counting procedures.

Interactive learning. The way the children worked together during the classroom discussion appeared to greatly contribute to their understanding of double-structuring. This became evident, for example, when the teacher asked the children to fill the 10-egg carton with five yellow and five white eggs in a way that would make it easier to see that there are ten eggs. First the children unitarily counted the eggs. One girl alternated one yellow and one white egg. Although this did not contribute to an abbreviated way of determining the total number of eggs, it did reflect the first instruction activity in which patterning was also associated with structure. Yet, when the teacher asked another girl to help her, the two girls arranged one white row and one yellow row of five eggs. This shows that they had come up with a way to represent the double-structure as two independent rows in the egg box. Their combined effort resulted in the successful construction of a double-structure.

Comparing spatial structures. To ensure the continuity of a focus on spatial structuring throughout the activities, the teacher started a discussion that related the structure of the egg cartons to other structures in the Trick Box (Fig. 7.6).

Fig. 7.6 Video frame of the children exploring double-structures in the egg cartons and compare them to other objects in the Trick Box

One girl interpreted the structure of a 6-egg carton by tracing the two rows of three in the air with her finger and recalling that the double-structure of three makes six. Then a boy took the large die from the Trick Box and placed it next to the egg cartons with five dots faced upwards. As such, double-structuring, which was to become taken-as-shared in this activity, was concisely associated with the dice dot configurations that the children had encountered previously in the Trick Box and Giant Cards activities. This comparison is a valuable tool for translating insight into familiar structures to relatively unfamiliar structures.

Although not all the children in the class reached this level during the activity, they were at least exposed to ways of structuring the eggs as they witnessed how the two girls rearranged the eggs. As such, this activity provided all the children with opportunities to practice double-structuring at their own pace, whether that involved practicing counting, recognizing double-structures, using double-structures or constructing double-structures. For instance, as the teacher asked the children to use their fingers to show the number of eggs that were missing or that were left in the carton. This was good practice for children who still had difficulty showing eight on their fingers. Just like at the end of the Giant Card Game, the teacher involved the whole class by asking all the children to raise the number of fingers that corresponded to the number of eggs that were left in the carton. In this way, she explicitly tried to relate two types of structures (i.e., double-structures and finger patterns) to their structured representation of one quantity (i.e., eight).

In summary, although the children tended to prefer their familiar counting strategies, as the activity progressed, it appeared that they increasingly appreciated spatial structuring. This implies that they understood better that the resultant structures would contribute to "easy ways" of finding out how many eggs were missing or were still left in the carton. This insight was stimulated by how the children worked together to solve a problem. Moreover, through comparing egg cartons to the objects in the Trick Box, the children again associated familiar with relatively unfamiliar spatial structures to foster flexible spatial structuring strategies.

7.3.5 "Huts and Castles": Spatial structure and three-dimensional constructions

HLT for the activity

In the Huts and Castles activity, the children were stimulated to make use of the structures of the objects in the Trick Box to determine the number of blocks in a structured (symmetrical) and in an unstructured (asymmetrical) 3-D block construction. For this, the children were expected to draw upon their insight into part-whole relationships to decompose each 3-D construction into its component parts and subsequently compare the structures. Hence, the objective of this activity was to translate children's insight into patterning and structuring to a more spatial, 3-D setting. This was to help stimulate children's flexibility in manipulating structures and in applying them to abbreviate and facilitate numerical procedures.

The children were expected to either count each block or to partially count them and determine the rest of the blocks through reasoning without counting. According to the HLT, we expected that the children would have more trouble determining the number of blocks in the ten-block construction compared to the five-block construction, regardless of whether they were counting or using the structure of the constructions. Those children who use structure for the five-block construction, could even revert to counting strategies for the ten-block construction.

By encouraging the children to compare their experiences for the two types of constructions and between the two sizes of the constructions, the teacher could support children's insight into what it is that makes a construction "easy" or "difficult" to count. Taking this one step further, the children in the focus group built one construction that was "easy to count" and another that was "difficult to count". The aim was to see whether the children could represent what they understand to mean "easy" or "difficult" to count and whether this implicitly or explicitly involves structure. The discussion that followed was meant to inspire the children to rebuild the blocks into more or less structured (i.e., "easy" and "difficult") constructions.

Analysis and illustrations

(De)composing the structure. The teacher played an important role in this activity by how she guided children towards decomposing the constructions to elucidate their structure (Fig. 7.7, see also section 8.3). She simplified the question by first asking the children how many blocks there were in the top part of the structure, and then encouraged them to use structure to determine the quantity:

Teacher:	How many blocks are at the top?
Becky:	(examining the structure) Four.
Teacher:	Four, very good. How do you see that so quickly?
Becky:	There (tracing with her finger along the vertical edge of the two layers of two blocks) there is a line and two plus two is four.
Teacher:	Very clever. So you don't have to touch them (pointing to the blocks one by one), 1, 2, 3, 4, but (pointing to the two layers of two blocks) you see two and two. Very good. So there are four at the top. How many are on the bottom?
Daria:	(after several children have had a chance to count the blocks) Six.
Teacher:	How did you see that? Very good.
Daria:	(pointing to each row of three) Because here there are three and here there are three and that makes six.
Teacher:	Very clever of you. Here there are three and there are three, let me help you.

(removes the top four blocks from the bottom) Do you see that these are six blocks? (spreads the six blocks apart so the structure becomes more apparent)

Children: Yes, six!

Teacher: Now I'm going to put it back together and I'm going to ask you how many there are altogether.

Simon: That makes ten.

Teacher: Right, and how do you know? How many were on the bottom?

Simon: Six.

Teacher: And how many are on top?

Simon: Four.

Teacher: Right, and that makes...

Simon: Ten!

James: And you know (pointing to the blocks), if this is four and this is three and this is three then that makes ten. And then they are also like two fives (showing all fingers on both hands)!

This episode highlights a perceptual way of analyzing the configuration of the blocks that is effective and efficient for determining the quantity. By revoicing the children's responses, attending to the inconvenience of counting "1, 2, 3, 4", and offering the alternative of counting by, for example, twos, the teacher showed the children that questions about determining quantities do not always require unitary counting. She offered spatial structuring as a valuable, alternative approach. This exercise also appealed to children who were at different levels of understanding because it offered them the opportunity to count smaller structures (four and six), to combine these structures into a larger whole, as well as to think about how this structure compares to other structures (e.g., two fives).

Fig. 7.7 Video frame of Lara determining the number of blocks in the structured ten-block construction

This method of decomposing a construction was also helpful for determining the number of blocks in the unstructured 10-block construction. Many of the children were confused in their counting by the orientation of the blocks and by how blocks were covered by other blocks. One girl suggested to start counting the blocks from the top so the teacher first removed the top two blocks. When the girl was still confused and another girl came to count the blocks, the teacher asked her to remove more blocks. With only six blocks left in the construction, the girl was then able to count each of the blocks. Together, the class subsequently determined that six and four blocks makes a total of ten. Although in this case the children counted unitarily, this exercise again showed the children how a large unstructured construction can be decomposed into its component parts to facilitate the counting procedure.

Creating a shared vocabulary. The decomposition of the blocks was important because the children had trouble determining the blocks of even the structured construction. As a result, they did not notice enough difference between the two constructions to help them become more aware of the role of structure in determining a quantity. Even those children who typically spontaneously made use of structure in the other contexts, reverted to counting the blocks unitarily. A reason for this may be found in the directive wording of the teacher's questions. When the teacher asked a girl how many blocks were in the construction, for example, she also asked the girl to think of "an easy way, an easy trick for how she could count it" and she encouraged the girl to come up to the construction "to count it, or to point to it, or to look at it and to think of an easy trick". As the girl was counting, however, the teacher referred more to the structure of the blocks and told her to "look at which blocks you can see". The following episode is an interesting interaction that illustrates how important the wording of a question can be for the type of answers to expect. At the start of the activity, one boy was particularly keen to show the class how he determined the number of blocks in the construction.

Teacher:	How many blocks do you think these are?
James:	Five.
Teacher:	You think there are five.
James:	(pointing and counting from where he is sitting) Yes, see, 1, 2, 3, 4, 5.
Teacher:	Allright, come and count them. Go and have a look.
James:	Here there are 1, 2, (...) but actually I can also show you how I could easily see that there are five. (Pointing to the blocks) Because here there are two and there are two and one on top so there are five.

Hence, although the teacher asked James to come and "count" the blocks, as he started counting he seemed to experience a moment of reflection and changed his mind to share with the class how it would be easier to analyze the structure instead of unitarily counting each block. In this case James was confident to suggest more than just a counting strategy and he actually showed the teacher what she was looking for. This contrasts with many children who did not answer beyond the literal question, even though they had originally used more sophisticated strategies than unitary counting.

Children's interpretations of an "easy or difficult way to count something" were reflected in the focus group activity. James, for example, had constructed "a rocket" (i.e., three blocks with space in between on the bottom, followed by three blocks, two blocks and finally one block on top). He explained that it was easy to count the rocket because "1, 2, 3, 4, 5, 6, 7, 8, 9" as he pointed to each block on the front face of the construction. His counting procedure was indeed an easy one because the construction was only a single layer of blocks. In James' terms, a "difficult" structure is one "with too many blocks where you lose track of count". Apparently James had found an "easy way" to count the number of blocks, but the question had not been accurate enough, because James only made use of the spatial organization (i.e., an arrangement that supports unitary counting) of the blocks rather than the spatial structure (i.e., an arrangement that can be read off) of the blocks.

Lisa, Mark and Becky in the focus group had other interpretations of what makes a construction "easy" or "difficult" to count. Lisa explained that her "difficult" construction was difficult because "the blocks are a little mixed up", and because she had trouble keeping it together. Towards the end of the activity, Mark had constructed two equal sized constructions, one of which he considered more "difficult" to count than the other. The reason was that for the "easy" construction he could "see that three and three makes six but for the other it's harder to see". Throughout the activity Becky made several pairs of constructions using exactly six blocks for both constructions. This indicates that to her the difference between the constructions did not only depend on the number of blocks that they were made up of. For one of her "difficult" constructions, she explained that "if they're all on top of each other then it's really difficult".

The focus group was also successful at identifying which of someone else's two structures represented the "easy" or the "difficult" constructions. In particular, they considered an "easy" construction to be one that could literally be counted easily because the blocks were visible to them. A "difficult" construction, then, is one with blocks that are less perceptually available. In practice this means that an "easy construction" is less layered than a "difficult" one and that a "difficult" construction is also more complicated to build because of its asymmetry. This outcome resembles that of the "Giant Cards" activity because children sometimes require a stronger motivation for structuring if their counting strategies are effective enough to answer the questions correctly.

Comparing spatial structures. Similar to the previous activities, to become more aware of the evolving practice of spatial structuring, the teacher brought the Trick Box into the discussion and connected this instruction activity to the rest of the instructional sequence. For example, the children were asked whether they knew which egg carton resembled the bottom layer of the large structured construction. The teacher had taken apart the construction to highlight two rows of three and emphasize the composition of the structure. Indeed, one boy placed six eggs as two rows of three in the egg carton, explaining that this structure was necessary because “otherwise you lose track of how many you counted”. The way he connected the block construction to the rows in the egg cartons, suggests that he understood structure to be the common factor in both activities. Moreover, he may have set an example for other children in the class who had not yet associated the block structures with other structures in the Trick Box.

Taken together, the class showed implicit and sometimes explicit insight into the structure of a 3-D construction in this instruction activity. The teacher was particularly helpful in guiding the children to physically take apart the construction and analyze its component parts. This stimulated the children’s ability to (de)compose the 3-D block construction by the end of the session, which meets the HLT of this activity. Importantly, the teacher’s guidance also highlights the central role that language plays in stimulating children’s learning. Hence, to prevent the formulation of questions that guide the children in an undesired direction, it helped to ask children to compare the constructions to the Trick Box, and to construct, rather than tell about, their own “easy” and “difficult” structures. As a result, many of the children had gained more insight into various types of structure and their use for determining quantities in a variety of (3-D) settings.

7.3.6 “Picking Flowers”: Using and applying spatial structures

HLT for the activity

In the previous instruction activities, the children progressed from gaining insight into part-whole relationships (Activity 1), to exploring and comparing different types of structures (Activity 2, 3, 4, 5), to learning to make use of the structures in larger structured arrangements (Activity 3, 4, 5). This final instruction activity challenged the children to apply their own structure goal-directedly for abbreviating and facilitating numerical procedures such as determining, comparing and operating with small quantities.

The activity involved a garden (i.e., sheet of green paper) with flowers (i.e., ten uniformly colored plastic chips). The teacher told a child to look away while she picked a number of flowers from the garden. The child was to determine how many flowers were missing; for every correct answer, the teacher returned the flowers and planted an extra one, but for every incorrect answer, the teacher kept the flowers. The challenge for the child was to find a way to arrange the flowers so that it would be easier to keep track of count. The game-like setting was expected to motivate the child to search for a strategy to win the flowers. At the same time, with more wins and more flowers added to the garden, the child was expected to experience the need to structure unstructured quantities. Rather than imposing a particular structure on the child, the teacher was to acknowledge the children's strategies and encourage them to look for more effective strategies that could involve structuring the flowers.

Analysis and illustrations

Spatial structuring as a socio-mathematical norm. At the start of the activity, the children remembered that the Trick Box contained "things that we can count" and the aim was to "think of a way to be able to count correctly" or, in other words, to "think of an easy way for counting the things in the box". This episode shows how, from an individual perspective, the children understood the intention of spatial structuring. From a social perspective, it compiled many of the terms (e.g., "a trick", "a useful way", "to count it easily", "to quickly see how many there are", "structure", "predict") that the teacher has been using and that have become part of the shared vocabulary to indicate the convenience of spatial structuring. This highlights the extent to which spatial structuring had become a socio-mathematical norm for determining and comparing quantities, as a result of the children's experiences throughout the previous instruction activities.

Spatially structuring the flowers. According to the HLT, an important aspect of children's learning, is to acknowledge their own structures and to guide them towards more effective structures. As expected, at first several children counted the flowers unitarily, with differing outcomes. When one girl finally correctly said that there were twelve flowers, the teacher asked her to arrange the flowers "in a different way so that the rest of the class could easily see that there are twelve". The girl's solution was to place the flowers next to each other in a long S-shaped line. This was her own structure, one that she thought made it easy to see that there are twelve flowers. This structure agrees with how, in the previous activities, the girl primarily preferred to count the objects. Hence, this shows the girl's primary use of structure as a way to organize objects for unitary counting.

Becky showed a more sophisticated strategy for spatial structuring in placing the chips into two rows of six (Fig. 7.8). When the teacher asked her how this structure helped her "to count the chips easily", Becky showed the class how she could count the chips by twos. It is not clear whether Becky originally intended to count the chips by twos, or whether she could also recognize the two rows of six as twelve from earlier experiences with, for example, the egg carton activity. Possibly, the teacher's question about "counting" motivated her to show a counting procedure rather than the structuring procedure that she may have had in mind. Nevertheless, Becky's structure proved to be an effective structure for reading off a quantity because, after the teacher picked some flowers out of the garden, she successfully determined that there were three chips missing. Becky had clearly made use of her structure because she was very quick at answering and she confidently pointed to the empty spaces where the chips were missing. The teacher used this as a welcome example to show the objective of the activity to the class, and it contributed to the evolving sociomathematical norm of spatial structuring.

Fig. 7.8 Video frame of Becky as she is arranging the flowers in her garden into two rows of six so that she will easily be able to determine how many flowers the teacher picked from her garden

Noting the general progression in children's spatial structuring, the teacher was quick to summarize the insight that she hoped the children had gained from the first two examples:

Teacher:	So how come you were able to see how many flowers were missing? Simon, do you know? (Simon does not answer)
Teacher:	I'll ask Jamal. How do we know so quickly how many are missing? (Several children are keen to answer)
Jamal:	Because we saw it.
Teacher:	Because you saw it. But how can I see it quickly? What should we do first? (walks over to the garden) If I do this (moves all the chips together), and I would ask someone to close their eyes, and I take these two away (removes two chips from the garden), Lara, could you then easily see where I had taken them away?
Most children:	Yes.
Lara:	It's not so easy.
Teacher:	And why not then?
Lara:	Because you moved them.
Teacher:	I didn't move them. But are they placed easily?

Lara: No.

Teacher: (Looks around the circle) Their arrangement is not easy for counting. That's why I just asked you to make a "nice" figure out of them. Rick made a group like this and like this (arranges the chips into two groups of 4). But if it's like this (makes one group again), then you can't count so easily, and then you won't know where they were taken away. So it's best to think of an "easy way" to arrange them, so that you don't have to count them.

From this episode it appears that the children did not actively respond to the teacher's explanation because they were primarily listening and filling in some of the teacher's sentences. Although they had explored their own types of structures and showed progression, this illustrates how they did not actually get a chance to fully experience structure. Moreover, when the teacher summarized the activity by saying that "it's best to think of an easy way to arrange them, so that you don't have to count them", her choice of words was not very clear because it mentioned ways to "count" the number of chips rather than ways to "determine" the number of chips. This again illustrates the subtle difference that choice of words can make on children's choice of strategies.

The children in the focus group showed more meaningful structures (Fig. 7.9). They first arranged the chips in ways that they thought were "fun" or attractive; placing the chips in rows, along the edge of the paper, all in one pile or spread out across the paper. James, for example, used the counting-on strategy. He pointed and unitarily counted each chip, remembered that originally there were twelve and concluded that there were four chips missing.

Fig. 7.9 Video frame of the children in the focus group who are ready to test their spatial structures

The researcher pointed out that it was quite difficult to know how many chips were missing in James' garden. She then asked James to think of "an easy way, a clever way to properly see how many chips are taken away". He responded immediately by aligning the chips into a figure that resembled the capital letter "T", saying that he could make a "t" out of it because "then it's even easier". This illustrates how the children organized the flowers but not always with the aim of abbreviating the counting procedure because they would still count each of the flowers unitarily.

Building on the children's original structures, each turn was presented as an opportunity for the children to improve the configuration. As the focus group activity progressed, more children succeeded at using structure to determine the numbers of flowers in a gardens. When the children were asked whether it was easy to remember the structure that Mark had placed his chips in (i.e., one chip in every corner and two rows of four lined up vertically in the middle), Becky remarked that it would be easier if he placed the chips closer to each other. She showed him what she meant by moving the chips in the middle into two rows of four. After four chips were removed, the children first all thought that two chips were missing, referring to one chip on the corner and one in the middle of the centre structure. Suddenly Becky realized that there were four missing and she pointed to where the two chips on the top of the two rows of four originally were. Similarly, Lisa made three groups of four and one while Jenna made four rows of three chips. Independently of one another, the children used their structures to determine the number of chips that were missing. This shows how, towards the end of the focus group activity, children of various abilities in the focus group successfully devised and applied structures that were effective for determining the number of missing flowers.

Interactive learning. The children were repeatedly guided with questions that stimulated them to "think of a way of arranging the chips that would be easier to remember" and see how structure may or may not have contributed to determining the number of missing flowers. This kept the children motivated to surpass their own and other's structures. Becky, for example, seemed to have taken note of confusions in Mark's structure, in which the flowers were spread apart, because she arranged her own flowers that were placed on the edge of the paper closer together. She also wanted to "take a good look first" at her structure before looking away. This indicates that she realized the importance of remembering the configuration rather than counting the flowers. Indeed, Becky was the first to be successful at using the arrangement of her chips to determine the number that was missing without counting. Further, she intrigued the other children because they stopped playing with their chips to look and listen, and to see how their arrangements differed from Becky's structure and how they could apply more structure to their own arrangements.

In summary, throughout this activity, the children exhibited the socio-mathematical norm of spatial structuring in how they used the shared vocabulary to structure their flowers. Although many children started with arrangements that appealed to them without abbreviating counting procedures, the activity allowed them to experience the convenience of spatial structure. In fact, most children were not only using the structure to determine the number of missing chips in their garden, but, encouraged by the teacher, they were also challenging each other's structures and trying to come up with one that would make it easiest to read off a quantity. This suggests that they were (becoming) aware of the advantage of spatially structuring their chips to determine a quantity. Other children may still have been at the stage of practicing their counting skills, but they were at least exposed to various ways of spatial structuring and to spatial structuring as an alternative to unitary counting. These outcomes conclude a trajectory of instruction activities that have guided the children from unitary counting towards learning to recognize and apply spatial structure for abbreviating procedures for determining, comparing and operating with small quantities.

7.4 RME in a Kindergarten setting

The results of each instruction activity were described in the previous section. To understand how these observations could contribute to refining the conjectured local instruction theory and to improving the instruction activities for a second round of the instruction experiment, we organized the video data using ATLAS.ti (see also section 3.4). As more and more clippings of the videos were created and discussed with experts, several patterns emerged that gave insight into the role of the learning ecology (Cobb et al., 2003) in children's learning. These patterns can be generalized as follows:

- The context within and between the instruction activities must not only be meaningful and inspiring to the children, but also offer productive and situation specific strategies.
- A shared vocabulary, the type of language use of the teacher, and the way children interpret language is essential.
- The instruction activities relate the children's levels of understanding to a learning goal while acknowledging children's own productions.
- The teacher is an essential factor in promoting interaction between the children.
- The socio-mathematical norm of spatial structuring promotes a shared awareness of spatial structure in mathematical practices.

In what follows, we explain how these patterns influenced the shared awareness of spatial structuring, and how they may supplement RME principles for a Kindergarten setting (see section 2.5).

Context, shared vocabulary and language use in a Kindergarten setting. Realistic Mathematics Education principles state that an instruction activity should be embedded in a context that creates opportunities for the children to engage themselves in the problem (Treffers, 1987). Such an overarching context can help bridge the activity with the child's meaningful reality because the design of the activity starts from the perspective of the child and reaches towards the curricular expectations rather than vice versa. Moreover, an overarching context contributes to establishing a shared vocabulary that can help to bridge the topics of discussion between two subsequent activities. As such, an overarching context can create a shared frame of reference for recognizing the role of spatial structures in a particular activity.

From the observations of this round of the instruction experiment, it appeared that the children neither always understood the problem, nor did they seem to experience the advantage of spatial structuring over unitary counting strategies. Apparently, the classroom mathematical practices for determining a quantity were still predominantly related to unitary counting, because the socio-mathematical norm for spatial structuring was not established well enough yet. Moreover, the instructional sequence lacked an overall context to connect the activities together. The context had to be appealing to the children if they were to be motivated to take part in the activity. More importantly, however, is that it had to offer mathematical content in terms of giving rise to productive and situation specific strategies: productive in the sense that the strategies help to abbreviate numerical procedures, and situation specific in the sense that the strategies derive from a particular context (cf. De Lange, 1987).

Next to a shared vocabulary, it became clear that the teacher should be aware of how children may interpret the formulation of a question to make sure that they properly understand the activity. For example, the children often counted unitarily if the teacher asked them to find a way to "easily count" the objects. Yet, they tended to look more for structure if the teacher talked about finding a way to "easily find out" or "see" the number of objects. The classroom mathematical practice of spatial structuring therefore strongly depended on the socio-mathematical norms that are embodied in the shared vocabulary.

Children's own productions. In line with RME principles, the observations show how the teacher must remain aware of the children's own solutions to problems and welcome these solutions as a first step towards developing more effective or efficient strategies. This awareness shifts the role of the teacher from showing and teaching children about structure, to letting the children themselves find out what kinds of structures appeal to them. During the "Picking Flowers" activity, for example, some children started arranging the flowers into one large flower. This was an effective structure for a small number of flowers, but as the number of flowers in the garden increased, the children experienced the shortcomings of their structures and consequently learned about more effective spatial structuring. The children would probably have had more difficulty grasping the need to spatially structure a set of objects if the teacher had immediately shown the children how to arrange the flowers into rows, for example.

Guided reinvention in a Kindergarten setting. While the observations above generally find support in RME principles, other observations also give the principle of “guided reinvention” a different character in the Kindergarten setting. The children’s interaction with the teacher exemplified “guided reinvention” in the way that the teacher asked the children to explain each other’s solutions and in the way she supported the children’s solutions by pointing to the objects and by taking apart a block construction to elucidate its structure. Further, by asking two children with different mathematical abilities to work together, the children could compare spatial structuring strategies and the relatively high-achieving child could set an example for the relatively low-achieving child.

Still, given the age and relatively short attention spans of these children, the teacher had to do more than just “guide” the children. The teacher had to play a more directive role in encouraging children’s interaction with each other and with the teacher, and in stimulating children to reflect on the activities to understand spatial structuring (cf. Leseman, Rollenberg, & Rispens, 2001). This is because a Kindergarten instructional setting involves more hands-on activities and vertical interaction (i.e., between the teacher and one child or a group of children) than the verbal and horizontal interactions (i.e., between the children) that typically occur in classes with older children (Hatano & Inagaki, 1991; Nelissen, 2002). Hence, although children of different levels of mathematical abilities benefited from interacting with the teacher and with each other, these interactions were dependent on the teacher’s stimulation to encourage children to mutually share and compare their strategies. This illustrates how language and influences of a classroom culture play a greater role in Kindergarten than is typically described in the RME principles.

Establishing socio-mathematical norms. The guided interaction between teacher and child stimulated kindergartners’ awareness of spatial structuring and helped to establish the socio-mathematical norm of spatial structuring. The most important socio-mathematical norm that spanned across this sequence of instruction activities is what the teachers considered to be appropriate explanations for children’s answers. The teachers were aware that they should note any instances of spatial structuring that the children may have used to solve the instruction activities. Hence, the teachers tried to gain insight into the children’s strategies by guiding them in learning to formulate explanations that would involve spatial structuring. Unfortunately, in some cases where the children had not clearly used spatial structuring strategies, this norm defeated its purpose because it confused the children or frustrated the teacher who was expecting a more “structured” answer. At other times, however, the teacher’s explicit focus on using structure stimulated children’s awareness of structure in solutions. As such, this socio-mathematical norm greatly influences the teacher’s type of questioning and highlights episodes in the activity during which the teacher and the child made explicit reference to spatial structure in their solutions.

In the remainder of this chapter, we describe how these patterns were taken into account to improve the instructional sequence for Round 2 of the instruction experiment. One of the most outstanding revisions, the overarching context, is discussed in the next section.

7.5 An overarching context

In light of what was learned from the RME principles in a Kindergarten setting, we developed a context that united the activities in the sequence in terms of both an appealing and a mathematical content. The Trick Box was taken as the starting point because it already served as an important source of reference with a mathematical content in Round 1; the teacher repeatedly asked the children to “think of a trick in the Trick Box” that could help them “see how many there are”. Yet the teachers noted that the Dutch word “trucje” (i.e.,“trick”) could confuse the children because it is associated more with magic than with mathematics. They suggested the Dutch word “maniertje” (i.e., “way”) to denote a convenient procedure for determining a quantity.

As such, the name of the Trick Box was changed to “ManiertjesDoos” (i.e., “Tool Box”; Van Nes & Doorman, 2009). Moreover, we set out to devise a context that could connect the activities in the sequence based on (a) a continuous story, and (b) shared references to spatial structure within and between the various activities. Indeed, the teachers remarked that the activities involved “too many stories” that were not interrelated, and that this could confuse the children. To connect to the children’s levels of mathematical understanding, we assumed that children should build on their existing body of knowledge. As the teacher referred to the Tool Box during every activity, asking the children for various “ways” to “conveniently determine how many of something there are”, the children were expected to make reference to the spatial structures in the box and to begin to make use of their new insights into spatial structures for abbreviating the numerical procedures in the instructional activities.

In addition to the Tool Box, the character “Miertje Maniertje” (i.e., “Ant with his Tools” in Dutch) was developed to accompany the children throughout the activities (Van Nes & Doorman, 2009). Ant has a name that rhymes and is easy to remember in Dutch, and the character itself appeals to the children's imagination. Another important consideration was that Ant has six legs, which highlights the important symmetrical six-structure on dice and in egg cartons. Finally, ants are known for their physical strength, and that, in theory, they would be able to carry such a box into the classroom (Fig. 7.10).

Fig. 7.10 Miertje Maniertje and the Maniertjesdoos (Ant and its Tool Box)

Ant's introduction to the class was intended to impress the children so that they would be motivated by the context throughout the subsequent activities (Fig. 7.11). Before the children came to class, the box was set in the middle of the circle, Ant was hidden on a bookshelf, and several sheets of paper were placed on the floor leading from the door of the classroom to the box and from the box to the book case where Ant was hiding. Six black dots were drawn on each paper to represent the footprints that Ant had left behind. The children started questioning the setting as soon as they entered the classroom. One of the children followed the footprints to see what she could find. This was the first encounter with Ant. The children then investigated Ant and studied whether the footprints belonged to Ant, and whether Ant had carried the box into class. Subsequently, the children unpacked the box to see what Ant had brought into the classroom. This set the stage for the mystery that underlied the rest of the activities: why did Ant bring this Tool Box to class and what are these Tools for?

Fig. 7.11 The class is excited about the footprints on the floor and the colorful box in the centre of the circle

Each instruction activity could now be embedded in a context that relates to Ant and its Tool Box and that makes reference to previous activities that involve Ant. This quickly focused the children on the ongoing topic of discussion (i.e., structuring and counting), it tied the activities together (i.e., having the children refer to new insights from the previous activities), and it encouraged the children to make references to the "tools" (i.e., the structures) in the box. In this way, the overarching context not only offered an appealing content, but also a mathematical content that offered productive and situation specific strategies (cf. De Lange, 1987). In the next section we elaborate on what other revisions were made to each instruction activity.

7.6 Improvements to the instruction activities for Round 2

The previous sections describe the patterns and issues that came up as the sequence of instruction activities in Round 1 was tried out in the classroom. In this section, we present the revisions that were made to each activity based on the outcomes of the first round, including the overarching context of Ant and its Tool Box. The manual with the final instructional sequence can be found in Appendix 5.

7.6.1 Activity 1: The Tool Box and Ant Steps (i.e., De Maniertjesdoos en Mierenstapjes)

Round 1 of the instructional sequence started with a patterning activity to build on the importance of children's patterning ability for spatial structuring. This was expected to underpin the subsequent activities by guiding the children towards thinking in terms of spatial structuring; the children could continue to make use of the context of "predicting" when they analyze the structures in the original Trick Box activity. Yet our observations suggest that the contextual links between the activities did not extend beyond two subsequent activities. Such an overarching link is important because the children were expected to make explicit or implicit reference to earlier activities. That did happen with the introduction of the Trick Box because the tricks in the Trick Box motivated the children and created a shared frame of reference. Still, the Trick Box served more as an illustration than as an overarching context. In contrast, Ant and its Tool Box were to be introduced in the first activity and subsequently woven into the remaining four activities. This overarching context provided a recognizable setting that the teacher could refer to in every activity by making use of a vocabulary that the children and teacher came to share.

The second revision that was made to the original Trick Box activity has to do with the observation that there were too many sample "tricks" (i.e., objects representing spatial structures) in the box. It took too long to discuss each object and not all objects were equally related to spatial structures. It was therefore decided to include only the most familiar structures (i.e., the finger pattern flashcards, the egg cartons, the bead necklaces and two large dice) and to leave out the less common structures (i.e., the blocks and the set of butterflies). This abbreviated the original Trick Box activity, so for Round 2 it could be combined with the original Highest Card activity.

The cards in the Highest Card activity were no longer related to giants, but rather to the six (two rows of three) footprints that represent the many footsteps that Ant left behind as he carried the box into the classroom. This resulted in one activity that first introduced Ant and the Tool Box to the children, and that then showed them how they could make use of the Tool Box. This sparked the mystery of exploring why and how Ant wanted to help the children with the contents of the box. Consequently, Round 2 encompassed five instead of six activities.

The contribution of the Tool Box activity to the research questions is that it was fundamental to the effects of the rest of the instruction activities in the research. The assumption was that, once the children were intrigued by the context and excited to learn more about the "tools" in the Tool Box, then they would be more motivated to discover the use of these “tools”. The mathematical content of the Tool Box should also challenge the children to try to make use of the spatial structures in the rest of the instruction activities. As such, the context allowed for meaningful and mathematical connections between the activities.

The Ant Steps part of the activity had a more direct contribution to answering the research questions. Just like in Round 1, this activity was expected to stimulate children’s ability to recognize spatial structures in relatively larger structured arrangements. To support them in this, the children were encouraged to make use of the “tools” that they had explored in the Tool Box earlier. Children may recognize five dots on a card, for example, as the same arrangement as five on a die. Such comparisons between various structures for one particular quantity should support children in learning to flexibly recognize spatial structures.

Since only a two month period separated the second round of the instruction experiment from the first, we assumed that the children would recognize some of the props and questions in the activity. Yet, the children were still expected to be intrigued by the new version of the activity (i.e., meeting Ant and finding out why he left the Tool Box in the classroom). It was particularly important for the teacher to introduce the new context in such a way that the children would be keen to go along with it and to substitute the idea of the Trick Box with the Tool Box. Again, the reason for repeating the activities is not merely to test whether the context works, but more to find out how it works or how it may be improved to interweave better with the children’s interests and conceptual understanding.

7.6.2 Activity 2: Filling Egg Cartons (i.e., Eierdozen Vullen)

The main issue that came up in this activity after Round 1 was that the children did not seem to experience the need to use (double-)structures to determine the number of eggs in the carton. Hence, the challenge for Round 2 was to motivate the children to make use of the egg cartons’ structure. The story about Ant helping Farmer John continued the context about how Ant was showing the children “ways” to quickly see how many of something there are. The first question involved six eggs to relate to most children’s familiarity with the six-structure (after subitizing). The discussion that followed was expected to encourage the children to think more about spatial structures and to experience the convenience of structuring compared to counting the eggs unitarily. As such, the children were repeatedly stimulated to associate the egg carton structures with the contents of the Tool Box.

The aim of the revised activity was to support the children in experiencing the convenience of using double-structures. Hence the focus would be less on recognizing a structure and adding on the rest (e.g., five with six, seven, and eight), and more on recognizing the components of a structure to subsequently recognize the structure as a whole (e.g., two rows of four). Such conceptual subitizing could help bridge children's natural ability to perceptually subitize quantities up to four, with formal addition and subtraction abilities (Clements, 1999a). This requires it to be clear for the children whether the goal of the activity is to find easy methods for adding and subtracting, or whether it would be better to try to recognize the overall structure so that just the structure can be read off. Therefore, more than in Round 1, the questions were formulated in ways that asked the children to read off the structures.

Another revision was to include a game-setting such as in the Ant Steps activity. The children were to structure the eggs themselves and to anticipate how someone else would interpret the arrangement of the eggs in the carton. We expected that the responsibility of arranging the eggs for someone else would add to the children's excitement and motivation to find "clever ways" of arranging the eggs that were to be counted.

7.6.3 Activity 3: Marching in a Procession (i.e., In Optocht)

The main difference between Round 1 and Round 2 for this activity was its position in the instructional sequence. Given the importance of patterning for developing spatial structuring ability (Papic & Mulligan, 2005; Waters, 2004), the activity was initially planned as the first of the six in Round 1 to set the stage for "predicting" the rest of a structure rather than unitary counting. The main concern that arose after performing this activity, however, was that it was unclear whether the children had really come to understand the essence of a pattern or whether they merely enjoyed repeating two colors in a sequence. To gain more insight into children's understanding of patterns, the children were shown part of a pattern and were asked whether they thought that more of one particular element or the other would be necessary to extend the pattern. The children who understood a pattern were expected to know that a pattern consists of a structure that is repeated and that the structure is the foundation for the rest of the pattern. When determining whether more of one element is needed than another, it would be sufficient to study that particular part of the pattern that is repeated. We expected children who lacked such insight to depend more on the physical availability of the whole pattern to determine of which of the elements there are more. They could also base their answer on all the elements that were available rather than only on the repeating structure.

For example, given a necklace with two red, three white, two red, three white and two red beads, more white than red beads are necessary to continue the necklace. Children may give this answer if they see that the structure that makes up this pattern is the repetition of two red and three white beads. Children who lack this insight, however, may be distracted by the last two red beads and base their answer on the entire collection of beads rather than on the structure of the necklace. They will see that there are now six red beads and six white beads and they may therefore conclude that just as many red as white beads are necessary. This answer may indicate that the child has not focused on the structure of the pattern, but rather on the individual elements.

To connect the activity to the rest of the instructional sequence, the introductory story was about ants who typically do not walk alone because they prefer to walk in line (a procession) with other ants. The children were to act out how red and black ants walk in a procession together and to try to make the procession longer according to the “special way” in which the ants were arranged (Fig. 7.12). After adding several children to the procession as ants, the children were asked whether they knew what type of ant was represented more in the procession than the others. This combination of extending the line and predicting the rest of the sequence was expected to support the children in understanding the essence of repeated parts of a whole that are inherent to a pattern (e.g., Papic & Mulligan, 2005). In light of this context, this activity was planned to start with some examples of lines that are made up of children (the ants), followed by lines that are characterized by the colored papers that the children are holding, and finally back to colored chips to symbolize the lines that the children had been a part of earlier. Each pattern in these lines could then be compared to the tools of the Tool Box, and particularly to the necklaces with beads in the box that were also arranged according to a pattern.

Fig. 7.12 The children on the right are lined up like ants marching in a procession

With the patterning activity set now as the third activity in the instructional sequence, the role of the activity in Round 2 was less fundamental to the rest of the activities. Instead, it was based on the children's familiarity with the contents of the Tool Box and with the children's identification with the context of ants marching in a procession. This answered better to the need for an instructional sequence that unites children's present knowledge with what they are expected to learn: after having studied the basic structures in the box and after working with such structures in relation to large cards and egg cartons, the children then explored the idea of regularity and connected it to the tools in the Tool Box.

7.6.4 Activity 4: Building Ant Hills (i.e., Mierenhopen Bouwen)

This activity was the most difficult to design, interpret and perform in the instructional sequence; the challenge was to present symmetry and regularity in a 3-D construction to the children as a convenient way for understanding its structure and for manipulating its components. More practically, the aim of the activity was to support children in becoming aware of the symmetry of a block construction and to help them apply this insight to abbreviate procedures for determining the number of blocks in the construction. Such insight could help those children who tend to lose track of count or who tend to become confused by how apparently unsystematically the blocks are structured. The focus on abstracting a structure from an arrangement and applying the structure to abbreviate numerical procedures coincides with the previous activities.

Although the children were aware of the difference between a structured and an unstructured construction, they found it confusing to bridge this insight with efficient methods for determining the number of blocks in the construction. In Round 1 the teacher tried to highlight the significance of the difference between the two types of constructions by asking the children to compare "easy ways" for counting these blocks. She referred to the "tricks" in the Trick Box to trigger the idea of spatial structure. The focus group also built and compared structures that were "easy" or "difficult" to count. In this way, the activity again illustrated children's insight into spatial structure. What was missing was motivation for the children to spontaneously make use of the structure to determining a quantity; the activity did not provide enough reason for the children to relate the structuring with blocks to the structuring with egg cartons or dot arrangements from the Trick Box.

The revised activity was meant to help the children translate their experiences with spatial structures in the Tool Box to this more spatial, 3-D context. In a game-like setting, two children at a time were asked to examine a construction of blocks that was placed in the middle of the class. The constructions were all structured (i.e., symmetric and with a pattern). The children were to determine as fast as possible how the construction could be made taller to help Ant make his ant hill taller. They were encouraged to analyze the structure of the construction to determine what the next layer in the construction would look like. This consistently kept the focus of the activity on the structure of the construction. Children with insight into the structure of the construction were expected to relatively easily determine what the next layer of blocks should look like. The process of analyzing the construction was expected to involve reasoning about the number of blocks in each layer and comparing the layers to each other, and that this type of reasoning may underpin structured ways for determining how many blocks make up a construction.

The first construction was made up of five layers with two blocks each. This was a simple construction that introduced the children to the idea of looking at the construction, determining its structure and deciding what the next layer should look like. The second construction was one with a layer of two, a layer of one, and again a layer of two blocks. The pattern that the children were to explicate was the alternating two-one-two pattern. The last construction had three blocks at the bottom, followed by two blocks, and one block and again two blocks and finally three blocks. This was a more challenging construction to see whether and how the children made use of their patterning insight to identify the structure of the construction for determining the next layer of the construction.

7.6.5 Activity 5: Picking Flowers (i.e., Bloemen Plukken)

From the children's perspective this activity stayed the same as in Round 1. What the analyses highlighted, however, is the importance of the teacher's choice in deciding which chips to remove from the children's gardens. In practice, it was difficult for the teacher to spontaneously decide which of the chips played a significant part in the arrangement and should be removed to instigate the need for spatial structuring. For example, if a child arranged four lines of four chips, then it would be less effective to remove one chip out of the middle of every line because the children can notice the empty spaces and subitize the total of chips that are missing. Yet, if one or two chips are removed from the edges of the lines, then the children must know the exact length of their lines. In this case the children would first have to refer to the structures before they can judge the number of chips that are missing.

One way to challenge the children is to confront them with the effectivity of their arrangement by taking greater care in choosing those chips that compose the structure. Therefore, the children were not only asked how many chips were missing, but also how many chips were left. The key was to determine this amount by making use of the arrangement rather than unitary counting. A structure with four lines of four and three chips missing, for example, can be approached in various ways. The child can count the number of chips that are left, or determine the number that is missing and subtract that from the original amount, or reason that two lines of three, one line of four, and one line of two are left so twelve chips are left. We expect this method to be a step towards understanding more formal addition and subtraction procedures, because the child should gain insight into the composition of the total and therefore remember sums and differences more easily (Van Eerde, 1996). This also accommodates various levels of learning because children can practice their counting abilities while others may go as far as applying formal addition and subtraction strategies.

Activity 1: The Tool Box and Ant Steps

- Establish the context
- Explore spatial structures in the Tool Box
- Compare representations of quantities on structured dotted cards

Activity 2: Filling Egg cartons

- Compare representations of quantities in double-structures
- Compare double-structures to other structures in the Tool Box

Activity 3: Marching in a Procession

- Explore structures of 2-D patterns
- Compare structures in patterns to other structures in the Tool Box

Activity 4: Building Ant Hills

- Explore structures of 3-D patterns
- Compare structures of block constructions to other structures in the Tool Box

Activity 5: Picking Flowers

- Apply structures to unstructured arrangements
- Compare representations of structures in the Tool Box
- Explore the role of structures in the (de)composition of quantities

Fig. 7.13 Operationalization of spatial structuring in terms of the five instruction activities

Rather than spending too much time introducing the context of the activity, the teacher was advised to, right from the start, ask the children to determine the number of flowers that are missing, just like in the original Robbie de Rover activity (Speciaal Rekenen, 2003).

This was expected to motivate the children to rearrange the chips and find suitable ways of efficiently and effectively determining and comparing quantities. The teacher was also alerted to use language that focuses on “determining how many chips are missing” rather than on “guessing how many chips are missing”, to refine the shared vocabulary so that the children would understand that they were expected to think of strategies for determining a quantity. The revisions of this activity, with more attention to which chips are removed from the garden and increased time pressure, should help the children recognize and make use of spatial structures in unstructured configurations. Next to contributing to the HLT, these revised instruction activities refine the operationalization of spatial structuring in the research, as illustrated in Fig. 7.13.

In the next chapter we highlight how the outcomes of the revised instructional sequence in Round 2 combine with observations from Round 1 to contribute to the emerging local instruction theory.

8 Analysis of Round 2 of the Instruction Experiment

The analyses of the second round of the instruction experiment supplement the insights that were gained from the first round which are discussed in the previous chapter. Based on observations from the exploratory studies, the learning goals of the instruction experiment, and the four phases regarding the development of children's spatial structuring ability, the following five conjectured learning moments were outlined in section 6.2:

(1) organize objects as a step towards becoming aware of spatial structuring to simplify counting procedures;
(2) create a motivation for spatially structuring objects;
(3) use spatial structuring to elucidate numerical relations;
(4) develop abstract spatial structures that are less context- or task-dependent;
(5) use spatial structuring in a goal-directed way outside the instruction experiment.

This chapter begins with a retrospective analysis of observations from the classroom and focus group discussions that support and elaborate on these conjectured learning moments with respect to the global hypothetical learning trajectory (HLT). In section 8.2, some unexpected observations are presented that required attention when interpreting the instruction experiment because they supplement the learning moments. The impression of what constitutes an effective learning ecology for this instruction experiment is completed in section 8.3 with illustrations of the teacher's role in children's constructive learning processes. Together, the observations lead to section 8.4, in which the outcomes of a general retrospective analysis are presented to summarize how children progressed in their development of spatial structuring ability as they participated in the instruction experiment.

Following the constant comparison method (Glaser & Strauss, 1967), we used ATLAS.ti to code each observation (i.e., quotation) and to define various memo-types that took the description of an observation to a more theoretical level so as to encompass both the confirmatory and challenging observations (section 3.4). The most illustrative of the observations at a theory-type memo level are discussed in this chapter. The result of the ATLAS.ti analyses, together with discussions with experts and literature studies, is summarized into nine learning insights that appear essential to children's progression along the HLT. As such, this chapter contributes to answering the second research question in the study: to create a local instruction theory about how the development of young children's spatial structuring ability may be supported.

8.1 Retrospective analyses of the HLT

Each of the following paragraphs discusses one of the conjectured learning moments listed above. The learning moment is first projected onto the global HLT (see section 6.1) and then illustrated with observations that highlight important patterns in children's responses.

8.1.1 Organize objects as a step towards becoming aware of spatial structuring

The global HLT
The HLT of each instruction activity is built on the assumption that children initially tend to organize objects so that their counting procedures become easier to perform and less error-prone. An example of such a type of organization is when children arrange objects in a line so that when the objects are counted unitarily, it is easier to keep track of which ones have already been counted (cf. Battista et al., 1998). This is an important starting point for introducing the children to advantages of spatial structuring strategies.

Retrospective analysis and illustrations
Children generally progressed from unitary counting to spatial structuring strategies through organizational (e.g., grouping or lining up) strategies. This coincides with the first learning moment that was outlined in paragraph 6.2.1, namely organizing the objects (and practicing counting skills) as a step towards spatially structuring them. In the interviews and instruction activities, children often lined up objects because "then (pointing to each object unitarily) you can count them easily". During the "Picking Flowers" activity in Round 1, for example, almost everyone in the focus group placed the chips on the edge of the paper. This was effective for reading off the number of chips on the paper because the children could note the number of empty spaces where chips were missing. The next step towards spatial structuring is for the children to experience how not just the empty spaces, but also the position of the chips is important to keep track of the missing quantity. Hence, the children applied their own type of organization to the task, but they were confronted with the effectiveness of this type of organization as they evaluated it with the teacher. This is a meaningful starting point for developing spatial structuring strategies.

