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Connecting engagement and focus in pedagogic task design

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Teachers of mathematics face a particular tension, which the authors call the *planning paradox*. If teachers plan from objectives, the tasks they set are likely to be unrewarding for the pupils and mathematically impoverished. Planning from tasks may increase pupils' engagement but their activity is likely to be unfocused and learning difficult to assess. By seeking inspiration from research in the areas of curriculum design, the nature of authenticity in the classroom and the use of tools, and by looking retrospectively at the design of computer-based tasks that have underpinned their research for many years, the authors recognise a theme of purposeful activity, leading to a planned appreciation of utilities for certain mathematical concepts. The authors propose *utility* as a third dimension of understanding, which can be linked to purpose in the effective design of tasks. The article concludes with a set of heuristics to guide such planning.

Preamble

There is a strong sense in which the ideas presented in this article are inspired by the notion of the *play paradox* (Noss & Hoyles, 1992), which recognises an inherent tension in the teaching–learning process. Play can facilitate learning and so there is a desire to incorporate play-like freedom into more formal school-based learning, even for older pupils. However, such a strategy transfers control over what is learned away from the teacher to the pupils themselves. This is unsatisfactory if the teacher has an agenda in which certain specific knowledge should be assimilated.

The notion of the play paradox stemmed from analysis of pupils'[1] activity[2] when using computer-based microworlds. A feature of microworld design is to insert into the microworld mathematical concepts that the designer expects pupils to concretise through use (Wilensky, 1991). Papert (1982) has described this process as planting nuggets of mathematical knowledge into the microworld. The designer is intending that the pupil will stumble across these concepts whilst engaged in play-like,

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informal activity. However, Noss and Hoyles (1992) point out that 'involvement in the microworld can allow the learner to ignore just those mathematical nuggets that are placed there so carefully' (p. 451).

Noss and Hoyles (1992) recognise that their own interventions during the research had a major impact on how the activity developed and that these interventions were necessary to trigger reflection by the pupils. Microworlds are generally designed to be used by pupils in ways which are relatively unstructured, through the development of their own projects. In this article we consider a broader range of contexts for mathematical teaching and learning, but see similar tensions arising between the teacher's intentions and the experiences of pupils. In particular, we consider the design of pedagogic tasks. How might the teacher plan tasks in such a way that pupils are likely to engage with intended mathematical knowledge in mathematically meaningful ways?

We begin by problematising the procedure of setting learning objectives in mathematics lessons, and later we look retrospectively at the design of tasks that have played a significant part in our own research in order to propose two constructs that can guide the connecting of objectives to the design of tasks.

The planning paradox

With our emphasis on task design, we reinterpret the play paradox as what we call the *planning paradox*: if teachers plan from tightly focused learning objectives, the tasks they set are likely to be unrewarding for the pupils, and mathematically impoverished. If teaching is planned around engaging tasks the pupils' activity may be far richer, but it is likely to be less focused and learning may be difficult to assess.

To elaborate a little, we offer two contrasting examples. The teacher plans a lesson which focuses on adding two-digit numbers together. The lesson proceeds with some explanation of procedures and then a range of tasks which enables pupils to practise this calculation. The tasks have been determined by the objective in a narrow and constrained way, and even an imaginative teacher will find it difficult to make the tasks interesting in more than a superficial way. Such a teacher has fallen foul of the first part of the planning paradox.

Now consider a teacher who, as the starting point of a mathematics lesson, asks the pupils to design their ideal bedroom. The pupils may become highly engaged in meaningful activity, which could lead them into several areas of mathematics, but the teacher may find it difficult to take advantage of such opportunities, or to monitor any mathematical thinking.

We identify three levels across which a resolution of the planning paradox might reside. At one level, there is the curriculum. The curriculum is likely to set out essential content with which pupils are expected to engage. Rethinking how we approach that content might offer ways of simultaneously addressing content focus and motivation. More broadly, the ways in which a teacher contextualises mathematical activity could support or obstruct a resolution of the planning paradox. In particular we are interested in how contextualisation may relate school and 'real-world' experiences, and how this may stimulate a sense of purpose. Finally we are interested in the sorts of tools that teachers offer pupils whilst they work on the task. The affordances of tools shape the way that pupils are able to pursue the teacher's plans and understanding this in turn changes the way that teachers construct the plans themselves.

