Beneath the Tip of the Iceberg:

Using Representations to Support Student Understanding

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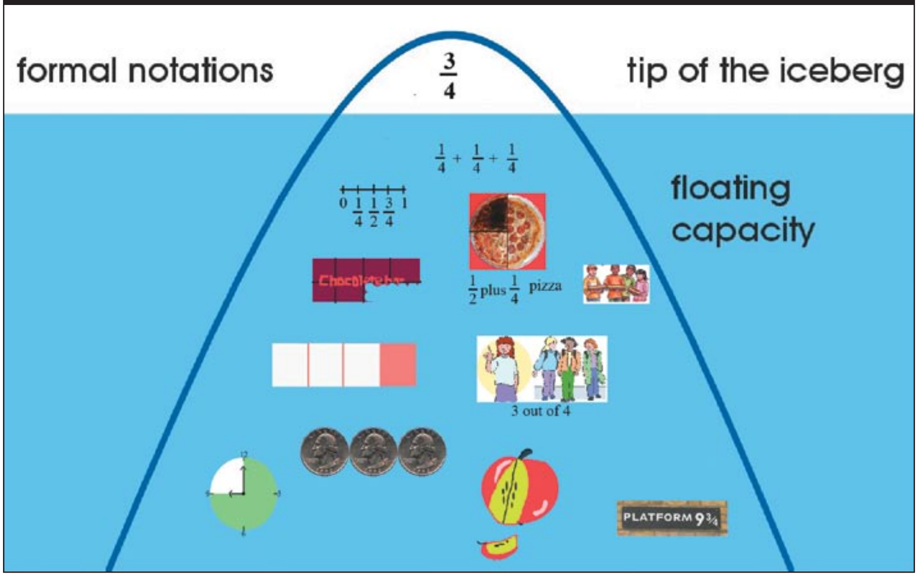
A common challenge for middle-grades mathematics teachers is to find ways to promote student understanding of mathematics. When an algorithm for adding or subtracting fractions is explained as clearly as possible and students have opportunities to practice it, the reality is that many students will continue to confuse the procedures and forget how they work. The primary intervention used to address student confusion is to reteach common procedures and give additional practice in the hope that the students will understand over time.

Over the past decade, the reform of mathematics teaching and learning and the implementation of NSF-funded, Standards-based instructional materials have broadened the options for teaching and learning mathematics. Ideally, these new instructional materials should increase opportunities for all students to learn mathematics. However, both regular and special education teachers have a difficult time making instructional decisions to address the needs of underachieving students. In particular, to address the instructional needs of students who are already multiple grade levels behind, teachers need to make informed decisions regarding the selection of experiences that can close the achievement gap. However, without a framework for selecting essential representations and experiences, teachers are dependent on the design and sequencing of activities in the instructional materials they use in their classrooms.

How can teachers use representations to increase student access to mathematics? In this article, we will describe how an "iceberg model," developed by the Freudenthal Institute for teachers in the Netherlands, was used by middle-grades teachers in the United States. This model supports the selection of more accessible instructional interventions and student-centered instructional sequences. (For more information about the Dutch approach to mathematics education, see Case 2006.)

Fig. 1 The iceberg model



AN ICEBERG FULL OF REPRESENTATIONS

Researchers at the Freudenthal Institute for Science and Mathematics Education at the University of Utrecht developed the iceberg model to support teacher thinking about learning processes and strategies used by students (Boswinkel and Moerlands 2001). This model has proved to be a powerful metaphor for illustrating how students need to experience a broad range of mathematical models to make sense of formal mathematical representations (i.e., the tip of the iceberg).

A rich professional development experience can be based on the construction of representational icebergs. Groups of teachers can work together with the iceberg model, which offers a context for exploring and discussing essential representations and the sequencing of activities in their instructional materials. This model is a metaphor, distinguishing the role of informal, preformal, and formal representations used by students.

