P.H.M. Drijvers

# Learning algebra in a computer algebra environment

## Design research on the understanding of the concept of parameter

LEARNING ALGEBRA IN A COMPUTER ALGEBRA ENVIRONMENT

*Design research on the understanding of the concept of parameter*

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# Learning algebra in a computer algebra environment

*Design research on the understanding of the concept of parameter*

# Algebra leren in een computeralgebra omgeving

*Ontwikkelingsonderzoek naar het inzicht in het begrip parameter*

(met een samenvatting in het Nederlands)

PROEFSCHRIFT
TER VERKRIJGING VAN DE GRAAD VAN DOCTOR
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OP GEZAG VAN DE RECTOR MAGNIFICUS PROF. DR. W. GISPEN
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Paulus Hendrikus Maria Drijvers
geboren op 16 april 1958 te Breda

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## Preface

I first encountered computer algebra software in the 1980s. During in-service and pre-service courses for mathematics teachers, my colleagues and I would demonstrate the MuMath software package. The course participants were impressed, and we told them (proudly, as though we had developed the software ourselves) that this was the future of mathematics education.

I was fascinated with the phenomenon of computer algebra – and surprised that a machine was able to carry out sophisticated procedures (for example, for calculating limits and derivatives) and algebraic simplifications, techniques I considered to be at the very heart of mathematics. The implementation of these techniques also seemed to trivialize mathematics: if one can explain to a machine how to 'do' it, surely it cannot be conceptually difficult. Furthermore, I was puzzled by the possible influence on mathematics education: what would happen if students were to use such powerful tools? How would this affect their conceptual development and their repertoire of techniques?

Much has happened since then. The first computer algebra system for educational purposes – Derive – was developed in the late 1980s. I attended the first conferences on computer algebra use in education, where I met colleagues with a similar fascination and similar interests. Then handheld devices appeared, first with only graphical and numerical features but later with full computer algebra options. I became involved in the implementation of graphing calculators in Dutch upper-secondary education. However, computer algebra was not being integrated into mathematics education as quickly as had been predicted, and the fundamental questions concerning the relation between computer algebra use, by-hand algebraic skills and conceptual understanding were still to be answered.

Thus I was happy when, in 1999, I got the opportunity to conduct a PhD research study on the effects of using computer algebra on the learning of algebra. This thesis – the result of that study – once more reveals the intrinsic tension between using a computer algebra machine's routines and constructing conceptual understanding. However, the perspectives of instrumentation and symbolization, which are presented in this book, offer means to focus on this issue and to deal with it. A theoretical framework on learning mathematics by using cognitive tools such as computer algebra systems seems to be developing. Taking part in this process has been both an informative and a compelling experience.

Although this research study was essentially a one-man project, many persons contributed to it. I should like to express my gratitude to the following. First, my supervisors Koeno Gravemeijer and Jan de Lange, who supported me with their suggestions and confidence, and provided me with the freedom I needed. The advisory board – Kees Hoogland, Gerard Huls, Dirk Janssens, Gellof Kanselaar, Anne van

Streun and Agnes Verweij – stimulated me to clarify my intentions and ideas. My colleagues in the PROO project 'IT and mathematics' – Dirk Hoek, Monique Pijls and, in particular, Arthur Bakker and Michiel Doorman – were inspiring and helpful sparring partners during the four years this study took. Aad Goddijn and Martin Kindt contributed to the design of the student activities.

Five teachers – Jonneke Gerrits, Gerard Huls, Peter Lanser, Rina van Opstal and Harrie Smits – were kindly willing to involve their students in the teaching experiments; we enjoyed an intensive and fruitful collaboration. Texas Instruments provided symbolic calculators for the teaching experiments and Pieter Schadron programmed additional applications. Eva van Dam, Danny Dullens, Vera Emons, Onno van Herwaarden and Matthias Visser contributed to the quality of the research by performing classroom observations and data analysis.

Nathalie Kuijpers and Jeremy Rayner corrected my written English, and Ellen Hanepen and Betty Heijman provided editorial support while I was finishing the manuscript.

Finally, I thank Else, Jefte and Manu for their support. My sons Jefte and Manu were willing to discuss mathematical topics and, on many occasions, to serve as guinea-pigs, thus allowing me to observe their conceptual growth. I thank Else, my partner and colleague, both for her support and confidence in me and for the elbow-room she provided. Her competent and to-the-point reflections on my work helped me to sharpen my thinking and to focus on the essential aspects of this study.

# 1 Research questions

## 1.1 Introduction

This thesis reports on the research project ‘Learning algebra in a computer algebra environment’, which was granted by the Netherlands Organization for Scientific Research (NWO) under number 575-36-003E and was carried out in 1999 - 2002. In this study we investigate whether the use of computer algebra can contribute to the learning of algebra in general, and to the understanding of the concept of parameter in particular. As such, the project is a specific case of research on the integration of information technology (IT) into education, and in particular its influence on the learning of mathematics.

This research is part of the umbrella project ‘IT as a means for self-reliant learning of mathematics’ (NWO grant number 575-36-003). The latter project focuses on the question how reinvention and independent learning using IT can be integrated. In the sub-study reported on here, the IT component is computer algebra, and the reinvention concerns the concept of parameter.

In this chapter we define the study’s research questions and sketch the outline of the thesis. Section 1.2 presents a global inventory of the difficulties in the learning of algebra, and of the way in which IT in general and computer algebra in particular might contribute to their resolution. This leads to the definition of the focus of the study. Section 1.3 introduces the main research question, which is specified into two subquestions: the first addresses the learning of the concept of parameter, and the second concerns the instrumentation of computer algebra. The notions that appear in the research questions are explained and expectations are formulated. Then we motivate the relevant decisions that were made at the start of the research project and describe the desired outcomes (1.4). A classroom observation from a previous study that provoked the author’s interest illustrates how students encounter conceptual difficulties while manipulating variables in a computer algebra environment. Then we indicate the main components of the theoretical framework used in this study (Section 1.5). As a guideline for the reader – whether a researcher, mathematics educator or mathematics teacher – Section 1.6 sketches the structure of the book in the form of a graph.

## 1.2 What is the problem?

This section presents a global inventory of difficulties in the learning of algebra, and of the way in which IT in general and computer algebra in particular can contribute to resolving them.

### *Difficulties of learning algebra*

It is well known that algebra is a difficult topic in the school mathematics curriculum, and is often experienced as a stumbling-block. The Discussion Document for the 12th ICMI Study entitled ‘The Future of the Teaching and Learning of Algebra’ phrases this as follows:

> As the language of higher mathematics, algebra is a gateway to future study and mathematically significant ideas, but it is often a wall that blocks the paths of many.
> (Chick et al., 2001, p. 1)

Students often make mistakes while executing algebraic operations, and they find it hard to trace their errors or improve their performance. Even students who are able to perform specific algebraic procedures may attach a very limited meaning to them, and may be lost as soon as the problem situation is slightly changed.
A well-known example of student difficulties with algebra is the ‘student-professor problem’ (van Amerom, 2002; Malle, 1993; Rosnick, 1981). The task was:

> Write an equation, using the variables *S* and *P* to represent the following statement: “At this university there are six times as many students as professors”.
> (Rosnick, 1981, p. 418)

As a solution to this task, many students wrote the equation $6S = P$ rather than $6P = S$. This reveals a limited understanding of the concept of variable that is seen as a label referring to an object, in this case a student or a professor. The letter $S$ is used as an abbreviation for ‘Student’, and not as a variable that refers to the *number* of students. Many other studies address the difficulties of the concept of variable (e.g. Küchemann, 1981; Usiskin, 1988; Wagner, 1981).
Another classical problem that is relevant to our study was presented by Wenger (1987). Wenger described that students were unable to solve the equation $v\sqrt{u} = 1 + 2v\sqrt{1+u}$ with respect to $v$, although it is linear in $v$. The conclusion is that while students may be able to apply isolated algebraic techniques in simple situations, they are often not able to apply the appropriate techniques from their repertoire in a more complex situation. In this example, they are not able to see the sub-expressions $\sqrt{u}$ and $\sqrt{u+1}$ as objects, as entities. Instead, they perceive the square root sign as a strong cue that asks for manipulation.
The revision of algebra education recently received a lot of attention worldwide. The above-mentioned ICMI Study is devoted to the future of algebra (Chick et al., 2001). Many teachers, educators and researchers criticized the traditional method of teaching algebra: too much emphasis on symbols, on manipulative skills and on algebraic facts rather than on concept development and problem-solving skills (O’Callaghan, 1998). New approaches have been elaborated and implemented in educational systems (Boers-van Oosterum, 1990; Fey et al., 1995; Mathematics in Context Development Team, 1998; van Reeuwijk, 1995). In the Netherlands, the reconsideration of algebra in mathematics education is an issue as well. Examples of new approaches

to algebra have been developed and implemented in a new mathematics curriculum for 12- to 16-year-old students, in which concrete reality is the starting point for mathematization (Kindt, 1980, 2000, 2003; Team W12-16, 1992). However, teachers at the upper secondary level (eleventh and twelfth grades) often complain that their students need abstract algebra skills that they did not learn in this new curriculum. This is an issue in particular for pre-university students who are studying the exact sciences. Bringing about the transition from the informal reality-bound algebra of the curriculum for 12- to 16-year-olds to the more abstract algebra at pre-university upper-secondary level is one of the challenges for algebra education in the Netherlands. Recently, Goddijn and Kindt suggested two other approaches to algebra, a discrete approach and a geometrical approach (Goddijn & Kindt, 2001; Kindt, 2000, 2003). The authors stress the importance of productive rather than reproductive practice.

Several difficulties in the learning of algebra that are described in the literature are relevant to this study. First, students often cannot relate the formal, algorithmic character of algebraic procedures to informal and natural approaches that are meaningful to them. A second difficulty is the abstract level at which problems are solved, compared to the concrete situations they arise from, and the lack of meaning that the students attribute to the mathematical objects at the abstract level. A third factor is the need to keep track of the overall problem-solving process while executing the elementary algebraic procedures that are part of it. A fourth difficulty is the compact algebraic language with its specific conventions and symbols. A fifth obstacle may be the object character of algebraic formulas and expressions, where students often perceive them as processes or actions, and the accompanying 'lack of closure' obstacle. The main difficulties addressed in this study are the first, second and fifth issues. Our ambition is to help students in developing meaningful algebraic objects and operations that are not only related to real-life contexts, but are also embedded in a meaningful framework of mathematical relations. In particular, formulas and expressions are to be considered as objects that derive meaning from this framework of algebraic relations, and not only as processes or as prescriptions for numerical calculations. These issues are elaborated in Chapter 3.

### *Can IT be of any help?*

One of the directions in which solutions to these problems with the learning of algebra can be sought is the integration of IT into mathematics education. IT tools for mathematics education are being developed at high speed and along different avenues: on the one hand towards dedicated handheld tools, and on the other hand towards general digital learning environments (Kanselaar et al., 1999). Many research projects and teaching experiments are being conducted, because it is expected that IT can contribute to the visualization of concepts, or can free the students from the need to perform operations, thus directing the attention towards concept development or problem-solving strategies. IT is then used to facilitate the learning of the 'traditional' curriculum. In the meantime, however, the integration of technology raises questions concerning the goals of algebra education and the relevance of paper-and-pencil techniques, now that they can be left to a technological device. For example, is it necessary to be able to calculate the roots of a quadratic equation by hand, if a powerful digital algebra tool is available that will do it for you, fast and without errors (see Fig. 1.1)?

$$\text{solve}\left(x^2 + 5\cdot x + 4 = 0, x\right) \quad \to x = -1 \text{ or } x = -4$$

$$\text{solve}\left(x^2 + 5\cdot x + c = 0, x\right)$$

$$\to x = \frac{-\left(\sqrt{25 - 4\cdot c} + 5\right)}{2} \text{ or } x = \frac{\sqrt{25 - 4\cdot c} - 5}{2}$$

Figure 1.1 Solving quadratic equations in a CAS environment

Although originally not developed for educational purposes, a computer algebra system (CAS) is an IT tool that can be integrated into the learning of algebra. Besides offering the execution of algebraic calculations, a CAS usually also provides options for numerical calculations, graphing and data handling. Word processing and Internet access are made available too in many CAS environments.

In the literature, several capacities of IT and computer algebra for mathematics education are mentioned, such as addressing rich and realistic problem contexts, saving time on the development of paper-and-pencil skills, visualizing mathematical concepts, introducing dynamics and deepening the insight into algebraic concepts (Heid, 2002; Hillel et al., 1992; O'Callaghan, 1998). For the purpose of this study, the main affordances or perceived opportunities of computer algebra environments for the learning of algebra are expected to be:

- Variables and parameters in a CAS do not refer directly to numerical values, but stand on their own. This is expected to foster the development of a concept of variables and parameters that is more mature than the limited view of variables as labels or pseudo-numbers.
- In the CAS environment, expressions have an object character rather than a process character; expressions are 'things' rather than 'actions'. This is expected to enhance the development of a more structural view of algebraic expressions rather than a purely operational one. This may improve the students' symbol sense.
- The CAS can serve as an environment for experimentation that generates examples that elicit generalization. This is expected to stimulate students to make generalizations and to 'discover the general from the particular'.

- The CAS offers facilities for visualizing the dynamic effect of a sliding parameter. This is expected to reinforce the relation between formula and graph, thus making the formula more meaningful to the students.

From these expectations, which are elaborated on in Chapter 5, one might hope that the integration of a computer algebra environment would contribute to solving the difficulties of learning algebra. However, things are not that easy. As is the case for the four-function calculator (de Moor & van den Brink, 2001; Team W12-16, 1992), the integration of computer algebra into mathematics education provokes many questions concerning the curriculum and the pedagogy, such as: what do students need to learn and what do they not need to learn, if the machine 'does everything' for them? What is the relevance of paper-and-pencil skills? How can CAS use be integrated in a didactical way? What skills are needed to correctly interpret CAS outputs, such as the result presented in Fig. 1.1? Can conceptual development take place thanks to or in spite of working in a computer algebra environment? These questions, which are discussed among mathematics researchers and educators all over the world, involve the view of algebra, the intended learning process, the pedagogy of computer algebra use and the characteristics of CAS. The main focus of this study is to investigate how the use of computer algebra can promote the insight into algebraic concepts and operations.

## 1.3 Research questions

In this section, we first define the research topic more precisely. Then the research questions, which emerge from the global problem description, are stated and explained. After that, some prior expectations are discussed.

*Focus on the concept of parameter and on instrumentation*

Algebra as such is too extensive to investigate in the framework of this study. We need to confine ourselves to a topic within algebra that allows us to address the relevant issues mentioned above. With the Dutch issue of the difficult transition from middle school algebra to upper secondary algebra in mind, it is interesting to focus our study on the interface between the two types of education, where the more informal approach in the curriculum for 12- to 16-year-olds meets the more formal and abstract approach at the upper-secondary pre-university level. Therefore, we choose to investigate computer algebra use in the ninth and tenth grades.

A suitable topic, then, seems to be the concept of parameter, which on the one hand may emerge quite naturally from concrete contexts, and on the other hand may be a means of generalization and abstraction. Therefore, the general question of the influence of computer algebra use on the development of insight into algebraic concepts and operations is specified into the question whether the development of the concept of parameter as it appears in algebraic expressions and functions can be im-

proved by CAS use. The concept of parameter is discussed in detail in Chapter 4.
Previous research shows that the integration of computer algebra into teaching and learning often turns out to be more complex than expected (Drijvers, 2000; Lagrange, 2000; Guin & Trouche, 2002; Trouche, 2000). The idea that technology carries out the operations so that the students can concentrate on conceptual understanding is too simplistic. The question is how to ensure that the student gives proper meaning to the operations within the CAS environment. The construction of meaning is often related to the mastering of the technical skills that using the CAS requires. The instrumental approach to IT use, which will be described in Section 1.5 and in Chapter 5, provides a framework for the study of this dual development. It states that the machine techniques that need to be developed are related to the mental concepts that students are constructing. This simultaneous development is not an easy process and the CAS environment may present barriers. For example, the black box character of the CAS tool and the rigid demands on syntactically correct input may limit the opportunities for the progressive formalization of informal strategies that is proposed by the theory of Realistic Mathematics Education. To benefit from the potential that computer algebra offers, such obstacles have to be overcome. As a second focus of this study, we therefore consider the instrumentation process of computer algebra, and the relation between the development of techniques in the computer algebra environment and the development of mathematical concepts.

*Statement of the research questions*

In Section 1.2, we hypothesized that the use of IT might contribute to the learning of algebra. Because of their algebraic power, computer algebra environments seem promising. The basic aim of this study, therefore, is to investigate whether computer algebra use can contribute to the understanding of algebra. This leads to the following main research question:

> *How can the use of computer algebra promote the understanding of algebraic concepts and operations?*

This question needs further specification. For reasons mentioned in the previous section, the focus is on the concept of parameter. Therefore, the first research subquestion is:

1. *How can the use of computer algebra contribute to a higher level understanding of the concept of parameter?*

Above we described the tension between the potential of computer algebra use and the barriers that a – novice – user has to overcome in order to work efficiently with a CAS. The acquisition of the skills to properly use a CAS has both a technical and a conceptual aspect. The second research subquestion concerns this dual process:

2. *What is the relation between instrumented techniques in the computer algebra environment and mathematical concepts, as it is established during the instrumentation process?*

### *The main research question word by word*

Let us discuss the meaning that we attribute to the key words in the three research questions. We briefly define them here and refer to further chapters for more details.
The main question starts with the words *How can*. This indicates that the question is not whether computer algebra use can promote algebraic understanding, but what conditions and approaches optimize the affordances that computer algebra offers. Although we do not expect this to happen, it is possible that the results of the study will show that computer algebra use does not contribute at all to algebraic understanding. In that case, the answer to the 'how-question' will be that it can't.
*Computer algebra* refers to software that performs algebraic calculations and formula manipulations. Such software is available for PCs and handheld calculators, and offers options for numerical calculations and graphing. More information on computer algebra can be found in Chapter 5.
The word *understanding* will be explained in the discussion of the first subquestion below.
The main *algebraic concept* in this study is the concept of parameter as a 'meta-variable'. The rationale for this choice lies in the idea that the parameter is a suitable concept to foster the transition from informal, context-related algebra to formal algebra in the framework of mathematical relations. The concept of parameter may emerge quite naturally from concrete contexts, and may enhance generalization and abstraction. The concept of parameter is addressed further in the discussion of the first subquestion.
Other *algebraic concepts* that are important in this study are those of formula and expression. Formulas and expressions are to be perceived as entities, as objects rather than calculation processes. In particular, expressions should be accepted as general solutions of equations that contain parameters, and as objects that can be substituted into other equations. This issue is elaborated on in Section 3.4.
The repertoire of *algebraic operations* is limited in this study to three operations: the isolation of variables, the solving of equations that contain parameters, and the substitution of expressions into equations.

### *The first research subquestion word by word*

The first research subquestion specifies the general main question. As in the main question, the first words *How can* imply that we conjecture that there are possible contributions. We assume affordances of computer algebra use, and hope to be able to find out how these can be exploited. On the other hand, the conclusion may be that there are no contributions at all. The main aspect of the *concept of parameter* is the notion that a parameter is an ‘extra variable’ or a ‘changing constant’ in a formula or function that generates a class of formulas, a family of functions and a sheaf of graphs. Therefore, the parameter unifies a class of problem situations and solutions, and can be used for generalization. Variation of the parameter leads to global rather than local change. The concept of parameter is elaborated on in Chapter 4.

A central issue in the first research subquestion is the *higher level understanding* of the concept of parameter. The higher level understanding is defined as an extension of the understanding of the parameter to the parameter as a changing quantity, as a generalizer and as an unknown. The changing quantity affects the problem situation, the formulas and the graphs as a whole. The parameter as a generalizer unifies a class of situations, and the parameter as an unknown selects a subset of these situations that fulfil a condition. A theoretical basis of the concept of levels of understanding and a further elaboration of the higher level understanding into the concept of parameter is described in Section 4.5.

### *The second research subquestion word by word*

The second research subquestion concerns the instrumentation of computer algebra, which provides the dual process of constructing meaning and mastering technical skills in the IT environment with a promising perspective that goes beyond the somewhat naive idea ‘skills for technology, concepts for the students’. Instrumentation is the learning process that users go through while using a new tool for specific purposes. The *instrumentation process* or instrumental genesis takes place through developing utilization schemes and instrumented techniques. A utilization scheme integrates the technical skills for using the machine, and the conceptual meaning that is attached to these manipulations, including both mathematical understanding and insight into the way the technological tool deals with the mathematics. These schemes give meaning to the use of the tool. This process leads to the development of a useful instrument (Artigue, 1997a; Lagrange, 1999a; Guin & Trouche, 2002; Trouche, 2000).

An *instrumented technique* has a technical side that consists of an integrated series of machine acts that has become a routinized way of dealing with a specific type of regularly occurring tasks. The conceptual side of the machine technique concerns the development of the accompanying mathematical objects and of the insight into the relation between those objects and the technical part of the technique. In this study, the most important instrumented techniques are the solving of equations and systems of equations, and the substitution of expressions.

The *mathematical concepts* involved are the students’ mental images of the techniques that they carry out and of the mathematical objects and the procedural knowledge that play a role in these techniques. In this study, the most important mathemat-

ical objects are the parameter and the formula; the procedural knowledge involved concerns solving equations and substituting expressions.
The start of the question, *What is the relation*, suggests that there is indeed a relation between the instrumented techniques and the mathematical concepts. The theory of instrumentation that is discussed in Chapter 5 strongly suggests this. The question is whether such a relationship can be observed in this study, and if so, how the instrumented techniques and the conceptual development influence each other.
The instrumentation of computer algebra concerns all mathematical topics and is not limited to algebra; therefore, the second research subquestion has a broader scope than the first one. Because of the focus on algebra, however, in this study we confine ourselves to the instrumentation of algebraic techniques, and in particular to those mentioned above: solving equations, possibly containing parameters and substituting expressions.

#### *Expectations*

What kind of answers did we expect at the start of the study? As far as the main research question is concerned, our initial feeling was that computer algebra use offers opportunities for algebra activities that may support the development of algebraic understanding. In particular, we supposed that computer algebra use can contribute to the perception of variables and parameters as objects rather than numbers, to the perception of expressions and formulas as objects, and that the examples generated in the computer algebra environment may elicit generalizations that can be carried out in the same environment as well.
We have already pointed out some possible pitfalls of computer algebra use, such as the black box ‘oracle’ character of the CAS, the difficult instrumentation process that is required, and the abstract style of working that the CAS environment elicits.
In short, we saw opportunities for the learning of algebra in a CAS environment, and in particular for the development of the concept of parameter. Meanwhile, we noticed that the instrumentation process may be a difficult one that requires attention, and that this may frustrate the learning process if the obstacles encountered are too hard to overcome.

## 1.4 Decisions, outcomes and motive

In this section we first motivate some relevant decisions that were made at the start of the research project. Then the targeted outcomes are described. The section ends with a classroom observation from a previous study that can be seen as the motive for this research project.
The observation provoked our interest in the algebraic aspects of working in a computer algebra environment; it shows how students encounter conceptual difficulties while manipulating variables and parameters with a CAS.

*Decisions*

After determining the research questions, many decisions had to be taken before we could proceed. We will mention the most important ones.

The research questions focus on algebraic understanding supported by computer algebra and on the instrumentation of computer algebra. Our first decision was to investigate this by means of teaching experiments, because they provide good opportunities for gathering relevant data for these questions (Steffe, 1983; Steffe & Thompson, 2000).

As we see the development of understanding and the instrumental genesis as qualitative processes, our second decision was that the main sources of data would be observation protocols of student behaviour and interviews with students throughout the teaching experiments.

We wanted to get results that could be extrapolated to other students, other schools and other teachers. Therefore, our third decision was to carry out teaching experiments at regular schools with teachers who are not very experienced as far as the integration of technology is concerned.

The fourth decision concerned the level of education at which the teaching experiments would be held. Because of the difficult interface between the Dutch mathematics curriculum for 12- to 16-year-olds and that of upper-secondary mathematics, we choose the ninth and tenth grades as the target groups. Furthermore, we did not focus only on students who are very gifted in mathematics. At the start, we doubted whether students of this age and level would be able to overcome the obstacles that using a CAS may present. Would the balance between efforts and opportunities be positive? An extra argument for choosing students of 14- to 16-year-olds is the fact that most of the research studies in this field focus on higher grades or university level (Artigue, 1997a; Guin & Trouche, 1999; Heid, 1988; Hillel et al., 1992; Lagrange, 1999b). Exceptions are the studies by Brown (1998), Hunter et al. (1994) and Klinger (1993), who did involve ninth-grade students.

The fifth decision concerned the computer algebra system to be used. We decided, mainly for practical reasons, to use handheld computer algebra machines. Such symbolic calculators influence the classroom settings to a lesser extent than do PCs, which tend to dominate the classroom. Symbolic calculators are always available, which is not the case for the computer lab at most schools in the Netherlands. An advantage of the personal handheld device is that the students' familiarity and confidence with it develops faster than with PC software (Lagrange, 1999b). A disadvantage of the symbolic calculator can be its private character (Doerr & Zangor, 2000; Monaghan, 1994). An advantage of using PCs is that they provide students with the opportunity to write down their comments on and explanations of their calculations. However, preliminary classroom experiences indicated that students were reluctant to make full use of this facility.

Compared to other studies in this domain, new elements in this research are the age and level of the students and the focus on the concept of parameter and the instrumentation of algebraic techniques.

### *Outcomes*

The research has several aims. The first is to gain insight into the learning of the concept of parameter, embodied in a local instruction theory and a hypothetical learning trajectory that make full use of the opportunities that computer algebra offers. The learning trajectory is to be based on a conceptual and pedagogical analysis of the concept of parameter, and is made concrete in instructional activities that are tested in the classroom. The results from the teaching experiments lead to an answer to the first research subquestion.

The second aim is to describe the relation between instrumented techniques and mathematical concept, as it is observed in students' behaviour. This will lead to a better understanding of this relation and to the identification of factors that play a role in the instrumental genesis. These factors are expected to frustrate the learning process in the computer algebra environment, but may also stimulate reflection and concept development. These findings will be used to answer the second research subquestion.

The third aim is to answer the main research question by extrapolating the conclusions on the two research subquestions.

The fourth is to evaluate the theoretical framework that is used. A brief outline of this framework is given in the following section. The evaluation involves the confrontation of the data with the theoretical perspectives and the formulation of conclusions about the applicability of the theories in this specific research context. The methodology will be evaluated as well. The generalizability of the results to other mathematical topics and to other kinds of computer algebra tools or IT tools in general will be considered.

Finally, the study will yield suggestions for teaching mathematics by using computer algebra, for designing software and for further research in this domain.

### *Motive*

Experiences in previous research projects provoked our interest in obstacles that students encounter while working in a computer algebra environment (Drijvers, 1999a, 2000). One of the observations in one of these projects, which is described below, revealed the difficulties that students can have with the algebraic concept of variable, and with generalized solutions. Observations like this stressed the relevance of algebraic insights while working in a computer algebra environment; they also show that the phenomena the students encounter in the CAS work can be starting points for the learning process.

The observation concerned the well-known problem of a pipe that is to be carried horizontally around a corner in a corridor (see Fig. 1.2). The question is: how long can the pipe be? After the problem was solved for concrete dimensions, the situation was generalized to corridors of width $p$ and $q$ metres, respectively. The final question was to express the maximum length of the pipe that can pass in terms of $p$ and $q$.

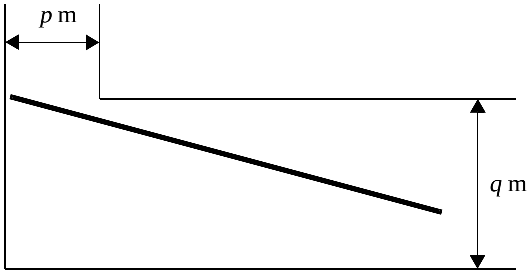



Figure 1.2 The pipe passing the corner

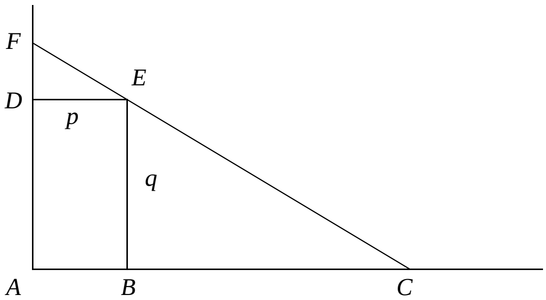



Figure 1.3 Dennis and Niels' approach

Two students in the eleventh grade, Dennis and Niels, called $BC$ $x$ and derived that $CE = \sqrt{x^2 + q^2}$ (see Fig. 1.3).
Then they realized that triangle $CBE$ is similar to $CAF$.

They noticed that the factor of multiplication was $\frac{x+p}{x}$, so $CF = \frac{x+p}{x} \cdot \sqrt{x^2 + q^2}$.

Before this, they had entered $CE$ as function $y_1$ in their computer algebra environment (in this case, the TI-92 symbolic calculator), and $CF$ as $y_2$, but with numerical values for $p$ and $q$. While they were working, the teacher passed by and suggested that they made these functions more generic.

*Teacher:* *You can use p and q in $y_1$ and $y_2$, and then give p and q the values you want.*

*Dennis:* *[some time later] That goes very nice, with that p and q and then give them values. We make the function super-general!*

They managed to express the value of *x* in terms of *p* and *q* by means of differentiation and solving in the CAS (see Fig. 1.4).

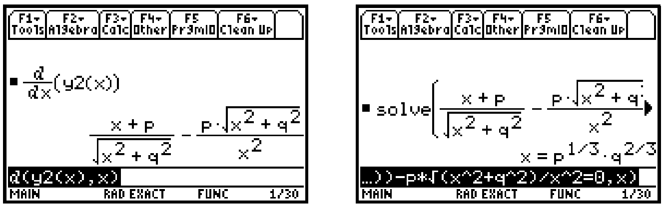



Figure 1.4 Dennis and Niels' TI-92 screens

However, although they succeeded in solving the problem in terms of *p* and *q*, they did not include this in their written report. During the classroom discussion afterwards, it became clear why. The teacher asked Dennis to explain his method. He wanted Dennis to give the general solution, but Dennis said he couldn't.

*Dennis:* *You first have to fill in values for p and q, don't you?*
*Teacher:* *You can also solve immediately. Then you get the answer expressed in p and q.*

This observation shows that the students didn't perceive the formula for *x* as an answer, as a solution to the generic problem. They probably suffered from the 'lack of closure' obstacle, which is known from research on algebra education in general (Collis, 1975; Küchemann, 1981; Tall & Thomas, 1991). This obstacle refers to the fact that students are unable to proceed as long as the result of the process is not a 'closed form', such as a numeric value, but a formula that still contains operators, such as $a+b$. The result in the latter case is an algebraic object, but the student wants to carry out a procedure to find a closed result.
But what is the specific role of computer algebra in this? Does the computer algebra environment simply make the difficulties with variables and general solutions more explicit? We assume that adequately operating with symbols using computer algebra requires that students are aware of the different roles of the different letters, and that they really understand the concept of variable and can interpret expressions as general solutions.
This observation and similar ones provoked our interest in the learning of algebra in a computer algebra environment, and in the concept of parameter in particular.

## 1.5 Theoretical framework

Now that the research questions have been formulated and some important decisions made, the question is what theoretical framework can be used. Criteria for a useful theoretical framework are its relation with the research questions and its applicability to the design of instructional activities and to the interpretation and analysis of student observations.

A ready-to-use theoretical framework for the study of learning algebra in a computer algebra environment is not available. Therefore, the approach in this study is to select theoretical elements from research on learning algebra and mathematics in general that seem promising for application, and adaptation to this situation.

As a consequence, these elements are taken out of their usual scope and localized for the aim of this study. This eclectic approach is called 'theory-guided bricolage' (Gravemeijer, 1994). On the one hand, this is a test for the theoretical elements: can they be applied to this new context? On the other hand, the findings from this study may contribute to the development of the theoretical elements that are used here.

We choose to describe the theoretical elements of the framework when appropriate for the content of the book; as a consequence, the theoretical elements appear at different instances in Chapters 3, 4 and 5. For the sake of an overview, we now list the elements of the theoretical framework and refer to the sections where they are elaborated.

- *The domain-specific instruction theory of Realistic Mathematics Education*

  In the domain-specific instruction theory of Realistic Mathematics Education (RME), the development of mathematical knowledge by means of progressively mathematizing informal but meaningful strategies is an important instruction principle (Treffers, 1987a). The question is how this relates to the idea that a CAS is a black box, and that working in the computer algebra environment has a top-down character (Drijvers, 2000).

  In the theory of RME horizontal and vertical mathematization are distinguished, the first referring to describing and organizing realistic problems by mathematical means, and the second to the emergence of a framework of mathematical objects and relations (Treffers, 1987a). Gravemeijer (1994, 1999) elaborated on this by distinguishing different referential levels of working with emergent models.

  The RME principles of guided reinvention and didactical phenomenology are used for developing a hypothetical learning trajectory and for designing instructional activities. More about RME can be found in Sections 3.6 and 4.5. In Chapter 5, RME principles are related to computer algebra use.

- *Level theories*

  The first research subquestion speaks about 'a higher level understanding of the concept of parameter'. To make explicit what is meant by that, several perspectives are considered. First, the level theory of Van Hiele distinguishes between a ground level, a second level and a third level of insight (Van Hiele, 1973, 1986). A second approach is the concept of emergent models and levels of activity that are part of the theory of RME (Gravemeijer, 1994, 1999). Third, generalization and abstraction are important for the higher level understanding of the concept of parameter. Finally, the process-object duality (see below) is an aspect of the targeted level of understanding. This is elaborated in Section 4.5.

- *Theories on symbol sense and symbolizing*
  Theories on symbol sense concern the learning of algebra and gaining insight into the structure and meaning of algebraic formulas and expressions in particular (Arcavi, 1994; Zorn, 2002). The character of insight into formulas seems to change while working in a computer algebra environment, but it does not become irrelevant. For the understanding of the concept of parameter, symbol sense seems to play an important role.
  How to acquire symbol sense? The theories of symbolization stress the parallel development of symbols and meaning by means of a process of signification (Gravemeijer et al., 2000). Because giving meaning to algebraic techniques, formulas and expressions in the computer algebra environment is an important aspect of the instrumental genesis, the perspective of symbolization may be fruitful in this study. It will be elaborated in Section 3.5.
- *The process-object duality*
  The process-object duality is important for the learning of mathematics: a mature conceptual understanding involves the ability to see both the process aspect and the object aspect of the concept. Often, students first experience the process aspect; on the basis of this they may develop the object aspect, which is a prerequisite for progress. This development is called reification (Sfard, 1991) or encapsulation (Dubinsky, 1991) and leads to proceptual understanding (Tall & Thomas, 1991). These theoretical notions concern the learning of mathematics in general, but are also applied to the learning of algebra. In this study we consider the reification of algebraic formulas and expressions. This seems to be of interest because computer algebra use might affect the process-object view of formulas and because the reification of formulas and expressions is important for the development of the concept of parameter. The process-object duality is discussed in Section 3.4.
- *The theory of instrumentation*
  The theory of instrumentation has been addressed briefly in Section 1.3. Its focus is the development of utilization schemes or instrumented techniques for the use of IT tools (Guin & Trouche, 2002). Such schemes combine technical skills and conceptual insights. The theory of instrumentation is applied to computer algebra environments, and is expected to help in answering the second research

subquestion. The theory is elaborated in Section 5.6.

Concerning the research methodology, the two main theoretical concepts are the methodology of design research, and the hypothetical learning trajectory as a means of planning the learning trajectory and making explicit the expected student behaviour. These methodological issues are addressed in Chapter 2.