We now consider each of these levels in turn, and from this discussion our own proposal for resolving the planning paradox emerges. At each level, we identify and discuss specific research that has influenced our thinking.

Curriculum: the contribution of 'Realistic Mathematics Education' (RME)

We first turn to a Dutch approach, Realistic Mathematics Education (RME), which has for many years explored curriculum design with similar objectives to our own. Freudenthal (1979) recognised mathematics as 'a natural and social activity which develops according to the growth needs of an expanding world' (p. 324). He criticised what he called 'new math', describing it as a form of 'mathematical abstraction, detached from meaning or context, interpreted as subject matter and concretised in a preposterous way' (p. 324). This begs the question of how suitable contexts for mathematical learning can be created.

Evolving out of Freudenthal's philosophy of mathematics, RME sets out a curriculum based on principles (Van den Heuvel-Panhuizen, 2000) of which we shall explicitly discuss two that have particular relevance to this article. The *reality principle* relates to Freudenthal's (1968) premise that mathematics must be learnt 'so as to be useful'. This learning occurs throughout the process of 'progressive mathematization' (Gravemeijer, 1994). In addition to this, the rich contexts that afford mathematisation are a prerequisite of the tasks in the RME curriculum. The *guidance principle* is concerned with giving students a 'guided' opportunity to 'reinvent' mathematics through teacher-led tasks that meet the intended learning trajectories. By teachers providing tasks that allow for reinvention, pupils are able to 'construct mathematical insights and tools by themselves' (Van den Heuvel-Panhuizen, 2000).

The use of 'realistic' within the name of the approach comes from the Dutch 'to imagine': 'zich REALISEren' (Van den Heuvel-Panhuizen, 2000). The emphasis within the RME approach is to offer students a context problem: a 'problem situation that is experientially real to [them]' (Gravemeijer & Doorman, 1999). This may come from a real-world scenario, a fantasy or from a pure mathematical problem, as long as they are 'real' to the pupils. Gravemeijer and Doorman explain the purpose of context problems being more than just usefulness and motivation. They believe that context problems should be used from the very start of a course as an anchoring point for reinventing mathematics. There are two aspects to this. The first is to help pupils to elaborate on their informal understanding, thus leading to a formal understanding. This is what Treffers (1987) identifies as 'horizontal mathematization'. It is also to preserve the connections between concepts and the contexts which concepts describe, or 'vertical mathematization' (Treffers, 1987).

We find this research most interesting in the way that it builds curricula out of specified design principles. However, our aim is somewhat more modest though possibly more direct. We aspire to provide teachers with tools that they can use to construct their own approach to a predefined curriculum. Pragmatically this is the situation in which teachers often find themselves. They may have little control over the curriculum and, without some guiding constructs, we believe that there is little likelihood of teachers, even those operating with an RME designed curriculum, inventing resolutions of the planning paradox. RME may provide interesting guidelines that might generate a coherent curriculum through a structured sequence of tasks. For us though, there is a further concern that tasks taken in isolation should nevertheless offer a similar coherent sense of completeness. We wish to avoid coherence being only in the mind of the curriculum designer with the result that the pupils experience a series of disconnected tasks. We aim instead to provide a means by which teachers can offer meaningful tasks even when operating within a curriculum that might for them lack the coherence of one based on RME principles. Hence we turn our attention to the second level, that of the setting, where we discuss attempts to introduce authenticity into the classroom.

Contextualisation: authenticity and situated cognition

In recent years, mathematics educators have taken a great interest in situated cognition research (for example, Lave, 1988; Lave & Wenger, 1991; Nunes *et al*., 1993). Such studies have drawn attention to the importance of a sense of purpose which is characteristic of out-of-school learning contexts (*street* mathematics). For example, Lave and Wenger (1991) claim that, 'learners, as peripheral participants, can develop a view of what the whole enterprise is about'. We are struck by the similarities between Lave and Wenger's descriptions of *legitimate peripheral participation*, and the ways in which young pupils' learning is described in research into the acquisition of literacy. Teale and Sulzby (1988) include the following in a summary of research about literacy development in early childhood: 'Literacy develops in real-life settings for real-life activities in order to "get things done". Therefore the functions of literacy are as integral a part of learning about writing and reading as they are to other forms of literacy' (p. 259).