The development from organizational skills for unitary counting to recognizing spatial structures, was stimulated further in Round 2 through the introduction of the Tool Box. Key to this development was the shared vocabulary that the teacher and children developed, and that revolved around spatial structuring and finding "easy" or "clever" ways to determine a quantity. When, for example, the teacher asked the class why they could "easily see" how many dots are on the cards and whether they recognized the dots from anything else, the children immediately recalled the dice. For six dots, the children not only mentioned the dice, but they also pointed to the finger pattern card, the card with dots on it, and the egg cartons. Further, for the card with ten dots, the class called out that they recognized it in the large egg cartons or in two dice as “five, five”. They also spontaneously showed the two fives using their fingers. This shared vocabulary is essential for the instruction experiment because it creates a foundation on which to build experiences with and shared knowledge about spatial structuring. Such a foundation is what should help children translate their organizational skills to spatial structuring strategies.

Children’s developing understanding of spatial structuring became particularly apparent during the Picking Flowers activity in Round 2. The class was asked why it was difficult for Dori to determine the missing number of chips, and Becky answered:

Becky:	Because they were first lying there differently. And then it’s really hard for Dori. And if...
Teacher:	Can you think of a way, then, a way to very easily see how many will be missing?
Becky:	Ok, ehm...(walks over to the garden)
Teacher:	Can you think of an easy way to arrange them so that Ant can take some away and you quickly see how many flowers are missing?
	(Becky starts rearranging the flowers and the other children are eager to show the teacher their “easy” ways of arranging the flower)
Teacher:	Becky, you think this is an easy way so that you can quickly see how many flowers Ant will have taken away?
Becky:	I’m not ready yet... (continues to arrange the flowers into four touching rows of two, four, two, and two chips) Ready.
Teacher:	Allright, well, let’s see. Simon will close his eyes now. Becky has just thought of an “easy” way to arrange the flowers. And now we’ll see if Simon can close his eyes and if Ant can take some away. Then Simon will see how many are missing. And *where* they’re missing.
	(Ant comes to remove two chips, one from each of the two bottom rows in the structure)
Teacher:	Have a look. How many do you...
Sam:	Three gone.
Teacher:	You think three. Where do you think they’re missing?
Sam:	(points to the empty spaces and to one space on the end of the third row) Four, four.
Teacher:	Do you think there are four missing? Where were they lying then?
Sam:	No, one, two, two!
Teacher:	Where were the two then?
Sam:	(pointing to the two empty spaces) There and there.
Teacher:	Well done, because look (replaces the missing chips), this is how it was arranged.

Becky’s first response (that the teacher had moved the flowers) suggests that she was aware of the role of the position (i.e., organization) of the flowers in the garden for determining a missing quantity. Further, Sam’s successful use of Becky’s arrangement showed for the first time how the children used a type of organization to minimize the difficulty of determining a large number of originally unstructured objects. It also illustrates the difference between organizing and spatial structuring because organizing objects only supports, rather than abbreviates, a unitary counting procedure.

In summary, several observations in Round 1 and Round 2 support the first learning moment and suggest that children were becoming more aware of the advantages of organizing, and in particular, spatially structuring objects to determine and compare quantities. Hence, children's organizational skills are a meaningful starting point that, through a shared vocabulary, can help to develop goal-directed spatial structuring strategies.

8.1.2 Creating a motivation for structuring objects

The global HLT
One of the main goals of the instruction experiment is to support children in the development of their spatial structuring ability so that they may ultimately succeed in recognizing spatial structure in relatively larger and unstructured settings. This builds on the previous learning moment in the sense that children who organize objects and who are familiar with spatial structures are expected to be ready to explore more efficient (spatial structuring) strategies. The challenge, then, was to create an instructional setting that can motivate the children to make use of spatial structures without imposing spatial structuring strategies on them. Since counting strategies are typically sufficient for Kindergarten mathematics, children cannot be expected to actually experience a necessity for spatial structuring. Such a necessity is associated more with, for example, large multiplication and division. Nevertheless, at this stage of developing children's spatial structuring ability, the goal is to create an awareness of spatial structuring strategies as an efficient and effective alternative to unitary counting. Regarding the advantage of spatial structuring over unitary counting, such an awareness was expected to induce a motivation for beginning to recognize and use spatial structuring strategies.

Retrospective analysis and illustrations
The first step towards spatial structuring was to create a motive for looking for alternative strategies to unitary counting. Such a motive is illustrated by Mark during the block construction activity in Round 1, for example, when he explained that his construction was easy to count because "1, 2, 3, 4, 5, 6" whereas "it's harder to count the other construction because you can't really see what's under it". Lisa added to this that "it's more complicated". These comments suggest that the children were experiencing a dissonance between how easily they could count structured compared to unstructured blocks.

As introduced in the first learning moment, it appears that children who are motivated to use spatial structures have typically experienced the convenience of spatial structuring before. An example comes from the focus group in Round 2. Although Becky was first to recognize the nine dots on the card, it was not clear how she used structure to recognize the quantity. At the same time, James demonstrated how he counted the dots one by one. Becky then counted the dots and this time answered eight. Both children agreed that there were eight dots and that their counting procedure overruled the structure that Becky thought she had recognized in the card. Apparently, the children were so familiar with counting procedures, that they relied more on the outcome of their counting than on their insight into spatial structures. Hence, even children who showed spatial structuring abilities in the other tasks, needed to experience the convenience of spatial structuring as an alternative to more familiar counting procedures. Similarly, the class seemed to understand that the contents of the box had something to do with “counting easily”. The next step, however, was for the children to become aware of the difference between “easy” and abbreviated counting. Therefore, as the class was studying the egg cartons, the teacher asked:

Teacher:	Does anyone know why Ant put the egg cartons into the Tool Box?
Rick:	(picks up a carton and opens it) Because, you can count in it how many there are. And here there are six. I did that by counting.
Teacher:	Yes, but Ant puts things in the box that give you a very quick way to count. That you don’t have to go 1, 2, 3, 4, 5, 6 with your finger. (The class joins in to count).
Daria:	Because then you can’t see it. Then you won’t know where the 1 and the 2 is.
Teacher:	He has thought of “ways” hasn’t he. And the “way” for this box, what do you think it is? Dori?
Dori:	There are three there and three there and then it’s six.

This is a rich episode in how it not only illustrates the children’s familiarity with the six-structure, but it also shows how several children were (becoming) aware of the convenience of using a spatial structure to determine a quantity. Rick started off by counting each egg, but he agreed with the teacher that that was not “an easy way”. The teacher’s explanation was supported by Daria who added in her own words that that strategy makes it difficult to keep track of count. This explanation suggests counting strategies (such as three and three making six) that are more organized and coherent than unitary “easy counting” strategies.

Another example of differences in children’s experience with spatial structuring, comes from the following classroom discussion during Round 2. Mark arranged the ten chips in the garden into two rows of five chips:

Teacher:	What an easy way. Why did you arrange the flowers like that?
Mark:	Because they’re in a row.
Teacher:	And so how many are there?

	(Mark looks at the flowers, looks at his fingers while he seems to be counting the flowers, looks up at the teacher)
Teacher:	How many did you arrange? How many flowers are in the garden?
The class:	(calling out that they know the answer) Ten!
Teacher:	Ten, very good. (turns back to Mark) what a good "way". Ali, can you think of another good way to quickly see how many there are?
	(Several other children call out that they want to show another way)
Ali:	(places the flowers into one long line)
Teacher:	You already know how many there are, don't you? Do you think this is an easy way?
	(Several other children call out that they know how many there are)
Teacher:	But Ali, if I add these flowers (adds another four flowers to the end of the line), now you can't just point to them, how many flowers do you think there are now?
Ali:	I don't know.
Teacher:	No, is this really an easy way?
	(Ali shakes his head)
Teacher:	It's fine if you put the flowers like that, but can you quickly see how many there are? Dori, if you look at the way Ali put the flowers, is it easier to see how many there are? Or do you think Mark's way is easier?
The class:	Mark's.
Teacher:	Why, Daria, is Mark's arrangement easier?
Daria:	Because (pointing with her finger in the air) the two lines are easier to see in a garden.
Teacher:	Do you remember anything from the Tool Box that also makes it easy to see how many there are?
Daria:	Yes! The egg cartons, because that also has 10.

In this episode, Mark showed insight into the purpose of the activity ("now you can easily see how many there are") and yet he had trouble determining the number of flowers without counting them unitarily. Perhaps he held on to the mathematical practice of counting unitarily rather than making use of spatial structure. Similarly, when Ali placed the ten chips into one line, he said that this was a "good" structure because this was a structure that made it "easy" for children to "count" how many chips there are. This shows how not all children understood the norm of spatial structuring in terms of finding an "easy way" to read off the number of chips. The teacher, however, played an important supportive role in encouraging children to make use of the mathematical practice of spatial structuring. When she asked Daria, for example, to explain Mark's answer, Daria simplified the counting procedure by associating the flowers with "two lines". To Daria, determining the number of flowers does not merely involve unitary counting, but also rearranging the flowers so that the procedure can be abbreviated. The teacher then highlighted this idea for the rest of the class by asking Daria to related Mark's structure to a tool from the Tool Box. Daria showed how the two lines resemble the egg carton and therefore how she was able to translate the idea of double-structures to an originally unstructured arrangement of ten flowers in the garden.

As such, this episode illustrates various phases in the development of spatial structuring ability and how some children understood the activity while others were still confused: Ali was attached to his counting strategies, while Mark seemed to be aware of the use of structures, but still had to experience the advantage of spatial structuring over unitary counting, and while Daria recognized the similarity between different types of spatial structures that can help to read off a quantity. This relates to a variety of interrelated factors that are involved in the ability to determine a quantity (e.g., a development from unitary counting to recognizing and naming the number of items in a small configuration, to learning names and eventually ordered lists of number words with insight into symmetry, to cardinality, and eventually to numerical relations; section 2.1).

Children's general progress in understanding the role of spatial structures in counting is illustrated by how they explained the presence of the box in class. In Round 1, the class explained that the Trick Box contained things that "we can count" and that the aim was to "think of a way to be able to count correctly" or, in other words, to "think of an easy way for counting the things in the box". In contrast, in Round 2, the class was less focused on the act of counting: Mark said that the contents of the Tool Box helped them "to do it themselves" (i.e., independently use the structures that were left by Ant in the Tool Box to find out how many of something there are), and Tim added that now they could do "counting games". Mark also said that Ant helped them "to look" (i.e., find structures in various configurations) and James said "to learn the numbers". James' answer came closest to the advantage of recognizing spatial structures for gaining insight into numerical relations.

The teacher can let children of relatively higher spatial structuring abilities set an example for the rest of the class by letting the children explicate ways of recognizing and making use of spatial structure. Hence, based on their own experience rather than taught insights, these children can convey the advantage of spatial structuring and inspire other children to explore spatial structuring procedures. This illustrates the importance of interactive learning that is required to synchronize mathematical ideas (see paragraph 2.5.2) through establishing a shared vocabulary and socio-mathematical norms that acknowledge spatial structuring as "easy ways" for performing numerical procedures.

Overall, the instruction activities seem to have supported children's motivation for using spatial structuring rather than unitary counting procedures. This suggests that, although children may be familiar with spatial structures, they may not spontaneously apply them because they have not yet experienced the convenience, or sometimes the necessity, of spatial structuring.

8.1.3 Using spatial structures to elucidate numerical relations

The global HLT

According to the HLT, spatial structuring can be used to mentally or physically organize a relatively large structured or unstructured configuration in a way that can help develop insight into numerical relations. This insight can contribute to the ability to read off a quantity and ultimately, to abbreviate numerical procedures such as determining, comparing and operating with small quantities. The “Picking Flowers” activity is designed for the children to explore ways for making use of spatial structures to elucidate numerical relations. By applying spatial structure to the flowers in the garden, we expected the children to gain insight into the (de)composition of quantities so that they would conveniently determine how many flowers there are and how many are missing. As illustrated by the observations below, after performing Round 2 of this instruction experiment, the children seemed more familiar with spatial structures and showed greater ability to read off a quantity.

Retrospective analysis and illustrations

The “Picking Flowers” activity challenged the children in Round 2 because, as illustrated by Mark and Becky, they were keen to try new structures and to see whether these structures improved their previous ideas. Mark had already been successful with a structure that had several chips on the edge of the paper and several chips in the middle, but this time he chose to arrange all the chips into a triangle. He had more trouble determining the number of missing chips in this structure, however, so the researcher encouraged him to compare this structure to how he previously arranged the chips on the edge of the paper. Becky could also determine the number of missing chips in her new structure, but she had more trouble identifying where the chips had been removed. Nevertheless, Becky appeared to have realized the advantage of placing chips close together. It may have taken one or two more turns for her to come to a most effective spatial structure.

Hence, by the end of Round 2, the children were exploring and improving their own structures and strategies. This is important because children probably understand these structures better than those that are imposed on them. Meanwhile, the teacher encouraged them to compare their successful structures to other structures and to reflect on the differences between them. The key is for the children to become aware of what characterizes a useful structure, and for them to learn to recognize these characteristics in unstructured settings.

The most notable spatial structuring episode occurred at the end of the “Picking Flowers” classroom discussion during Round 2 when Lisa was asked to arrange the chips in such a way that “would make it easy to see how many chips are missing”. Without hesitation, Lisa proceeded to arrange the chips into three sets of five, with each five structured as on dice (Fig. 8.1). This illustrates how she improved from her classification in the Recognition phase (phase 2) in the pre-interviews to how she now showed that she had understood the advantage of using the contents of the Tool Box for abbreviating her counting procedure. In the focus group, Lisa's final structure consisted of several chips placed on the edge of the garden and chips in a flower-like circle in the middle of the garden. This again is an example of how Lisa had experienced an evolution in the way she used structure first as an attractive looking flower and then as a functional structure that helped her to keep track of how many chips were missing.

Fig. 8.1 Lisa is arranging the flowers into three groups of five

Not only Lisa understood the effectiveness of the structure that she created in the classroom discussion. Several other children in the class picked up on her structure and showed an advancement in their understanding of spatial structuring. Sam, for example, called out that Lisa was making "three fives", and Rick explained that Lisa's structure was effective because it "looks like on a die". The variety of structures that the children applied by the end of the "Picking Flowers" focus group activity in Round 2, also illustrates the children's improved insights into abbreviating procedures using spatial structures. After arranging the chips into unorganized or large configurations that were difficult to read off, James' last structure consisted of three piles of four chips, Mark's structure was a triangle with all the chips touching, and Jenna made several sets of two chips. Becky started with a structure similar to Lisa's, but then she changed it to make two touching triangles. James and Lisa were most successful with their structures; James immediately saw the difference in height of his piles of chips and Lisa could identify the number of missing chips by comparing the before and after configurations (Fig. 8.2).

Fig. 8.2 Video frame showing how the children in the focus group arrange their flowers (colored chips) into spatial structures that they think could help to easily determine how many are missing

Taken together, the different types of structures that the children created throughout the instruction activity suggest that they had been experiencing the confusion of one type of structure (e.g., too long rows or too large circles) compared to relatively organized other types of structures (e.g., chips touching each other in smaller sets) that helped to determine a missing quantity in an abbreviated way. Such dissonances were expected to help the children gain insight into the composition of quantities, which in turn underlies insight into numerical relations.

8.1.4 Developing context- and task-independent spatial structures

The global HLT

Building on children's ability to recognize and make use of spatial structures, the next learning moment involves learning to think about and work with spatial structures in a flexible (i.e., context- and task-independent) way and in various (spatial) contexts. We expected children to improve in their spatial structuring ability if they compared various types of spatial structures. The Tool Box is intended to provide the children with input for comparing spatial structures. Such comparisons are important for children to be able to think more flexibly about different types of spatial structures so that their ability to recognize or use a particular spatial structure becomes less context- and task-dependent (i.e., working with spatial structures exactly as they are, such as analyzing dice configurations and egg cartons) and therefore more secured. Hence, context- and task-independence involves the ability to use the essence of spatial structures (e.g., five-structures or double-structures) and to apply them to various quantities and in other settings (e.g., five on dice or finger patterns, patterning in block constructions).

Retrospective analysis and illustrations

Comparing representations of structures for one quantity. The Trick Box and Tool Box proved to be valuable contexts for helping the children become familiar with how various spatial structures can represent one particular quantity, and for translating a type of spatial structure to another setting. In Round 1 the children already showed that they could compare various structures that they had explored in the Trick Box activity. For example, when the teacher asked the children whether they could recall an object from the box that looked like how the dots were arranged on the card, Mark showed the teacher the face of the large die with six dots on it. He also recognized the ten dots on a card and explained it by saying that "five and five makes ten". He used both hands to point to two dots on one side and two dots on the other side explaining "here are two, here are another two, and one in the middle makes five". James related the next card, with four dots on it, to his own age. Subsequently, Simon showed the teacher that the four dots look like the four on the die.

Fig. 8.3 Video frame of the class as they are (a) comparing the egg cartons to other representations in the Tool Box, and (b) showing different ways of representing six using various finger patterns

Similarly, in Round 2 the teacher repeatedly encouraged the children to compare structures in the Tool Box to see how various quantities are represented and to relate one type of structure to another (Fig. 8.3). When, for example, Matt did not see how the arrangement of flowers in the garden activity resembled the dice configuration for five, the teacher placed a large die next to the garden and guided him towards recognizing the configuration in the garden. Assuming that Matt was familiar with the dice configuration, this direct comparison helped him to translate a familiar structure to an unfamiliar context.

Comparing structures for one quantity within one representation. Several observations support how children recognized fundamental spatial structures in relatively larger structured settings. For example, the teacher asked the children whether a particular finger pattern was the same "tool" as the other "tools" that were in the box, and whether it was "also three, three" (referring to the children's conception of the structure for six). Initially, the children all agreed, but when they started putting up their own fingers, they discussed with each other that even with their fingers they could create different structures for one quantity (e.g., four and two or three and three). This discussion offered important illustrations of the different spatial structures that exist within one representation for a particular quantity and that contribute to thinking flexibly about structures so that they can be recognized more easily in structured and unstructured settings. Similarly, in the following focus group activity about egg cartons, Becky explored different arrangements of seven eggs:

Teacher:	(holding a 10-egg carton that has five eggs in the top row, and two in the centre of the bottom row) So what does the seven in the egg carton look like? (The focus group is loud and distracted but Becky is keen to answer the question)
Becky:	I know! Look, these, these (pointing to the right side of the top row, and then taking the middle egg in the top to place it next to the two eggs in the bottom row) these are two (pointing to the upper right), these are two (pointing to the upper left), and this is one (pointing to the lower right), this is one (pointing to the lower left) and this is one (pointing to the lower middle egg).
Teacher:	Yes, and how many is that altogether?
Becky:	(thinking and counting mentally) Eight.
Teacher:	Eight? How did you see that?
Becky:	But I did it in my head.
Teacher:	Did you? Can you think of an easier way?
Becky:	(spontaneously places the lower middle egg in the middle of the top row, places the lower left egg in the lower left most space and the lower right egg in the lower right most space) There! (touches each egg) 1, 2, no!
Teacher:	You don't need to count them do you? Don't you know an easy way to quickly see how many there are?
Becky:	(arranges the eggs into two rows of three and one more in the bottom row) Like this (the rest of the focus group is attending again). Seven.
Teacher:	How did you see that?
Becky:	Well, (moving her hand across the two rows) because it was like this before.
Teacher:	What was like this? How can you see this is seven without having to count them?

Becky:	Look (arranges the eggs into a group of four on the left and a group of three on the right of the carton) These are four and those are three.
Teacher:	And how much is that altogether?
Becky:	Uhm (rearranges the eggs back into the two rows of three and one egg structure) That is seven.
Teacher:	Yes, but can you see that there are seven in the way they are arranged now?
Becky:	Yes (pointing) these are four and that is three and that makes seven.

The significance of this episode is that Becky seemed to have trouble reading off the total of seven eggs in the first arrangements that she made, since she was tempted to count the eggs unitarily. Only when she continued to rearrange the eggs, did she finally explain that she now knew that there were seven eggs because “that is three and that is four”. This sequence of rearrangements is illustrative of how Becky was comparing different types of structures for seven, some of which were more familiar to her than others. It seems that Becky was most familiar with the grouping of four and three. Perhaps this experience with rearranging and comparing different structures for seven can supplement her reference to different structures for seven and improve her insight into the composition of seven. In fact, after Mark explained that he saw seven as three and two and two, Becky spontaneously arranged the eggs into a group of four and three. This suggests that Becky had translated the group of three and four to Mark’s type of structure. Indeed, later Becky explained that three eggs were missing from the box because she needed two more to make a group of four on one side of the box and one more to make a group of three on the other side of the box. As such, Becky seems to have become familiar with various structures for seven.

The class in general showed an improved ability in making use of complex structures to show how they could easily see the number of dots on the cards. This was especially striking with Dori who thus far had not seemed to attend much to structures and yet who, in this classroom activity, recognized the ten dots on the card as two rows of four and a row of two in the middle. She also interpreted a card with seven dots on it as a five “like on dice with two dots next to it”. Even Jenna, who thus far had stayed most attached to unitary counting strategies could identify different dice configurations in various cards during the classroom activity. It was a great improvement to the first round of the instruction experiment when, during the focus group activity in Round 2, Becky was fastest at recognizing the number of dots on a card. Moreover, she said she recognized it from the classroom discussion, where there were two rows and two dots in the middle. She also recognized the five in a card with a set of five dots and two sets of two dots on the side. Similarly, James explained how he saw the five in the middle with two sets of two dots on both sides. Therefore, motivated by the time limits, these children succeeded in decomposing the dot structures into recognizable structures rather than counting each dot unitarily.

Patterning for task- and context-independent structuring. In patterning, children learn to decompose a whole into its composite parts. This is a valuable ability for learning to decompose any type of structure into more recognizable parts. In turn, insight into part-whole relations should help to develop task- and context-independent structures for translating a familiar spatial structure to different settings. We describe children's learning progress in patterning across Round 1 and Round 2, and then explain how children translated their patterning skills to the "Building Ant Hills" activity.

In Round 1, the children gradually succeeded in extending a pattern that alternated two colors. However, although the children understood the "rule" of a particular pattern (i.e., alternating two colors), it is unclear to what extent they actually understood the "regularity" of the pattern (i.e., repeating each chunk of two differently colored chips). For example, after discussing a line that had no apparent rule to it, the teacher rounded this part of the activity off by asking the children to explain why the last sequence was more difficult to "predict" than the other two sequences. In Round 1, Lara had answered that "it wasn't a clear line". Round 2, however, was more dynamic and Daria quickly called out that it "didn't make sense" because there were more boys than girls, which made it "unfair". Rick added that the line "is crazy" because "there are not the same number" (of boys and girls). Lara was convinced that the line had "a way" so she added boys to the line. Other children interrupted and changed the line again to balance the number of boys and girls. Finally, the children agreed that the line had "a way" which helped them to see how it should be lengthened. The teacher took this conclusion to summarize that it can help to take part of the line away to correct the rest of the line.

The search for what characterizes the pattern of a line, marked the beginning of a search for structure because it made the children more aware of the relationship between the individual elements in the line and their composite structure. The children were balancing the number of boys and girls, but at the same time, they were taking into account that the numbers were repeating because after two girls, three boys, two girls and another three boys, the line was to continue with two girls and again three boys. Not only were the children searching for structure within a pattern, in Round 2 they were also keen to compare patterns across types of structures. When the teacher asked whether they remembered similar activities, for example, the children recalled beading activities. Rick even mentioned the word "patterns". Apparently, then, the children linked this activity to other games. This underlies their understanding of what characterizes a pattern. James verbalized what seemed to be most children's understanding of a pattern of alternating colors when he asked "are we going to make a line again with different, different, different, with two figures?" Hence, at that point, most children were familiar with the method of recalling the beginning of the line and using the rhythm to extend the line.

The challenge in this activity was to move away from only recalling the rhythm of the line to, instead, try to analyze what made up the rhythm and, then, what structure made up the pattern. In the focus group, James, for example, moved the green chip of one of the groups from the end back to the beginning to show that the two groups are really the same. Similarly, for the next line, the children concisely said that "each time there were two brown chips and then three blue chips". When Lisa was asked to separate the line in such a way that each part would be the same, she split the line up in the middle. James explained that this was correct because "here there's a brown one and there there's a brown one and here there are three and there there are three". In short, through quickly analyzing two patterns, the focus group showed that they knew how to lengthen the pattern without having to recall the rhythm of the pattern, and that they knew how to split the pattern into equal parts (Fig. 8.4).

Fig. 8.4 Video frame of the other four focus group children as they are studying whether Mark's sequence of chips can be split into equal parts to show that it is a pattern with repeated units

What is not clear yet is whether the children related these equal parts to the idea of a repeated structure of a pattern. For example, when in the focus group James explained how he saw what the next color should be, he said "because here there is one, there is one, there is one, and then it's a pattern". He seemed familiar with a pattern of alternating two colors, preferably with only one element of each. On the other hand, James was also familiar with extending a "pattern" (i.e., sequence of differently colored chips) just by repeating the elements that were already present. This became apparent from the pattern that he had made which didn't contain any repeating elements. It only started repeating when the children lengthened it by adding the same order of colors to it as from the beginning. Hence, James considered blue-red-green to be a pattern while, essentially, it is the extension of structure that ultimately makes up a pattern. To study whether the children understood what elements made up a structure of a pattern, the focus group was asked to split up the pattern into several identical parts. Despite the researcher's suggestion to split up the line into more parts, the children all split the pattern into two equal parts rather than on the basis of repeating elements.

The "Building Ant Hills" activity was intended for children to translate their patterning abilities to a 3-D setting. In the classroom activity, Becky, for example, analyzed the block construction conscientiously before adding blocks to make a new layer (Fig. 8.5). Similarly, Daria pointed to each of the layers in the construction as she said "here like this and here and here like this and here also like this". This suggests that they were examining the beginning of the construction to find a pattern and to continue that pattern upwards. In the focus group activity, Mark said the next layer would need "three" blocks because (pointing to the construction) "that's also what's on top of here". Similarly, Becky was asked how many she thought would come after the two blocks in the top layer and she explained "three, because it starts over again every time". These explanations suggest that these children indeed made use of their insight into patterning to complete the constructions.

Fig. 8.5 Video frame of the children studying the structure of the blocks to find a pattern so that they can make the ant hill taller

Although not all the children could identify the pattern, when the teacher asked the class to summarize how they had investigated the constructions, Lisa said that "you have to look at how something is put together". The teacher used this remark to relate the activity to the previous patterning activity by adding that "you have to look at how the beginning is put together and then keep looking so you know how you can finish it". Taken together, for this part of the activity the children appeared to have (implicitly) understood the connection between patterning and the continuation of a block construction. The children were able to translate their insight into patterns, which was the topic of the previous activity, to a more applied context. This is an important step towards recognizing and making use of spatial structures in various structured and unstructured contexts.

In summary, in studying their progression from Round 1 to Round 2, it appears that most children became more familiar with spatial structures through comparing them to other spatial structures from the Tool Box and through applying them to new settings. The observations also underline the importance of being able to translate spatial structures to different settings to improve flexibility in recognizing and using spatial structures. This was illustrated with the “Marching Ants” and “Building Ant Hills” activity. The children clearly progressed in their patterning ability because they were able to search for the repeated structure in a pattern and they took the initiative to compare the structure to other structures that they had encountered in the instruction experiment. What is still unclear, however, is whether the children were able to relate the rule of a pattern to its regularity; they succeeded in extending patterns, but they were less able to create their own patterns with more than two alternating colors.

The way children explored the structure of a pattern to discover how this structure can help to extend the pattern, is an example of how the instruction activities offered the children opportunities for learning to apply the spatial structures that were explored in the Tool Box to various settings.

8.1.5 Goal-directed spatial structuring outside the instruction experiment

The global HLT

To this point, the observations highlight general improvements in children’s spatial structuring ability with respect to specific instruction activities. The fifth learning moment encompasses how some of the children spontaneously translated and applied structures to contexts and settings outside of the instruction experiment. Such transfer effects are necessary for the Applied phase (phase 4) of the spatial structuring trajectory (i.e., recognizing, using and applying spatial structures to goal-directedly abbreviate numerical procedures) to prepare for higher-order mathematical procedures.

Retrospective analysis and illustrations

Several transfer effects were noted in between Round 1 and Round 2 outside of the instruction experiment. The significance of this is that it (a) highlights children’s continued learning and spontaneous reference-making to the instruction activities, and it (b) indicates the extent to which the children not only improved their spatial structuring ability, but were also able to generalize it and translate it to other contexts and settings.

Only three days after performing the patterning activity in Round 1, for example, several children were connecting colored plastic puzzle-like pieces to make a “mat”. The mats all had a particular pattern to them. The teacher reported that the children recognized the alternating colors (red-green and blue-green-red), the alternating numerosities (two colors versus three colors), the alternating orientation of the pieces (wide versus long), and the alternating connecting pieces (one versus two connections). Although this still does not clarify whether the children understood the regularity in a pattern, it does illustrate their improved awareness of patterning and symmetry that, according to the teacher, they had not shown much of before. Even the youngest children continued this activity and made very simple, but regular patterns with the plastic pieces (Fig. 8.6).

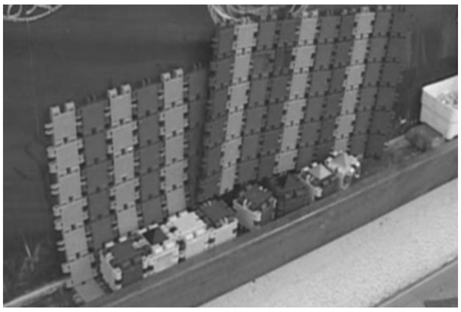


Fig. 8.6 The patterns that children created in their mats

About a week after the first “Giant Cards” activity, James spontaneously established that a toy duck in the classroom had six ducklings because the two rows of three ducklings make six ducklings (Fig. 8.7). The teacher was very surprised at James’ initiative, and she said that she did not think that James would have made such an observation if the children had not encountered such structures in an activity like in the “Egg Carton” activity. This suggests that James was very aware of various structures and their uses, and he seemed to be exploring structures in his surroundings.

Similarly, during Round 2, as the teacher was telling the children about how they were going to walk like ants in a procession, James studied his green-yellow striped t-shirt and discussed with his neighbors what the color of the next line on his shirt would be. The teacher noticed this and opened the discussion to the rest of the class. This illustrates how the children were thinking about the patterns and were trying to generalize the pattern from the Tool Box to their own familiar patterns. This could contribute to their general understanding of patterns and their relation to other spatial structures.

Fig. 8.7 The duck with its six ducklings which could be read off as two rows of three ducklings

The observations in this section illustrate several episodes from Round 1 and Round 2 of the instruction experiment, in which children were successful at translating the spatial structures with which they were familiar, to other contexts and settings both within and outside of the instruction experiment. While the observations in each paragraph illustrate the five conjectured learning moments for the development of spatial structuring ability (section 6.2), several other observations have shed a different light on the course and influence of the instruction experiment on children's spatial structuring ability. In the next section we discuss how these observations contributed with new insights that supplement earlier conjectures. Together, the observations from this and the next section culminate towards tracing several crucial learning insights that appear to underlie the HLT. These crucial insights will be discussed in a general retrospective analysis in section 8.4.

8.2 Unexpected issues in learning

In the process of analyzing the two rounds and describing notable observations that could contribute to a local instruction theory, several observations alerted us to unexpected issues that require attention because of their close relation to the conjectured learning moments. Although these issues are not new in this area of research, they are discussed in this section to highlight their influences on the design of an effective instructional setting:

(1) Children should at least be competent counters and familiar with certain structures before focusing on spatial structuring strategies.
(2) Children are persistent in using counting strategies.

(3) Children’s own spatial structures should be acknowledged.
(4) The interpretation of children’s learning progress can be affected by children’s relatively short attention spans and sometimes limited verbal communication, combined with the teachers’ type of language usage.
(5) For the instruction activity to proceed as planned, teachers’ interpretation of the instruction activity must be accurate and thorough.
(6) The effect of the instruction activity can be stimulated by an overarching context, an appealing setting and a shared vocabulary.

In what follows, we illustrate each issue with observations from Round 1 and Round 2 of the instruction experiment. As in the previous section, the relevance of the observations is not restricted to one particular issue.

8.2.1 Competent in counting and familiar with certain spatial structures before focusing on spatial structuring strategies

In paragraph 8.1.1, we elaborated on the role of organization skills in preparing children to develop spatial structuring strategies. What the observations also emphasized, is that this development requires that children count synchronously and resultatively and that they are at least familiar with certain spatial structures. The significance of these abilities for spatial structuring is illustrated by differences in children’s spatial structuring ability. Some children prefer to perceptually subitize and count as a familiar and secure way of determining a quantity and other children recognize the advantage of conceptual subitizing and evolve from unitary counting procedures to spatial structuring strategies. For example, while Beth could show two ways of using structure to determine how many dots were on her card, Lisa and Matt counted the dots unitarily. Children who have not mastered an accurate method for determining a quantity in the first place, will not be ready for goal-directed use of spatial structures. These children must first come to experience the advantage of strategies that simplify the often time-consuming and more error-prone unitary counting procedures.

The teacher played an important role in encouraging children to reflect on the efficiency of their counting procedures compared to spatial structuring strategies. When Ali was not convinced that there were six dots on the card, for example, instead of showing Ali his counting error, the teacher invited him to count the dots again. This not only gave Ali the opportunity to realize that counting can be an error-prone procedure, it also gave the teacher a chance to comment on the different strategies that the children could use to determine a quantity. Similarly, the teacher offered the youngest children opportunities for developing their counting strategies in the context of spatial structuring. In Round 2, for example, to accommodate the younger children who were still focusing on improving their counting abilities, the teacher removed chips in a way that made it easy for the children to count the empty spaces. Hence, with a little help from the teacher, Laila was able to practice her unitary counting skills to determine the missing quantity.

What the observations also underlined, is that, in order for children to start making use of spatial structures, they must at least be familiar with certain spatial structures. Without this background knowledge, they will not know what to look for in structured or unstructured configurations or what “easy ways” means other than an invitation to count. In contrast, children who are familiar with structures, can be encouraged to recognize and use these structures in various contexts and settings. This is illustrated by several (older) children who already in the pre-interviews showed awareness of spatial structure, and who used these insights to practice and improve spatial structuring during the instruction activities. James, for example, noted that he knew the number of fingers on the flashcard because he “didn’t even have to count them”. During Round 1 of the instruction experiment, as he was determining the seven dots on a large playing card, James responded so quickly that he most probably used (a combination of) spatial structures to recognize the number of dots on the card. Similarly, when Mark quickly determined five dots on a card, he explained that “two and three more is five” and that he had seen it on dice. These children show how their familiarity with structures could be translated to goal-directed use of the structures for abbreviating numerical procedures.

8.2.2 Persistent use of counting strategies

This issue is closely related to children’s motivation to apply spatial structuring strategies in a goal-directed way (paragraph 8.1.2). In Round 1, several children firmly held on to their (perceptual or pointing) counting strategies, regardless of the teacher’s suggestions and questions about “easier ways”. Given a card with two sets of five dots arranged like on dice, Lisa, for example, still pointed and counted each dot unitarily. When the teacher took the card away, Lisa guessed eight dots. Lisa’s answer is curious because in the pre-interview, Lisa had repeatedly recognized the dice structure for five. The question is why she did not appear to recognize that structure in this card or, if she did recognize the structure, what prevented her from making use of this issue to circumvent the counting procedure like the teacher had asked her to. One reason may be that Lisa was not ready to use structure; her repertoire of strategies in the pre-interview suggests that she fit in the Recognition phase (i.e., recognizing structures but not using or applying them yet). We discuss two possible explanations.

The first explanation is that children tended to revert to counting procedures because that was their routine for determining quantities. Children typically talk about counting and they realize that teachers usually applaud their counting ability because this is a well-established classroom mathematical practice. Consequently, to them that was "an easy way" for determining a quantity. They therefore needed a strong motivation to explore and explain spatial structuring strategies. In Round 1, for example, Jenna repeatedly counted and recounted the objects. Apparently Jenna did not feel the need to use any other strategy if she felt confident enough to use a counting procedure that resulted in the correct answers anyway. In explaining how she saw the seven dots, she said that "it looks like the seven and if you count it, then it is the seven". Nevertheless, Jenna showed surprising progress in Round 1 when a card with seven dots was presented to her. She almost instantly called out seven and when she was asked to explain what "clever trick" she had used, she said that "here is two and here is five and I think five and two is easy". Her use of structure illustrates her progress towards less use of unitary counting procedures and possibly more structuring strategies. At the same time, Jenna's strategy repertoire exemplifies the need for a context that justifies the use of spatial structures as an alternative to unitary counting strategies.

A second explanation is that although some children have used spatial structuring strategies, the difficulty for them was to use the proper wording to describe these strategies. A reason for this is that they may have not been familiar enough with the shared vocabulary to use phrases such as "easy ways", "tools", and "reading off" to explain their spatial structuring strategies. Throughout the activity, some children, for example, described their perceptual grouping strategy by saying that they had "counted in their head". James gave a similar response when he saw the ten dots on a card too quickly to have counted them, but still explained that "we have to count them". In the egg carton activity, when the children were asked whether they had an "easy trick" to know how many eggs were left in the box, Mark also said "by counting", despite the structuring that he had shown throughout the activity. These instances reflect the difficulties that are inherent to researching young children's mathematical development (Hughes, 1986).

It was valuable to see what role the teacher could play in guiding the children towards using a more effective structure. For example, a challenge for the teacher in the "Picking Flowers" activity in Round 2 was to help Ali understand the difficulty that his type of structure (placing objects into one line) could cause if he had to count a large number of chips. The teacher added four more chips to Ali's line and then asked him whether he could still see how many chips there were in total. He was not allowed to point to the chips and count them. Considering these constraints, Ali agreed that it was going to be difficult to determine the number of chips. Together with the teacher, Ali summarized that this structure was not as effective as he had thought.

8.2.3 Acknowledging children's own spatial structures

The significance of acknowledging the spatial structures that children devised and developed on their own has been discussed in terms of the principles of RME (section 2.5). Several observations regarding children's organizational skills (see also paragraph 8.1.1) emphasize the importance of this issue to the design of an effective instructional setting. For example, the trouble with how Ali and the teacher studied Ali's structure in the "Picking Flowers" activity (see the previous paragraph) is that it was too directive; Ali still appeared concerned with practicing his counting procedures and with organizing objects to keep track of count (indicating a Unitary phase spatial structuring ability), so he may not have been ready for the next phase by merely being told that spatial structuring is a useful strategy for keeping track of large quantities.

In line with the RME principles, children must "reinvent" spatial structuring to experience the value of spatial structuring themselves. Ali probably had not yet encountered situations where counting procedures were insufficient, so he would not be motivated to try relatively unfamiliar strategies. In everyday situations, he could still have counted the fourteen chips accurately, regardless of how much time that would have cost. The design of the instruction activity should place Ali in a situation in which his counting procedures are less useful, to inspire him to develop other (spatial structuring) strategies. This calls for situations that "beg to be mathematized" (Freudenthal, 1973, 1991).

When in Round 1 the teacher asked the children to place the chips into an arrangement that "would make it easy to count how many chips there are", Simon noted that he could also make a "real flower" out of it. This highlights how the aim of the activity should not be to focus on the structures that children learn and master in formal mathematics. Rather, the instructional sequence must be approached from the children's perspective. That includes thinking about structures in the way that children explore and interpret them. One such structure may indeed be a flower with the chips forming the petals, the stem and the leaves of the flower. Other structures that the children referred to are a house, a tree, the sun and an alignment of the chips with the edge of the piece of paper. All the structures are acceptable for the activity because they can help the children read off quantities and determine the number of chips that are missing, without unitary counting. With more practice, the children may experience the differences between the types of structures and come to increasingly more effective structures that can help them abbreviate numerical procedures such as determining, comparing and operating with small quantities.

The importance of children's own spatial structures in learning to use spatial structuring strategies is further illustrated in the "Picking Flowers" activity in Round 2. Lisa was asked to make a structure that would help her keep track of how many chips were missing. She explained that she had arranged the chips "in this way (a flower) because that's easy to count" and she showed how she counted the chips as she pointed to each of them. Then she was asked to point to where she thought the chips had been removed, in order for her to see the difficulty with her structure. When she was encouraged to think of "an easier way to properly see how many chips are missing", she immediately started rearranging the chips into a different flower. This suggests that she realized the difficulty of keeping track of a number of chips in long rows (the stem) and large circles (the flower). Therefore, the next flower she constructed had a shorter stem and chips that were placed closer together. She also placed one chip in every corner of the garden. By the end of the activity she succeeded in using her flower structure to determine how many chips were missing without counting.

8.2.4 Effects of children's relatively short attention spans and sometimes limited verbal communication and the teachers' type of language usage

As mentioned in paragraph 8.2.2, the challenge of research with young children is that young children are disadvantaged in verbally expressing their thinking and in the amount of attention that they require to perform a task (Hughes, 1986). The instructional sequence had to fit within a short enough time frame to keep drawing children's attention. Moreover, it had to be accessible to all the children so that they would not be distracted by other activities. These constraints set limits on what could be achieved within one instructional activity. As such, they led to the design of five hour-long instruction activities that started with a classroom discussion and ended in the focus group setting. The teacher rounded off the activity as soon as the children became too tired or distracted.

The difficulty of communication in a Kindergarten setting is reflected in dissonances between the children's answers and the teacher's expectations. In the patterning activity, for instance, Jenna explained that a "good" sequence is one that has *two* colors in it. Curiously, however, she gave the example of a pattern that one of the older children had constructed at the end of the classroom discussion with one red, one grey, and two white chips. Hence, Jenna may have picked up more from the activity than what is revealed by her explanation of a "good" pattern. This illustrates the classroom practice of using structure for patterning that was in the process of being established (see paragraph 8.1.4).

Although Lisa also showed progress in her spatial structuring ability, her responses sometimes seemed to suggest otherwise. In the classroom discussion during the Trick Box activity, Lisa repeatedly counted each of the dots, despite the teacher asking her to try to use a “trick” to easily see how many dots there are. This time, however, Lisa recognized the six as being “three and three in a row” and she saw the ten as being “five and five”. Lisa did not make reference to other structures and she counted the five dots to show that there were indeed five. The way she pointed to the collection of dots, however, and the way she hesitated and subsequently reverted to counting as a way of describing how she saw the two fives that made up the ten dots, gives the impression that she actually was using spatial structuring to determine the number of dots. Indeed, when she was later presented with a card with nine dots, Lisa first counted each of the dots, but when she was repeatedly encouraged to use a “clever trick”, she explained how she also found “a two and a five and another two which makes nine”. This episode shows how Lisa was aware of and regularly made use of various spatial structures, but that she had difficulty describing the structures she used and reverted to counting to simplify the question and show why the number of dots was correct.

Next to the difficulty that children may have had in finding the proper words to explain their strategies, the children’s confusion may have been spurred on by the teachers’ choice of words. The teacher may have confused the children in Round 1 by, on the one hand, positively reinforcing them as they correctly continued a pattern with two alternating colors, while on the other hand, asking them to construct a pattern with more colors and a “different rule” than the examples. The teacher only asked the children to make a “good” sequence. The children therefore tended to make sequences with two alternating colors because that fit their definition of a “good” sequence. Confusion due to the teacher’s language also occurred when, rather than consistently asking the children to use “easy ways of finding out how many there are”, in her enthusiasm, the teacher sometimes asked the children to use “easy ways for *counting* how many there are”. This choice of words could steer the children towards unitary counting strategies rather than other ways of determining a quantity.

Another important reason why children continued to count objects was that they were keen to respond exactly to what the teacher was asking. Earlier in the activity, for example, Lara showed the teacher that she could count the dots on the die. The teacher asked her to show the class how, and Lara pointed to each dot as she counted the six. This shows how Lara (who according to the pre-interview was already familiar with spatial structures and often used them to abbreviate her counting) was directed to responding to the teacher’s question by literally showing her how she could count the dots. Hence, regarding these communication issues, in preparing for Round 2, the teachers were encouraged to refine the shared vocabulary (“determine” rather than “count”) and context (using the Tool Box) for minimizing dissociations between what children seem to know and what they really understand.

8.2.5 Teacher’s interpretation of the instruction activity must be accurate and thorough

The instructional sequence is only effective if the teacher succeeds in translating its underlying theory to the instructional setting. As explained in Chapter 2, the teachers were provided with a manual with information about each activity to prepare for the session (see Appendix 5). The observations in this paragraph show how even after discussing each activity beforehand with the researchers, the teachers sometimes had trouble implementing the activity in the way that it was originally planned.

In the “Guess My Rule” activity in Round 1, for example, the intention was for children to not only notice the “rule” of the sequence but also its “regularity”. However, the teacher missed the opportunity to discuss the “rule” and regularity in the pattern, because she was satisfied with the children alternating the colors of the pattern (e.g., yellow, blue, green, yellow, blue, green...). Although the children succeeded in continuing a pattern that alternated two colors, the teacher only repeated the elements of the pattern together with the children and subsequently moved on to another pattern. Apparently, not only was the context of the activity not strong enough to motivate the children to make patterns with more complicated and varied structures, but the teacher had not understood the essence of the activity well enough to (try to) motivate children in this.

In the “Picking Flowers” activity in Round 1, the children also reflected less on spatial structuring strategies than what is expected from a bottom-up, reinvention perspective (section 2.5). The teacher collected all the chips into one group, removed two and asked the children whether they thought that now the chips were easy to count. The children were primarily listening and filling in some of the teacher’s sentences. That illustrates how they had not really had the chance to experience and find out for themselves that using structure can indeed be advantageous. In addition, the teacher’s wording was not very clear because she talked about ways to “count” the number of chips rather than ways to find out how many chips there are. The teacher also mentioned how “not structuring can make it difficult to know which chips to take away”, although she probably meant that the structure can help clarify the number of chips that are missing. Again, these subtle differences in choice of words can have very directive effects on the strategies that the children use.