## 1.6 The structure of the book

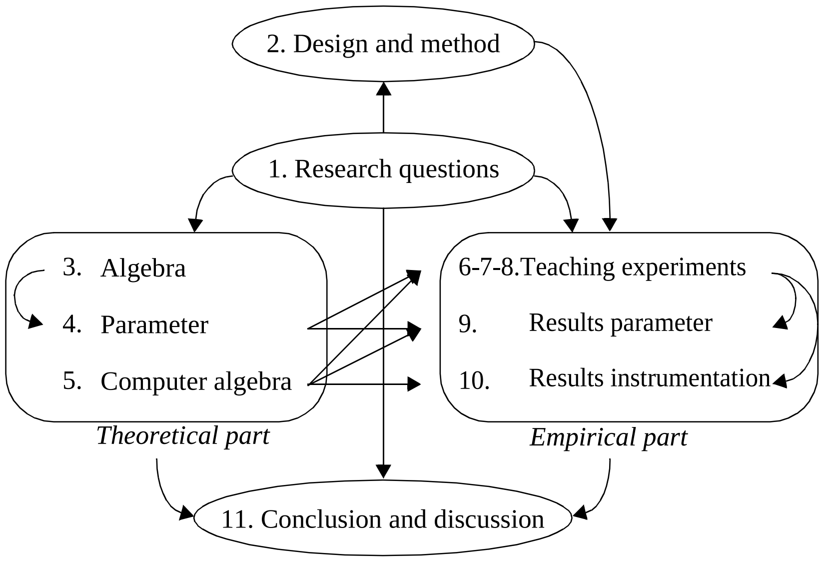



Figure 1.5 Structure of the book with abbreviated chapter titles

Fig. 1.5 shows the structure of this book. The present chapter, Chapter 1, contains the research questions and explains the aims and backgrounds of the study. In Chapter 2 the research design and methodology are described. Key words are design research and hypothetical learning trajectory. Chapters 1 and 2 together indicate what the research is about and how it is conducted.
Chapters 3, 4 and 5 form the theoretical part of the thesis. They treat the main themes of the study: algebra in general, the concept of parameter in particular and the possible roles of computer algebra. Chapter 3 concerns algebra in general. It sketches different views on algebra and describes the standpoint of this study. The theoretical issues of symbol sense, symbolizing, the process-object duality and Realistic Mathematics Education are addressed.
In Chapter 4, we zoom in on the concept of parameter. After a brief historical per-

spective, a conceptual analysis of the parameter is given. Then we describe what we consider a higher level understanding of the concept of parameter. This is connected to the theoretical notions from Chapter 3.

Chapter 5 deals with the tool that students use in this research project: computer algebra. Besides an overview of previous research in this domain, it contains a description of the theory of instrumentation that will be used in Chapter 10 in particular.

Chapters 6 - 10 form the empirical part of the dissertation. Chapters 6, 7 and 8 describe the development of the hypothetical learning trajectory and the classroom experiences during the three subsequent research cycles.

Chapter 9 concerns the contribution of computer algebra use to the understanding of the concept of parameter, and thereby answers the first research subquestion.

In Chapter 10, the results concerning the instrumentation of computer algebra are presented. The second research subquestion is answered here.

Chapter 11, finally, answers the main research question. After that, we look back on the study and discuss the results and the methodology. Also, the relevance of the theoretical framework and the generalizability of the findings are evaluated. The chapter ends with recommendations for teaching, for software design and for further research.

Now that the research questions have been stated, the main decisions have been taken and the theoretical framework has been specified, the next step is to describe the research design and the methodology. These are the topics of Chapter 2.

# 2 Research design and methodology

## 2.1 Introduction

In the previous chapter we defined the research questions. The next step, which we describe in the present chapter, is to decide on the research design and methodology. The research questions, the first decisions on the focus of the research and the theoretical framework led to using the design research methodology. In Section 2.2 we describe the main characteristics of the design research paradigm, why it was chosen for this study and how we guarantee the reliability and validity of the results.

One of the characteristics of the design research methodology is the cyclic research design. Within each research cycle the phases of preliminary design, teaching experiment and retrospective analysis are distinguished. We describe these phases successively. The first phase includes the development of a hypothetical learning trajectory (Section 2.3) and the design of instructional activities (2.4). The second phase, the teaching experiment, is addressed in 2.5. In particular, we pay attention to data sampling techniques within the teaching experiments. The final phase, the retrospective analysis, includes the analysis of the data and is described in Section 2.6.

This research study consists of three full research cycles and one intermediate mini-cycle. This arrangement is presented in Section 2.7. Section 2.8 contains some practical information on the way the research was carried out, such as on the school where the experiments took place, on the teachers involved and on the way data was gathered; it also provides an overview of the data from the teaching experiments.

## 2.2 Design research

The research methodology that we use in this study is the design research paradigm. In this section we first explain this methodology. Then we motivate why design research is appropriate for this study. Finally, we address the issues of reliability and validity and we discuss the measures that were taken to meet these criteria.

### 2.2.1 *Characteristics of design research*

Design research – also known as developmental research or development research – is a research methodology that aims at developing theories, instructional materials and an empirically grounded understanding of ‘how the learning works’ (Research Advisory Committee, 1996). The main objective of design research is understanding and not explaining (Bruner, 1996). This objective implies different norms of justification than would be the case in comparative empirical research.

One important feature of design research is the adaptation of the learning trajectory throughout the research; based on previous experience, the instructional sequences and teaching experiment conditions are adjusted. Therefore, design research is particularly suitable in situations where a full theoretical framework is not yet available and where hypotheses are still to be developed.

The methodology of design research is addressed in many recent publications (e.g. van den Akker, 1999; Brown, 1992; Edelson, 2002; Freudenthal, 1991; Gravemeijer, 1993, 1994, 1998; Gravemeijer & Cobb, 2001; Leijnse, 1995; Treffers, 1993). In spite of varying interpretations of the notion of design research, there is agreement on the identification of two key aspects: the cyclic character of design research and the central position of the design of instructional activities. We now address these two issues.

*The cyclic character of design research*

Design research has a cyclic character: a design research study consists of research cycles in which thought experiments and teaching experiments alternate. We distinguish macro-cycles that concern the global level of the teaching experiments, and micro-cycles that concern the level of subsequent lessons. Gravemeijer argued that the cycles lead to a cumulative effect of small steps, in which teaching experiments provide ‘feed-forward’ for the next thought experiments and teaching experiments (Gravemeijer, 1993, 1994).

A macro-cycle of design research consists of three phases: the preliminary design phase, the teaching experiment phase, and the phase of retrospective analysis. In the last-mentioned phase, the reflection captures the development of the insights of the researcher. Following Goffree (1986) and Schön (1983), Gravemeijer called this ‘reflection-in-action’ (Gravemeijer, 1993, 1994). As a result, new theories, new hypotheses and new instructional activities emerge, that form the feed-forward for the next research cycle that may have a different character, according to new insights and hypotheses. The process of the researcher’s thinking should be reported, to ensure the trackability of this development for others (Freudenthal, 1991; Gravemeijer, 1994).

As far as the role of theory in design research is concerned, the term ‘theory-guided bricolage’ is used (Gravemeijer, 1994). The researcher is like a tinkerer, who tries to combine and integrate global and local theories, which may be issued from other domains, to develop a learning trajectory and a local instruction theory for a specific topic. This local instruction theory contributes to the development of the domain-specific instruction theory.

In our study, three full macro-cycles – indicated as G9-I, G9-II and G10-II – and one intermediate cycle were carried out. The first phase of preliminary design includes two related parts, the development of a hypothetical learning trajectory (HLT) and the design of instructional activities. This phase is followed by the teaching experiment and the retrospective analysis. Fig. 2.1 shows the three full research cycles. Cycle 1 started with a conceptual analysis that is described in Chapter 4. Each of the phases is elaborated on in Sections 2.3 - 2.6, whereas specific information on each of the cycles is presented in 2.7.

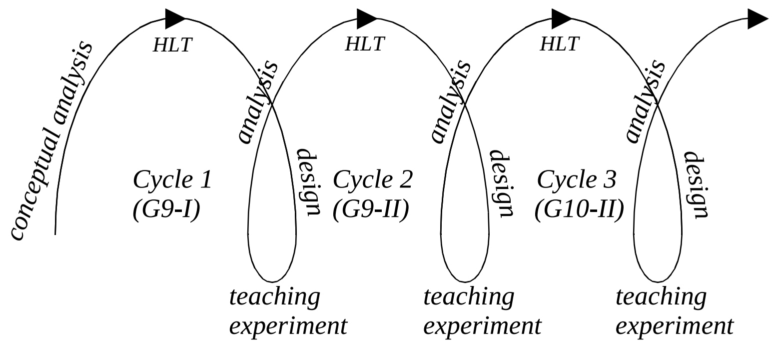



Figure 2.1 Design research cycles within this study

*The role of design*

A second characteristic of design research is the importance of the development of a learning trajectory that is made tangible in instructional activities (Gravemeijer, 1994). The design of instructional activities is more than a necessity for carrying out teaching experiments. The design process forces the researcher to make explicit choices, hypotheses and expectations that otherwise might have remained implicit. The development of the design also indicates how the emphasis within the theoretical development may shift and how the researcher's insights and hypotheses develop. We agree with Edelson, who argued that design of student texts is a meaningful part of the research methodology:

> (...) design research explicitly exploits the design process as an opportunity to advance the researchers' understanding of teaching, learning, and educational systems. Design research may still incorporate the same types of outcome-based evaluation that characterize traditional theory testing, however, it recognizes design as an important approach to research in its own right.
> (Edelson, 2002, p.107)

This is particularly the case when the theoretical framework involved is under construction:

> (...) it [the research] started with only a partial theory and has proceeded with the explicit goal of elaborating that theory before attempting any summary evaluation. The lessons that are emerging from this effort are being shaped by the concrete, practical work of design.
> (Edelson, 2002, p. 112)

### 2.2.2 *Why design research?*

The design research methodology is used in this study for several reasons. The research questions that were defined in Section 1.3 start with 'How can...' and 'What is the relation...'. This illustrates that we are interested not just in knowing whether computer algebra affects the learning of algebra, but specifically in understanding how. This character of the research questions links up with the general objectives of design research that we described in the previous section.

A second reason for choosing the design research methodology is that a ready-to-use theoretical framework for dealing with the research questions is lacking: the use of technology, and of computer algebra in particular, in mathematics education is a relatively new phenomenon; hardly any elaborated theoretical views are available. In Section 1.5 a provisional theoretical framework was 'tinkered' from elements from diverse sources by a 'bricolage' process. Therefore, the study has an explorative character and we need a research design that allows for revising theories, hypotheses and instructional activities during the subsequent research cycles. Design research meets this requirement.

Third, the integration of computer algebra into algebra education affects many aspects of teaching, such as the tasks that are set, instructional activities and assessment. New teaching materials need to be designed. In the design process, many choices are made that are relevant for the outcomes of this study. Therefore, we see the design process as an integrated part of the research, which is in line with the principles of design research methodology.

In a similar situation, Hillel et al. argued that integrating a computer algebra environment into teaching affects education as a whole, which makes the research design of control group - experimental group inappropriate:

> Our description of the experimental setting in the previous section should have made apparent that there would have been little sense in using a traditional functions class as a control group – too many aspects of the course were changed in the experimental class.
> (Hillel et al., 1992, p. 137)

For the above reasons the design research methodology was chosen. In the terminology of van den Akker, this study is a didactics design research, as it aims at developing a local instruction theory (van den Akker, 1999).

One approach to design research is to immediately adapt the instructional activities after every lesson according to the experiences. This leads to many short micro-cycles and can be an effective method, as shown by the following quotation that describes a situation where time constraints led to a micro-cycle method:

> The time pressure was considerable, but it had one advantage: experiences in the classroom could be incorporated immediately. Consequently, practical experience assumed great significance as a 'feed forward'; the components of the curriculum which were still to be developed could be adjusted directly on the basis of classroom experiences.
> (Gravemeijer, 1994, p. 36)

In this study, however, we started the teaching experiments with a complete set of instructional activities. Due to practical circumstances, adaptations were limited to decisions on skipping tasks or stressing other assignments. Therefore, the cyclic character of the research primarily concerns the macro-cycles.

### 2.2.3 *Validity and reliability*

This section addresses the reliability and validity of the research methodology. The question is how these criteria are met in this design research study, where observations and interviews are the main sources of data, and interpretation and coding are the main techniques of analysis. As the goals and methods of design research with qualitative data are different from those of comparative empirical research, the operationalization of reliability and validity is different as well (Edelson, 2002; Gravemeijer, 1993, 1994; Gravemeijer & Cobb, 2001; Research Advisory Committee, 1996; Smaling, 1987, 1992; Swanborn, 1996). For design research, the justification often has an argumentative character: the explanatory power, grounded in empirical evidence, is essential. We briefly address internal and external reliability and validity.

*Internal reliability* concerns the reliability of the methods that were used within the research project. Our measures for obtaining internal reliability included systematically gathering data by means of prior identified key items in student activities, and processing the data using consistent coding systems. Also, the protocol analysis and coding process were carried out by a second researcher. Crucial observations were shared with colleagues (peer examination).

For the *external reliability*, the criterion is virtual replicability by means of trackability (Gravemeijer, 1993, 1994; Gravemeijer & Cobb, 2001; Smaling, 1987, 1992). This means that the research is reported in such a way that is can be reconstructed by others. This requires transparency and explicitness about the learning process of the researcher and justification of the choices that are made within the research project. Raw data should be made available. The following quotation addresses the need for trackability:

> Developmental research means: ‘experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience.’
> (Freudenthal, 1991, p. 161)

In this study we ensured the external reliability by reporting extensively on the research methodology, the process of data reduction and the learning process of the researcher, by means of justifying the decisions and by making available the raw data.

The *internal validity* refers to the quality of the data collection and the soundness of the reasoning that led to the conclusions. The measures we took to improve the internal validity included finding a proper balance between involvement and distance during teaching experiments, and 'playing devil's advocate' while analysing the data, in order to find counter-examples of the hypotheses and to develop alternative explanations of the findings. These activities were also shared with colleagues. Data triangulation was used to consider observed phenomena from different perspectives.

*External validity* is realized by means of reflecting on the generalizability of the conclusions. Also, the quality of the reasoning and the conclusions was controlled by means of submitting papers and conference contributions that were reviewed during the research period.

## 2.3 The hypothetical learning trajectory

Within each macro level research cycle we distinguish three phases, the preliminary design phase, the teaching experiment phase and the phase of retrospective analysis. The first phase of preliminary design includes two related parts, the development of a hypothetical learning trajectory (HLT) and the design of instructional activities. In the following four sections we elaborate on each of these phases. Of course, each phase has its particular circumstances within the three cycles, but here we describe their common aspects. The specific features of each of the phases in the three cycles are presented in Chapters 6, 7 and 8.

The first phase of each research cycle includes the development of a hypothetical learning trajectory. The term 'hypothetical learning trajectory' is borrowed from Simon, who described how teaching takes place according to a mathematics teaching cycle that includes stating the goals of the teaching, developing a hypothetical learning trajectory and the actual teaching (Gravemeijer & Cobb, 2001; Simon, 1995). The most important part of the mathematics teaching cycle here is the hypothetical learning trajectory. The development of an HLT involves the assessment of the starting level of understanding, the end goal and the development of a chain of student activities that bring about a movement towards that goal. The student activities in this chain are designed to foster productive mental activities by the students, and are accompanied by the designer's description of why the instructional activity is supposed to work and what kind of mental development is expected to be elicited. This sequence of activities, motivations and expectations makes explicit the hypothetical learning process in terms of student activities and cognitive development.

For the design of the student activities, their motivation and the estimation of their mental effects, the designer makes full use of his domain specific knowledge, his repertoire of activities and representations, his teaching experience, and his view of the teaching and learning of the topic. After a field test by means of a teaching experiment, the HLT usually will be adapted and changed. These changes, based on the experiences in the classroom, start a new round through the mathematical teaching cycle, and, in terms of the design research method, the next research cycle.

The concept of the HLT may seem to suggest that all students follow the same learning trajectory at the same speed. This is not how the HLT should be understood. Rather than a rigid structure, the HLT represents a learning route that is broader than one single track and has a particular bandwidth. The students can go through it at different speeds. As a side remark we notice that the expression ‘learning trajectory’ is also used in a different context, namely as longitudinal strands across several grades (van den Heuvel - Panhuizen, 2001).

Because of its stress on the mental activities of the students and on the motivation of the expected results by the designer, the HLT concept is an adequate research instrument for monitoring the development of the designed instructional activities and the accompanying hypotheses within design research. It provides a means of capturing the researcher’s thinking and its development throughout the research and helps in progressing from problem analysis to design solution. Therefore, each of the Chapters 6, 7 and 8 will take the development of the HLT as its starting point for the description of the research cycle that is addressed.

How did we use the HLT concept in this research? The first phase of each cycle started with posing the problem, with analysing the concepts and with reflecting on possible solutions. The problem definition for the second and the third research cycle was based on the feed-forward of the previous cycle. The proposed solutions then were made concrete in the form of student activities that define the hypothetical learning trajectory. The HLT finally was condensed into a table that contains the chain of student activities that are supposed to lead to the next step in the learning process, the expected mental activity that it stimulates and the role of computer algebra in it.

Originally, Simon used the HLT for designing and planning short teaching cycles of one or two lessons. In our study, however, we developed HLTs for teaching experiments that lasted for 15 - 20 lessons. As a consequence, the HLT in this study comes close to the local instruction theory of the learning of the concept of parameter, and can be seen as a concrete embodiment of this local instruction theory (Gravemeijer, 1995). Also, this interpretation of the HLT is close to the scenario concept as it is used by Klaassen (1995). A second difference between Simon’s approach and ours is that Simon took a teacher’s perspective, whereas we take a researcher’s perspective.

## 2.4 Design of instructional activities

The preliminary design phase of the design research cycles includes the development of the HLT and the design of instructional activities. Of course, HLT development and design of instructional activities are closely related: the HLT guides the design of instructional activities, and the ordering of instructional activities affects the HLT. Choices made in the design process may lead to reconsidering the HLT. The expectations concerning the students’ mental activities that were established in the HLT are elaborated into specific key activities in the instructional materials. The HLT guides the design of the instructional activities, but choices made in the design process may lead to reconsideration of the HLT. In this section, we describe our design method and principles.

The design of instructional activities in this study included the development of student text booklets, teacher guides, solutions to the assignments, tests and software for use within the computer algebra environment. While designing these materials, choices and intentions were captured and motivated, to inform the teacher and to keep track of the development of the designer's insights. When the materials were about to be finalized, these aims and expectations were described at the task level. Key items that embodied the main phases in the HLT were identified by means of a prior coding system. This coding system reflected the relevant aspects of the intended learning process and was based on the conceptual analysis of the concept of parameter and on the HLT. This guaranteed the links between HLT and key items in the teaching materials, and was a means of verifying whether the intentions in the HLT were indeed realized in the teaching materials. Meanwhile the identification of key items guided the observations and prepared the ground for the data analysis. Finally, teacher guides as well as observation instructions were written, to make intentions and expectations clear to teachers and observers. During the design phase, products were presented to colleagues, teachers and observers. This led to feed-back that forced the researcher to be explicit about goals and aims, and that provided opportunities for improving the instructional activities.

While designing instructional activities, the key question is what meaningful problems may foster the students' cognitive development according to the goals of the HLT. Three design principles guided the design process: guided reinvention, didactical phenomenology and mediating models.

The design principle of *guided reinvention* involves reconstructing the natural way of developing a mathematical concept from a given problem situation. One way to do this is to try to think how you would approach a problem situation if it were new to you, or, as Gravemeijer phrases it, 'think how you might have figured it out yourself' (Gravemeijer 1994, p. 179). In practice, this is not always easy to do, because as a domain expert it is hard to think as though you were a freshman.

The second design principle, *didactical phenomenology*, was developed by Freudenthal (Freudenthal, 1983; Gravemeijer, 1994; Gravemeijer & Cobb, 2001). Gravemeijer explains it as follows:

> Didactical phenomenology points to applications as a possible source. Following on the idea that mathematics developed as increasing mathematization of what where originally solutions to practical problems, it may be concluded that the starting points for the re-invention process can be found in current applications.
> (Gravemeijer 1994, p. 179)

Didactical phenomenology aims at confronting the students with phenomena that beg to be organized by means of mathematical structures. In that way, students are invited to develop mathematical concepts (see Section 3.6). Meaningful contexts, from real life or ‘experientially real’ in another way, are sources for generating such phenomena (de Lange, 1987; Treffers, 1987a). The question, therefore, is to find meaningful problem contexts that may foster the development of the targeted mathematical objects. Streefland and van Amerom (1996) suggested that the context should be more than ‘dressing up the mathematics’, and that it should be appropriate for playing a metaphorical role in a bottom-up approach. The context should be perceived as natural and meaningful, and offer an orientation basis for vertical mathematization (see Section 3.6) and for the construction of algorithms (Treffers, 1993).

The last remark leads to the third design principle, the use of *mediating models* (Gravemeijer, 1994). In the design phase we try to find problem situations that lead to models that initially represent the concrete problem situation, but in the meantime have the potential to develop into general models for an abstract world of mathematical objects and relations. We come back to this model shift in Sections 3.6 and 4.5.

As far as the use of realistic contexts is concerned, in the design phase of the first and the second research cycle we supposed that the students in ninth grade had had enough experience with informal strategies in real-life contexts in the previous grades. We hoped that representations such as graphs and formulas in the computer algebra environment would be meaningful to them on the basis of this previous experience. We tried not so much to pay attention to real-life contexts, as to make the student experience the ‘world of the computer algebra device’ as a meaningful environment that is ‘experientially real’ to them. The work in the CAS would lead to the intended process of mathematization. Therefore, real-life contexts play only a limited role in the instructional activities. However, as the findings in Chapters 6 and 7 will indicate, these assumptions were not supported by the data of the first and second teaching experiment. Therefore, the design for the third research cycle included more real-life contexts as starting points.

## 2.5 Teaching experiments

The second phase of the design research cycle is the phase of the teaching experiment, in which the prior expectations embedded in the HLT and the instructional activities are confronted with classroom reality. The term ‘teaching experiment’ is borrowed from Steffe (Steffe, 1983; Steffe & Thompson, 2000). The word ‘experiment’ does not refer to an experimental group - control group design. In this section we explain how the teaching experiments were carried out; in particular, we pay attention to the data sampling techniques during the teaching experiments.

Before the start of the teaching experiments we spoke with the teachers and observers about the aims of the experiment, the teaching materials and the schedule of the teaching sequence. We preferred to have more observers than only the researcher himself: other observers might see different aspects, and discussing the observations and the observation method was useful for improving the internal reliability. In the classroom the observers participated by means of joining in classroom discussions and explanations. Also, they approached students with questions concerning key items, or answered questions from students.

The research questions share a process character: they concern the development of the understanding of the concept of parameter and the process of instrumentation of computer algebra. Therefore, we focused on data that reflected that process and provided insight into the thinking of the students. The main sources of data, therefore, were observations of student behaviour and interviews with students. The observations took place at three levels: classroom level, group level and duo level. Observations at classroom level concerned the classroom discussions, explanations and demonstrations that were audio- and videotaped. These plenary observations were completed by written data from students, such as handed-in tasks and notebooks.

Observations at group level took place while the students were working in pairs on the instructional activities. The observers often took the initiative to interview pairs of students about their answers to key items. As the whole class was too big to manage this way, we selected a number of student pairs in each class who would receive special attention. These selected pairs (in total about half of the students) were divided among the observers. Selection took place during the first lesson according to practical criteria such as the position of the pair in the classroom, the willingness to discuss their solutions, communicative skills and mathematical level. The questions that the observers posed during these mini-interviews had been prepared beforehand. The aim was to ask them about their solution process soon after they finished the task. These mini-interviews were audiotaped. In addition to this, the observers made field notes on an observation form that contained information that was not recorded, such as student notations or commands in the computer algebra environment. In practice, the mini-interviews on some occasions were guided so much by the observer that they had a learning effect. Also, the timing of the mini-interviews was important: if the key assignment was part of the homework, or had been discussed by the teacher, the mini-interviews lost their relevance. As the tasks got harder, students were tempted to see the observer more and more as an assistant teacher. It was difficult for the observer to reject that role, but in the meantime their help influenced the learning process. Although the students' questions were informative, sometimes the concept of the mini-interviews about key items required the observer to refrain from answering.

Observations at the duo level concerned one pair of students in each of the classes. This pair had been selected by the teacher such that the two students were neither very good nor very poor in mathematics, were willing to take this role, had a good attitude and were communicative and willing to talk about their work. The two stu-

dents were encouraged to discuss their work by having them share one computer algebra calculator. The screen of this calculator was videotaped.
The lessons were evaluated with the teachers. These discussions influenced the subsequent lessons, as is characteristic of design research micro-cycles. In particular, the organization of the next lesson and the content of the plenary parts were discussed. Also, decisions were taken about skipping tasks because of time pressure. Such decisions were recorded in the teaching experiment logbook.
For the teaching experiments we aimed at finding a regular school with teachers who were not specifically skilled in using technology in their teaching. This would enable generalization of the results to regular educational situations.

## 2.6 Retrospective analysis

The fourth and final phase of a design research cycle comprises the retrospective analysis. It includes the data analysis, reflection on the findings and the formulation of the feed-forward for the next research cycle. In this section we focus on the data analysis method of this study. As the main sources of data are observations and interviews, the data analysis concentrates on that.
The first step of the retrospective analysis concerned *elaborating on the data*. A selection from video- and audio-tapes was made by event sampling. Criteria for the selection were the relevance of the fragment to the research questions (understanding of the concept of parameter and instrumentation of computer algebra) and to the HLT of this teaching experiment in particular. Data concerning key items was always selected. This selection was protocolled verbatim. From the audiotaped mini-interviews, more than half of the recorded period was typed out. From the video-taped work of the pair of students, only a limited part of the registered time was typed out verbatim, as students often proceeded slowly while working on their own, and did not always explain their method. In such cases, the tapes were summarized into descriptions of machine manipulations and overall strategy. Also fragments of mini-interviews that were long-winded and unrelated to essential aspects of the study were summarized so that if any doubt were to arise they could be traced later. Long plenary explanations were summarized as well, unless there were interesting student interventions. Registrations where students were not working at all, or where organizational matters were discussed, were not elaborated on. The written work from the students was surveyed and analysed, especially the work on key items, tests and hand-in tasks. Results were summarized in partial analyses.
The first phase of the analysis consisted of *working through the protocols* with an open approach that was inspired by the constant comparative method (Glaser & Strauss, 1967; Strauss, 1987; Strauss & Corbin, 1998). Remarkable events or trends were noted and were confronted with the expectations and with the coding system that had been developed before the teaching experiment, based on the HLT and the instructional activities. The question now was whether the proposed coding system

was appropriate for covering the remarkable events and trends that were found by the open approach. Adaptations to the coding system and to the interpretation of categories were made to fine-tune this match.

The second phase of the data analysis was *the first round of coding*. In practice, it was necessary to memo decisions on specific categories of observations that were not foreseen, or on dealing with overlap between categories. The first round of coding took a lot of time; for example, previous codes needed to be revised in accordance with new decisions. After a while, the coding advanced faster and this indicated the completion of the coding system and its additional decisions.

The third phase of analysis concerned *looking for trends* by sorting events with the same code and analysing these categories. This led to the distinction of subcategories that needed to be coded accordingly. The findings were summarized for each of the categories and illustrated by prototypical observations. These preliminary conclusions were tested by surveying the data to find counter-examples or other interpretations, and by data triangulation: we analysed the other data sources, and in particular the written student material, to find instances that confirmed or rejected the preliminary conclusions. The analysis of the written materials often provoked a reconsideration of the protocols. The analysis was continued in this way until saturation, which meant that no new elements were added to the analysis and no conclusions were subject to change.

The fourth phase of the data analysis included *the second round of coding*: a value judgement was added to the codes. Was the observed student behaviour in the protocol correct, neutral or did it inhibit the progress of the task in question? The results of this coding process are presented in Chapters 9 and 10.

The coding process was partially *repeated by a second researcher* who was not involved too closely in the research, in order to achieve an inter-subjective agreement. The procedure for this was that the final coding system was explained to the second researcher. The second researcher then carried out a coding trial on a first part of the data, after which the two researchers discussed the results of this trial; the second researcher often needed further information on the sometimes implicit decisions that were made during the first coding process. Also, the second researcher's remarks sometimes led to corrections of the first researcher's codes. The two researchers worked through the first part of the data until they agreed on the coding. Then the second researcher coded a second part of the data, which was discussed again. Gradually, the agreement between the two researchers grew, and discussions were no longer needed. This process led to coding results that were attributed in a clear way, were not biased by the perception of the first researcher and contained fewer mistakes than would have been the case without a second researcher.

The next phase in analysing the data was the *interpretation of the findings* and the comparison with the preliminary expectations. Also, explanations for the differences between expectations and findings were developed. These conclusions and interpretations functioned as feed-forward for the formulation of new hypotheses for the next cycle in the research. During the data analysis phase, the decisions and findings were tracked in a chronological analysis logbook.

## 2.7 Research cycle arrangement

In this section we describe the arrangement of the design research macro-cycles in this study. The design of the study provides three research cycles, indicated by G9-I, G9-II and G10-II, and one intermediate cycle, called G10-I. The G stands for 'grade', and the 9 and 10 refer to the grade in which the teaching experiment was conducted. The I and II refer to the first and second cohort of students involved in the study. Fig. 2.2 visualizes this arrangement.

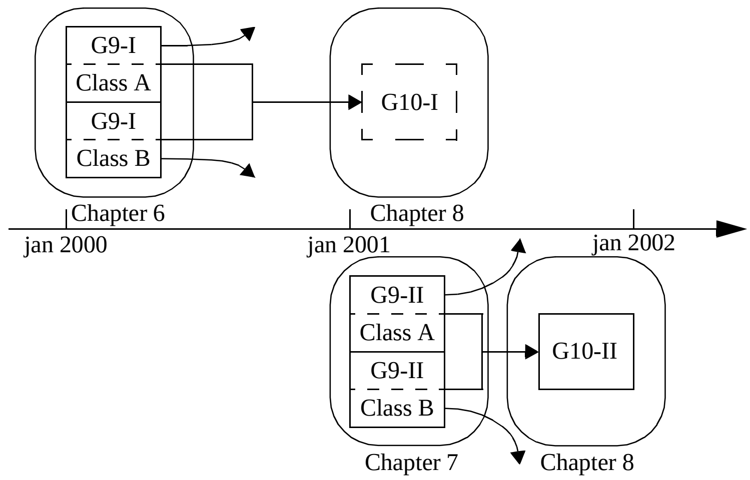



Figure 2.2 Arrangement of research cycles and teaching experiments

The arrows in Fig. 2.2 indicate that about half of the subjects from G9-I also participated in G10-I whereas the other half was no longer involved; the same applies to G9-II and G10-II. The three main cycles are discussed in Chapters 6, 7 and 8. The intermediate cycle G10-I is briefly addressed in Section 8.2. Below, we give more detailed information about the subsequent research cycles, and in particular about the teaching experiments.

### *The first teaching experiment G9-I*

The G9-I teaching experiment started in January 2000 in two ninth grade classes. The 53 students (14- to 15-year-olds) were at the highest level of secondary education, the pre-university level (in Dutch: vwo). Streaming according to interest had not yet taken place: future students of the exact sciences and those intending to study

languages or the social sciences were mixed up together. Therefore, the population consisted of pupils who were gifted in general, but not specifically in mathematics. The experiments ran for about five weeks with four mathematics lessons of 45 minutes each per week. The time for these teaching experiments, which did not fit well into the regular curriculum, was created by means of one extra mathematics lesson during half of the school year. This additional time was paid for by the research project. The students were not aware of the relation between the teaching experiment and the extra mathematics lesson during the second half of the school year.
During the experiments the students had a symbolic calculator, the TI-89, as computer algebra tool. The rationale for this choice is explained in Section 5.5. The students used the TI-89 both in school and at home.

### *The intermediate experiment G10-I*

One year later, the population of the G9-I research cycle had reached the tenth grade and had been streamed into a ‘society stream’ (in Dutch: maatschappij profiel), directed towards social studies and language, and an exact stream (in Dutch: natuur profiel), aiming at medical and exact studies. The G10-I experiment was carried out with the class of the exact stream of 14. All students but one participated in the G9-I teaching experiment, so the G10-I population consisted of those students from G9-I who were interested in mathematics and science. The G9-I students who opted for the social stream in the tenth grade were outside the scope of the study. The rationale for the choice for the exact stream lies in the perception that the concept of parameter is more relevant to them than it is to students in the society stream, and that computer algebra use fits their curriculum better.
In G10-I there were no facilities for extra mathematics lessons. This, in combination with the full programme for tenth grade mathematics and some loss of teaching time because of the teacher’s absence, led to having only five lessons available for the experiment, which was less than scheduled. Also, the teacher felt the need to stick close to the regular curriculum. Because of these constraints, the research purposes could not be dealt with as intended. Therefore, we consider G10-I as a short and meaningful experience, but not as a fully developed research cycle.

### *The second teaching experiment G9-II*

The G9-II teaching experiment was similar to G9-I. It involved two ninth-grade classes, with a total of 54 students. It concerned the cohort of students that came one year after the G9-I cohort. This experiment consisted of 19 lessons in class A and 22 in class B over a period of 5 weeks, and it was facilitated by extra mathematics lessons during the school year. The G9-II population was a mixture of students who would be joining the social stream and the exact stream.

*The third teaching experiment G10-II*

Just as in cohort I, the students of the G9-II population could choose the exact stream in the tenth grade. The 28 students that did so were involved in the G10-II teaching experiment. The other students from G9-II were no longer involved.
Contrary to the situation for G10-I, in G10-II we managed to find time for a series of 15 lessons entirely devoted to the research goals. As there were no extra lessons available, the students had to work through the regular curriculum during the rest of the school year at a somewhat higher speed.

All together, 110 students were involved in the teaching experiments: 53 in G9-I, 54 in G9-II and three students who entered the experiments in tenth grade, due to a failure to move on to the eleventh grade. Over 100 lessons were observed: 46 in G9-I, 41 in G9-II, 5 in G10-I and 15 in G10-II.

## 2.8 Design and methodology in practice

In this section we discuss some practical matters concerning the research design and the methodology. We address the choice of the experiment school and the teachers involved, the teaching styles and organization of a lesson, the classroom setting for video registration and an overview of the data.