When pupils learn to read and to write their native language, they can immediately use these skills for (some of) the same purposes as adults: they can read for entertainment and for information, they can write messages, lists, birthday cards, email. The same is true for learning street mathematics, but not for learning school mathematics. Indeed, it is clear that many pupils do not always understand the purposes of even those aspects of mathematics which might be regarded as most practical, such as measurement, when they learn them in classroom contexts (Ainley, 1991).

A possible implication is that we should attempt to offer the authenticity of street mathematics to pupils in classrooms. Whilst we would agree with Schliemann (1995) that 'for meaningful mathematical learning to take place in the classroom, reflection upon mathematical relations must be embedded in meaningful socially relevant situations', we see the provision of authentic tasks as inherently problematic.

Teachers often provide pupils with tasks that may superficially offer authenticity by resembling out-of-school activities. For example, a teacher of young pupils may set up a play shop in the corner of the classroom to encourage some mathematical learning. However, the structuring resources provided by this situation will be very different from those offered when the child is really shopping with a parent. For example, in the play-shop task, emphasis may be placed on number, and so the prices of items on sale are simplified to an extent that even young pupils will recognise as unrealistic. Getting the correct change will not be the same level of concern for customers in the play shop as it is for customers in a real shop. Even with an element of role-play, the social interactions of the play shop will not provide the structure and constraints experienced in a real shopping trip (Brenner, 1998). In Walkerdine's (1988) words: 'The practical context is a foil for the teaching of certain mathematical relations, so that everything about the task is different from shopping. In school mathematics the goal of the task is to compute the answer rather than to make a purchase' (p. 146).

Attempts to provide 'authenticity' through the contextualisation of school mathematical tasks in out-of-school settings will not resolve the planning paradox. Such settings will not ensure engagement with the task, or more importantly, engagement with mathematical ideas. Indeed, as Cooper and Dunne's (2000) research has indicated, in order to engage appropriately with the mathematical focus of a contextualised task, pupils have to understand complex but implicit rules about the extent to which they should attend to features of the real-world setting. Adler (2000) argues that school mathematics is necessarily a hybridisation of contextualised (street) mathematics, and academic mathematics, and claims that this presents teachers with: 'The important challenge of whether and how to be explicit about mathematical purposes … and thus about where meanings need to be located to facilitate sense-making, access, and success in school mathematics' (p. 208). Schliemann (1995) concludes a discussion of the problems of bringing everyday mathematics into the classroom with the statement that 'we need school situations that are as challenging and relevant for school children as getting the correct amount of change is for the street seller and his customers'.

We argue that the use of 'authentic' settings in an attempt to capture the levels of engagement characteristic of street mathematics within the classroom context will not meet this challenge, and will fail to resolve the planning paradox since the more they are shaped to have clear mathematical focus, the further removed they become from socially meaningful contexts. In the second part of this article we will argue that an alternative analysis of street mathematics, which focuses on the purposeful nature of mathematical activity, and the usefulness of mathematical ideas, offers a more successful approach to combining engagement and focus in the design of classroom tasks.

Tools: the influence of constructionism

Much situated cognition research has studied the activities of master and apprentice, and some mathematics educators have attempted to use this model to examine the activities of teachers and pupils (e.g. Masingila, 1993). In out-of-school situations, the master and the apprentice have a common goal, which in the short term is typically to make some product (so the tailor and his apprentice may aim to make a waistcoat), and in the longer term is to make a profit. We question whether teachers and pupils can have such common goals. The teachers' agenda must be to focus on their pupils' learning (and the pupils know this), whereas the pupils' agenda will be to complete the task, hopefully to the satisfaction of the teacher. Although the production of (real or virtual) artefacts appears to be a fruitful context of meaningful mathematical activity, this leads us to question what kinds of products teachers and pupils might make together. We now consider approaches that do place product creation at the forefront of pupils' activity, though in ways which differ significantly from the tailoring workshop. Our focus over many years has been on the use of technology in the learning of mathematics, and so it is natural for us to look towards the literature in that field for inspiration.