The challenge for teaching and implementing an instruction experiment is that teachers are required to both follow the instructions for the activities, and improvise to adequately respond to the children’s unanticipated solutions, all within the time constraints of the activity. This happened when Lara made a side-step towards trying to find a structure in the patterns of the necklaces. Her action was very promising because Lara was trying to relate repeated colors to a convenient way for determining a quantity. However, the teacher was not prepared to recognize this insight, so she did not continue with it, nor did she refer back to it during the subsequent patterning activity. What it does show, however, is that Lara was on her way to discovering a link between a structure that makes up a pattern and a structure that gives insight into a quantity. Had Lara had more opportunity to compare how she counted the yellow and subsequently the green beads, with how the teacher added the totals together, or rather to examine a necklace with two differently colored sets of five beads, then she may have been able to clarify the association for herself and indeed compare the egg cartons with another type of structure from the box.

The effects of the overarching context of the activities depended largely on how well the teacher could introduce it and recall it in every activity. For example, although in Round 2 the teacher thoroughly recalled the context with the children, she did not make as much use of this context as she could have done when she introduced the garden in the whole class discussion. She told the children that Ant was looking for some flowers and twigs, but she did not explicitly connect this to the ant hills that played such a prominent role in the previous activity. The children may have been more prepared to make use of structures in this activity if the teacher had made reference to the “way” the children had used the structure of the beginning of the construction to determine the rest of the construction, just like in the patterning activity. This may have tempted the children more to use this “way” of structuring the chips into repeated arrangements such as lines and rows. The significance of these types of structures is that they relate to basic repeated addition and can therefore help the child towards learning formal addition and multiplication procedures.

Notwithstanding such missed opportunities and miscommunications, the teachers also contributed greatly to the development of the activities. As indicated in Chapter 3, they offered many suggestions for the design of the activities during the preparatory talks. An example of the teacher’s contribution during the activity itself occurred during the “Picking Flowers” activity in Round 1. When Lara determined the correct missing quantity, on her own initiative, the teacher gave Lara white chips to place in the empty spaces. This was an effective way of making the empty spaces of the structure more visible to the children so that they could actually see the structure that helped Lara determine the number of missing chips.

8.2.6 Effects of an overarching context, an appealing setting, and a shared vocabulary

In Chapter 7, we elaborated on five patterns of observations that seemed to influence the effectivity of an instructional sequence. One of these was the significant role of an overarching context with its shared vocabulary, embedded in an appealing setting. The significance of this element was supported by many observations throughout Round 1 and, especially, Round 2 of the instruction experiment.

From the start of the first activity in Round 1, for example, the context proved to be effective in that the children indulged themselves in the activity; the teacher's tone of voice, the way she explored the box from the outside, and the way she finally let one child unpack the box, created a somewhat mysterious setting that greatly contributed to the children's excitement. Even though the children were already familiar with the idea of a box filled with "tricks" from the previous round, they were still intrigued and concentrated as they followed along with the teacher's story about the ant in Round 2. The children were quiet and focused. This was the first important step towards creating a shared vocabulary in this new overarching context. The new context also sparked the children's curiosity because they were still keen to know what was in the box. As Matt was unpacking the box, other children were getting up to see what else was in the box and to look at the objects a little closer. The children also spontaneously started counting the number of egg cartons that were in the box and connecting it to other settings ("that's how old I am!").

In Round 1, the teacher remarked how difficult it was to count all the objects that were in the box. This intrigued the children and they began to think about ways to organize the objects. Further, the game-setting in the card activity not only engaged the two competing children, but also the children who were sitting around them. This excitement extended the children's attention spans for more than the average twenty minutes. Moreover, in the patterning activity, the children had no trouble extending the first sequence of children. They were very excited about the activity and they took their role as ants very seriously. In assessing the activity afterwards, the teacher noted that the children's excitement greatly contributed to the success of the activity. Moreover, the teacher excited the children in how she put the green paper in the centre and placed the chips on it, and animatedly told them about the garden and its flowers. The highlight of the final part of the "Picking Flowers" activity came when Mark set off to rearrange his chips and called out "this is going to work!". This showed how the design of the activity motivated the children to improve their spatial structures.

The mathematical content of the overarching context in Round 2 is exemplified by how the dots on the papers (i.e., Ant's footprints leading to the Tool Box) triggered a numerical perspective in the class. When the teacher first asked the children what they thought that the dots on the floor could be, the children called out "six". It seems the children thought that the teacher expected them to tell her how many dots were on the cards. After the teacher clarified that she wanted to know what the dots were or where they came from, Rick first answered that they were “from dice”. This is also a mathematical answer. Similarly, after children suggested examples such as “a ladybird” and “little balls”, Simon called out “eggs”. It seems that Simon may have attended to the structure of the egg cartons in the Trick Box since eggs are not the most typical suggestion that could explain the dots on the cards. Even if it was the researcher’s presence in the classroom that may have steered the children towards thinking in terms of numbers, it was exciting to note during Round 2 that some children were thinking in terms of structures that they had encountered in the box two months earlier. Apparently, the activities in Round 1 with the structures in the box had created an awareness for spatial structuring.

Regarding the overarching aspect of the context, the introduction of the new context at the beginning of the instructional sequence in Round 2 was essential for subsequent activities in that it created a foundation for the shared vocabulary and the socio-mathematical norm about the convenience and sometimes necessity for spatial structuring. By positively reinforcing children’s use of structure and saying that this indeed was “an easy way”, the teacher contributed to this classroom mathematical practice of spatial structuring. Further, after several children participated in the counting game, the teacher ended the activity by reflecting again on the possible reasons why Ant had left the box behind in the class. This was an important part of the activity, because it required the children to think critically about the activity and to try to understand the underlying meaning of trying to find easy ways to determine an amount other than counting unitarily. This type of reflection was also expected to spark the children’s curiosity where one question may lead to more questions that would motivate the children to find out more about the “tools” in the box.

The effect of the shared vocabulary in the overarching context became apparent in the subsequent activities, when, as the class recalled the context, the children remembered that the ant was called Ant and that he had left behind many small footsteps and a large box. In fact, the children remembered exactly that there were “six steps every time”. Someone called out “3, 3” and “that’s six” and when the teacher picked up Ant, the children could once again see that this ant has six legs that left footprints on the floor of the classroom. Further, when the teacher asked the children what they had found in the Tool Box, the children remembered each of the different objects. This start of the activity is motivating because it illustrates how the children could relate to the context and how it helped them to focus on the structure of six. A focus on at least one such fundamental structure was expected to serve as the necessary basis for exploring more types of spatial structures; by relating to the children’s already available knowledge of and familiarity with the structure for six, the teacher could move towards analyzing analogous structures such as those for five, eight and ten.

The observations in this and the previous section mostly concern the children and their performance on the instruction activities. In the next section, we turn to another component of the learning ecology to examine the pro-active role of the teacher in performing the instruction activities. This will shed more light on what characterizes a learning ecology that can help children become more aware of spatial structuring and use spatial structures to abbreviate numerical procedures (i.e., the second part of the second research question).

8.3 The pro-active role of the teacher

In Chapters 3 and 7 and earlier in this chapter, we alluded to the important role that the teachers played in the instruction experiment. In this section, several observations are clustered into themes to illustrate how the teachers (and researcher) fostered constructive learning. The essence of constructive learning is to help the children “construct” their own mathematical knowledge (see paragraphs 2.5.1 and 2.5.2). As explained in section 7.4, however, in a Kindergarten setting this involves more “guidance” than is generally implied by the RME principle of guided reinvention. The observations show how the effect of instruction activities on kindergartners’ learning greatly depends on teachers’ (and the researcher’s) effort to stimulate children’s constructive learning.

Guiding the reinvention process. From the observations it becomes clear that the teachers (and researcher) play an important role in guiding the learning process so that the children do not have to reinvent mathematics from scratch. The teacher in the reinvention process challenges children to think more in the direction of the desired mathematical construct, particularly in a Kindergarten setting (section 7.4). For the first trial of the “Picking Flowers” activity, for example, several single chips were removed. This was expected to be an easy trial because the children could simply count the number of empty spaces in the structure. The confusion that the children experienced in using their arrangements was intended to encourage them to make structures that they could remember and use to determine the missing quantities. During the patterning activity in Round 2, the teacher explicitly guided the reinvention process as she helped the children interpret and lengthen a line with a pattern:

Teacher:	How should we lengthen the line? Jamal, take a look at the line (...) We’re going to make the line longer, just like we did before. Well, take a good look at the children. Go ahead and say it out loud, you don’t have to call out the names, just whether it’s a girl or a boy. We start with Salih: (pointing, the rest of the class joins in) boy, boy, girl, girl (Jamal says boy) Becky is a girl right?
Jamal:	Ah, yes.
Teacher:	We’ll start again. Boy, boy, girl, girl. What comes after that?
Jamal:	Boy!
Teacher:	Yes! Boy, boy ...

Jamal: Girl, girl.
Teacher: (with the rest of the class) Boy, boy, girl, girl!

Besides these directions, more subtly, the teacher's accompaniment suggested a rhythm that Jamal could follow and continue independently to extend the line. The result of this episode was that the class successfully continued patterning in the activity.

Interactive problem solving. Another way that the teachers cultivated the children's learning, was by encouraging them to interact with each other as a way to build on each other's insights. They did this cleverly through the type of questions they asked, the types of responses they gave and the type of feedback they offered. One type of question was to ask about children's thinking processes ("how did you see that?" or "can you think of an easier way?"). Sometimes the teachers used revoicing techniques to evoke reactions from the children ("you think there are five?" or "you say the next color in the pattern should be red").

The teachers' enthusiasm was also an important part of children's learning because it sparked children's curiosity and motivation to participate in the activity. For example, as a result of the teachers' excitement about the different ways for seeing how two rows of three objects makes six, the children spontaneously started counting out loud and talking to each other about how they saw the six. The positive feedback that the teachers offered, helped to further stimulate children's explorations ("that's right!" or "you found a very clever way"). For example, the teacher expressed surprise when the children were fast to determine or compare a quantity. This positive feedback motivated the children to look for more strategies to quickly and efficiently determine a quantity.

An example of the teachers' role in guiding interactive problem solving occurred when the teacher supported a particular strategy by saying that that "is a really useful trick" and then asked Lara to share her trick with the class. Likewise, instead of pointing out a child's mistake, the teacher invited the child to come forward and count the dots on the card again. This highlighted the different strategies that the children used to determine a quantity. For example, while one boy counted the dots, some children said they knew the number of dots just by looking at the card. One child who had not counted the dots unitarily, spontaneously explained "yes, because three and three is six". The way the teacher encouraged children to share their strategies exemplifies constructive learning that capitalizes on the children's own responses and differing levels of learning.

More instances of such constructive learning occurred in Round 2. For example, before the teacher removed chips from a configuration of three sets of five flowers, arranged like the five on dice, she asked Matt:

Teacher:	Matt, have you taken a good look at it? Now Ant will come to pick some flowers from the garden. (Several children join Matt and close their eyes while Ant picks four flowers from the garden)
Teacher:	(The children call out that they know the answer and that Ant has been very naughty) Matt, come sit by the garden and point to the places where you think the flowers are missing.
Matt:	(points outside of one of the sets of five) Here.
Teacher:	Ok, so you think one's missing over there. Where else?
Matt:	Here (points again to the outside of one other set of five)
Teacher:	Think about this (takes the die out of the Tool Box and shows Matt the face with five dots)
Matt:	Here (points to the middle of one of the sets of five where one chip had been taken away) (The other children try to take a look at the die)
Teacher:	There's one missing, you're right! Where else are the chips missing? Take a look at this die.
Matt:	(points outside a set of five) Here.
Teacher:	(places the die next to the garden) Look at this, where do you think one is missing?
Matt:	(points outside another set of five) Here?
Teacher:	Daria, let's see if you can show us where Ant picked his flowers.
Daria:	(carefully pointing to the correct empty spaces) There, there, there and there.
Teacher:	Very clever of you. But how do you know?
Daria:	Because here there are three and you need two more.
Teacher:	But I can also add them here (points outside of the set of five)
Daria:	Yes, but then you can't see it easily.
Teacher:	That's right, see, if we look at the die, then you know that it's missing a flower in the middle. (Daria nods in agreement)

This episode illustrates how the teacher built on the children's responses to show them how they could compare a configuration to a familiar structure like on a die. She used the discussion with Matt to introduce the dice to the class and to encourage the children to connect the dice configuration to the arrangement of flowers. This resulted in successful uses of the dice configurations for completing the structured arrangements of flowers.

Comparing spatial structures. Many observations in this chapter have already illustrated how the teachers compared spatial structures to help children recognize structures in relatively larger structured and unstructured configurations of objects. More specifically, in the focus group, the researcher took the teachers' example and encouraged the children to reconnect the spatial structure that was topic of discussion to the structures in the Trick/Tool Box. In the focus group activity, for example, Mark was asked to explain how he had seen the five on the card, and the rest of the group was asked where else they may have seen the same kind of five. By explicitly placing one familiar structure in a familiar context (i.e., dots on a die) next to a familiar structure in a relatively unfamiliar context (i.e., dots on the playing card), the researcher tried to help the children translate spatial structures across various contexts.

The teachers and researchers not only encouraged the comparison between different types of spatial structures, but also between different ways of representing a quantity within one type of spatial structure. In the focus group Egg Carton activity, for example, Mark explained that he saw the seven because "one more than six is seven". Becky, on the other hand, showed how she perceptually counted each egg. To that, the researcher asked whether this could be considered "an easy way" to determine the number of eggs in the carton. Mark then pointed to a row of four and a row of three and Becky agreed that she saw the difference between these ways of looking at the eggs. This opened up a discussion in the focus group about different ways of looking at one arrangement, which is valuable because these different ways could again inspire children to read off rather than unitarily count a quantity.

Highlighting spatial structure. Another way the teachers supported the children's constructive learning was by bringing spatial structure more to the fore. This occurred, for example, when the teacher tried to capitalize on James' structuring by asking him to show the class how the blocks could also be counted. At first this sequence of questions seemed very contradictory and confusing to the evolving classroom mathematical practice of structuring. Yet, when the teacher summarized this part of the activity for the class, she tried to clarify to the children how James' structuring strategy contrasted with the counting strategies that they were more familiar with. James re-explained his strategy and this time he said "because here there were two and there there were two so there has to be one in the middle". The "one in the middle" could allude to the dice dot structure for five that James was very familiar with. To support his point, the teacher subsequently took apart the construction so that the children could also see the structure of the two and two blocks. This episode was an important start to the activity because the children were presented with two different strategies for determining the number of blocks in this structured construction.

Likewise, when Daria spontaneously answered that the construction was made up of six blocks, she explained (pointing) "here there are three and here there are three". To support this structured observation, the teacher first took the top part off and then spread out the six bottom blocks so that the children could see the two rows of three that Daria observed (Fig. 8.8). Next, she rebuilt the construction and Simon deduced that the total number of blocks is ten. In other trials, the teacher also took off layers of blocks to help children who had difficulty determining the number of blocks. The challenge for the instruction was to appeal to the children's insight into layering and unitary counting of the blocks in a context that invited the children to explore how structure of a construction can simplify and abbreviate numerical procedures. In the focus group patterning activity, the children were asked to separate the patterns into equal parts and explore different arrangements that help to physically see the component parts.

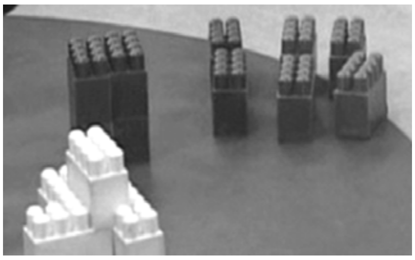


Fig. 8.8 Video frame of the blocks: the teacher removed the top block and spread out the bottom six blocks to show the children the structure of the structured 10-block construction

The themes discussed above illustrate how the teachers supported the children's constructive learning processes. The essence of the instructional setting was that it would stimulate the children to focus on what constitutes an efficient and effective structure, and that it helps them to learn to recognize these characteristics in unstructured settings. Such insight into structure can help them gain a better understanding of the composition and decomposition of quantities (i.e., numerical relations), which should ultimately contribute to the development of higher-order mathematical operations. The results of the instruction experiment which were discussed in Chapter 7 and in the previous sections are summarized in the next section.

8.4 General retrospective analysis of the instruction experiment

In sections 7.3 and 8.1, we documented the analyses of Round 1 and Round 2 of the instruction experiment in terms of the contribution of the instruction activities to children's development of spatial structuring ability. The observations that supplement the conjectured learning moments were discussed in section 8.2. Finally, through creating theory-type memos in ATLAS.ti, we organized our conjectures about these outcomes of the instruction experiment. This general retrospective analysis resulted in nine broad learning insights: these learning insights describe children's progression along a developmental trajectory for gaining awareness of spatial structures and learning to use and apply them for abbreviating numerical procedures (Fig. 8.9). Although the insights are listed in a generally cumulative order, they are intertwined like the strategies in the inventory (section 5.3).

	Learning insight	Brief description and examples	References
1	Preparatory skills including synchronous and resultative counting skills, familiarity with fundamental spatial structures, and basic organizational skills for counting quantities	• can use counting strategies to accurately determine a quantity • recognizes dice configurations and finger patterns • rearranges objects into a line to keep track of which are counted	• 7.3.2 • 8.1.1 • 8.2.1
2	Understanding the characteristics of a pattern	• understands the regularity of structure that forms a pattern • abstracts the elements that make up the structure of a pattern • "it's all the time red, white..."	• 7.3.1 • 8.1.4
3	Awareness of spatial structures in composed structural settings	• abstracts structure from a relatively larger structure • recognizes 6 as two rows of 3 within a structure of 10 dots	• 7.3.2 • 7.3.3 • 7.3.4 • 8.1.3
4	Recognizing and alternating different types of spatial structures to represent one particular quantity	• can associate 6 on a die with 6 fingers in a finger pattern	• 7.3.2 • 7.3.3 • 8.1.4
5	Recognizing and alternating different configurations in one type of spatial structure to represent a particular quantity	• perceives 6 in an egg carton through (de)composing it as rows of 5 and 1 or as rows of 3 and 3	• 7.3.4 • 8.1.4
6	Recognizing and using spatial structure in spatial (3-D) settings	• recognizes and abstracts structure from a 3-D block construction	• 7.3.5 • 8.1.3 • 8.2.6
7	Alternating unitary counting strategies with goal-directed spatial structuring strategies	• understands the convenience of using spatial structure to read off quantities as an alternative to unitary counting strategies	• 7.3.3 • 8.1.2 • 8.2.2 • 8.2.4 • 8.2.6
8	Constructing adequate, effective and efficient spatial structures in (un)structured settings	• explores effective structures to increase chances of winning the Picking Flowers game • compares structures such as 3 piles of 3 chips to rows of 2, 3, 2, 2 chips	• 7.3.6 • 8.1.2 • 8.1.3 • 8.1.4 • 8.2.3
9	Generalizing and applying fundamental spatial structures: transfer	• makes spontaneous references to spatial structures outside the planned instructional activity • "like on dice"	• 7.3.6 • 8.1.4 • 8.1.5 • 8.2.6

Fig. 8.9 The nine learning insights with brief descriptions and examples, and references to paragraphs in Chapter 7 and 8 that relate to each insight

In summary, many observations from the two rounds of the instruction experiment have supported and supplemented the conjectured learning moments. This resulted in nine crucial insights that underlie a general developmental trajectory for spatial structuring ability. In the next chapter, the analysis of the instruction experiment is concluded with a quantitative and qualitative reflection on the outcomes of the post-interviews and teacher evaluations. The post-interviews are interpreted in light of the pre-interviews, and the teacher evaluations supplement our analysis of the pro-active role of the teacher with the teachers' own thoughts and ideas about the instruction experiment. The outcomes of these analyses give more insight into how the instructional sequence influenced children's approach to spatial structuring and numerical tasks after the instruction experiment.

9 The Post-interview and Teacher Evaluations

In Chapter 4 we described the development of the spatial structuring and number sense tasks for the pre-interviews. These interviews were held to gauge the children's levels of spatial structuring ability and number sense in terms of a developmental trajectory for spatial structuring ability that was outlined in four phases. The outcomes of the pre-interview are presented in Chapter 5, showing that, as expected, the youngest children were mostly classified in the lowest phase, and that the phase classifications generally agreed with the children's LVS scores.

This chapter is about the post-interviews that were conducted with the children shortly after Round 2 of the instruction experiment. These interviews are interpreted against the outcomes of the pre-interviews. The aim of the post-interviews is to gain insight into whether and how the instructional sequence influenced the performance of the intervention group (IG). Although the non-intervention group (NG) is not intended to be a control group, we make use of their post-interview outcomes to gain more insight into the influence of the instruction experiment on the development of IG children's spatial structuring ability. We discuss quantitative differences between the levels of strategy use that the IG used and interpret them in light of the strategies that the NG used during the post-interviews compared to the pre-interviews. Qualitatively, we study the influence of children's participation in the instruction activities on their post-interview strategies compared to their pre-interview strategies, and on how the strategies that the IG children used differed from those of the NG.

The quantitative outcomes of the post-interviews are discussed in relation to the pre-interviews and LVS scores in section 9.1. The qualitative analyses are presented in section 9.2. We conclude in section 9.3 that the results from the post-interviews point to a greater awareness of spatial structure and an improved spatial structuring ability, including flexible use of spatial structuring strategies, for the intervention group. Further, the results highlight the role of language in supporting the development of children's spatial structuring ability. These outcomes are confirmed by the IG teacher evaluations of the instruction experiment which are discussed in section 9.4.

9.1 Quantitative comparison of the post-interview to the pre-interview and the LVS scores

The post-interviews are analyzed in the same way as the pre-interviews (section 5.4). Using ATLAS.ti, each response to an interview question was scored with a category from the strategy inventory. This resulted in a phase classification for each child which could subsequently be compared to the phase classifications for the pre-interviews. The aim of this comparison is to see to what extent these children may have developed in their spatial structuring ability. We supplemented these comparisons with differences that were noted between the IG and NG regarding the change in phase classifications before and after the instruction experiment. Although the quantitative comparisons are not decisive (the setting was not controlled and the number of children in each group is too low for statistical analyses, see also paragraph 10.3.3), we made these comparisons to convey any tendencies in changed spatial structuring ability as a result of children's participation in the instruction experiment. This supplements the analysis of the instruction experiment (Chapters 7 and 8) and the qualitative analysis of the post-interviews that are discussed in section 9.2.

9.1.1 The post-interviews related to the pre-interviews

In section 5.2 the following four phases were outlined to describe a general developmental trajectory in children's spatial structuring ability for the interview tasks:

> Phase 1 (Unitary phase): The child recognizes almost no spatial structures and consequently neither uses nor applies structures to abbreviate numerical procedures.

> Phase 2 (Recognition phase): The child recognizes several fundamental spatial structures, but rarely uses or applies spatial structures to abbreviate numerical procedures. Instead, the child may rationalize the use of spatial structures in hindsight.

> Phase 3 (Usage phase): The child recognizes and uses most available spatial structures, but rarely shows initiative in constructing and applying its own spatial structures as a means to abbreviate numerical procedures.

> Phase 4 (Application phase): The child uses spatial structures in a goal-directed way and spontaneously constructs and applies spatial structures as a means to abbreviate numerical procedures.

In Fig. 9.1, a comparison is made between the number of children per class (IG or NG) and per grade (K-1 or K-2) who, in the post-interview showed a particular repertoire of strategies that coincides with one of the four phases (see also the post-interview graph in Fig. 9.2). The distributions for the pre-interview (section 5.4) are stated in parentheses.

	Phase 1 (Unitary)	Phase 2 (Recognition)	Phase 3 (Usage)	Phase 4 (Application)	Total number
IG K-1	(9) 3	(2) 3	(2) 5	(0) 2	13
IG K-2	(1) 0	(3) 0	(1) 4	(3) 4	8
Total IG	(10) 3	(5) 3	(3) 9	(3) 6	21
NG K-1	(6) 3	(1) 2	(1) 3	(0) 0	8
NG K-2	(1) 0	(1) 1	(1) 3	(6) 5	9
Total NG	(7) 3	(2) 3	(2) 6	(6) 5	17

Fig. 9.1 The number of children per Intervention group (IG) and Non-intervention group (NG) and per Kindergarten 1 (K-1) and Kindergarten 2 (K-2) showing a repertoire of strategies in the interviews that coincides with one of the four phases. The pre-interview distributions are stated in parentheses.

General trends in the combined IG and NG results. Analogous to the pre-interviews, the results of the post-interview supported our expectations that the youngest children in both the IG and NG were categorized relatively more often in the lower phases than the older children and vice versa (Fig. 9.1). Those children who, after the post-interview, were still classified in the Unitary phase (phase 1) or the Recognition phase (phase 2), were mainly K-1 children who could benefit from more practice for developing their counting strategies. Most children (15 out of 38) were classified in the Usage phase (phase 3) after the post-interview. This indicates that these children were by now familiar with the spatial structures that were discussed in the interviews. Although they used these structures when they were readily available to them, these children were not at the stage yet of spontaneously and goal-directedly applying spatial structure to abbreviate numerical procedures. Many of the older children (9 out of 17) were at the Application phase (phase 4) by the post-interview and seemed ready to begin to learn to use more higher-order procedures such as formal addition and multiplication.

Comparing the IG results on the pre- and post-interviews. A large number of IG children (13 out of 21) shifted from lower to higher phases between the pre- and the post-interview (Fig. 9.2). This means that three of them improved from recognizing none of the spatial structures to recognizing, for example, eight as two rows of four or six as two sets of three. They still, however, counted the objects to determine the total and were therefore classified in the Recognition phase (phase 2). Four IG children improved from the Recognition phase (phase 2) to the Usage phase (phase 3) because in the post-interview they demonstrated that they could make use of the available structures. Two IG children improved from the Usage phase (phase 3) to the Application phase (phase 4) in preparation for more formal mathematical operations such as addition and multiplication. Three IG children shifted two phases (from phase 1 to phase 3) and one IG child shifted three phases (from phase 1 to phase 4). The fact that three other IG children were classified in the Application phase

(phase 4) for both the pre- and the post-interviews, can be interpreted as ceiling effects (i.e., the children scored maximally for the pre-interviews because the tasks were not challenging for them) because they already were making the most goal-directed use of spatial structures to abbreviate their procedures (Fig. 9.2).

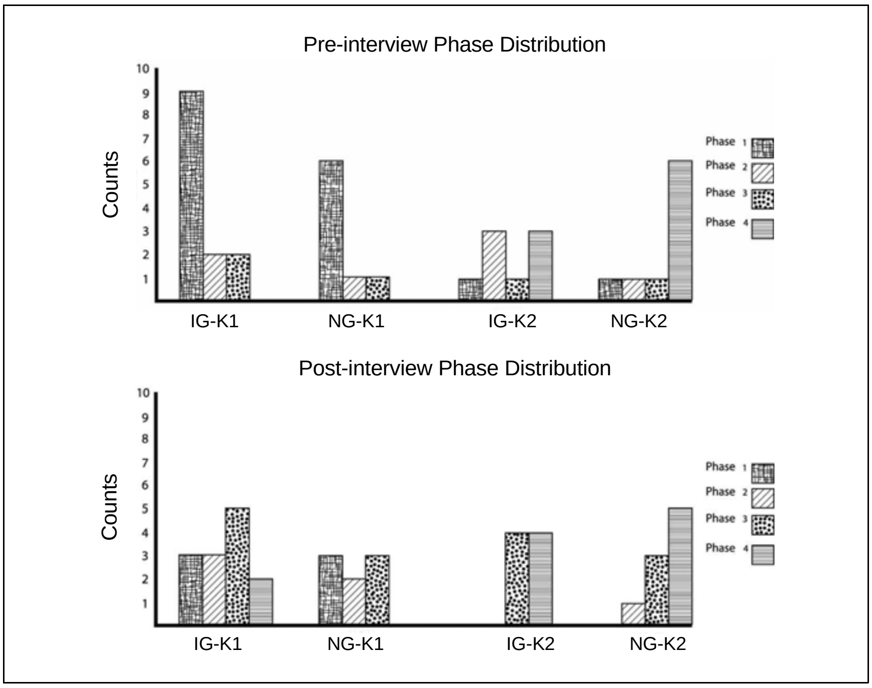


Fig. 9.2 A comparison between the pre-interview and post-interview for the number of (K-1 and K-2 from the IG and NG) children who were classified as Phase 1 (Unitary), Phase 2 (Recognition), Phase 3 (Usage), or Phase 4 (Application)

Comparing the IG pre- and post-interview results to the NG. In general, more improvements in performance occurred in the IG compared to NG (Fig. 9.2). In the IG (N= 21), 13 children shifted towards a higher phase while 8 children stayed in the same phase. Still, when taking the nuances (i.e., children whose repertoire of strategies tended strongly towards the next phase, but not enough to be classified into those phases) into account, 5 of these 8 children showed strategies that tended towards the next phase. More specifically, the IG K-1 group experienced a relatively strong decrease (6 children less) in the number of children who were classified in the Unitary phase (phase 1), while there was a relatively strong increase in the higher phases with 3 children more in the Usage phase and with two children now classified in the Application phase (phase 4). Similarly, while the IG K-2 children were classified in all four phases in the pre-interview, they had all improved towards either the Usage phase or the Application phase (4 children each).

In the NG (N = 17), 5 children improved, 11 children stayed constant (5 of whom showed ceiling effects because they belonged to the Application phase in both the pre- and post-interviews) and one child declined from phase 4 to phase 3. Taking the nuances into account, 3 of the 11 children whose phase classifications remained constant, showed strategies that tended towards a higher phase. Like the IG K-1 group, the NG K-1 group also experienced a decrease in classifications in the Unitary phase (3 less), but all these children were now classified in the Recognition phase (phase 2) and the Usage phase (phase 3) and not, in contrast to the IG K-1 group, to the Application phase (phase 4).

These quantitative outcomes must be interpreted in light of the fact that there were more IG children than NG children (21 compared to 17) and that, analogous to the ceiling effects, floor effects (i.e., the children scored minimally on the pre-interview tasks because the tasks were too challenging for them) may have influenced the pre-interview outcomes of some of the youngest IG children. Nevertheless, the outcomes show that (a) all the NG K-1 children were classified in either the Unitary phase (phase 1), the Recognition phase (phase 2), or the Usage phase (phase 3) in both the pre- and the post-interviews, while (b) the IG K-1 children were mostly (9 out of 13) classified in the Unitary phase (phase 1) in the pre-interview but, in the post-interview, they distributed over the Recognition phase, the Usage phase, and even the Application phase (phase 4). Regarding the K-2 children, the distribution of classification for the IG K-2 children shifted from all four phases to an equal distribution across the two highest (Usage and Application) phases. In contrast, the post-interview classifications for the NG K-2 children were similar to the pre-interviews apart from two children who improved to a higher phase and one who declined from the highest phase.

9.1.2 The post-interviews related to the LVS scores

To gain insight into the validity of the post-interviews, we related the quantitative outcomes to the children's LVS scores. For the post-interviews, four K-1 children were left out of the comparison with LVS scores because they had not taken an LVS test. The general distribution of the LVS scores did not change from the pre- to the post-interviews, because these scores are standardized and normed to the children's age. This makes it difficult to compare the frequency distribution across the phases, to the distribution across the LVS scores; the LVS distribution of scores seem to have stayed constant while the distribution of classifications in phases shifted. Nevertheless, most children who scored LVS *A*, were also classified either in the Usage (phase 3; 4 out of 14 children) or the Application phase (phase 4; 8 out of 14n children). Most children who scored LVS *B* were classified in the Usage phase (5 out of 7 children), while the repertoire of strategies was distributed equally across all four phases for those who scored LVS *C*. These outcomes agree with the general trend that was found in comparing the pre-interview classifications to children's LVS scores. This supports the validity of the interview assessments because they are in line with the normed LVS test scores.

Further insight into the validity of the interviews came from comparing how the pre- and post-interview outcomes related to the LVS scores. For the pre-interviews, the normed test scores for the IG children were relatively higher than those of the NG children, while particularly the youngest IG children had more trouble with the interview tasks than the NG children did. Hence, the pattern of phase classifications contrasts with how the LVS scores of the IG were initially higher than the scores of the NG. This suggests that floor effects (i.e., the children scored minimally on the pre-interview tasks because the tasks were too challenging for them) occurred in the IG. Yet, although the children in the IG K-1 group were perhaps more challenged by the activities in the pre-interview than the NG K-1 children were, they greatly improved their performance from the pre- to the post-interview. This could indicate that the IG K-1 children were not necessarily less capable than the NG K-1 children, but that the instruction experiment may have provided them with opportunities to improve their understanding of spatial structuring. For example, an IG K-1 boy was classified in the Unitary phase (phase 1) in the pre-interviews even though he received the highest score on the LVS test. Hence, he performed strongly on the LVS test, and yet he had trouble with the questions in the interviews. This outcome encouraged us to see how the interview tasks could be revised to improve their accessibility for the younger children who otherwise perform well in regular classroom activities.

The comparisons between children's level of strategy use and their LVS scores also highlights children who may require more attention in their mathematical development. Two IG K-2 children, for example, showed a repertoire of strategies that corresponded to the Usage phase, while they performed relatively poorly with a "*C*" score on the LVS test. These children had demonstrated interesting insights during the interviews, which may not have come to the fore in a standardized paper-and-pencil test. This outcome carefully shows limitations of the LVS test because it regenerates an awareness of the advantages and disadvantages of performing a regular paper-and-pencil test (section 3.3). It also shows how these children could benefit from additional work with spatial structures as a means to support their numerical development. In the next section we focus on specific observations in the post-interview that were analyzed qualitatively to investigate influences of the instruction experiment on children's spatial structuring ability.

9.2 Qualitative analysis of the post-interviews

In this section we present observations that show how the instructional sequence influenced the children's spatial structuring ability. These observations were also analyzed with the help of ATLAS.ti, but for the qualitative analyses of the interviews, the quotations were coded and interpreted in the same way as for the instruction experiment. This means that we used the method of constant comparison to code each observation (i.e., quotation) within each interview question. These codes were subsequently used to define various memo-types that took the observation to a more theoretical level. We discuss two themes that arose from this analysis, which differentiate the Intervention Group (IG) from the Non-intervention Group (NG) after their participation in the instruction experiment:

(1) The IG showed greater awareness of spatial structure and improved spatial structuring ability from the pre-interview to the post-interview than the NG did.
(2) The IG made explicit references to the classroom activities, while the NG had relatively more trouble understanding the interview tasks because they did not share the vocabulary that was developed in class to discuss concepts relating to spatial structure.

9.2.1 Awareness of spatial structure and development of spatial structuring ability

Support for shifts in phase classifications. In general, spontaneous and improved responses of the IG to the post-interview tasks were noted. The observations provide support for the four phases that were identified to describe children's development in spatial structuring ability (Chapter 4). As expected, the children who seemed to have benefited least from participating in the instruction activities were the youngest children in the class. These children were mostly concerned with learning how to count properly in the first place, and it may be that the teacher challenged them less than the older children. The focus on counting is what characterizes children whose repertoire of strategies corresponds to the Unitary phase (phase 1). Still, two children from the IG K-1 group showed significant improvement in their level of strategy use. Both children were classified in the Unitary phase (phase 1) in the pre-interview, while they performed according to Usage phase (phase 3) and Application phase (phase 4) standards in the post-interview. This improvement may suggest positive influences of the instruction activities on their spatial structuring strategies.

Less unitary counting in the IG. Although the number of children in each group is small and the differences are not statistically underpinned, we observe that, compared to the NG, the strategies that the IG children used were less varied, while they tended less to unitary counting procedures and more to spatial structuring with the specific intention of reading off a quantity. We illustrate this using the tasks where the children were asked to (1) determine which group of flowers contained eight flowers, and to (2) arrange the flowers in a way that could show others how many there are without counting (Fig. 9.3). These two interview tasks are representative for gauging children's ability to recognize, make use of and apply spatial structures to abbreviate procedures for determining and comparing quantities. Regarding the first task, 7 out of 17 NG children, compared to 14 out of 21 IG children, made use of the structure of the group to find out that there were eight flowers. Eight NG children compared to 5 IG children counted the flowers. The rest of the children had trouble answering the question or used a strategy that was unclear but that tended to structuring. For the second task, 6 out of 17 NG children compared to 11 out of 21 IG children goal-directedly structured the flowers. Seven NG children compared to 8 IG children organized the flowers in a way that simplified counting procedures. Further, 1 of the 7 NG children and 3 of the 8 IG children who organized the flowers, organized them in a way that tended to spatial structuring for abbreviating counting procedures.

Fig. 9.3 Lisa is counting the flowers unitarily as she is looking for the group of exactly eight flowers

More awareness of spatial structure in the IG. For the same flower task as above, compared to the NG, the IG seemed more aware of spatial structuring as an alternative to unitary counting. For example, 15 out of 21 IG children (including all the Application phase children), compared to 6 out of 17 NG children (including 4 of the 5 Application phase children), spontaneously recognized and made use of available structures such as two sets of three or two sets of four to determine six and eight. Although not all of these children came to the correct answer, for several of these children (7 out of the 15 IG children compared to 1 of the 6 NG children) this was a notable improvement compared to their primarily unitary counting strategies (i.e., the Unitary and Recognition Phase) in the pre-interview.

One IG K-1 boy, for example, related two sets of four as well as two sets of three to a total of eight. Although the two sets of three structure is not correct, his focus on double-structures illustrates an improved awareness of spatial strategies compared to his performance on the pre-interview. Nine of these 15 IG children belonged to the K-1 group (in comparison, all 6 NG children belonged to the K-2 group) which shows how even relatively young IG children started to use double-structures to recognize two sets of three or four and to try to deduce the total quantity using these structures.

Next to the general development in phase classification for the IG which was noted in the quantitative and qualitative analyses, the NG stayed more constant in the phase classifications (11 out of 17 compared to 8 out of 21 in the IG). This was not only because of ceiling effects, since it seems that, regardless of the researcher's encouragements to attend to spatial structures in the tasks, the NG children preferred unitary counting strategies or more formal addition procedures rather than advancing in spatial structuring strategies. In fact, one NG K-2 child was classified in the Usage phase (phase 3) in the post-interview, despite his Application phase (phase 4) performance in the pre-interview. From the qualitative analyses it appears that this child was somewhat careless in the post-interview and that he was having trouble applying formal arithmetic to the tasks.

Reasoning about and flexible use of structures. An important indication for children's understanding of spatial structuring, is whether they can reason or act flexibly with structures. One IG boy, for example, had become very flexible in the types of strategies that he used, explaining that "if you put them like this (i.e., two rows of four), then you will always see that there are eight". Another child had been experimenting with her own types of spatial structures in the pre-interview (e.g., arranged in the shape of a flower), while in the post-interview she explained that she "saw six (i.e., two rows of three) and two makes eight". The children also explicated that "five looks like four with one in the middle", and regularly made reference to the dice configurations, "like on dice". For determining the number of blocks in a construction, one girl explained the structure by referring to "its rectangles" and by showing how she could count the blocks by twos. As she was analyzing one of the patterns in the patterning activity, another child asked herself "does it make sense or not?", making reference to the beginning of the pattern and checking whether its structure occurred regularly throughout the pattern.

9.2.2 Explicit references to classroom instruction activities and the role of shared vocabulary in understanding the task

References of the NG. As expected, the NG made no explicit references to instruction activities while the examples that several IG children gave were specifically related to the activities. The NG children did make reference to daily situations, however. One girl, for example, explained that she could "see" the structure of the block construction because she sometimes played with blocks at home. Another girl said she "knows six because when you have six people, then there are three and three sitting around a table". Such references give important insight into how children encounter and learn about spatial structures in informal and formal learning ecologies.

Transfer effects. The value of explicit references that the IG made to the classroom instruction activities is that they indicate how involved the children were in the activity. The children who recalled a specific activity or insights from an activity during the post-interview were demonstrating transfer effects that show the meaning and effectivity of the activity in supporting the children's learning. An example of the references that IG children made to the instruction experiment occurred when the researcher was arranging the four groups of flowers in the first interview task. The girl instantly remarked how she liked the game with the gardens that they had played in class. Apparently, this girl overtly connected the flowers in the interview to the flowers that Ant had picked from the children's gardens in the classroom instruction activity. Such references provide support for how the overarching context of Ant and its Tool Box appealed to the children and stimulated their performance in the instruction activities.

Some of the children showed spontaneous transfer effects when the interview question did not explicitly ask them to relate the task to an instruction activity. For example, on their own initiative, several IG children decided which color was most necessary for extending the pattern, before they had received any instructions for the task (see paragraph 7.6.3). They were not relating the task to the pre-interview task because these are the instructions from the instruction activity. Moreover, the children made reference to the patterning activity from the instruction experiment in performing this particular interview task. This demonstrates how actively involved they had been in the classroom activity and how patterning was becoming part of their cognitive repertoire.

Specific explanations of the IG. The IG children also gave more specific explanations than the NG children did. One girl, for example, repeatedly referred to "three, three" to denote the spatial structure of two rows of three during the post-interview in the same way that she had referred to such double-structures during the classroom activities. Similarly, another girl spontaneously used phrases from the shared vocabulary such as "you can see it in an even easier way" or "you can arrange it easily like eight", placing her hand in between two groups of four to explain how the spatial structure helps her to determine the quantity.

Awareness of structures. Judging from the children's questions and responses, the NG children seemed less aware of spatial structuring strategies as a way to approach the post-interview tasks than the IG children were. One NG girl, for example, asked whether she was allowed to count because she did not know of another way. Apparently, she was at least at that moment not aware of an approach that could involve spatial structuring for determining a quantity. This is noteworthy because the girl's overall performance improved in the post-interview while, despite the researcher's repeated references to spatial structures, she kept to her counting strategies.

Role of a shared vocabulary. A reason why the NG children had difficulty understanding the task was because these children were not used to thinking and talking about "convenient ways" to "see" or to "determine how many there are" and to "arrange things neatly". This is not surprising because these terms belong to the shared vocabulary that the teacher, the researcher and IG children had developed throughout the instruction experiment. Nevertheless, what the episodes show is that even the NG children with high LVS scores did not always spontaneously revert to spatial structuring procedures to abbreviate and improve their performance. It would be expected that high-achieving children make use of spatial structures in a goal-directed way if they really understand the convenience of spatial structuring. Moreover, this outcome once again highlights the important role of a shared vocabulary in stimulating children to focus on spatial structuring as an alternative and more convenient approach to a numerical task than unitary counting.

9.3 Conclusions from the post-interviews

Taking the qualitative and quantitative analyses of the post-interviews together, we conclude that the IG children showed signs of having benefited from participating in the instruction activities. Despite the small groups, the observations show relatively more use of spatial structuring strategies and explicit references to the instruction activities for the IG compared to the NG. IG children's phase classifications and LVS scores also provide support for the marked shift towards using and applying spatial structures for abbreviating numerical procedures. This was interpreted against the way the NG approached the interview questions; although the NG children performed relatively strongly on the pre-interviews, they showed less advancement in spatial structuring ability in the post-interviews than the IG children did.

The two themes (awareness of spatial structures and explicit references to spatial structuring) that were discussed in this section for a qualitative analysis of the influences of the instructional sequence on the IG, are reflected in the post-interviews that were held to gauge the IG teachers' evaluation of the instruction experiment. The outcomes of these interviews are presented in the next section.

9.4 Teacher evaluations

As described in the previous chapters, the two teachers who performed the instruction experiment played a key role in the design of the HLT and the instructional sequence. They helped bridge the theoretical perspectives with educational practice, and they both conducted and actively reflected on the instruction activities to contribute to their improvement.

9.4.1 Comparing mind maps

Before introducing the instructional sequence and the focus on spatial structuring to the teachers, these IG teachers as well as the teachers of the NG were asked to create a mind map about the constructs of number sense and spatial thinking. The aim was to see what the teachers' conceptions of these constructs were at the start of the research. Discussions with the teachers about these mind maps gave insight into their perspective on the relative importance of the constructs for mathematical development and on what kinds of instruction activities and materials they associate with these constructs. Moreover, the intention was to compare the mind maps of the IG teachers to those of the NG teachers to see whether, at the start of the instruction experiment, there were any significant differences in the way the teachers approached these constructs and how they taught them in class. The teachers were purposely asked to brainstorm about spatial thinking rather than spatial structuring, because the reference to spatial structuring could prime the teachers about the expectations of the instructional sequence. Instead, it was interesting to see what role spatial structuring may play in their conceptions of spatial thinking.

Since the teachers' mind maps were very similar to each other, there was not much to compare them on. The teachers all noted the characteristics that are most commonly associated with number sense and spatial thinking (Fig. 9.4). For number sense this included patterning activities, dice, recognizing quantities, counting, counting songs and games, numerical symbols, comparing, sequencing, and estimating. Some of the characteristics of spatial thinking that were noted are constructing, distance, time, orientation with maps, patterning, measurement, crafts, shapes, perspective taking, physical activity, and technology. In short, the teachers associated number sense and spatial thinking with similar topics and materials. With more knowledge about what materials the teachers used in class, the mind maps provided a frame of reference in which the instruction activities could be discussed, and in which to evaluate each session with the teachers during the instruction experiment.