*The choice of the experiment school and the involved teachers*

For the teaching experiment we chose a regular educational setting with facilities that were limited to the availability of computer algebra machines and extra lessons in ninth grade. The rationale for this choice was that it would enable generalization of the experiences to other secondary schools. The fact that teaching experiments performed in a ‘real-life’ educational setting may include noise, due to the large number of factors that influence the results, was taken for granted.
The teaching experiments took place in the years 2000 and 2001 at a school called ‘Werkplaats Kindergemeenschap’ (‘Child Community Workplace’) in Bilthoven. We were in contact with one of the teachers at this school, and through him we heard that there were more teachers willing to be involved in experiments. If we take into consideration the size of the experiments and the spread over several school years, a team of teachers who were willing to participate in experiments was an important factor; often, teachers who are interested in doing so have an isolated position within their school.
The Werkplaats Kindergemeenschap is a school in a small town not far from Utrecht. Some of the – mainly white – students live in Bilthoven, but many come from Utrecht. Apparently, a school outside the city is attractive to students and their parents, as is the particular pedagogical background of the school: it originated from a private initiative and describes its vision on education as follows:

> We are a school where the learning comes first. But learning for a certificate alone is not enough. Our results in national examinations are fine, but we want to give children more than that. The most important is the way in which the children, the workers, and the adults, the co-workers, treat each other. Students are taken seriously and are stimulated to give their opinion. Students learn to deal with adults and acquire self-confidence by means of taking part in discussions, giving presentations and carrying out projects. In the out-of-classroom activities such as sport, musicals and open stages, the children learn to co-operate and to appreciate each other's performance. This gives them an advantage in many respects at the start of a study at a university or a vocational college, because social skills such as collaboration, presentation, independence, initiative taking and being able to formulate your opinion are very important in higher education.
> (Retrieved 2 July 2002 from www.wpkeesboeke.nl, translation PD)

Our impression is that this vision on education did not influence the way the teaching experiments took place. The population is diverse and the students do indeed express themselves in the classroom, towards each other and towards the teacher.
Carrying out all the teaching experiments within one school had the advantage that the teachers could participate twice, and that students could be monitored during two subsequent experiments. The disadvantage could be that the school situation is too particular to allow for generalization. We have no indication that this is the case.
The two student cohorts that took part in this study were considered as weak cohorts by the teachers. Evidence for this was the low student participation in the exact stream in the eleventh and twelfth grades.

| Teaching experiment | Class | Teacher | Observers |
|---|---|---|---|
| G9-I | A | Teacher A | 3 |
| | B | Teacher B | 3 |
| G9-II | A | Teacher A | 2 |
| | B | Teacher B | 2 |
| G10-I | | Teacher C | 1 |
| G10-II | | Teacher D | 1 |

Table 2.1 Teachers and observers in the teaching experiments

The teachers who took part in the teaching experiments were those who normally taught the ninth and tenth grades, and who were willing to experiment. The two teachers who taught in G9-I both repeated the experiments in G9-II. Table 2.1 provides an overview the arrangement of teachers and observers. In G9-I there were two extra observers besides the researcher, and in G9-II one.

Teacher C, who was involved in the intermediate experiment G10-I, was the only one who was experienced in using graphing calculators and computer algebra software in his teaching. The other teachers had hardly any experience in using technology in their classroom teaching.
Beside the three teaching experiments G9-I, G9-II and G10-II, and the intermediate experiment G10-I, at the Werkplaats Kindergemeenschap, two other schools were involved in the study. In the autumn of 1999, pilot interviews about the concepts of variables and parameters were held with students in the ninth grade at St. Gregorius College in Utrecht. In the spring of 2000, a five-lesson teaching experiment on the concepts of variable and parameter was held in a ninth-grade class at J.S.G. Maimonides in Amsterdam using the Studyworks software package. Other project work (the project 'Algebra investigations in a digital learning environment' and the 'Teacher network TI-Interactive') provided a frame of reference, and in particular allowed for observing in other schools and with other computer algebra tool phenomena similar to those observed in this study.

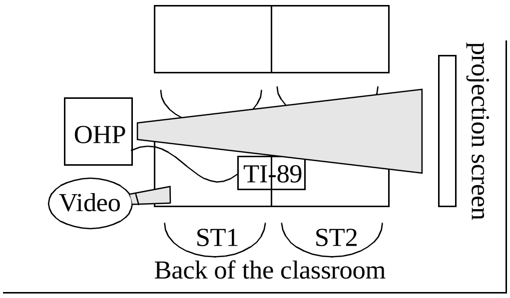
### *Teaching style and the organization of a lesson*

In spite of the differences between the participating teachers, the global organization of the lessons in the teaching experiments showed many similarities. Often, a lesson started with a plenary discussion of home-work assignments. The teacher reacted to questions from students, or discussed issues that he or she found relevant to stress. In some cases a global line for the student activities was sketched. The TI-89 viewscreen projection could be used for demonstrations by the teacher or by students. The second phase of the lesson often consisted of student work. The students usually worked in pairs, although they could choose to work individually. Exchange between pairs took place often. Solutions to the assignments were available, either in the student text books or at the front of the class. The third phase of most lessons consisted of feedback on the students' work, either through a plenary discussion or explanation that was guided by the teacher, or by students who presented their work.
However, there were differences in teaching styles between the participating teachers. Whole-class discussions or explanations were not held very often by teachers B and D. All teachers seemed to find it very important for the students to work in pairs and reserved much time for that. As a consequence, the whole-class exchange of opinions and findings took place only to a limited extent and data analysis concentrated on the observations of the students.

### *The classroom setting for video registration*

Video registration of the screens of the TI-89 symbolic calculator was used for two kind of observations: for capturing the work of the selected pair of students and for registering classroom demonstrations by the teacher or the students.

The classroom setting for this registration is shown in Fig. 2.3. One of the two machines of the selected pair of students was connected to a viewscreen on top of an overhead projector. A video camera registered the projection of the calculator screen, and the discussion between the two students. Because other students could see the projection screen as well, the selected pair was situated at the back of the classroom. In practice, sometimes students watched the screen of the selected pair, but this did not disturb the students' work.
The same arrangement was used for classical demonstrations. Often, the teacher stood at the blackboard while one of the students operated the machine (cf. the 'Sherpa-student', Guin & Trouche, 1999; Trouche, 2000). Each of the students had a computer algebra machine that could be connected to the viewscreen.



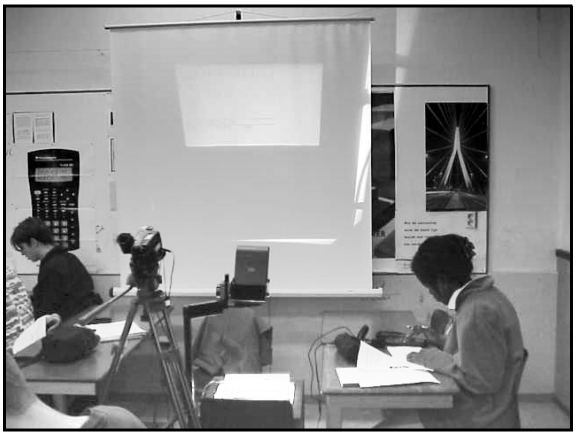



Figure 2.3 Classroom arrangement for video registration

*An overview of the data*

Several kinds of data were collected in each of the teaching experiments. Table 2.2 provides an overview.

| Data collection technique | G9-I | G9-II | G10-II |
|---|---|---|---|
| Pretest - posttest | + | + | − |
| Mini-interviews during lessons | + | + | + |
| Observation of classroom discussions | + | + | + |
| Permanent observation of pair of students | − | + | + |
| Students' written notebook materials | + | + | + |
| Students' written hand-in tasks | + | + | + |
| Observers' logbook | − | + | + |
| Post-interviews with students | + | + | − |
| Written evaluations by students | + | − | + |
| Post-interviews with teachers | + | + | + |

Table 2.2 Overview of data

Pretest and posttest data were collected in two of the three teaching experiments. The pretest served as an indication of the entrance level of the students. The posttest assignments from G9-II were matched to the pretest assignments in order to determine the progress during the teaching experiment. This approach averted the situation in G9-I, where the pretest and the posttest were similar. In this case the pretest discouraged the students because it was too hard.

We considered using software for the analysis of the qualitative data. Packages such as Atlas-TI, Nudist and Mepa support the coding, retrieving and sorting of qualitative data. After estimating the balance between required effort and expected benefits, we decided not to do so. Once the data were systematically entered in a word processor, the text editing software provided support for an important part of the functionality of the dedicated software. In particular, the search function could do most of the tasks for us.

Onno van Herwaarden (Universiteit Wageningen) carried out the second coding of the data for G9-I, Vera Emons (Katholieke Universiteit Leuven) did so for G9-II and Danny Dullens (Universiteit Utrecht) did so for G10-II.

# 3 The learning and teaching of algebra

## 3.1 Introduction

Understanding algebra is a central issue in the main research question. To elaborate on this, the present chapter concerns the learning and teaching of algebra. Together with Chapters 4 and 5, it forms the theoretical part of this thesis.

In this chapter we first address the different approaches to algebra that are distinguished in the literature. This categorization helps us to define our position (Section 3.2). Next, we identify the main difficulties of learning algebra (3.3). One of these difficulties concerns the process-object duality. This duality is relevant to our study because we conjecture that working in a computer algebra environment affects it and may reinforce the object view of algebraic formulas and expressions (3.4).

Dealing with algebraic language, which consists of symbols and formulas, is another important aspect of learning algebra. Therefore, we discuss the concept of symbol sense, which includes the ability to understand the meaning and the structure of formulas and expressions. The theories on symbolizing offer ideas on how to acquire these aspects of symbol sense (Section 3.5).

Notions on how a meaningful framework of mathematical relations is constructed are part of the theory of Realistic Mathematics Education. In Section 3.6 we discuss this theory and define our position for this study.

In order to be able to understand the educational context of the teaching experiments that are presented in Chapters 6, 7 and 8, we describe some characteristics. of algebra education in the Netherlands in Section 3.7.

## 3.2 Approaches to algebra

Algebra is a versatile and comprehensive topic within school mathematics. In order to get more insight into the structure of school algebra and the views that exist on it, we consider the different approaches to school algebra that are described in the literature. These approaches can be considered as a phenomenological categorization of algebra and will help us to clarify our position towards algebra in this study.

*Four approaches*

In the educational research literature, different approaches to algebra and to the learning of algebra are distinguished (Bednarz et al., 1996; Usiskin, 1988). These approaches don't form a hierarchy, a sequence or a level structure; rather, the categorization provides an overview of the different characteristics of school algebra. Although differences can be found, the following inventory of approaches to algebra is quite common:

- The problem-solving approach;
- The functional approach;

- The approach by generalization, pattern and structure;
- The language approach.

The *problem-solving approach* sees algebra primarily as a means of solving problems that are formulated in equations. The question is what value(s) of the variable, which plays the role of an unknown, fulfil(s) the required conditions.

The *functional approach* sees algebra primarily as the study of relations and functions, which explains the term 'functional'. Algebra, then, is a means to formulate and investigate relations between variables. This involves covariance and dynamics: how does a change in one variable affect the other? The variables have the character of changing quantities.

The *generalization approach* focuses on the generalization of relations, and the investigation of patterns and structures. We distinguish generalization *over* and generalization *to*. The latter involves transfer, but this study concerns generalization *over* classes of situations, graphs and formulas. Variables are generalized numbers.

The *language approach* stresses the language aspect of algebra. Algebra is a means to express mathematical ideas and for that syntax, symbols and notations are needed. The language approach to algebra views algebra as a system of representations, as a semiotic system (Drouhard, 2001). Variables in this view of algebra are merely symbols that do not refer to a specific context-bound meaning.

### *History of algebra*

A fifth approach to algebra that is often mentioned is the historical one (e.g. Bednarz et al., 1996). Some research studies take the historical development of algebra as a source of inspiration for the development of a learning trajectory (e.g. van Amerom, 2002). The rationale for this is that the historical development and the struggle of mathematicians throughout the centuries indicate where the difficulties of a particular topic lie. For example, Rojano (1996) uses the historical perspective to stress the role of problem-solving in the development of algebra and discusses the danger of a too rapid transition to symbolic manipulation.

The historical development can serve as a design heuristic in two ways: it can point out conceptual difficulties within a specific domain, and it can provide an outline of the learning trajectory. The learning trajectory can follow the global historical development, and historical examples and strategies can serve as contexts used in student texts (Harper, 1987).

### *Position in this study*

Distinguishing different aspects of algebra can be fruitful for the researcher, the designer and the teacher. In educational practice, however, these approaches cannot be separated, because one and the same problem situation often provokes algebraic activities from different approaches.

Let us illustrate this with a simple example. If we represent the price of a taxi ride by the formula $price = 5 + 2 \cdot distance$, we are in ‘functional mode’. However, quite naturally, the question arises how far we can get with 10 euros, so the problem-solving aspect comes in. If the prices quoted by several taxi companies are compared, generalization is a logical next step. To formulate the findings we use algebraic language. In short, the four approaches are often integrated. One can focus temporarily on one of the aspects, but by preference the approaches are interwoven.
In this study, we therefore choose an integrated approach. The point of departure is the functional approach: algebra as the study of relations, as a means to represent and investigate relations between variables that have a dynamic character. This point of departure is chosen because it links up with the Dutch curriculum for 12- to 16-year-olds and with the preliminary knowledge of the students in the ninth and tenth grades who are involved in the study. Quite soon, the functional approach will be combined with the problem-solving approach, which is natural, as was stated above. A next step is the generalization of relations and solutions by means of parameters. Algebra in this study is a study of relations that leads to problem-solving and generalization. Language aspects are important, particularly while working in a computer algebra environment that puts high demands on syntax and notation.
The historical development of the concept of parameter in this study is used to indicate the main conceptual difficulties involved (Section 4.3). However, we do not use the historical development as a guideline for the learning trajectory. The learning trajectory does not necessarily follow the historical line, and in our case the functional approach, which historically was developed later than the concept of the algebraic parameter, is the main approach in the preliminary education of the students involved in the study.

## 3.3 Difficulties with learning algebra

Algebra is a powerful and indispensable tool for doing mathematics because it ‘makes hard things simple’:

> There is a stage in the curriculum when the introduction of algebra may make simple things hard, but not teaching algebra will soon render it impossible to make hard things simple.
> (Tall & Thomas, 1991, p. 128)

The learning of algebra is not easy. Along the line of the difficulties mentioned in Section 1.2, we identify the following five difficult aspects of the learning of algebra:

1 The formal, algorithmic character of algebraic procedures that the student can not relate to informal and meaningful approaches;
2 The abstract level at which problems are solved, compared to the concrete situ-

ations they arise from, and the lack of meaning that the student attributes to the mathematical objects at the abstract level;

3 The need to keep track of the overall problem-solving process while executing the elementary algebraic procedures that are part of it;
4 The compact algebraic language with its specific conventions and symbols;
5 The object character of algebraic formulas and expressions, where the student often perceives them as processes or actions, and will have problems with the accompanying 'lack of closure' obstacle.

We now elaborate on items 1 - 4; the last item is addressed in Section 3.4.

*1. Informal and formal*

The students' first approach to algebraic problems is often a natural and informal one, related to the specific problem situation. In education, however, we often focus on developing more formal routine methods that are supposed to be automatized. Formal routine methods are efficient and, once mastered, free the students from re-inventing the solution method at each instance. According to Kindt, the transition from informal to formal algebra is a level jump that students find difficult to make (Kindt, 1980, 2000). The lack of time spent on the informal phase and on the students' schematization are responsible for this. Too soon, strategies are shortened, automatized and condensed into compact algebraic forms.

Once the students acquire some formal and routinized method, they often make errors while executing the algebraic procedures. They are unable to trace or correct their errors (e.g. Rosnick, 1981; Wagner, 1983; Wenger, 1987). Students who are able to perform routine algorithms at a formal level correctly, often show a limited idea of the meaning and are lost as soon as the problem situation is slightly changed. There is a lack of mathematical flexibility to adapt the problem-solving procedure to the new situation, unless the students are able to refer to the informal, natural method (Küchemann, 1981). This indicates that the development of routine formal methods is full of risks, but allows for standard problems to be dealt with efficiently. Freudenthal formulated the problem with automized formal methods as follows:

> I have observed, not only with other people but also with myself (...) that sources of insight can be clogged by automatism. One finally masters an activity so perfectly that the question of how and why is not asked any more, cannot be asked any more, and is not even understood any more as a meaningful and relevant question.
> (Freudenthal, 1983, p. 469)

Therefore, the first bottleneck in the learning of algebra is the tension between the informal and natural approach to algebraic problems and the formal, routinized algorithms. This issue is elaborated in Section 3.6 on Realistic Mathematics Education.

*2. Concrete and abstract*

The second bottleneck, which is related to the first one, concerns the need to jump back and forth between the concrete level of the realistic problem situation and the global, general level that is more abstract (Goddijn, 1979; Kemme, 1990; de Lange, 1987). On the one hand, the symbols and operations acquire their meanings in realistic problem situations; on the other hand it can be efficient to 'forget' this reality during the execution of the problem-solving strategy and think within the 'world of algebra', where the meaning of the objects is related to the algebraic rules and properties. This distinction can be personified by Descartes, who considered the connection between symbolizations and the setting from which they were derived as critical, and Leibniz, who stressed the use of symbolizations that were detached from their origin (Gravemeijer et al., 2000). As Malle (1993) indicates, the separation of meaning and form, of semantics and syntax, is one of the difficulties of algebra.

While learning algebra the student needs to be able to switch between those two perspectives, which is not easy, in particular when the algebraic framework has not acquired meaning in itself for the student. The second bottleneck, therefore, is the tension between concrete and abstract: algebra tends towards abstraction, but its meaning originates in concrete problems. If the student sticks to the concrete too much, the power of generalization and abstraction that algebra offers, will remain unappreciated. The ability to see this power requires the development of meaningful algebraic objects and operations in the world of algebra itself.

This point is also related to the theory of Realistic Mathematics Education and is discussed further in Section 3.5 on symbol sense and symbolizing, and in Section 4.5 on levels of understanding.

*3. Elementary procedures and overall strategy*

A third obstacle while learning algebra is the need to keep track of the overall problem-solving process while executing the elementary algebraic procedures that are part of it. Often students are able to perform isolated elementary algebraic techniques. In more complex situations, however, the execution of the subprocesses that the students do master, requires so much of the students' attention that they lose sight of the complete problem-solving process. They no longer have an overview of how the series of partial procedures forms a whole, integrated problem-solving strategy. This issue is elaborated in Section 5.4.

*4. Natural language and algebraic language*

The fourth difficulty of the learning of algebra is the language aspect. As Philipp (1992) puts it, algebra is syntactically strong but semantically weak. Algebraic language has many symbols, conventions and notations that are compact, powerful and unambiguous; however, it is difficult for students to learn and differs from natural language in many respects (Pozzi, 1994). Therefore, the learning of the algebraic language is a bottleneck for students. The language issue is elaborated in Section 3.5 on symbol sense and symbolization.

This inventory of difficulties of the learning of algebra is not exhaustive; it does, however, contain the main issues that are addressed in this study. The main difficulties addressed in this study are issues 1, 2 and 5. Our ambition is to help students to develop meaningful algebraic objects and operations that are not only related to real-life contexts, but are also embedded in a meaningful framework of mathematical relations. Furthermore, formulas and expressions are to be considered as objects that derive meaning from this framework of algebraic relations, and not only as processes or as prescriptions for numerical calculations. This raises the issue of the process-object duality, which we will now address.

## 3.4 Process and object

One important difficulty in learning algebra is the process-object duality. A mathematical concept often has two faces: an operational process side and a structural object side. For students, the process aspect initially dominates the concept. A mature understanding, however, includes the object side and the flexibility to switch between the two views. This process-object development is hard to realize.
In this study we consider the process-object duality in algebraic formulas and expressions, because we conjecture that working in a computer algebra environment affects it and may reinforce the object view of algebraic formulas and expression. This requires overcoming the lack of closure obstacle.
The following two quotations indicate that the construction of mathematical entities or objects is the 'heart' of mathematics:

> (...) the whole of mathematics may therefore be thought of in terms of the construction of structures, . .. mathematical entities move from one level to another; an operation on such "entities" becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by "stronger" structures.
> (Piaget, cited in Dubinsky, 1991, p. 101)

> What characterizes the level structure may be expressed in a few words by saying that the operational matter of a lower level may become subject matter on a higher level.
> (Freudenthal, 1962, p. 27)

However, students often focus on the operational aspects of mathematical concepts. Several theoretical notions have been developed to deal with this duality and to describe the transition from process to object. The most important ones are reification, encapsulation and the sequence action-process-object-schema (APOS) and procept.

*Reification, encapsulation and procept*

A common starting point of these three ideas is the notion that mathematical concepts usually have two sides: an operational side, which concerns the process aspect and the actions, and a structural side, which stresses the static, object character. In short, the concept can be considered as both a process and an object.

For example, a function such as $f(x) = x^2 - 7$ has a process aspect that may consist of the calculation of function values, the drawing of its graph or the solving of equations such as $f(x) = 9$. The function is a prescription for a calculation process. Meanwhile, this function also has structural aspects: it is a member of the family of quadratic functions, represented by $g(x) = a \cdot x^2 + b \cdot x + c$, it can be submitted to processes such as substitution or differentiation, and it can be the solution of a differential equation.

According to Sfard, concept development often starts with the operational side: by doing something (Sfard, 1991; Sfard & Linchevski, 1994). Gradually, by means of practice and by the need to apply higher order processes to the concept, the process character is complemented by the object aspect. This ‘objectification’ of processes is called *reification*, ‘thing-making’:

> To sum up, the history of numbers has been presented here as a long chain of transitions from operational to structural conceptions: again and again, processes performed on already accepted abstract objects have been converted into compact wholes, or reified (from the Latin word res - thing), to become a new kind of self-contained static constructs.
> (Sfard, 1991, p. 14)

The notion of reification comes close to what Dubinsky calls *encapsulation* (Cottrill et al., 1996; Dubinsky, 1991, 2000). Dubinsky refers to Piaget when he states that encapsulation is an important step in reflective abstraction. Encapsulation is the transition from process to object within the comprehensive chain action-process-object-schema. Encapsulating a concept allows it to be submitted to processes of a higher order. The encapsulation or reification is crucial for level raising and abstraction in mathematical thinking, as it allows concepts to be organised into structures that can be organized into higher level structures, and so on.

The flexibility to shift between the process and the object perspective is indispensable for mature mathematical thinking (Graham & Thomas, 2000; Gray & Tall, 1994; Tall et al., 2000; Tall & Thomas, 1991). This is expressed in the term ‘*procept*’, a contamination of ‘process’ and ‘concept’ (Gray & Tall, 1994). The idea here is that one needs to be able to combine both perspectives adequately according to the problem situation. Gray and Tall (ibid.) describe a procept as a triad of process, object and symbol:

> An elementary procept is the amalgam of three components: A process which produces a mathematical object, and a symbol which is used to represent

> either process or object. (...) A procept consists of a collection of elementary procepts which have the same object.
> (Gray & Tall, 1994, p. 121)

The power of proceptual thinking and of the sequence process-object-higher order process lies in the so-called mental compression that it allows:

> Proceptual thinking is characterized by the ability to compress stages in symbol manipulation to the point where symbols are viewed as objects that can be decomposed and recomposed in flexible ways.
> (Gray & Tall, 1994, p. 132)

The process-object duality is still in discussion. We agree with Artigue (1996) that there is a qualitative gap between the process level and the object level of conceptualization. For example, students have difficulties in perceiving equivalent functions as identical if their calculation processes are different, such as $f(x) = 3 \cdot (x - 2)$ and $g(x) = 3 \cdot x - 6$. According to Artigue, it is hard for students to surpass the process level and to integrate the process and the object perspective (Artigue, 1996). For the purpose of observing reification, Dörfler's comment is useful. He states that reification has a unifying aspect, and that it can be expressed by language (e.g. by using nouns for the objects) and by graphical or diagrammatic means, such as drawing ovals around objects (Dörfler, 2002). Noss and Hoyles argue that the theory of reification in its present form leads to the inescapable conclusion that the process of understanding is a one-way, hierarchically-structured, uphill struggle (Noss & Hoyles, 1996). We agree that the reification view is a hierarchical one that may tend to neglect the forth-and-back movements that are involved in developing a relational framework. However, for the purpose of this study we think the reification theory offers an appropriate means to investigate the development of symbol sense and insight into algebraic formulas and expressions.

*The reification of formulas and expressions, and the lack of closure obstacle*

The reification of algebraic formulas and expressions is important for the learning of algebra, and is related to the lack of closure obstacle that is described in research literature on the learning of algebra (Collis, 1975; Küchemann, 1981; Tall & Thomas, 1991). While learning algebra, students need to extend the process view of algebraic expressions and formulas with an object view: contrary to the situation in arithmetic, in algebraic expressions there is often no process to be carried out in algebra. The inability to deal with this is called the lack of closure obstacle. This means that students feel uncomfortable as long as an expression or variable has no numerical value and contains operators, so that the process is not closed. This can prevent the students from proceeding. To them, the job is not finished. This is related to the expected answer obstacle, which concerns the expectation of numerical answers. Tall and Thomas explained the lack of closure obstacle as follows:

> It [the expected answer obstacle] causes a related difficulty, which we term the lack of closure obstacle, in which the child experiences discomfort attempting to handle an algebraic expression which represents a process that (s)he cannot carry out.
> (Tall & Thomas, 1991, p. 126)

Although algebra sometimes is considered as generalized arithmetic, many researchers stress the differences between arithmetic and algebra (van Amerom, 2002; Charbonneau, 1996; Mason, 1996; Wagner, 1983). The process-object duality and the lack of closure obstacle can help to clarify the differences, and thereby the 'didactical cut' between arithmetic and algebra (van Amerom, 2002; Filloy & Royano, 1989). An important aspect of this transition is that the operation view that dominates the field of arithmetic is extended with the process view in algebra. In order to allow the flexibility between the process and the object character, algebraic notation is often ambiguous, which is not the case in arithmetic. For example, the + sign in arithmetic refers to the process of addition. Many students tend to consider the + in 3+5 as a cue to perform the addition and will start the calculation. The + sign is procedural, operational and asymmetric: adding 5 to 3 is an action that differs from adding 3 to 5. In algebra, on the other hand, the + sign has a more ambiguous meaning (Freudenthal, 1962, 1983; Tall et al., 2000). Calculating $x$+3 or $a+b$ is not possible unless the values of the variables are known. As a result, $a+b$ is only meaningful if it signifies an algebraic object to the student and if the lack of closure obstacle is overcome. Then adding ($a+b$) and ($a+2b$) becomes possible. The + sign in algebra has an operational but also a structural, relational and symmetric character.
Similar remarks can be made for the = sign. In 3+5 =, the = signifies the process of calculating the result. In algebra, it can be that as well. For example, in $(x+5)\cdot(x-3) =$, the = sign may be a cue for a process, in this case expanding the parentheses. In $\sin(2x) = \cos(x)$, the = sign may ask for the process of solving the equation. In $a+b=b+a$, however, the = sign denotes the commutative property, and in $\sin(2x) = 2\cdot\sin(x)\cdot\cos(x)$ it indicates equivalence (Hoogland, 1995). As Graham and Thomas argue, an arithmetic expression asks for a process, which is often not the case for an algebraic expression (Graham & Thomas, 2000). The transition from arithmetic to algebra therefore involves the reification of processes into objects, in other words the extension of the process view of formulas and expressions with an object perception of these algebraic entities, and overcoming the lack of closure obstacle.

### *Process and object in this study*

In this study, the process-object issue is relevant to the understanding of formulas and expressions. The reification of these concepts is important for success in alge-

bra, and for the understanding of the concept of parameter in particular.
The study aims at facilitating students to perceive an expression or a formula not only as a calculation prescription but also as an algebraic object, and at helping them to overcome the lack of closure obstacle. Students should accept expressions and formulas as solutions of equations, as outcomes of processes that can be submitted to higher order processes such as substitution (Freudenthal, 1983). Reification of expressions and formulas is a prerequisite for dealing with them efficiently in the computer algebra environment; on the other hand, the confrontation with expressions as solutions of parametric equations in the CAS and its substitution may stimulate the reification process (Section 5.5). The use of a noun to indicate an expression and of descriptive rather than action language may evince this reification, as will overcoming the lack of closure obstacle and the acceptance and further processing of expressions as solutions to equations.
Let us make this concrete by means of an example. Suppose the task is to find the equation of a line through the point (0, -2) that 'touches' the parabola with equation $y = x^2 - 4x + 5$ (see Fig. 3.1).

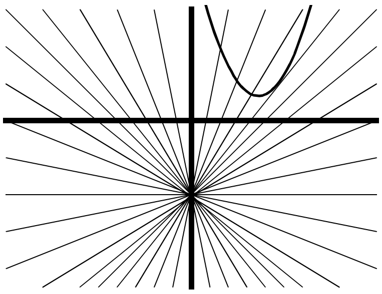


Figure 3.1 Which line 'touches' the parabola?

To tackle this problem, one can solve the equation $a \cdot x - 2 = x^2 - 4x + 5$ with respect to *x*. The solution consists of two expressions in *a*, $\frac{1}{2}a + 2 + \frac{1}{2}\sqrt{a^2 + 8a - 12}$ and $\frac{1}{2}a + 2 - \frac{1}{2}\sqrt{a^2 + 8a - 12}$ that have to be acceptable to the student as objects that represent the solution. Next, the two objects need to be equal, because only one solution is required for the tangent.
Therefore, solving the equation $a^2 + 8 \cdot a - 12 = 0$ with respect to *a* yields the two (!) solutions for *a* (Fig. 3.2). This example shows that reification of algebraic formulas is important, because they are treated as objects in the problem-solving procedure. A second issue in this example is the shift of roles that the parameter plays. This will be addressed in Chapter 4. The reification of formulas and expressions is related to the reification of the function concept. Overcoming the lack of closure does not imply that the object *x*+3 is considered as a function; reification of function

involves more aspects (Slavit, 1997). The function concept is so versatile that its reification is an issue that we find too comprehensive to deal with within the scope of this study.

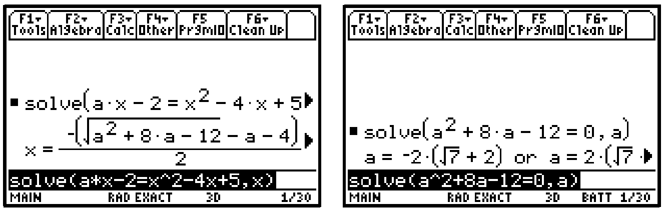



Figure 3.2 Solution procedure using the TI-89

Therefore, we concentrate on reifying formulas and expressions, which includes overcoming the lack of closure obstacle and seeing expressions and sub-expressions as entities. A second reason to focus on the reification of formulas and expressions is the availability of procedures in the computer algebra environment that may stress the object character of the algebraic expressions, such as a generic solver and a command for algebraic substitution.

The process-object duality links up with Van Hiele level theory, which will be discussed in Section 4.5. In fact, the second Van Hiele level concerns the reflection on processes of the first level that thereby get an object character (Gravemeijer, 1994).

## 3.5 Symbol sense and symbolizing

The reification of expressions and formulas is an important issue in the learning of algebra. Related to this is the ability to understand the meaning and structure of expressions and formulas. This is part of the general concept of symbol sense. After a discussion of symbol sense we address theories of symbolizing, which offer ideas on how to acquire these aspects of symbol sense.

### *Symbol sense*

The acquisition of symbol sense is an important aspect in the learning of algebra (Arcavi, 1994; Fey, 1990; Zorn, 2002). The term ‘symbol sense’ is not defined precisely, but refers to the ability to give meaning to symbols, expressions and formulas and to have a ‘feeling’ for their structure. This ability is not self-evident. It relates to the notational aspect of algebra:

> Mathematical notation is a wonderful and powerful tool. It is also subtle and difficult to learn. When we have mastered it, we often forget just how hard it was and just what went into the development of our understanding.
> (Schoenfeld & Arcavi, 1988, p. 426)

Arcavi (1994) considers symbol sense as the analogy of number sense. He distinguishes several aspects of symbol sense, such as the skill to set up a formula, the ability to recognize equivalent expressions and to see how they may ‘tell a different story’. Arcavi states that symbol sense is essential for success in algebra, and for mathematics in general: ‘Symbol sense is the algebraic component of a broader theme: sense-making in mathematics’ (Arcavi, 1994, p. 32).
Arcavi refers to the example that we already mentioned in Section 1.2, the equation $v\sqrt{u} = 1 + 2v\sqrt{1 + u}$ (Gravemeijer, 1990; Wenger, 1987). Although this equation can be seen as $v \cdot A = 1 + 2v \cdot B$, students usually miss the ‘Gestallt view’ to notice the linearity in *v*. They are not able to see the sub-expressions $\sqrt{u}$ and $\sqrt{u+1}$ as objects, as entities. They do not have the ‘global substitution principle’ at their disposal that would make them consider $\sqrt{u+1}$ as a ‘thing’ that they can move without caring about its content. Instead, they perceive the square root signs as a strong cue that asks for manipulation in order to get rid of them. Such a global look at sub-expressions can be stimulated by putting ‘tiles’ on or ovals around the objects so that the above equation can be rephrased as $v \cdot \square = 1 + 2 \cdot v \cdot \bigcirc$ (Freudenthal, 1962, Team W12-16, 1992). Clearly, this kind of symbol sense is related to the reification of expressions and formulas that was addressed in the previous section.
A similar example can be found in Wagner, Rachlin and Jensen (1984). The question ‘What is the value of $(2z + 1)/2$ when $5 \cdot (2z + 1) = 10$’ leads to efforts to calculate *z*. However, a global view of the expression indicates that this is not necessary; just considering $2z + 1$ as an object, a ‘Gestallt view’, would do.

The two examples illustrate the importance of being able to ‘see through’ the structure of a formula or expressions and to be able to see sub-expressions as entities. We conjecture that this is easier to achieve in cases where the formulas or expressions emerge from contexts that give meaning to them. Therefore, we confine the interpretation of symbol sense in this study to the understanding of the meaning and structure of algebraic formulas and expressions. This links up with Zorn’s description of symbol sense:

> By symbol sense I mean a very general ability to extract mathematical meaning and structure from symbols, to encode meaning efficiently in symbols, and to manipulate symbols effectively to discover new mathematical meaning and structure.
> (Zorn, 2002, p. 4)

This interpretation of symbol sense comes close to the concept of structure sense, which focuses on abilities such as using equivalent structures of an expression flexibly and creatively (Hoch, 2003; Linchevski & Livneh, 1999). An important question is what kind of symbol sense appropriate use of computer algebra requires (Pierce & Stacey, 2002).

*Symbolizing*

Algebraic expertise in the form of symbol sense is indispensable for the learning of mathematics. But how do students acquire it?

According to the representational view of the learning of mathematics, knowledge is tied up in external representations or inscriptions that are supposed to bring about the construction of internal, mental representations by the students (Cobb, 2002; Cobb et al., 1992). The problem with this conception is, however that the use and interpretation of the external representations already require a mental frame of reference and an internal representation, which is part of what needs to be developed by means of the external representations. This problem, a kind of 'chicken-egg problem', is the so-called learning paradox (Bereiter, 1985).

Kaput (1994) suggests an exploratory approach that aims at exploring conventional mathematical symbolizations in commission with experientially real settings. In such settings technology can play a major role, as is the case in Kaput's example of the MathCars. The technology can bridge the gap between the 'island of formal mathematics and the mainland of authentic experience'. Nemirovsky (1994) considers the appropriation of symbols from a sociocultural perspective. He distinguishes the symbol system from symbol use: the symbol system is a rather static, non-referential object, whereas symbol use is a meaningful, situated activity that also involves a process of reasoning.

In contrast to the representational view, theories on symbolizing take as their point of departure the reflexive relation between the development of symbols, models and notations on the one hand, and the construction of meaningful mental objects on the other (Cobb et al., 1997; Gravemeijer et al., 2000; Meira, 1995). The process of symbolizing concerns the simultaneous and interactive development of symbols and meaning. In this signification process, the student creates, uses and improves symbolizations in relation to the development of a meaningful framework of mathematical relations (Nelissen, 1998). However, for this process to take place the students must be confronted with phenomena that can foster this reflexive co-construction.

The perspective of symbolizing is relevant for this study because the learning of algebra involves the joint development of symbols and meaning. Furthermore, the relation with the theory of instrumentation, which is discussed in Section 5.6, is interesting: the instrumentation theory also concerns the dialectic relation between mental objects and external representations, but focuses in particular at representations that are not developed by the students themselves, but exist in the IT microworld.

## 3.6 RME and the learning of algebra

The domain-specific instruction theory of Realistic Mathematics Education (RME) provides ideas on how a meaningful framework of mathematical relations and objects is developed. In this section we first describe some of the key elements of this theory. Then we focus on RME and algebra. Finally, the role of RME in this study is explained.