See **figure 1** for an illustration of the iceberg model. It consists of the tip of the iceberg and a much larger area underneath, called the floating capacity. The tip of the iceberg represents the targeted formal procedure or symbolic representation. The bulk of the iceberg that sits under the water is represented by a combination of informal, including context-bound, representations (e.g., time, coins, part of an apple, and so on), which transition to preformal strategies and models (e.g., fraction strips, a number line, and so on). This metaphor can be used across many different problems. In the case of operations with fractions, for example, even before students can meaningfully discuss various strategies, they need to have an understanding of what a fraction represents.

In general, the progressive formalization suggested by the different levels within the iceberg's floating capacity implies that more formal representations build on less formal ones. However, this does not mean that as soon as a student has reached a formal understanding that he or she will never return to the use of preformal representations. Rather, a student should be able to revisit preformal representations, especially when new and unfamiliar contexts are encountered. In fact, it is reasonable to expect that some special education students may not understand the formal representation at all, and yet they may be able to solve problems using informal or preformal approaches.

INFORMAL AND PREFORMAL REPRESENTATIONS

Mathematics teachers are quite familiar with formal representations and strategies for manipulating them. The common algorithms, which demonstrate efficiency and fluency, are the formal strategies for numeric computation. Unfortunately, such formal strategies are often presented in ways that require students to make connections to other strategies or representations, which is difficult for students who struggle with different models.

One example of a formal representation would be a proportion with one unknown (e.g., $x/6 = 10/18$). This type of proportion suggests a formal and somewhat efficient solution strategy: the cross-multiply-and-divide method. Imagine students who have not had experience using preformal representations, such as a ratio table, percent bar, or double number line. In the cross-multiply-and-divide method, there are few opportunities for these students to relate the proportion to other meaningful strategies that promote number sense and proportional reasoning (e.g., 6 is one-third of 18, so x should be one-third of 10).

For students, diagrams and explanations are informal representations, which are often grounded in student experiences with real or imagined contexts. An informal explanation for adding "three-quarters" and "one-half" would be to reason, "Two quarters and a half dollar is one dollar. This leaves one quarter. So it's a dollar and a quarter." An informal explanation using patterns, for example, might relate odd and even numbers to a V-pattern. When students are asked, "Will a V-pattern ever have an even number?" a student might reason informally using the V-formation used by geese. "On each side of the V, the geese will have a partner, but the leader will always be on its own. So the number will always be odd."

Preformal representations build on students' informal representations, or reasoning, and offer greater mathematical structure. Some examples are a recursive formula to describe numerical patterns; a double number line to solve a problem involving scale; and an area model for multiplying whole numbers, mixed fractions, or binomials. Most preformal representations are rarely developed by students on their own to solve a problem. Instead, students are guided by teachers or instructional materials to use preformal representations and strategies that can be applied across many situations and contexts. Preformal representations offer greater opportunities to empower students' sense-making, but they often have limitations in the scope of problems that can be solved using the chosen representation. For example, the area model helps students understand how multiplication of mixed numbers relates to the distributive property, but it is not very practical for computing

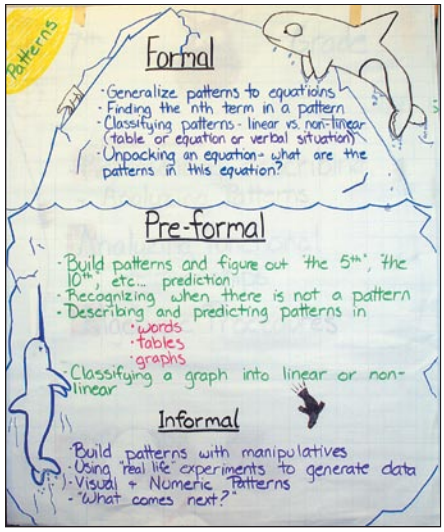
$$5\frac{7}{12} \times 3\frac{1}{7}.$$

Some students arrive at the formal level faster than others. However, students should not be forced to use formal strategies if they have not had experiences with essential informal and preformal representations. The time invested in sense-making experiences at the preformal level will substantially reduce the time needed to reteach and practice at the formal level.