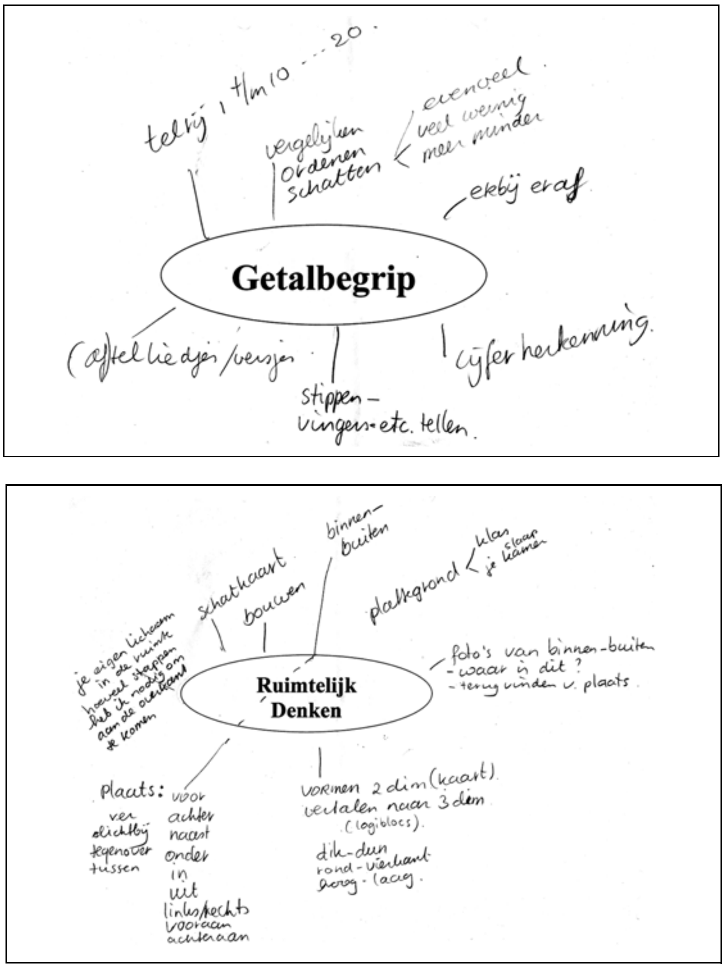


Fig. 9.4 A mind map for number sense ("getalbegrip") and spatial sense ("ruimtelijk denken")

What is most striking about these mind maps, is that they were very focused on procedures. In other words, the teachers mostly seemed concerned with children's ability to perform certain procedures such as counting, determining locations, and identifying and comparing shapes. What is particularly relevant to this study is that, although "recognizing quantities", "comparing, organizing, and estimating", "patterns", and "dice configurations", were mentioned, none of the mind maps alluded to connections between these concepts, nor did they include words that relate to spatial structuring or numerical relations. Hence, what is missing from the teachers' conceptualizations of number sense and spatial thinking are relationships between the components of the constructs. If these conceptualizations are indeed translated to their educational practice, then it may explain why spatial structure is often underexposed in class and why children tend to experience difficulties in learning mathematics that is taught as separate constructs.

9.4.2 Post-interviews with the teachers

As described in Chapter 3, the aim and design of the activity was discussed with the teacher before each session, and the teacher evaluated each activity in a debriefing immediately after the end of the session. Apart from the many relatively informal discussions that were held in between the activities, the two IG teachers were interviewed two months after the instruction experiment. These interviews serve as a final evaluation of the instructional sequence, to see what influences the instruction experiment had on these teachers' perspectives on number sense and spatial structuring in teaching and learning about mathematics. We present the teachers' answers to each of the five interview questions.

Question 1. In the first question, the teachers were asked whether they thought that their perspective on number sense and spatial thinking had changed as a result of the instruction experiment. The teachers responded convincingly. Teacher Alice said that her perspective had not changed as a result of the instruction experiment, because she felt she had always been aware of things that have to do with number sense and spatial thinking in the classroom. What she said that definitely did change, however, was her perspective on how to teach mathematics and spatial structuring. She declared that "*her* own world had opened up". Throughout the instruction experiment, she experienced how mathematics can be taught in a much simpler way using, for instance, patterning activities. She said that the instructional sequence highlighted the important issues in mathematics and that she is now more conscious about spatial structuring. This has given her confidence in recognizing and attending to spatial structures in class. One example of when this happened that she could recall, was when the class was celebrating a birthday and she discussed with the children how the six candles were conveniently arranged into two rows of three.

What frustrated Teacher Alice, is that the mathematics materials in the classroom and mathematics in general always seem subordinate to language lessons in their school. What she said this intervention helped her with, is to see "how the mathematics that is embedded in daily activities can be brought more to the fore with everyday language and meaningful situations". She could, for example, encourage children to look for structure in the playground and introduce new words to discuss this structure with the children. In this way, mathematics can gain more attention without requiring a significantly greater amount of class time or more domain-specific materials.

Teacher Tracy noted that she had always understood the importance of spatial structures. Yet, because this instruction experiment emphasized the role of spatial structuring in learning mathematics, spatial structures became tools that deepened her understanding of spatial thinking. The dice configurations, for example, are used often in class, but since the instruction experiment, these spatial structures are perceived and used in a more purposeful rather than incidental way. Teacher Tracy also explained that she was satisfied with an instructional sequence that consists of only five instruction activities because each activity repeats many of the insights about spatial structuring. These insights can be recalled again during other classroom activities.

Another point that Teacher Tracy made was that she realized once again how important a strong context is for children to learn mathematics. She said that Ant (the figure that is central to the overarching context of Ant and Ant's Tool Box) demonstrated how if something is presented in different way to children, then "children also pick it up in another way". The knowledge "will be established better". This confirmed what Teacher Tracy said she already knew, but often had trouble realizing in her lessons. The instructional sequence gave her more inspiration for using the already available materials and regular daily situations to perform mathematics in a relatively simple activity.

Question 2. In the second question, the teachers were asked whether the instruction experiment had changed their perspective on (a) the importance of number sense and spatial thinking in daily life, and/or (b) the role of number sense and spatial thinking in the development of early mathematical abilities. Teacher Alice answered that it had become more clear to her how "Kindergarten mathematics is not only related to numbers, but just as well to mathematical concepts". This includes an important overlap with language and it offers a natural way of integrating mathematics education with language education. For example, she could combine language lessons with contexts that highlight spatial structures. This kind of relation can stimulate mathematics education in school in parallel with language education. Moreover, the intervention encouraged Teacher Alice to work with spatial structures on her own. She was confident that she would attend to the topic of spatial structuring in the new school year and help the children become more aware of spatial structures in their surroundings.

Teacher Tracy explained how she had learned more about the mathematical development of the children. She had become more aware of the different levels of spatial structuring ability in class; how some children had never encountered particular spatial structures and how these children familiarized themselves with structures throughout the instructional sequence, while others were learning to make use of or apply the structures to abbreviate numerical procedures. Teacher Tracy noted that the children especially seemed to have improved in their structuring ability after participating in the “Guess my Rule” patterning activity. She thought that, by the end of the activity, most children understood the essence of a pattern and were able to continue the pattern on their own on the basis of its structure.

Teacher Tracy also said she considered the significance of the intervention to be more of a long term issue. In her opinion, spatial structuring is especially important for numerical procedures such as formal addition and multiplication. These young children, however, are still developing their ability to count and understand number, so Teacher Tracy thought that they would only benefit from the activities if they can “experience the *need* to spatially structure”. Her remarks are valuable because they agree with the theoretical background of the research. In line with RME principles, children must experience the necessity of spatial structuring to be able to start making goal-directed use of these structures. Teacher Tracy noted that many young children are satisfied with performing their counting procedures because they may not be ready to concern themselves with spatial structures, or because the activities were not designed properly enough to guide children towards prioritizing spatial structuring over unitary counting. Nevertheless, she agreed that the significance of exposing children to spatial structures at such an early age is that it can support children in gaining insight into numerical relations. These, in turn, are valuable tools for understanding higher-order mathematical operations such as addition and subtraction.

Question 3. To see whether the instruction activities had effects after the instruction experiment, the teachers were asked whether during or after the intervention they had seen children continue the activity, and whether this differs from how the children continue other classroom activities. The teachers referred to the evaluations that they had completed for each activity in which they described how some children made reference to spatial structures outside the classroom. For example, Teacher Alice explained how, during a regular lesson, the children had spontaneously made patterns in their mosaics (see also paragraph 8.1.5). The children were able to show her the structure of the patterns and express an understanding of a pattern. Further, Teacher Tracy noted that the children are now doubtlessly familiar with the egg cartons and dice structures. She recalled how, when she first asked one boy to determine the number of dots on a card, he did not answer. Yet, when she encouraged him to think about an egg carton, he immediately said “six!”. Taken together, Teacher Tracy said that she had become more aware of the importance of teaching mathematics to kindergartners and of structures in their surroundings. She also added that she thought that five activities were sufficient for establishing an awareness of spatial structuring, although it would be important for the teachers to continue making reference to spatial structure in the regular classroom activities. She thought that in this way, one run through the instructional sequence would be just as effective as the two rounds in the instruction experiment.

Question 4. For the fourth question, the teachers were asked whether during or after the intervention, they had gained new insights about how children think and learn about number sense and spatial thinking, and whether the intervention had changed their idea about children's learning abilities. Teacher Alice responded with three main points. First, she realized how clear the introduction to an activity must be for children to understand the instructions and the aim of the activity. Second, the activities emphasized the importance of properly formulated questions. The way a question is formulated strongly depends on what the teacher intends to gain by asking it. This became clear in the controversy of using phrases such as "counting" versus "determining" a quantity, and words such as a "trick", or a "way", or a "tool". Finally, Teacher Alice acknowledged that teachers must attend to the children's experiential world, but that that was difficult in this setting because the teachers were required to conduct the activities as they were described in the manual.

Teacher Tracy also mentioned the difficulty of understanding the activity and performing it exactly in the way that it was described in the manual. She said that she normally "lets things go the way they go" and that she intervenes whenever she sees an interesting learning moment, but that this was more difficult now that she had to follow the instructions. Nevertheless, Teacher Tracy also answered that what she observed during the activities confirmed what she knew about the general development that children experience in mathematics. She concluded that children ultimately go through the same phases, even though children may start at different phases. Teacher Tracy expected the developmental steps (i.e., learning moments) to take longer or shorter depending on the child, while some children may even skip a step.

Question 5. According to Teacher Alice, the most important outcomes and gains from the instruction experiment were that she thought that the children started thinking and looking differently at spatial structures, given that the activity appealed to the child. She also appreciated the changes that were made to the activities; the shift from a "Trick Box" to a "Tool Box" was appealing and effective in their classroom setting. Most importantly, however, Teacher Alice explained how the instruction experiment had changed her perspective on teaching kindergartners mathematics because it showed her "how it is possible to teach mathematics in a playful and fun way that extends beyond the curriculum and regular classroom materials". The focus group setting reminded her of how she usually works with children in smaller groups and it inspired her to encourage these groups to focus more on a particular mathematical topic. Teacher Alice said that she used to focus her mathematics lessons on what materials she had at hand, and that it took too much time and effort to devise new activities. For her, the local instruction theory and the instructional sequence offer an outline for an unprecedented way of teaching mathematics. The activity with patterning children and comparing structures to each other, for example, showed her how such activities and materials are more accessible than she thought.

Similar to Teacher Alice, Teacher Tracy responded that the instruction experiment had made her more alert. The curriculum regularly focuses on language, but this experience has emphasized the importance of fostering early mathematical abilities. It motivates her to try to find a better balance between the two subject areas by, for example, attending to the mathematics that is interwoven in language and vice versa. With regard to the children, Teacher Tracy found that they attended more to structures. Although she could not determine this objectively, she said that the activities seemed to have helped them gain insight into quantities and numbers. Her impression complements the conclusions that were drawn from the qualitative and quantitative analyses of the instruction experiment and the pre- and post-interviews.

9.4.3 Conclusions from the teacher evaluations

The teachers' responses to the interview questions reflect their positive attitudes towards the instruction experiment. Most importantly, even though spatial structuring per se is not new to a Kindergarten classroom, the outcomes of the instruction experiment highlight two "revolutionary" insights that the teachers said they gained and that are analogous to the themes that were identified in the post-interview analyses. First, the work that is typically done with spatial structures in class (e.g., dice and finger patterns), usually does not extend beyond learning to recognize the spatial structures. What this research adds, is motivation for teachers to help the children learn how to actually make use of the spatial structures to gain insight into numerical relations. The teachers had not realized that such insight can provide children with tools to understand the meaning behind number, numerical symbols and more formal mathematical procedures.

The second contribution of this research to the teachers' practice is that it offers them ways (i.e., the instructional sequence with its overarching context) to help the children translate one type of spatial structure to another. The teachers were used to using domain-specific materials that teach the children to recognize finger patterns and dice configurations as spatial structures. Yet, these materials rarely connect spatial structures to each other. The context of the Tool Box and the repeated references to the contents of the Tool Box throughout the instructional sequence, however, fostered more flexible thought about spatial structuring. It introduced the children to various types of spatial structures for a particular quantity, as well as to the various ways in which one type of spatial structure can represent a particular quantity.

An important requirement for a context of an instruction activity to be effective, however, is that the teachers are aware of the role of their language use and a shared vocabulary in children's mathematics learning (Van Eerde, Hayer, & Prenger, 2008). As such, the teachers learned to shift from questions about "how to count", to questions about "how to determine" a quantity because they became more aware of how such a difference in wording translated to a difference in children's interpretation of the question and consequently to the kind of strategy that the children used (i.e., unitary counting or spatial structuring). Moreover, the teacher's introduction of phrases such as "easy", "quick", or "convenient ways to find out how many there are", stimulated a shared vocabulary that created a shared understanding amongst the teacher and the children regarding the socio-mathematical norm of spatial structuring.

9.5 Summary

In this chapter we discussed the outcomes of the post-interview compared to the pre-interview and LVS scores, and the teachers' evaluations of the instruction experiment. The conclusion that can be drawn from analyzing the post-interviews, is that the children in the intervention group showed signs of having benefited from participating in the instruction experiment. In the interviews with the teachers, the teachers explained how the instruction experiment changed their perception of kindergartners' mathematics development and the role of spatial structuring in it. These two outcomes contribute to answering the second research question about how young children can be supported in learning to use spatial structures for abbreviating numerical procedures; it provides supporting evidence for the instructional effects of the instructional setting that was designed to stimulate children's spatial structuring ability, and it provides insight into the role of the learning ecology in children's learning. In the last chapter of this thesis, we reflect on the answers to both research questions and discuss implications for the outcomes with suggestions for future research.

10 Conclusions, Discussion, and Future Research

The purpose of the research that is documented in this thesis is to gain insight into the relationship between young children's spatial structuring ability and their emerging number sense, and to develop an empirically grounded local instruction theory about how to support children's spatial structuring ability. This purpose is divided into two research questions which will be answered in this chapter. First, we discuss young children's spatial structuring abilities and how these relate to their numerical performance (section 10.1). This insight was used to develop the local instruction theory about how to support children in learning to make use of spatial structuring strategies for abbreviating numerical procedures (section 10.2). In reflecting on the outcomes of the two research questions, the following topics are discussed in section 10.3: relating early spatial sense to emerging number sense, components of an effective learning ecology, limitations to the research, and implications of the research. This discussion leads to suggestions for future research in section 10.4.

10.1 Research question 1: Early spatial structuring ability

The first research question is stated as follows:

1. *What strategies for solving spatial and numerical problems characterize young children's spatial structuring abilities?*

To answer this question, we constructed a strategy inventory that can function as an interpretative framework for the strategies that characterize young children's spatial structuring abilities for solving the interview tasks (Chapters 4 and 9). The original design of the strategy inventory was based on literature reviews, input from experts, exploratory studies, and the first versions of the conceptual schema (section 4.1). As the development of the interviews progressed, these strategies were revised to contribute to a reliable, more thorough and encompassing instrument for gauging kindergartners' insight into numerical relations in terms of their spatial structuring ability. In light of the research focus on children's spatial structuring and number sense, the strategies pertain to children's ability to recognize, use and extend spatial structures within the domain of spatial structuring, and to children's ability to determine, compare, and operate with quantities within the domain of number sense. In addition, an activity for gauging spatial orientation was included and a method was defined for measuring children's mastery of the tasks (i.e., accuracy).

Four phases in the development of spatial structuring ability. The strategies are listed in a cumulative order from a strong tendency to count (asynchronously to synchronously and resultatively), to a tendency to organize the objects, to a tendency to spatially structure the objects, and finally to the goal-directed use and application of spatial structure to abbreviate numerical procedures such as determining, comparing and operating with small (up to 10) quantities. Through condensing the scoring procedure of each question for each ability in both domains (Chapter 5), general trends in strategy development were identified in terms of four levels of sophistication in children's spatial structuring ability. Consequently, a child's repertoire of strategies for the interviews could be classified as one of the following four phases:

Phase 1 (Unitary phase): The child recognizes almost no spatial structures and consequently neither uses nor applies structures to abbreviate numerical procedures.

> e.g., counts all the dots and finger patterns on the flashcards; counts flowers on the table one by one and leaves them spread apart as an "easy way" to read off how many there are

Phase 2 (Recognition phase): The child recognizes several fundamental spatial structures, but rarely uses or applies spatial structures to abbreviate numerical procedures. Instead, the child may rationalize the use of spatial structures in hindsight.

> e.g., recognizes most flashcards but counts flowers on the table unitarily even if they are already structured; recognizes structure only when explicitly guided to attend to structure

Phase 3 (Usage phase): The child recognizes and uses most available spatial structures, but rarely shows initiative in constructing and applying its own spatial structures as a means to abbreviate numerical procedures.

> e.g., reads off a structure and uses the structure to abbreviate a numerical procedure if it is available ("there are two rows of three so that's six"), but does not structure unstructured sets of objects (leaves them bunched in a group or spread apart)

Phase 4 (Application phase): The child uses spatial structures in a goal-directed way and spontaneously constructs and applies spatial structures as a means to abbreviate numerical procedures

> e.g., "I know it's 8 because here is the 6 like on dice, and 2 more makes 8"; rearranges unstructured sets of objects into structures such as smaller, subitizable groups or in dice configurations because "now it's easier to see and you don't have to count"

The general cumulativity of the phases implies that children who use one particular strategy, are also capable of using the previous strategies in the inventory. These phases have overlapping starting and end points; children are assumed to gradually shift from one phase to the next in the way that their main tendency to use a particular repertoire of strategies shifts towards a more sophisticated repertoire of strategies (Siegler, 2002, 2005; section 5.3).

The four phases in a broader perspective. The four phases that were identified in our study on the basis of the interview tasks, may globally be associated with Mulligan et al.'s

(2004, 2005) stages of spatial structuring ability and the levels of learning distinguished by Gravemeijer (1994, 1999) in the emergent modeling perspective (Fig. 10.1).

	Stages of spatial structuring ability (Mulligan et al., 2004)	Phases identified in the present research	Four learning levels in emergent modeling (Gravemeijer, 1994)
1	Pre-structural stage	Unitary	
2	Emergent stage	Recognition	Situational
3	Partial structural stage	Usage	Referential
4	Structural development	Application	General
			Formal

Fig. 10.1 A global comparison between the four phases of spatial structuring ability in this research (centre column) with the four stages of spatial structuring ability (Mulligan et al., 2004; left column) and with the four levels of learning in the emergent modeling perspective (Gravemeijer, 1994; right column)

Mulligan et al. used a broad set of tasks to explore children's use of mathematical and spatial structure in number, measurement, space and data. The *pre-structural* stage, where "representations lack any evidence of mathematical or spatial structure" (Mulligan et al., 2005, p. 1), coincides with the Unitary phase where children are not familiar with most spatial structures and therefore do not tend to make use of them. The *emergent* stage, where "representations showed some elements of structure such as use of units", and the *partial structural* stage, where "some aspects of mathematical notation or symbolism and/or spatial features such as grids or arrays are found", correspond to the Recognition and Usage phases, respectively, in terms of the development of the ability to make use of and apply spatial structures in mathematical contexts. Finally, the stage of *structural development*, where "representations clearly integrate mathematical and spatial structural features" matches the Application phase because both levels of spatial structuring ability imply a degree of spontaneous and goal-directed use of spatial structure to abbreviate a numerical task.

Similarly, our four phases can be associated with the levels of learning in emergent modeling (Gravemeijer, 1994; section 2.5). The Unitary phase precedes Gravemeijer's levels because children recognize almost none of the familiar structures and are predominantly focused on developing unitary counting strategies. Children who are classified in the Recognition phase, however, may be related to the *Situational* level of learning because the spatial structures that young children in this phase recognize are usually strongly related to specific contexts. For example, a child may know how to show four as a quantity using a finger pattern because the child is familiar with finger patterning as a way of communicating age. The child may also recognize six from its representation on dice used in games.

Although the children in the Recognition phase recognize particular structures, they require experience and practice to recognize the structures in a different task setting (e.g., six on dice translated to six as two rows of three flowers). Children who begin to make use of available spatial structures that are less situation-dependent, may be classified in the Usage phase. These children are learning to translate the spatial structures that they recognize from situational settings, to a specific mathematical task. The Trick/Tool Box created this learning setting in the instruction experiment. The instructional activities provided the children with opportunities to practice translating the spatial structures that they recognize from the objects in the box (e.g., “the six like on dice”) to other contexts and settings for the mathematical task of abbreviating numerical procedures such as determining and comparing quantities. Children in this phase also compare spatial structures and evaluate their use for abbreviating procedures in various contexts and settings. For example, children may read off a quantity according to how the objects are already structured, and they come to recognize that two rows of three represent the same spatial structure as three rows of two and that the two rows of three dots on dice reflects the same quantity as three fingers on each hand or as five and one finger. This relates to the *Referential* level in emergent modeling because the model (e.g., dice and egg cartons) derives its mathematical meaning from how it is referred to (e.g., “easy ways to see how many there are”) in the activity.

In the Application phase, children use the model in a goal-directed way. They understand that spatial structuring strategies are usually more convenient than unitary counting procedures, and they can generalize structures which helps to spontaneously make use of structure for abbreviating a numerical task. Moreover, their experience with spatial structuring contributes to their insight into numerical relations (e.g., 6 is 3 and 3 but also 5 and 1). Analogously, the model that is developing at the *General* level in emergent modeling, relates to spatial structures in how it derives its meaning from understanding relationships between types of structures, their association with numerical relations, and their use in solving mathematical tasks.

Some of the older children in our instruction experiment were classified in the Application phase, while, at the same time, they sometimes already exemplified insight into higher-order mathematical abilities (e.g., “it’s nine because I know 3 rows of 3 makes 9 and 3 times 3 is 9”). This illustrates how the insight that children gain by the time they reach the Application phase prepares them for understanding and performing higher-order operations such as addition, subtraction and multiplication without depending on physically available spatially structured objects. This is analogous to the *Formal* level of learning in emergent modeling, where children learn to reason with conventional symbols and they no longer depend on the original situations (e.g., dice and egg cartons) that underpin these spatial structures.

Taken together, the strategy inventory is an important outcome for answering the first research question in two ways. On the one hand, it functioned as an instrument for determining children's insight into numerical relations in terms of their level of spatial structuring ability in the interviews. As such, the classification of the children's strategy repertoires helped to gain insight into the influence of the instruction experiment on children's learning. On the other hand, the strategy inventory itself is an important interpretative framework that reflects a developmental trajectory in the types of strategies that young children applied to solve the interview tasks.

Validity and reliability of the strategy inventory. The association between the phases that were identified in our research and those of Mulligan et al. (2004) and Gravemeijer (1994), contributes to the validity of the strategy inventory and the interview tasks as an instrument for gauging children's progress in developing spatial structuring ability. The reliability of the strategy inventory and the interview tasks is supported by its high (0.87) interrater reliability. Further, the extensive documentation of the development of the interview tasks and the strategy inventory, of the analysis of the children's responses, of the condensation of the scores and of the classification of children's strategy repertoires into four phases contributes to the trackability of the research (Gravemeijer, 1994; Gravemeijer & Cobb, 2006; Maso & Smaling, 1998). This should make the research "virtually replicable" and therefore more reliable (Maso & Smaling, 1998).

The insight into young children's early spatial structuring ability that was gained from developing, performing and analyzing the interviews and the strategy inventory, formed the foundations for a hypothetical learning trajectory and an instruction experiment to investigate how children may be supported in their development of spatial structuring ability. This intervention part of the research is covered by the second research question which we answer in the next section.

10.2 Research question 2: Developing a local instruction theory

The second research question is stated as follows:

2a. How can young children be supported in learning to recognize and make use of spatial structures for abbreviating numerical procedures?

2b. What characterizes a learning ecology that can facilitate the development of children's spatial structuring ability?

Through retrospective analyses of the HLT in combination with the patterns that were observed in children's behavior during the analyses of the instruction experiment (Round 1 and Round 2) and analyses of the children's pre- and post-interviews, conclusions were drawn about how the instruction experiment helped children become aware of spatial structures and of the convenience of using and applying spatial structure to abbreviate numerical procedures in various contexts and settings. In the following paragraphs, we present outcomes of the instruction experiment and explain each of the components of the local instruction theory. The theory is summarized at the end of this section.

10.2.1 Children's developing spatial structuring ability

Although the non-intervention group (NG) is not intended to be a control group, we made use of their post-interview outcomes to gain more insight into the influence of the instruction experiment on the development of the intervention group's (IG) spatial structuring ability. Despite the small groups and the uncontrolled setting, from the comparisons between children's performance on the pre- and post-interviews, it appears that the IG children benefited from having performed the instruction activities: 13 out of 21 IG children shifted from lower to higher phases between the pre- and post-interviews (section 9.1). Relatively more improvements occurred in the IG compared to the NG where 5 out of 17 children improved, 11 children stayed in the same phase (5 of whom had already been classified in the highest phase in the pre-interviews), and 1 child declined a phase. Further, while all the Kindergarten-1 NG children were classified in one of the first three phases in both the pre- and the post-interviews, the classifications of the IG children shifted to the second, third, and even fourth phase in the post-interview. Finally, the qualitative analyses of the interviews show that the IG children were more aware of spatial structuring as a means to abbreviate numerical procedures than the NG children were, and that they made more spontaneous references to spatial structuring than the NG children did (section 9.2).

Not all children could be expected to reach the Application phase (phase 4) by the end of the instruction experiment, but they benefited from participating in the instructional sequence in their own way (e.g., practicing counting skills, learning to recognize structures, learning to use structures or learning to apply structures) for a greater awareness of spatial structuring. Moreover, as the teachers remarked (section 9.4), it is unlikely that children will improve in spatial structuring from participating only in the five instruction activities. Rather, the sequence will be most effective if used as an instructional sequence for establishing the socio-mathematical norm of spatial structuring, and for introducing related classroom mathematical practices. The teacher then has the important task of consistently referring to these practices, making use of the shared vocabulary to keep children aware of spatial structuring in other classroom activities.

10.2.2 The local instruction theory

The local instruction theory that resulted from the instruction experiment builds on children's reasonable counting skills, organizational abilities, and familiarity with basic spatial structures. The learning trajectory begins with introducing the children to the context of the instructional sequence, while the children explore fundamental spatial structures such as dice configurations, finger patterns and double-structures. Next, children are encouraged to use spatial structures that are readily available in various settings. This should prepare them for learning to spontaneously construct their own spatial structures in flexible ways for abbreviating numerical procedures. As such, the learning trajectory can help children to apply their insight into spatial structures to understanding higher-order numerical procedures such as addition, subtraction and multiplication. In what follows, we explain each component of the emerging theory.

The preliminary situation. The first learning moment assumed that children begin by organizing the objects that are to be counted in order to maximize their counting accuracy (paragraph 8.1.1). The observations from the instruction experiment added that some children were still learning to count synchronously and resultatively and that children at this level of learning cannot be expected to consider other ways of determining a quantity, such as by spatial structuring (paragraph 8.2.1). Children must at least be proficient at counting to be able to accurately determine a quantity in the first place (apart from subitizing quantities up to 4). This explains the connection between "counting quantities" and "spatial structuring" in the conceptual schema (section 4.4).

Further, from the interviews and instruction experiment, it became apparent that children should be familiar with spatial structures such as finger patterns so that they may compare these to less familiar structures such as double-structures. This should contribute to a flexible ability to recognize and make use of spatial structures in various contexts and settings. These three prerequisites constitute the starting points for the learning trajectory, so as to maximize the influence of the instructional sequence on helping children to compare strategies, recognize structures, and understand the convenience of spatial structuring as an alternative to unitary counting strategies.

Introducing the context and exploring fundamental spatial structures. Although context is not specifically related to spatial structuring, it was accentuated in this learning trajectory because of its prominent role in developing the socio-mathematical norm of spatial structuring. This norm challenged the initial classroom mathematical practice of unitary counting. In the instruction experiment, the introduction to the overarching context of the instructional sequence was the first step towards developing children's awareness of spatial structuring as a convenient way to determine a quantity. In this first step, the children became familiar with the overarching context of Ant and its Tool Box. As the children explored the objects in the Tool Box, the teacher encouraged them to compare the objects and to find out how the objects could function as tools for conveniently determining a quantity.

The development of the socio-mathematical norm for spatial structuring was further stimulated by the teacher's introduction to phrases that were to become part of a shared vocabulary for helping the class communicate about spatial structure. Already, the teacher used phrases such as "tools" and "easy ways" for "seeing how many of something there are". These were to become part of a shared vocabulary for discussing spatial structure throughout the instructional sequence. Further, the mystery about why Ant brought the box into class set the stage for the rest of the instructional sequence. It embedded each instruction activity into one appealing context and its mathematical content offered productive and situation specific strategies (paragraph 8.2.6).

Using readily available spatial structures in various settings. The mystery of Ant's Tool Box motivated children and kept spatial structuring as the topic of discussion. Hence, in the first two activities after the introduction of the context, the children worked at becoming more familiar with large structured dot configurations (i.e., the "Ant Steps" activity) and double-structures (i.e., the "Filling Egg Cartons" activity) to see how they could use readily available spatial structures as an alternative to unitary counting procedures (paragraph 8.1.2). The teacher encouraged the children to repeatedly associate the activity with the contents of the box so that, by translating relatively unfamiliar structures to more familiar structures and settings and vice versa, the children could improve their ability to recognize and subsequently make use of different kinds of spatial structures in various settings. This involved making use of the "tools" in the Tool Box to read off a quantity.

As such, through comparing, for example, symmetrical double-structures as represented by egg cartons to dot configurations as represented by dice configurations, the children were expected to recognize the common underlying structure of two rows of three. Such insight could help children to flexibly manipulate and use a greater variety of structures. All the while, the teacher consistently made use of the shared structuring phrases (e.g., "easy way") to create and maintain an awareness of spatial structure. As the children compared and manipulated the structures, they began to make use of these phrases themselves to discuss their approach to the activity. This marked the beginning of a vocabulary for spatial structuring that is shared between the children and the teacher.

Applying spatial structures to various settings. The next step in the learning trajectory builds on children's familiarity with certain spatial structures in structured arrangements (e.g., objects in the Tool Box that represent dice configurations and double-structures), to encourage them to recognize structure in more composite settings. In the "Marching Ants" activity, children decomposed a pattern into its constituent elements. These elements form the structure of the pattern. The teacher was expected to use the shared vocabulary to guide the children towards decomposing the pattern. Since the repetition (i.e., its regularity) of a particular structure composes a pattern, the activity required the children to think in terms of part-whole relationships. This draws upon children's experiences with the previous activities in which they explored how to recognize (i.e., decompose) structure in (larger) already structured configurations (e.g., 6 is 2 rows of 3 or 3 rows of 2 on dice or in the egg cartons).

Children must first realize that they should start by studying the order of the elements from the beginning of the line because that sets the example for the rest of the line (paragraph 8.1.4). Next, they must find out how the beginning of the line continues throughout the rest of the line. One basic way of discovering this continuation is by relying on a rhythm that is elicited when the elements of the pattern are repeated out loud. This rhythm can bridge the already present elements of the pattern with the elements that are to be added. The next step, then, is for children to gain insight into what unit of the pattern is repeating (i.e., the structure). The challenge in this activity was to guide the children towards constructing patterns that alternate more than two colors so that they would come to understand increasingly complex patterns that vary in the number of colors as well as in the number of elements per color (e.g., "every time red, white, white, red...").

The "Building Ant Hills" activity took patterning a step further by translating the concept of recognizing a repeated structure, to a 3-D block construction. Using a shared vocabulary to encourage spatial structuring strategies, the teacher guided the children towards applying their insight into part-whole relationships to a 3-D setting in which they were to abstract structure from a block construction. It was expected that the children could note the regularity in the layers of blocks in the construction, and therefore could determine what the next layer should look like. By comparing the block constructions to the structures in the Tool Box, the children could also compare unfamiliar to familiar structures. Again, this approach to a block construction put spatial structuring forward as a strategy for determining the number of blocks in a layer and in the whole construction in an abbreviated way (section 8.3). This contributed to establishing awareness of spatial structuring.

Insight into spatial structures to prepare for higher-order mathematics. We conjecture that the experiences that children had in familiarizing themselves with various spatial structures, in learning to use these structures, and in learning to apply these structures to larger structured arrangements, should establish and secure their awareness of spatial structure and prepare them for learning to apply structures to larger unstructured arrangements. The activity that was designed to encourage children to apply spatial structures to unstructured arrangements is the “Picking Flowers” activity (Speciaal Rekenen, 2003). The game-like setting of this activity stimulated the children to look for effective ways of arranging the flowers so that they could win more flowers for their garden. This setting put the children into a position in which they encountered the advantages of using spatial structures. The children who succeeded in applying spatial structures to unstructured arrangements with the intention of simplifying and abbreviating numerical procedures, were considered to have captured the essence of spatial structuring. This was apparent from their insight into numerical relations (e.g., referring to six as two sets of three or three sets of two or five and one; paragraph 8.1.3). Such insight is expected to prepare them for higher-order arithmetic procedures (e.g., Griffin, 2004b), such as learning and understanding the principles of multiplication (i.e., repeated addition, Anghileri, 1989; Buijs, 2008) and, ultimately, algebra (Battista et al., 1998; Carraher et al., 2006).

By repeatedly associating the Tool Box with the activity, the teacher emphasized the socio-mathematical norm of spatial structuring. She helped the children shift their mathematical practice of unitary counting to spatial structuring by reminding the children to try to arrange the flowers in “easy ways” using spatial structures such as the “tools” that had been the topic of discussion throughout the previous four instruction activities. As such, starting from the children’s own structures, the game helped the children experience that, as soon as the number of chips increases, arranging chips in the shape of a house, for example, is less effective than applying dice configurations or double-structures. The children started by arranging the flowers in structures that are meaningful to them (e.g., flowers or houses). Subsequently, they revised these structures throughout the activity to come to structures that seem most effective to them (e.g., arrangements of five or two rows of three or in piles, paragraph 8.2.3). This activity shows how in this learning trajectory, spatial structuring was not imposed on children. Rather, guided by the shared vocabulary and socio-mathematical norm, the children “reinvented” the convenience of using spatial structure themselves, and they were encouraged to apply spatial structure to the instruction activities in a way that was meaningful to them (Freudenthal, 1973, 1991; Gravemeijer, 1998).

10.2.3 Summary of the local instruction theory

In Fig. 10.2 we summarize how the children are expected to respond to the mathematical tool in an activity, how the activity builds on the previous activities, and how the activity is expected to influence children’s development of spatial structuring ability. This highlights the relationship between the learning goals, the instruction activity, and the conceptual development of the children. It also underlies the design of the final sequence of instruction activities after they were tried out in Round 1 and Round 2 of the instruction experiment (Appendix 5).

The table is adapted from Gravemeijer et al.'s (2003) outline for describing a learning trajectory for measurement and flexible arithmetic. It includes the following components (Gravemeijer, 2004):

- Mathematical tool: the practical instrument for solving a mathematical problem (see section 6.3), or, in our case, the instrument that is the subject of the instruction activity.
- Imagery: a summary of what prior experiences and knowledge children are assumed to have for validating the use of a particular mathematical tool in the problem. Children's existing knowledge is expected to be a foundation for what the mathematical tool means to them in the instruction activity.
- Activity: offers children the opportunity to learn to make use of the mathematical tool, and to learn to understand and use the shared vocabulary for establishing the socio-mathematical norm of spatial structuring.
- Potential mathematical discourse topics: the topic of a discussion that is initiated as children come across several solution strategies (e.g., several ways of determining a quantity) in performing an activity. The discussion can contribute to the development of the socio-mathematical norm of spatial structuring. The teacher must anticipate these topics to effectively support the discussions.

The activity and discourse topic columns in the table (Fig. 10.2) share a component about the important role of a shared vocabulary and content-related motives for establishing the socio-mathematical norm of spatial structuring (cf. Doorman & Gravemeijer, 2009).

10.3 Discussion

In the first two sections of this chapter we answered the two research questions. This led to a description of the local instruction theory about how young children can be supported in developing an awareness of spatial structure for improving their spatial structuring ability. In this section we reflect on the significance of the research outcomes. The topics that are discussed concern how the components of spatial sense may support numerical development, what constitutes an effective learning ecology, limitations to the research, and implications for educational practice. We conclude the chapter with suggestions for future research.

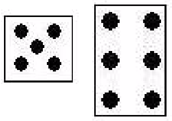
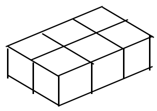
<table>
<tr><th>Mathematical tool</th><th>Imagery</th><th>Activity</th><th>Potential mathematical discourse topics</th></tr>
<tr><td rowspan="2">An appealing context with a box that is filled with ordinary objects that represent familiar spatial structures

</td><td rowspan="2">The objects in this classroom setting relate to real-life situations (e.g., egg cartons in the kitchen, dice for use in games)</td><td>Ant and the Tool Box:
Introducing the context and the mystery of the Tool Box</td><td>Introducing the context and exploring fundamental spatial structures: why and how Ant wants to share the contents of the Tool Box with the children. What are the contents and how do they relate to Ant?</td></tr>
<tr><td colspan="2">Introduction to phrases that express the idea of “tools” helping to “easily see” how many there are. Introduction to thinking about alternatives to unitary counting procedures.</td></tr>
<tr><td rowspan="2">The Tool Box including dice representing dot configurations, egg cartons representing double-structures, and fundamental finger patterns

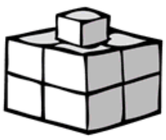


</td><td rowspan="2">The “tools” in the box refer to familiar spatial structures and support connections such as the structure for 6 on dice is like the 6 Ant feet, which is also similar to the two rows of 3 eggs in the 6-egg cartons. This can be compared to ways of showing 6 using finger patterns.</td><td>Ant Steps:
Recognize dice dot configurations in relatively larger structured configurations of dots

Filling Egg Cartons:
Compare the number of eggs in an egg carton to other structures</td><td>Exploring and comparing dice egg cartons, and finger patterns as “tools”:
Recognizing the underlying structures in can help to simplify and abbreviate the process of determining the number of dots on the card or eggs in an egg carton. These structures can be compared to each other and to familiar finger patterns.</td></tr>
<tr><td colspan="2">Greater awareness of dice dot configurations, double-structures and finger patterns and the vocabulary that is to become taken-as-shared, as the structures are compared to other spatial structures in terms of ways to read off a quantity using the symmetry in the structure (e.g., 3 and 3 makes 6).</td></tr>
</table>

<table>
<tr><th>Mathematical tool</th><th>Imagery</th><th>Activity</th><th>Potential mathematical discourse topics</th></tr>
<tr><td rowspan="2">Abstracted spatial structures

</td><td rowspan="2">From context-dependent to context-independent structuring: 2 rows of any 3 elements can be recognized as the 6 on dice or as eggs in an egg carton or as two hands showing 3 fingers</td><td>Ant Steps: Determining the number of dots in a large structured configuration of dots

Filling Egg Cartons: Determining the number of missing eggs in an egg carton</td><td>Making use of the “tools”: Recognizing that relatively large structured dot configurations are composed of dot configurations such as on dice, and that the egg cartons resemble the double-structures (such as on dice), illustrates how spatial structures can abbreviate counting procedures in various contexts. The key is to abstract the underlying structure by recognizing it from more familiar structures (such as dice configurations or finger patterns).</td></tr>
<tr><td colspan="2">Greater awareness of general characteristics of spatial structures and their role in abbreviating numerical procedures. This becomes a topic of discussion using phrases that the teacher introduced to create a taken-as-shared vocabulary.</td></tr>
<tr><td rowspan="2">Patterning with children and with colors

</td><td rowspan="2">Translating the idea of (de)composing structures to patterning: Abstracting spatial structure helps to decompose a pattern into its elements and identify the structure that makes up the pattern</td><td>Marching in a Procession: Abstracting the structure of a pattern in order to extend the pattern</td><td>Abstracting structure from, and applying structure to, relatively large and more complex configurations: Recognizing the part-whole relationship between structure and pattern</td></tr>
<tr><td colspan="2">Greater awareness of how a repeated structure composes a pattern. More own use of the shared vocabulary to discuss the structure of a pattern and “easy ways” to extend it.</td></tr>
<tr><td rowspan="2">Structured 3-D block constructions
</td><td rowspan="2">Decomposing a 3-D structure into its constituent parts that signify previous spatial structures helps to gain insight into the construction and into numerical relations (e.g., layers of 4, 4, and 1 blocks makes 9 blocks)</td><td>Building Ant Hills: Abstracting the pattern in a block construction in order to extend the construction</td><td>Patterning as a “tool” for abstracting the structure of a 3-D construction to conveniently determine what the next layer of blocks should look like</td></tr>
<tr><td colspan="2">Uses the shared vocabulary to relate the block construction to patterning and to spatial structures from the previous activities</td></tr>
</table>

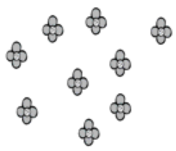
<table>
<tr><th>Mathematical tool</th><th>Imagery</th><th>Activity</th><th>Potential mathematical discourse topics</th></tr>
<tr><td rowspan="2">A randomly arranged set of uniformly colored chips (i.e., flowers in a garden)
</td><td rowspan="2">Applying spatial structure helps to arrange real-life objects that are to be counted, organized, or compared to other objects</td><td>Picking Flowers: Applying structure to unstructured objects in order to determine and compare quantities</td><td>Making use of the contents of the Tool Box as “tools” for spatial structuring: the larger the set of objects, the more difficult it is to keep track of the quantity, and the more convenient it is to find a way to structure the objects and avoid unitary counting procedures</td></tr>
<tr><td colspan="2">Goal-directed and spontaneous use of spatial structuring strategies and the shared vocabulary to contrast the abbreviated numerical procedures to unitary counting procedures</td></tr>
</table>

Fig. 10.2 Outline of the local instruction theory

10.3.1 Relating early spatial sense to emerging number sense

The conceptual schema that underlies the study (section 4.4), evokes questions about whether young children’s early spatial sense (in terms of the three components of visualization, orientation, and shape) could be supported with the specific purpose of fostering the development of their emerging numerical abilities. Considering children’s general improvement in spatial structuring for determining, comparing and operating with quantities, the results of the instruction experiment underline the important role of spatial structuring ability in helping children understand how to determine, compare and operate with quantities (see the central position of spatial structuring in Fig. 10.3). This suggests that spatial structuring may play a binding role in the development of spatial sense and number sense for higher-order arithmetic abilities. The analyses do not, however, provide enough information about how each of the three components of spatial sense specifically support numerical development. This is because children performed similarly on both of the interview domains (spatial structuring and number sense).

As such, we may only speculate about how spatial visualization, spatial orientation and shape played a role in children’s spatial structuring ability and their numerical development. Regarding the component of spatial visualization, it appears that children made use of several types of *mental images* throughout the activities (Owens & Clements, 1998; Presmeg, 1986). Concrete imagery could have been used to determine the number of corners in one of the shapes shown on the flashcards. Dynamic imagery relates to how children rearranged the flowers to come to effective spatial structures. Pattern imagery helped the children understand the structure of the patterns that they were asked to extend. Some children may have used action imagery to determine a number of flowers by mentally rearranging them. Finally, procedural imagery could refer to how, once children were able to apply structure to a set of objects, they were often able to repeat the procedure in a later task. These types of mental images closely relate to children's early ability to perceive *gestalts* (Quinn et al., 1993, 2002), as explained by the component of shape. Hence, we expect that children made use of the gestalt laws to recognize (composites of) shapes and figures. This suggests an intricate relationship between spatial visualization and shape regarding what components of spatial sense underlie children's ability to spatially structure (see the relative proximity of spatial visualization and shape in Fig. 10.3).

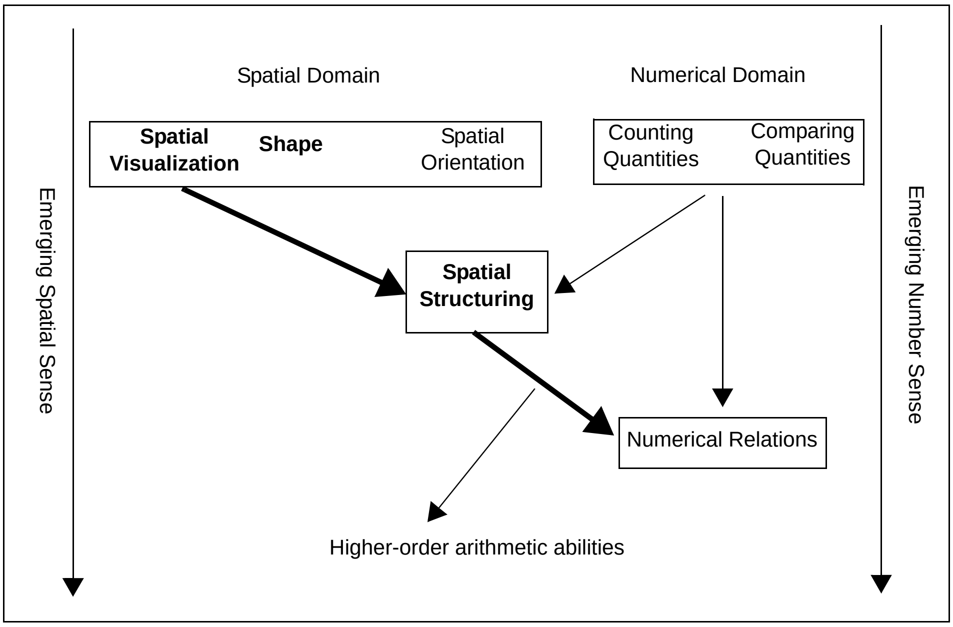


Fig. 10.3 The revised conceptual schema: a relatively reduced influence of spatial orientation on the development of spatial structuring ability, with spatial structuring as a binding factor between the development of spatial and number sense for the development of higher-order arithmetic abilities

It is still unclear how the component of spatial orientation may influence spatial structuring. Throughout the development of the research, the spatial structuring factor in spatial orientation seemed too broad to operationalize properly; the spatial orientation interview task involved localization and navigation skills and the instruction activities incorporated spatial orientation in terms of spatially structuring a spatial object such as the block construction. This was not sufficiently relevant to the research focus on (de)composing spatial objects and quantities. For the interview task, the children were first asked to show on a map of their Kindergarten how they would walk from one particular room (e.g., their classroom) to another location (e.g., the playground). Then they were asked to point in the air in the direction of a particular room (e.g., the gym, the kitchen). Half of the IG and the NG children either did or did not succeed at pointing in the correct directions, irrespective of their accuracy or level of strategy use for any of the interview tasks. As such, children's spatial orientation (i.e., localization and navigation) skills did not have strong effects on their performance on the interview tasks. In contrast, spatial visualization and shape were operationalized more strongly in the interview tasks and instruction activities. Hence, we have not yet found convincing support for the conjecture that the spatial orientation component influences spatial structuring ability as strongly as spatial visualization and shape could do (Fig. 10.3).