The constructionist movement (Harel & Papert, 1991a) has proposed that tasks in which pupils make products, generally through programming computers, are particularly conducive to learning. Thus, in one sense the constructionists replace the 'waistcoat' with a product that is programmed by the child into the computer. In our experience (for example, as reported in Pratt & Ainley, 1997, and Ainley *et al.*, 2000), such programming tasks can generate activity that has some of the characteristics of everyday activity studied in the research of situated cognitionists. For example, teachers often become engaged in working with the child to make the virtual product, reminiscent of the tailor and apprentice collaborating to make the waistcoat. The task, rather than the externally set objectives, takes on the role of being the arbiter of what counts as progress. Furthermore, any mathematical learning that takes place is contextualised within the activity of making the product, which provides meaning for the mathematics, but perhaps limits its apparent range of applicability. Harel and Papert (1991b) also recognise the connection between constructionism and situated cognition, and at the same time signal some differences in emphasis:

> We see several trends in contemporary educational discussion such as 'situated learning' and 'apprenticeship learning' … as being convergent with our approach, but different in other respects … our emphasis [is] on developing new kinds of activities in which children can exercise their doing/learning/thinking … [and] on project activity which is self-directed by the student. (p. 42)

We believe that the constructionist approach recognises that the classroom is not the marketplace, and does not attempt to place emphasis on authenticity. However, by placing emphasis on the creation of products, it positions consideration of meaningfulness and motivation high on the agenda for the design of tasks that are likely to promote mathematical learning. The constructionist literature promotes the use of microworlds, targeted domains within which pupils are able to explore safely and freely, designed in such a way that they are likely to encounter powerful mathematical ideas. Within the literature on microworld design, we have found a second important idea that relates to our thinking about task design.

Microworld settings provide the possibility of pupils working with concepts that they do not yet understand. Understanding emerges through activity. Noss and

Hoyles and Noss (1987) have called this *using before knowing*. Mathematics teaching tends to start from definitions and explanations of the mathematical ideas. Microworlds offer the possibility of providing mathematical knowledge as ready-built on-screen instantiations. Thus the child using the RIGHT and LEFT commands in Logo *uses* angle to *learn* about angle. Papert (1996) refers to this idea as the *power principle* and argues that it re-inverts an inversion imposed by the conventional pedagogy in mathematics. By learning through use, pupils are empowered to learn mathematics in much the same way as they learn naturally in other contexts; for example, the ways in which they learn to read and write. This approach stands in contrast to a conventional approach in which the pupil rarely experiences using mathematics in meaningful ways.

Constructionists then advocate learning through making and in particular through building with virtual artefacts. From the perspective of task design and our desire to resolve the planning paradox, however, we are left with a number of unanswered questions. Microworld designers pay very careful attention to the tools that are offered to pupils since they can direct attention towards or away from the targeted mathematical domain. Those designing pedagogic tasks are faced with a similar issue. We aim to provide a vocabulary that will enable articulation and conceptualisation of this concern. Teachers have a similar problem when planning tasks and yet we do not have a sufficient vocabulary to talk about and conceptualise their problem. The power principle asserts that pupils *can* learn about mathematics through use but it provides no advice as to *how* teachers might facilitate this process through task design. What is it about tasks based on making products that encourages mathematical learning? Surely it is not the case that any product will do?

Purpose and utility

We recognise that RME principles, constructionist approaches and out-of-school settings can provide challenging, relevant and powerful contexts for learning mathematics. However, our analysis of where the power of these learning environments lies suggests that it is not in the explicit features (an end product, real-world activity) but rather in two less visible characteristics which we have called *purpose* and *utility*. We now describe these constructs in detail, and discuss how they may be used to inform the design of tasks in ways which offer a resolution to the planning paradox.