### *Key elements of RME*

The principles that underlie RME are strongly influenced by Freudenthal's concept of 'mathematics as a human activity'. Freudenthal felt that students can not be considered as passive recipients of ready-made mathematics; rather, education should guide the students towards reinventing mathematics by doing it themselves. Although a clear-cut definition of RME is not easy to provide, the following characteristics are crucial for its role in this study:

*a* Guided reinvention and progressive mathematization
*b* Didactical phenomenology
*c* Horizontal and vertical mathematization
*d* Emergent modelling.

*a Guided reinvention and progressive mathematization*
According to the reinvention principle, students should be given the opportunity to experience a process similar to that by which a given mathematical topic was invented (Freudenthal, 1973). Thus a route has to be designed that allows the students to develop 'their own' mathematics. This process, however, needs guidance from the teacher, to help to further develop sensible directions, to leave 'dead-end streets' and to ascertain convergence towards the common standards within the mathematical community.
The point of departure for this process are the informal strategies that students come up with that gradually develop into more formalized methods. Mathematical knowledge is developed by progressively formalizing informal but meaningful strategies (Gravemeijer, 1994). This is called progressive mathematization. It includes mathematizing procedures and shortening them under the influence of context problems (Treffers, 1987a, 1987b). Typical of progressive mathematization is that students in every phase can refer to the concrete level of the previous step in the mathematization process and infer meaning from that.
*b Didactical phenomenology*
How can we arrange guided reinvention through progressive mathematization? Didactical phenomenology can serve as a heuristic for designing student activities that encourage students to develop their strategies. The principle of didactical phenomenology was developed by Freudenthal (1983). Freudenthal distinguishes the 'nooumenon' – the thought object – and the 'phainomenon', the phenomenon. Didactical phenomenology concerns the relation between thought object and phenomenon from the perspective of teaching and learning. In particular, it addresses the question how mathematical 'thought objects' can help in organizing and structuring phenom-

ena in reality.
The word 'reality' in this description can be interpreted in two ways. First, it can refer to real-life contexts that offer opportunities for concept-building, model-building, application and exercising (de Lange, 1987). However, reality is not synonymous with real life. It can also refer to mathematical situations that students experience as realistic. The essential point here is the word 'experience'. Therefore, Gravemeijer (1999) speaks about 'experientially real' situations, which can refer both to real life and to mathematics.
For example, if students have developed a mathematical reality in which linear functions are meaningful objects, an assignment that starts with 'A linear function *f* has the property...' may be perceived as realistic. Essential in the word realistic, therefore, is that the activities and concepts involved are meaningful and natural to the students, no matter whether the meaning is derived from a real-life situation, from mathematics or from another topic. Didactical phenomenology, therefore, is the study of the way in which phenomena can be organized by means of specific mathematical activities or concepts.

*c Horizontal and vertical mathematization*

We see mathematizing as the organization of a kind of reality with mathematical means. In the previous section we stated that the word 'reality' can refer to real life in the sense of a tangible reality as well as to the experientially real world of mathematics.
Within RME a distinction is made between horizontal and vertical mathematization. The following quotations explain this difference.

> Thus, through an empirical approach - observation, experimentation, inductive reasoning - the problem is transformed in such a way that it can be approached by strictly mathematical means. The attempt to schematize the problem mathematically is indicated by the term 'horizontal' mathematization.
> (...)
> The activities that follow and that are related to the mathematical process, the solution of the problem, the generalization of the solution and the further formalization, can be described as 'vertical' mathematization.
> (Treffers, 1987a, p. 71)

As we see, horizontal mathematization concerns organizing, translating, and transforming realistic problems into mathematical terms, in short, mathematizing reality. Vertical mathematization concerns reflection on the horizontal mathematization from a mathematical perspective, in short, mathematizing the mathematical activities and developing a framework of mathematical relations (see Fig. 3.3).Instruments that can be helpful for vertical mathematization are models, schemes, symbols and diagrams (Treffers, 1987a). In realistic mathematics education, horizontal and vertical mathematization should complement each other. De Lange (1987) states that the horizontal aspect does not necessarily precede the vertical component. As a consequence, mathematization can follow different routes.

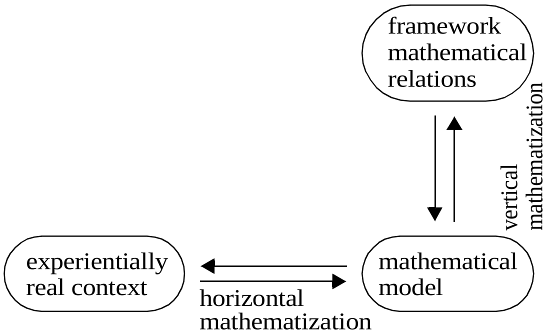



Figure 3.3 Horizontal and vertical mathematization

*d Emergent modelling*

Above, we described progressive mathematizing as the formalizing and schematizing of informal problem-solving strategies. Horizontal and vertical mathematization were distinguished. As a means of supporting the process of progressive mathematization in which a new mathematical reality develops, Gravemeijer proposes the instructional design heuristic of emergent modelling. According to this approach, a model may play different roles during different levels of activities (Gravemeijer, 1994, 1999; Gravemeijer et al., 2000).

A model initially is context-specific: it refers to a meaningful problem situation that is experientially real for the student, and is a model *of* that situation. Then, through working with the model, the model gradually acquires a more generic character and develops into a model *for* mathematical reasoning that is possible because of the development of new mathematical objects in a more abstract framework of mathematical relations that the model starts to refer to. The development from *model-of* into *model-for* is elaborated by Gravemeijer into a four-level structure (see Fig. 3.4) that represents levels of mathematical activity.

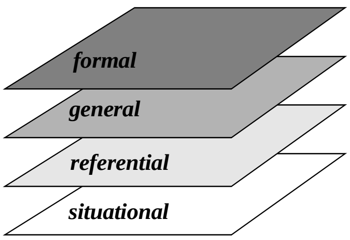



Figure 3.4 Four levels of mathematical activity (Gravemeijer, 1994, 1999)

These four levels are described as follows:

> *Level 1:* Activity in the task setting, in which interpretations and solutions depend on understanding of how to act in the setting (often out of school settings)
> *Level 2:* Referential activity, in which models-of refer to activity in the setting described in instructional activities (posed mostly in school)
> *Level 3:* General activity, in which models-for make possible a focus on interpretations and solutions independently of situation-specific imagery
> *Level 4:* Reasoning with conventional symbolizations, which is no longer dependent on the support of models-for mathematical activity
> (Gravemeijer et al., 2000, p. 243)

The model-of/model-for terminology, which was first used by Streefland (1985) in a slightly different sense, indicates how a model becomes the basis of mathematical reasoning that is not directly related to the original context; rather, it becomes a referential base for the level of formal mathematics.

#### *RME and the learning of algebra*

What is the meaning of the presented key elements of RME for the learning of algebra? The principle of didactical phenomenology raises the question what phenomena may lend themselves for organization by means of algebra. The answer to this links up with the approaches to algebra that were described in Section 3.2. First, the functional approach to algebra allows for the organization and investigation of quantitative relations with algebraic formulas. Second, the problem-solving approach offers algebraic equations to solve questions concerning unknown values effectively. Third, algebraic expressions can be used to describe and generalize patterns, and solution methods can be generalized by using parameters, as is suggested by the generalization approach. Algebra offers a compact language and powerful general methods to deal with these phenomena efficiently. The didactical phenomenology now asks for problem situations that naturally lead to such algebraic mathematizations.

The concept of guided reinvention suggests that the students will be given opportunities for developing informal problem-solving strategies, and that guidance is available during the process of progressive schematizing towards conventional and efficient algebraic methods.

The notion of horizontal and vertical mathematization and that of emerging modelling contribute to understanding the relation between context-bound and context-free work. Translating a realistic problem situation into algebraic models such as formulas, equations and graphs, has a horizontal character. Vertical mathematization concerns the mathematization of the mathematical activities in the context, and results in the construction of a relational framework with algebraic objects as junctions that infer their meaning from algebraic properties and procedures. If procedures are routinized, the meaning of the procedures and objects is no longer inferred from the original real-life problem situation. Instead, the model refers to the more general level of the algebraic relational framework. This once again shows the difficult relation between concrete versus abstract that was addressed in 3.3 and was illustrated by the positions of Descartes and Leibniz: the meaning of symbols is often related to a concrete problem situation by means of horizontal mathematization. However, in order to deal with the algebraic translation of the problem it is efficient to infer the meaning of the symbols and procedures from the algebraic world that results from vertical mathematization.

> In fact, from one point of view, this is one of the strengths of symbols – they enable us to detach from, and even "forget", their referents in order to produce results efficiently.
> (Arcavi, 1994, p. 26)

An algebraic model can be a formula, an expression, a set of equations. Such a model initially refers to a concrete problem situation; within this context, informal strategies are developed. Later, routine procedures emerge as methods that can be applied to classes of situations. This brings about generalization, a shift of attention from the context to the mathematical relations involved. This way, a framework of mathematical relations is constructed: the algebraic models no longer refer to original context but to the algebraic world of relations, rules and properties. One of the goals of algebra education is that the students develop a meaningful mental 'world of algebra'.

*RME in this study*

In this study the theory of RME is a frame of reference that is constantly present in the background. In three ways, its role is more evident: RME in the design phase of the research cycles, RME to make explicit the aims of the study, and RME as a means to characterize the use of computer algebra.

In the design phase of the research cycles the RME principles of didactical phenomenology and guided reinvention are used as design heuristics in the research phase of developing the hypothetical learning trajectory and designing instructional activities.

The second point – RME to make explicit the aims of the study – refers to the notions of horizontal and vertical mathematization, and emergent models that can be used to characterize the focus of the study. As we describe in Section 3.7, algebra education in the seventh and the eighth grade in the Netherlands uses mainly 'models-of' at the referential level and pays a lot of attention to horizontal mathematization. The concrete experiences that the students have offer opportunities for progressive mathematization and vertical mathematization that can lead to developing a meaningful framework of algebraic objects, relations and procedures, and to routinizing such procedures. However, in grades 10 - 12, especially in the pre-university exact stream, students often perform algebraic procedures in a meaningless and erroneous way. Therefore, this study focuses on the development of general and formal models, on vertical mathematization, and on the generalization of relations, procedures and solutions using parameters in particular. Reinvention plays a role here, too: from specific cases the student is supposed to discover and formulate the general aspects. In its third role, RME is a means to characterize the use of computer algebra. A computer algebra tool has an abstract and formal character, and embodies a ‘microworld’ of mathematical objects, relations and procedures without references to real life outside this environment. These characteristics and the CAS’s strict requirements concerning syntax raise the question of whether the use of computer algebra is appropriate for reinvention and symbolizing, and if it can be perceived as an experientially real environment (Drijvers, 2000). Does computer algebra use leave enough room for the referential level? This issue is elaborated on in Section 5.4.

## 3.7 Algebra education in the Netherlands

In this section we situate the study in the setting of algebra education in the Netherlands. This enables the reader to position the proposed hypothetical learning trajectory and the student materials that are described in Chapters 6, 7 and 8, in their context.

In the early 1990s the algebra curriculum in the Netherlands for grades 7 to 9 was changed considerably. The reason for this was that students were having difficulties with algebra, and manipulated symbols in a meaningless way. Algebraic formalism was an obstacle and students were not flexible with algebraic expressions and solution procedures. The team developing the new curriculum (Team W12-16, 1992) chose a functional approach in which relations are represented in different ways:

- in problem situation descriptions
- in tables
- in graphs
- in formulas

The aim was that together these representations would develop into an integrated whole. This approach fits with the RME ideas that integrated and flexible knowledge is relevant, that concrete problem situations are the starting point, and that attention is needed for the progressive mathematizing of informal strategies.

As a result of this approach, the introduction of algebra during the first years of secondary education (grades 7, 8 and 9; i.e. 12- to 15-year-old students) takes place in a careful way. Much attention is paid to the exploration of realistic situations at the referential level, to the process of horizontal mathematization, to the translation of the problem situation into mathematics and to the development of informal problem-solving strategies. Variables, for example, are often labelled with words that have a

direct relationship with the context from which they are taken.
These concrete experiences can be the basis for entering the general level and for vertical mathematization, formalization and abstraction. However, this is often delayed in grades 7 - 9. As a consequence, algebraic skills such as formal manipulations are developed relatively late and to a limited extent. The relational framework of algebraic objects, properties and procedures is not yet developed in a way that gives meaning to formal and routinized procedures. Hence, by the time of carrying out the teaching experiments in ninth grade the students' knowledge of algebraic techniques is limited. For example, the general solution of a quadratic equation is taught only at the end of ninth grade.
Especially for the students who opt for the exact streams at pre-university level, there seems to be too little attention to vertical mathematization in grades 7 - 9. As a result, they do not meaningfully master routine procedures as the teachers of grades 10 - 12 expect them to do. Therefore, in addressing the general level this study focuses on capitalizing on the informal experiences at the referential level from the seventh and eighth grades.
Now we should be more specific about how these aims are elaborated. As far as algebra is concerned, we focus on the concept of parameter. The technological tool used is a computer algebra environment. The general view of algebra presented in this chapter is elaborated on by zooming in on the concept of parameter in Chapter 4, and by considering the role of computer algebra in Chapter 5.

# 4 The concept of parameter

## 4.1 Introduction

The first research subquestion of this study concerns the contribution of computer algebra use to the learning of the concept of parameter. After the global reflections on algebra that we presented in Chapter 3, this chapter elaborates on the concept of parameter.

First, we explain why the study focuses on the concept of parameter. The main reasons are that parameter use can contribute to the reification of expressions and formulas, that the parameter is a means of generalization, that parameter use can make explicit the different roles that literal symbols can play, and finally that parameter use can improve symbol sense (Section 4.2).

The historical development of the concept of parameter may indicate the conceptual barriers that students have to take. We globally sketch this long and difficult historical development. An important step in this trajectory was the jump from Diophantus – who did not formulate general solutions – to Viète, who developed solutions that contain parameters (4.3).

With this historical development in mind, we draw up a conceptual analysis of the concept of parameter (4.4). A parameter can have different meanings, and may act as generalizer over and as a 'dynamizer' of situations. The parameter roles that are distinguished are the parameter as placeholder, as changing quantity, as generalizer and as unknown.

Finally, we address the issue of the higher level understanding of the concept of parameter that is mentioned in the first research subquestion. How do we define that in this study? Reflections on different theoretical perspectives on levels of understanding lead to a dichotomy with the placeholder level on the low-level side and the other parameter roles on the high-level side. As a result, five indicators of a higher level understanding in the concept of parameter are identified (4.5).

The conceptual analysis of the concept of parameter, the distinction of the different parameter roles and the definition of the higher level understanding of the concept of parameter guide the development of the hypothetical learning trajectory in Chapter 6.

## 4.2 Why parameters?

In this section we motivate why the concept of parameter is adequate for investigating the learning of algebra in a computer algebra environment from the perspective of the difficulties of learning algebra that were presented in Chapter 3.

The understanding of literal symbols, and of variables in particular, is essential in algebra. Therefore it has been the focus of many research studies (Bills, 1997, 2001; Rosnick, 1981; Schoenfeld & Arcavi, 1988; Trigueros & Ursini, 1999; Ursini &

Trigueros, 1997, 2001; Wagner, 1983). Often, several roles of variables are distinguished where the parameter role is one of them (Janvier, 1996; Küchemann, 1981; Philipp, 1992; Usiskin, 1988). Parameters may emerge quite naturally from concrete contexts; on the other hand parameters may be a means of generalization and abstraction. We see the parameter as an 'extra variable' in a formula or function that makes it represent a class of formulas, a family of functions and a sheaf of graphs.
To avoid any confusion, from now on we exclude the parameter role from the concept of variable; by a variable we mean a literal symbol that does not play the role of parameter. We use the word variable for all the other roles, and we do not specifically refer to the variable as a varying quantity or changing quantity in particular, as is sometimes done in the literature.
We have four arguments for focusing on the concept of parameter in this research. First, we conjecture that parameter use can foster the reification of algebraic formulas and expressions. Second, we suppose that parameter use is a means of generalization, which is an important vertical mathematization activity. Third, the concept of parameter allows the students to revisit the different roles of variables that they have encountered before. Fourth, developing the concept of parameter may improve students' insight into the meaning and structure of algebraic formulas and expressions, which is an element of symbol sense. We now elaborate on these four arguments.

### *Parameter to foster reification*

In Chapter 3 we identified the process-object duality as one of the main difficulties of learning algebra. This resulted in the intention to reinforce the object character of algebraic expressions and formulas.
We conjecture that working with algebraic expressions and formulas that contain parameters stimulates the reification of the expressions and formulas that are unified by the parametric form. For example, the perception of the parametric relation $y = a \cdot x + b$ as symbolizing the complete set of formulas such as $y = x + \frac{1}{2}$, $y = -3x$ and $y = 100000x - 200000$ may facilitate the idea that these formulas form a set that is represented by the parametric form. The three formulas are objects that are elements of that set. The use of the parameter unifies a set of 'things', and by that these 'things' – in this case linear formulas – may be considered as objects. Sedivy formulated this as follows: '... one form with variables *a*, *b*, *x* represents a whole family of forms with the only variable *x*' (Sedivy, 1976, p. 123).
While using the parametric form, students probably will have in mind a 'representative' of the reference set and will see the parametric form as an object in itself, especially when it is submitted to subsequent processes. For example, solving parametric equations leads to expressions as solutions that may be substituted in other formulas. Such activities are expected to reinforce the object character of the alge-

braic expressions as an addition to its process character. This is in line with the observation made by Sfard that the concept of parameter is suitable to foster reification of the function concept (Sfard, 1991).
Variation of the parameter value acts at a higher level than variation of an ‘ordinary’ variable does; it affects the complete formula, the complete equation, and that also stimulates the object view of formulas and expressions.

### *Parameter as a means of generalization*

In Chapter 3 we identified vertical mathematization, emergent modelling and symbolization as processes that give meaning to algebraic objects and procedures that refer to a framework of mathematical objects and relations. generalization is regarded as an activity of vertical mathematization and the general level is the third level in the four-level structure of Gravemeijer (1994, 199). The relevance of generalization can hardly be overestimated. As Borges (cited in Arcavi, 1994, p. 24) phrased it: ‘To think is to forget differences, to generalize, to abstract.’
The second reason to focus on the concept of parameter is that the use of parameters can facilitate generalising and building up a framework of algebraic relations. After having worked on a number of analogous concrete situations, the student can use the parameter to generalize over a class of situations, functions, formulas, equations and solutions, and reverts from particular cases to a generic level. By introducing parameters, the student is invited to enter the ‘meta-level’ of general relations and solutions. This generic level is part of the world of algebraic relations and objects. This issue is revisited in Section 4.5 on higher level understanding.

### *Parameter to revisit roles of variables*

It was stated above that variables can play different roles in algebra, such as the role of placeholder, generalized number, changing quantity or unknown. An important skill in algebra is the ability to deal with these different roles in a conscious and flexible way, as these roles have consequences for the algebraic procedures, and can change during the problem-solving process (Arcavi, 1994; Bills, 1997; Bloedy-Vinner, 1994, 2001). While working in a computer algebra environment, this skill is particularly relevant, as for the CAS ‘all letters are equal’. It is indispensable for making productive use of the CAS that the user distinguishes the roles of the different literal symbols in the problem situation.
The point now is that the parameter can play roles similar to those of the ‘ordinary’ variable, but at a higher level, which makes it more complicated (Heck, 2001; Sedivy, 1976). This means that by working on the concept of parameter, the concept of variable is revisited and deepened. Developing the concept of parameter, therefore, provides a ‘second round’ for developing the concept of variable.

*Parameter to improve insight into formulas*

The fourth argument for focusing on the concept of parameter is the expected benefit to students' insight into formulas and expressions. In Section 3.5 it was argued that students should be able to see sub-expressions as entities within a complete formula, 'boxes' as it were, whereas in other instances they should be able to 'open the box' and to consider the sub-expression in detail. This flexibility requires the skill to deal with 'mixed cues' (Arcavi, 1994; Wenger, 1987). By this is meant that symbols are interpreted by the students as invitations to undertake an action that is not always appropriate. For example, the square sign in $(x + 3)^2 = 16$ may invite expansion, but that is not the easiest way to solve the equation. According to Wenger, situations that involve parameters are especially suitable for teaching students how to deal with such mixed cues, because these situations often offer different directions to take.

## 4.3 Historical perspective

In this section the historical development of the concept of parameter is briefly sketched. Insight into this development can help in identifying conceptual obstacles, and can offer heuristics for the construction of a learning trajectory. Because a parameter is a special kind of variable, we also consider the concept of variable.

Several researchers distinguish three phases in the historical development of the concept of variable: the rhetoric phase, the syncopated phase and the symbolic phase (Boyer, 1968; Harper, 1987). In the phase of *rhetoric algebra*, which lasted from ancient times until the time of Diophantus (ca. 250 A.D.), words were not abbreviated and there were no symbols for unknowns. Problems and solutions were described in natural language.

The *syncopated phase* is the period from Diophantus until Viète (1540 - 1603). In this phase, shortened notations and abbreviations were introduced, for example to represent powers. Literal symbols were now used to represent unknowns. For example, Diophantus used $\zeta$ to denote the unknown. The algebraic activity was aimed at finding the numerical value of the unknown; solutions were numbers. Eventually different letters were used for different variables, but general solutions in terms of parameters did not yet appear. Van der Waerden (1983) sketches how the Babylonian writings and the first book of Diophantus treated problems in which the number of unknowns is equal to the number of equations. This yields a finite number of numerical solutions. Diophantus often used the Rule of False Position. This is a kind of 'trial and improve' method, in which one starts with an estimation of the solution, often on the basis of a fair-share principle. By substitution, the 'error' is calculated and used to correct the initial estimation (Radford, 1996). However, in his later books, Diophantus solved systems with more variables than equations. Such systems have an infinite number of solutions, and to write them down in a compact way requires parameters. These so-called undetermined systems of equations were a prelude to the use of parameters. In the syncopated phase, problem-solving procedures were demonstrated by means of numerical examples, which the reader could adapt to other instances of the equivalent problem. However, the syncopated notations did not allow for the formulation of general solutions.

In the thirteenth century Jordanus Nemorarius (1225 - 1260) published his book De Numeris Datis. In it he used literal variables for unknowns, but also for known values, in fact for parameters. This opened the horizon for the general solutions that are lacking in Diophantus' work and to symbolic algebra (Rojano, 1996).

The third phase – that of *symbolic algebra* – started with François Viète (see Fig. 4.1, retrieved from www-history.mcs.st-and.ac.uk/history/). Viète (1540 - 1603) was a French lawyer and member of the king's council. In his work literal variables also refer to given quantities. These are in fact parameters, 'changing constants' that change at a higher level than the 'ordinary' variables, namely at the global rather than the local level. This allows for generalization and for the notation of general, algebraic solutions of parametric equations. As the values of the parameters could be filled in afterwards, his work was more general than the exemplary results from Diophantus.

Figure 4.1 François Viète

The work of Viète is very important in the development of algebra, as he was the first to distinguish the different roles of literal symbols (Boyer, 1968). An unknown (the 'cosa', the 'thing') was represented by a vowel, and consonants symbolized the known or given quantities, the parameters. This had important consequences: an expression in terms of the parameter could now occur as the solution of an equation, and could be submitted to a subsequent process, although its numerical value was still not known (Sfard & Linchevski, 1994). This led to the conception of formulas and expressions as entities, as objects; it was an important step towards the reification of algebraic expressions and formulas. One generation later, Descartes started using the letters from the beginning of the alphabet for given and known values, and letters from the end of the alphabet for unknowns, a convention that is still used nowadays.

Viète aimed at developing a 'homogeneous' algebra that was independent from arithmetic and geometry (Charbonneau, 1996). The first part of Viète's book 'Introduction to the analytic art', called 'Zetitics', contains a set of rules for the manipulation of algebraic expressions. The unknowns and the coefficients (the parameters) were treated in a similar manner.

Viète's distinction of unknowns and parameters, his notation and his general solutions of parametric equations were a great step forward in algebra. It was only after the work of Viète that Newton, Leibniz and others were able to develop the function concept, where the variable has the character of a dynamically changing quantity.

What consequences can we draw from this historical sketch for the purpose of this study? The study focuses on students' development of the concept of parameter, and in fact on the transition from syncopic to symbolic algebra, or, in other words, the transition from Diophantus to Viète. The lesson that we take from history concerns the slow and apparently difficult development of the concept of parameter (Furinghetti & Paola, 1994). In particular, the development of symbolizations for parameters, their distinction from ordinary variables and the perception of expressions as objects that represent general solutions are obstacles that had to be overcome during this historical development. These obstacles are cues for the difficulties that students encounter while developing conceptual understanding in parameters.

As we indicated in Section 3.2, we use the historical development of the concept of parameter as an indication of cognitive obstacles; however, it is not a guideline for the learning trajectory, as the students' preliminary knowledge of the concept of variable has its roots in the functional approach that was not developed at the time of Diophantus or of Viète.

## 4.4 Conceptual analysis of variable and parameter

In Section 1.3 we described a parameter as an extra variable or a changing constant in a formula or function that makes it represent a class of formulas, a family of functions and a sheaf of graphs. The parameter unifies a class of problem situations and solutions, and can be used for generalization. In this section we first address the concept of variable, upon which the concept of parameter as a higher order variable is based. Second, we analyse the concept of parameter further, resulting in the identification of four different parameter roles.

### *The concept of variable*

Many research studies address various aspects of the concept of variable, and in particular the different roles and meanings variables have in school mathematics (e.g. Bills, 1997, 2001; Kemme, 1990; Küchemann, 1981; MacGregor & Stacey, 1997;

Malle, 1993; Phillip, 1992; Rosnick, 1981; Stacey & MacGregor, 1997; Usisikin, 1988; Wagner, 1981, 1983). We now present an overview of the commonly distinguished roles.

In Section 1.2 the student-professor example was presented (Rosnick, 1981), where students translated the sentence ‘At this university there are six times as many students as professors’ into the equation $6S=P$ rather than $6P=S$. This indicates the conception of a variable as a label referring to an object, in this case a student or a professor, rather than to a *number* of students or professors. We consider the label view of variables as a pre-conception that is not productive.

The first sensible conception of a variable in our opinion is that of a placeholder for numerical values. A variable is seen as an ‘empty place’, a kind of lettered box into which numerical values can be inserted and from which they can be retrieved. This conception is sometimes appropriate in computer science (Heck, 2001; Kemme, 1990). Computer algebra software is different in this aspect, as variables do not necessarily refer to numerical values. We think that the placeholder view of variables is appropriate in some situations, but is limited in the context of algebraic understanding.

Malle (1993) distinguishes three aspects of variables: a situational aspect, in which the variable acts as unknown or generalized number; a substitution aspect, which concerns the variable as placeholder; and a calculation aspect, in which variables are meaningless objects that can be manipulated according to certain rules.

Freudenthal described variables as ambiguous, polyvalent names (Freudenthal, 1962, 1982, 1983). They are ambiguous because the variable can act as unknown and as undetermined number, and polyvalent because they can refer to different numerical values. In itself this ambiguity and polyvalence is natural:

> We are familiar with the use of ambiguous, or rather polyvalent, names – one name for many objects. When my daughter was at the age when children play the game of “what does this mean?” and I asked her what is “thing” she answered: Thing is if you mean something and you do not know what is its name.
> (Freudenthal, 1983, p. 474)

Freudenthal distinguished the variable as unknown and the variable as undetermined quantity. In addition to this, Kindt stressed two aspects of the concept of variable: the generalising aspect and the dynamic aspect (Kindt, 1980). The variable as generalizer is close to the undetermined quantity of Freudenthal; the variable as a dynamic, changing quantity is a new element. Kindt argued that variables in school algebra too often have a static character, and that not enough attention is paid to the dynamics. We consider the dynamic view of the variable as fitting to the operational aspect in the sense of Sfard (1991), whereas the generalising aspect links up with the structural side.

These different roles and meanings of variables depend on the problem situation and on the question under consideration; furthermore, the roles can change during the problem-solving process. As indicated in Section 3.2, the roles are related to the approaches to algebra in the following way:

- the variable as placeholder for numerical values;
- the variable as changing quantity in functional algebra;
- the variable as generalized number in algebra of pattern and structure;
- the variable as unknown in problem-solving;
- the variable as symbol in algebraic language.

The variable as changing quantity reflects the dynamic aspect, and corresponds to the functional approach. The variable as generalized number fits to the generalization aspect of algebra, and the variable as unknown goes with the problem-solving approach. The variable as symbol without references, finally, is the role corresponding to the view of algebra as a language.

One of the main difficulties with the concept of variable is to distinguish the different roles and meanings, and to deal with them in a flexible way. In addition to this, other difficulties with the concept of variable have been identified (Bills, 1997, 2001; Küchemann, 1981; Tall & Thomas, 1991; Tall et al., 2000; Usiskin, 1988; Wagner, 1981, 1983). For example, the similarities and differences between variables, numbers and words are seen as a source of misconceptions (Wagner, 1983). Students sometimes relate the order of the numbers to the alphabetical order of literal symbols; for example, as the successor of 37 is 38, the successor of $m$ may be thought to be equal to $n$ rather than $m$+1.

Other obstacles that students encounter while learning the concept of variable are the expected answer obstacle and the lack of closure obstacle, which were discussed in Section 3.4 (Collis, 1975; Küchemann, 1981; Tall & Thomas, 1991; Tall et al., 2000). These issues are supposed to account for the ‘didactical cut’ between arithmetic and algebra (Filloy & Royana, 1989).

To summarize this section, we argue that the concept of variable is a multi-faceted one, because of its ambiguous and polyvalent character, and because of the diversity of roles variables can play, and the diversity of meanings variables can have. The student who is learning algebra needs to acquire skills to identify the different meanings and roles of the variable and to deal flexibly with them.

### *The concept of parameter*

So far, we have not mentioned the parameter role that a variable can have. Because a parameter is essentially a kind of higher order variable, the above discussion on the concept of variable can partially be applied to the concept of parameter. Like the variable, the parameter can play several roles and have different meanings. As in the case of variable, dealing adequately with these roles is complicated, all the more so because these roles often remain implicit and may change during the problem-solving process.

In addition to these similarities between the concept of variable and that of parameter, there are also differences. Specific to the concept of parameter are the following three aspects: the higher level at which variation and generalization act, the hierarchical position of the parameter compared to the variable, and the complexity of the different literal symbols and their roles.

First, variation and generalization of the parameter take place at a *higher level* than variation and generalization of the variable. Variation of $x$ in $3 \cdot x + 5$ leads to variation in the numerical value of the expression. Graphically, this means 'tracing' the graph of $x \to 3 \cdot x + 5$. However, variation of the $a$ in $a \cdot x + 5$ results in variation of the expression as a whole, and, graphically, in changing the complete graph of $x \to a \cdot x + 5$. Concerning generalization, the identity $(x + 1)^2 = x^2 + 2x + 1$ is a generalization over arithmetical identities such as $(4 + 1)^2 = 4^2 + 2 \cdot 4 + 1$, $(100 + 1)^2 = 100^2 + 2 \cdot 100 + 1$, et cetera. $(x + a)^2 = x^2 + 2 \cdot x \cdot a + a^2$, however, is the generalization over algebraic identities such as $(x + 1)^2 = x^2 + 2x + 1$, $(x + 2)^2 = x^2 + 4x + 4$, et cetera. In this sense, the parameter acts as a 'meta-variable'. Variation and generalization using 'ordinary' variables brings about change of or generalization over arithmetic relations; variation and generalization of the parameter concern the changing of or generalization over algebraic relations. This is expected to complete the reification of algebraic expressions with one variable and formulas that contain two variables.

A second, related aspect of the concept of parameter is the *hierarchical* position of the parameter compared to the variable. A parametric expression or formula stands for a class or family of expressions or formulas. As such, a parametric form can be considered as a second-order function with the parameter as argument and the corresponding expressions or formulas as function values. For example, each value of $a$ determines a linear function $x \to a \cdot x + 5$ in $x$, as is symbolized by the long arrow in Fig. 4.2. The values of this function are non-parametric functions in $x$, in which the parameter $a$ is replaced by a constant numerical value. This mixture of the parameter as a constant at the first level, and as a variable at the second level, is reflected in the expression 'varying constant'. In that sense, the parameter has a hierarchically higher position than the variable (Bills, 1997, 2001; Bloedy-Vinner, 1994, 2001).

$$a \longrightarrow (x \to a \cdot x + 5)$$

Figure 4.2 The parameter as argument of a second-order function

A third aspect of the concept of parameter results from the fact that the use of a parameter, an ‘extra letter’, complicates *the roles of the different literal symbols* involved. Parametric expressions and formulas, such as $a \cdot x + 5$ or $y = \sqrt{a^2 - x^2}$, contain also ‘ordinary’ variables, so at least two or three literal symbols. This means that distinguishing the roles and meanings of each of these symbols is a more complicated affair than it is in situations with just one or two variables. Moreover, the roles are not fixed and may change during the solution process. For example, the parameter *a* in $y = x \cdot (a - x)$ may acquire the role of unknown if the equation of the curve through the vertices of the family of graphs is asked for.
This third issue is related to the stereotyping of literal variables (Bills, 2001). This refers to conventions that exist to clarify the roles of the literal symbols. For example, variables are often indicated by *x* and *y*, and parameters are called *a*, *b*, *p*, *q*, *m*, and *n*. In fact, this was already done by Viète. The advantage of stereotyping is that it may help students to distinguish the different roles of the literal symbols:

> The first feeling emerging is that the sentence by William Shakespeare “What’s in a name? That which we call a rose by any other name would smell as sweet” (Romeo and Juliet, act II, scene II) does not always apply to algebra, as students perceive different names as labels for different ‘objects’.
> (Furinghetti & Paola, 1994, p.374)

However, stereotyping may hinder the flexibility to change the role of a literal symbol, for example to give a parameter the role of unknown, as was argued by Bloedy-Vinner (1994, p. 89): ‘Moreover, the meaning of a letter as a parameter or as an unknown or variable, might change throughout the process of solving a problem.’
As a result of this conceptual analysis, two main conceptual difficulties of the concept of parameter are identified. The first is the difficulty of understanding the roles of the parameter, which may change during the problem-solving process. The second is the hierarchical relation between the ‘ordinary’ variables and the parameter, particularly when the parameter is not a placeholder for a constant value. The concept of parameter and the concept of variable on the one hand share the different roles and meanings, and on the other hand address different algebraic levels. We believe that the different roles of the concept of variable can be applied to the concept of parameter as well. Therefore, we distinguish the parameter roles of placeholder, changing quantity, generalizer, and unknown. These parameter roles are explained below.

- *The parameter as placeholder*
  In the placeholder conception, the parameter is seen as a position, an empty place into which numerical values can be inserted and from which they can be retrieved. The value in the ‘empty box’ is a fixed value, known or unknown. The focus is not on finding the unknown value. Once a value is filled in, it can be re-

placed by another one, but systematic variation of the parameter value is not considered, nor is the similarity perceived between the situations that different parameter values represent: each new parameter value is seen as a new situation. This is the ground level of understanding the concept of parameter: the parameter as placeholder for a constant value that does not change. For the following three roles this is different.

- *The parameter as changing quantity*
  In the conception of the parameter as changing quantity, the parameter still represents a numerical value, as was the case for the parameter as a placeholder. However, there is systematic variation of this value, and the parameter acquires a dynamic character: it is a 'sliding parameter' – terminology from van de Giessen (2002) – that smoothly runs through a reference set. This variation affects the complete situation, the formula, the global graph, whereas variation of an 'ordinary' variable only acts locally. The concept of the parameter as a changing quantity is related to the reification of graphs and formulas. Research evidence suggests that the parameter as 'positioner' of graphs is the most accessible conception for students (Furinghetti & Paola, 1994).