10.3.2 Components of an effective learning ecology

Part of the process of designing an instruction experiment (research question 2a), involves reflecting on what characterizes an effective learning ecology (research question 2b). A learning ecology includes elements such as the instruction activities, the role of the teacher and the researcher, and the classroom setting (Cobb et al., 2003). In section 10.2 we highlighted the important role of the instructional setting in designing, performing and evaluating the instruction experiment. In this paragraph we elaborate on section 7.4 and the IG teacher's evaluations of the instruction experiment (section 9.4) to highlight several components that characterize the pro-active role that the learning ecology can play in children's learning (Fig. 10.4).

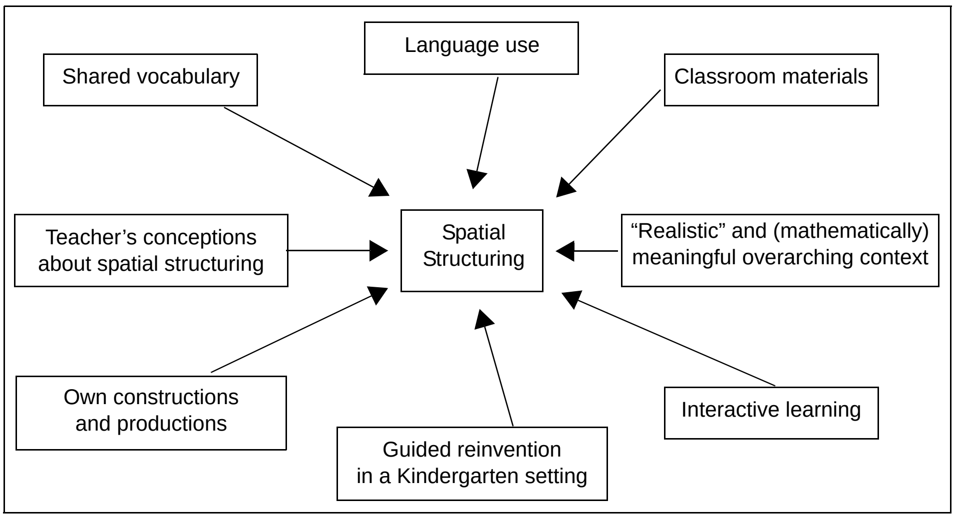


Fig. 10.4 Components of a learning ecology that influence the extent to which children may be supported in the development of their spatial structuring ability

An overarching realistic context. In line with the RME design principles, a realistic context was created to overarch the instruction activities and (implicitly) connect the conceptual knowledge that was expected to emerge within and between each activity. Ant and its Tool Box made the activities more experientially real for the children, motivating them to participate and improve their approach to the activities (see section 9.2). It not only provided an inspiring context in the form of an Ant and its Tool Box, but the Tool Box itself also served as a mathematical context that offered productive and situation specific strategies (De Lange, 1987). This shows how even such young children can be offered an engaging context that, at the same time, has a mathematical content.

The Tool Box context embedded the transition from a model-of to a model-for spatial structuring (Gravemeijer, 1994, 1999). Hence, from exploring the contents of the Tool Box (i.e., models of spatial structures), the children were encouraged to investigate how the tools could be used as an alternative to unitary counting procedures. This is how the structures became models for determining, comparing and operating with small quantities. In the end, the aim of the instruction activities was to support the children in reaching the level at which it would no longer be necessary for the models to be physically present (i.e., situation independent strategies).

Building on children's present understanding. In the "Picking Flowers" activity, the children first created flowers and houses, rather than, as would be expected, placing the chips in, for example, rows to abbreviate counting procedures. Still, the instructional sequence had to increase awareness of spatial structuring rather than unitary counting strategies according to RME principles (e.g., didactical phenomenology, section 2.5). In light of vertical mathematization, the instruction activities were designed to form an instructional sequence that guides children from a "realistic" problem to a problem that may be interpreted more in "mathematical" terms. At the same time, the children were expected to create a stronger foundation for understanding spatial structuring by constructing and working with their own spatial structures. Therefore, the instructional sequence was designed such that, after exploring examples of familiar spatial structures for inspiration, the children were encouraged to develop their own spatial structuring strategies in a variety of contexts and settings. This required the teacher to welcome children's own constructions and productions.

Interactive learning with a shared vocabulary. The importance of interactive education (Treffers, 1987) for children's learning is underlined by the role of a shared vocabulary in helping children shift from one repertoire of strategies to another and consequently from one phase of spatial structuring ability to the next (Chapter 5). An overarching context contributes to creating a shared vocabulary, because the teacher presents the activities in a way that evokes discussions about essential mathematical topics which could spur the children's understanding about spatial structuring (i.e., potential discourse topics in the local instruction theory). This can help to establish shared conceptions and synchronize differing opinions, strategies, and mathematical ideas about spatial structuring (paragraph 2.5.2). As such, the mathematical practice of unitary counting can be substituted with spatial structuring strategies to encourage the socio-mathematical norm of spatial structuring.

The socio-mathematical norm implies that children's ways of thinking and reasoning about spatial structuring coincides with the teacher's expectations. This highlights the importance of teachers' conceptions of the role of spatial structuring in the development of number sense. Any confusion in the shared vocabulary offers the teacher insight into children's understanding of spatial structuring. In the patterning activity, for example, the teacher noticed that the children had difficulty extending the line when she asked them what the "rule" or "way" of the line was. By guiding the children along the line so that they could decompose it into its structure, the teacher succeeded in helping the children understand what the "rule" of a line means and that finding this "rule" offers a "way" to quickly see how the line can be extended.

It became clear from the instruction experiment that the teacher plays a crucial role in guiding the children towards interacting with each other and towards comparing spatial structures, so that they can evaluate and subsequently improve the efficiency of their own constructions (section 7.4). Hence, guided reinvention in a Kindergarten setting puts a different accent on the traditional RME definitions of interactive learning (Treffers, 1987), because young children require more explicit guidance from the teacher to work together and discuss the mathematics (Leseman, Rollenberg, & Rispens, 2001; Nelissen, 2002). This is reflected in how the instruction activities had to be introduced, discussed and monitored within centralized classroom and focus group discussions that were closely guided by the teacher or the researcher.

In the next paragraph we discuss several issues that were encountered as the research was being designed and performed, and that may influence the interpretation of the research outcomes.

10.3.3 Limitations to the research

In this paragraph we summarize several methodological issues that were discussed throughout this thesis, and that may be taken into consideration for interpreting the outcomes of the research. First, it is important to emphasize that the local instruction theory pertains to the observations that were made with the children, teachers, and researchers in this specific instruction experiment that was conducted with the teachers and children of this particular Jenaplan Kindergarten class. Since the intervention and non-intervention groups are very small, and since the non-intervention group is not strictly a control group (section 3.1), our analyses are exploratory rather than confirmatory, offering trends that can be tested in future research.

Moreover, since the hypothetical learning trajectory that was set out in preparation for Round 1, was partly based on observations in the pre-interviews, it specifically fit the situation and the children that were studied. Hence, we do not propagate that the four phases that were defined to describe the development of spatial structuring, and the local instruction theory will describe the learning processes of other children. Conclusions can only be drawn about the children who were involved in the research, regarding how they performed the interviews and the instruction activities, how they responded to the revisions, how they interacted with the teacher and with each other, how the teacher guided the class, and what the instructional setting was like. Such conclusions acknowledge the many factors that influence children's development (e.g., type of schooling, language background, intellectual capacities).

The second issue concerns an often heard comment of empirical scientists about how conclusions cannot be drawn about the effects of an intervention if the same children participate to study revised versions of the intervention. Such a methodology would measure practice and developmental growth effects. To explain why it was decided to work with the same children in both rounds of the instruction experiment, we refer again to the aims of this type of research. The purpose of design research is not to set out one generalizable learning trajectory. Rather, the yield is a theory about the learning processes of the children who participated in this particular design experiment, with insight into how these learning processes may be stimulated in the instructional setting in which this design experiment took place (Gravemeijer, 2004). Hence, considering the complexity of a learning ecology, a coherent theory about the learning processes of children can only be developed if one and the same group of children is studied in an as consistent as possible instructional setting (i.e., the same teachers, materials, school). This emphasizes how it is not the performance itself that is the main interest to the research, but rather *how* the children approached the tasks and whether and how this approach changed between the two rounds of the experiment as a result of performing the instructional sequence. Such a theory informs teachers about *how* a theory worked in one particular setting. It can then motivate teachers to test the theory in their own instructional setting and add on to the cumulative nature of design research in the way that macro cycles add on to the instruction theory (section 7.1).

Notwithstanding these limitations, several practical implications could be distilled from the local instruction theory for fostering children's spatial structuring ability to support children's insight into numerical relations. These implications are presented in the next paragraph.

10.3.4 Implications for educational practice

The strategy inventory and the interviews. The strategy inventory that was developed to evaluate children's performance on the interviews, can offer teachers a valuable instrument for gauging children's insight into numerical relations in terms of their spatial structuring ability, in an appealing way. In the post-interviews with the IG teachers, they expressed their excitement about a form of assessment that they found to be more ecologically valid than the more traditional, paper-and-pencil LVS tests that they used. Future studies may, however, include more challenging interview tasks to minimize ceiling effects. Nevertheless, the significance of this inventory for educational practice is that it can enable teachers to highlight potential delays in children's development of spatial structuring ability and number sense at a very early stage. This answers to the need to identify and stimulate delayed developmental trajectories as soon as possible (see also Leseman, 2004).

From the analyses of children's approaches to the interview tasks, it became clear that children differ in the extent to which they use spatial structures rather than unitary counting strategies for abbreviating numerical procedures. A practical implication of such a developmental trajectory, is that it can highlight those children whose level of spatial structuring ability does not coincide with what is expected on the basis of their age and experience. Relatively older children who do not recognize or use spatial structuring strategies, for example, may be at risk of experiencing a delay in the development of higher-order arithmetic abilities. On the other hand, relatively young children who are already making use of spatial structuring strategies, could benefit from extra challenges to foster their relatively advanced mathematical development.

The local instruction theory and instructional sequence. The local instruction theory offers teachers a framework of reference for planning a Kindergarten mathematics curriculum that highlights spatial structuring. This can encourage teachers to consistently acknowledge the importance of spatial structuring in their classrooms, and to answer to the need for instruction activities that promote spatial structuring strategies rather than unitary counting procedures (Clements, 1999a; Clements & Sarama, 2007). First experiences with in-service teacher training activities are promising.

The significance of such a curriculum is also that it gives attention to how children's awareness of spatial structures and improved understanding of numerical relations can help bridge their informal mathematical abilities with the formal mathematical learning that is required as soon as they enter first grade. Indeed, the teachers remarked in the interviews that they considered this instructional sequence to be a valuable tool for helping to prepare children for the type of mathematical teaching and learning they encounter after Kindergarten (section 9.4). This could help minimize the dissonance between children's pre-school mathematical interests and understanding, and the relatively top-down formal mathematics education (e.g., Clements & Sarama, 2007).

The learning ecology. The instruction experiment also highlighted the influence of the learning ecology (i.e., the instructional setting with the instruction activities, the materials and the role of the teacher) in supporting children's mathematical development (see paragraph 10.3.2). The main educational implication of these influences is for teachers to become more aware of the crucial role that they play in children's learning. This includes the influence of their own perspectives on the importance of spatial structuring in early mathematical development. As described in section 8.3, the teacher played a central role in how the children interpreted the activity, how they approached the problem, how they verbalized their response, and how their response was interpreted by the teacher and adapted for formulating the next question in the activity. In fact, children's mathematics learning in school depends on what the teacher considers relevant to the field of mathematics. Indeed, in the post-interview with the teachers, they noted their surprise about the significance of spatial structuring ability for understanding numerical relations and for stimulating mathematical performance. Their changed perspective on spatial structuring can therefore initiate important shifts in classroom mathematical practices.

More specifically, as noted in section 9.4, the teachers had to get used to consistently having to use particular phrases that relate to the context and to spatial structuring (i.e., "tools" and "easy ways"), but they realized how essential their choice of words was when words such as "counting" resulted in different strategies than phrases such as "finding out how many there are". As such, the teacher's language use was essential for guiding children towards developing a shared vocabulary that underlies a socio-mathematical norm for spatial structuring.

The teachers also came to acknowledge how relatively simple activities and materials can appeal to the children and support children's spatial structuring abilities. To their surprise, even ordinary objects such as dice and egg cartons can give rise to a mathematical discussion about spatial structures. The teachers also became aware of spatial structures in their surroundings, outside the instruction activity, and remarked that they were therefore better able to integrate mathematics teaching and regular daily activities. They said this could improve the balance between the greater emphasis that they find is typically paid to language in Kindergarten curricula compared to mathematics.

As such, the observations provide concrete examples of practical situations that many teachers can recognize. Although the teachers in this instruction experiment were already aware of their role in the children's learning, it was not until we discussed the intentions of the activity and the effect of, for example, their choice of words or spatial structuring with the teachers, that they explicitly related such aspects of their role in the children's learning trajectory. This insight for teachers is essential because it influences the extent to which the potential mathematical discourse topics are acknowledged which, in turn, contribute to the development of shared ways of thinking and reasoning about spatial structuring. It can be a valuable step towards educational practice that acknowledges the importance of spatial structuring for fostering children's numerical insight (Clements, 1999a).

10.4 Suggestions for future research

The local instruction theory that is developed on the basis of the instruction experiment, is only the start of an important research trajectory regarding children's development of spatial structuring ability. We present several suggestions for future research in terms of content, methodology and educational practice.

Suggestions concerning content. This research has focused on one particular factor in spatial sense, namely spatial structuring, and its association with developing insight into numerical relations. As illustrated in the conceptual schema and in the analysis of the learning ecology, many more components of spatial sense and factors in an instructional setting are involved in the development of spatial sense and number sense. Now that we have gained more insight into the role of spatial structuring in developing numerical relations, it may be valuable to investigate what other spatial factors may influence children's insight into numerical relations, and number sense in general. Hence, future research could continue to study each of these factors to disentangle their relationships and interdependencies. This may, for example, shed more light on the role of spatial orientation on spatial structuring and whether it could influence numerical development. Other studies may also investigate how, for example, visualizing shapes may relate to the development of geometrical abilities, or what role language specifically plays in this context.

Future research could also extend the focus on spatial structuring ability to grade 1 and beyond. This would involve formulating a HLT that continues into formal mathematics to explore the effects of such an instructional sequence on children's development of higher-order mathematical concepts and procedures. This could bridge our research with studies that have focused on spatial structuring for children's arithmetic and algebraic development starting from the first years of formal schooling (e.g., Battista & Clements, 1996; Battista et al., 1998; Buijs, 2008; Van Eerde, 1996).

Suggestions concerning methodology. Future research could continue this design research by extrapolating the outcomes of the micro cycles to macro cycles. This means implementing the instructional sequence in another setting (i.e., another Kindergarten class at another school with a different school philosophy) to provide new impetus for adjusting the activities and for revising the hypothetical learning trajectory. After several of such instruction experiments, the instructional sequence may be suitably grounded to be tested in a larger-scaled, more experimental research setting with formal experimental and control groups. Such a confirmatory study would give insight into the effect of the activities on children's spatial structuring ability in comparison to how children are typically taught to count and prepare for formal mathematics.

Longitudinal studies could shed more light on the long-term development of children's mathematical thinking. This would involve comparing the mathematical performance of children who participate in the instructional sequence to children who take part in the regular mathematics curriculum, at different time intervals after performing the instructional sequence. Such a comparison could highlight the extent to which the instructional sequence supports children's spatial structuring ability at different points in their mathematical development. It may well be, for example, that although the instruction activities support children's ability to read off quantities at a Kindergarten level, this insight does not differentiate these children from the non-intervention group when both groups enter grade 1 and are taught higher-order mathematical procedures. Hence, a longitudinal study would contribute to manifesting the effects of the instructional sequence.

Suggestions concerning educational practice. The story about Ant's Tool Box has the potential to be extended to more contexts involving a figure with, for instance, two sets of two legs (e.g., cat, elephant) or two sets of four legs (e.g., spiders). Further, the Tool Box may be filled with objects that represent other spatial structures (e.g., dominoes instead of dice). New contexts and "tools" can lead to more instruction activities that may challenge children with a greater variety of spatial structures and provide more opportunities for children to practice spatial structuring.

Taken together, this research has illustrated the development and outcomes of an instructional sequence for fostering young children's early numerical insight. It sets the stage for future research that may contribute to ways of stimulating young children's early spatial and numerical abilities. In the end, the earlier we may understand (parts of) children's mathematical learning trajectories, the better we may furnish a supportive instructional setting to foster their mathematical development and offer them a head start in their formal mathematics education.

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Summary

Introduction

It is disconcerting that "early childhood education, in both formal and informal settings, may not be helping all children to maximize their cognitive capacities" (National Research Council, 2005, p. 3) and that a dissociation is perceived between young children's informal, intuitive knowledge and interests, and the formal learning opportunities at the start of their schooling. Many early elementary mathematics curricula, for example, recognize the importance of number sense, but appreciate young children's early *spatial sense* (i.e., insights that relate to spatial visualization, spatial orientation and shape; cf. Clements & Sarama, 2007) to a lesser degree.

The research that is documented in this thesis is aimed at investigating the role of young children's spatial structuring ability in the development of number sense, particularly in terms of insight into numerical relations, in order to improve early mathematics education. Insight into *numerical relations* involves the structuring (e.g., (de)composing) of quantities (e.g., understanding 6 to be 3 and 3, but also 5 and 1 or 4 and 2, Hunting, 2003; Steffe, Cobb, & Von Glasersfeld, 1988). This is considered essential for the development of higher-order mathematical abilities (Van Eerde, 1996; Van den Heuvel-Panhuizen, 2001). We take Battista and Clements' (1996) definition to define the act of *spatial structuring* as:

> The mental operation of constructing an organization or form for an object or set of objects. Spatially structuring an object determines its nature or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. (Battista & Clements, 1996, p. 503)

Through exploring and comparing, for example, symmetrical double-structures as represented by egg cartons to dot configurations as represented by dice configurations, children can come to recognize the common underlying structure of two rows of three. We propose that such insight can help to establish children's awareness of spatial structures, to support children's ability to recognize and manipulate such structures in various contexts and settings, and to anchor their use of spatial structures for abbreviating numerical procedures.

Background and research questions

The literature study (Chapter 2) that was conducted first, suggests that children's ability to (de)compose quantities is essential for the development of numerical insight. Numerical insight, in turn, underlies higher-order arithmetic abilities such as counting and grouping for part-whole knowledge in addition, multiplication and division (e.g., 8 + 6 = 14 because 5 + 5 = 10 and 3 + 1 = 4 so 10 + 4 = 14), for using variables in algebra, for proving, predicting and generalizing, and for determining the structure of a shape in order to subsequently mentally rotate or manipulate it (Anghileri, 1989; Buijs, 2008; Carraher et al., 2006; Papic & Mulligan, 2007; Steffe, Cobb, & Von Glasersfeld, 1988).

The ability to (de)compose quantities also involves insight into part-whole relationships. In describing spatial sense, we note a relationship between the three components of spatial sense (spatial visualization, spatial orientation, and shape) and part-whole relations in the (de)composition of spatial objects (cf. Clements & Sarama, 2007). First, in spatial visualization, the ability to manipulate mental images can support children in rearranging objects to explore their composition. Second, the spatial structuring factor in spatial orientation involves integrating previously abstracted items to form new structures. Third, insight into shapes helps children perceive, for example, parts and wholes, congruence, symmetry, and transformations. We suggest that these three components share a spatial structuring ability. This spatial structuring ability is the focus of our research, in which we study how it may influence young children's ability to (de)compose quantities for gaining insight into numerical relations.

The advantages of being able to recognize spatial structure and being able to apply spatial structure to abbreviate numerical procedures are evident, for instance, when reading off a quantity (i.e., recognizing six as three and three, Steffe et al., 1988; Van den Heuvel-Panhuizen, 2001; Van Eerde, 1996; Van Nes & De Lange, 2007), when comparing a number of objects (i.e., one dot in every one of four corners is less than the same configuration with a dot in the centre, Clements, 1999a), when extending a pattern (i.e., repeating the structure, Papic & Mulligan, 2005, 2007), and when building a construction of blocks (i.e., relating the characteristics and orientation of the constituent shapes and figures to each other, Battista et al., 1998; Van den Heuvel-Panhuizen & Buijs, 2005). In fact, children who focus on non-mathematical features and who continue to prefer to count objects unitarily without using any form of structure, may be prone to experiencing delays in their mathematical development (Butterworth, 1999; Mulligan, Mitchelmore, & Prescott, 2005).

Researchers who have studied children's spatial structuring ability and its influence on mathematical performance, call for more insight into the characterization of the developmental trajectory for spatial structuring, as well as into how an instructional setting may stimulate this development for supporting young children's insight into numerical relations. Therefore, the purpose of our research is to (a) contribute to an understanding of the development of young children's spatial structuring ability, and to (b) design a local instruction theory about *how* an instructional setting may foster this development and support children in learning to use spatial structures for abbreviating numerical procedures. This can encourage teachers to acknowledge the importance of spatial structuring in their classrooms, and answer to the need for instruction activities that promote spatial structuring strategies (Clements, 1999a; Clements & Sarama, 2007). The principles of Realistic Mathematics Education (RME; Freudenthal 1973, 1991; Gravemeijer, 1994; Treffers, 1987) and socio-constructivism (Cobb & Yackel, 1996) offered guidelines for designing, conducting, and interpreting the research. The research questions are defined as follows:

1. *What strategies for solving spatial and numerical problems characterize young children's spatial structuring abilities?*

2a. *How can young children be supported in learning to recognize and make use of spatial structures for abbreviating numerical procedures?*
2b. *What characterizes a learning ecology that can facilitate the development of children's spatial structuring ability?*

Methodology

Participants and setting. The study was conducted in a Kindergarten class at a local Jenaplan elementary school. The children at the school had mixed social and cultural backgrounds. The intervention group (IG) was a combined grade 1 and grade 2, for a total of 21 children ranging in age from four to six years. The class was taught by two teachers. The non-intervention group (i.e., the group that did not participate in the instruction experiment; NG) was one of the three other Kindergarten classes of the school and it consisted of 17 four- to six-year old children. This group was not intended as a control group, but rather to provide additional data for developing the strategy inventory and for analyzing the interviews. We also used their pre- and post-interview outcomes to supplement quantitative and qualitative analyses of the results of the instruction experiment.

The strategy inventory and the interviews. In order to answer the first research question, a set of tasks was designed to gauge children's spatial structuring and numerical ability in a one-to-one clinical interview setting (Van Eerde, 1996; Chapters 4, 5, and 9). These tasks were inspired by the literature study, experiences from several exploratory studies and consultations with experts. The strategies that the children used to solve these tasks were charted to create a strategy inventory. This inventory included a cumulative list of strategies for three components of spatial structuring (recognizing, applying and extending a structure) and number sense (determining, comparing and operating with small quantities). The data consisted of video recordings of the interviews, researcher's notes of the children's responses, and children's standardized school based assessment scores (LVS).

Considering the high interrater reliability with a Cohen's Kappa value of 0.87, the inventory was a reliable instrument for evaluating children's performance on the interviews. Further, the children's pre-interview and post-interview strategy repertoire coincided with the children's standardized test scores, which supports the validity of the interview outcomes and highlights the potential of the interviews as instruments for gauging children's insight into numerical relations in terms of their spatial structuring ability. The IG and NG were interviewed once before (the pre-interview) and once after the instruction experiment (the post-interview) to investigate any differences between both interviews regarding children's le-vel and type of strategy use.

The instruction experiment. Cumulative cyclic, classroom-based, design research (Gravemeijer & Cobb, 2006) was used to answer the second research question. This involved formulating, testing and refining a hypothetical learning trajectory (HLT) and a corresponding sequence of instruction activities for the instruction experiment (Chapters 6, 7, and 8). The HLT included testable conjectures that outlined how the instruction experiment was expected to influence the children's learning processes and provide empirical evidence for a local instruction theory about how young children can be supported in the development of their spatial structuring ability.

The instructional sequence consisted of five instruction activities that were tried out with the IG during two rounds of the instruction experiment. Each instruction activity started as a classroom discussion that was guided by the teacher. Then the researcher took five children aside (the focus group) for more in-depth discussions and detailed observations of their approaches to the activity. The data of the instruction experiment consisted of video recordings of each of the instruction activities, the questionnaires that the teachers completed for debriefing, the log that was written about what happened during the activity, and additional notes from discussing the activity with the teacher before and after the session.

After performing the instructional sequence, we analyzed our observations to see how the instruction activities influenced children's awareness of spatial structuring. Children's responses and behavior were compared to the observation criteria that were formulated as part of the development of the hypothetical learning trajectory. The dissonances between our observations and how the children were expected to respond to the instruction activities, led to adjustments in the hypothetical learning trajectory and revisions of the instructional sequence. After the analyses of Round 1, the revised instructional sequence was tried out in the same class during Round 2 of the instruction experiment.

Shortly after the second round, the IG and NG children performed the post-interviews. Their strategy repertoire for the post-interview was then quantitatively and qualitatively compared to the strategy repertoire on the pre-interviews, which they performed before the instruction experiment. This was to provide more insight into whether and how the instructional sequence influenced the children's development of spatial structuring. The teachers were interviewed shortly after Round 2 to evaluate how the instruction experiment influenced their perspectives on teaching about spatial structuring and on the role of spatial structure in young children's early mathematical development.

Data analysis. The qualitative data analysis followed the principles of constant comparison (Glaser & Strauss, 1967; Strauss & Corbin, 1998) with the help of the multimedia data analysis program ATLAS.ti. This program provides a format for organizing the raw data into clips that simplify the process of tracing and analyzing patterns in children's behavior and responses. In this way we were better able to establish how the children were solving the problems, how they were developing in their conceptual understanding, and what role the proactive instructional setting had played in this development.

The conceptual schema

The literature study, the strategy inventory, our exploratory studies and the consultations with experts contributed to the development of a conceptual schema. The schema associates the development of young children's spatial structuring ability (derived from three components of spatial sense, Clements & Sarama, 2007) with number sense (based on the combined ability to count and compare quantities; Griffin & Case, 1997; Griffin, 2004b), particularly in terms of their insight into numerical relations (understanding the (de)composition of quantities; Chapters 4 and 10). As the strategy inventory began to take shape, it served as a theoretical model that provided more insight into how these constructs could be related (Chapter 5). The final conceptual schema that underlies the research is outlined in Fig. S.1.

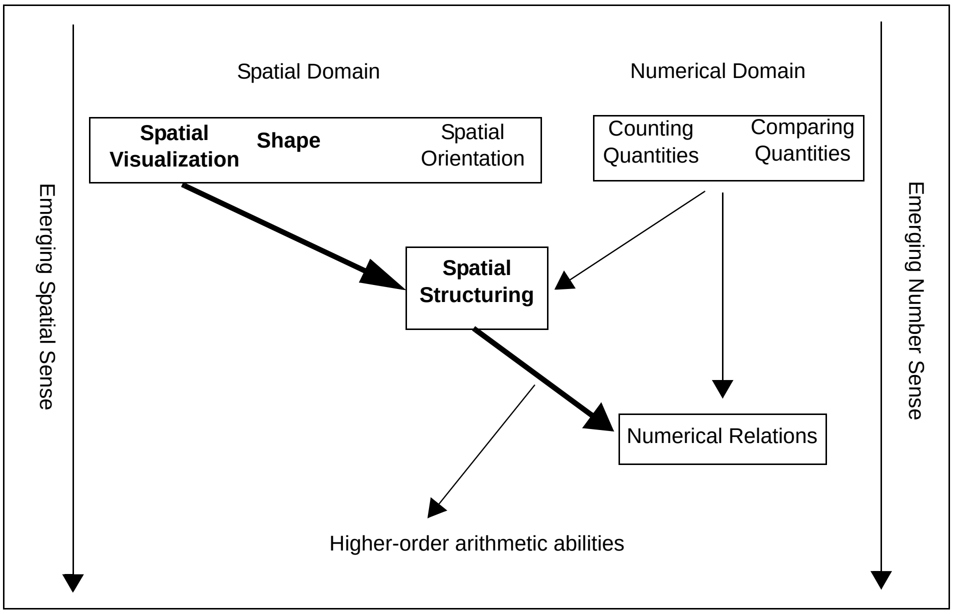


Fig. S.1 The conceptual schema of the research

This conceptual schema shows how the main focus of the research is on children's ability to spatially structure quantities and on the role of this ability in supporting insight into numerical relations. This should help improve children's learning and understanding of more higher-order arithmetic abilities such as addition, subtraction, and multiplication. The ability to "count quantities" is also connected to "spatial structuring" to indicate that children must at least be able to count resultatively before focusing on ways to abbreviate numerical procedures. In practice, children's ability to recognize spatial structures is expected to help them read off a quantity rather than use unitary counting procedures.

Phases in a developmental trajectory for spatial structuring ability

As the strategy inventory became an instrument for gauging young children's insight into numerical relations in terms of their spatial structuring ability, the scoring procedures had to be condensed for the instrument to become easier to use and more reliable (Chapter 5). As such, all the strategies that a child used to answer the interview tasks were collected and interpreted as a repertoire of strategies. This repertoire fit one of the following phases which describe cumulative levels of sophistication in the child's spatial structuring ability in this particular interview setting:

Phase 1 (Unitary phase): The child recognizes almost no spatial structures and consequently neither uses nor applies structures to abbreviate numerical procedures.

e.g., counts all the dots and finger patterns on the flashcards; counts flowers on the table one by one and leaves them spread apart as an “easy way” to read off how many there are

Phase 2 (Recognition phase): The child recognizes several fundamental spatial structures, but rarely uses or applies spatial structures to abbreviate numerical procedures. Instead, the child may rationalize the use of spatial structures in hindsight.

e.g., recognizes most flashcards but counts flowers on the table unitarily even if they are already structured; recognizes structure only when explicitly guided to attend to structure

Phase 3 (Usage phase): The child recognizes and uses most available spatial structures, but rarely shows initiative in constructing and applying its own spatial structures as a means to abbreviate numerical procedures.

e.g., reads off a structure and uses the structure to abbreviate a numerical procedure if it is available (“there are two rows of three so that’s six”), but does not structure unstructured sets of objects (leaves them bunched in a group or spread apart)

Phase 4 (Application phase): The child uses spatial structures in a goal-directed way and spontaneously constructs and applies spatial structures as a means to abbreviate numerical procedures

e.g., “I know it’s 8 because here is the 6 like on dice, and 2 more makes 8”; rearranges unstructured sets of objects into structures such as smaller, subitizable groups or in dice configurations because “now it’s easier to see and you don’t have to count”

These phases have overlapping starting and end points, and children are assumed to gradually shift from one phase to the next, according to how their main tendency to use a particular repertoire of strategies shifts towards a more sophisticated repertoire of strategies (cf. Siegler, 2002, 2005).

Strategy scores on the interview tasks

↓ condensed into

Four phases in the development of spatial structuring ability:

1. Unitary Phase
2. Recognition Phase
3. Usage Phase
4. Application Phase

Fig. S.2 Operationalization of the development of insight into numerical relations

The outcomes of children’s performance on the interview tasks (i.e., their strategy scores) and their corresponding strategy repertoire (i.e., phase classification) reflected children’s spatial structuring ability. As such, these phase classifications could be used to compare children's repertoire of strategies in the pre-interview to the post-interview.

This gave an impression of whether and how the instruction experiment influenced the children's approach to the interview tasks. In light of the proposed influence of children's spatial structuring ability on insight into numerical relations (Chapter 2), the phase classification operationalized the development of insight into numerical relations in the research as illustrated in Fig. S.2.

Development and results of the HLT and instruction experiment

Based on observations from our exploratory studies in several Kindergarten classrooms at a local elementary school, the learning goals of the instruction experiment, and the four phases in children's spatial structuring ability, we hypothesized initially that the children would encounter the following interrelated learning moments as they progressed along the learning trajectory (see Chapter 6):

(1) organize objects as a step towards becoming aware of spatial structuring to simplify counting procedures;
(2) create a motivation for spatially structuring objects;
(3) use spatial structuring to elucidate numerical relations;
(4) develop abstract spatial structures that are less context- or taskdependent;
(5) use spatial structuring in a goal-directed way outside the instruction experiment.

These learning moments inspired the outline of the hypothetical learning trajectory (HLT) for Round 1 of the instruction experiment. Considering the importance of patterning in gaining awareness of spatial structures (e.g., Papic & Mulligan, 2007), the first activity involved a patterning activity in which the children were to identify the structure of a pattern in order to extend it. The children were expected to use such insight into part-whole relationships to explore and compare various types of spatial structures that were presented to them in a so-called Trick Box (i.e., tricks to "easily see" how many of something there are) during the second activity. The third activity required the children to make use of part-whole relationships to recognize and read off structured quantities in relatively large structured arrangements on dotted cards. An alternative activity was developed that involved recognizing and reading off double-structures in egg cartons. In the fourth activity, children's awareness of spatial structure was taken to a 3-D setting in which they were asked to try to recognize and make use of the structure of a block construction. The children were expected to use and apply their experiences with spatial structuring in the fifth activity. They had to spatially structure a set of randomly arranged flowers in order to keep track of how many there are without counting them unitarily.

As more and more episodes of the videos of Round 1 were created, analyzed, and discussed with experts, several patterns emerged that gave insight into the role of the learning ecology (Cobb et al., 2003) on children's learning:

- The context within and between the instruction activities must not only be meaningful and inspiring to the children, but also offer productive situation specific strategies.
- A shared vocabulary, the type of language use of the teacher, and the way children interpret language is essential.
- The instruction activities relate the children's levels of understanding to a learning goal while acknowledging children's own productions.
- The teacher is an essential factor in promoting interaction between the children.
- The socio-mathematical norm of spatial structuring promotes a shared awareness of spatial structure in mathematical practices.

These patterns influenced the design of the instructional sequence for Round 2 of the instruction experiment (Chapter 7). In Round 1 each of the instruction activities had appealing contexts of its own. For Round 2 we introduced Ant and its Tool Box as an overarching context (Fig. S.3; Van Nes & Doorman, 2009). The creation of this overarching context was essential for making the instruction activities experientially real, for connecting the learning issues of the instruction activities, and for providing productive situation specific strategies in a mathematical context (De Lange, 1987). It also helped to create a shared vocabulary that is fundamental to establishing the socio-mathematical norm of spatial structuring (Gravemeijer & Cobb, 2006).

The Tool Box contained enlarged cards with finger patterns, large dice, enlarged cards with structured dotted configurations, egg cartons for six and ten eggs, and patterned bead necklaces (cf. Clements, 1999a; Mulligan et al., 2004; Van den Heuvel-Panhuizen, 2001; Van Eerde, 1996). The reason for choosing an Ant as the main character in this context, is that an ant has six legs (i.e., a fundamental and familiar spatial structure), it is strong and possibly able to carry this box into the classroom, the ant's name in Dutch conveniently relates to the name of the box ("Miertje Maniertje" and its "Maniertjesdoos"), and, finally, ants appeal to children's imagination. The story is that the Ant had "tools" ("handige maniertjes" in Dutch, translated as "clever ways") that it wanted to share with the class because the tools could help the children to conveniently determine a quantity.

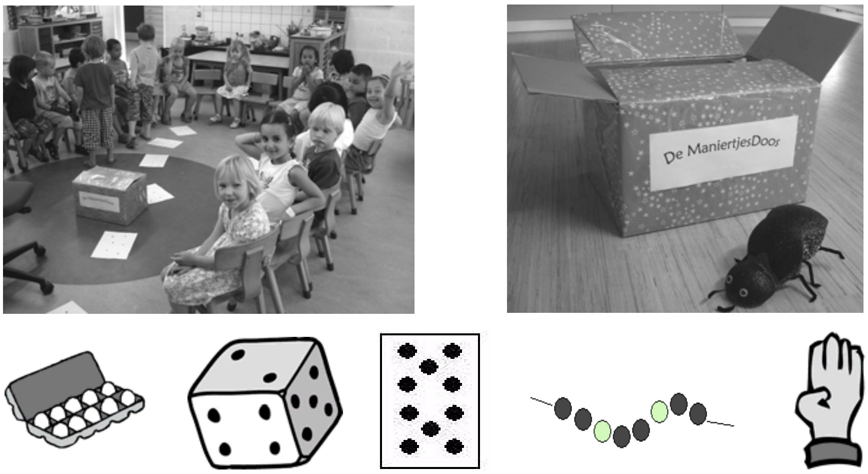


Fig. S.3 Ant and its Tool Box

The teacher created a shared vocabulary with phrases such as Ant's "useful tools for determining a quantity", to refer to the contents of the Tool Box. This helped to guide the children in exploring and comparing spatial structures, and to make spatial structuring and insight into numerical relations the topic of discussion throughout the instruction experiment (Chapter 8). The overarching context of Ant and its Tool Box was one of the main revisions that were made to the instructional sequence for Round 2. As such, the revised instructional sequence refined the operationalization of spatial structuring ability as conveyed in Fig. S.4.

Results of the instruction experiment. The analyses of the instruction experiment after Round 2 showed the benefit of an instructional setting that supports awareness of spatial structuring for fostering young children's insight into numerical relations (Chapter 8 and 9). The post-interviews showed that 18 out of the 21 intervention group (IG) children tended towards or were classified into a higher spatial structuring phase. This means that these children improved in their ability to use spatial structuring strategies for solving the interview tasks. The non-intervention group (NG) experienced relatively less improvements (5 out of 17) and more constants (11 out of 17, and 5 of whom showed ceiling effects) with one child who declined from the Application (phase 4) to the Usage (phase 3) phase. The IG increasingly started referring to spatial structures and making use of the shared vocabulary to discuss the conveniences of spatial structuring strategies over unitary counting procedures. This helped to establish a norm for spatial structuring. Moreover, the teachers who participated in the instruction experiment reported that they themselves had gained awareness of spatial structures as well as a greater appreciation for the importance of spatial structuring ability in young children's mathematical development.

Activity 1: The Tool Box and Ant Steps

- Establish the context
- Explore spatial structures in the Tool Box
- Compare representations of quantities on structured dotted cards

Activity 2: Filling Egg cartons

- Compare representations of quantities in double-structures
- Compare double-structures to other structures in the Tool Box

Activity 3: Marching in a Procession

- Explore structures of 2-D patterns
- Compare structures in patterns to other structures in the Tool Box

Activity 4: Building Ant Hills

- Explore structures of 3-D patterns
- Compare structures of block constructions to other structures in the Tool Box

Activity 5: Picking Flowers

- Apply structures to unstructured arrangements
- Compare representations of structures in the Tool Box
- Explore the role of structures in the (de)composition of quantities

Fig. S.4 Operationalization of spatial structuring in terms of the instructional sequence

Conclusions and Discussion

In this research, we first conducted interviews to identify a learning trajectory in the development of spatial structuring ability (research question 1). Through formulating a HLT and creating an instruction experiment, we then investigated how this development may be supported in an instructional setting (research question 2). This resulted in the following answers and topics for discussion:

Research question 1. Using the strategy inventory, young children's spatial structuring ability in this particular instructional setting can be classified as one of four phases. These phases describe a general developmental trajectory that ranges from no awareness of spatial structures, to learning to recognize spatial structures, to using such structures, to spontaneously applying them in a goal-directed way for gaining insight into numerical relations and for abbreviating numerical procedures such as determining, comparing and operating with small quantities. In this way, the interview tasks and the strategy inventory offer a tool for gauging young children's insight into numerical relations in terms of their spatial structuring ability. The strategy inventory itself is also an important interpretative framework in how it reflects a developmental trajectory in the types of strategies that young children applied to solve the interview tasks. This framework was an important inspiration for the development of the hypothetical learning trajectory for answering the second research question.

Research question 2a. The sequence of five instruction activities provides support for a local instruction theory that builds on children's reasonable counting skills, organizational abilities, and familiarity with basic spatial structures. The final learning trajectory begins with introducing the children to the context of the instructional sequence, while the children explore fundamental spatial structures such as dice configurations, finger patterns and double-structures. Next, children are encouraged to use readily available spatial structures in various settings. This should prepare them for learning to spontaneously construct their own spatial structures to abbreviate numerical procedures. In this way, the learning trajectory secures children's spatial structuring ability as a means to improve their insight into numerical relations, which in turn should prepare children for higher-order arithmetic procedures.

Research question 2b. From the instruction experiment, it appeared that an effective learning ecology involves building on children's present understanding, introducing a meaningful overarching context, creating a shared vocabulary, and including interactive learning. The factors in a learning ecology that appeared to influence the development of children's spatial structuring ability are pictured in Fig. S.5.

RME in a Kindergarten setting. Several observations gave the principle of "guided reinvention" a different character in the Kindergarten setting. The children's interaction with the teacher exemplified "guided reinvention" in the way that the teacher asked the children to share and compare their spatial structuring strategies and in the way she supported the children's spatial structuring strategies by, for example, taking apart a block construction to elucidate its structure. By asking two children of different mathematical abilities to work together, the children could compare spatial structuring strategies and set examples for each other. Still, given the age and relatively short attention spans of these children, the teacher had to do more than just "guide" the children. The teacher had to play a more directive role in encouraging children's interaction with each other and with the teacher, and in stimulating children to reflect on the activities to understand spatial structuring (cf. Leseman, Rollenberg, & Rispens, 2001; Nelissen, 2002). This illustrates how language, vertical interactions (Hatano & Inagaki, 1991) and influences of a classroom culture play a different role in Kindergarten than is generally implied by the RME principles.

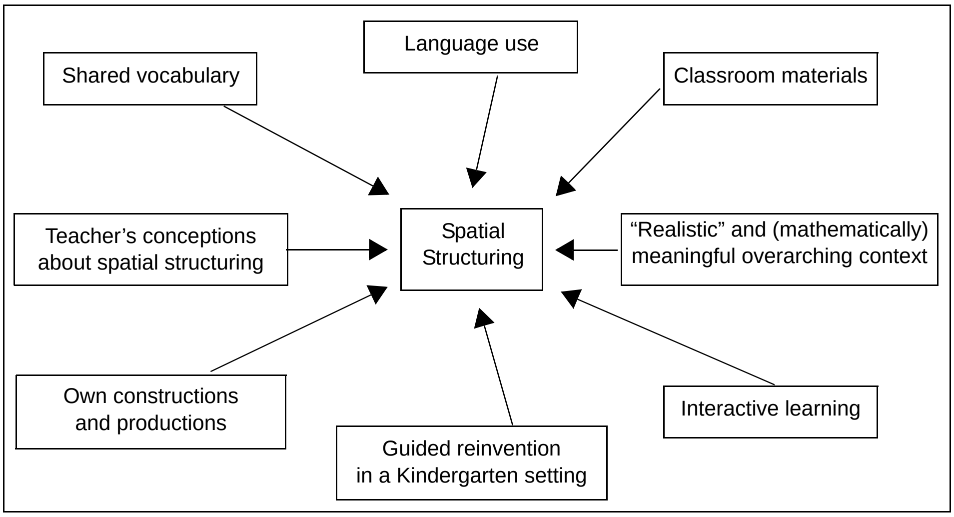


Fig. S.5 Factors in a learning ecology that influence the development of children's spatial structuring ability

Spatial sense related to number sense. Although an instructional sequence concerning spatial structuring helped children gain insight into numerical relations, more research is needed to disentangle specific influences of the spatial sense components on children's numerical development. In particular, no convincing support was found for influences of children's spatial orientation on their performance on the interview tasks. This explains the relative proximity of spatial orientation compared to the more closely related spatial visualization and shape components in the conceptual schema (Fig. S.1). Still, children's counting ability, organizational skills, and familiarity with spatial structures appear to be prerequisites for learning to use and to apply spatial structures.

Limitations to the research. Several issues had to be taken into account for developing, performing and properly interpreting the research. First, the local instruction theory pertains only to the observations that were made in this specific instruction experiment, which was conducted with the teachers and children of these particular Jenaplan Kindergarten classes. Second, the intervention and non-intervention groups were very small, and the non-intervention group was not strictly a control group, so confirmatory statistical data analyses could not be conducted. Notwithstanding these limitations, the value of this research is that it generates knowledge about *how* the instruction experiment influenced children's learning processes. The exploratory character of design research allows for a greater understanding of the complex learning processes that arise during instructional interventions (Gravemeijer & Cobb, 2006). This has offered several practical implications of the research.

Implications of the research. First, the interview tasks and strategy inventory can be taken as valuable instruments for gauging children's insight into numerical relations in terms of their spatial structuring ability. The phase classifications can help to highlight those particular children whose level of spatial structuring ability does not coincide with what is expected based on their age and experience. Second, the local instruction theory offers teachers a framework of reference for planning a Kindergarten mathematics curriculum that integrates spatial structuring activites. First experiences with in-service teacher training activities are promising. It also supports teachers in gaining insight into their role in supporting children's learning through establishing socio-mathematical norms; the teachers' choice of words, their flexible use of ordinary materials and meaningful contexts, and their perspectives on spatial structuring, greatly influence children's approach to the instruction activity. These implications appeal to the need for educational practice that acknowledges the importance of spatial structuring for children's developing numerical insight (Clements, 1999a).

Suggestions for future research. Future research can extrapolate this local instruction theory to other contexts and settings and contribute to its generalizability. The outcomes of the research welcome longitudinal research on long-term effects of the instructional sequence on children's learning. At a more practical level, the context of Ant and the Tool Box can be elaborated and tried out in various classroom settings. Taken together, this research sets the stage for future studies on stimulating children's early spatial and numerical abilities. In effect, the earlier we may understand (parts of) children's mathematical learning trajectories, the better we may furnish a supportive instructional setting to cultivate children's mathematical learning and offer them a head start in their formal mathematics education.