Purpose, as we use the term, refers to the perceptions of the pupil rather than to any uses of mathematics outside the classroom context. We define a purposeful task as one that has a meaningful outcome for the pupil, in terms of an actual or virtual product, or the solution of an engaging problem. There is considerable evidence of the problematic nature of pedagogic materials which contextualise mathematics in supposedly real-world settings, but fail to provide *purpose* to which the learner can relate, either in terms of the overall task, or the ways in which mathematical ideas are used within it (see, for example, Cooper & Dunne, 2000; Ainley, 2000).

The *purpose* of a task, as perceived by pupils, may be quite distinct from any objectives identified by the teacher. In a classroom situation, this may be true in a trivial sense: pupils may construct the purpose of any task in ways other than those intended by the teacher (Ainley, 1991). However, we are saying something more than this: within our framework for task design, *purpose* is a distinct element that needs to be considered separately from, but in parallel with, learning objectives.

Alongside purpose, we identify a second feature of the learning which takes place in both constructionist approaches and out-of-school settings, which we have called the *utility* of mathematical ideas. By this we mean that the learning of mathematics encompasses not just the ability to carry out procedures, but the construction of meaning for the ways in which those mathematical ideas are useful. At first glance, the need for mathematical learning to include this feature may seem obvious, but even a brief look at the typical content of school mathematics makes it clear that opportunities to appreciate *utility* are not common, even within tasks which relate to real-world settings. For example, what understanding of the *utility* of multiplication and division might learners be expected to develop from questions such as these (all taken from the English National Numeracy Strategy Framework [Department for Education and Skills, 1999])?

> A beetle has 6 legs. How many legs have 9 beetles?
>
> 196 children and 15 adults went on a school trip. Buses seat 57 people. How many buses were needed?
>
> Four people paid £72 for football tickets. What was the cost of each ticket? How much change did they get from £100?

Within our framework, *purpose* and *utility* are closely connected. Appreciation of the utility of mathematical ideas can best be developed within purposeful tasks, and consideration of the need to plan for such opportunities is an important factor in the resolution of the planning paradox. A focus on purpose in isolation may produce tasks which are rich and motivating, but fall foul of the second part of the planning paradox, by lacking mathematical focus. Within pedagogic tasks that are designed to have purpose for learners, we have found that it is possible to plan for opportunities for learners to appreciate the *utility* of mathematical concepts and techniques. Whilst engaged in a purposeful task, learners may learn to use a particular mathematical idea in ways that allow them to understand how and why that idea is useful, by applying it in that purposeful context. This parallels closely the way in which mathematical ideas are learnt in out-of-school settings.

Illustrative tasks designed with purpose and utility

We offer two examples to clarify the related constructs of purpose and utility.

The spinner task

A task which we have used on a number of occasions is that of designing a paper spinner (Ainley *et al.*, 2000). In this task the purpose for the pupils is clear: to make a spinner that will stay in the air for as long as possible. In investigating aspects of the design, for example, by changing the length of the wings, pupils record results of test flights on a spreadsheet (the wing length and time of flight). Their activity offers opportunities to use a number of mathematical ideas, including measurement of length and time, decimal notation, graphing and the use of measures of average. From these we now describe two examples of *utility*.

Initially it is difficult for pupils to see patterns in the numerical data on the spreadsheet, partly because the data is not usually collected in a systematic way, and partly because of experimental inaccuracy. Using a scatter graph to display the results at intervals during the experiment makes it easier to see emerging patterns in the time of flight as the wing length varies. Information from the scatter graph is used to make conjectures about the effects of changing the wing length, and which spinners will prove most efficient, and also to identify further areas for experimental investigation. Using a scatter graph in this purposeful way offers opportunities to learn about the conventions of this particular graph, but also to understand that graphing is an analytical tool, which can inform the process of doing an experiment: that is, the pupils are given clear opportunities to construct a *utility* of graphing.

Discussion of the experimental inaccuracies in the activity leads to the introduction of the idea of taking the mean value of several experiments with each wing length to produce a 'better' graph. This can be done quickly and easily using the 'Average' function of the spreadsheet. Pupils are able to use their everyday knowledge of the meaning of 'average' to understand enough about this process to appreciate a *utility* of average (which does indeed produce a clearer graph), even though they do not know the detail of how the mean was calculated. Thus they learn about the mathematical idea of average through using averages in a purposeful task.