- *The parameter as generalizer*
  The parameter as generalizer is used to generalize over classes of situations, of concrete cases, of expressions, formulas, and solutions. By doing so, this 'family parameter' (van de Giessen, 2002) represents these classes, and unifies them. The parameter is no longer a specific number, but stands for an exemplary number or a set of numbers. The generic representation allows for seeing the general in the particular, for solving categories of problems and for formulating solutions at a general level. The ability to solve all concrete cases at once by means of a parametric general solution requires the reification of the expressions and formulas that are involved in the generic problem-solving process.
  The conception of the parameter as generalizer is close to the role of undetermined. We use the term generalizer here, because it indicates that the parameter *does* something: it allows for organizing and unifying mathematical phenomena, whereas the term 'undetermined' is somewhat vague.

- *The parameter as unknown*
  If the parameter generalizes over a class of situations, a natural next step is often to take the opposite direction and to select particular cases from the general representation on the basis of an extra condition or criterion. In such situations, the parameter acquires the role of unknown-to-be-found (Bills, 2001) or, shorter, unknown. To students, this means a shift of perspective that is often found to be

hard to make: usually, the variable plays the role of unknown, so the parameter taking the role of unknown affects the hierarchy between parameter and variable. For example, Sfard and Linchevski (1994) found that students in the tenth grade had difficulties with the task of investigating whether the system of equations $k - y = 2$, $x + y = k$ has a solution for every value of $k$.

In some cases, the result of solving an equation with respect to the parameter will be one or more numerical values. In other cases, the result will be an expression, so that the reification of algebraic expressions is involved once more.

We argue that the parameter can play similar roles with similar meanings as the variable does, but that it acts at a higher level, which complicates the situation and the distinction of the roles of each of the literal symbols in particular. The four roles of the parameter are not presented in a random order: in the placeholder role, the parameter behaves as though it were a constant. That is the starting level of this study. In the three other roles, the parameter acts like a ‘changing constant’, and the objects that this change affects are at a higher level than is the case for ‘ordinary’ variables. In the following section, we address the transition from the placeholder level to the other levels.

## 4.5 Levels of understanding the concept of parameter

A central issue in the first research subquestion is the *higher level understanding* of the concept of parameter. In the previous section, we distinguished the parameter as placeholder from the three ‘higher’ roles of changing quantity, generalizer and unknown. In fact the intended higher level understanding concerns the jump from the placeholder conception to the higher roles, or, in historical terms, the jump from Diophantus to Viète. This jump requires the reification of expressions and formulas.

In this section, this level jump is first founded theoretically in the level theory of Van Hiele, the emergent models from RME, theories on generalization and abstraction, and theories on reification. Then, we make the higher level understanding concrete for the case of the concept of parameter, and suggest means to reach for the higher level.

### *The level theory of Van Hiele*

Van Hiele’s level theory is a theory of cognitive levels. The point of departure is the idea that learning takes place at various levels, ‘plateaus in the learning curve’ (Van Hiele, 1973, 1986). Van Hiele distinguishes a hierarchy of three thinking levels: the ground level or first level, the descriptive level or second level, and the third level. In some publications, this numbering starts at zero rather than one. The ground level is the level of the concrete, visual understanding of the concept. At this level the students have not yet developed a relational framework. The conceptual understanding

is at the second thinking level, when the concept has been developed into a mathematical object with properties that are vertices in a meaningful relational framework. When the elements of the relational framework are subjected to logical organizing and reasoning, the conceptual understanding reaches the third thinking level. This process of level-raising is iterative and relative: the second level understanding of a concept can act as the ground level for the development of another concept. This is called level reduction. For example, the insight that 5 is an object on its own, and not only a counting process, can be considered as a second level understanding of the number concept. For algebra, however, this is the ground level. This example shows the relation between Van Hiele's theory and the concept of reification, which is described in Section 3.4 and below. Reification concerns the transformation of processes at the lower level into objects at the higher level, which are submitted to higher order processes. The level theory of Van Hiele also addresses the forming of object, although in a more static level hierarchy.
In this study, we use Van Hiele's level theory to specify the intended level-raising of the understanding of the concept of parameter. The parameter as placeholder represents the ground level. The understanding of the parameter as changing quantity, generalizer and unknown involves building up a new relational framework, in which parametric expressions and formulas, sheaves of graphs and classes of situations are the new objects. This hypothetical level structure is presented in Fig. 4.3. We conjecture that the parameter as generalizer affects this relational framework the most, which is why it is positioned a little higher than the other two roles.

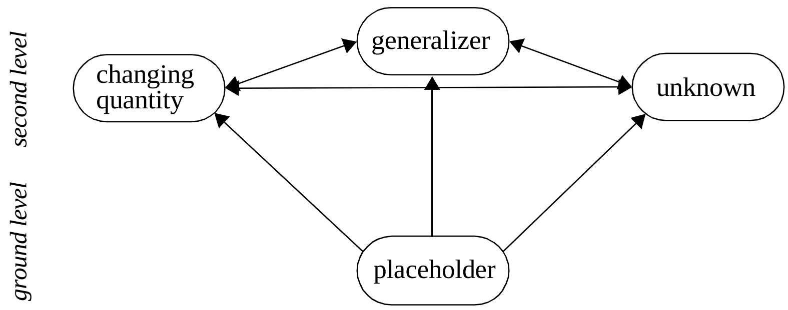



Figure 4.3 Levels of understanding the concept of parameter

### *Emergent models*

A didactic way of looking at levels of formalization that takes into account the problem contexts and the development of symbols is the notion of emergent models. In Section 3.6 (which is on the theory of RME) we described the model-of/model-for development and the four-level structure of student activities with models. The higher level here refers to the development of a model-of into a model-for, in which new

objects and meanings are constructed that refer to a mathematical relational framework. The construction of meaning is related to the development of language, notations and symbols and therefore to the process of symbolizing (Section 3.5).
In this study, the four-level structure provides a means of defining the higher level understanding: the jump from Diophantus to Viète can be considered as the transition from the referential to the general level by means of parameters (Gravemeijer et al., 2000). The placeholder role of the parameter is considered as referential to one specific situation, whereas the parameter as changing quantity and as generalizer refers to a more general level. The hypothetical learning trajectory will describe the activities that are supposed to foster this development.

### *Generalization and abstraction*

In both the level theory of Van Hiele and the concept of emergent models, the higher level is characterized by the reference to a relational framework of meaningful mathematical objects and relations, rather than to a concrete problem situation. In the case of algebra, key issues in this building up process are generalization and abstraction. *generalization* can mean generalization *over* classes of situations and generalization *to* other situations or transfer. Here we consider generalization *over*, which involves exceeding the local level of the particular and realising the generality of a relation, pattern or property. As such, generalization can embody the transition from the ground level to the second level in Van Hiele's terms, or from referential level to general level, in Gravemeijer's terms. As de Lange stated, generalization can be considered as the top level of vertical mathematization (de Lange, 1987). We agree with Mason, who argued that generalization is the heart of algebra and mathematics (Mason, 1996). As with the levels that we saw above, generality is relative; we already illustrated how an algebraic formula is a generalization over a class of arithmetical relations, whereas an algebraic formula containing a parameter is a generalization over a class of algebraic formulas.
Several researchers (e.g. Mitchelmore & White, 2000) distinguish empirical generalization and theoretical generalization. The first refers to pattern-seeking without intrinsic motivation of the patterns that are discovered. The latter stands for considering structural similarities, which identify inner connections.
Mason formulates a link between generalization and abstraction:

> One way to work at developing awareness of generality is to be sensitized by the distinction between looking through and looking at, which leads to the primal abstraction and concretization experiences, namely seeing a generality through the particular and seeing the particular in the general.
> (Mason, 1996, p. 65)

Abstraction is addressed in many publications (Dubinsky, 1991; Mason, 1996; Mitchelmore & White 2000; Sfard, 1991; Sfard & Linchevski, 1994; Tall et al., 2000). Often, one refers to Piaget (Piaget, 1980, 1985), who distinguished empirical abstraction, pseudo-empirical abstraction and reflective abstraction. Reflective abstraction is considered to be the highest form of abstraction, because it concerns the internal coordination of actions in which (again!) processes can become objects. Reflective abstraction is the construction of schemata with mental objects and processes that act upon these objects.

Dubinsky (1991) elaborates on Piaget (Piaget, 1980, 1985) when he states that reflective abstraction consists of different phases: the interiorization of operations and actions into processes, the coordination of processes to construct new processes, the encapsulation of processes into objects, the generalization of schemata by means of assimilation, and the reversal of the process to construct new processes. generalization is part of reflective abstraction, and we note the similarities with reification.

For the purpose of this study, we see generalization as an important level-raising activity that involves encapsulation and reification. We address these notions below.

### *Theories on reification*

As we argued in Section 3.4, Dubinsky’s ideas on encapsulation share the stress on the process-object duality with the reification concept of Sfard and the procept theory of Tall and Thomas: abstraction often involves the transition from processes to objects, which is the completion of the procept, or the integration of the ‘two sides of the coin’ (Dubinski, 1991; Sfard, 1991; Tall & Thomas, 1991).

Sfard argued that reification can be completed by means of submitting the new objects to higher level processes (Sfard, 1991). Van Hiele also pointed at the importance of the construction of objects for level-raising, when he remarked that the thinking and acting at the lower level is subject matter for the thinking at the higher level (Van Hiele, 1986). Gravemeijer formulated this as follows:

> The next level is unlocked when the first-level processes are accessible for reflection and thus become thinking objects for the second level.
> (Gravemeijer, 1994, p. 23)

In short, the targeted higher level understanding in this study includes reification. In the case of the concept of parameter, the mathematical objects that need to be built up for the understanding of the parameter as generalizer and the parameter as unknown are expressions and formulas that may contain parameters.

These formulas and expressions represent general relations or solutions. Their reification can be fostered by means of submitting them to further processes, such as substitution.

### *Higher level understanding of the concept of parameter*

So far we have used four theoretical elements for the elaboration of the higher level understanding of the concept of parameter. However, the four perspectives share common features. These features include the relativity of the levels, the importance of the construction of a mathematical framework of relations that is not directly related to the context, the need for ‘objectivation’ of processes, and the relevance of generalization. These common features determine our position towards the level-raising. The targeted higher level understanding thus consists of the development of a meaningful relational framework that includes the parameter as changing quantity, as generalizer and as unknown, and parametric formulas and expressions as objects. Therefore, in this study we consider the following points as indicating a higher level understanding of the concept of parameter than the placeholder ground level:

*a* *The parameter as changing quantity*
This concerns the notion that variation of the parameter affects the global situation, the formula and the graph as a whole, whereas the change of an ‘ordinary’ variable affects only the position of one point of the graph.

*b* *The parameter as generalizer of relations*
This concerns the notion that a formula containing a parameter represents a class of situations, a family of formulas and functions, a sheaf of graphs. The parameter represents a reference set as a whole, rather than one element from that reference set.

*c* *The parameter as generalizer of solutions*
This concerns the notion that the expression that is the solution of a parametric equation stands for the general solution of the class of problems that is represented by the parametric equation.

*d* *The parameter as unknown*
This concerns the notion that the selection of a particular solution from the general solution based on an extra condition leads to a solution process in which the role of the parameter shifts towards the role of unknown. The solution may be a set of parameter values or an expression.

*e* *The reification of algebraic formulas and expressions*
This concerns the notion that solving an equation containing a parameter results in an expression rather than a numerical value, and that this expression can be considered as an object that can be submitted to other procedures such as substitution.

Point *a* concerns the ‘meta-dynamics’ of the parameter as changing quantity. The points *b* and *c* concern the parameter as generalizer, split up into the generalization of situations and relations, and the generalization of solutions. Point *d* concerns the parameter as unknown. Point *e* formulates the reification of expressions and formulas as a prerequisite for understanding the general solutions.

In Fig. 4.3 we positioned the generalizer role of the parameter a little higher than the other two higher parameter roles. This was done because we conjecture that the parameter as generalizer is the hardest issue, where the reification of the formulas and expression is crucial.

### *How to achieve the higher level understanding?*

Now that we have defined the higher level understanding of the concept of parameter, the question is what instructional activities might foster the development of the framework of meaningful algebraic relations and objects. A first inventory leads to the following ideas:

- Graphical models can mediate between the placeholder view and the higher parameter roles. For the parameter as changing quantity, the sliding graph may be an appropriate model. For the parameter as generalizer, the sheaf of graphs is adequate. For the parameter as unknown, a specific feature of the graph or a condition in graphical terms may bring to the fore that aspect of the concept of parameter. Graphical IT tools can be used for these purposes.
- For the parameter as generalizer, it may be effective to have students work through similar examples that vary only with respect to the parameter value. Repetition of the problem-solving procedure may make them sensitive to the overarching method. Computer algebra can be a suitable medium for such activities.
- In order to mediate between the parameter as placeholder and the parameter as generalizer, it may be worthwhile to work with an 'exemplary value' of the parameter that is a specific numerical value that allows for concrete results, but that has 'nothing special', so that it can be considered as an example for every possible parameter value. The exemplary parameter value is a pars pro toto for the class of situations.
- The calculation of general solutions of parametric equations, by hand or in a computer algebra environment, may enhance the perception of formulas and expressions as objects.

These instructional activities that might bring about transitions to the higher level parameter roles form the basis of the hypothetical learning trajectory and the teaching materials that are described in Chapters 6, 7 and 8. First, though, Chapter 5 focuses on the potentials of computer algebra use for carrying out these activities as well as on its pitfalls.

# 5 Computer algebra in mathematics education

## 5.1 Introduction

In this chapter we investigate the possible contributions of computer algebra use to the learning of algebra.

First, as a background we consider the use of IT tools in mathematics education in general. We note that for the learning of algebra, most information technology (IT) tools provide limited algebraic facilities and notations (Section 5.2). Then, we focus on computer algebra systems (CAS). We specify what computer algebra is and review some didactic issues that arose in early research projects into computer algebra use in mathematics education. In this way we sketch the state of the art on which this study elaborates (5.3).

Next we draw up an inventory of the opportunities and pitfalls that computer algebra use presents to mathematics education in general (5.4), and to learning the concept of parameter in particular (5.5). These opportunities and pitfalls are related to the theoretical framework of the study and to the different parameter roles that were identified in Chapter 4. The choice for using the handheld TI-89 symbolic calculator in the teaching experiments is motivated.

Finally, we describe the theory of instrumentation that focuses on students developing utilization schemes and instrumented techniques that integrate technical and conceptual aspects (5.6). This is the main theoretical framework we use in Chapter 10 for analysing the interaction between student and computer algebra environment.

## 5.2 Information technology and the learning of algebra

In this section we briefly address the general issue of the learning of mathematics, and particularly algebra, in a technological environment. It offers a background to our focus on the use of computer algebra for the learning of algebra.

First, we present Pea's view of the use of cognitive technology as amplifier and organizer. Second, we discuss the opportunities that computer use offers for visualization and for the integration of multiple representations. Finally, we focus on the use of technology in the algebra learning process in general and in gaining an understanding of the concept of variable in particular. Computer algebra technology is addressed in Section 5.3.

### *Amplifier and organizer*

Pea suggested an interesting approach to the use of technology in education (Pea, 1985, 1987). He considered cognitive technologies as media that help 'transcend the limitations of the mind... in thinking, learning, and problem-solving activities' (Pea, 1987, p. 91). He distinguished two functions of IT tools: the amplifier function and the organizer function. The former refers to the amplification of possibilities, for example by investigating many cases of similar situations at high speed. The complexity of the examples can be amplified as well, thus allowing for a wider range of examples with an increasing variation. In this way, IT makes accessible insights that otherwise would be hard for students to gain. We agree with Berger that the use of IT thus enlarges the possible 'zone of proximal development', to use the Vygotskian terminology (Berger, 1998; Vygotsky, 1978). The amplifier function of an IT tool may also be used to invite algebraic generalization (Mason, 1996).

As Heid (2002) indicated, the amplifier function does not affect essentially the curriculum. In its second role, the IT tool can also function as organizer or reorganizer affecting the organization and the character of the curriculum. For example, Hillel et al. (1992) describe how the possibility to simultaneously display graphical and symbolical windows changed the instructional approach to functions. Heid's study, which is addressed below, shows similar effects (Heid, 1988).

The IT function of generic organizer that is described by Tall (Tall, 1989; Tall & Thomas, 1991) is, surprisingly, close to the amplifier function. Technology, according to Tall, can help to generate a wide scale of examples and non-examples that provide the opportunity for the student to gather data and to develop a cognitive structure on the basis of these experiences.

The amplifier-organizer distinction can be helpful in explaining the role of technology in a specific educational setting. In this study computer algebra primarily functions as amplifier. For example, the computer algebra tool is used to generate examples that are a source for generalization. In the meantime, the computer algebra tool acts as an organizer as well: available options, such as the slider for dynamically changing graphs, affect the traditional instructional sequence for the learning of the concept of parameter.

### *Visualization and multiple representations*

As soon as technology became available for integration into mathematics education, its main opportunities were surveyed by many researchers (Hillel et al., 1992; O'Callaghan, 1998). These perceived opportunities are called affordances. For example, the affordances of the use of graphing calculators from the perspective of RME have been identified (Doorman et al., 1994; Drijvers & Doorman, 1997). The graphing calculator was supposed to allow for:

- the use of realistic contexts;
- an exploration approach to problem situations;
- visualization and the integration of different representations;
- the experience of dynamics within a problem situation;
- a flexible way of doing mathematics.

In similar lists that also concern other technological tools, the issues of visualization and the integration of multiple representations, and exploration are often mentioned.

Visualization within functional algebra involves drawing and manipulating graphs. In a technological environment this can be done in a fast, flexible and dynamic way. This offers opportunities for linking multiple representations of a relation, and in particular for linking graphical and algebraic representations (e.g. Doerr & Zangor, 2000; Janvier, 1987; Ruthven, 1990; van Streun, 2000; van Streun et al., 2000; Yerushalmy, 1991, 1997). The 'algebra of graphs' example in Fig. 5.1 illustrates how the graphing calculator can be used to explore how changes in the formula affect the graphs (Doorman et al., 1994; Drijvers, 1994). It is an example of an exploratory approach that aims at combining the graphical and the algebraic representation.

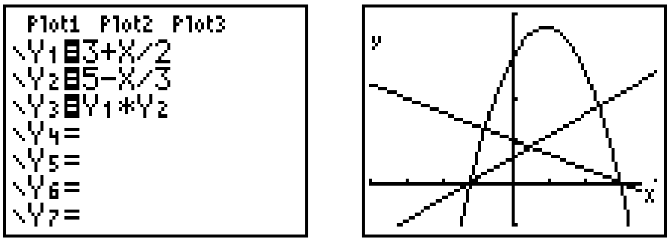



Figure 5.1 The algebra of graphs example

The function list in the left-hand screen of Fig. 5.1 shows two linear functions. The rule is that the product function Y3 is locked and cannot be changed. The 'game' is now to change the formulas of Y1 and Y2 to generate a required visual effect on the graph of the product function Y3, shown in the right-hand screen. For example, the graph of Y3 is required to be a 'valley parabola' rather than a 'mountain parabola', or it is supposed to touch the horizontal axis, or we want its vertex to be exactly above the intersection point of the two lines. This game could be played with graphical software on a desktop PC as well as with the graphing calculator.

Examples such as the algebra of graphs illustrate how working in the IT environment can stimulate the perception of different representations as different views of the same mathematical object, and the linking of the visual and algebraic properties of the functions involved. Furthermore, it shows that an IT environment allows for exploration and variation. Dynamics in the sense of sliders and gradually moving graphs are not included in the algebra of graphs example.

Note that the effect of tasks such as the algebra of graphs does not depend only on the affordances of the technological tool, but rather on the exploitation of these affordances embedded in the educational context and managed by the teacher (Doerr & Zangor, 2000).

In this study, we intend to use technology for visualization, dynamics and for combining graphical features with algebraic properties.

*Learning the concept of variable with IT tools*

Technology can be used to foster the understanding of the concept of variable. This is relevant to our study, as the concept of parameter builds on the concept of variable. We discuss two examples: the work of Graham and Thomas with the graphing calculator and the Freudenthal Institute's computer applet 'number strips'.

Graham and Thomas (1999, 2000) propose the following assignment to address the relation between a variable and its numerical value:

> Store the value 2.5 in A and 0.1 in B.
> Now predict the results of the ten sequences listed below.
> Then press the sequences to check your predictions.
> A+B, B+A, A – 5B, 2A+10B, A/B, AB, BA, 2A+2B, 2(A+B), 4(A+5B)
> (Graham and Thomas, 2000, p. 270)

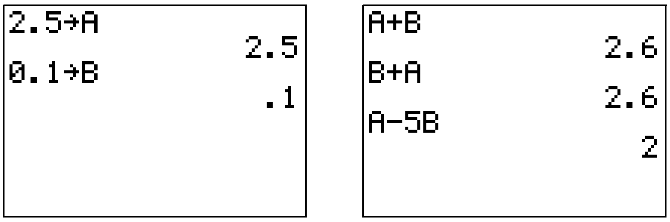



Figure 5.2 Screens for the Graham and Thomas task

The graphing calculator screens that students obtain while working on this task are shown in Fig. 5.2. In the left-hand screen, values are attributed to the variables. After that, the student is supposed to predict the values of the expressions in the right-hand screen. This task addresses the variable as placeholder, as a 'box' in which numerical values can be stored, as was described in Section 4.4 (Heck, 2001; Kemme, 1990). When a student shows only the right-hand screen to his neighbour, the task for his colleague is to reconstruct the values of *A* and *B*. In that case the variables play the role of unknowns. The authors report that the graphing calculator proved to be an instrument for achieving a significant improvement in student understanding of the concept of variable, and of the roles of placeholder and unknown in particular. The approach in this study is similar to the one described by Tall and Thomas (1991).

The Graham and Thomas example focuses on technology use for enhancing the concept of a variable as a placeholder for numerical values, and as unknown. The graphing calculator as the chosen IT tool restricts the references of the variable to decimal numbers; fractions, radicals or other exact values are not involved. The task does not address the understanding of the variable as generalizer or as changing quantity, which are the more difficult variable roles. Therefore, our criticism of this study would be that its focus is very limited, viz. to the variable as a placeholder for decimal values. The graphing calculator merely acts as a solution generator that allows

the students to check their answers.
The second example of IT use for learning the concept of variable concerns the Java applet ‘number strips’. This applet addresses the concept of a variable as a generalizer that represents a set of values. The environment allows for manipulating a complete column of integer values. In Fig. 5.3 (retrieved 26 March 2002 from www.wisweb.nl) the column of integers is multiplied by 5; then 1 is added. These operations on the column as a whole are supposed to mediate between operating on individual values and the generic operation symbolized by the formula 1+5*n*, where *n* no longer refers to one numerical value but represents the complete set of integer values. Directly entering the formula 1+5*n* is possible in the headings of the columns. Classroom trials suggest that it is difficult to have students reflect on the results of their work with the applet, which is a general concern for IT use.

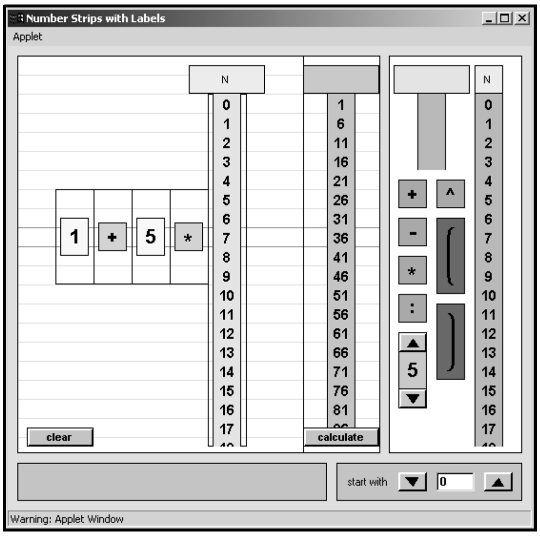



Figure 5.3 The applet ‘number strips with labels’

The two examples show how IT tools can be used to enhance the understanding of different aspects of the concept of variable. In this study, we focus on the concept of parameter, and in particular on the higher roles of the parameter as changing quantity, as generalizer and as unknown that go beyond the parameter as placeholder (Section 4.5). Therefore, we want to use an IT tool that allows for using literal symbols that do not refer to numerical values. Also, exact values such as fractions and radicals should be available, and from the perspective of symbol sense and insight into formulas a ‘pretty print’ formula editor is required, as is the possibility to manipulate such formulas as objects. Here, graphical and numerical software packages and graphing calculators have limitations. However, computer algebra environments offer these options, and are therefore considered in the following section.

## 5.3 Research on computer algebra in mathematics education

This section addresses previous research on computer algebra use in mathematics education. First, we explain what computer algebra is. Then we discuss two didactic concepts that have dominated the debate on computer algebra use in mathematics education for some time. After that, we review some important studies on computer algebra use in mathematics education, first on mathematics education in general and later focusing on algebra. The section ends with a list of questions that often arise in discussions on computer algebra use in mathematics education.

### *What is computer algebra?*

Computer algebra is software that performs algebraic calculations and formula manipulations. For example, algebraic expressions can be manipulated and simplified, equations can be solved exactly and functions can be differentiated and integrated analytically. Such software comes with options for numerical calculations and graphing, and is available for both PCs and handheld devices such as symbolic calculators. Well-known computer algebra packages for PCs that are used for educational purposes are Maple, Mathematica, Mathcad, Mupad, Derive, TI-Interactive, Scientific Notebook and Studyworks, among others. Well-known handheld devices that offer computer algebra facilities are Casio Cassiopeia, Casio ClassPad 300, Hewlett-Packard 40G, Hewlett-Packard 200 LX, Texas Instruments TI-89 and Texas Instruments Voyage 200. The handheld devices are compatible with the software packages as far as algebraic power is concerned; however, screen dimensions, screen resolution and memory capacity of the handheld devices are limited, compared to the desktop computers.

To give an impression of the algebraic options of a CAS, Fig. 5.4 shows some examples in TI-Interactive. The left-hand expression is entered by the user, the right-hand side after the = sign shows the response of the software. The screen shows an automatic simplification (where the case $c = 0$ is neglected), the substitution of the expression $a$+1 for $x$ (with immediate rewriting of the result), finding the common denominator of rational algebraic expressions, solving the lens equation with respect to $b$, expanding parentheses and factorization. Other computer algebra software provides similar options.

Computer algebra systems were developed in the 1950s for scientific purposes. From dedicated software packages the CASs gradually developed into complete environments that now also offer options for numerical calculations and graphics. Recently, computer algebra software has become embedded in environments for word processing and for communication via the Internet. Electronic books that students can work through in the environment have been developed. Handheld devices and PCs can be connected to each other and to dedicated websites. The effects of these fascinating developments are beyond the scope of this study.

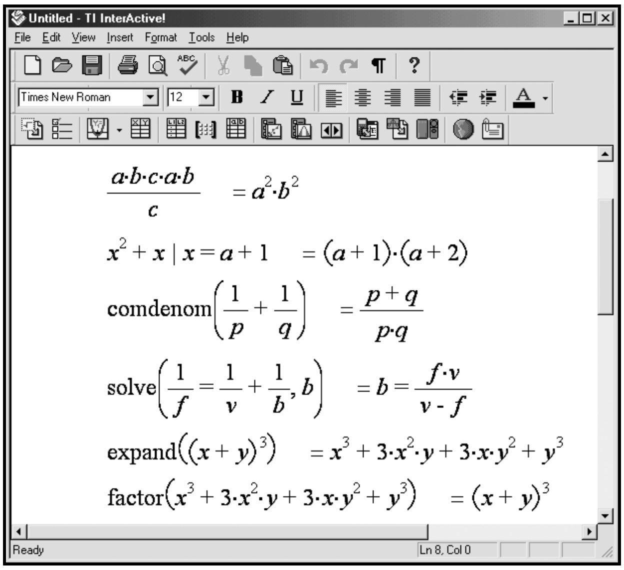



Figure 5.4 Examples of algebraic features of TI-Interactive

### *Didactic concepts of computer algebra use in education*

In the 1980s, mathematics educators started to seriously consider using computer algebra for educational purposes. Those who were interested in integrating technology into mathematics education often were not involved in the research on mathematics education. We conjecture that this is the reason for the loose relation between the early 'computer algebra community' and the current didactic trends, concepts and opinions. This resulted in the development of specific didactic concepts for the integration of computer algebra into mathematics education. We consider the two most important ones that dominated the debate during the 1990s: the resequencing concept and the white box / black box issue.

*Resequencing*

The resequencing concept was addressed in the paper 'Resequencing skills and concepts in applied calculus using the computer as a tool' by Heid (1988). In this study, which is considered as the start of research on computer algebra in mathematics education, Heid showed that the integration of computer algebra into a calculus course for first-year university students in business, architecture and life sciences allowed for resequencing the course, so that a 'concept-first' approach could be followed. This links up with the reorganizer function that was identified by Pea (1985, 1987). The remarkable result of the Heid study is that the students in the computer algebra group outperformed the control group on conceptual tasks in the final test, and also did nearly as well on the computational tasks that had to be performed by hand, and which, in the experimental course, were treated only briefly at the end. The subjects in the experimental group reported that the use of computer algebra took over the calculational work, made them feel confident about their work and helped them to concentrate on the global problem-solving process.

Our own experiences with resequencing a course on optimization problems indicated that although students did develop an understanding of the global problem-solving strategy, some felt uncomfortable with trusting the derivative functions provided by the CAS, which they were not yet able to verify (Drijvers, 1999a). This probably has to do with the black box character of the CAS, which is addressed below. For the purpose of this study, the issue of transfer from concept development in the computer algebra environment to paper-and-pencil techniques is very interesting.

*The white box / black box issue*

The white box / black box issue addresses the role that computer algebra can play in the learning process, and in particular the relation with paper-and-pencil work. In his paper 'Should students learn integration rules?', Buchberger (1990) argued that while learning a new mathematical concept or technique the student should do the relevant operations by hand. The danger in using a computer algebra system in this phase is that the student may lose control and understanding. Once the new subject was mastered, computer algebra could be used to carry out the (now trivial) work, especially while studying a new topic at a higher level. According to Buchberger, the CAS was then used as a black box that enabled the student to focus on new aspects without being distracted by details that are already known. In addition to this, the CAS could also be used for scaffolding of gaps in the students' preliminary knowledge (Kutzler, 1994).

The white box / black box sequence can also be inverted. Proponents of the black box / white box approach suggested using the CAS as a generator of examples and as an explorative tool that confronts the student with new situations (e.g. Berry et al., 1993, 1994). This black box phase can lead to discoveries that form the basis of the explanation phase, where the findings are sorted and proved, or lead to the development of new concepts, which is the white box phase. This approach can be categorised as the amplifier function of computer algebra according to the Pea distinction. Our experiences with a black box / white box approach to discovering the chain rule for differentiation showed that it was successful in the sense that students discovered the procedure; however, this discovery had the character of pattern recognition without intrinsic meaning related to the concept of derivative (Drijvers, 1992). In cases where the discovery of the pattern can hardly be done without explanation, the black box / white box approach can be fruitful. An example of such a task is that of finding out how many zeros there are at the end of ‘big’ numbers such as 1997! or 1999! (Drijvers, 1999b; Trouche, 1997, 1998).

In addition to these two approaches, Macintyre and Forbes (2002) suggested a grey box approach, which consists of a step-by-step problem-solving procedure in which students touch on black box operations, but do not rely on the technological device in a totally dependent manner. Subprocesses are left to the computer algebra environment, but the overall strategy is determined by the user.

The white box / black box issue distinguishes two possible ways of using computer algebra in mathematics education. It is a teaching rather than a learning concept, and addresses the important question of the relationship between work in the computer algebra environment and paper-and-pencil work (Drijvers, 1995a). In this study, we consider establishing this relation as part of the instrumentation process, which will be discussed in Section 5.6.

*Research on CAS use in mathematics education*

Two early studies into CAS use in education indicated a development towards bridging the gap between the CAS research and research on mathematics education in general. These two studies were Repo’s study on differentiation and Pozzi’s study on algebra. The research by Repo (1994) reported in the paper ‘Understanding and reflective abstraction: learning the concept of the derivative in the computer environment’ is based on the concept of reflective abstraction, which is addressed in Sections 3.4 and 4.5 (Dubinsky, 1991; Piaget, 1980, 1985). By means of a genetic decomposition of the concept of derivative, a learning trajectory for this topic is developed, in which the computer algebra system Derive plays an important role. Repo reported that the conceptual understanding in the experimental group was significantly better than in the control groups. The Repo study integrates a theoretical framework from mathematics education with the, at that time, new approach to using computer algebra.

In his paper ‘Algebraic reasoning and CAS: Freeing students from syntax?’ Pozzi (1994) described how two students (16/17-year-olds) who were not high performers in mathematics discovered the mechanisms of the product and quotient rule for differentiation by using Derive. Besides the already mentioned problem of superficial pattern recognition rather than intrinsic generalization, the students paid considerable attention to the peculiarities of the computer algebra software, such as the specific means of representing and simplifying expressions. Pozzi argued that this can be a source of misconceptions:

> Without a clear basis for explaining particular manipulations, students may spontaneously develop their own models for what the computer is doing, partly or wholly based on surface features of the generated algebra. This can result in misconceptions unless the teacher supports the student in making sense of these moves at some semantic level.
> (Pozzi, 1994, p. 187)

New elements in the Pozzi study are the qualitative methodology, the age and level of the students and the attention to obstacles that CAS can present while developing algebraic insight. Especially the last-mentioned aspect is interesting for the purpose of our study, because it concerns the relation between CAS use and conceptual development, which is essential in the instrumentation theory.

### *Research on CAS use for the learning of algebra*

In this section we describe some relevant research on the use of computer algebra for the learning of algebra. It sketches the state of the art on which this study elaborates. In the Pozzi study, algebra was not the primary topic of investigation, although algebraic issues did come up during the research. In a similar way, algebraic aspects show up in the work of Artigue and of Heid et al. discussed below. After that, we consider three studies that really focus on the learning of algebra in a computer algebra environment.

In her survey of French research on the integration of computer algebra, Artigue described the phenomena of pseudo-transparency and double reference (Artigue, 1997a; Guin & Trouche, 2002). Pseudo-transparency means that the technique in the computer algebra environment is close to the paper-and-pencil technique, but is not exactly the same and sometimes has quite subtle differences. This is particularly relevant to algebra. For example, if one enters ($x$+5)/3, in many CASs one needs to use parentheses, but on the screen the formula appears without parentheses. In fact, the horizontal fraction bar in $\frac{x+5}{3}$ can be considered as a special notation for parentheses in the nominator, but the students often are not aware of that. In a similar way the parentheses used to enter the square root of an expression disappear immediately after entering. As a consequence of this pseudo-transparency, students working in a computer algebra environment sometimes are not 'discovering mathematics', as we would expect, but may be discovering the software with all its peculiarities. This is what is meant by double reference: referring to specific representations of the computer algebra tool rather than referring to mathematical concepts. The difficulties that are reported in the Pozzi study can be considered as results of this double reference.