Samenvatting

Introductie

Het is zorgelijk dat “vroegschools onderwijs, in zowel formele als informele settings, jonge kinderen wellicht niet ondersteunt in het maximaliseren van hun cognitieve capaciteiten” (National Research Council, 2005, blz. 3), en dat er een dissociatie bestaat tussen de infomele, intuïtieve kennis en interesses van jonge kinderen, en de formele leermogelijkheden aan het begin van hun scholing. Veel reken-wiskundecurricula voor de basisschool bijvoorbeeld, erkennen het belang van getalbegrip, maar hebben minder aandacht voor de vroege *ruimtelijke inzichten* van jonge kinderen (bijvoorbeeld inzichten die te maken hebben met ruimtelijke visualisatie, ruimtelijke oriëntatie en vorm (Clements & Sarama, 2007).

Het onderzoek in dit proefschrift heeft als doel het bestuderen van de rol van het vermogen van jonge kinderen om ruimtelijk te structureren bij de ontwikkeling van hun getalbegrip, met name hun inzicht in getalrelaties, om zo het reken-wiskundeonderwijs te verbeteren. *Inzicht in getalrelaties* betreft het structureren (samenstellen en splitsen) van hoeveelheden (begrijpen dat 6 zowel 3 met 3, 5 met 1 als 4 met 2 is; Hunting, 2003; Steffe, Cobb, & Von Glasersfeld, 1988), wat essentieel is voor de ontwikkeling van meer gevorderde reken-wiskundige vaardigheden (Van Eerde, 1996; Van den Heuvel-Panhuizen, 2001). We passen Battista en Clements’ (1996) definitie toe om de handeling van het *ruimtelijk structureren* te omschrijven als

> De mentale operatie voor het construeren van een organisatie of vorm voor een object of een verzameling van objecten. Het ruimtelijk structureren van een object bepaalt zijn kenmerken of vorm door zijn ruimtelijke componenten te identificeren, door de componenten te combineren tot ruimtelijke samenstellingen, en door interrelaties vast te stellen tussen componenten en tussen samenstellingen. (vrij vertaald uit Battista & Clements, 1996, p. 503)

Door het onderzoeken en vergelijken van, bijvoorbeeld, symmetrische dubbelstructuren zoals gerepresenteerd wordt door eierdozen, met stipconfiguraties zoals gerepresenteerd wordt door dobbelsteenconfiguraties, kunnen kinderen de gedeelde onderliggende structuur van twee rijtjes van drie gaan herkennen. We veronderstellen dat zulk inzicht kan helpen om kinderen bewust te laten worden van ruimtelijke structuren, hen ondersteuning te bieden voor het herkennen en handelen met zulke structuren in verschillende contexten en settings, en voor het leren gebruiken van ruimtelijke structuren om numerieke handelingen te verkorten.

Achtergrond en onderzoeksvragen

De literatuurstudie (Hoofdstuk 1 en 2) die als eerste is uitgevoerd, suggereert dat het vermogen van kinderen om hoeveelheden samen te stellen en te splitsen essentieel is voor de ontwikkeling van inzicht in getalrelaties. Inzicht in getalrelaties is fundamenteel voor meer gevorderde reken-wiskundige vaardigheden, zoals tellen en groeperen, voor kennis over deel-geheelrelaties in optellen, vermenigvuldigen en delen (8 + 6 = 14 want 5 + 5 = 10 en 3 + 1 = 4 dus 10 + 4 = 14), voor het gebruikmaken van variabelen in algebra, voor bewijzen, voorspellen en generaliseren, en voor het bepalen van de structuur van een vorm om die vervolgens mentaal te roteren en ermee te handelen (Anghileri, 1989; Buijs, 2008; Carraher et al., 2006; Papic & Mulligan, 2007; Steffe, Cobb, & Von Glasersfeld, 1988).

Het kunnen samenstellen en splitsen van hoeveelheden heeft ook te maken met inzicht in deel-geheelrelaties. In de omschrijving van ruimtelijk inzicht, merken we een relatie op tussen de drie componenten van ruimtelijk inzicht (ruimtelijke visualisatie, oriëntatie en vorm) en deel-geheelrelaties in het ontleden van ruimtelijke objecten (vgl. Clements & Sarama, 2007). Ten eerste kan ruimtelijke visualisatie het vermogen om te handelen met mentale beelden kinderen ondersteunen in het rangschikken van objecten om zo hun samenstelling te onderzoeken. Ten tweede betreft ruimtelijke oriëntatie onder andere het integreren van al geabstraheerde objecten, om op die manier nieuwe structuren te creëren. Ten derde helpt het inzicht in vormen kinderen om delen en gehelen van geometrische patronen, congruentie, symmetrie en transformaties waar te nemen. We stellen daarom voor dat de drie componenten een ruimtelijk structureervermogen met elkaar delen. De focus van ons onderzoek is dus ruimtelijke structuren, om te begrijpen hoe ze van invloed kunnen zijn op het vermogen van jonge kinderen om hoeveelheden samen te stellen en te splitsen voor het krijgen van inzicht in getalrelaties.

Het voordeel van het kunnen herkennen van een ruimtelijke structuur en het toepassen ervan om reken-wiskundige handelingen te verkorten, blijkt bijvoorbeeld bij het aflezen van een hoeveelheid (6 herkennen als 3 met 3; Steffe et al., 1988; Van den Heuvel-Panhuizen, 2001; Van Eerde, 1996; Van Nes & De Lange, 2007), bij het vergelijken van een hoeveelheid (vier hoeken met ieder een stip is minder stippen bij elkaar dan dezelfde configuratie met nog een stip in het midden; Clements, 1999a), bij het verlengen van een patroon (de structuur herhalen; Papic & Mulligan, 2005, 2007) en bij het bouwen van een constructie met blokken (eigenschappen en oriëntatie van vormen en figuren aan elkaar relateren; Battista et al., 1998; Van den Heuvel-Panhuizen & Buijs, 2005). Zelfs is het zo dat kinderen die gefocust zijn op niet reken-wiskundige eigenschappen en een voorkeur blijven houden voor het één-voor-één tellen zonder gebruik te maken van enige vorm van structuur, gevoelig zijn voor achterstanden in hun reken-wiskundige ontwikkeling (Butterworth, 1999; Mulligan, Mitchelmore, & Prescott, 2005).

Dergelijk onderzoek naar het ruimtelijk structureervermogen van jonge kinderen en de invloed ervan op hun reken-wiskundige prestaties, vraagt om meer inzicht in de eigenschappen van een ontwikkelingstraject voor ruimtelijk structureren, en hoe een leeromgeving deze ontwikkeling zou kunnen stimuleren om het inzicht van jonge kinderen in getalrelaties te ondersteunen. Het doel van ons onderzoek is om (a) bij te dragen aan het begrijpen van de ontwikkeling van het ruimtelijk structureervermogen van jonge kinderen, en (b) een lokale instructietheorie te ontwikkelen over *hoe* een leeromgeving deze ontwikkeling kan stimuleren en kinderen kan ondersteunen in het leren gebruik te maken van ruimtelijke structuren voor het verkorten van reken-wiskundige procedures. Dit kan leerkrachten aanmoedigen het belang van ruimtelijke structuren in hun klas te herkennen. Het beantwoordt ook aan de vraag naar instructie-activiteiten die ruimtelijke structureerstrategieën, in plaats van telstrategieën, benadrukken (Clements, 1999a; Clements & Sarama, 2007). De principes van realistisch reken-wiskundeonderwijs (RME; Freudenthal 1973, 1991; Gravemeijer, 1994; Treffers, 1987) en socio-constructivisme (Cobb & Yackel, 1996) bieden aanwijzingen voor het ontwerpen, uitvoeren, en interpreteren van het onderzoek. Op basis van de literatuurstudie, definiëren we de onderzoeksvragen als volgt:

1. *Welke strategieën voor het oplossen van ruimtelijke en numerieke problemen kenmerken de ruimtelijke structureervermogens van jonge kinderen?*

2a. *Hoe kunnen jonge kinderen ondersteund worden in het leren herkennen en gebruikmaken van ruimtelijke structuren voor het verkorten van numerieke procedures?*
2b. *Wat kenmerkt een leeromgeving ('learning ecology') die de ontwikkeling van het ruimtelijk structureervermogen van kinderen kan faciliteren?*

Methodologie

Deelnemers en setting. De studie is uitgevoerd in een kleuterklas op een lokale Jenaplan basisschool. De sociale en culturele achtergrond van de kinderen op deze school is gemengd. De interventiegroep (IG) was een gecombineerde groep 1 en 2, met in totaal 21 kinderen die in leeftijd variëren van vier tot zes jaar. Twee leerkrachten waren verantwoordelijk voor de klas. De non-interventiegroep (NG) was een van de drie andere kleuterklassen van de school en bestond uit zeventien vier- tot zesjarige kinderen. Deze groep was niet bedoeld als controlegroep, maar voor het verzamelen van aanvullende data voor het ontwikkelen van het strategie-instrument en het analyseren van de interviews. De pre- en post-interviewuitkomsten zijn ook gebruikt als aanvulling op de kwantitatieve en kwalitatieve analyses van de resultaten van het instructie-experiment.

De inventarisatie van strategieën en de interviews. Voor het beantwoorden van de eerste onderzoeksvraag is een set taken ontwikkeld om het niveau van ruimtelijk structureren en getalbegrip van kinderen te peilen in een één-op-één klinische interviewsetting (Van Eerde, 1996; Hoofdstukken 4, 5 en 9). Deze taken zijn geïnspireerd door literatuurstudie, ervaringen uit een aantal exploratieve studies en consultaties met experts. De strategieën die de kinderen voor de taken gebruikten zijn verzameld om een inventarisatie van strategieën te maken. Deze inventarisatie bestaat uit een cumulatieve lijst strategieën voor drie componenten van ruimtelijk structureren (het herkennen, toepassen en verlengen van een structuur) en getalbegrip (het bepalen, vergelijken en handelen met hoeveelheden). De data bestonden uit video-opnamen van de interviews, aantekeningen van de onderzoeker over de handelingen van de kinderen en de gestandaardiseerde leerlingvolgsysteem (LVS) scores van de kinderen.

Gezien de interbeoordelaarsbetrouwbaarheid met een Cohen's Kappa waarde van 0.87, is de inventarisatie van strategieën een betrouwbaar instrument voor het beoordelen van de prestaties van kinderen op de interviews. Verder kwam het repertoire aan strategieën van kinderen uit de pre- en post-interviews overeen met de gestandaardiseerde LVS-scores, wat de validiteit van de interviewuitkomsten ondersteunt en het potentieel van de interviews als instrument voor het peilen van inzicht in getalrelaties in termen van ruimtelijk structureervermogen onderschrijft. De IG en NG zijn één keer eerder geïnterviewd (pre-interview) en één keer na het instructie-experiment (post-interview) om te onderzoeken of er verschillen zijn tussen beide interviews en wellicht van invloed zijn op het niveau en soort strategiegebruik van de kinderen.

Het instructie-experiment. Cumulatief cyclische, klassikale design research (Gravemeijer & Cobb, 2006) is gebruikt om de tweede onderzoeksvraag te beantwoorden. Dit betreft het formuleren, testen en verbeteren van een hypothetisch leertraject (HLT) met corresponderende instructie-activiteiten voor het instructie-experiment (Hoofdstukken 6, 7 en 8). Het HLT bestond uit testbare hypothesen die een beeld geven van hoe verwacht werd dat het instructie-experiment de leerprocessen van de kinderen zou beïnvloeden. Dit zou empirische ondersteuning kunnen leveren voor een lokale instructietheorie over hoe jonge kinderen gesteund kunnen worden in de ontwikkeling van hun ruimtelijk structureervermogen.

De vijf instructie-activiteiten werden door de IG in de klas uitgevoerd tijdens twee ronden van het instructie-experiment. Elke instructie-activiteit begon met een kringgesprek dat werd geleid door de leerkracht. Vervolgens nam de onderzoeker vijf kinderen apart (de zogenaamde focusgroep) om meer diepgaande discussies te voeren en gedetailleerde observaties van hun aanpak te verkrijgen. De data bestonden uit video-opnamen van elke instructie-activiteit, vragenlijsten die de leerkrachten invulden, het verslag over de les dat na afloop door de onderzoeker werd geschreven en aantekeningen van de discussie met de leerkracht voor en na de les over de voorbereidingen voor en reflectie op de instructie-activiteit.

Na Ronde 1 van het instructie-experiment zijn de observaties geanalyseerd om te zien hoe de instructie-activiteiten invloed hadden op het bewustworden van ruimtelijk structureren. De reacties en de handelingen van de kinderen zijn vergeleken met de observatiecriteria die geformuleerd zijn tijdens de ontwikkeling van het HLT. De verschillen tussen onze observaties en de verwachting hoe kinderen zouden reageren op de instructie-activiteiten, hebben geleid tot aanpassingen van het HLT en herziene instructie-activiteiten. De gereviseerde serie instructie-activiteiten is in dezelfde klas tijdens de tweede ronde van het instructie-experiment uitgeprobeerd.

Kort na de tweede ronde van het instructie-experiment zijn de post-interviews met de IG en NG kinderen gehouden. Hun repertoire aan strategieën voor de post-interviews werd kwantitatief en kwalitatief vergeleken met het repertoire voor de pre-interviews die voorafgaand aan het instructie-experiment zijn gehouden. Dit zou meer inzicht moeten geven in hoe de instructie-activiteiten de ontwikkeling van het ruimtelijk structureren van kinderen zouden kunnen beïnvloeden. De leerkrachten zijn kort na Ronde 2 geïnterviewd om te bepalen of en hoe het instructie-experiment hun perspectief op het lesgeven over ruimtelijk structureren en de rol van ruimtelijke structuren op de ontwikkeling van vroege rekenvaardigheden, heeft beïnvloed.

Data-analyse. De kwalitatieve data-analyse volgde de principes van constante vergelijkingen (Glaser & Strauss, 1967; Strauss & Corbin, 1998) met behulp van het multimediadata-analyseprogramma ATLAS.ti. Dit programma biedt een format voor het organiseren van ruwe data in de vorm van clips, die het proces van het nagaan en analyseren van patronen in de handelingen van de kinderen kan vereenvoudigen. Op deze manier zijn we beter in staat geweest vast te stellen hoe kinderen de problemen aanpakten, hoe hun conceptuele kennis ontwikkelde en hoe de pro-actieve leeromgeving een rol heeft gespeeld bij deze ontwikkeling.

Het conceptuele schema

De literatuurstudie, inventarisatie van strategieën, explorative studies en consultaties met experts hebben bijgedragen aan de ontwikkeling van een conceptueel schema. Dit schema associeert de ontwikkeling van ruimtelijk structureervermogen (ontleend aan drie componenten van ruimtelijk inzicht; Clements & Sarama, 2007) met getalbegrip (gebaseerd op de integratie van het vermogen om te kunnen tellen en hoeveelheden te vergelijken; Griffin & Case, 1997; Griffin, 2004b), met name in termen van hun inzicht in getalrelaties (begrijpen van de samenstelling van hoeveelheden; Hoofdstukken 4 en 10). Toen de inventarisatie van strategieën vorm begon te krijgen, ontstond tegelijkertijd een theoretisch model dat inzicht gaf in hoe deze constructen aan elkaar gerelateerd kunnen zijn (Hoofdstuk 5). Het uiteindelijke conceptuele schema dat onderliggend is aan ons onderzoek is als volgt uitgelijnd (Fig. S.1).

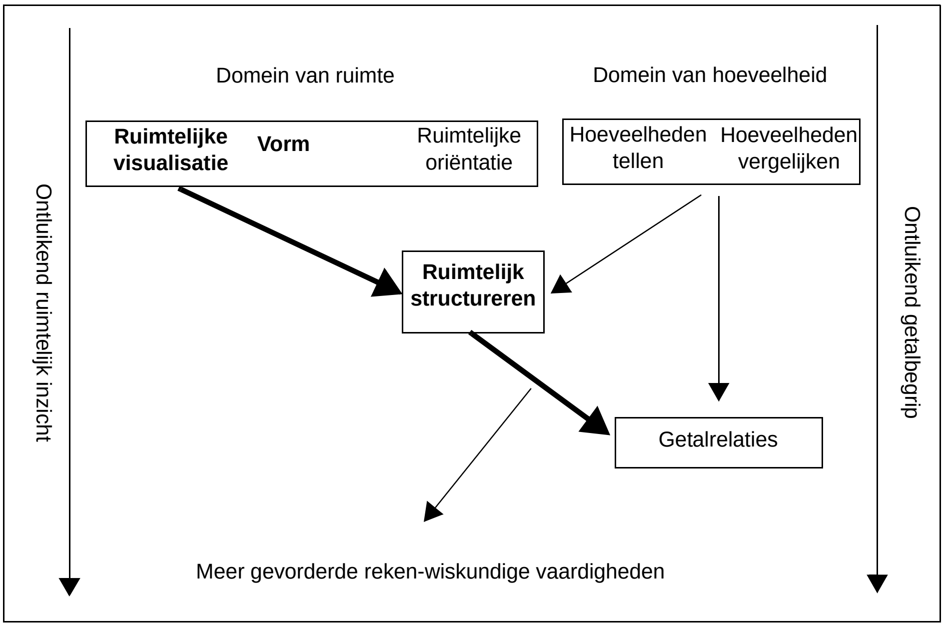


Fig. S.1 Het conceptuele schema

Dit schema laat zien dat de focus van het onderzoek ligt op het vermogen van kinderen om hoeveelheden ruimtelijk te structureren en de rol ervan bij het ondersteunen van inzicht in getalrelaties. Dit zou kunnen helpen om hun leren en begrijpen van meer gevorderde reken-wiskundige vaardigheden, zoals optellen, aftrekken en vermenigvuldigen te ondersteunen. Het vermogen om "hoeveelheden te tellen" is ook verbonden met "ruimtelijk structureren" om aan te geven dat kinderen minstens resultatief moeten kunnen tellen voordat ze zich richten op manieren om numerieke procedures te verkorten. In de praktijk stellen we dat het vermogen van kinderen om ruimtelijke structuren te herkennen hen zou kunnen helpen bij het aflezen van een hoeveelheid in plaats van één-voor-één te blijven tellen.

Fasen in de ontwikkeling van ruimtelijk structureervermogen

Naarmate de inventarisatie van strategieën een instrument werd voor het peilen van inzicht in getalrelaties in termen van ruimtelijk structureervermogen, moesten de scoringsprocedures gereduceerd worden, zodat het instrument makkelijker in gebruik en meer betrouwbaar zou worden (Hoofdstuk 5). Hiervoor werden alle strategieën die een kind gebruikt om de interviewtaken te beantwoorden geïnventariseerd en geïnterpreteerd als een repertoire aan strategieën. Dit repertoire past bij een van de volgende fasen die een omschrijving geven van cumulatieve niveaus van geavanceerdheid in het ruimtelijk structureervermogen van een kind in deze specifieke interviewsetting:

> Fase 1 (Eenheidsfase): Het kind herkent bijna geen ruimtelijke structuren en maakt daardoor geen gebruik van structuren en past ze niet toe om numerieke procedures te verkorten.
>
> Bijvoorbeeld: telt alle stippen en vingerpatronen op de flitskaarten; telt bloemen op de tafel één-voor-één en laat ze verspreid liggen als een “makkelijke manier” om af te lezen hoeveel het er zijn.
>
> Fase 2 (Herkenningsfase): Het kind herkent een aantal fundamentele ruimtelijke structuren, maar maakt zelden gebruik van structuren en past ze zelden toe om numerieke procedures te verkorten. In plaats daarvan, zou het kind het gebruik van structuren achteraf kunnen rationaliseren.
>
> Bijvoorbeeld: herkent de meeste flitskaarten, maar telt de bloemen op de tafel één-voor-één, zelfs als ze al gestructureerd zijn; herkent structuur alleen als de aandacht expliciet op structuur wordt gericht.
>
> Fase 3 (Gebruiksfase): Het kind herkent en maakt gebruik van de meest aanwezige ruimtelijke structuren, maar laat zelden initiatief zien bij het construeren en toepassen van eigen ruimtelijke structuren als een manier om numerieke procedures te verkorten.
>
> Bijvoorbeeld: leest een structuur af en maakt gebruik van een reeds aanwezige structuur om een numerieke procedure te verkorten (“ik zie twee rijen van drie, dus dat is zes”), maar structureert ongestructureerde verzamelingen niet (laat ze bijvoorbeeld op een hoop of verspreid liggen).
>
> Fase 4 (Toepassingsfase): Het kind maakt gebruik van ruimtelijke structuren op een doelgerichte manier en construeert en past spontaan ruimtelijke structuren toe als een manier om numerieke procedures te verkorten.
>
> Bijvoorbeeld: “Ik weet dat het acht is want hier is zes, zoals op de dobbelsteen en twee meer maakt acht”; herschikt ongestructureerde verzamelingen naar structuren, zoals kleinere, te subiteren groepen of in dobbelsteenconfiguraties, want: “Nu is het makkelijker om het te zien en je hoeft niet te tellen”.

Deze fasen hebben overlappende begin- en eindpunten. We veronderstellen dat kinderen geleidelijk van de ene naar de volgende fase overstappen, afhankelijk van hoe hun tendens om een bepaald repertoire aan strategieën toe te passen verschuift naar een meer geavanceerd repertoire (vgl. Siegler, 2002, 2005). De classificaties van deze fasen zijn gebruikt om het repertoire aan strategieën van kinderen, nodig voor de pre- en post-interviews, te vergelijken. Hiermee kregen wij een indruk of en hoe het instructie-experiment van invloed is geweest op de manier waarop kinderen de interviewtaken hebben aangepakt. De prestaties van de kinderen op de interviewtaken (hun strategiescores) en het corresponderende repertoire aan strategieën (faseclassificatie) werden een weergave van hun ruimtelijk structureervermogen. Tegen het licht van de voorgestelde invloed van het ruimtelijk structureervermogen op inzicht in getalrelaties (Hoofdstuk 2), maakte dit tevens de ontwikkeling van inzicht in getalrelaties in het onderzoek duidelijk. Dit wordt geïllustreerd in Fig. S.2:

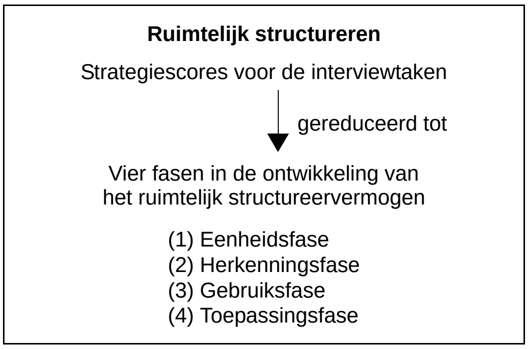


Fig. S.2 Operationalisatie van de ontwikkeling van inzicht in getalrelaties

Ontwikkeling en resultaten van de HLT en het instructie-experiment

Op basis van de observaties van onze exploratieve studies in verschillende kleuterklassen op een lokale basisschool, de leerdoelen van het instructie-experiment, en de vier fasen in de ontwikkeling van ruimtelijk structureervermogen, was onze oorspronkelijke verwachting dat de kinderen gedurende het leertraject de volgende aan elkaar gerelateerde leermomenten zouden tegenkomen (Hoofdstuk 6):

(1) organiseer objecten als een stap naar de bewustwording van ruimtelijk structureren voor het vereenvoudigen van telhandelingen;
(2) creëer een motivatie voor het ruimtelijk structureren van objecten;
(3) gebruik ruimtelijke structuren om getalrelaties zichtbaar te maken;
(4) ontwikkel abstracte ruimtelijke structuren die minder context- en taakafhankelijk zijn;
(5) gebruik ruimtelijk structureren op een doelgerichte manier buiten het instructie-experiment om.

Deze leermomenten hebben de beschrijving van het hypothetisch leertraject (HLT) voor de eerste ronde van het instructie-experiment geïnspireerd. In het kader van het belang van patronen leggen voor het bewustworden van ruimtelijk structuren (vgl. Papic & Mulligan, 2007), was de eerste activiteit een “patroonactiviteit” waarin kinderen de structuur van een patroon moesten herkennen en vervolgens verlengen. Tijdens de tweede activiteit, was de verwachting dat kinderen zulk inzicht in deel-geheel-relaties zouden gebruiken om verschillende typen ruimtelijke structuren die in een zogenaamde “Trucjesdoos” (trucjes om “makkelijk te zien” hoeveel er van iets zijn) zaten, te onderzoeken en te vergelijken. De derde activiteit vereiste dat de kinderen gebruik gingen maken van deel-geheel-relaties om gestructureerde hoeveelheden in relatief grote gestructureerde stipconfiguraties op kaarten te herkennen en af te lezen. Een gelijke activiteit betrof het herkennen en aflezen van dubbelstructuren van eierdozen. In de vierde activiteit werd het inzicht in ruimtelijke structuren van kinderen naar een 3-D setting vertaald, waarbij ze de structuur van een blokkenbouwsel moesten proberen te herkennen en te gebruiken. De verwachting was dat zij hun ervaringen met ruimtelijk structureren zouden gebruiken en toepassen tijdens de vijfde activiteit. Daarin moesten ze een verzameling ongestructureerde bloemen structureren en bijhouden hoeveel bloemen er waren zonder ze één-voor-één te hoeven tellen.

Terwijl de video-opnamen verder geanalyseerd en bediscussieerd werden met experts, ontstonden er patronen die inzicht gaven in de rol van een leeromgeving (“learning ecology”, Cobb et al., 2003) bij het leren van kinderen:

- De context binnen en tussen de instructie-activiteit moet niet alleen betekenisvol en inspirerend zijn voor de kinderen, maar het moet ook productieve, situatie-specifieke strategieën bieden.
- Een gedeelde woordenschat, het type taalgebruik van de leerkracht, en de manier waarop kinderen taal interpreteren is essentieel.
- De instructie-activiteiten overbruggen het niveau van begrip van kinderen met een leerdoel terwijl de eigen producties van kinderen worden erkend.
- De leerkracht is een essentiële factor in het aanmoedigen van interactie tussen de kinderen.
- De socio-mathematische norm van ruimtelijk structureren moedigt een gedeeld bewustzijn van ruimtelijke structuren aan in reken-wiskundige praktijken.

De patronen die in Ronde 1 zijn geobserveerd, zijn van invloed geweest op het design van de serie instructieactiviteiten voor Ronde 2 in het instructie-experiment (Hoofdstuk 7). In Ronde 1 hadden alle activiteiten aansprekende contexten. Voor Ronde 2 hebben we Miertje Maniertje en zijn Maniertjesdoos als overkoepelende context geïntroduceerd (Fig. S.3; Van Nes & Doorman, 2009). Deze context was essentieel om ervoor te zorgen dat de activiteiten betekenisvol waren en voor het bieden van productieve, situatiespecifieke strategieën in een reken-wiskundige context (De Lange, 1987). Het heeft ook geholpen om een gedeelde woordenschat te ontwikkelen die fundamenteel is voor het vaststellen van de socio-mathematische norm van ruimtelijk structureren (Gravemeijer & Cobb, 2006).

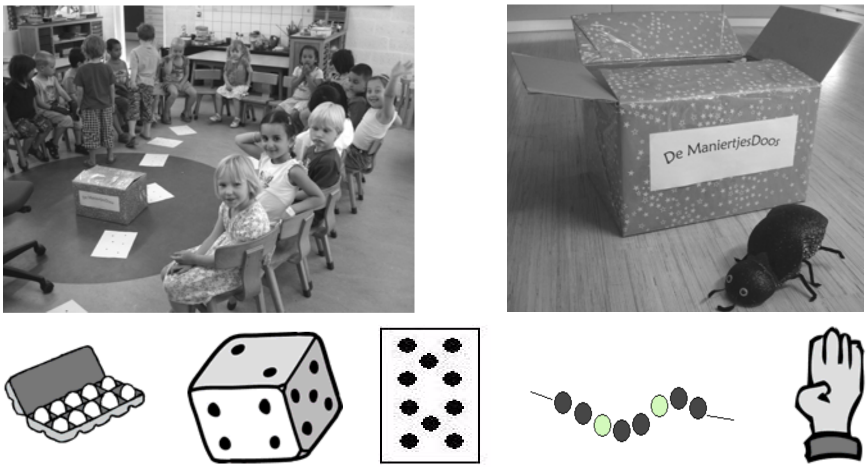


Fig. S.3 Miertje Maniertje en de Maniertjesdoos

De Maniertjesdoos bevatte uitvergrote kaarten met vingerpatronen, grote dobbelstenen, uitvergrote kaarten met gestructureerde stipconfiguraties, eierdozen voor zes en tien eieren en kralenkettingen (vgl. Clements, 1999a; Mulligan et al., 2004; Van den Heuvel-Panhuizen, 2001; Van Eerde, 1996). De reden waarom we Miertje Maniertje voor deze context hebben gekozen, is dat een mier zes poten heeft (een fundamentele en bekende ruimtelijke structuur), sterk is, wellicht een doos kan dragen, zijn naam rijmt met die van de doos en, tot slot, spreken mieren jonge kinderen aan. Het verhaal was dat Miertje Maniertje zijn “maniertjes” met de klas wilde delen, omdat hij de kinderen wilde helpen manieren te vinden om een hoeveelheid handig te kunnen bepalen zonder de voorwerpen één-voor-één te hoeven te tellen.

De leerkracht ontwikkelde een gezamenlijke woordenschat aan uitspraken, zoals Miertje Maniertjes “handige maniertjes om te zien hoeveel het er zijn”, voor het verwijzen naar de inhoud van de doos. Dit hielp om de kinderen te leiden naar het verkennen en vergelijken van ruimtelijke structuren om deze en het inzicht in getalrelaties gedurende het instructie-experiment tot onderwerp van discussie te maken (Hoofdstuk 8). De overkoepelende context van Miertje Maniertje en de Maniertjesdoos was een van de belangrijkste revisies die voor Ronde 2 gemaakt zijn. De herziene serie instructieactiviteiten verfijnde de operationalisatie van ruimtelijk structureren als volgt (Fig. S.4).

Activiteit 1: De Maniertjesdoos en Mierenstapjes
- Introduceer en verken de context
- Verken ruimtelijke structuren in de Maniertjesdoos
- Vergelijk representaties van hoeveelheden op gestructureerde gestipte kaarten

Activiteit 2: Eierdozen vullen
- Vergelijk representaties van hoeveelheden in dubbelstructuren
- Vergelijk dubbelstructuren met andere structuren uit de Maniertjesdoos

Activiteit 3: In optocht
- Verken structuren van 2-D patronen
- Vergelijk structuren in patronen met andere structuren uit de Maniertjesdoos

Activiteit 4: Mierenhopen bouwen
- Verken structuren van 3-D patronen
- Vergelijk structuren van blokkenbouwsels met andere structuren uit de Maniertjesdoos

Activiteit 5: Bloemen plukken
- Structuren op ongestructureerde verzamelingen toepassen
- Vergelijk representaties van structuren in de Maniertjesdoos
- Verken de rol van structuren in het samenstellen en splitsen van hoeveelheden

Fig. S.4 Operationalisatie van ruimtelijk structureren in termen van de serie instructieactiviteiten

Resultaten van het instructie-experiment. De analyses van het instructie-experiment na Ronde 1 duiden op voordelen van een leeromgeving die ondersteuning biedt bij het bewustworden van ruimtelijk structureren voor het ondersteunen van inzicht in getalrelaties van jonge kinderen (Hoofdstuk 8 en 9). Uit de post-interviews is gebleken dat achttien van de 21 (IG)kinderen van de interventiegroep naar een hogere fase neigeden, dus dat deze kinderen relatief meer gebruik hebben gemaakt van ruimtelijke structuren in hun aanpak van de interviewtaken. De NG-kinderen vertoonden relatief minder verbeteringen (vijf uit zeventien) en zijn vaker gelijk gebleven (elf uit zeventien, waarvan er vijf al in de pre-interviews plafondeffecten vertoonden), met één kind dat van de Toepassingsfase (fase 4) naar de Gebruiksfase (fase 3) is gezakt. Verder begonnen de IG-kinderen steeds meer te verwijzen naar ruimtelijke structuren en gebruik te maken van de gedeelde woordenschat om de voordelen van ruimtelijk structureren versus tellen te bediscussiëren. Zo ontstond een norm voor ruimtelijk structureren. De IG-leerkrachten lieten weten dat zij zich meer bewust zijn geworden van ruimtelijk structureren en waardering hebben gekregen voor het belang ervan voor de ontwikkeling van een in een vroeg stadium ontstaan van reken-wiskundige vaardigheden.

Conclusie en discussie

In dit onderzoek hebben we eerst interviews gehouden om een leertraject in de ontwikkeling van het leren ruimtelijk te structureren vast te stellen (onderzoeksvraag 1). Met het formuleren van een HLT en het ontwikkelen van een instructie-experiment, hebben we vervolgens onderzocht hoe deze ontwikkeling in een leeromgeving ondersteund kan worden (onderzoeksvraag 2). Dit resulteerde in de volgende antwoorden en discussiepunten.

Onderzoeksvraag 1. Gebruikmakend van de inventarisaties van strategieën, hebben we het ruimtelijk structureervermogen van jonge kinderen in deze leeromgeving in een van vier fasen geclassificeerd. Deze beschrijven een algemene ontwikkeling van geen bewustzijn van ruimtelijke structuren, tot het leren herkennen en gebruikmaken ervan, het spontaan en doelgericht toepassen ervan om inzicht te krijgen in getalrelaties en voor het verkorten van numerieke procedures, zoals het bepalen, vergelijken en handelen met kleine hoeveelheden. Op deze manier bieden de interviewtaken en de inventarisatie een instrument voor het peilen van inzicht in getalrelaties in termen van ruimtelijk structureervermogen van jonge kinderen. De inventarisatie van de strategieën zelf is ook een belangrijk interpretatief model dat een ontwikkelingstraject omvat voor de typen strategieën die jonge kinderen toepassen op de interviewtaken. Dit model was een belangrijke inspiratiebron voor de ontwikkeling van het hypothetisch leertraject, nodig voor het beantwoorden van de tweede onderzoeksvraag.

Onderzoeksvraag 2a. De serie van vijf instructie-activiteiten biedt ondersteuning voor een lokale instructietheorie, die stoelt op de telvaardigheid van jonge kinderen, hun organisatievermogen en hun bekendheid met fundamentele ruimtelijke structuren. Het uiteindelijke leertraject begint met een introductie van de overkoepelende context van de instructie-activiteiten, terwijl de kinderen fundamentele ruimtelijke structuren, zoals dobbelsteenpatronen, vingerpatronen en dubbelstructuren verkenden. Vervolgens worden ze aangemoedigd om reeds aanwezige structuren in verschillende settings te gebruiken. Dit kan ze voorbereiden op het leren spontaan hun eigen ruimtelijke structuren te construeren voor het verkorten van numerieke procedures. Zo verankert het leertraject het ruimtelijk structureervermogen van jonge kinderen op een manier die hun inzicht in getalrelaties bevordert, hetgeen hen weer kan helpen bij het leren en begrijpen van hogere orde reken-wiskundige vaardigheden.

Onderzoeksvraag 2b. Uit het instructie-experiment is gebleken dat een effectieve leeromgeving voortbouwt op het reeds aanwezige begrip van kinderen, op het introduceren van een betekenisvolle context, het ontwikkelen van een gezamenlijke woordenschat en op het bevorderen van interactief leren. Factoren die van invloed zijn geweest op de ontwikkeling van het ruimtelijk structureervermogen van kinderen staan afgebeeld in Fig. S.5.

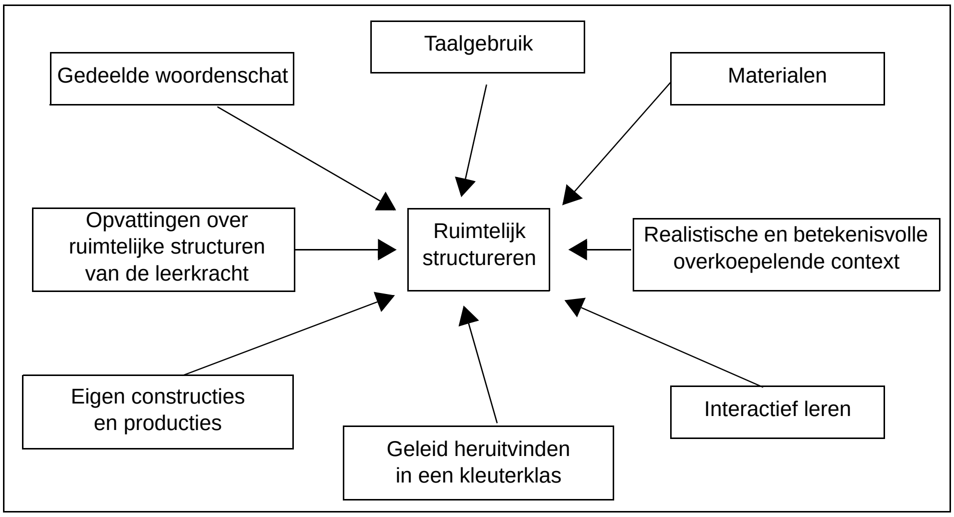


Fig. S.5 Factoren in een leeromgeving die de ontwikkeling van ruimtelijk structureervermogen kunnen beïnvloeden

RME in de kleuterklas. Verschillende observaties hebben het principe van geleid heruitvinden in de kleuterklas een ander karakter gegeven. De interactie tussen kinderen en leerkracht lieten geleid heruitvinden zien op een manier waarop de leerkracht de kinderen vroeg hun ruimtelijk structureerstrategieën met elkaar te delen en te vergelijken, en hoe ze hun ruimtelijk structureerstrategieën ondersteunden door bijvoorbeeld een blokkenbouwsel uit elkaar te halen om de structuur zichtbaar te maken. Door twee kinderen van verschillende niveaus samen te laten werken, konden de kinderen hun strategieën vergelijken en een voorbeeld voor elkaar zijn. Toch, gegeven de leeftijd en relatief korte aandacht van deze kinderen, moest de leerkracht meer doen dan alleen begeleiden. De leerkracht moest een meer directieve rol spelen in het aanmoedigen van interactie tussen kinderen en leerkracht, en in het stimuleren te reflecteren op de activiteiten voor het begrijpen van ruimtelijk structureren (vgl. Leseman, Rollenberg, & Rispens, 2001; Nelissen, 2002). Dit illustreert hoe taal, verticale interactie (Hatano & Inagaki, 1991) en invloeden van de klassencultuur in de kleuterklas een andere rol spelen dan wat doorgaans door de RME-principes belicht wordt.

Ruimtelijk inzicht in relatie tot getalbegrip. Alhoewel deze serie instructieactiviteiten over ruimtelijk structureren kinderen geholpen heeft inzicht in getalrelaties op te doen, is meer onderzoek nodig om specifieke invloeden van componenten van ruimtelijk inzicht op numerieke ontwikkeling te onderscheiden. Er is bijvoorbeeld geen overtuigende ondersteuning gevonden voor invloeden van ruimtelijke oriëntatie op de prestatie van kinderen op de interviewtaken. Dit verklaart de relatieve afstand tussen ruimtelijke oriëntatie vergeleken met de meer verbonden ruimtelijke visualisatie- en vormcomponenten in het conceptuele schema (Fig. S.1). Toch lijken het vermogen van kinderen om te tellen en te organiseren en hun kennis van fundamentele structuren, vereisten te zijn voor het leren gebruiken en toepassen van ruimtelijke structuren.

Beperkingen van het onderzoek. Er moest met een aantal kwesties rekening worden gehouden voor het ontwikkelen, uitvoeren en interpreteren van het onderzoek. Ten eerste verhoudt de lokale instructietheorie zich alleen tot de observaties uit dit specifieke instructie-experiment, dat is uitgevoerd met leerkrachten en kinderen van deze Jenaplan kleuterklassen. Ten tweede waren de interventie- en non-interventiegroepen erg klein en de non-interventiegroep was geen strikte controlegroep. Daarom waren bevestigende statische data- analyses niet mogelijk. Toch is de waarde van het onderzoek dat het kennis genereert over *hoe* het instructie-experiment de leerprocessen van de kinderen heeft beïnvloed. Het exploratieve karakter van design research biedt de mogelijkheid om meer kennis op te doen over de complexe leerprocessen die instructie-interventies oproepen (Gravemeijer & Cobb, 2006).

Implicaties van het onderzoek. Ten eerste kunnen de interviewtaken en de inventarisatie van strategieën worden beschouwd als waardevolle instrumenten voor het peilen van inzicht in getalrelaties in termen van het ruimtelijk structureervermogen van jonge kinderen. De faseclassificaties kunnen helpen om bepaalde kinderen te identificeren wiens niveau van ruimtelijk structureervermogen niet overeenkomt met wat verwacht zou mogen worden op basis van hun leeftijd en ervaringen. Ten tweede biedt de lokale instructietheorie leerkrachten een model voor het plannen van een reken-wiskundecurriculum voor kleuters dat verweven is met ruimtelijke structureeractiviteiten. Voorlopige ervaringen met nascholing zijn veelbelovend. Deze lokale instructietheorie ondersteunt leerkrachten ook in het verwerven van inzicht in hun rol bij het leren van kinderen door socio-mathematische normen vast te stellen; de woordkeuze van de leerkrachten, hun flexibel gebruik van dagelijkse materialen en betekenisvolle contexten, en hun perspectief op ruimtelijk structureren, zijn van grote invloed op hoe kinderen de instructie-activiteiten aanpakken. Deze implicaties sluiten aan op de vraag naar onderwijspraktijken die het belang van ruimtelijk structureren voor de ontwikkeling van inzicht in getalrelaties van jonge kinderen meer erkennen (Clements, 1999a).

Aanbevelingen voor toekomstig onderzoek. Toekomstig onderzoek kan deze lokale instructietheorie extrapoleren naar andere contexten en settings teneinde de generalisatie ervan te bevorderen. De uitkomsten van het onderzoek verwelkomen longitudinale studies voor het bestuderen van langetermijneffecten van de serie instructieactiviteiten op het leren van kinderen en op de gerichtheid van de leerkracht op ruimtelijk structureren in de klas. Op een meer praktisch niveau kan de context van Miertje Maniertje en de Maniertjesdoos in verschillende settings uitgebreid en uitgeprobeerd worden.

Samenvattend zet dit onderzoek de toon voor meer onderzoek naar het stimuleren van vroege ruimtelijke en numerieke vaardigheden. Immers, hoe eerder we (delen van) leertrajecten van jonge kinderen in kaart kunnen brengen, des te beter we in staat zijn om een ondersteunde leeromgeving in te richten die het reken-wiskundige leren van jonge kinderen kan handhaven en ze een voorsprong kan geven in hun reken-wiskundige opleiding.

Appendices

1 Spatial Structuring for Automatic Quantity Processing [1]

In this appendix, we first explain what motivated the interdisciplinary character of the MENS research project. Then we turn to the neuroscientific component of the project and present the study on automatic quantity processing. “Automatic” implies that the meanings of Arabic numerals (i.e., the magnitudes of numerical symbols) are instantly retrieved even if the retrieval process is not task-relevant. Finally, we explain the difficulties that had to be negotiated for planning, performing and analyzing the combined research. Although the combination of the two research disciplines remains theoretical, an important outcome of the combined research is that the explorations have set the stage for future interdisciplinary investigations so that they may build on our two studies and lay a stronger foundation for the early prevention and identification of developmental delays.

1.1 Motivation for interdisciplinary research

The significance of the collaboration between the two research disciplines lies in the grounding of the research in educational mathematics theory as well as in neuroscientific theory, while at the same time providing the neuroscientific research with a strong practical basis from which testable predictions can be made (Jolles et al., 2005). Many recent publications have emphasized how scientists from the disciplines of mathematics education and neurosciences can and should contribute to each other’s research (Berninger & Corina, 1998; Byrnes & Fox, 1998; Davis, 2004; Griffin & Case, 1997; Jolles et al., 2005; Siegler, 2003; Spelke, 2002). With research literature increasingly advocating the need to integrate results from neuroscientific research and education research, (mathematics) education researchers are becoming more aware of the benefits of applying neuroscientific findings to (mathematics) education (e.g., Campbell, 2006; De Jong, 2008). Neuroscientific brain imaging techniques, for example, offer educational researchers a tool for gaining insight into cognitive processing that extends beyond behavioral data. This provides another level of analysis that can help to validate behavioral data.

As Cobb (2000) points out, comparing and contrasting research from various perspectives has the added benefit of deepening understanding of the phenomena that is being studied and of broadening the practicality of the results. As Berninger and Corina (1998) posit, rather than a unilateral conception of interdisciplinarity, bidirectional collaboration between cognitive neuroscience and educational research can enrich both educational re-

1. This chapter is partly based on an earlier publication in EPAT (Van Nes, in press)

search as well as the neurosciences. Neuroscientific research can help educational researchers gain insight into brain-cognition relations that underlie cognitive processes and the effects of instructional interventions. At the same time, the behavioral patterns that educational researchers study before and after a particular instructional intervention, can contribute to understanding the nature of possible changes in brain activation that result as a function of learning (Berninger & Corina, 1998). Thus, whereas "neuroscience can investigate synaptic connections, but not conceptual ones" (Bennett & Hacker, 2003, p. 405), educational researchers can extend the research with conceptual investigations for *understanding why*, for example, certain interventions may or may not work, and for formulating new research questions (Ansari & Coch, 2006). Such a dialogue between the two disciplines is what underlies the motivation for finding ways in which the neuroscientists and the educational researchers can collaborate in the MENS research project.

Interdisciplinary collaboration in the MENS research project is also particularly relevant to its focus on young children's mathematical development. Despite the body of research on the remarkable mathematical competencies of young children, what is disconcerting is that studies and subsequent instructional interventions often stay focused only on children's numerical skills (Baroody, 2004). This overshadows young children's excellence in less trivial mathematics areas such as patterning, spatial perception, reasoning, classification and problem solving (Griffin & Case, 1997; Ness & Farenga, 2007).