Mending gadgets

By way of further illustration of purpose and utility, consider a microworld designed for pupils to construct probabilistic meanings (Pratt, 2000). This microworld consisted of various gadgets, small computational simulations of everyday random generators such as coins, spinners and dice. The gadgets were programmed in such a way that some contained a bias. For example, the dice gadget was programmed to generate more sixes than a fair dice would. The pupils were challenged to play with the gadgets and decide which were working correctly. The notion of 'working correctly' was intentionally vague though the researcher expected that the pupils might regard a biased gadget as working incorrectly. The gadgets could be opened up to reveal the *workings box*. For example, the workings box for the dice gadget read: choose-from [1 2 3 4 5 6 6 6]. The workings box could be edited by the child and so provided a means for the child to control the operation of the gadget. At the same time, the workings box represented the probability distribution. Other tools included facilities to generate many results quickly and to graph those results as a pie chart or as a pictogram.

When the pupils found a gadget that they felt was not working properly, they were further challenged to use the tools in order to 'mend' the gadget. This task was purposeful for the pupils as they found the aim of mending familiar, concrete and challenging. At the same time, the tools with which they had to operate required them to engage with a representation of distribution and to contrast the consequences of using large numbers of trials versus using small numbers of trials.

The pupils began to articulate how the appearance of the graphs was controlled by the numbers of trials, and by the workings box. In this sense, they constructed utilities (albeit highly situated ones) for crucial probabilistic notions. Mathematicians and statisticians recognise the central importance of the *law of large numbers*. This law specifies how, if a random experiment is repeated many times, the actual outcomes, as a proportion of the number of trials, gradually settle down to the 'theoretical' probabilities of those outcomes. These probabilities make up what is referred to as the distribution. The task of mending the gadgets seemed not only to provide motivation but also targeted that purposeful activity on two centrally important concepts in probability.

A third dimension of learning

We wish to emphasise here that the opportunity to understand the utility of these ideas arises because of the purposeful nature of the task set, and of the learners' activity in response to these tasks. Without the underlying purpose of producing an efficient spinner or mending gadgets, concepts such as graphing, average and the law of large numbers might only have been introduced in a disconnected, isolated fashion. The usefulness of these concepts might have been described through imagined applications, but may not have been experienced in ways that allowed learners to construct rich meanings for the mathematical ideas.

It is widely recognised that constructing meaning for a mathematical idea involves many related elements. The distinction is often made between those elements relating to procedures or techniques, and those concerned with conceptual or relational understanding. At this grain size, we still find Skemp's (1976) seminal ideas on instrumental and relational understanding powerful. We propose here a third cluster of elements: those relating to the utility of an idea, why that idea is useful, how it can be used and what it can be used for. We argue that a rich understanding of a mathematical idea involves all three of these elements.

Let us offer a quick illustration of the relationship between instrumental, relational and utility-focused understanding. Consider a pupil who is learning about the subtraction of numbers. The pupil may hold an instrumental understanding of subtraction based on the routine execution of an algorithm, such as decomposition. That understanding may have been influenced by the teacher's initial emphasis on many examples and paper and pencil calculations. On the other hand, the pupil may appreciate the role of place value in subtraction algorithms. Such a relational understanding would provide a means of appreciating errors in executing an algorithm and even inventing alternative algorithms, and may have been influenced by the teacher's explanation and the use of concrete materials. Neither of these types of understanding, though, offers the pupil an appreciation of why he should be at all interested in subtraction. In contrast, a pupil may recognise the power of subtraction to calculate efficiently the remaining target in a darts-based game. An appreciation of such a utility is, we believe, an important and often ignored aspect of mathematical understanding. Skemp might have argued that it is in fact part of relational understanding, but, even if that is the case, it is sufficiently significant, in our view, to merit attention that it has not yet received.

We conjecture that understanding mathematical ideas without an understanding of their utility leads to significantly impoverished learning. Unlike street mathematics, ideas in school mathematics are frequently learnt in contexts where they are divorced from aspects of utility. Within the classroom, opportunities to understand utility can only be provided through purposeful tasks.