Another study that reports on similar algebraic difficulties is that by Heid et al. (1999). The authors report that college-level students, while working on non-routine problems, had difficulties in interpreting the output of the computer algebra device,

in this case a TI-92 symbolic calculator. To be able to deal with this, students needed to develop the ability to recognize the typical idiosyncrasies of the computer algebra tool concerning the representation of formulas and expressions.
The small-scale study by Boers-van Oosterum (1990) investigated whether students could develop a rich understanding of the concept of variable by means of a computer-intensive approach. Beside computer algebra, the ninth-grade students also used spread sheets and graphical software. They showed a better conceptual understanding of variable than students from the control group did.
In a similar, computer-intensive approach, O'Callaghan (1998) investigated college-level algebra students' understanding of the function concept. He reported that students showed an improved understanding of the function concept and better problem-solving skills. No significant improvement was found, however, as regards reifying the function concept. The author reported that these students found it very difficult to reify functions.
The final relevant study is the research on the concept of variable using computer algebra by Brown (1998). Students involved in this study (13/14-year-olds) worked on 'THOAN's ('think of a number'-problems). Algebra in this study is primarily generalized arithmetic; numerical values are substituted into variables. This approach is close to the one of Graham and Thomas (1999, 2000), which was described in the previous section: the variable acts as placeholder or as unknown. The facilities that computer algebra offers are confined to the ones that support this placeholder conception, and the higher variable roles are not addressed.
The studies reviewed here show that doing algebra in a computer algebra environment is not as simple as it might seem. The findings indicate that adequately using a CAS requires insight into notational, syntactic and conceptual matters and is not just a question of 'leaving the work to the computer'. The technical work is related to conceptual understanding. As a second remark, we notice that the studies described here do not address the concept of parameter, nor do they seem to make full use of the opportunities that computer algebra offers. In our study, we try to go one step further, but without ignoring the findings from previous research.

### *Questions*

In spite of the original optimism manifest in many of the early publications on the integration of computer algebra into mathematics education, a number of questions are still unanswered (e.g. see Drijvers, 1997). For example, what is the relation between work with paper-and-pencil and work in the computer algebra environment? How does computer algebra use affect the curriculum? How does it influence conceptual understanding? What preliminary knowledge is required in order to use a CAS in a productive way? The Discussion Document for the 12th ICMI Study entitled 'The Future of the Teaching and Learning of Algebra' raised the following questions:

- *For which students and when is it appropriate to introduce the use of a computer algebra system? When do the advantages of using such a system outweigh the effort that must be put into learning to use it? Are there activities using such systems that can be profitably undertaken by younger students?*
- *What algebraic insights and 'symbol sense' does the user of a computer algebra system need and what insights does the use of the systems bring?*
- *A strength of computer algebra systems is that they support multiple representations of mathematical concepts. How can this be used well? Might it be overused?*
- *What are the relationships and interactions between different approaches and philosophies of mathematics teaching with the use of computer algebra systems?*
- *Students using different computational tools solve problems and think about concepts differently. Teachers have more options for how they teach. What impact does this have on teaching and learning? Which types of system support which kinds of learning? Can these differences be characterized theoretically?*
- *What should an algebra curriculum look like in a country where computer algebra systems are freely available? What 'by hand' skills should be retained?*

(Chick et al., 2001, p. 4)

It is clear that the integration of computer algebra into mathematics education raises many questions that are still to be answered. For the purpose of this study, interesting elements from the ICMI list are the question concerning the age of the students, the question of the relationship with an educational philosophy, the question of prerequisite symbol sense, and the issue of the 'by hand' skills. These questions are addressed in the following section.

## 5.4 Opportunities and pitfalls of computer algebra use

In this section we first address the opportunities that computer algebra offers for the learning of algebra and calculus. Then some pitfalls of the use of computer algebra are discussed. In Section 5.5, we specify the opportunities that are relevant to the learning of the concept of parameter.

### *Opportunities*

In Section 5.2 we identified opportunities of IT use in general for the learning of mathematics. Most of these opportunities also hold for computer algebra use. However, there are two additional issues that distinguish computer algebra environments from other technological tools and offer educational opportunities.

The first one is obvious: *computer algebra adds algebra*. Computer algebra tools offer a wide range of algebraic features that allow for a flexible and sophisticated approach to algebraic problems in a way that is not possible in other technological environments. This has far-reaching consequences. For example, notations and results of the CAS are mathematically correct and students feel confident about them (Heid, 1988). Exact solutions can be obtained, and expressions can be manipulated. The complete repertoire of algebraic procedures allows for more than one problem-solving strategy, thus providing opportunities for different approaches and for an exploratory approach to algebraic problems (Hillel et al., 1992). The results of such explorations can be sources for generalization and vertical mathematization. In this sense, computer algebra can act as an amplifier. Furthermore, the algebraic representations and manipulations can be linked with graphical and numerical representations and manipulations, thus stimulating a more integrated view of the mathematical concept. Another consequence is that more complex algebraic problems can be considered and worked out, such as problems with realistic data (Hillel et al., 1992; O'Callaghan, 1998).

The second point is a consequence of the availability of algebraic facilities. By freeing the students from the algebraic calculations, the computer algebra offers opportunities to *concentrate on the problem-solving strategy and on concept building*. While working in the computer algebra environment, students can think about the next step in the problem-solving process without worrying about the actual performance, which in a paper-and-pencil environment would require attention and might lead to disturbing errors. While using a CAS, the student can leave it to the computer to carry out the procedures and concentrate on the strategy and the concepts involved, as is reflected in the following quotations:

> By passing the majority of algebraic manipulation over to the computer, CAS can be used to redress the balance by moving away from the rote-learning of manipulation skills and towards conceptual understanding.
> (Pozzi, 1994, p. 173)

> Computer algebra systems seem to be ideal technologies for fostering growth in the level of understanding of mathematical concepts. Using the CAS, students can signify (and execute) actions on mathematical entities without needing to carry out the procedures by hand.
> (Heid, 2002, p. 107)

The use of computer algebra affects the relation between concepts and skills. In the previous section we discussed the idea of resequencing concepts and skills (Heid, 1988). Monaghan argued that CAS use helps to distinguish concepts and skills (Monaghan, 1993). Many other studies indicate that learning in a computer algebra envi-

ronment allows for emphasising conceptual understanding (Hillel et al., 1992; O'Callaghan, 1998). Students in Heid's study (1988) reported that CAS use helped them to keep track of the line of the overall problem-solving process.
A frequently mentioned benefit provided by the use of computer algebra is an increase in efficiency and a reduction in the time spent on algebraic calculations. In secondary education, where computer algebra is used by novice users, this can be questioned. The students lack both the mathematical expertise and the experience in using the CAS to be able to use it efficiently (Artigue, 1997a).

### *Pitfalls*

What are the most important pitfalls of integrating computer algebra into the learning of algebra? From the literature and from our own experience, we see the following three somewhat related difficulties (Drijvers, 2000, 2002b).
First, the computer algebra environment tends to have the character of a *top-down tool*. Because a CAS contains so much mathematical expertise, a risk of using it for an educational purpose is that results are obtained in a top-down manner. Everything 'is already there', is already invented. Furthermore, the CAS is often quite rigid as far as syntax is concerned, and does not support informal notations. In that sense, a CAS is rather an explorative than an expressive tool (Doerr, 2001; Doerr & Zangor, 2000). From the perspective of RME, these are unfavourable characteristics that may frustrate a student's motivation for construction and reinvention, unless adequate didactic measures are taken. Heck (2001) suggests the following approach to this pitfall for the case of symbolism and variables:

> Make a virtue of necessity: use the explicit symbolism of computer algebra to make pupils conscious of the versatile use of variables and use surprising results or bugs as opportunities to discuss mathematical topics further.
> (Heck, 2001, p. 215)

A second pitfall, which is related to the top-down issue, is the *black box character* of CASs. Like many other technological tools, a CAS usually does not provide an insight into the way it obtains its results. The software is a black box that does not show its methods. Specific to computer algebra tools is that these methods are often, even in simple problems, far more sophisticated than the methods students themselves would use. This may be manifested in unexpected outcomes or representations. As a result, students may perceive a lack of transparency: they feel unable to 'look through' the way the CAS arrived at its results, and cannot relate them to their own experience with paper-and-pencil techniques. This can lead to experiencing a lack of congruence between the computer algebra environment and the paper-and-pencil environment: the paper-and-pencil technique and the computer algebra technique are seen as unrelated, rather than as different implementations of the same technique

(Drijvers, 2002ab, in press).
The third pitfall is described by Guin and Trouche who argued that a CAS is a *closed microworld*, and that students do not automatically link this microworld to the mathematical or real life world 'outside' (Guin & Trouche, 1999). Of course, this pitfall also holds for other technical tools, but usually to a lesser extent. The perception of CAS as a microworld on its own may lead to the phenomena of pseudo-transparency and double reference, which were discussed in the previous section (Artigue, 1997a). Pseudo-transparency concerns the subtle differences between techniques in the computer algebra environment and paper-and-pencil techniques. Double reference refers to the fact that students are discovering the CAS microworld rather than developing mathematical understanding.
As a fourth pitfall of computer algebra use one might consider the risk that students will just *press the buttons* and thus no longer need to think. Our experiences are that students sometimes do indeed exhibit such 'fishing behaviour': they try several commands, hoping that one of them will lead to the correct answer. However, this tactic rarely works. Working in the CAS environment requires precise and well-formulated expression of what needs to be done, and therefore necessitates thinking (Malle, 1993). Zorn (2002, p. 1) stated: '... as machines do more and more lower level symbolic operations, higher level thinking and deeper understanding of what is really happening become more, not less, important'. The following remark by Heck is in line with these opinions:

> With a powerful repertory of tools at hand, it becomes even more important for the user to realize what goals she has, what results already have been obtained in this direction, and what steps can come next. Stated simply in a one-liner: computer algebra is not hands-on, but brains-on computing.
> (Heck, 2001, p. 213)

We think it is important to pay attention to these pitfalls. In the early years of using computer algebra systems in mathematics teaching and learning, there was much optimism among educators concerning the possible benefits of computer algebra tools for the learner, but questions concerning the pitfalls and obstacles were hardly addressed (see, for example, the topics addressed in Heugl & Kutzler, 1994). However, we agree with Artigue, who argued in 1997 that an analysis of constraints is relevant to the understanding of the potentials of a technological tool:

> The analytical work of identifying the constraints is essential for understanding the possible functions of the knowledge that a given software package allows for, for analysing the necessarily existing differences with the usual school functioning of this knowledge and identifying the conflicts and the problems of institutional legitimacy that may result from this.
> (Artigue, 1997a, p. 139, translation PD)

Guin and Trouche (2002) elaborated on this by distinguishing different types of con-

straints. Our point of view is that the relation between the potentials of computer algebra environments and the obstacles they may generate is important, because students' difficulties and errors can be opportunities for real learning, and also may provide insightful data for the teacher or the researcher.
Now that we have identified some of the opportunities and pitfalls presented by computer algebra use in mathematics education, in the following section we will focus on the specific potentials of computer algebra for the learning of the concept of parameter.

## 5.5 Computer algebra in this study

In this section we first discuss the potentials of computer algebra for the learning of the concept of parameter, which is the central issue of the first research subquestion. In the second part of this section, we motivate our decision to use the TI-89 handheld symbolic calculator as the computer algebra device in our teaching experiments.

### *Potentials of computer algebra in this study*

For the purpose of this study, we distinguish four important potentials of computer algebra environments that may enhance the understanding of the concept of parameter.
The first potential concerns *the object character of algebraic expressions and formulas*. In the CAS environment, expressions and formulas have an object character rather than a process character. They represent 'things' rather than 'actions', entities that can act as input for subsequent procedures such as substitution or factorization, and that show up as solutions of parametric equations. Therefore, a prerequisite for dealing effectively with expressions and formulas in the computer algebra environment seems to be their reification. On the other hand, computer algebra use is supposed to enhance the development of a more structural view of algebraic expressions rather than a limited operational view, because it forces the user to treat the formulas and expressions as objects (Section 3.4). In the case of function, Weigand (1995) argued that working on chains of functions and composite functions in the computer algebra environment supports reification. Monaghan (1994) stated that there is a danger that CAS use might even stress the object character of function too much, so that the function as input-output machine is neglected. For the case of algebraic expressions and formulas we do not share his concern, as these will still be considered as processes by means of actions such as substituting numerical values.
The second potential concerns *variables and parameters*, which in a cas environment do not refer directly to numerical values, but are objects that represent a reference set of possible values. The literal symbols in the cas therefore are conceptually different from their counterparts in numerical and graphical software packages that do not do algebra. This opens the way to go beyond the conception of variables as

placeholder, as pseudo-number, and to address the higher parameter roles of changing quantity and generalizer. Computer algebra goes beyond the level of the syncopic algebra and reaches for symbolic algebra (Monaghan, 1994). As a consequence, 'all literal symbols are equal' within the computer algebra environment. This allows for changing the roles that the user attributes to them, so that for example a parameter can act as unknown later in the problem-solving process. These features of computer algebra environments are expected to foster the understanding of the higher parameter roles and the flexibility to deal with them.
The third potential concerns *algebraic exploration*. The CAS can serve as an environment for algebraic experimentation that generates examples which can be easily changed and recalculated within the context of the CAS. These examples invite generalization and vertical mathematization. In this way, computer algebra is used as an amplifier of the possible range of examples, and – in Tall's terms – as generic organizer (Tall, 1989). This is expected to stimulate students to make generalizations and to discover the general in the particular. Such activities are expected to provide access to the concept of the parameter as generalizer.
Finally, the fourth potential is not exclusive to CAS, as it is provided by other technological tools as well. It concerns the facilities for visualizing the *dynamic effect of a sliding parameter*. The investigation of the way the dynamically changing parameter value affects the graph and other properties of a function is expected to reinforce the relation between formula and graph, and to improve the insight into the parameter as changing quantity at a higher level than the changing variable. The possibility to relate the effect of the sliding parameter to algebraic properties makes the computer algebra environment even more suitable for this purpose.

In the previous section we stated that computer algebra use offers not only opportunities but also pitfalls. For example, gradual formalization, as proposed within the RME framework, is difficult to realize within a computer algebra environment, which requires complete, well-formulated procedures as input. The process of symbolizing – which was described in Section 3.5 as the building up of a dialectic relation between notation, symbol and meaning – is not easy to carry out within a CAS. What is needed, though, is for the student to develop meaning for the conventional notations and commands that exist in the computer algebra environment. This process, which inevitably has a somewhat top-down character, can be considered as part of the instrumentation process addressed in the following section.

### *Which computer algebra device to use?*

At the start of the research, we had to decide which computer algebra tool would be the best one to use for the purpose of this study.
The first decision was whether to use PC software or a handheld device. Using a

desk top PC would offer the opportunity to have students work entirely at the computer by typing in their explanations with the mathematical output. Our first experiences with this approach, however, were not very successful. Although this may change in the nearby future, the students preferred to write down results and comments in their conventional paper notebook. Furthermore, in most secondary schools in the Netherlands it is hard to arrange an instructional sequence of several lessons in a computer lab. These labs are often not available for mathematics lessons, and usually do not offer opportunities for classroom discussions and demonstrations using projections of the computer screens. Finally, it is difficult to arrange the use of PC software at home, because students have different configurations at home and home licences are lacking.
For these mainly practical reasons, we decided to use a *handheld device*. In the literature, the private character of handheld technology – which makes students feel free to try things and to make errors, and makes them perceive the technology as personal – is seen as an advantage (Monaghan, 1994). There is a risk, however, of students working too individually when using handheld technology (Doerr & Zangor, 2000).
In order to overcome the limitations of the small screen of the handheld calculators we organized two lessons in the computer lab in the last teaching experiment.
Now, which of the available handheld symbolic calculators would be used? We chose the Texas Instruments TI-89 (Fig. 5.5). The main reasons for this were the completeness and high quality of the algebraic features that are also embedded adequately in the other modules of the machine. An additional advantage of the TI-89 was that the students of the school where the experiments took place would be using the Texas Instruments graphing calculator TI-83 in tenth grade, and we assumed that the change to the new machines would be easier it were the same brand[1].

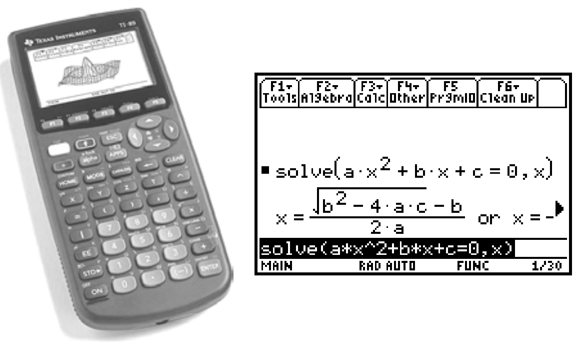



Figure 5.5 Front and screen of the TI-89 handheld symbolic calculator

1. We thank Texas Instruments for providing the machines for this research.

The screen on the right in Fig. 5.5 shows the bidimensional 'pretty print' output of the TI-89. The input in the black bar at the bottom is linear. That is a disadvantage, and the question is whether this would present the students with difficulties. The aim of the research is not to evaluate one specific computer algebra device or to compare it with others, but to draw conclusions that are as independent of the chosen platform as possible. In the meantime, it is inevitable that at the micro-level of syntax and command structure, the findings will be partially machine-specific.

## 5.6 Instrumentation of computer algebra

This section discusses the theory of instrumentation of IT tools, which plays a central role in the second research subquestion. It was developed in the context of educational research on the integration of computer algebra into mathematics education by mainly French researchers, who prefer to call it the 'instrumental approach' rather than the theory of instrumentation ('l'approche instrumentale', Trouche, 2002, p. 187).

First, we briefly address the roots of this approach: the Vygotskian perspective of tools, and the Vergnaud view of schemes. Then, the main elements of the instrumentation theory are presented: the duality tool - instrument, instrumental genesis, utilization schemes and instrumented techniques. The classroom aspects of instrumentation are discussed briefly, and finally we address the role of the instrumentation theory in this study.

### *Background*

One of the backgrounds to the instrumentation theory is the work by Vygotsky on tool use (Vygotsky, 1978). A central issue in Vygotsky's work is the idea that tools mediate between human activity and the environment. These cultural-historical artifacts can be material tools, but also cognitive tools, such as language or algebraic symbols. More recently, Noss and Hoyles (1996) argued that technological tools also can mediate between the learner and the knowledge that is supposed to be learned. From the Vygotskian perspective, the question in this study is how technological tools can act as mediators in the construction of new knowledge. An other recent development in the line of Vygotsky is the so-called cultural-historical activity theory (Engeström et al., 1999). According to this theory, learning is not considered to be an individual activity, but to be related to the 'community of practice' (Wenger, 1998).

A second background of the instrumentation theory is the notion of schemes as developed by Vergnaud, who elaborated on Piaget. According to Vergnaud, a scheme is 'une organisation invariante de la conduite pour une classe donnée de situations', an invariant organization of behaviour for a given class of situations (Vergnaud, 1987, 1996). For example, the scheme of counting is described as follows:

> Counting a set is a scheme, a functional and organized sequence of rule-governed actions, a dynamic totality whose efficiency requires both sensory-motor skills and cognitive competencies ...
> (Vergnaud, 1987, p. 47)

A mental scheme has an intention, a goal and it contains different components, such as ‘operative invariants’, the often implicit knowledge that is embedded in a scheme in the form of ‘concepts-in-action’ or ‘theorems-in-action’ (Guin & Trouche, 2002a; Trouche, 2000).

*Tool and instrument*

The theory of instrumentation emanates from cognitive ergonomics (Rabardel, 1995) and concerns handling tools. It builds on the notions of tool use in the work of Vygotsky. A key point in the theory of instrumentation is the idea that a ‘bare’ tool or artifact is not automatically a mediating instrument. As a simple example, let’s consider the hammer. A hammer may initially be a meaningless thing to a prospective user, unless he/she has used it before, or has seen somebody else using it. Only after the need to have something like a hammer is felt, and after the novice user has acquired some experience in using it, does the hammer gradually develop into a valuable and useful instrument that mediates the activity. The experienced user has developed skills to use it in a handy way and knows in what circumstances a hammer is useful.

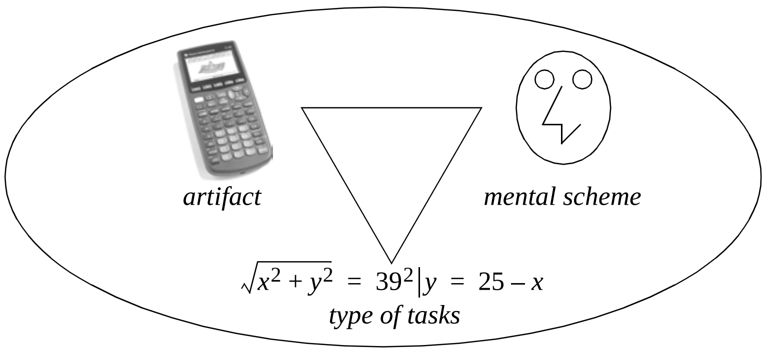



Figure 5.6 The instrument: a triad of artifact - mental scheme - task

A similar distinction between tool and instrument can be made for other artifacts, such as calculators and computer algebra environments. Rabardel speaks of an instrument when there is a meaningful relationship between the artifact – or a part of the artifact – the user and a type of tasks, in our case mathematical tasks (see Fig. 5.6).

The instrument, according to this view, consists not only of the part of the tool that is involved – in our case, for example, a module of the symbolic calculator or CAS – but also of accompanying mental schemes of the user – in our case the student – who knows how to make efficient use of the tool to achieve the intended type of tasks, which is the third component of the instrument.

### *Instrumental genesis and schemes*

The question now is how the availability of a tool or artifact leads to the development of a useful and meaningful instrument. The process of developing means to use the tool in an appropriate and sensible way is called the instrumental genesis. The results of this instrumental genesis are condensed in the form of utilization schemes; differently stated, the instrumental genesis consists of building up utilization schemes (schèmes d'utilisation in French). The idea of utilization schemes is close to the Vergnaud - Piaget schemes that were described above. The word 'scheme' can also be considered a synonym for 'schema' that is represented by the S in the APOS acronym for Action - Process - Object - Schema (see Section 3.4):

> A schema is a coherent collection of actions, processes, objects and other schemas that are linked in some way and brought to bear on a problem situation.
> (Cottrill et al., 1996, p. 172)

Two kind of utilization schemes can be distinguished (Guin & Trouche, 2002; Rabardel, 1995; Trouche, 2000). The first category comprises the *utility schemes* (schèmes d'usage) for adapting an artifact for specific purposes and thereby changing or extending its functionality. For example, a PC can be upgraded with a new version of a word processing software package, calculator menus can be customized, and additional programs and applications can be downloaded into a graphical calculator. These schemes concern the instrumentalization of the artifact. By instrumentalization we mean that the user adapts the tool.

The second category comprises the *schemes of instrumented action* (schèmes d'action instrumentée), which we define as coherent and meaningful mental schemes for using the technological tool to solve a specific type of problems. For example, the 'problem' of moving a text block while writing in a word processing environment can be solved by means of the cut-and-paste scheme. An experienced user applies this cut-and-paste scheme quickly and accurately by means of a sequence of key strokes and/or mouse clicks. A novice user, however, has to deal with the technical and conceptual aspects, such as the fact that the text block that he/she wants to move elsewhere is invisible for a moment after it has been cut. The instrumentation process leads to the building up of schemes of instrumented action that are useful for fulfilling a specific kind of task.

Of course, the instrumentation and instrumentalization – the development of utility schemes and schemes of instrumented action – are related, and in some cases it is not easy to decide what kind of schemes are developed. However, in this study we concentrate on the second category of schemes – those of instrumented action – because we focus on using the computer algebra tool to achieve specific tasks that will contribute to an improved understanding. To avoid the somewhat long term 'schemes of instrumented action', we abbreviate it to '*instrumentation scheme*'.

An instrumentation scheme has an external, visible and technical part, which concerns the machine actions. However, the most important aspect of the scheme is the mental, cognitive part. For example, suppose a student wants to solve the equation $2x^2 - 5 = 3^x$ with a graphing calculator. The graphical solving scheme involves the mental step of seeing both sides of the equation as functions $y_1(x) = 2x^2 - 5$ and $y_2(x) = 3^x$, which can be drawn. Furthermore, applying the scheme involves the conception of a solution as the *x* coordinate of an intersection point of the two graphs. Those are mental activities that give meaning to the technical actions such as entering functions, drawing graphs and calculating intersection points.

A well-known example that illustrates the relation between conceptual and technical aspects concerns the difficulties that students have with scaling the viewing window of a graphing calculator (Goldenberg, 1988). The instrumentation scheme that needs to be developed involves the technical skills of setting the viewing window dimensions, but also the mental image of the calculator screen as a relatively small window that can be moved over an infinite plane, where the position and the dimensions of a relevant rectangle need to be chosen. We conjecture that it is the conceptual part of this scheme that causes the difficulties.

These examples illustrate that an instrumentation scheme is an interplay between acting and thinking, and that it integrates machine techniques and mental concepts. Technical skills and algorithms on the one hand and conceptual insights on the other are inextricably bound up with each other within the instrumentation scheme. In the case of mathematical IT tools, the mental part consists of the mathematical objects involved, and of a mental image of the problem-solving process and the machine actions. Therefore, the conceptual part of instrumentation schemes include both mathematical objects and insight into the 'mathematics of the machine'. During the instrumental genesis, such mental mathematical conceptions can develop parallel to the development of the scheme itself.

Instrumentation, in short, concerns the emergence and evolution of utilization schemes. The instrumentation schemes to which we confine ourselves can have different scopes: some are elementary, simple schemes that serve as basic components for other, composed schemes. For example, the scheme to approximate exact answers by decimal values is a simple scheme. It can be integrated into a composed scheme, such as the scheme for calculating the zeros of a function that involves the approximation of the outcomes of a solve procedure.

In practice, the instrumental genesis – that is, the construction of utilization schemes with technical and conceptual sides – is not easy for the students. Often, this process requires time and effort. Students may construct schemes that are not appropriate, not efficient, or that are based on inadequate conceptions. Examples of difficulties with instrumentation schemes can be found in Drijvers (2001abc, 2002ab, in press) and Drijvers and van Herwaarden (2000, 2001). In Chapter 10, similar difficulties that we observed in this study are addressed in detail.

### *Instrumented techniques*

The technical side of an instrumentation scheme may seem the less interesting part, as the primary interest in mathematics education concerns the cognitive, conceptual development.
However, the two quotations below indicate that the technical work in a computer algebra environment is not per se reduced, and that an eventual reduction of the technical work does not automatically induce conceptual reflection.

> In fact, the view on the computer environment as an environment that would allow for 'unbalancing' the relation between technical work and conceptual work, allowing for a kind of economy of the technical work left over to the machine, in order to concentrate on the conceptual work, does not seem to be supported by our observations.
> (Artigue, 1997a, p. 164, translation PD)

> We found a common assumption: CAS lightens the technical work in doing mathematics, and then students will focus on application or understanding. (...) Our survey of the French classrooms showed neither a clear lightening in the technical aspects of the work nor a definite enhancement of pupils' conceptual reflection (Lagrange 1996). Technical difficulties in the use of CAS replaced the usual difficulties that pupils encountered in paper/pencil calculations. Easier calculation did not automatically enhance students' reflection and understanding.
> (Lagrange, 1999c, p. 144)

Lagrange elaborated on the idea of instrumented techniques, in particular for the case of computer algebra environments (Lagrange, 1999abc, 2000, 2002). He describes a technique as follows:

> In this link with concepts, the technical work in mathematics is not to be seen just as 'skills and procedures'. The technical work in a given topic consists of a set of rules, and methods and, in France, we call such sets 'techniques', as they are less specific and imply less training than 'skills' and more reflection than 'procedures'.
> (Lagrange, 1999c, p. 144)

In line with this, we define a technique as a set of procedures in a technological or non-technological environment that is used for solving a specific type of problems. Lagrange argues that techniques are still important in the computer algebra environ-

ment, because they are related to the conceptual aspect by means of the instrumentation schemes. He states that the character of the instrumented techniques in the computer algebra environment differs from that of paper-and-pencil techniques. Therefore, developing computer algebra techniques adds new aspects to the paper-and-pencil techniques and makes the students' repertoire richer and more multiform.
As is the case with instrumentation schemes, techniques can be elementary or more complex and composite. The elementary instrumented techniques, such as the direct application of one single command, are called 'gestures', and a composed instrumented technique usually consists of a set of such gestures. The word 'gesture' here has a figurative sense, and is not limited to physical movements.

*Schemes and techniques*

Now, what is the difference between an instrumentation scheme and an instrumented technique? According to Lagrange (1999abc, 2000, 2002), the instrumented technique is more than a sequence of key strokes; in fact, the instrumented technique is close to the instrumentation scheme.

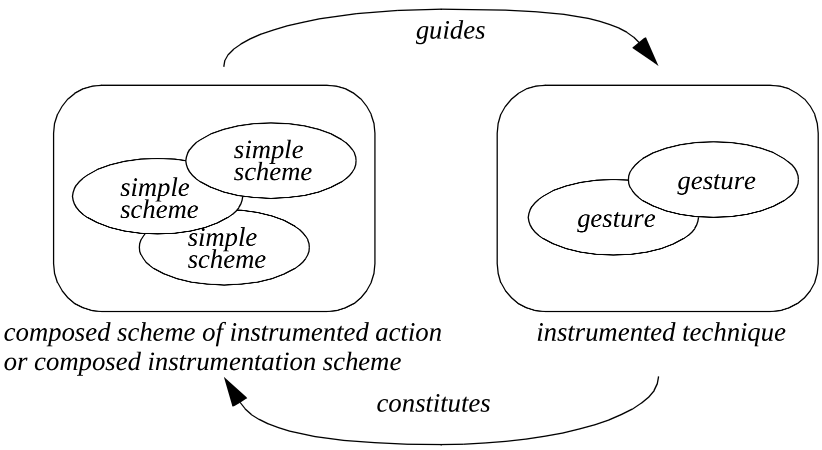



Figure 5.7 Instrumentation scheme and instrumented technique

The main difference, however, is that the instrumented technique concerns the external, visible and manifest part of the instrumentation scheme, whereas in the instrumentation scheme, the invisible mental and cognitive aspects are stressed.
In fact, instrumented techniques can be observed, discussed and taught, whereas the more abstract instrumentation schemes are more difficult to observe. The visibility of instrumented techniques is why they – rather than the instrumentation schemes, which have a more internal and personal character – are the gateway to instrumental genesis.

The relation between instrumentation scheme and instrumented technique is symbolized in Fig. 5.7, adapted from Trouche (2002, p. 208).

### *Instrumentation and classroom practice*

So far instrumental genesis has been presented as the development of mental instrumentation schemes that have an individual character. However, the instrumental genesis is not only a personal process (Docq & Daele, 2001). In education it takes place in an educative setting, such as a classroom, and the instrumentation thereby acquires a collective character as well. The way in which the teacher takes care of the orchestration of the IT tools will influence the instrumentation, as will the discoveries or mistakes made by other students. As indicated above, this orchestration takes place by teaching the 'tangible' instrumented techniques. The research studies by Kendal and Stacey (1999, 2001) clearly indicated that the teacher may strongly influence the collective instrumentation by privileging specific techniques and disprivileging others. We argue that the successes that were reported in the studies by Doerr and Zangor and by Trouche can be attributed for an important part to the effective and adequate attention that was paid to the collective instrumentation process and the development of common instrumented techniques by the teacher (Doerr & Zangor, 2000; Trouche, 1998).

### *Instrumentation in this study*

The instrumentation theory was originally developed in the domain of cognitive ergonomics, but (mainly French) mathematics educationalists have applied it to the learning of mathematics using IT tools and computer algebra in particular (Artigue, 1997b, 2002; Guin & Trouche, 1999, 2002; Lagrange, 1999abc, 2000, 2002; Ruthven, 2002; Trouche, 2000, 2002). Of course, that is the role of instrumentation in this study as well.

In this study, the instrumental approach offers a framework for investigating the interaction between student and technological environment. The combination of technical and conceptual aspects within the instrumentation scheme makes the theory promising for this purpose, because it goes beyond the somewhat naive idea of reducing skills and reinforcing concepts.

In addition, the theory of the instrumentation of IT tools can be related to the theories on symbolizing (Section 3.5). Both symbolizing and instrumentation concern the simultaneous development of meaning, notation and external representation. In the case of instrumentation, these are condensed into utilization schemes and instrumented techniques that also include the technical implementation within the computer algebra environment. The theory of instrumentation is a local, technology-specific addition to the perspective of symbolizing, and will be used in Chapter 10 as a theoretical framework for analysing, understanding and interpreting the interaction

between student and CAS.
The examples presented above show that the theory of instrumentation of IT tools concerns schemes for different kinds of tasks. However, in this study we confine ourselves to instrumentation schemes that address algebraic topics. In particular, we study simple instrumentation schemes for solving equations, for expressing variables in others and for substituting expressions, as well as the combining of these simple instrumentation schemes into composed instrumentation schemes.
This ends the theoretical part of this thesis. In Chapter 6 we will describe how the theoretical points of departure are transformed into a hypothetical learning trajectory and instructional activities, and report on the results of the first teaching experiment.

# 6 The G9-I research cycle: learning trajectory and experience

## 6.1 Introduction

This chapter is the first of the empirical part of the thesis. In Chapter 2, we described that this design research study consists of three main research cycles, indicated as G9-I, G9-II and G10-II, and one intermediate cycle, G10-I. At the heart of each of these cycles is a teaching experiment. The arrangement of these teaching experiments is shown in Fig. 6.1.

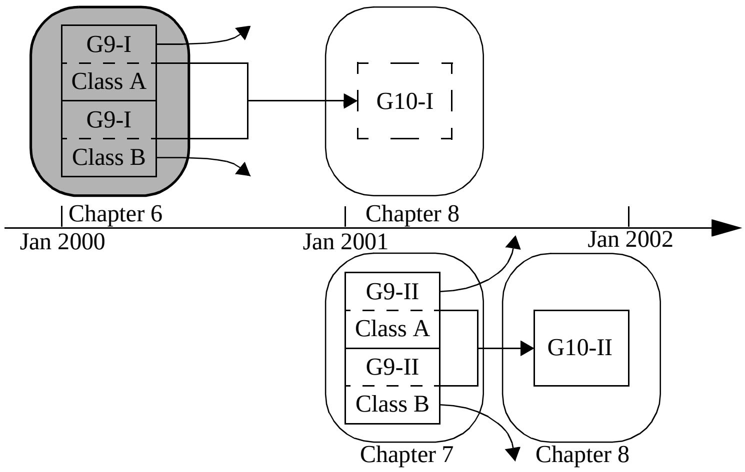



Figure 6.1 Teaching experiment arrangement

This chapter addresses the first research cycle in ninth grade, G9-I. First we describe the development of the hypothetical learning trajectory (HLT), which was based on the analysis of the concept of parameter, on the analysis of the opportunities of computer algebra and on the theoretical framework indicated in the previous chapters. The HLT includes the global learning line, the intended mental activities of the students and expectations concerning the contribution of computer algebra (Section 6.2). The key instructional activities and their expected effects are explained in Section 6.3, which also provides an overview of the G9-I teaching materials.

Then the experiences during the teaching experiment are described. The educational setting, the pretest, the classroom experiences, the posttest, interviews and reactions are presented consecutively. We briefly compare the different data types (6.4).

Finally, we reflect on the expectations that we had at the start of the teaching experiment, and formulate the feed-forward of the G9-I research cycle for the next cycles (6.5).

These follow-up research cycles will be discussed in a similar manner in Chapters 7 (G9-II) and 8 (G10-I and G10-II). Together, Chapters 6, 7 and 8 thus provide a chronological description of the development of the HLT throughout the research cycles. After that, Chapters 9 and 10 will revisit these cycles and their associated teaching experiments from the perspective of the two research subquestions, the first on the concept of parameter (Chapter 9) and the second on the instrumentation of computer algebra (Chapter 10).

## 6.2 The G9-I hypothetical learning trajectory for the parameter concept

In Chapter 2 we described the role of the hypothetical learning trajectory in this study. The design process of HLT and instructional activities makes explicit the choices that are made during the research. These choices illustrate the shift of focus during the subsequent research cycles and the development of the local instruction theory. In the HLT, the intended learning trajectory is described in terms of the mental activities of the students and the expected effect on concept development.

In this section we first describe the starting points for the HLT and the expectations that are investigated in this first teaching experiment. Then we motivate the global trajectory through the different parameter roles. The activities that are supposed to bring about the transitions between the different parameter roles are described next. This leads to the global HLT. Finally, we address some instrumentation aspects in the learning trajectory.