Both research disciplines can in their own ways gain knowledge about how to identify and support young children who may be experiencing delays in their mathematical development as well as those who welcome challenges to optimize their mathematical development. As explained in this thesis, the purpose of the mathematics education study is to gain insight into young children's spatial structuring ability and its role in emerging number sense, particularly with regard to insight into numerical relations. What concerned the neuroscientists in the research project is that, compared to for example studies on literacy deficits, relatively little is known about the cognitive processes that are involved in the development of mathematical competencies and that may underlie numerical deficits (Ansari & Karmiloff-Smith, 2002). As such, one of the main objectives of the neuroscientific study was to augment current diagnostic and remedial practice with new insights into prerequisites for the development of mathematical abilities.

In what follows, we describe the theoretical background of the neuroscientific study in the MENS research project. This leads to an explanation of the cognitive study that was performed to prepare for the neuroscientific study. At the time of writing this thesis, the outcomes of the neuroscientific study were not ready for documentation yet. We continue with a discussion about the various differences that have been encountered between the research methodologies. Although our search towards interdisciplinary collaboration has not resulted in data-driven research outcomes, our explorations for ways to integrate the two research disciplines may inspire future interdisciplinary research initiatives in the field of mathematics education.

1.2 Neuroscientific theoretical background

This section begins with a description of several neuroscientific research techniques that make it possible to chart behavioral and neurophysiological processes in young children. The outcomes of recent neuroscientific research regarding the development of quantity processing are then discussed to sketch the setting of the neuroscientific study in the MENS research project. Although this is not an exhaustive literature review, it is intended to give an impression of neuroscientific theories and research outcomes that (can) influence current mathematics educational practice and this interdisciplinary research.

1.2.1 Neuroscientific research techniques

Neuroscientists make use of cognitive behavioral tasks to investigate explicit and implicit behavioral changes with respect to changes in subjects' reaction times (see Hubbard, Piazza, Pinel, & Dehaene, 2005). Although behavioral tasks are easy, inexpensive and non-intervening, they provide little information about the nature of the cognitive processes and the brain activity that occurs in response to the tasks. For that reason, behavioral studies are often supplemented with modern brain imaging techniques. The added value of modern brain imaging techniques compared to experimental behavioral studies, is that extraordinary behavioral observations can be retraced in the neuronal networks. As such, the premise of neuropsychological research is that activations in one location of the brain implies that the same processes are occurring, whereas activation in different locations of the brain indicates different processes.

The electroencephalography (EEG, Fig. A.1) technique highlights the location and the reaction and processing speed of the activated brain areas. The electrocortical activity that occurs at and after the presentation of a stimulus, is registered by electrodes that are attached to the head. This activation is subsequently summed and averaged by a computer to result in Event Related Potentials (ERPs). The graph of these ERPs distinguishes several positive and negative peaks (components) that characterize certain types of responses. A P3 component (a positive peak at approximately 300 ms after stimulus onset), for example, identifies the type of response that is required, in contrast to the P2 component that is task-related and associated more with the physical characteristics of the stimulus. The advantage of this technique is that it is non-invasive, that it can be used with young children, and that it has a stronger temporal resolution than other brain imaging techniques. A disadvantage, however, is that the technique is not sensitive enough for precise localization of brain activity.

Fig. A.1 A sample EEG set-up (Rocha, Rocha, Massad, & Menezes, 2005)

Neuroimaging techniques such as functional Magnetic Resonance Imaging (fMRI) can not only highlight the structure, but also measure the activation of specific areas in the brain. The fMRI technique measures relatively strong blood flow. The assumption is that active brain areas require more blood and therefore use up more energy in the form of glucose. Hence, compared to the EEG technique, the advantage of these techniques is that their spatial and temporal resolutions are high. The disadvantage is that they are expensive and that the statistical methods that are used to localize active brain areas can result in rather ambiguous information. To limit the effects of these disadvantages, researchers are increasingly combining several techniques within one study (Spelke, 2002).

In the next section we discuss several neuropsychological studies that make use of behavioral and imaging techniques to gain insight into the development of mathematical thinking and learning.

1.2.2 Numerosity in the brain

Based on what is known about the activation of certain brain areas, Dehaene, Piazza, Pinel, and Cohen (2003) and Dehaene, Molko, Cohen, and Wilson (2004) described two dissociable systems that represent numerosity differently in the brain (see also Lemer, Dehaene, Spelke, & Cohen, 2003). One system is based on a circuit that becomes active for general visuo-spatial functions and that is associated with understanding number as well as with estimation and the manipulation of quantities. The horizontal segment of the intraparietal sulcus (the HIPS), in particular, is considered to be the core area in the brain where this type of numerosity is processed. The second system is based on a circuit in the brain that is associated with language and with the storage and retrieval of factual mathematical information. These tasks are said to activate the angular gyrus.

The researchers have incorporated the neurological experimental support for these two processing systems into the Triple Code Theory (Dehaene et al., 2003). In this theory, apart from the HIPS and the angular gyrus in the parietal lobes, the posterior superior parietal system is involved in number related processes. This third system is less domain-specifically activated than the HIPS is, because it is associated with the attention and eye orientation that is necessary for functioning in space. For this reason, this area is especially activated during tasks that relate to the spatial manipulation of quantities, such as counting a sequence. According to the Triple Code Theory, these three conjugated systems summarize the organization of number related functions in the brain and support the idea of mathematical intuition (Dehaene et al., 1999, 2003; Wynn, 1998; but see Ansari, Donlan, Thomas, Ewing, Peen, & Karmiloff-Smith, 2003).

The ability to differentiate quantities appears to develop before the manifestation of language. Starkey and Cooper (1980) were one of the first to show that babies as young as four to six months can differentiate arrays of different quantities of black dots that are visually presented. In Wynn's research (1998) it appeared that young babies are not only able to differentiate groups of objects, but also temporal sequences of sounds or events, for example. Xu and Spelke (2000) found that six-month old babies can note a change from eight to sixteen objects (a 2.0 proportion), but not a change from eight to twelve objects (a 1.5 proportion), given that spatial factors such as surface area, density and brightness remain constant. These studies have been expanded and varied (e.g., Lipton & Spelke, 2003; Mix, Huttenlocher, & Levine, 2002; Wood & Spelke, 2004, 2005), and the results support the idea that the ability to differentiate quantities improves with age and that this improvement occurs before the development of language or symbolic counting.

Rivera, Reis, Eckert, and Menon (2005) studied the development of mental mathematical abilities in particular. Their brain imaging studies demonstrated that older subjects showed more activation in brain areas such as the left parietal lobe than children. This activation suggests that mental mathematical procedures require more working memory and attention from children than for older subjects. Likewise, their declarative and procedural memory is loaded more. The researchers conclude that certain brain areas are functionally specialized for mental mathematical procedures, and that, during the development of these areas, the load on memory and attention decreases. Characterizations of patients with abnormal behaviors have also pointed to how the brain has separate processing mechanisms for facts that pertain to addition and multiplication, and that the executive system particularly plays an important role in simple mental procedures (Kaufmann, 2002; Kaufmann, Lochy, Drexler, & Semenza, 2004).

The studies above are examples of research that, in line with the Triple Code Theory, suggest that the human brain is naturally designed to represent and process number. What has been investigated less, however, is how the various systems in the brain are behaviorally and neurophysiologically related and how this relationship translates to (the development of) magnitude processing. For instance, correlations between visuo-spatial deficits and a delayed or deficient development of mathematical abilities have illustrated the prominent role that visuo-spatial abilities play in numerical processing (Rourke & Conway, 1997).

Cantlon, Brannon, Carter, and Pelphrey (2006) made a neurophysiological connection between non-symbolic and symbolic numerical processing of adults. This suggests that complex, symbolic mathematical abilities share a neurobiological and developmental origin with non-symbolic mathematical abilities. Since the HIPS of four-year old children was just as activated as that of the adults, Cantlon et al. conclude that the non-symbolic activity in the HIPS may be the source of adult mathematical knowledge. This provides more insight into the neurophysiology of mathematical development. Further, it underlines the development of automatic processing of non-symbolic and symbolic quantities as a prere-quisite for the ability to perform more complex mathematical procedures (Butterworth, 1999; Rousselle & Noël, 2007). Children who lag behind in the automaticity of simple arithmetic facts, tend to continue to use unsophisticated strategies (i.e., counting; Butterworth, 2001). This suggests that they have not gained enough insight into numerical relations to be able to operate with quantities at a more abstract level. This has also been discussed in the motivation for the ME research on spatial structuring for insight into numerical relations (see Chapter 2).

The Numerical Stroop Paradigm (NSP) is often used to gauge children's ability to process quantities automatically. In this paradigm, subjects determine whether a numerical stimulus is physically or numerically larger than a certain other numerical stimulus. Two stimuli (number symbols for the symbolic NSP and sets of dots for the non-symbolic NSP) are simultaneously projected on a screen (Fig. A.2). The stimuli vary in their physical or numerical size (i.e., magnitude) and participants are asked to respond to one of these dimensions.

The expectation is that if the magnitude of the stimulus (symbols or dots) is automatically activated (even if the dimension is not relevant to the task), then facilitation effects occur for congruent stimuli, interference effects occur for incongruent stimuli, and no effects occur for neutral stimuli. Facilitation and interference effects (measured in terms of reaction times and accuracy) are called Size Congruency Effects (SCE). This effect occurs when at least one of the dimensions is automatically activated (even if the dimension is task irrelevant). This activation improves participants' performance when the dimensions are congruent, but (task-irrelevant) information must be inhibited if the stimuli are incongruent. Inhibition takes time (higher reaction times) and is error-prone (lower accuracy).

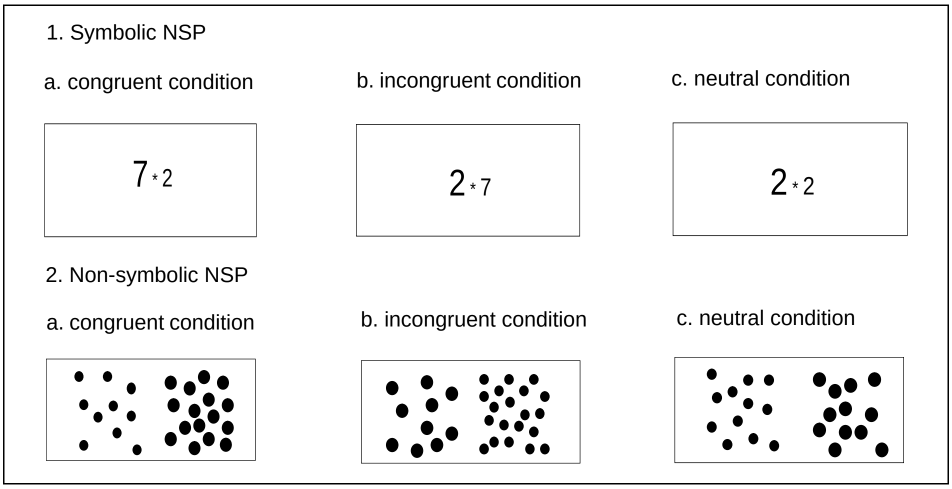


Fig. A.2 Example of the stimuli in the (1) symbolic NSP and (2) non-symbolic NSP. Participants respond either to the physical size or to the numerical size of the stimuli. (a) The congruent condition: the physically larger stimuli are also greater in number. (b) The incongruent condition: the physically larger stimuli are less in magnitude. (c) The neutral condition in the physical size comparison task: the physical size of the stimuli differs, but the numerical size is the same.

Very young children show no interference effects when the physical size contrasts with the magnitude of the numerical stimulus. This contrasts with older children who automatically process the meaning of a numerical symbol and therefore need more time to think about possibly contrasting physical size or magnitude (Girelli, Lucangeli, & Butterworth, 2000; Rubinsten, Henik, Berger, & Shahar-Shalev, 2002). The significance of investigating automatic quantity processing as an influential factor in the development of mathematical abilities becomes apparent from studies with dyscalculic children. These children had difficulty with automatically activating the magnitude of a number and they therefore required more attention than normal for understanding the meaning of a numerical symbol (Rubinsten & Henik, 2005).

By measuring the brain activity of subjects in the NSP, Szùcs and Soltész (2007a, 2008) and Szùcs, Soltész, Jármi, and Csépe (2007b) studied how the interference and facilitation effects of children compare to those of adults. The researchers found that facilitation and interference effects occur at different moments in the processing and responding to the stimulus. Further, children experienced relatively more interference than adults because they had more difficulty suppressing the irrelevant response. This agrees with Ansari, Garcia, Lucas, Hamon, and Dhital's (2005) fMRI research which showed that, throughout the development, the parietal lobes become increasingly involved in symbolic number processing and that this can eventually contribute to an automatic and thereby a faster ability to associate numerical symbols with the magnitudes that they represent.

1.3 The NS component of the MENS research project

The literature review above outlines what is already known about the development of mathematical abilities in terms of the brain areas that are involved and the processing of symbolically represented quantities. What is not yet understood, is how the early non-symbolic representations become integrated with later acquired symbolic representations of quantities. The NS study of the MENS research project builds upon the research above to gain insight into the development of automatic quantity processing (Gebuis, Cohen Kadosh, De Haan, & Henik, 2008; Gebuis, 2009).

The assumption of the NS study, is that automatic processing of numerical symbols develops with age. For this, the researchers compared kindergartners' performance (comparable in age to the Kindergarten 2 children in the ME study, see Chapter 3) to that of adults. SCE effects for the processing of non-symbolic stimuli and symbolic stimuli could be compared by using both symbolic (i.e., Arabic numerals) and non-symbolic stimuli (i.e., sets of dots; Gebuis et al., 2008). The participants were instructed to press the button on the side of the computer screen that corresponded to the side where the physically larger (non-symbolic or symbolic) stimulus was presented (see Fig. A.2).

The results of the adults showed SCE effects for both the symbolic and non-symbolic tasks. The researchers concluded from this that the non-symbolic task is appropriate for investigating children's ability to automatically process non-symbolic quantities. While in the symbolic task, the children revealed no SCE (i.e., young children may be familiar with number symbols, but they do not yet have automatic access to their meaning), in the non-symbolic task, children did show SCE. This is interpreted to indicate that they had access to non-symbolic numerosities and that they did have knowledge about magnitude and how magnitudes relate to each other (Gebuis et al., 2008).

Since the automatic processing of symbolically presented quantities can influence mathematical development, one practical implication of this study could be that it emphasizes the importance of attending to numerical *relations* and the *meaning* of quantities and numerical symbols in education. In fact, Butterworth, Zorzi, Girelli and Jonckheere (2001) state that children's ability to organize arithmetical facts in terms of magnitude, is essential for developing arithmetic ability. Education could expose children to numerical symbols as a way to repeatedly stimulate their ability to process magnitudes more automatically.

This study has set the stage for an EEG study on children's non-symbolic and symbolic quantity processing. A difference in neuronal activation between young children (Kindergarten) and older children (grade 2) and adults on a symbolic but not a non-symbolic task, should highlight the neuronal processes that play a role in the development of automatic quantity processing. To better be able to study these neuronal processes, the researchers measured Event Related Potentials (ERPs) as the participants performed the non-symbolic and symbolic size congruity tasks described above. This contributed to insight into children's inhibition mechanisms, and whether the congruency effects occurred during the act of processing or during the act of responding to the stimulus (see Gebuis, 2009).

1.4 Relating ME and NS

The interdisciplinary character of the MENS research project was intended to answer to the call for more interdisciplinary knowledge about how young children's developmental potentials in the area of spatial thinking and number sense may be cultivated (Jolles et al., 2005). In this section we describe the difficulties that were encountered in realizing this interdisciplinarity. The proposed theoretical relationship between the two studies, can be seen as a preparation for future interdisciplinary initiatives in the field of mathematics education.

1.4.1 ME versus NS

The challenge in the overarching MENS research project was to explore ways for relating the differing research paradigms of the two research disciplines to each other. This search involved continuous discussions during which the researchers first had to chart the theoretical and methodological differences before looking at how these differences could be assimilated. In what follows, we discuss three general issues that had to be negotiated.

First, the NS study was limited to a laboratory setting to minimize noise in the data. The consequence of this is that it dismissed research in the classroom setting. However, part of the aim of the ME study is to grasp characteristics of an instructional setting that may stimulate children to spatially structure. By definition, this involves the simultaneous manipulation of more (difficult to control) variables than what is permitted for sound experimental research. A distinction between experimental groups would be less valuable to an educational setting because any differences in results between the two groups could be attributed to variables that are difficult to control (e.g., teaching styles or children's individual differences, Schoenfeld, 2000).

Second, limitations to research techniques determine the type of activities that may be performed in research, as well as the types of questions that could be answered using these techniques (Varma, McCandliss, & Schwartz, 2008). The tasks that were developed for the NS study relied on a computer with an EEG set-up for measuring participants' realtime information processing mechanisms. Children must be able to respond almost instantly without being distracted, and their motor activity must be limited to prevent noise in the EEG data. Such experimental tasks differ greatly from the tasks that are generally accepted in and relevant to educational research. Hence, the question is whether the results of research performed in a strictly controlled setting with downsized tasks, could actually be extrapolated to a level of cognition that is relevant to educational practice (Davis, 2004).

The third issue that we encountered is a more practical one. The neuroscientific study depended on many external variables that made the study vulnerable to delaying the MENS research trajectory. The research could not be performed, for example, without the consent of the Medical Ethical Testing Committee, and the EEG apparatus malfunctioned regularly. Further, it took at least four hours per participant to gather data, so it was difficult to include the many participants that were necessary for such neuroscientific research within the time frame that was available for the research project. As such, it was no longer possible to come to combined data-driven conclusions. However, our struggles in negotiating such theoretical and methodological differences have provided experiences that set a valuable example for future interdisciplinary investigations in the field of mathematics education research. Moreover, as discussed in the next paragraph, the outcomes of the two studies provide input for a theoretical relationship between spatial structuring, automatic quantity processing and the development of children's numerical insight.

1.4.2 ME & NS

One interpretation of the NS study is that young children's insight into numerical relations is fundamental to knowledge of numerical symbols, and therefore to their mathematical development. The fact that the Kindergarten children in this study were already familiar with numerical symbols, highlights how early these children are taught to recognize the numbers. Yet, in the NS study it became clear that although these children recognized numerical symbols and had a sense of ordinality (i.e., they were able to indicate whether the magnitude of a certain number is greater or smaller than another number), they had not yet automatized access to the meaning of these symbols. A limited understanding of the meaning of the symbols could suggest a lack of insight into numerical relations that help to understand the composition of a quantity (e.g., 6 is greater than 2 because 2 fits into 6 three times) which may be represented by a numerical symbol. The importance of not only recognizing but also understanding the meaning of these symbols (i.e., through insight into numerical relations) for the development of mathematical abilities, has repeatedly been addressed in other research (e.g., Butterworth, 2001).

At the same time, the ME study sheds light on the role of spatial structuring in children's development of number sense, particularly for insight into numerical relations. In theory, this association can offer an explanation for the differences in congruency effects that were found between the two groups of participants in the NS study; children's lack of understanding the meaning of numerical symbols may be the result of limited insight into numerical relations, while this may be supported by a greater awareness of spatial structures and the development of spatial structuring ability. As such, we propose that the instruction activities which were developed in the ME study for helping children become aware of spatial structures for abbreviating numerical procedures, may be implemented in classrooms to foster children's insight into numerical relations (i.e., how magnitudes relate to each other for (de)composing quantities). Fig. A.3 summarizes the proposed relationship between spatial structuring ability for stimulating insight into numerical relations, and the development of automatic access to symbolic meaning. This trajectory should help children prepare for higher-order arithmetic abilities.

This theoretical outcome of the collaborative research highlights the need for more attention in education to stimulate insight in numerical relations that elucidates the *meaning* of the quantity (i.e., magnitude) which may be represented by a particular numerical symbol. The ME study suggests that one viable method for fostering this understanding is through supporting children's spatial structuring ability. Future research could investigate whether improvements in children's spatial structuring ability in the ME study, affects their insight into numerical relationships. This may, in turn, support them in understanding the numerical symbols so that access to the meaning of symbolically presented number may be more automatized.

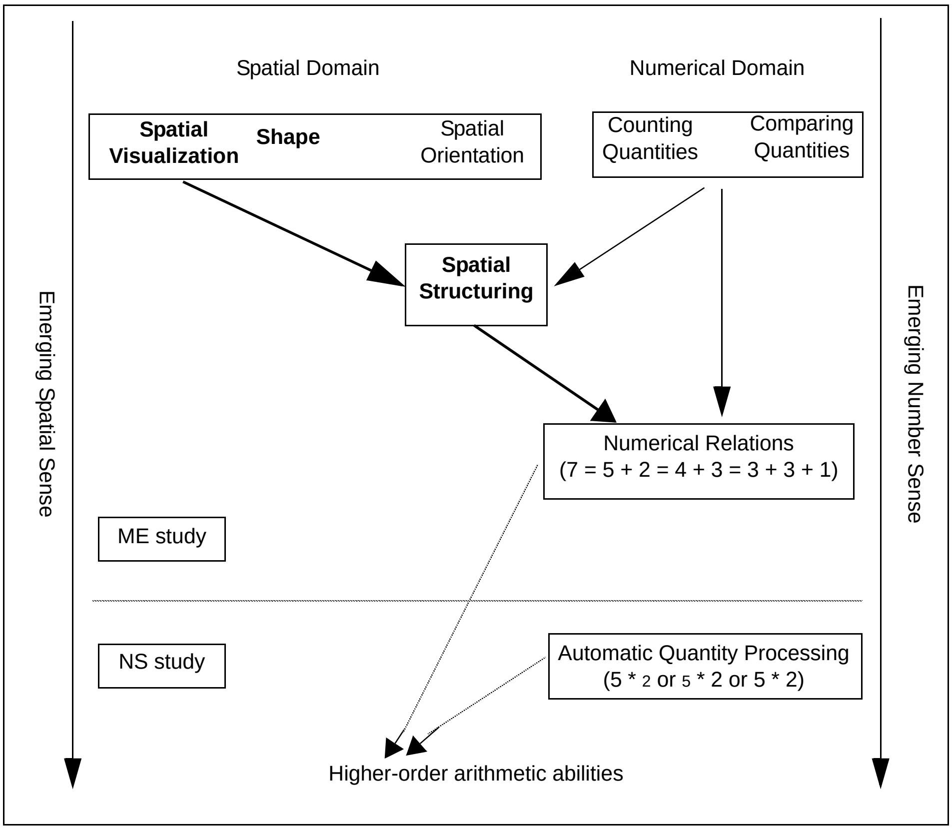


Fig. A.3 A summary of how the mathematics education study and the neuroscientific study in the MENS research project may be related: the role of automatic quantity processing for supporting formal arithmetic abilities is incorporated into the conceptual schema

Children with insight into numerical relations are ultimately expected to automatically retrieve magnitudes of numerical symbols, to abbreviate numerical procedures, and to develop strong foundations on which to build more formal mathematical insights. A longitudinal study could involve the same children in both studies for a longitudinal investigation into possible correlations between the learning trajectory on children's spatial structuring ability and neurophysiological development of non-symbolic and symbolic quantity processing. This may help to further disentangle the role of spatial structuring ability for the development of number sense, insight into numerical relations, automatized access to the meaning of numerical symbols, and ultimately, formal mathematical abilities.

The search for ways to combine the two disciplines in the MENS research project has shown that interdisciplinary success depends on how well the research disciplines that are involved communicate with each other so that the research perspectives can mutually be respected. Davis (2004) states that research is only truly interdisciplinary if the parties that are involved understand each other's specialist contributions. Only when neuroscientists and education researchers stay involved with each other's research, will they be able to work together rather than past each other (Ansari & Coch, 2006). Throughout the development of the MENS research project, we tried to emphasize that neuroscience is not educational research and that educational research is not neuroscience. This insight underlined the added value of interdisciplinary research: by approaching the research questions from two perspectives, we could perform more enriched research with the outlook of better grounded research results. This underlines how the more researchers from different research domains work together, the better the research results may contribute to reliable and more generalizable models about learning (Kaufmann & Nuerk, 2005).

1.5 Conclusion: MENS

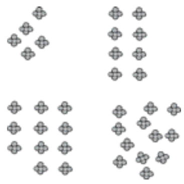
In the previous section, we proposed a theoretical relationship between the outcomes of the NS study regarding the role of insight into numerical relations for automatic quantity processing, and the outcomes of the ME study regarding the role of spatial structuring ability for stimulating insight into numerical relations. From this we conclude that (a) insight into numerical relations is an important prerequisite for understanding numerical symbols, that (b) attention to spatial structuring can support this insight, and that (c) despite several theoretical and methodological differences, the MENS research project is a valuable initiative for realizing interdisciplinary research to improve mathematics education.

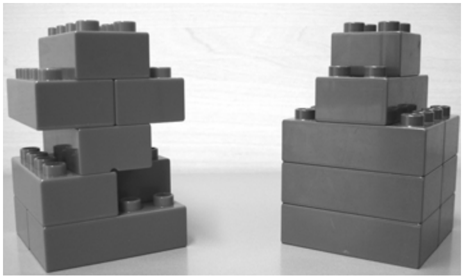
Although the combined research outcomes of the MENS research project are still very tentative, the trajectory along which the theoretical and methodological differences as well as domain-specific questions, considerations and decisions were negotiated, offers valuable resources and learning opportunities for researchers in similar interdisciplinary investigations. Hence, an important outcome of the overarching MENS research project is that it set an example for how certain theoretical and methodological differences in interdisciplinary investigations may be negotiated (Van Nes & Doorman, 2006; Van Nes, in press). In effect, the two disciplines performed preparatory work for future research by setting up and performing the two studies. This has culminated in a research design that has the potential for combining the outcomes of the instruction experiment with the outcomes of the EEG experiment through, for example, involving the same groups of participants or through performing a longitudinal correlational study.

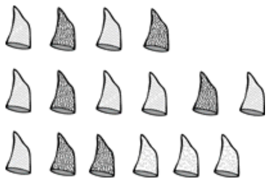
Taken together, the MENS research project offered a unique opportunity to explore the characteristics, including the relative limitations, of the methodologies of the two research disciplines. We conclude that although the outcomes are speculative, they pave the way for future interdisciplinary research initiatives in the field of mathematics education. In the meantime, the outcomes of the mathematics education study that have been discussed in this thesis, highlight the important role of spatial structuring ability in mathematical development. Moreover, the instructional sequence offers practical means to improve instructional settings for cultivating young children's early spatial structuring ability. In effect, the earlier young children's mathematical development is fostered, the higher the chances of long-term success in preventing delays as well as in encouraging advanced mathematical understanding.

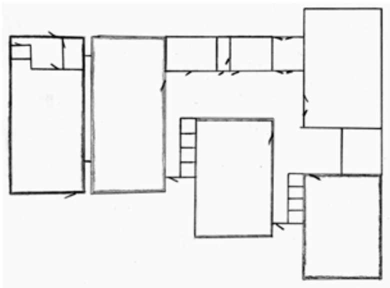
2 Script of the Tasks For the Pre- and Post-interviews

Note: this script is translated from Dutch and stated briefly here

Interview Task 1	**Picking Flowers with Noddy**
Set-up	8 flowers in between 5, 9, and 11 flowers. The 8 are arranged as 2 rows of 4, the 11 as 3 rows of 3 and another 2. The 5 and 9 flowers are bunched together. 
Introduction	This is Dwarf Noddy (*show the toy figure*). Have you seen him before? Today he is here because he needs your help. Would you like to help him?
Question 1	When Noddy last visited the forest, he was looking for flowers for decorating his house. He had just turned 8 dwarf years old. He was going to celebrate that with his dwarf friends. Can you use your fingers to show me how much 8 is?
Solution	
Question 2	Noddy thought: what if I could quickly see which bunch of flowers has 8 in it, then I'll be finished in no time! Can you explain to Noddy how, without counting, you can know which bunch has 8 flowers in it?
Solution	
Question 3	Which bunch of flowers has more than 8 in it? Can you spot another bunch that has more than 8 flowers?
Solution	
Question 4	How many flowers do we need to add to this bunch so that it also has 8 flowers?
Solution	
Question 5	How can you arrange the flowers so that it is easier for someone else to see, without counting, that there are 8 flowers? (*the 8 flowers are now arranged randomly)*
Solution	

Interview Task 2	**Dwarf Houses**
Set-up	Two block constructions each made up of 8 uniformly colored blocks; the first is unstructured (asymmetrical) and the second is structured (symmetrical). 
Introduc-tion	When Noddy came home, he met his neighbor. His neighbor is always complaining. This time, he was complaining about how he thought that his house was smaller than Noddy's house. Noddy wanted to cheer him up by showing the neighbor that his house is not so small at all.
Question 1	For which house is it easier to see out of how many blocks it is made? How do you know?
Solution	
Question 2	How many blocks do you think you will need to build that house? And how many for the other house?
Solution	
Question 3	Now you can use these blocks (*give the child a different set of another 8 blocks*) to rebuild this house. Can you rebuild the other house as well?
Solution	
Question 4	Which house did you find easier to rebuild? Why?
Solution	

Interview Task 3	**Party Hats**
Set-up	Set of red, white and blue party hats (cut out of paper) arranged in the following patterns a. abab b. abaaba c. abbccc 
Intro-duction	The dwarf friends took their hats off when they came into Noddy's house. Noddy put the hats in a long line on a table outside (*show the child the first pattern of hats*). The party was a lot of fun. Many friends came by and they liked Noddy's flowers. But then...a strong gust of wind came through the window and blew some of the party hats off the table. What a mess! How can we now know which hat belongs to which dwarf if the dwarves only remembered where Noddy had put their hat in the line...?
Question 1	Can you show Noddy how he can place the hats in the right order so that the dwarves can find their own hat again?
Solution a.	
Solution b.	
Solution c.	
Question 2	How can you explain to Noddy what the line looks like so that he can make the line himself?
Solution a.	
Solution b.	
Solution c.	

Interview Task 4	**Map of the School**
Set-up	A picture of a simple outline of the school 
Introduc-tion	Before Noddy leaves to go home again, he would like to take a look around your school (*show the map of the school*). Here is a map of your school. Look, this is your classroom. Can you point to the other classrooms on the map?
Question 1	Can you tell Noddy where the gym is on the map? Can you use your finger to point to where the gym is?
Solution	
Question 2	Can you point with your finger in the air to where the front door of the school is? And in which direction is the playground?
Solution	

3 Table of Phases and Condensed Scores

	Recognizing structure		Applying structure		Determining quantities		Comparing quantities		Accuracy		Operating with quantities		Extending structure	
	Part 1	Part 2	Part 1	Part 2	Part 1	Part 2	Part 1	Part 2	Part 1	Part 2	Part 1	Part 2	Part 1	Part 2
Phase 1 (Unitary)	1, 3	1, 3, (4)	1, 3, 4, 5	1, 3, 4	1, 3, 4, 5	1, 3, 4, 5	N/A	1, 3, 4, (5)	1, 2, (3)	N/A	1, 3, 4	N/A	N/A	1, 3, 4, (5)
Phase 2 (Recogni-tion)	(3), 4, 5	(1, 3), 4	(5), 6	4, 5	5, (6)	5	N/A	5, (6)	1, 2, (3)	N/A	4, (5)	N/A	N/A	5, 6
Phase 3 (Usage)	5, 6	5, 6	6, 7	6, 7	5, 6	5, 6	N/A	5, 6, (7)	2, 3	2, 3	5, 6	5, 6	N/A	6, 7
Phase 4 (Applica-tion)	7, 8	7, 8	7, 8	7, 8	7	7	N/A	6, 7	2, 3	2, 3	6, 7	6, 7	N/A	7, 8

Degree of priority of a particular component

Key to using the table

- The categories “recognizing structure”, “applying structure”, “determining quantities”, “comparing quantities”, “operating with small quantities”, and “extending spatial structure” are the components within the domains of spatial structuring and number sense
- Part 1: The flashcards that make up the first part of the interviews
- Part 2: The tasks that make up the second part of the interviews
- Codes 1-8: Refer to codes that identify each strategy in the strategy inventory
- N/A means that this part of the classification is not applicable to the part of the interview
- Code 2 (i.e. “the strategy is too difficult to interpret”) is possible in all phases for all the questions in the interview.

4 The Strategy Inventory

Spatial Structuring			
Code	Recognizing spatial structures	Applying spatial structures	Extending spatial structures
1	Does not seem to know how to approach the problem		
2	The strategy is ambiguous, yet the result is acceptable		
3	Does *not spontaneously recognize* spatial structures • (Usually) does not recognize spatial structures at first sight and must therefore count quantities e.g., Counts all the finger patterns; Has trouble producing own finger patterns; Counts the dots on all the flashcards; Has no (grounded) preference for the structured construction and finds the structured house easier to count because (e.g.) "it's prettier" or because "it doesn't have to be turned around"	Acts without spatial structuring • Acts unsystematically even though a structure is provided in the task • Leaves the objects unstructured *or* arranges them without mathematical intentions e.g., Shows, without having tried to count them, a wrong number of fingers; Doesn't make use of readily available rows to count systematically; Arranges chips into a particular shape because "that's fun"; Counts both the structured and unstructured constructions without keeping track of count; Rebuilds a block construction without a plan, and/or the construction differs greatly in size/shape from the example	Does not pattern • Attends neither to the *rule*, nor the *regularity* of a pattern e.g., After 'abcabc' in a pattern, comes 'dcabf' or 'cbabacb' • Cannot verbalize the regularity of a pattern e.g., Points and says "this one and then this one and this one"; Makes own rule like "pile up the hats and put them away"

Spatial Structuring			
Code	Recognizing spatial structures	Applying spatial structures	Extending spatial structures
4	Recognizes structures that are related to an *experience* • Recognizes certain spatial structures on the flashcards that are based on experiences with small quantities e.g., Recognizes finger patterns ("I just see it") for a certain age ("that's how old I am") or for 5 and 10; Recognizes either the 5 or 10 on the dice; Recognizes certain shapes based on experiences with them	Can apply an *arrangement* to objects • Tries to keep track of count by applying a kind of arrangement to the objects e.g., Carefully counts each finger to show a particular finger pattern; Moves objects apart because "that's how you can count them easily"; Spreads out already structured objects to be able to count them easier; Counts the blocks of a structured construction in a systematic, but not yet accurate, way; Refers mostly to the front face when rebuilding a block construction and tries to at least rebuild the same size and shape as the example; Requires much guidance to complete the construction	Recognizes *certain characteristics* of a pattern • Shows part of the regularity of a pattern by only examining the beginning of the pattern. Does not appear to understand the essence of a pattern yet e.g., Literally copies the pattern by placing elements above or below the example; Repeats the previous elements of the pattern without showing insight into how the pattern should continue • Has trouble verbalizing the regularity. Names only the colors of the elements in the example e.g., "red, white, blue, red, white, blue, red, white..."

Spatial Structuring			
Code	Recognizing spatial structures	Applying spatial structures	Extending spatial structures
5	Is developing towards making use of readily available structures • (Usually) recognizes spatial structures for various quantities within a specific setting, as well as various types of spatial structures for certain small quantities. (Often) requires guidance in this. Can verbalize in what ways spatial structure may be useful. e.g., Recognizes several of the 7, 8, and 9 finger patterns; Recognizes both the 5 and 6 on dice; Can read off the number of corners of the pentagon *or* hexagon; Arranges a 5-structure only when asked retrospectively; Perceives structure implicitly in a set of objects ("that group has one row more") or in a construction ("there are less blocks because it's easier to count, that's the way it's built")	Can apply a type of *organization* • Uses readily available structure to act systematically and accurately, but not yet for abbreviating the procedure e.g., Counts objects in a particular order; Counts each finger to accurately show a particular finger pattern; Unitarily counts the blocks of a construction in a systematic way; Rebuilds a construction by not only examining its front, but also by taking the relationship between the blocks into account; Requires guidance to complete the construction properly • (Often) applies a type of organization by arranging the objects into a. a single row or circle b. several short rows c. a coherent figure e.g., Arranges 10 flowers as 2 rows of 4 and 2 below, or 2 rows of 5, for *counting unitarily rather than for abbreviating the procedure;* Arranges the objects in the shape of a flower or house and subsequently counts them unitarily	Can make use of part of the regularity of a pattern • (Often) uses the correct colors but (mostly) without showing insight into the regularity of the pattern e.g., Arranges "red, white, purple" while the pattern is "red, white, white, purple, purple, purple"; Continues the pattern in a mirrored sequence so "abcabc" becomes "bacbac" • Verbalizes the pattern by reading off the colors of the elements that were added to the example. Still has trouble explaining the regularity of the pattern e.g., "Red, white, blue, red, white...blue, red, white, blue"

Spatial Structuring			
Code	Recognizing spatial structures	Applying spatial structures	Extending spatial structures
6	Can spontaneously use readily available structures • Recognizes the standard spatial structures (usually) in different types of representations and outside of the (familiar) flashcard context e.g., Now also recognizes the finger patterns for 6, 7, 8, and 9; Reads off the number of corners for both the pentagon and hexagon; Spontaneously recognizes the 5-structure in the way that objects are arranged on the table; Not only recognizes dot configurations, but also clovers on a playing card, or 2 rows of 3 eggs in an egg carton; Can explicitly explain the structure of a group of objects or a construction (“that group has 1 more row, so it’s larger”)	Is developing the ability to use and apply spatial structure as a way to abbreviate a numerical procedure • (Sometimes) tries to determine a quantity by using spatial structure rather than unitary counting procedures. Is not always successful and consequently has to count anyway e.g., Reads off a 5-structure in some settings (e.g., flashcards), but doesn’t use structure with the intention of abbreviating a counting procedure; Can spontaneously present certain finger patterns that are connected to an experience (e.g., age), but still has to count other quantities (e.g., 8) unitarily • (Usually) does not apply structure spontaneously, but rather retrospectively e.g., Counts all the objects but explains, when asked, that there are 4 and 4 and that makes 8; Structures objects only after having determined the quantity by counting; Steadily and independently rebuilds a block construction by attending as much as possible to the structure of its parts and by comparing the rest of the construction block by block to the example	Is developing towards recognizing not only the structure but also the regularity of a pattern • Adds the correct elements in the correct proportions to the pattern e.g., Looks back at the beginning of the pattern and continues the pattern based on the perceived regularity • Describes the rule of a pattern by first reading off all the elements and then repeating them in the proper sequence. (Usually) requires some encouragement to start this. Only explains that the pattern has a rule retrospectively. e.g., Reads “red, red, blue, red, red, blue...red, red, blue...”; “The rule is 1 red, 2 blue, 3 yellow”

Spatial Structuring			
Code	Recognizing spatial structures	Applying spatial structures	Extending spatial structures
7	Spontaneously makes use of a *bottom-up* strategy to abbreviate procedures involving large quantities • Reads off small (≤6) structured quantities and combines these to determine relatively larger (≤10) quantities (i.e. a constructive approach) e.g., Counts by twos; First reads off two groups of 4 and reasons that together that makes 8; "6 with 3, that makes 9" • Is (usually) not familiar enough with larger (>10) structured quantities to recognize them instantly and counts them unitarily	Spontaneously makes use of the structure of small (≤6) quantities (subitizing) to abbreviate a procedure • Can abstract a structure from a small (≤6) structured or unstructured arrangement in order to abbreviate a procedure e.g., Perceives two groups of 3 and knows that that is 6 altogether, so there are 6 dots; Makes use of the structure of the structured construction to abbreviate the procedure of determining the number of blocks that it consists of; Determines the number of blocks in a construction as 2 times 3 blocks and another 2 makes 8 • Makes use of experientially real structures in unstructured settings, but still has to count on to find the total. (Sometimes) makes use of structure retrospectively to determine a relatively larger (>6) quantity e.g., Arranges 5 objects like the dots on dice and reads off the 5-structure, but has to spread out 7 objects or arrange them in rows in order to count them in a systematic and organized way; Makes use of the (structure of the) example to rebuild the construction in a goal-directed and independent way	Can describe the *rule of a pattern* • Can, on own initiative, describe the rule of a pattern in a concise way without having to repeat each element of the pattern in a routine-like manner e.g., "Every time 1 blue, 2 red, 1 blue, 3 red, 1 blue"; "Every time there are 2 purple, 2 purple, 2 purple and 1 red one in the middle"; Names the elements of a pattern rhythmically to denote the regularity

Spatial Structuring			
Code	Recognizing spatial structures	Applying spatial structures	Extending spatial structures
8	Spontaneously makes use of a *top-down* strategy to abbreviate procedures that involve larger (≤10) quantities • Convincingly recognizes large (≤10) quantities and can reason about their arrangement retrospectively (i.e. a top-down approach) e.g., “9 because there are 3 rows of 3”; “10 because 3 and 2 makes 5 and there are 2 rows of 5”; “10 because it’s 2 times the 5 like the dots on dice” • Uses convenient strategies or formal operations to determine relatively large (>10) quantities in an effort to abbreviate the procedures (i.e. a bottom-up approach) e.g., See code 7	Can goal-directedly and spontaneously apply spatial structure to unstructured settings • Spontaneously applies structure or reads off a structure mentally from an unstructured arrangement in order to abbreviate a particular numerical procedure e.g., “I see it because here there are 3 and there are 2, and there are 2, and together that makes 7”; Can determine the number of blocks in the construction in an abbreviated way rather than unitary counting • Applies structure to a set of objects with the intention of abbreviating a particular procedure e.g., Can, without hesitation, show a particular quantity in terms of a finger pattern; Arranges objects in smaller subitizable groups or in dice configurations; Explains that 2 rows of 5 makes 10 because “3 × 3 + 1 = 10”; “2, 4, 6, and 1 makes 7”	Can describe the *regularity of a pattern* • Can concisely summarize the rule and regularity of a pattern without having to name each element. May come up with variations to the pattern e.g., “Every time it’s boy-girl-boy-girl” so the next one is boy”; “Every time it’s 1 red, 2 blue, 3 green, so the next one can be 1 red, but it can also be 4 red, 5 blue, 6 green”

Number Sense			
Code	Determine a quantity	Compare quantities	Operate with quantities
1	Does not seem to know how to approach the problem		
2	The strategy is ambiguous, yet the result is acceptable		
3	Counts asynchronously and unitarily by: a. Moving each object aside b. Pointing to each object c. Perception • Counts unsystematically and therefore has trouble keeping track of count e.g., Does not count beyond 10 and has not automatized counting yet; Cannot show 4, 6-10 as finger patterns yet; Counts 10 flowers one-by-one and concludes that there are 11	Compares quantities perceptually • Compares quantities only on a perceptual level, without actively reorganizing the objects e.g., Compares only general surface areas and/or the relative positioning of the groups on the table	Operates with quantities in a context-dependent way • Exaggerates or overgeneralizes a quantity in simple addition tasks or does 'trial and error' with the available objects e.g., "Some are missing so there must be 10 missing!"; Adds a number of objects and removes some in order to come closer to a particular quantity, but cannot keep track of how many objects are finally added to the group

Number Sense			
Code	Determine a quantity	Compare quantities	Operate with quantities
4	Counts synchronously and unitarily by: a. Moving each object aside b. Pointing to each object c. Perceptive • Counts context-dependently[a] and (often) systematically and can therefore keep better track of count • Knows the counting sequence up to 15 and counts by naming objects 1-by-1[b] • Despite possibly an incorrect result, the intention of the procedure indicates a development towards resultative counting	Can apply an arrangement to a group to estimate quantities • Can compare quantities not only at a perceptual level, but also by explicitly or implicitly linking objects to each other e.g., “That one has 1 more than this one so that group must be larger”; Understands that counting offers a way to check the accuracy of an estimation; Places every flower next to every butterfly and determines the difference in quantity	Can “count all”[c] • Adds by making use of the objects and own fingers to count all the objects. (Sometimes) has trouble keeping track of how many were added e.g., Recounts the fingers on one hand of a flashcard, and counts on to include the 3 fingers on the other hand; Recounts the 5 flowers on the table and counts on by using fingers or by placing another 3 flowers in the group to come to the required number of flowers

Number Sense			
Code	Determine a quantity	Compare quantities	Operate with quantities
5	Counts resultatively by a. Moving each object aside b. Pointing to each object c. Perceptive • Is developing a generalized counting ability[d] because counting depends less on the context and can be performed in more than one way	Can compare groups based on their quantity • Knows that a certain number is larger or smaller than another number. Has therefore integrated counting abilities with the ability to compare quantities[e] e.g., Counts a group of 8 and 6 and knows that 8 is larger than 6 so that group must be larger than the other group • Explores and compares small quantities more with regard to number than quantity e.g., Seems to pick the group of 5 randomly as being larger in comparison to the group of 8, counts the group and concludes that “that one has 5 and that’s less than 8 so it’s the smaller group”	Is developing the ability to “count on”[f] • Performs addition by adding on to the group while keeping track of how many were added to come to the required amount. (Usually) does this with the help of the objects or fingers. e.g., Recognizes 5 fingers on a hand and adds a number of fingers on the other hand to it; Given 5 flowers, places another 3 flowers in the group and adds on from 5 to 8 flowers.