The examples we have given of tasks designed using the constructs of purpose and utility exemplify the role of technology in supporting a shift in pedagogic emphasis, which we see as lying at the heart of opportunity for powerful mathematical learning offered by this approach. Mathematical ideas (such as average, graphing, distribution or geometrical construction) are rich and complex, composed of different elements, which here we categorise very roughly as *procedures* (techniques and algorithms, specific rules of formulae), *relationships* (links within mathematics, internal structure and consistency), and *utilities* (why, how and when the idea may be useful). As a pupil constructs meaning for a new mathematical idea, mental connections will be made with existing knowledge, but the pedagogic emphasis placed on the different elements will affect the ways in which those links are made.

It is generally acknowledged that pedagogic approaches that focus mainly, or exclusively, on procedures will result in impoverished learning. However, even approaches that emphasise relationships tend to give little attention to utilities. The pedagogic tradition, embodied in textbooks around the world, is to begin with procedures and relationships, and to address utilities as the final stage in the pedagogic sequence (if at all). We suggest that this results in mathematical knowledge becoming isolated as weak connections are made to the pupil's existing knowledge of the contexts in which it may be usefully applied.

In learning mathematics in out-of-school contexts, we believe that immediate connections between the learner's existing knowledge and the utilities of the new idea are established in ways which enrich mathematical learning. In school mathematics, the initial links are generally made to procedures and relationships. Pedagogic design based on the framework of purpose and utility, with the support of technology, inverts the pedagogic tradition of school mathematics by placing the emphasis primarily on the utilities of a new mathematical idea. Thus the learner is able to construct meanings that are shaped by strong connections to the application of that idea: in Lave and Wenger's terms, to develop a view of what the whole enterprise is about.

This inversion is made possible largely (though not exclusively) by the power of technology to offer opportunities for *using* a mathematical idea before you learn about its procedures and relationships. Technology affords the possibility of pursuing purposeful tasks by working with mathematical tools, instantiated on the screen, whilst simultaneously coming to appreciate the utility of those tools, in ways which lead to powerful mathematical learning. Ongoing research is developing and refining this framework for the design of pedagogic tasks in various areas of the mathematics curriculum.

Resolving the planning paradox

The two constructs of purpose and utility offer a framework for task design that may resolve the planning paradox. Designing tasks that are purposeful for learners ensures that the activity will be rich and motivating. Such purposeful tasks provide opportunities to learn about the utility of particular ideas, which will give the focus that may otherwise be absent.

However, the design of tasks that offer both purpose and utility is challenging. It requires the teacher to imagine the trajectory of a pupil's activity, taking both a mathematical and a learner-centred perspective. In order to tease out aspects of the design of such tasks, we will discuss our own struggle to create a task with purpose and utility in a particular environment (for an extended discussion of this task, see Pratt & Ainley, 1997).

As part of a long-term project, a group of pupils in our research school had been given access to dynamic geometry software (the original version of Cabri Geometry). This software provides the basic components of geometry, such as lines, points, circles, and tools for construction, but allows the user to create dynamic images which can be dragged around on the screen. The pupils had explored the software with little intervention from their teacher, but had effectively used it as a drawing package. The pupils had explored many of the features of the software, and produced impressive drawings, but had actually made no use of the construction tools, and so had not engaged with the important mathematical idea of *construction*.

Geometry is fundamentally about properties of shapes and the relationships between them. A shape that is constructed might be seen as a figure that has been defined through the declaration of various properties and relationships. Thus, a square that has been merely drawn will soon lose it shape when one vertex is dragged. This is because the 'squareness' of the shape has not been established as a set of predefined properties in creating the shape. On the other hand, a shape which has been *constructed* as a square will remain a square even if the vertices are dragged to different positions. This is because the equality of the sides and the right angles has been built into the definition of the shape through the way that it was created.

For these pupils, their (self-selected) tasks were purposeful, with clear end products (drawings of a football pitch, a clown, and motorcycle and rider), but we felt that the pupils were not learning any mathematics. We wanted to design a task that pointed them towards the utility of construction whilst at the same time being seen as purposeful. This proved problematic. It was easy to design tasks that involved construction (such as creating a square which couldn't be 'messed up') but such tasks had no purpose for the pupils—except to satisfy the teacher. We knew that the pupils found creating drawings purposeful, but for such tasks placing points by eye, rather than constructing them, was perfectly satisfactory.