### *Starting points and expectations*

The starting point for the development of an HLT for the concept of parameter is the level structure that was developed in Section 4.5. There, we identified the placeholder as the ground-level parameter role, and the generalizer, changing quantity and unknown as the three higher roles of the concept of parameter. The reification of algebraic expressions and formulas was considered to be important in that conceptual development (Section 3.4). Chapter 4 ended with some suggestions as to how this higher level understanding might be brought about. Assuming that students have a conception of the parameter as a placeholder, the HLT takes this as the starting level and its goal is to extend the concept of parameter towards the three higher roles.

In Section 5.5, the main potentials of computer algebra use for the purpose of the higher level understanding of the concept of parameter were identified. The work in the computer algebra environment was supposed to accentuate the object character of algebraic expressions and formulas, to surpass the limited concept of variables and parameters as placeholders, to allow for algebraic exploration and to visualize the dynamic effects of the parameter change on graphs.

If we combine these means of achieving the higher level understanding with the computer algebra potentials, we come to the following expectations that inform the development of the HLT for G9-I:

1 Within the computer algebra environment, students can easily work through similar examples that vary only with respect to the parameter value. Repetition of the problem-solving procedure is expected to make them aware of the overarching method and of the power of *generalization*. The specific cases can be considered as examples for every possible parameter value and for the generic class of situations. The computer algebra device also allows for repeating the procedure for the general case. This is expected to facilitate the transition from the parameter as placeholder to the parameter as generalizer.
2 The calculation of general solutions of parametric equations in the computer algebra environment and the substitution of algebraic expressions and formulas are expected to make students *reify* formulas and expressions.
3 *Graphical models* are expected to make students extend their placeholder conception of the parameter towards the view of the parameter as generalizer (a sheaf of graphs) and changing quantity (a sliding graph or a sequence of graphs).
4 A specific feature of the graph or a condition in graphical terms may cause the parameter to take the role of *unknown.* The flexible way in which computer algebra environments deal with variables and parameters is expected to support the students in making such *shifts of roles*.

As we described in Section 5.6, the instrumentation of the computer algebra tool requires our attention as well. As a consequence of the above-mentioned expectations, we expect that the most important instrumentation issues concern solving equations and substituting expressions. The solve instrumentation scheme is expected to draw the students' attention to the variable and parameter as unknown. The substitution instrumentation scheme, when applied to algebraic expressions, is expected to foster the reification of formulas and expressions. A composed instrumentation scheme that combines solving and substituting – the isolate-substitute-solve scheme (ISS) that is described below – is expected to improve the understanding of the problem-solving strategy and to generate a more flexible way of dealing with the different roles of variables and parameters.

*A trajectory towards the higher level understanding of the parameter concept*

The first design question to answer while developing the learning trajectory concerns the order of the higher parameter roles. Which of the three higher parameter roles should be addressed first from the placeholder role? What activities invite the student to advance towards the higher level? How can the use of computer algebra support or foster this process?

In the G9-I teaching experiment, we chose the line placeholder - generalizer - changing quantity - unknown. The following arguments motivated this choice.

First, we consider the unknown to be best addressed as the last parameter role. For

'ordinary' variables, the role of unknown is often taken as the starting point of the concept development (e.g. Bills, 2001). It is argued that this was the historical development of the concept of variable, and that the question of which value(s) of a variable make(s) a condition become true is a natural one for students. The next step is often the variable as undetermined. In the case of parameter, however, this does not seem to be the adequate trajectory, because the parameter as 'changing constant' first appears as a constant. It is only after that, that the parameter value changes. This change involves the concept of the 'sliding parameter', which leads to a new but similar situation. Only after the student realizes that the 'family parameter' generalizes over all these situations is the question appropriate to 'filter' out parameter values that fulfil specific conditions within this class. This is why we think that the parameter as unknown should be addressed only after addressing the generalizer and changing quantity role of the parameter.
The next question, then, concerns the order of the generalizer and the changing quantity roles of the parameter. If we think of the process-object duality, and in particular of the idea that the operational precedes the structural, the optimal order seems to be to start with the parameter as changing quantity, which has a more dynamic and operational character than the parameter as somewhat static generalizer. An argument for the inverse order, the line generalizer - changing quantity, is that for the concept of variable the changing quantity role, which is characteristic of functional algebra, developed historically after the variable as generalizer. Also, the generalizer role of the parameter is considered to be the most important in this study, so why not address the primary focus directly? For the G9-I learning trajectory, we found the latter arguments decisive and chose the order generalizer - changing quantity.

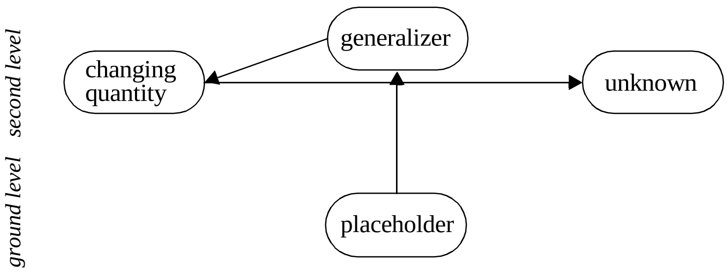



Figure 6.2 Learning trajectory through parameter roles in G9-I

The conclusion so far is that the global learning trajectory for G9-I follows the line generalizer - changing quantity - unknown. This is sketched in Fig. 6.2, which is a variant of the level structure in Fig. 4.2. The design problem now is to define student activities that bring about the conceptual transitions from the placeholder role of the parameter to the generalizer, from the generalizer to the changing quantity, and from the changing quantity to the unknown.

*How to bring about the transitions?*

The first step towards solving the design problem is to globally describe the activities that foster the transition to the higher parameter roles.

The transition *from the parameter as placeholder to the parameter as generalizer* is expected to be brought about by confronting the student with several situations that are mathematically similar but have different parameter values. The idea is that the student perceives the similarity of the problem-solving procedure in spite of the different numerical values. This is supposed to elicit the wish to 'solve the problem for once and for all', viz. to find a solution that includes all the different numerical cases. To represent such a general solution, algebraic expressions and formulas are used that need to be treated as mathematical objects, that do not refer to a calculation recipe, but represent the solution of a class of problems. This involves the mental reification of expressions and formulas. Computer algebra use may support this development by solving equations algebraically, by allowing for easy recalculation of similar problem situations with different parameter values, and by generating general solutions. Graphs can be used for visualization in this process, and sheaves of graphs may visualize the class of situations.

The transition *from the parameter as generalizer to the parameter as changing quantity* is expected to be brought about by a graphical approach, in which the parameter value changes gradually and systematically. The student is asked to study the effect of the changing parameter value on the graph. Mentally, the student realizes that changing the parameter value affects the problem situation as a whole, as well as the complete graph. Computer algebra can be used in this transition for substituting numerical values and for graphing a sequence of 'shifting' graphs.

The transition *from the parameter as changing quantity to the parameter as unknown* is expected to be fostered by applying additional criteria or properties to the situation or to the graph. Such an additional criterion requires the filtering of appropriate parameter values out of the set that the sliding parameter goes through. For example, a criterion might be 'For which values of $p$ does the graph have only one zero?'. In the context of a sliding graph, such questions may arise quite naturally. Often, the condition will concern graphical properties (intersection points, tangent points, vertices), as they are easy to imagine for the students. Such questions are expected to lead to a mental shift of parameter role, so that the parameter will act as unknown rather than as changing quantity. Computer algebra allows for this shift, because for the CAS, all literal symbols can play any role, and equations can be solved with respect to any literal symbol. The parameter as unknown plays only a limited role in the learning trajectory for G9-I. The teachers discouraged stressing this role, because they considered the shift of roles as too difficult for the students.

An intermediate step towards the parameter as unknown might be the parameter as a second, independent variable. In that case, three-dimensional graphs can be drawn

in the computer algebra environment. An argument for this approach is the experience from previous experiments that students find three-dimensional graphs quite fascinating. This intermediate step is addressed briefly in the HLT.
The global learning trajectory for the G9-I teaching experiment is sketched in Fig. 6.3. The learning trajectory runs from top to bottom. The columns subsequently describe the role of the parameter, the algebraic and graphic meaning of this parameter role, the students' overt and mental activity that brings about the transition, and the way computer algebra supports this activity.

| role of parameter | algebraic meaning | graphic image |
|---|---|---|
| placeholder | one numerical value | one graph |
| generalizer | a set of numerical values | a sheaf of graphs |
| changing quantity | a changing numerical value | a changing graph |
| unknown | subset of numerical values | a subset of graphs in a sheaf |

| student activity | mental activity | role of computer algebra |
|---|---|---|
| explore similar situations for different values | see similarity, generalize, formula as object | graph, solve equations, general solutions |
| change parameter values | see dynamical changes, graph as object | substitute parameter values, animate graph |
| filter specific parameter values | change parameter role | solve with respect to parameter |

Figure 6.3 HLT scheme for the parameter concept in the G9-I teaching experiment

### *Instrumentation aspects within the HLT*

The instrumentation of relevant algebraic procedures requires attention in the learning trajectory. The instrumental genesis of the solve instrumentation scheme involves the awareness that an equation is always solved *with respect to* a variable or parameter and that the solution of a parametric equation is an expression. This aspect is addressed by means of solving equations with respect to different variables, and by solving equations containing parameters.
The instrumentation of the substitution instrumentation scheme is addressed by means of having students substitute algebraic expressions into other formulas. The intention is for this to promote the reification of the expression that is submitted to substitution.
The instrumentation of the composed instrumentation scheme that combines solving

and substituting is stimulated by comparing the execution in the computer algebra environment with the performance by hand.

## 6.3 Key activities in the teaching materials

In this section we describe the elaboration of some of the key activities of the HLT of the concept of parameter into assignments in the teaching materials. These activities were designed in interaction with the HLT development. In Chapter 2 we described the design principles of guided reinvention, didactical phenomenology and mediating models that guided the design process.

For the G9-I teaching experiment the teaching materials consist of two booklets, ‘Introduction TI-89’ and ‘Changing algebra’. The first booklet aims at introducing the students to using the TI-89 symbolic calculator. Such an introduction is needed, because a successful completion of the learning trajectory demands some preliminary machine skills. Furthermore, it aims at directing the students’ attention to CAS techniques that are important in the HLT, such as substitution, solving equations and manipulating expressions and formulas. The second booklet, ‘Changing algebra’, aims at the development of the concept of parameter in the indicated way. Fig. 6.4 provides the tables of contents of both booklets; the complete manuscripts (in Dutch) are available at www.fi.uu.nl/~pauld/dissertation.

| ***Introduction TI-89*** | ***Changing algebra*** |
|---|---|
| 1 Arithmetic | 1 Sum and difference |
| 2 Tables | 2 Algebraic solution |
| 3 Graphs | 3 The power of algebra |
| 4 Algebra | 4 Rectangles with a fixed perimeter |
| 5 Investigation task | 5 Substitution |
| | 6 A network of graphs |
| | 7 A sheaf of graphs |
| | 8 Three-dimensional graphs |
| | 9 Graphs with Derive |
| | 10 Summary and exercises |
| | 11 Investigation tasks |

Figure 6.4 Contents of the two teaching units

The first three sections of ‘Introduction TI-89’ address the elementary skills of operating this device. Section 4 concerns algebra, and solving equations and substituting expressions in particular. It prepares for the future topic of the teaching experiment. The goal of the substitution assignment in Fig. 6.5 is to make students realize that a formula within the computer algebra environment can be ‘picked up’ as a whole to replace a variable. This is expected to prepare them for the reification of expressions and formulas. The question is whether students understand what happens when an expression is substituted for a variable.

**4.17** The volume of a cylinder is equal to the product of the area of the bottom and the height, in short a * h.
The area of the bottom equals π times the square of the radius.
So enter: a*h | a = π * r^2
You can get π with 2nd ^.
This way, the formula for the area is substituted into the formula of the volume.

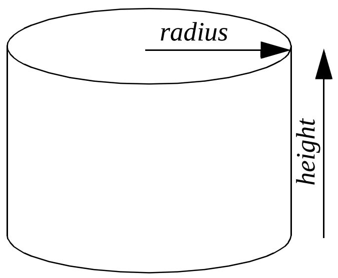



Figure 6.5 Substitution task in 'Introduction TI-89'

Section 5 of 'Introduction TI-89' contains a series of tasks on pattern recognition, and prepares for the transition from placeholder to generalizer. The aim of the assignments (Fig. 6.6) is that students experience the power of the computer algebra environment for generating examples that are the source materials for generalization. The question is whether students can attach meaning to the pattern or whether they will simply take it as a superficial recognition task. Fig. 6.6 shows that the introductory booklet contains screen dumps of the computer algebra device, so that students can verify their results.

**5.1** Enter $(x + y)^2$
Have the brackets expanded.
**5.2** Do the same for $(x + 2{\cdot}y)^2$, $(x + 3{\cdot}y)^2$, $(x + 4{\cdot}y)^2$, and so on. What patters do you discover in the outcomes?
**5.3** Verify the pattern you found by expanding $(x + c{\cdot}y)^2$.

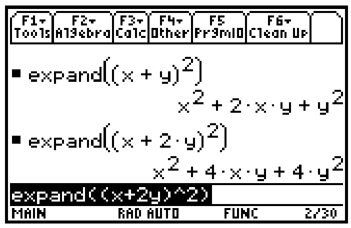



Figure 6.6 Pattern matching task in 'Introduction TI-89'

The first three sections of the booklet 'Changing algebra' focus on sum-difference problems. The task is to find two numbers when their sum and difference are given. This historical problem seems suitable for the transition from placeholder to generalizer. Students approach this problem by means of a fair share method that is close to the Rule of False Position that is described in Section 4.3 (van Amerom, 2002; Radford, 1996; Stacey & MacGregor, 1997). Students are invited to describe their approach in natural language as a means to temporarily avoid algebraic formalism (see Fig. 6.7). If they are not able to do so, it may help to ask them how they would solve the same problem with 800 rather than 700 and with 500 rather than 600.

**2.1** Two numbers, $x$ and $y$, are 700 together, and $x$ is 600 bigger than $y$:

$$x + y = 700$$
$$x - y = 600$$

**a** How big are these numbers?
**b** Try to write down a recipe for the solution of this problem. Indicate which steps you carry out to calculate $x$ and $y$.

Figure 6.7 Sum-difference assignment

In order to help the students with machine operations, TI-89 help frames are included in the teaching material. Fig. 6.8 shows such a frame that belongs with assignment 2.1 shown in Fig. 6.7.

| *What do you want to do?* | *How do you do that with the TI-89?* |
|---|---|
| Isolate a variable in an expression, for example solve $y$ in $x + y = 700$ | choose F2 option 1: solve<br>complete: solve (x + y = 700, y)<br>finish with ENTER |
| Carry out a substitution, for example replace $y$ in $x - y = 600$ by 100 or by $700 - x$ | enter: x – y = 600 \| y = 100<br>or x – y = 600 \| y = 700 – x<br>finish with ENTER<br>read the vertical bar \| as 'with' or 'wherein' |

Figure 6.8 TI-89 help frame

Section 4 of 'Changing algebra' addresses the sum-product problem in a similar way. The dimensions of a rectangle with a fixed perimeter and surface are to be found. The procedure is generalized into a generic one. Compared to the sum-difference problem, the solution formulas are more complex and harder to understand.

Section 5 contains a mixture of assignments that address substitution and generalization. Most of the problem situations have a geometrical context. Fig. 6.9 provides an example concerning a right-angled triangle. The idea is that solving the problem for two pairs of parameter values opens the way to the general solution in question **c**, so that the transition from placeholder to generalizer is addressed once more. The question is whether students are able to make this jump.

**5.4** The two right-angled edges of a right-angled triangle together have a length of 31 units. The hypotenuse is 25 units long.

**a** How long is each of the right-angled edges?

**b** Solve the problem also in case the total length of the two edges is 35 instead of 31.

**c** Solve the problem in general, that is without the values 31 and 25 are given.

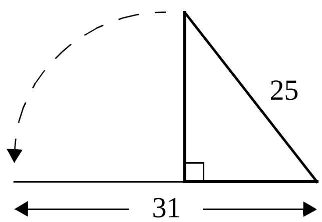
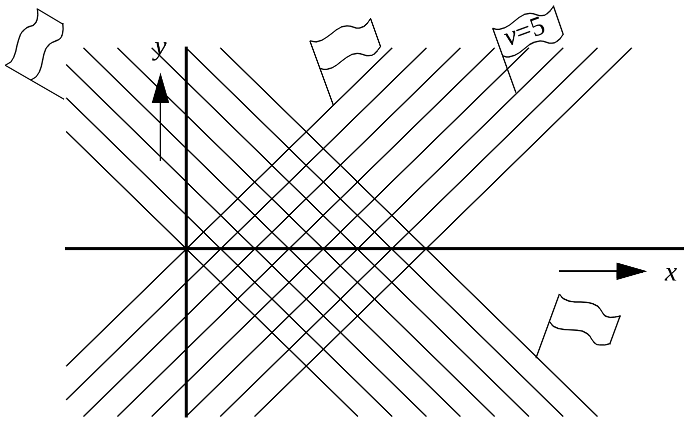



Figure 6.9 Generalization in the right-angled triangle situation

In Sections 6 and 7 the transition to the parameter as changing quantity is made. For example, Fig. 6.10 shows a task in which the effect of a ‘sliding parameter’ on the graph is asked. Can students connect the dynamics in the graph with the formula?




Above you see the graphs of $y = s - x$ and $y = x - v$ for several values of *s* and *v*: *s* varies from 0, 1, 2,... to 7 and for *v* the same. This yields a network of graphs. With ‘flags’ one can indicate which value of *s* or *v* determines the line.

**6.5 a** Make the drawing on the screen of your TI-89.

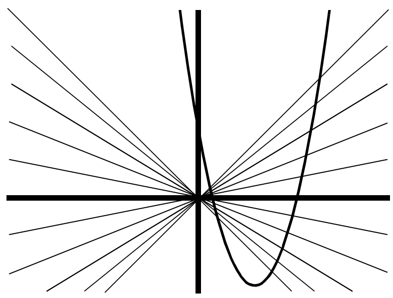
**b** What happens with the graph of $y = s - x$ when *s* grows?

**c** The same question for the graph of $y = x - v$ when *v* grows.

Figure 6.10 Assignment on the ‘sliding parameter’

Section 8 of ‘Changing algebra’ treats three-dimensional graphs that can be drawn with the TI-89, although this requires some patience from the user, and the screen resolution is limited. To overcome these limitation, Section 9 concerns a side-step to a computer algebra environment on the PC that allows for fast, high-quality graphing.
Section 10 summarizes the teaching unit and contains some assignments in which the parameter role shifts to the role as unknown. For example, in the context of the right-angled triangle (assignment 5.4, see Fig. 6.9), the length of the hypotenuse is no longer fixed but is called *k*. The question in task 10.10 is for what values of *k* the problem has no solution. Such assignments are expected to support the transition from the parameter as changing quantity to the parameter as unknown.

**11.1 *A ‘fan’ of lines***



The figure above shows the graph of the function *y* with

$$y = 4x^2 - 8x + 1$$

Also, a ‘fan’ of lines through the origin is drawn. As you see, the number of intersection points between these lines and the parabola varies.
Questions now are:

- Between which values can the number of intersection points vary?
- What determines the number of intersection points of line and curve?
- Find a rule that indicates how you can infer the number of intersection points from the equation of the line.

Figure 6.11 Investigation task on the sheaf of lines and the parabola

Section 11 contains two investigation tasks. The first one is shown in Fig. 6.11. The idea is that the student needs to use the different parameter roles. First, in order to find the general equation of the line through the origin, a parameter as generalizer is needed. Second, a dynamic view of this parameter as changing quantity is required to see how the number of intersection points changes as the parameter value increases. Third, the general coordinates of the intersection point(s) have to be calculated, which requires the acceptance of an algebraic expression as the solution to the equation. Finally, the two solutions to the equation are to be set equal, in order to have one single intersection point. This equation has to be solved with respect to the parameter, which thus acquires the role of unknown. We hope to see whether students are flexible enough to switch between the different parameter roles.

## 6.4 The G9-I teaching experiment

The aim of the G9-I teaching experiment was to investigate how the developed HLT would play out in the classroom and whether computer algebra use improved the understanding of the concept of parameter in the way it was expected to. Also, we wanted to know how the instrumentation of the computer algebra environment took place.

The method of carrying out the teaching experiments in general is described in Section 2.5; information on the school where the experiment was held is provided in Section 2.8. In the present section, we first sketch the educational setting of the G9-I teaching experiment (6.4.1). Then we describe the results of the pretest (6.4.2). The global findings during the experiment are presented in 6.4.3. Section 6.4.4 contains the results of the final test. In 6.4.5 and 6.4.6, the data from the evaluation interview with the teachers and the student reactions are presented. Finally, we compare the different kinds of data (6.4.7).

### 6.4.1 *Educational setting*

The G9-I teaching experiment took place in two ninth-grade classes, with 53 students all together. It lasted for 22 lessons in class A and 24 lessons in class B, including a pretest lesson and a posttest lesson. Class A (27 students: 13 females, 14 males) was a regular Dutch class with a good working attitude. The average report mark for mathematics on the previous report was 6.2 out of 10. The female mathematics teacher was qualified to teach mathematics to 12- to 16-year-old students, and had many years of experience. Her teaching style can be characterized as controlling. For example, she often wrote down a timetable for the lesson on the blackboard and she regularly checked student notebooks. During whole-class explanations and discussions she demanded the attention of all the students, and created a climate in which the students listened to each other. She invited students to contribute and was open to interaction. The classroom atmosphere was good. The teacher’s experience with IT use in teaching was limited to some incidental lessons using graphical software in the computer lab. For the students, using IT in the mathematics lessons was quite new as well.

Class B consisted of 26 students (16 females, 10 males). The average report mark for mathematics on the previous report was 6.0 out of 10. The students’ attitude was

not as good as in class A. Some students were often absent. The atmosphere was less quiet than in class A. Some of the students sometimes talked loudly in class. The class B mathematics teacher – also female and qualified to teach mathematics to 12- to 16-year-old students – was not very experienced. The organization of the lessons, and in particular of the whole-class explanations and discussions that require all the students' attention, was not easy for her. Interactive whole-class discussions on the topic were rare. The planning of the lesson was not always effectuated and the whole-class explanation sometimes lasted longer than estimated because of the students' lack of attention. When students were working individually or in pairs, efficiency was sometimes low. Neither the teacher nor the students had substantial experiences with using technology in mathematics lessons.
See Section 3.7 for an indication of the preliminary knowledge of algebra of the students from both classes. Information on data collection in this teaching experiment is given in Chapter 2.
Before the teaching experiment, the two teachers and the researcher discussed the aims and goals of the experiment. The teachers commented on the teaching materials, and a planning for the teaching experiment was agreed upon. During the experiment, each lesson was discussed by teacher and researcher, and decisions for the next lesson were taken.

### 6.4.2 *Pretest*

The G9-I teaching experiment started with a pretest that was taken by 49 of the 53 students. The aim was to determine the knowledge of the students concerning parameters, general solution and dynamics at the start of the experiment. Furthermore, some of the pretest items were intended to be matched with posttest items. A first version of the pretest had been tested in a pilot test at another school. The full text of the pretest is presented in Appendix A. Table 6.1 summarizes the results.

| Item | 1 | 2 | 3 | 4 | 5 | 6a | 6b | 6c |
|---|---|---|---|---|---|---|---|---|
| Correct | 47 | 35 | 10 | 39 | 45 | 47 | 43 | 35 |
| Incorrect | 2 | 14 | 39 | 10 | 4 | 2 | 6 | 14 |

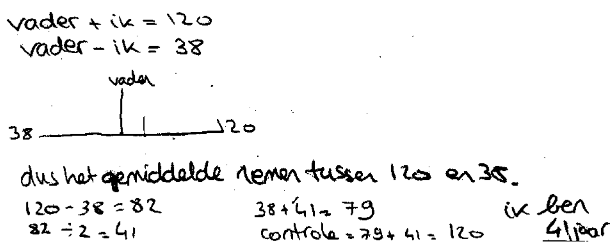
Table 6.1 Results of the G9-I pretest

We will now discuss the findings on the pretest items that involve preparation for generalization (2, 3), preparation for dynamics (1, 6b, 6c) and tasks that involve the meaning of the literal symbols (4, 5, 6a).
Items 2 and 3 addressed solving systems of two equations with two unknowns, which later were the subject of *generalization* in the teaching materials. In item 2, many students revealed that they had informal strategies for solving the concrete

sum-difference problem. Apparently, the problem was natural to them. Out of a total of 49, 19 students carried out a systematic method that essentially comes down to $(120 \pm 38)/2$ or $120/2 \pm 38/2$. The work of Student1 (see Fig. 6.12) is an example of this approach. He mentioned the average of 120 and 38 ('gemiddelde' in Dutch) and also checked his answer ('controle'). Some of the students, such as Student2 in Fig. 6.12, mixed up different approaches. An approach that is based on trial-and-improve was chosen by 15 students, sometimes with fair share as the starting point. One student used equations to solve the problem and 14 students did not find the correct solution.

***Pretest item 2***
If I add the age of my father and my own age, I get 120. If I subtract my age from that of my father, the result is 38. How old am I?
How did you find it?

*Student 1*




*Student 2*

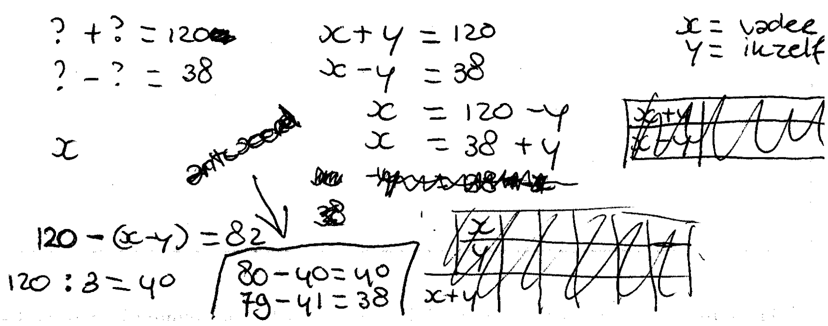



Figure 6.12 Sum-difference problem in pretest: different strategies

Let us consider the work of Student 2 in Fig. 6.12 once more. Remarkably, she usesd the symbol ? for the age of the father as well as for the age of the son. Some of the other students did the same. Later, Student 2 introduced the literal symbols *x* and *y*. In both equations she isolated *x*, but she did not equal the two expressions for *x*. Instead, she substituted $x - y$ for 38 and subtracted this from 120. Finally, she divided 120 by 3 (maybe she thought the father is about twice as old as the son?) and adapted the value of 40 until she found the correct solution. She started with formal strategies, but ended up with trial-and-improve when the formal method did not produce the result.

Item 3 revealed a different picture. In the more complicated situation of the right-angled triangle, most students did not know what to do with the equations ($a + b = 31$ and $a^2 + b^2 = 25^2$) that are more complex and probably do not mean anything to them.

The items that involve *dynamics* (1, 6b and 6c) revealed good results. Apparently, the students were able to see 'what moves' when the formula is meaningful to them. Answers such as 'the total costs get higher' or 'the slope remains the same' for item 6b were common. However, some of the students tended to compensate; they thought that a higher admission price should result in a lower attraction price, so that the total costs remain unchanged.

Attaching *meaning* to formulas and to the literal symbols that they contain is the issue of items 4, 5 and 6a. The results indicate that for linear relations, the students were able to give meaning to the formulas and letters involved. For item 4, for example, answers such as 'costs = 15 + price x number' were common. In 5, we find answers such as '*p* is the slope number and *a* is the starting number'. In 6a, many students found the right formula.

### 6.4.3 *Experiences during the teaching experiment*

In this section we describe the experiences during the teaching experiment, based on classroom observations and the written work of the students. First, we address the reification of expressions and formulas. Then we follow the line of the HLT: the parameter as generalizer, as changing quantity and as unknown. Finally, we discuss the concluding investigation task.

The protocols presented here are representative of the work of the students, unless indicated otherwise. Observations or explanations by the observer are added between square brackets [ ]. By (...) we indicate that a part of the text was skipped because it was not relevant. A (?) means that the registration was not audible. Dots ... indicate that the speaker was interrupted or goes on later. Each protocol ends with a code representing the teaching experiment during which the observation took place, the fragment of the audio- or video-registration and the assignment.

#### *Reification of expressions and formulas*

At the start of the teaching experiment the students worked fast and with enthusiasm through the tasks of the booklet 'Introduction TI-89'. The instrumentation of solve was difficult because of the specification of the unknown with respect to which the equation was solved. We supposed that substitution of formulas and expressions would stimulate the students to reify them. One of the first assignments in which this plays a role is the cylinder task (assignment 4.17, see Fig. 6.5). The expression $\pi \cdot r^2$ needs to be substituted for *o* in *a*h*. In TI-89 language: *a*h* | *a* = π * *r*^2. Many students seemed to acquire an understanding of what was happening and started considering the formulas as objects that can be substituted. The following fragment shows how Suzanne did not understand the situation completely at first, but then corrected her formulation. She considered the formula as an object that can be moved.

> *Observer:* *Do you understand what happens, Suzanne?*
> *Suzanne:* *Yes, this [points at the part after the vertical bar] is multiplied by this [points at part before the bar].*
> *Observer:* *Multiplied?*
> *Suzanne:* *Filled in.*
> (G9-I-B6, assignment 4.17)

A minority of the students found it difficult to give meaning to the substitution. For example, one of the students replied to the question whether she understood what was happening in *a*h* | $a = \pi \cdot r^2$ : 'No, in fact I'm just typing blindly.'
We did not give a name to the substitution bar |. Perhaps this would have helped the students to see the meaning of the substitution procedure. The impression was, however, that carrying out this substitution in the CAS fostered the perception of formulas and expressions as objects.

*The parameter as generalizer*

The introductory booklet ended with an investigation task on the recognition of the pattern in the expansions of $(x + c \cdot y)^2$ that is shown in Fig. 6.6. This was the first step towards the parameter as generalizer. Computer algebra acted as the generator of examples that were supposed to lead to pattern recognition.
Indeed, the examples generated in the computer algebra environment made students recognize the pattern that some formulated in a direct and others in a recursive way. One of the students formulated the pattern in natural language as follows: 'For the first number it doubles and for the second it goes times itself'. Another student: 'If you double the value of *y*, then the number itself doubles too, and the second number is squared'. Of course, in the latter formulation the student meant that the number *in front of the y* doubles, not *y* itself. Some of the students introduced a parameter for the formulation of the pattern before it was suggested in the task: 'The number *a* returns as *a**2 and as $a^2$'. It is remarkable that about half of the students did not explain the pattern they found, although they could perform the squaring by hand to see how it emerges. Insightful generalization was not observed very often. However, an explanation was not explicitly asked for. A practical obstacle in this task was the dot

for multiplication. Some students forgot this, entered $(x + cy)^2$, and got $x^2 + 2 \cdot cy \cdot x + cy^2$ as the expansion. Students did not always realize that this was not what they should get. More information about the result of this assignment can be found in Section 9.4.2.

The second generalization task concerns the sum-difference problem (see Fig. 6.7). Students did well in finding numbers with sum 700 and difference 600, in spite of some calculation errors. If we take into account the good results on a similar task in the pretest (see Fig. 6.12), this was no surprise. The students developed a sense of the general procedure, but had difficulty in formulating it by means of algebraic formulas. Rather than using the general solution, some students preferred to start from the beginning in a new case. For example, after the general solution of the sum-difference problem was given, one of the students found it confusing. She commented: 'This way I don't get confused so easily.' Maybe this situation was too simple to feel the need for the general solution and to make the effort to understand it; concrete cases could be solved easily.

The third instance of generalization was the sum-product problem: how can you find two numbers, given the sum and the product? This was hard for the students, even in concrete cases without generalization with a parameter. Many instrumentation problems with solve and substitute were encountered (Section 10.4). Apparently, the tasks in the introduction booklet were too easy compared to the way solve and substitute had to be applied in the sum-product problem. Also, students seemed to miss the intuition that guided an informal problem-solving strategy, as was the case with the sum-difference problem. Furthermore, the formulas in the sum-product problem, especially in the general case, were more complicated than those the students were used to handling. Students showed difficulties in 'looking through' the formulas and understanding their meanings. Because of these difficulties, the parameter as generalizer did not work out very well in this activity.

A fourth instance in which generalization was aimed at, was assignment 5.4 on the right-angled triangle (see Fig. 6.9). The task was first to solve systems of equations such as $x + y = 31$, $x^2 + y^2 = 25^2$, and later the general system $x + y = s$, $x^2 + y^2 = k^2$. The first, concrete system of equations can be solved by a composed instrumentation scheme that is called isolate-substitute-solve (ISS). This scheme is explained in detail in Section 10.4, but Fig. 6.13 gives an impression.

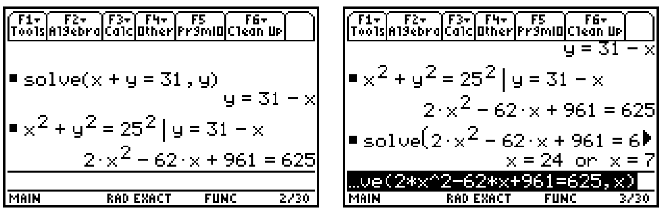



Figure 6.13 The isolate-substitute-solve scheme on the TI-89

The observations in the classroom indicate that students had difficulties with the instrumentation of the isolate-substitute-solve scheme. These difficulties occurred already in the concrete case, when the parameter was not yet involved. These difficulties concerned the solve procedure (e.g. solving with respect to the wrong unknown), the substitution (e.g. substitution of non-isolated expressions) and the integration of the simple instrumentation schemes into one composed scheme.
For the substitution, the students did not seem to care about substituting algebraic expressions rather than numerical values. This supports the idea that the reification of formulas and expressions was advancing. In the two classes, different problem-solving strategies and instrumentation schemes became popular. Apparently, the collective aspect of the instrumentation and the orchestration by the two teachers led to different classroom practices. These issues are addressed in more detail in Section 10.7.
Despite these instrumentation issues, the written work that 39 out of the 53 students handed in showed that the answers to the concrete situations were correct in most cases.
The instrumentation problems also seemed to hinder solving the general right-angled triangle problem. Out of the 39 students who handed in written work, 13 had answered correctly, 4 only artially and 22 incorrectly. Our interpretation is that in this phase of the teaching experiment, students were not very sensitive to the generalization of the problem. Once they saw the aim of the task, the execution was not too hard for them if they had managed the concrete cases before. However, the result of the general case contained formulas that were hard to understand. For example, many students had difficulties in copying the solution formulas correctly from the screen of the TI-89 into their notebooks. The work shown in Fig. 6.14 is an exception: this student copied the formula correctly, and then rewrote it at least partially by hand (this cannot be achieved with the TI-89!) in a form that she understood better.

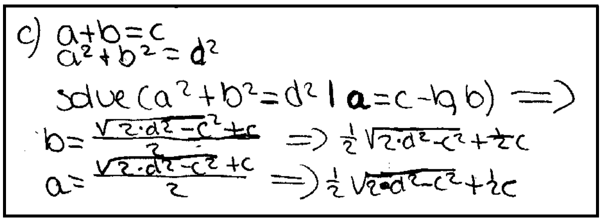



Figure 6.14 Solving by machine and rewriting by hand

*The parameter as changing quantity*

In Sections 6 and 7 of ‘Changing algebra’, the transition towards the parameter as changing quantity is addressed by studying the effect of parameter change on the graphs. Assignment 6.5 concernd the influence of parameters *s* and *v* on the graphs of $y = s - x$ and $y = x - v$. Most students saw the dynamics in the graphs, although it was not always easy to match the parameter values with the corresponding graphs. In the first protocol below, which was recorded during a classroom discussion, two students were unable to formulate the dynamics correctly. In the second protocol, one of these students started to see how it works.