CONSTRUCTING THE ICEBERG

Although professional development experiences in which teachers identify informal, preformal, and formal strategies and notations have been used extensively in the Netherlands, the mathematical iceberg activity was recently introduced to middle-grades mathematics teachers in the United States. The purpose of this activity is to encourage teachers to reflect on the representations found in their curricula and influence the selection of potential instructional interventions. The activity could be used for almost any mathematical topic that involves multiple representations, models, or strategies.

First, teachers begin with a diagram of an empty iceberg containing a water line. Then, the formal representation being studied is written in the tip of the iceberg, above the water line (for example, 3/4 to represent the use of fraction notation; $x/12 = 9/20$ to represent solving proportions; or $y = mx + b$ to represent the understanding of linear functions). Teachers are then asked to recall representations that their students have used and develop a collection of related informal and preformal representations that contribute to student understanding of the formal representation. This is followed by a scavenger hunt through district-adopted textbooks and other instructional materials to identify other informal and preformal representations.



The heart of the iceberg activity involves teachers working together to identify related representations and strategies and discussing how these representations support student understanding. Also, teachers discuss building on less formal understanding and deciding whether the representations are best categorized as informal, preformal, or formal. Given the scope of content that some topics involve, teachers will also need to consider whether a representation is distinctive enough to merit its own place in the iceberg. Depending on the mathematical representation at the tip of the iceberg, this activity might take anywhere from 45 to 120 minutes to complete.

Building on an iceberg model creates a context for a lively debate about representations, their connections to students' prior knowledge, and how collectively the teachers support the students' understanding of mathematics. It is also important to recognize questions about the usefulness of particular representations or strategies, and their potential drawbacks, that might emerge during group discussions. Time should be set aside to deliberate these real (or perceived) limitations.

The goal of constructing mathematical icebergs is to summarize the teachers' collective knowledge of representations and how those representations are interrelated. Formal representations are often considered the only important (end) goal. Unfortunately, they receive inordinate attention in classroom assessments even though less formal representations are invaluable in assessing a student's prior knowledge and revealing potential starting points for instruction and intervention.

FROM ICEBERGS TO REPRESENTATIONAL PATHWAYS

An indispensable follow-up activity to constructing icebergs involves reviewing instructional materials for representational pathways. For example, as part of a summer workshop focused on the development of rational number concepts and proportional reasoning, middle-grades teachers were given cards that included activities and representations from the two most commonly used mathematics curricula in the district. "Formal Addition of Fractions" was the concept named at the tip of the iceberg. Groups of teachers first discussed how to organize the cards vertically within an iceberg to represent the progressive formalization of the adding fractions. Then the groups deliberated over the appropriate sequencing of activities, offering suggestions for extra cards when they thought that something important was missing or deleting superfluous cards (see **fig. 2**).

Fig. 2 Teachers working on the design of representational pathways

Teachers found that the representational pathways in their instructional materials were either exclusively formal representations (found in one textbook series) or a mix of representations below the water line (as seen in another textbook series). Teachers also found that some groups constructed representational pathways that were dramatically different from their own, which led to a substantive discussion of the appropriate ordering of activities and options for alternative pathways. Another observation was that important representations seemed to be missing from the available cards, and so teachers were invited to describe what these missing activities should be. Crucial learning experiences in the pathway, which should not be skipped, were marked by teachers with a red box. A next step would be to add activities to the representational pathways around crucial learning experiences that seemed to be missing in the instructional materials.

When developing instructional plans for students who need individualized interventions, the representational pathways help teachers identify the appropriate starting points based on a student's prior knowledge. Although the Dutch project focused on supporting teachers of special education students, the construction and application of iceberg models and representational pathways is useful to teachers of all students.

With respect to opportunities for teacher learning, this article is only the "tip of the iceberg" for the types of professional development activities that can be done using multiple representations. Such development can support collaborative instructional planning, curriculum mapping, and identification of appropriate interventions for struggling students.

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