Eventually, we came upon the idea of harnessing the pupils' interest in drawing by setting them the task of producing a 'drawing kit' for younger pupils to use. This entailed them writing macros (short sequences of instructions) to produce a range of basic shapes (triangles, squares, diamonds etc.) from which young pupils could create their own pictures. For this task, there was a real purpose behind constructing a square which could be reproduced many times and manipulated without being messed up by the younger pupils. As the pupils worked on making their drawing kits, the utility of construction for producing 'perfect' robust shapes became clear.

The design of the drawing kit task embodies some elements which we offer here as heuristics for creating purposeful tasks that have the potential to associate that purpose with defined utilities.

(a) It has *an explicit end product* that the pupils cared about. This is a common feature in all our examples. In the first case, the pupils were attempting to design a spinner that would perform optimally, and, in the second case, the pupils were mending gadgets that they felt should operate in certain normative ways. In each case it was critical that the design process involved the use of tools that somehow represented the mathematical concept. In the drawing kit task, the pupils used the geometric construction tools in Cabri. For the spinners task, the tool used was the spreadsheet-generated scatter graph, and for the mending task, the pupils used the workings box, which represented the distribution.

(b) It involves *making something for another audience* to use. In the case of the drawing kit, the fact that the pupils for whom the product is intended were younger added an implicit further dimension. The product to be made can focus directly on the concept with which you want the pupils to engage. However, the focus shifts from the teacher teaching the pupil to the pupil teaching others. In the mending gadgets task, a similar strategy was used. The pupils were told that the programmer of the gadgets was in the process of building the software (which was in fact true) and would like the pupils' opinions about which gadgets were working properly. The pupils were ostensibly mending the gadgets for the programmer. We have observed this heuristic being applied elsewhere in the constructionist literature. For example, Kafai and Harel (1991a, b) report how pupils first created products for other pupils to learn from and then became consultants when another class engaged in similar activity (see also McClain & Cobb, 2001).

(c) It was well focused, but still contained *opportunities for pupils to make meaningful decisions*. Although most of the drawing kits produced contained a similar set of basic shapes, many groups added their own designs, such as wheels or roofs. In the spinner task, the pupils made their own decision on what data to collect next in their experiment though they were required to make use of the scatter graph in order to make that decision. In the case of mending the gadgets, the pupils came to the problem with prior conceptions of how such random generators should work. They tested out their personal conjectures, which were sometimes found to be naive.

We do not claim that these are the only means of designing tasks that link purpose and utility. The spinners task is an optimisation problem and such questions are often sufficiently intriguing to generate purposeful activity. We have seen also examples in the literature of pupils arguing from a particular point of view and this apparently captured the interest of the pupils involved (see, for example, McClain & Cobb, 2001; Ben-Zvi & Arcavi, 2001).

In conclusion then, we have provided an expressive, though non-exhaustive, set of heuristics for task design. As a final remark, we wish to comment on the observation that all of our examples are technology-based. This is not entirely happenstance. Technology affords a particularly powerful means of making mathematics quasi-concrete. The impact of the power principle is to enable pupils to learn mathematics through use. In turn they are able to construct utilities, which we conjecture is an especially meaningful way to first encounter new mathematical concepts. We wish to emphasise, however, the significance of our notion of utility. Whereas the power principle explains how technology can re-invert the traditional pedagogy, the purpose–utility linkage offers a means for thinking about why such an inversion should have an impact on learning. At the same time, this linkage provides a construct for elaborating task-design; we see our set of heuristics as a first attempt in this direction. Although inspired through the use of technology, these heuristics are not tied to that technology. Indeed, technology offers the specific danger that if the task is not well designed then the utility might be trivialised to button-pressing, a danger that can be avoided by proper consideration of the purpose–utility linkage.

Notes

1. We have chosen to use the term 'pupils' rather than 'children', 'students' or 'learners', since this term seems to best capture the full age range involved in formal schooling.
2. For clarity, within this theoretical paper, we use the term 'task' for what is set by the teacher, and reserve the term 'activity' for what subsequently takes place in the classroom setting.

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