Number Sense			
Code	Determine a quantity	Compare quantities	Operate with quantities
6	Is *developing the ability to abbreviate* a counting procedure • Tries to make use of a familiar structure in a relatively unfamiliar setting (e.g., counting flowers) rather than counting objects unitarily. Is not always successful and consequently has to count anyway. e.g., Can goal-directedly show 6 and 10 as finger patterns, but has to count 8 fingers unitarily; Recognizes 6 on dice but counts 5 unitarily; Recognizes the number of corners in a triangle and square but has to count the corners of a pentagon and/or hexagon; Applies a spatial structure to a part of a whole ("it looks like 5 on dice"), and has to add on the remaining objects; Uses the structure of a structured construction to determine the number of blocks, but has to count the blocks in the unstructured construction unitarily • Counts a quantity but can point out a spatial structure that, in retrospect, could have abbreviated the procedure e.g., Subitizes quantities ≤ 4 or says "2 and 3 is 5 altogether"; Recognizes the 5 as 5 dots on dice but counts the same number of objects unitarily when they are placed in the same structure in a different setting	Compares quantities based on a *grounded estimation* of the number of objects in at least one of the groups • Depends less on the physical presence of the objects for comparing perceptually, and can therefore compare groups based on the quantities themselves e.g., First examines the groups, reasons about which is largest, counts both groups to be certain, and says "8 is more than 7 so it is 1 more"	Can "count on" • Adds by adding on a number of objects up to the required amount. Can (usually) do this mentally, but the explanation does not yet reflect insight into formal mathematical operations e.g., "I thought 5 and added another 3 in my head and that makes 8"

Number Sense			
Code	Determine a quantity	Compare quantities	Operate with quantities
7	Can *spontaneously abbreviate* a numerical procedure • Can recognize a spatial structure in both structured and unstructured arrangements and use this structure to abbreviate a procedure. Can also apply structure to unstructured objects e.g., Recognizes both 5 and 6 on dice; Can show a particular finger pattern without counting; "here are 3 and there are 3 so it's 6"; Tries to use structure as much as possible to determine the number of blocks in both the structured and unstructured constructions • Reads off a quantity by recognizing a structure in the arrangement and by counting on. Can do this mentally. e.g., "Like on dice" or "4 and 1 makes 5" or "2, 4, 6, and 1 is 7" or "one in the middle"; "I've counted it before using my fingers"; "It's 8 because 5 and 3 make 8"	Can make use of spatial structure • Can compare quantities by spontaneously (recognized mentally or by moving the objects) making use of their structure. May refer to more numerical procedures in explaining this strategy, even though these procedures were not actually used e.g., The dice structure for 5 is the same as for 4 "with 1 in the middle"; Sees that the structured group of 11 is more than the structured group of 8 and subsequently shows how the quantities could also easily be counted; Recognizes a 5-structure in an unstructured setting and sees that the other group contains more than 5 so that group must be larger	Is developing higher-order mathematical abilities and can verbalize the procedure • Has decontextualized the manipulation of quantities[g] and can therefore operate with quantities both mentally and symbolically[h]. This further underlies the development of formal mathematical procedures. e.g., "3 rows of 3, and 3 times 3 is 9, add another 3 makes 12" • Applies a particular sum or spatial relationship to solve a problem e.g., "5 plus 5 is 10, but 5 is more than 4 so now we have 4 plus 6 making 10"; "3 plus 3 plus 3 makes 9"; "See, this is how it looks using my fingers"; "4 with another 4 looks like 8" • Retrieval[i]: recognizes a familiar numerical relationship e.g., "5 is half of 10"; "Last time there were 5 left with 3 gone so now it's the other way around"

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5 Manual of the Final Instructional Sequence and Observation Criteria

Note: this manual is translated from Dutch and based on the instructional sequence that was tried out in the second round of the instruction experiment

5.1 Activity 1: Ant's Tool Box and Ant Steps

Objectives of Ant's Tool Box	• Become familiar with the overarching context of the instructional sequence • Gain experience with naming objects, verbalizing spatial structure, and comparing quantities in terms of their spatial structures • Discover how certain structures can help to abbreviate tasks that involve determining and comparing quantities
Objectives of Ant Steps	• Discover how simple spatial structures can help to recognize and determine relatively larger structured quantities • Become aware of insight into the composition of quantities • Become familiar with phrases that are to become shared (e.g., "more", "less"; "easy ways", "tools")
Setting	Introduction in the circle with an extension in the focus group
Requirements 	• Objects that represent fundamental spatial structures: – 3 egg cartons for 10 eggs – 3 egg cartons for 6 eggs – enough plastic eggs to fill the egg cartons with – 1 or 2 large dice – 3 sets of large playing cards with dots – 4 or 5 patterned bead necklaces (e.g., 3 blue-3 white or 1 red-3 green-1 red) – 1 set of flashcards with finger patterns • The objects are in a large decorated cardboard box: "Ant's Tool Box" • Several sheets of paper with 2 rows of 3 dots drawn on them (i.e. the Ant's footprints from when he carried the box into the classroom). The papers are placed on the floor from the classroom door to the box and from the box to Ant's hiding place • A large toy ant to represent Ant

Guidelines for the teacher	
Introduction	Have you seen the dots on the floor? What could they be? Maybe they're footprints of an animal. Can you see how many footprints the animal left behind on one paper? (...) How did you see that so quickly? Can you show how many there are using your fingers? Look, the footprints lead all the way to the box. It looks like the animal carried this box into class. Maybe the animal is still hiding in the classroom. Where do the footprints lead to? Can we find the animal there? [Let one of the children get Ant from his hiding place] Look, what is it? It's an ant! How many feet does the ant have? Is that the same as the number we saw on the floor? How do you know so quickly? Could he be the one that left all these foot prints on the floor? This is Ant and, like all ants are, Ant is so strong that he was able to carry this box into the classroom. What is this box? (...) It is Ant's Tool Box. What can we find in Ant's Tool Box? I noticed that the box is very heavy. I'll give someone a turn to come and see what's in the box. Look at how there are so many different things in the box! Very mysterious. Why do you think Ant brought this Tool Box into class? Maybe he has a message for us. They may be "tools" that can help us. Let's take a look at these large cards in the box. Do you know what these are? (...) It looks the ants even walked across these cards. Every dot looks like one of Ant's footprints, Ant Steps. We're going to play a game with these cards. Two children will come stand in the middle of the circle. I will pick a card from the pile. The first person who sees how many "Ant Steps" are on the card, may keep the card. But...we're not going to count the dots, are we? That will take too long. Maybe we can think of a "quicker way" of finding out how many Ant Steps are on the card. What "other ways" can you think of? Try to find the "fastest way" to see how many dots are on the card. Then you'll be the winner!
Sample questions	(1) We're first going to play the game with this die. I will throw the die and then you tell me as fast as you can how many dots come up.

	[Take the large die and throw to get a 5 or 6] How did you see that so quickly? Does anyone see it differently? Which way is easier? So how did this tool help us? Does this remind you of another tool from the Tool Box? [Hold up the playing card that has 2 sets of 5 dots on it] Now we're going to play the same game, but with these cards. Can you see how many Ant Steps are on this card without counting each dot? How do you see that so quickly? Do you know another way to see it easily without counting? (2) That's funny! Some of you don't count the footprints. So, how do you see how many there are? Does anyone else know an easy way to quickly see how many Ant Steps are on the card? (3) Do the Ant Steps look like anything in the Tool Box? Go and get something out of the box [e.g., the dice] that is also easy for seeing how many of something there are. Did you use the same "way" that you used to know the number of Ant Steps on the card? (4) So: some of you count to know how many there are, and some of you found an easy way to see how many there are without counting. Do you have a better idea now of why Ant put these cards in the Tool Box? How do you think Ant wants to help us?
The focus group	[Pick a card and place it on the table. Give two children a turn to determine the number of dots on the card. The child who sees the number the fastest (without guessing) may keep the card. Then the next pair of children has a turn. The rest of the children should pay attention because they may be allowed to explain their "easy way" afterwards] (5) How did you see how many Ant Steps are on this card so quickly? How many steps are there? How did you use this card as a "tool" to quickly see the number without having to count each step? Did anyone else see it differently? (6) Do you remember anything from the Tool Box that you could use in the same way to see how many of something there are (e.g., the dice)?

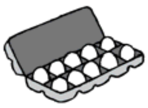
	(7) Now you can throw the die from the Tool Box. Who is first to recognize the face of the die (i.e. the dots that come up) somewhere in the card? So, who can find the die in the card? Can you find the die anywhere else in the card? (8) So now you have many ways of easily finding out how many Ant Steps are left on the cards. Could that be the reason for why Ant brought this box to class? Do you know what the tools in the Tool Box are for? We should try to remember these tools because we can take them out of the Tool Box and use them every time we want to know how many of something there are.
To simplify the activity	• Focus on supporting the children in learning to count synchronously and resultatively • Point to the objects or to the dots on the card, or physically pull the objects apart to make the structure more visible • Continue playing the game with two dice so the children begin with a more familiar context than the cards. Who is fastest in determining who threw the highest number of dots on the die?
To make the activity more challenging	• Increase the number of representations of structures (i.e. objects in the box) and let the children find their own examples in the classroom • Let the children more explicitly compare the structures to each other • Can you find another easy and quick way to see how many dots are on the card without counting? • How many more dots are on this card compared to the card? Do you know an easy way to find out? • Ask more challenging questions (e.g., if there are that many Ant Steps on the cards, and Ant has 6 feet, how many times did he have to hop to reach the centre of the circle?)

Background and observation criteria	This first activity in the instructional sequence caters to the different levels of mathematical development. Some children may still (asynchronously) count unitarily, while other children may already recognize and use various spatial structures to abbreviate numerical procedures (e.g., “that is 5 because it looks like the 5 on a die”). This will become apparent from the first discussions about Ant and his foot prints. Many children may start this activity by unitarily counting Ant’s legs. Other children may recognize a structure and share this with the rest of the class. This offers an important opportunity for the teacher to start a discussion about ways to determine a quantity and to turn children’s attention to the Tool Box. As the children unpack the box, the teacher should encourage the children to think of why the objects may be “tools” and why Ant left these tools to help them in class. The discussion should revolve around the difference between how some children count unitarily while others have “easy ways” to determine a quantity. For the Ant Steps activity, most children will unitarily count all the dots on the card without spontaneously looking for spatial structures. They may not be ready to integrate separate elements into one single structure (e.g., the children count all the dots unitarily because they cannot read off the amount at once, Steffe, Cobb & Von Glasersfeld, 1988) or because the children are so confident with counting procedures that they prefer to use such less challenging strategies to come to a correct answer. Again, the teacher should encourage the children to look for an “easy way” to conveniently and quickly determine how many dots are on the card. This introduces phrases that are to become part of a shared vocabulary. What may help is that the teacher posts the cards that have already been discussed, on the board in the classroom. The children can then refer to a card that has a certain number of dots on it and compare the dot configuration with other configurations that represent the same quantity. We expect that this will help children to associate different structures more with each other and they start to learn to recognize spatial structures in terms of different configurations (e.g., compare a 5-structure as 1 dot in four corners with one in the middle, to one row of 3 and another row of 2). The children will experience more pressure in the Ant Steps game when other children are faster at giving a (correct) answer and win the round (e.g., one child may count each of the dots while the other child may instantly see that there are five and two and seven in total). The teacher can help the children by referring to the “easy” or “clever” ways that they explored earlier in the Tool Box. That may stimulate the connection between the objects that represent a type of spatial structure in the box, and the structures that they should learn to abstract from the cards (e.g., “the dots on the die in the box look like the dots on this card”).

	The teacher can support the children by asking questions that guide them towards becoming more aware of the differences in the ways that they determine a certain quantity (e.g., "How do you see that so quickly? It was like you didn't even have to count the dots, or did you? Do you know of an "easy way" to find out how many dots there are?"). It is important that the teacher encourages the children to try to explain their strategies (e.g., the children point to the dots and tell each other how they determined the number of ant steps on the card and whether and how they see other structures) because the children who make use of "clever ways" of determining a quantity can set an example for the children who are still counting unitarily. The teacher should not only ask for an explanation, but also for a specific comparison (e.g., "do you know anything else in the box that resembles the "way" that you just used?"). This may help to prevent answers such as "I counted them really fast in my head" or "just because". Moreover, it is important for stimulating the translation between different types of spatial structures. The teacher can support the learning process by creating a shared vocabulary through repeatedly referring to the spatial structure as "an easy way". The teacher can choose her own formulations, but it will be most effective if the shared vocabulary relates to what the children are most familiar with. As the teacher will repeatedly be referring to "easy" or "clever" ways, it will help the children to make stronger, implicit and explicit, connections between the types of structures in the box and their use in simplifying and abbreviating numerical procedures. At the start of the activity, children may find it easier to read off particular quantities compared to others as how they are represented on the cards. The children may, for example, recognize a dice configuration easier on a card with two sets of five dots compared to a card with a set of five and a set of three dots, because it may be easier to recognize a double-structure than a quantity that is made up of two separate arrangements. Exchanging experiences and practice in looking for structure in various configurations, will help children to recognize more types of structures for one particular quantity (e.g., recognizing the ten as twice five, but also like on dice and like two rows of five), and to compare different types of structures (e.g., 6 as 5 and 1 or as 3 and 3). As a result, they will better be able to abstract relatively simple and complex structures, and they may begin to generalize particular structures to different contexts (e.g., seeing a structure as part of a "pattern" or recognizing dot configurations in playing cards that have pictures of clovers which are more irregular than dots). For these reasons, the teacher should begin the activity with relatively straightforward dot configurations (e.g., the configuration for six dots or the configuration of two sets of five dots). We conjecture that the more children search for different structures, the more they gain experience in determining and comparing quantities, and,

	the more they will be inspired to look for ways to quickly (faster than someone else) determine the quantity. Some children may already be able to read off (parts of) structures so that they no longer have to count the elements unitarily. Perhaps there may be children who recognize that the ten-structure is made up of two five structures (e.g., “there are ten eggs in the box and these are also two times five eggs”). Although not many children will have reached this stage at this point in the instructional sequence these children will already have the advantage of coming in touch with these different ways of determining and comparing quantities at such an early stage in their development. The ultimate challenge for the teacher is to guide the children towards an understanding of structures without imposing the structures on the children. The children must discover and compare the different structures on their own. This will stimulate them to explore how spatial structures can help them abbreviate numerical procedures. We conjecture that these experiences are what will help children to gain insight into numerical relationships (e.g., “they are 7 because 5 and 2 make 7”) and practice how to verbalize how structures can be used to determine quantities (e.g., “they are 7 because there is a 5 and there is the 2, but you can also see it as 3 and 3 and 1”). The activity can best be concluded in the way that it was introduced. Referring back to the context of Ant and the Tool Box should help the children gain more understanding of the activity and remember the lesson better. The aim of the context of the Tool Box is to give the children an impression of a box that is filled with tools (“easy ways”) for easily determining an amount. This is an important contribution to the shared vocabulary so that the teacher can keep referring to the examples in the box as the class continues to explore how spatial structures can abbreviate a numerical task (e.g., “do you remember what was in the Tool Box? Could we maybe use these ‘easy ways’ to find out how many of these there are?”). The teacher must make use of the by now well-established context to support the children to verbalize the connections between the various types of spatial structures (e.g., “so now you’ve thought of different ‘easy ways’ to find out how many ant steps are on the cards; those ‘easy ways’ were very useful for you to recognize the number of dots even faster than your partner. Ant wants us to remember the “ways” that he put in the Tool Box, so that later, during other activities, we can also quickly see how many of something there are”). This is how the teacher may help support the children in making connections between the conceptual knowledge that underlies the activities.

5.2 Activity 2: Filling Egg Cartons

Objectives	• Explore the structure of different kinds of egg cartons and discover how the structures can help to determine the number of eggs in a carton without unitary counting • Gain more insight into the composition of quantities • Practice determining and comparing quantities • Become more familiar with phrases of the shared vocabulary
Setting	Introduction in the circle with an extension in the focus group
Requirements 	• Ant and the Tool Box • Plastic eggs • Paper and pens for drawing the egg carton structure • Tape to post the papers on the board
Guidelines for the teacher *Introduction* *Sample questions*	 Do you remember what was in Ant's Tool Box? There are also egg cartons in the box. That's funny. Don't they belong in the kitchen? Why do you think Ant put these egg cartons into the Tool Box? How do you think this tool can help us? Ant sometimes visits Farmer John's farm. Farmer John has many chickens that lay so many eggs that he has trouble keeping track of how many egg cartons he needs to box all these eggs. This morning, his neighbor asked for a carton of 6 eggs. [First the empty egg cartons are placed in the centre of the circle] (1) Farmer John is always in a hurry, so he is looking for an easy way to quickly see which egg carton exactly fits 6 eggs. How can we know that without counting each cup in the egg carton? Do you know another way to easily and quickly see how many eggs fit in the egg carton? (2) Can you find another example in the Tool Box (e.g., the dice or the cards) and show how you can use the tool to quickly see how many of something there are? (3) If 6 eggs fit in this carton, how many eggs fit in the other cartons? Do you know an easy way to see that?

	[Depending on the children's level, they can show their "clever ways" or they can practice counting by pointing to the empty spaces in the carton or by filling the egg cartons with the plastic eggs]
	[Give one child a pen and a sheet of paper with the cups of the egg carton drawn on it. Post the Ant Steps cards for 6, 7, 8, 9, and 10 in a line on the board for the children to see]
	(4) Now your task is to color 5 eggs in the egg carton on this piece of paper. Try to do that in a way that makes it easy for someone else to see how many eggs are in the carton. Then you can post the paper on the board, right under the card that shows the same number of dots. Do you see "easy ways" for finding out how many dots are on the paper?
	[Place a number of plastic eggs in the middle of the circle] Farmer John is also always in a hurry to take the eggs to the market. He needs to quickly know how many eggs he has today.
	(5) Two children will come stand in the centre of the circle. One of you will be Farmer John and look away. The other will put the eggs in the egg carton in a way that makes it easy for Farmer John to, without counting, see how many eggs are in the carton.
	(6) Farmer John can come take a look. Does the "easy way" work? Can the farmer quickly see, without counting, how many eggs are in the carton? Who can draw the eggs on this paper and hang it up with the other papers on the board?
	(7) The cards with Ant Steps and the drawings of the egg cartons are posted on the board. Who knows now why Ant put the egg cartons in the Tool Box?
	[Take a 2 to 5 eggs out of the 10-egg carton]
The focus group	(8) Foxes also like to visit Farmer John's farm to steal eggs from the egg cartons. It looks like they've already done that to this carton. I am going to close the carton and point to someone who has to pay attention to the carton. Can you tell the other children what you saw in the carton so that they can guess how many eggs are still in the carton? Do you know how Farmer John can quickly see how many eggs are missing from the carton without having to count the eggs? Do you know which tool in the Tool Box could help us with this?
	(9) How many eggs still fit in the carton? Can you tell the other children how you can see that easily? Does anyone else know of another way of seeing it even quicker?

	[Repeat this several times so that children will compare different structures ("tools") with each other. Also change the roles so the children can practice with both manipulating and making use of structures]
To simplify the activity	• Use a plastic, see-through egg carton • Relate the egg carton structures to finger patterns; compare how children don't have to count the 5 on the hand, for example, either
To make the activity more challenging	• Keep the egg carton closed and let the children determine how many eggs are left in the carton based on their mental image of the inside of the carton. Let the child explain what the image looks like, open the carton and discuss any deviations.
Background and observation criteria	Some children may initially prefer to count the eggs unitarily. Other children may manipulate the eggs with abbreviated strategies by counting the eggs by twos, threes, or fives or by recognizing the egg structures from previous experiences. Some children may note that the number of eggs in the carton is the same as the number that is printed on the carton. The aim of the activity is that these different levels of counting and operating with small quantities will be practiced and developed further. The challenge is to stimulate the children into giving more than only the answer to the question. Previous exploratory studies have shown how the children will give very practical answers in return (e.g., "I just saw it that way or I already knew that or I just counted that really quickly in my head"). The focus of the questions should therefore not end at *how* the children know something (e.g., "how did you see that so quickly or how do you know?"). To gain more insight into a child's spatial structuring ability, it may be more fruitful to have the children explicitly relate the structure that they used to a structure that they recognize from the Tool Box. The teacher could ask the child to associate the strategy with a comparable structure in the box. If the child indeed used a spatial structure to determine the number of eggs in the egg carton, then the child is expected to pick a relevant object with a comparable spatial structure out of the box (e.g., the child reads off the number of eggs because it looks like the dot structure on the die). If, on the other hand, the child used little or no spatial structuring, then the child may pick any of the objects out of the box and lack an explanation that associates the structure with the egg carton activity (e.g., the child counts the dots on the cards or counts each of the fingers in the finger pattern flashcards, without relating them to the egg cartons). While the first part of the activity involves the introduction of the egg carton (double-) structures (e.g., "how many eggs fit into this carton?"), the second part of the activity is especially focused on using and applying the double-structures and the five-structures (e.g., "can you use the egg carton as an 'easy way' to see how many eggs it has?").

	Here too, the children have opportunities to practice their counting skills, but the main intention is that the children are stimulated to make use of the structures of the egg cartons. We anticipate that the game-component will inspire them to not only count the eggs, but to structure them so that someone else (the farmer) can read off the quantity. Some children may structure the eggs into groups on the floor. Although grouping is a way of structuring, it is not exactly what the task is aiming for since the activity is emphasizing double-structures and five-structures with the specific purpose of abbreviating the process of determining and comparing quantities. Hence, it is important that the children keep using the egg cartons and that the teacher continues to encourage the children to structure the eggs in the egg cartons. To determine which egg carton is for the neighbor, the children are expected to read off the quantities in the carton. Nevertheless, some children may already use early addition-strategies, while other children will adhere to the “counting all” strategy. These children first count all the eggs and then continue to count to find the difference (e.g., “I see 1, 2, 3, 4 eggs left and she wants another 5, 6, so we need 2 more”). Still other children may apply the “counting on” strategy whereby they count on from the number of eggs that they know are still in the carton (e.g., “I see that there are already four in the carton, and she wants another 5, 6, so there are still 2 eggs missing”). Ultimately, the discussions should stimulate the children to begin to make use of the structure of the egg carton for reading off the quantities or for determining the quantity in an abbreviated way (e.g., “I see two empty spaces so there are still 2 eggs missing” or “I see two rows of two so four eggs and there are still 2 eggs missing”). The teacher can also ask the children to show the number using their fingers as a way to relate the different kinds of structures for representing one particular quantity. By encouraging the children to try verbalize their strategies, the teacher can stimulate them to compare their strategies with each other (e.g., “I saw two rows of three so I knew that there are six”), using the contents of the Tool Box and the structures that the teacher had posted on the blackboard (e.g., it looks like the six on dice). This kind of comparison can help the children gain insight into the efficiency of their approach to a problem. That is important for judging the effectiveness of the structures that they used. The teacher can also relate the structures to, for example, finger patterns to put more emphasis on the role of structure in determining a quantity. A child answering faster than another child, may motivate children to look for ways to improve the efficiency of their strategies.

	The final challenge in this activity is for the focus group to determine the number of eggs without actually seeing the eggs (e.g., “I will close the carton and then you can try to tell me how the eggs were arranged in the carton so that someone else can guess how many eggs it contained”). This is should stimulate the children to work with the structure of the egg carton (e.g., two rows of three eggs). Although this will be difficult at first for most children, even an attempt at this task could support them in becoming more aware of the use of the structure of the eggs in the carton. If the children appear to have difficulty remembering the arrangements, then the teacher can show them the contents of the carton again and ask them the same question. The next time the children attempt such a task with the eggs out of sight, (guided by the teacher) they may be more prepared by taking note of the structure of the eggs in the carton. Some children may already be able to perform this task with the structure of the egg cartons as a mental image. That is an important step towards higher-order mathematics procedures such as addition, subtraction and multiplication.

5.3 Activity 3: Marching in a Procession

Objectives	• Explore the composition of a pattern to identify its structure • Gain more insight into the rule, and ultimately the regularity of the structure of a pattern • Practice constructing and extending a pattern
Setting	Introduction in the circle with an extension in the focus group
Requirements 	• Items that the children can hold that can be used to construct a pattern (e.g., colored papers, colored pencils) • Colored plastic chips, at least 12 for 5 colors • Ant and its Tool Box
Guidelines for the teacher	
Introduction	A few days ago we played a game with Ant Steps on large cards. Last time, we helped Farmer John fill his egg cartons. Ant had put the cards and the egg cartons in the Tool Box. Do you remember why he may have done that? Usually ants are not alone when you see them outside. How do ants walk around outside? They often walk in a long line, in a procession. They may be on their way to find food. That's what we're going to do too today.
Sample questions	(1) I am going to call up children to stand in line next to me. [After 4 children] Do we need more boys or girls to continue this line? How do you know that? Can you see that without counting? Do you know an easy way to do that? Examples of lines: - boy, girl, boy, girl... - taller, shorter, taller, shorter... - boy, girl, girl, boy, girl, girl... [Create a line of children that has no regularity] (2) Who should stand in line next? (...) So do you know what the rule of this line is? Why is it hard to see it? (...) Do you think we need more shorter or taller children to make this line longer? Do you know an "easy way" to see that?

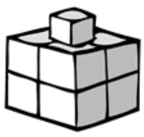
	(3) This time we're going to do it differently. I'm going to give you all a colored sheet of paper. I'll call up some children to stand in line next to me. [After 4 children] If we make this line longer, can you tell me if there will be more children holding blue papers or white papers? How do you know? Do you know an "easy way" to see that? Examples of lines: - red, white, red, white... - blue, white, white, blue, white, white... (4) Do we need more red or green papers to make this line longer? How do you know? Do you know an "easy way" to find out? [Create a line with an equal number of papers for each color (e.g., red, red, blue, blue, red, red etc.)] (5) What about his line? How do you see that we need just as many red papers as blue papers? (6) Do you remember something from the Tool Box that reminds you of the easy way that we used to make each of these lines longer? [e.g., the bead necklaces] [Show one of the bead necklaces from the Tool Box] (7) Take a good look at this necklace. If I want to copy this necklace, will I need more red or green beads? How do you know? Can you see that without counting each bead? [Show another bead necklace from the Tool Box] (8) To make this necklace longer, do I need more white or blue beads? How do you know that? Can you see that in an "easy way"? (9) Do you have a better idea now of why Ant may have put these necklaces in its Tool Box? How can you quickly see how many beads are in the necklace? Do you know an "easy way" for that? Is that the same "way" you use when you quickly want to know how many dots are on dice? Or does it look like the cards we used with the Ant Steps? Could we use the "easy way" that we used to find out how many eggs are in the egg cartons? Why do you think Ant put all these objects in the Tool Box?
The focus group	[Create a sequence of colored chips. Let the children determine whether they need more chips of one color or another color to extend the line]

	(10)Who thinks we need more blue or more yellow chips? What will the color be of the next four chips in the line? Show me how you would make the line longer. (...) Is that right? Who sees it differently? Why? [Pass around a number of colored chips that the children can use to create their own patterns. Give the children chips with, for example, two different colors of 2 different quantities, so that children may make a pattern like “abbabb”, or give the same number of chips of 3 colors so they may make “abcabcabc” patterns. Otherwise the children may stay with “ababab” or only “abc” arrangements.] (11)Now you can make your own line with the chips I gave you. Your line has to have a rule so that someone else can easily see how they can make the line longer. Who sees the rule of this line? Is that right? How do you know? Do you need more of this color or of this color to make the line longer?
To simplify the activity	• Let the children extend the line and discuss both the rule and the regularity in the line. Use a rhythm to name and continue the elements of the pattern • Work with an easy example using other bead necklaces or the colored chips • Lengthen the pattern starting from the other end of the line
To make the activity more challenging	• Create the line silently, without naming the elements so the children have to apply a rhythm on their own in order to find the regularity in the line • Keep the line short and ask for different possibilities for the rule of the line.
Background and observation criteria	In lengthening a pattern, the children will tend to recall the line from the beginning (e.g., “boy, girl, boy...” or “red, white, red, white...”) and to continue the line based on the rhythm and order of what they were saying (e.g., the children continue a particular rhythm or they may recognize the way the elements of the pattern are alternating). Some children may not continue the pattern of a line, but instead extend the line in a mirrored way by recalling the last elements of the pattern first (e.g., continue a line of “boy, girl, girl, boy, girl...” not with “girl, boy, girl, girl...” but with “boy, girl, girl, boy...”). This does not show insight into patterning if the children only read off the elements from back to front without taking note of the rule and regularity of the sequence.

	The children may extend the next line as the same pattern as the previous line (e.g., again “red, white, red...”). Here the teacher’s comments are important for convincing the children that the pattern involves a different “rule” and that the children should try to think of a way to discover the characteristics of the pattern that underlies each of these lines (e.g., look at who is standing in the line now, that is a different sequence than before. Now it’s two boys and a girl, so we must need more boys than girls to extend the line). As the children extend more lines, they are more often expected to experience more often that it helps to look at the beginning of the line to see how the end of the line should continue. In this way, the child may become more familiar with the structure of a pattern. An important part of this learning process is that for children to experience the difference between a line that does and a line that does not have a pattern (i.e. rule and regularity). A line without a pattern may first confuse many children. The teacher should discuss this confusion with the children so that they can become more aware of what characterizes a line that does have a pattern (e.g., “this line is difficult because it’s first boy, girl... and then I think it changes to small, big...”). It may not be sufficient to only ask the children to extend a pattern. The question that it leaves open is whether the children not only understood the “rule” of the pattern but also the regularity that is inherent to a pattern. Children’s ability to recall elements of a pattern and arrange alternating colors, does not say enough about what they understand about patterning. A question about beads of a particular color may support the children into thinking more in terms of the structure of a pattern. This is because the children can begin to experience that although they can count the elements of a pattern unitarily, they will find it more difficult to come to a correct and quick answer. The teacher can add to this by asking again for an “easy way” of finding out how to extend the line. Some children may come to understand that they do not have to count everything because they can make use of repeated structures in a pattern (e.g., “it isn’t red, white, red because you have to look at the rest of the line and then you see that it’s red, white, white, red, so there are just as many white as red beads in the pattern”). This should be analogous to “an easy way” of determining a quantity using the spatial structure that the objects are arranged in.

	To make the activity attractive for the children and to scaffold the generalization of the concept of a repeated structure, it is important that the children can encounter different kinds of patterns. The teacher should carefully guide the transition from one type of pattern to the next. Proper guidance is needed to take the large step from a concrete pattern to a more abstract pattern. As such, the teacher should carefully guide the children in the transition from analyzing patterns that are made up of children, to patterns that are made up of colors. This can be supported by giving each child a color (such as the color of a colored paper) that can be used to make a pattern with. As soon as the children succeed in extending the patterns with children, the teacher can begin to arrange the children according to the colored papers that they hold. The next step is to start to create lines that consist of colored chips. To smoothen this transition, the teacher should point out to the children that earlier on in the activity, they had been studying the children themselves, and that after that, they were studying the colors of the papers that the children were holding. Some children may then come to see that the next step is to study the colors of the colored chips that are arranged into a particular sequence. That is different from the characteristics of the children who first made up the structure of the pattern. This should prevent the children from constantly connecting the children's characteristics (e.g., gender or height) to the colors: "blue-white" may literally represent "boy-girl" to them. The teacher's guidance should help the children translate the way they extended the line that is made up of the children to ways to extend the line of colored papers and chips (e.g., from a pattern of boy, girl, boy, girl, to a pattern of red, white, blue, red, white, blue colored papers to a pattern of green, blue, white, green, blue, white colored chips). The children in the focus group are challenged to extend a pattern, to construct their own pattern by taking care to use structure, and to determine whether more or less of a particular color are required to extend the pattern. We conjecture that when children understand how the number of elements of one particular element relate to those of another element in the structure of a pattern, we may assume that the children not only understand the "rule" of a line, but also the regularity of the structure that characterizes a pattern. These children are expected to look back at the first elements of the pattern, recognize a repeated structure and compare quantities on the basis of the number of elements in the structure of the pattern (e.g., "it's always two red and one white so we need more red than white chips"). Later, these children may not only be able to illustrate this pattern in their own patterns, but they may also be able to verbalize the pattern in a more concise way (e.g., rather than repeating the color of each of the elements in the pattern such as "red, white, white, red, white, white", they may say that "it's always a red one and then two white ones").

5.4 Activity 4: Building Ant Hills

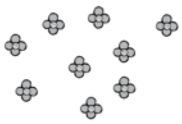
Objectives	• Explore how the structure of a block construction can help to abbreviate counting and building procedures • Practice constructing block constructions • Make use of terms such as “under” and “behind”
Setting	Introduction in the circle with an extension in the focus group
Requirements 	• 3 sets of rectangular Duplo blocks, built as two identical ant hills. There are three types that are made up of the following patterns of layers: (a) 2, 2, 2, 2, 2 blocks (b) 3, 2, 3, 2, 3 blocks (c) 3, 2, 1, 2, 3 blocks • Additional rectangular Duplo blocks to build with • A cloth to cover the ant hills with • 2 sets of 8 Duplo blocks for each child in the focus group. Every set is composed of one color • An illustrative picture of an ant hill
Guidance for the teacher *Introduction*	 Ant has been with us for a few days now. Do we know where he lives yet? (...) Ant lives in an Ant hill. Like the one in this picture [show the picture of an ant hill]. [Move the two ant hills (type (a)) into the centre of the circle] An ant hill is quite hard to build, especially when you want to make a tall one. Under this cloth are 2 ant hills. Let’s see if we can find a way to make these ant hills taller. Maybe our ideas will help Ant for the next ant hill that he is going to build. The first person who can describe how this ant hill is built and knows what the next layer of blocks should look like, can come build the next layer.

	But be careful! To make the ant hill taller, you first have to know how it is built. Think back at how we worked with the necklaces. Do you remember what we did then? (...) Can you give an example of the pattern of a necklace? How did you know what the next color in the necklace should be? (...) Can we use that easy way to find out what the next layer of blocks should look like?
Sample questions	(1) Raise your hand if you know what the next layer of blocks should look like. Do you know an easy way to find out? (...) Do you remember something from the Tool Box that can help you with this? Go ahead and build the next layer. (2) Can we maybe place one ant hill on top of the other one to make it even taller? Does that make sense? How can you see whether the layers of the ant hill are built properly now? Can you improve it? [Move the next two ant hills (type (b)) into the centre of the circle. Repeat the activity and perform it again with the type (c) ant hills] (3) Now we play the game again with these ant hills. (4) Can you give Ant some good advice so that he can have an "easy way" to see how he can make his ant hill taller?
The focus group	[Build a sample construction beforehand; two that are structured and one that is unstructured. The children in the focus group are going to rebuild the sample ant hills. Each child gets 8 uniformly colored blocks] (5) Here is one example of an ant hill. Do you think you can rebuild it easily? Why does it look easy? Can you explain how you will rebuild it? (...) Does anyone else know a faster way to see how I made this example? (...) Is there anything in the Tool Box that could help to see how this construction is built (e.g., the necklaces, de dice)? [Show the second example of a structured construction and then discuss the example that is unstructured] (6) Can you make this ant hill taller? How can you use the shape of this ant hill to see how to rebuild it in an even quicker "way"? How can you see that easily? (7) Which ant hill did you think was easiest to rebuild? Why? Which one was most difficult and why? So: how can we best build an ant hill so that someone else can easily see how the blocks are layered in the ant hill?

To simplify the activity	• Use ant hills with less blocks in a layer • Let the children use differently colored blocks so that they can identify the structure more easily by the color of the blocks
To make the activity more challenging	• Use ant hills that have more blocks in a layer • Let the children use more colors to show the structure of the construction in different ways
Background and observation criteria	Since this activity builds on the previous patterning activity, the children are expected to examine the bottom layer of the construction and to use the regularity in the layers to determine what the next layer of blocks should look like. Eventually they should be able to use the structure to abbreviate the way they determine the number of blocks in such a construction (e.g., “2, 4, 6, 8 so that makes 8 blocks in this structure”, or “3 and 2 is 5 and another 5 makes 10 blocks”). The task of trying to make the construction higher can also inspire the children to pay more attention to the structure of the blocks, rather than that the children stay too focused on counting the blocks in the structure. Only when the children have more insight into the structure of the construction, will they be able to independently add on another layer of blocks in a way that complements the structure of the rest of the construction (e.g., the regularity in the 3-2-3-2 construction is that it alternates layers of three and two blocks, so the next layer should also have three blocks). The game-like setting of this activity is expected to motivate the children to try to make use of the structure of the construction to determine how the construction is built. As the activity relates to the patterning activity, the teacher can attend the children to the repeating structure (i.e. the regularity) of the blocks in the layers of the construction. She may refer to the Tool Box for bridging the focus on structure of a construction in the present activity with the focus on structure of patterns in the previous activity. Another reason for the teacher to keep relating the activity to the Tool Box is to preventing children from giving very practical answers to practical questions (e.g., “how do you know that? Because, I just see it”). This should help children compare structures while it may make them less dependent on their ability to verbalize an explanation (e.g., “do you know how the Tool Box helped us to know how many there are in a very easy way?”). A construction with layers of five blocks, for example, may be compared to an egg carton with rows of five eggs. This insight may reflect the children’s spatial structuring ability better than their verbal descriptions of a structure.

	The challenge for the teacher is to motivate the children to focus on the shape (i.e. the rule or the structure) of the construction. If the children are able to describe the shape of the construction (e.g., “2, 2, 2, 2” or “every time 3, 2, 3, 2”), then they may be more motivated and able to continue the shape and to apply the same structure to the next layer. That is what is required in the focus group activity when the children construct their own towers and try to clarify the regularity for someone else. By the end of this activity, not many children may be able to verbalize the structure of the construction in a clear and concise way (e.g., “it has openings in it” or “it’s harder to count” or “you can’t see the blocks very well”). Nevertheless, their actions and explanations will have communicated their implicit insight into structure (e.g., determining the number of blocks in a construction by reasoning about the layers rather than unitarily counting the blocks). More importantly, the children are expected to advance in their understanding of patterning and of the structure that makes up a pattern (e.g., “every time 3, 2, 3”). This understanding of part-whole relationships is what is considered fundamental to identifying spatial structure and it is what seems to underlie the ability to use spatial structure as a way to abbreviate numerical procedures (e.g., rather than counting the blocks one-by-one, see that it is two rows of three and two on top, which makes 6 and 2 so 8 altogether).

5.5 Activity 5: Picking Flowers

Objectives	• Explore how structure can help to grasp relatively larger and unstructured quantities • Devise own effective and efficient spatial structures • Use spatial structures in a goal-directed way to simplify and abbreviate the process of determining and comparing quantities • Gain more insight into the composition of quantities • Make more spontaneous use of the shared vocabulary, such as "easy ways", "tools", "rows" or "in groups"
Setting	Introduction in the circle with an extension in the focus group
Requirements 	• One sheet of paper per child with a picture of a rectangular garden on it • Colored plastic chips (i.e. the flowers) • Items to emphasize the context (e.g., a basket to put the flowers in) • Possibly a hat or a cloth to cover children's eyes • Ant and its Tool Box
Guidelines for the teacher *Introduction*	[In the centre of the circle, place one garden and randomly arrange 10 uniformly colored chips in it] To make Ant's ant hill taller, like we did last time, he needs a lot of materials like sticks and flowers for decoration. Here you see a garden with flowers in it. How many flowers are in the garden? Do you know an "easy way" to find out so you don't have to count each of the flowers? Do you remember any other tools in the Tool Box that we could use here too? It's springtime so the flowers are growing fast. Ant is looking for flowers for his ant hill. This is [the name of a child]'s garden. Close your eyes. Ant is very secretly going to pick some flowers out of your garden. [Remove 3-5 chips from the garden] Now take a look. Can you say how many flowers have been picked out of your garden? If your answer is right, Ant will give you back your flowers and plant another one in your garden. But, if you're wrong, Ant keeps the flowers and uses them to decorate his ant hill.

Sample questions	(1) Do you know how many flowers were picked out of this garden? How did you see that so quickly? (2) How can you arrange the flowers so that next time you may see the number of missing chips in a better way? Do you know an “easy way” to quickly see how many flowers you had first, and how many are picked? How could the Tool Box help you? [Remove 3-5 chips from the garden each time. Choose chips that are crucial for the structure that the child may have arranged the chips in. Keep the removed chips if the child’s answer is incorrect, or replace the chips and add an extra one if the answer is correct] (3) Now someone else can come sit in the circle. While you close your eyes, Ant will come and pick some flowers. But first, see if you can arrange the flowers in a way that will make it easy for you to see, without counting, how many flowers Ant has taken away. Do you know an “easy way” for that? Could you use a tool from the Tool Box like we did before? [Keep encouraging the children to explore other types of structure or to take a look at another child’s structure if the child’s own structure does not seem to help determine the missing flowers] (4) It’s still quite difficult to easily see how many flowers have been picked out of the garden. Is there another way that could help you see it easily? [Make use of a successful structure as an example to show the other children that it did work. Challenge the child who made this structure further by adding more chips to the garden] (5) So you found a way to quickly see how many flowers were picked from your garden, without counting each of them. Now, how can you see how many flowers are left in your garden without counting? Do you know how you can arrange the flowers so that you can quickly see how many there are? Is there a tool in the Tool Box that can help you with this?
The focus group	[Each child has their own garden with 10 uniformly colored plastic chips in it. Choose a garden where Ant is going to pick his flowers. Ask the child to look away until the flowers have been picked] (6) How many flowers did Ant pick? Does your structure help to see how many flowers were picked? How can you arrange the flowers to that you can see it in an even “easier and quicker way” without counting each flower? (...) Does anyone else know a good way?

	[While flowers are being picked out of the gardens, the children look for ways to keep track of how many flowers are missing from their garden. Chips are either added to or removed from the garden depending on whether the child determined the correct number of missing flowers. In case the children become too distracted, the game can also be played centrally with one garden like in the classroom activity] (7) Can anyone give some advice about how we should arrange the flowers to quickly and easily see how many flowers Ant picked from a garden? (8) Now you found an "easy way" to see how many flowers have been removed from your garden. Can you use this way to see how many flowers are left in the garden? How can you arrange the rest of the flowers to see it better without counting each flower? [If there is time left, then the children can start comparing the structures across the gardens] (9) This time, Ant is going to pick flowers from each garden at the same time. Then we will see how many flowers are left in each garden each time. How can you quickly see who has the garden with most flowers in it? So which garden has the least flowers in it? Can you see that without counting? Do you know an "easy way" to see that?
To simplify the activity	• Start with a smaller number of flowers (5-8) • Play the game more often and encourage discussion
To make the activity more challenging	• Start with a larger number of flowers (12-16) • Let the children pick the flowers themselves so they can play the game with each other in twos • Discuss the structure of the remaining flowers and how the children may use this structure to determine how many flowers were left in the garden.
Background and observation criteria	Considering the experiences that the children will have had in the previous instruction activities with recognizing spatial structures and discovering the convenience of such structures, this activity is expected to challenge them to make more goal-directed use of the spatial structures. It is therefore a crucial activity in this instructional sequence for gauging what the influence of the previous activities is on children's insight into structures and their uses. The role of the teacher in this activity is to guide the children in such a way that the children experience a dilemma in the game that triggers them to search for their own (more structured) solutions.

	The children are expected to first judge by sight whether, and if so, how many flowers are missing from their garden. The way of doing this is by looking at where there may be large empty spaces (e.g., pointing and counting where the missing chips may have been laying). The game-aspect of this activity should stimulate the experience that more flowers in the garden can make it more difficult to quickly determine how many flowers are missing (e.g., the children will lose more and more flowers and become disappointed because they don't want to lose more flowers and risk losing the game). Previous versions of the activity have shown how important it is that the teacher strategically removes certain chips in the formation. The children will first be satisfied with a particular structure if that helps them see the number of empty spaces when chips are removed (e.g., in a structure of four rows of four chips, there is one missing out of every middle of a row and that is easy to see because of the resulting empty spaces). The key is to remove those chips that challenge the structure that the child appears to depend on (e.g., not the middle chip in a row, but one of the outer chips in each of the rows). To avoid losing all the flowers, the children will have to look for effective and efficient ways of finding out whether and, if so, how many flowers have been picked out of the garden (e.g., they can arrange the flowers into a line, in a circle, in several lines, in a dice configuration, or in composite structures such as a flower with petals and a stem). The children will be stimulated to not only focus on the structures that make it easy to determine how many chips are missing, but to also think of structures that help to see how many chips are remaining. Hence, the advantage of a particular spatial structure should become clear when children take the number of chips that are left into account (e.g., remembering that there were four rows of four and not four rows of three, which is the number of chips that are left after one outer in each row is removed). The teacher can guide the children towards arranging these structures and encourage them to look at each other's constructions as examples (e.g., highlighting that it seems to be more difficult to keep track of groups of twos and threes than groups of either two or three chips). The teacher can also stimulate a discussion by asking the children how they can know so quickly how many chips have been removed. This can help the children to make a connection between their "ways" of structuring the chips and the "clever and easy ways" that they will have encountered in the previous activities.

	The children can also, just like they did to determine the original number of flowers in the garden, begin to apply structure to compare the number of flowers that are left in the garden. This extends how the children will make more and more use of spatial structures as tools for not only determining the number that has been taken away but also for the number that is left. The teacher has the important task of connecting the number of flowers in the gardens to the ways that the children organized or structured the flowers beforehand (e.g., “you just saw that, when you arrange your flowers neatly, you can easily see how many are missing. Can you use that now to easily see in which garden more or less flowers are left?”). This may help the children to experience that structure is not only useful for determining a quantity but also for comparing quantities in an abbreviated way. Moreover, it makes a teacher alert about children who only count the empty spaces in an arrangement rather than actually making use of the structure of the arrangement; counting empty spaces only relates to organization and not to abbreviation of a numerical procedure. For this part of the activity, the children are expected to first count the number of flowers that remain in the garden (e.g., “there are six left in this garden and seven left in this garden, so more were removed from the garden that has six chips left”). Yet, when the teacher encourages the children to find, as fast as possible, the difference between the number of flowers in the gardens, then the children may come to make more use of the structures in which the chips are arranged (e.g., “I have a larger flower in my garden than you do, or my piles are taller, or I have more rows”). Although other children may already be able to compare quantities based on numerical differences (e.g., “I have six and you have five, but six is more than five so I have more”), it is important that the children are also able to determine the quantities in an abbreviated way (i.e. with “clever ways”). This should help them to note a difference in quantity faster than through counting or arithmetic and win the game. As the children gain more experience with counting the flowers, we conjecture that they will be able to spontaneously “arrange the flowers neatly” with the more goal-directed aim of abbreviating the way they determine and compare the number of flowers that have been removed or left in the garden (e.g., the children arrange the flowers into certain structures so that they can quickly, without counting, see how many flowers have been taken away).

Curriculum Vitae

Fenna van Nes was born on December 31st, 1982 in Jette, Belgium. After attending international schools in Columbia, Sweden, Ethiopia and Zimbabwe, she completed her secondary education with an International Baccalaureate diploma in 2000 at the American School of the Hague in the Netherlands. Four years later Fenna was awarded a Master's Degree in Cognitive Psychology (cum laude) at Leiden University. In the following academic year she took several mathematics university courses and spent six months working as an au pair in Canada. Subsequently, in September 2005, Fenna became a Ph.D. student (*Assistent in Opleiding*) at the Freudenthal Institute for Science and Mathematics Education. During this four year period, Fenna was also involved in several other research projects at the institute and she supervised Master's students. She has conducted numerous workshops for education practitioners, and she has presented her work to researchers at national and international conferences.

Index

L

M

N

O

P