*Teacher:* *Minke, what happens with the graph of* $y = s - x$ *when the sum gets bigger?*
*Minke:* *It goes steeper.*
*John:* *They get lower.*
*(G9-I-A14-15, assignment 6.5)*

*Observer:* *What happens with the graph of* $y = s - x$ *when s grows?*
*John:* *An additional line appears on the right side.*
*Observer:* *What happens with the graph of* $y = x - v$ *when v grows?*
*John:* *An additional line appears on the left side.*
*Observer:* *How is that?*
*John:* *When v gets bigger, y becomes smaller. [Looks at the graph again.] Hey that's not right, right-below.*
(G9-I-A15, assignment 6.5)

To achieve a better insight into the parameter as changing quantity, a slider bar to vary the parameter values might be more appropriate than the options for drawing sheaves that the TI-89 offers.

The Sections 8 and 9 of the booklet 'Changing algebra' on three-dimensional graphs with the TI-89 and with Derive were skipped during the teaching experiment for several reasons.

First, the other sections took more time than was foreseen. Second, we were uncertain about changing to the new interface of Derive. Third, our main argument was that the students were getting used to the hierarchy between variable and parameter, and that giving them similar roles in the three-dimensional graphs might confuse them. For the students this was a pity, because they enjoyed making three-dimensional graphs; some of them experimented with this option on their own.

*The parameter as unknown*

In some of the assignments in Section 10 the parameter got the role of unknown. For example, in the right-angled triangle task (assignment 10.10, situation from Fig. 6.9) the question was what values of the parameter *k* would lead to the system $x + y = 31$, $x^2 + y^2 = k^2$ having no solution. In class B only two of the students were able to answer this question. This change of perspective was really new to them. In some cases, they felt the need for the shift of roles, but were unable to carry it out because they got lost in the formulas and literal symbols.

### *The concluding investigation task*

The final section of the teaching materials consisted of two investigation tasks. The students worked on the first task in small groups. This task, 'A fan of graphs', is shown in Fig. 6.11. Besides observational data we got the written work from 14 members of the groups.

| | Equation $y = a \cdot x$? | Perception of dynamics? | Two general solutions for *x*? | Equal general solutions for *x*? |
|---|---|---|---|---|
| Yes | 9 | 9 | 8 | 4 |
| No | 5 | 5 | 6 | 10 |

Table 6.2 Group results on the final investigation task

The task turned out to be more complicated than foreseen. First, some of the students thought that only the graphs that were drawn in the figure had to be considered, rather than all possible lines through the origin.

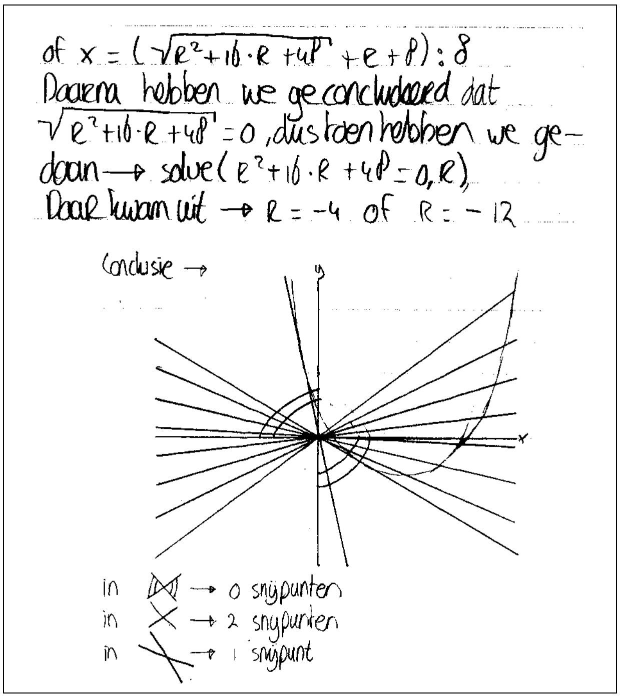



Figure 6.15 Students' work on the final task

They relied heavily on the illustration. A second complication was that some students did not understand the concept of a tangent line that ‘touches’ the parabola. This had not been taught in the class so far.
Third, some of the students did not know that a parabola always ‘wins’ from a line, and takes it over when the independent variable goes to infinity. These complications frustrated the students’ work in other aspects than we intended. Therefore, the findings do not provide a full insight into the capacities of the students.Table 6.2 gives an overview of the results of the handed-in group work. Out of the 14 groups, 9 were able to formulate the equation of the general line through the origin using a parameter and saw the dynamics of this line as the parameter value changed. These results indicate a sense for the parameter as generalizer and for the parameter as changing quantity.

The general solution of the equation $4x^2 - 8x + 1 = a \cdot x$ was found by eight groups. This shows that they accepted expressions as solutions. Finally, four groups noticed that the two solutions should be equal if the line was a tangent of the parabola, and thus considered the parameter also as an unknown. Fig. 6.15 shows the work of one of those successful groups. They found the ‘critical’ parameter values to be - 4 and -12 and related their findings to the graphical problem situation

### 6.4.4 *Posttest*

The teaching experiment ended with a written posttest that was taken by 51 students. The test lasted a whole lesson (45 minutes) and students used their TI-89 during the test. The full text of the posttest is presented in Appendix B. Table 6.3 shows the results. The row ‘Incorrect’ includes students who did not answer at all.

| Item | 1a | 1b | 2a | 2b | 2c | 3a | 3b | 3c | 4a | 4b | 4c |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Correct | 33 | 32 | 24 | 17 | 8 | 12 | 28 | 16 | 40 | 18 | 0 |
| Partial score | 3 | 1 | 2 | 0 | 0 | 17 | 7 | 3 | 0 | 8 | 10 |
| Incorrect | 15 | 18 | 25 | 34 | 43 | 22 | 11 | 32 | 11 | 25 | 41 |

Table 6.3 Results of the G9-I posttest

The average marks were 4.7 out of 10 for class A and 4.5 out of 10 for class B. We now discuss the items of the posttest in relation to the pretest items with which they match, following the line of the learning trajectory: the parameter as generalizer, as changing quantity and as unknown.
*The parameter as generalizer* appeared in items 2b and 4c. The results of task 2b on the length and the width of a rectangle with perimeter 20 and diameter *d* were not very good: 17 correct answers, 8 incorrect answers and 26 non-responses were given. The errors in most cases concerned the instrumentation of the scheme isolate-

substitute-solve. Similar errors occurred in the concrete task 2a, but less frequently: for 2a there were 24 correct answers. Apparently, introducing the parameter led to a decreasing number of correct answers. The number of non-responses is surprising if we consider the similarity of this task with item 5.4 in the teaching unit (see Fig. 6.9). Fig. 6.16 shows one of the better answers to question 2b. The student referred to task 2a where the length of the diagonal was 8, and wrote down: 'Now enter the same, but 8=*d'*. Then she carried out the solve command with a nested substitution. She wrote down the result correctly, and concluded: 'Now you can substitute a value for *d*'. This last remark indicates a tendency to see the general solution as a means of calculating numerical solutions. The pretest did not contain a comparable item.

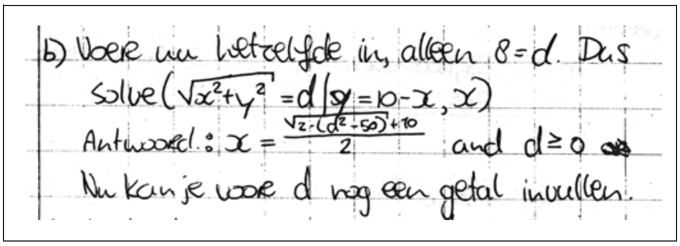



Figure 6.16 Student work on posttest item 2b

Item 4c involved the parameter as generalizer and as unknown, and will be discussed below. Here, we just note that the results of 4c were not good. Overall, the results from the posttest tasks that involve the parameter as generalizer were very modest.
Posttest items that involved *the parameter as changing quantity* were tasks 1a, 1b, 3b and, in a more complicated situation, 3c. The results of the items 1a, 1b and 3b were fairly good: 33, 32 and 28 correct, respectively. The results of task 3c were less favourable (16 correct), which is understandable considering the increased complexity; this question was completely new to the students. Overall, the results on the posttest tasks that involved the parameter as changing quantity were satisfying. The students had understood this parameter role and despite the limited capacities of the TI-89 on this issue, they grasped the idea of parameter change that affects the complete situation or graph. Apparently, the graphical visualization of this effect creates a strong mental image. Compared to the results of the pretest on this parameter aspect (items 1, 6a and 6b, see Section 6.4.2), we see that student considered graphs as objects, as manifested by such formulations as 'the complete graph shifts...'. Also, the parameter was used in the students' arguments more than in the pretest: 'if *a* gets bigger, then...', whereas in the pretest formulations such as 'if the 17.50 is replaced by a bigger number,...' were more frequent.
The *parameter as unknown* appeared in item 2c in a rather implicit way, and more explicitly in task 4c. The results on both items were bad: for 2c eight correct answers

and for 4c ten partial scores, which mainly concerned graphical, non-algebraic approximations of the lines tangent to the parabola. Possible explanations for these results are that 2c builds on the result of 2b, which was not answered correctly by many students. For 4c, the problem-solving procedure first required solving the general equation $-x-5 = -x^2 + c \cdot x - 7$ (the parameter as generalizer, reification of the solution formulas) and then setting the two solutions equal and solving with respect to *c*, the parameter as unknown. This involved the shift of the role of the parameter that students found hard during the teaching experiment. For 4c, the fact that it was the last question of the test probably influenced the results as well.

The reification issue played a role in tasks 2a, 2b, 4a and 4c, where it is required to accept expressions and formulas as solutions that can be processed further. The results on these items were not very good; however, we do not know whether it is the reification that is the bottleneck or other aspects, such as the instrumentation of the TI-89 or the generalization by means of the parameter.

We conjecture that the overall modest results of the test have to do with the interplay of instrumentation difficulties, conceptual barriers concerning parameter, and an incomplete reification of expressions and formulas. Also, a lack of practising during the teaching experiment may be a factor that resulted in a lack of overview and mastering of the techniques involved. The parameter as changing quantity is the role that scored best on the test, whereas the results on the items that address the parameter as generalizer and the parameter as unknown were not so good.

### 6.4.5 *Interview with the teachers*

An interview with the two teachers was held after the teaching experiment. In this interview, the teachers reflected on the students' learning process, the role of the computer algebra tool, the results of the posttest, and their own professional learning process.

The teachers were satisfied with *the students' learning process*. In particular, teacher A stated that 'The students think broader about letters', and that they 'Think about letters and try to figure out what they mean'. Concerning parameters, teacher A thought that sheaves of graphs and sliding graphs were the most appropriate models to stress the difference between parameters and variables. The parameter as changing quantity, in her view, was the most accessible parameter role. In general, graphical strategies are easier for the students than algebraic procedures are. Teacher B thought that the students learned the most from the isolation and substitution schemes. She argued that the benefit of the teaching experiment would pay off later when teaching the solution formula for the general quadratic equation. Teacher A said that the students still did not know how to isolate and substitute by hand, but that they had experienced how it can be used in the problem-solving process. The teaching experiment as a whole was considered to have too many new elements for the students, given the time available. For example, students did not have much experience with expressions and formulas, or with square roots and fractions. The lines in the teaching materials are long and require endurance.

The teachers were positive about *the role of the computer algebra tool*. Teacher A commented that students had to be conscious of various aspects, such as the window settings of the graph screen. The immediate feedback from the machine and the effect of changes on formulas and graphs were considered important affordances of the tool. Exploration and flexibility were stimulated and the machine corrected the students. Sometimes, however, the students seemed to use the machine without thinking, according to teacher B. Teacher A found that computer algebra use fostered insight, but she regretted the limited screen dimensions of the handheld TI-89. The students enjoyed using the machine and found it motivating.

Concerning *the results of the posttest*, the teachers were a bit disappointed. In their opinion, the students had performed better during the teaching than in the test. The teachers felt that they had not prepared the students for the test adequately, as they did not know how to do so. Teacher A argued that the students had not understood some of the questions, whereas a short explanation would have enabled them to demonstrate their ability. There was a big difference between the work during the lessons, which was often done in pairs, and the individual test. As far as the bad results concerning isolating and solving are concerned, the teachers admitted that the way they handled these schemes probably had not been optimal. Teacher A had promoted the nested form of the scheme, which turned out to be hard for the students, and teacher B had appreciated the combined form using ‘and’, which did not work for the posttest items.

This brings us to *the teachers’ professional learning process*. Both teachers indicated that participating in the experiment had meant a lot of work for them. Also, they tended to be guided by the computer algebra machine, and not by their professional experience. Teacher B admitted that she feared working with such a machine and with new teaching materials. Both teachers indicated that it was difficult to pick out the relevant aspects from the teaching materials and to guide the instrumentation in an efficient way. Using the TI-89 view screen during the whole-class teaching and discussions had been difficult for them, as they were used to using chalk and blackboard rather than technology, and the view screen use required extra attention from them.

Finally, the teachers said they had enjoyed taking part in the teaching experiment and were willing to participate the following year (see Chapter 7). The following words of teacher A show that she found participation in the experiment valuable:

> I myself found it an extremely nice experiment and I think the students are enriched by beautiful mathematical ideas that they maybe will apply when they get back to normal work.

Looking back at the teachers' role during the teaching experiment and their reactions in the interview, we note that despite the information we provided beforehand, it was difficult for them to distinguish the essential aspects of the learning trajectory and to stress these key issues in discussions with the students. In class B in particular, whole-class discussions that stressed the central problems in the student activities and revealed the main ideas in interaction with the students were rare. Collective instrumentation and conceptual development might have benefited from that.

### 6.4.6 *Student reactions*

In class A the teacher asked the students to write down their reactions after the first booklet 'Introduction TI-89' had been finished. Also in class A, a short post-experiment interview was held with six students. All students gave a brief written reaction to the teaching experiment after it was finished.

The reactions are quite coherent. Almost all students enjoyed working with the TI-89. In particular, they appreciated the graphical facilities. However, when asked what they had learned from the lesson series, most indicated that they had learned to use the machine, but did not mention any mathematical issues that they had learned. Arno's answer, in which he refers to the assignment in Fig. 6.9, reveals a mixture of technical skills and algebraic insight and therefore was quite exceptional:

> I learned how to write formulas more easily (solve, expand, factor). In the graph, I found it easy to use zoom square. Also, with three-dimensional graphs I learned a bit what happens if you change the value of a variable. For example, at the pretest we got the task on the right-angled triangle, I could not solve it then, but later I did.

Several students argued that the tasks were difficult and the working speed was too high.

### 6.4.7 *Combining different types of data*

Several kinds of data were collected in the G9-I teaching experiment. The hard points for the students seemed to be the generalization using parameters and the application of the isolate-substitute-solve (ISS) scheme (see Fig. 6.13). In this section we compare the different types of data on these two issues, in the expectation that this data triangulation will provide a better view of them.

The results of the posttest on the generalization and the application of the ISS scheme were not positive. If we compare these posttest data with the written work on the substitution assignments that students handed in, we see that 16 students out of the 53 did not show a good ISS scheme on either occasions. Apparently, the instrumental genesis of this scheme was not completed during the experiment. The situations of assignment 5.4 of the booklet and assignment 2 of the posttest are similar. In both cases, the generalization of the situation was done correctly by about 1/3 of the students. About half of the students did not show a good generalization in either the written work or the posttest. The data of written handed-in work was in line with the posttest results.

The information from the students' notebooks, classroom observations and mini-interviews, however, is somewhat more positive. The classroom observations and mini-interviews show many obstacles and barriers, but these often could be overcome. Furthermore, nice examples of (informal) generalizations and of appropriate applications of the ISS scheme were observed in discussions and written work, also in situations that are close to the tasks in the posttest. These findings are different from the data from the posttest and the hand-in tasks.

We notice an incongruence between data from hand-in tasks and posttest on the one hand, and observations and notebook work on the other hand. How can this be explained? We conjecture that the following causes play a role. First, during the lessons students often worked in pairs, but during the test they worked individually. As one of the students wrote in her evaluation: 'Two know more than one'. Also, during the classroom work the solutions to the assignments were available, and the teacher and the observers could be asked questions. There was no strict time pressure. These circumstances offered opportunities, which were not available during the individual test with limited time available, to overcome obstacles. We imagine that the relaxed classroom setting is more appropriate for informal generalization and instrumental genesis than the somewhat stressing test context. Of course, the observers were willing to appreciate the steps towards generalization and application of the ISS scheme, whereas at the posttest fully developed solutions were required.

## 6.5 Reflection and feed-forward

In this final section of the chapter we look back at our initial expectations and formulate the feed-forward of the G9-I research cycle for follow-up research cycles.

### *Reflecting on the expectations*

Section 6.2 started with expectations concerning the opportunities that computer algebra would offer for the learning of the concept of parameter. Did those expectations come true? We address them one by one.

1 The first expectation was that computer algebra use would make it easy for the students to work through similar examples that vary only with respect to the parameter value, thus making the students sensitive to generalization. This was expected to support the transition from the parameter as placeholder to the parameter as *generalizer*.

  Computer algebra indeed proved to be an appropriate medium for repetition, example generation and generalization. The task on expanding $(x + c \cdot y)^2$ indicated, however, that pattern recognition does not imply meaningful generalization. The sum-difference task showed that in simple cases students may not feel the need for formal generalization. The sum-product assignment and the right-angled triangle task made it clear that the students had difficulties in giving meaning to general solution formulas that were more complex than they were used to. Furthermore, the instrumentation problems that showed up while working on these tasks hindered the students' performance. In the final investigation task, general solutions were often found, but at the posttest this was not the case.

2 The calculation of general solutions of parametric equations in the computer algebra environment and the substitution of algebraic expressions and formulas were expected to support the *reification* of formulas and expressions.
The impression is that most of the students started to perceive expressions and formulas as entities that could symbolize a general solution, or that could be substituted. The cylinder assignment was the first indication of this. In other tasks, we hardly observed difficulties with accepting formulas as solutions; as mentioned earlier, however, the understanding and interpretation of such expressions was a hard issue for the students.

3 *Graphical models* were expected to mediate between the placeholder conception and the parameter as generalizer and as changing quantity.
Despite the limitations of the TI-89 on this issue, the graphical model of the sliding graph was helpful for visualizing the parameter as changing quantity. Many students addressed this in their reactions after the teaching experiment. It was not evident to them, however, that the parameter does not necessarily have an integer value, but can vary in a continuous way. The perception of the sheaf of graphs as representing a class of functions, symbolized by a parametric formula, was not so helpful. Rather, students seemed to see the sheaf as the result of the one-by-one drawing of the graphs.

4 The flexible way in which computer algebra environments deal with variables and parameters was expected to enable *shifts of roles*, so that the parameter can take the role of unknown.
As indicated in Section 6.2, the parameter as unknown was not addressed extensively in this teaching experiment. The modest experiences with this parameter role showed that the shift of roles can be carried out easily in the computer algebra environment, but that it requires a conceptual change of perspective that students found hard to achieve. For example, in the final investigation task eight groups found two general solutions, but only four groups solved the corresponding equation with respect to the parameter. The results of the posttest assignment that addressed the parameter as unknown were bad.

### *Feed forward*

The findings of the G9-I research cycle generated feed-forward for the next research cycles. We distinguish feed-forward that concerns the hypothetical learning trajectory, the instructional activities and the research methodology.

The *feed-forward concerning the HLT* addressed the global line through the parameter roles. The results from the teaching experiment suggested that the changing quantity is the most accessible parameter role for students, because of its dynamic character and the attractive visualization by means of sliding graphs. Therefore, we decided to address this parameter role earlier in the learning trajectory. A second argument for this was that the hierarchical relation between parameter and variable appears clearly in the 'sliding parameter', whereas this relation may look more symmetric in some of the generalization situations. A second issue concerning the HLT was the generalizer role that did not come across in a satisfying way. This can be attributed at least partially to difficulties with solving systems of equations, and the instrumentation of the ISS scheme in particular. It is important to master these techniques, so that they do not hinder in the generalization process. A third point for the HLT was the need for meaningful contexts that invite generalization. Such contexts should foster generalization in a natural and meaningful way. If the generalization takes place at an abstract level, students may not see its meaning because the model refers to a general level, which the students have not yet built up.
The *feed-forward concerning the instructional activities* followed from the feed-forward for the HLT. First, students needed more practice in applying techniques such as solving equations, substituting expressions, and the ISS scheme. The component simple instrumentation schemes required separated and integrated practice. Second, for a better visualization of the parameter as changing quantity, we aimed at using a better slider tool than was available in the TI-89. Within the learning trajectory sliding parameter - family parameter - parameter as unknown, the visualization could also be used for the sheaf of graphs that represent the parameter as generalizer. Third, in order to improve the understanding of substitution, we would give the vertical substitution bar | a name, the 'wherein-bar'. Finally, we wanted to try to include problem situations in the teaching materials that are meaningful to the students and more closely related to real-life phenomena than was the case in G9-I. As far as the role of the teacher is concerned, we hoped that the two G9-I teachers would be able to use their experience in G9-II for orchestrating the learning process in whole-class discussions.
The *feed-forward concerning the research methodology* addressed the teaching experiments. First, we aimed at a better match between pretest and posttest, so that the improvements in understanding during the teaching experiment could be monitored. Second, although the method of mini-interviews during the lessons on selected key items was appropriate and provided interesting data, the observer-interviewer may be guiding the discussion too much to observe the students' thinking. Also, in particular when tasks are difficult, students have many questions and the observer may get involved as a kind of assistant teacher. Therefore, in G9-II we wanted to try to avoid this role. Finally, informing and guiding the teachers were aspects of the preparation and performance of the teaching experiments that would require more atten-

tion. Soon after the G9-I teaching experiment, a short teaching experiment of five lessons was carried out at the J.S.G. Maimonides school in Amstelveen. In this experiment we used the PC software package Studyworks rather than the handheld TI-89. This software integrates text editing functions with computer algebra facilities, so that teaching materials could be presented to the students within the computer algebra environment, and students could add their answers in the same document. However, the results showed that students did not add much explanation to their solutions, whereas the students' notebooks in the G9-I experiments did contain comments on solutions, TI-89 commands and explanations. Furthermore, there were practical complications such as unavailable computer labs and classroom discussions that were difficult to hold in the computer lab. For these reasons, we decided to continue using the TI-89 in our teaching experiments rather than switching to PC software.

# 7 The G9-II research cycle: learning trajectory and experience

## 7.1 Introduction

In the previous chapter we described the first research cycle G9-I. This description focused on the hypothetical learning trajectory and on the global experiences during the teaching experiment. The present chapter describes in a similar manner the G9-II research cycle – the second cycle in ninth grade – and its teaching experiment. Fig. 7.1 shows how this cycle fits in the total arrangement.

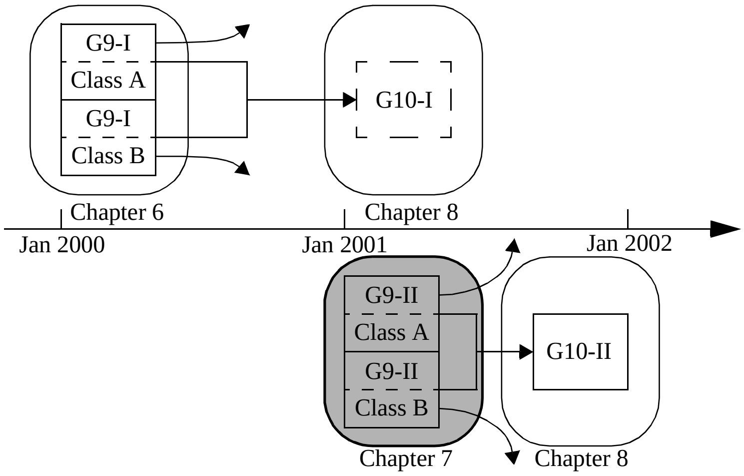



Figure 7.1 Teaching experiment arrangement

First, the development of the hypothetical learning trajectory (HLT) is described (Section 7.2). Then we present the key activities that are included in the teaching materials and their expected effects (7.3). The experiences during the teaching experiment are described in Section 7.4. The educational setting, the pretest, the classroom observations, the posttest, and the interviews with students and teachers are addressed. At the end of Section 7.4 we briefly compare the different types of data.
Finally, we reflect on the original expectations for this research cycle and formulate the feed-forward for the next research cycle (Section 7.5). This next cycle – G10-II – is the topic of Chapter 8.

## 7.2 The G9-II hypothetical learning trajectory for the parameter concept

In this section we first describe the starting points for the HLT of the G9-II research cycle and the expectations that were investigated in this second teaching experiment. Then we motivate the global trajectory through the different parameter roles. The activities that are supposed to bring about the transitions between the different parameter roles are described next. This leads to the global HLT. Finally, we address some instrumental aspects in the learning trajectory.

### *Starting points and expectations*

The starting point for the development of the HLT for the concept of parameter is the level structure that identified the parameter roles of changing quantity, generalizer and unknown as higher parameter roles (Section 4.5). As in G9-I, suggestions concerning how to achieve an understanding of these higher level parameter roles by using the opportunities offered by computer algebra (Section 5.5) guided the development of the HLT. Furthermore, the experiences and the feed-forward from G9-I informed the G9-II HLT.

In addition to the expectations that we formulated in the G9-I research cycle, we now state the following expectations that guide the development of the HLT for G9-II:

1 The parameter as *changing quantity* is a more natural starting point for developing insight into the higher parameter roles than the parameter as generalizer.
2 A better *slider tool* that allows for gradually changing the parameter value helps to perceive the parameter as changing quantity.
3 Practising the simple instrumentation schemes for solving and substitution and the isolate-substitute-solve composed instrumentation scheme will lead to a better *instrumentation* that creates room for generalization.
4 More attention to generalization that emerges from the dynamics of the 'sliding parameter' will lead to a better understanding of the *parameter as generalizer*.

These expectations are elaborated on in the HLT that is developed in the following sections.

### *The trajectory towards the higher level understanding of the parameter concept*

The first question concerns the order of the higher parameter roles in the learning trajectory. The main global change in the HLT for G9-II compared to the G9-I trajectory is that it starts with the parameter as changing quantity rather than the parameter as generalizer. Thus, the global G9-II trajectory follows the line of changing quantity - generalizer - unknown for the following reasons.

First, in G9-I we learned that students find the shift of parameter roles, which occurs when the parameter has the role of unknown, very difficult. Also, considering the parameter as unknown is seen as 'filtering out' specific parameter values from the complete 'family', so the parameter as generalizer is supposed to precede the parameter as unknown. Therefore, the parameter as unknown is addressed last.

Second, the G9-I findings suggested that the parameter as changing quantity is the most accessible for the students, and therefore the most suitable to start with. This is in line with the findings of Furinghetti and Paola (1994), who argued that 'The best results are achieved (...) where students have to understand the meaning of the pa-

rameter as an element that determines the position of geometrical entities.'
These two arguments led to the global learning trajectory according to the line that is visualized in Fig. 7.2.

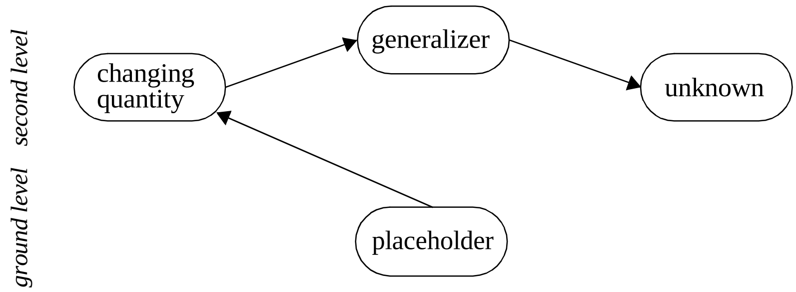



Figure 7.2 Learning trajectory through parameter roles in G9-II

*How to bring about the transitions?*

The next step now is to identify instructional activities that facilitate the transitions to the higher parameter roles along the lines sketched in Fig. 7.2.

The transition *from the parameter as placeholder to the parameter as changing quantity* is expected to be caused by using a slider tool that allows for systematic and gradual change of the parameter values. The effect that this has on the position and/or the form of the graph is supposed to help the students construe the mental conception of a parameter as a changing constant that acts at a higher level than an 'ordinary' variable does and affects the graph as a whole. Students are invited to make 'cartoon sequences' or 'movies' that capture the process of graphical change as a consequence of the sliding parameter. The computer algebra environment supports these activities by allowing for substituting different numerical parameter values and by generating animations.

The transition *from the parameter as changing quantity to the parameter as generalizer* is promoted by two means. The first approach is the graphical one: a number of graphs that were encountered in the 'movie' of graphs as the parameter was changing are now integrated into one figure, thus representing a sheaf or family of graphs. The corresponding parameter values form a 'family of values' and the parameter in the minds of the students thus acquires a set character rather than a numerical or a sliding character. The graphics module of the computer algebra environment allows for the drawing of such sheaves of graphs. The second means of transformation was already used in G9-I and concerns the repetition of similar problem situations for different parameter values, so that the student perceives the similarity of the problem-solving procedure. This is supposed to raise the need for a solution 'for once and for all' that includes all the different numerical cases.

To represent such a general solution, algebraic expressions and formulas are used that need to be treated as mathematical objects. This involves the mental reification of expressions and formulas. Computer algebra use may support this development by solving equations algebraically, by allowing for easy recalculation of similar problem situations with different parameter values, and by generating general solutions.

The transition *from the parameter as generalizer to the parameter as unknown* is fostered by additional criteria or properties that are set for the general solution or for the sheaf of graphs. Such a criterion requires filtering appropriate parameter values out of the set that the parameter as generalizer represents. For example, such a criterion might be: ‘For which values of *p* does the graph have only one intersection point with the horizontal axis?’ Often, the condition will concern graphical properties (intersection points, vertices, tangent points) that are easy for the students to imagine. Such questions are expected to invite a mental shift of the hierarchy between parameter and variable, so that the parameter will act as unknown. Computer algebra allows for this shift, because for the CAS, all literal symbols can play any role, and equations can be solved with respect to any literal symbol.

| role of parameter | algebraic meaning | graphic image |
|---|---|---|
| placeholder | one numerical value | one graph |
| changing quantity | a changing numerical value | a changing graph |
| generalizer | a set of numerical values | a sheaf of graphs |
| unknown | subset of numerical values | a subset of graphs in a sheaf |

| student activity | mental activity | role of computer algebra |
|---|---|---|
| slide parameter values | see global dynamics, graph as object | substitute parameter values, animate graph |
| explore similar situations for different values | see similarity, generalize, formula as object | graph, solve equations, general solutions |
| filter specific parameter values | change relation parameter - variable | solve with respect to parameter |

Figure 7.3 HLT scheme for the parameter concept in the G9-II teaching experiment

The global learning trajectory for G9-II is summarized in Fig. 7.3. The learning trajectory runs from top to bottom. The columns subsequently describe the role of the parameter, the algebraic and graphic meaning of this parameter role, the students’ overt and mental activity that brings about the transition, and the way computer algebra supports this activity.

*Instrumentation aspects within the HLT*

As we noticed above, the instrumentation of relevant algebraic procedures requires attention in the learning trajectory. The experiences from G9-I indicate that more practice is needed to complete the instrumental genesis. The solve instrumentation scheme involves the awareness that an equation is always solved *with respect to* a variable or parameter and that the solution of a parametric equation is an expression. This aspect is addressed by solving equations with respect to different unknowns, and by solving equations that contain parameters.

The instrumentation of the substitution instrumentation scheme is addressed by having students substitute algebraic expressions into other formulas. The intention is that this stimulates the reification of the expression that is submitted to substitution. Compared to G9-I, we now explicitly call the substitution bar | the ‘wherein bar’ or the ‘with bar’, and the substitution will be visualized by means of ovals.

We expect that a better instrumentation of the simple instrumentation schemes for solve and substitute will lead to a better instrumentation of the composed isolate-substitute-solve scheme.

## 7.3 Key activities in the teaching materials

In this section we describe the elaboration of some of the key activities of the HLT on the concept of parameter into assignments in the teaching materials.

***Introduction TI-89***
1 Arithmetic
2 Tables and graphs
3 Algebra

***Changing algebra***
1 Shooting and sliding with the TI-89
2 Fountains
3 A ‘strip cartoon’ of graphs
4 Sheaves of graphs
5 Changing rectangles
6 The algebraic solution
7 Perimeter and area
8 Graphs and other solution strategies
9 Comparing different methods
10 Quadratic equations
11 Exercises

Figure 7.4 Contents of the two parts of the teaching unit

Similar to the G9-I teaching experiment, the teaching materials for G9-II consist of two parts, ‘Introduction TI-89’ and ‘Changing algebra’, with similar aims as in G9-

I. The first part aims at introducing the students to using the TI-89 symbolic calculator, and in particular to CAS techniques that are important in the HLT such as substitution, solving equations and manipulating expressions and formulas. New in the first part are the self-tests at the end of each section, which provide students with the opportunity to practice the main machine skills.

**3.13** Enter: solve(a∗x + b = 5, x)
The answer of solve can be a formula: x is expressed in a and b.

**3.14** Change the last x in the solve command in the previous task into b. This way you solve the equation with respect to b and b is expressed in a and x.

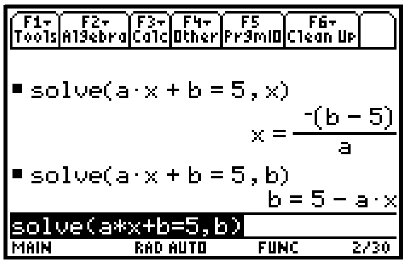
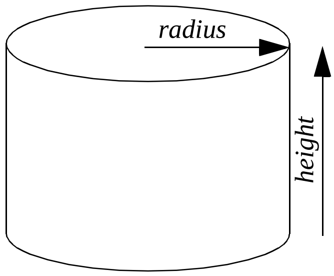



Figure 7.5 Solving with respect to different variables

**3.20** The volume of a cylinder is equal to the the area of the bottom times the height, in short v = a * h.
The area of the bottom equals π times the square of the radius r: a = π ⋆ r^2.
Enter these formulas and substitute the formula for the area in that of the volume.
You can get π with 2nd ^.
v = a * h | a = π ⋆ r^2 gives v = h·π·r$^2$




**3.21** If the height of the cylinder equals the diameter of the bottom, so if h = 2r, the cylinder looks square from beside. Express the area of this 'square cylinder' in the radius.

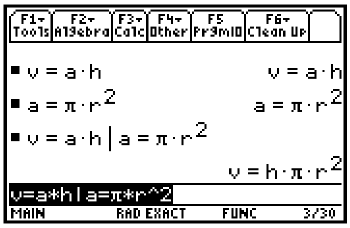



Figure 7.6 Cylinder assignment reformulated

The second part – 'Changing algebra' – aims at developing the concept of parameter according to the HLT. Fig. provides the tables of contents for both parts; the complete manuscripts (in Dutch) are available at www.fi.uu.nl/~pauld/dissertation.
In the part 'Introduction TI-89' somewhat less attention is paid to tables and graphs,

and more to algebra, as the experience of G9-I indicated that the algebra was the hardest part. The third section, on algebra, pays attention to the fact that equations are solved *with respect to* a specific unknown (see Fig. 7.5). The students are confronted with different answers, depending on which unknown they solve. This is expected to make explicit the role of the unknown in an equation and the fact that the solution of an equation can be an algebraic expression.
Some of the tasks from G9-I appear in the G9-II materials in a slightly revised form, such as the cylinder task (see Fig. 7.6). To visualize the idea of substitution, this time we put ovals around the expression that is substituted, as is also done by others with rectangular 'boxes' or 'tiles' (Freudenthal, 1962; Gravemeijer, 1990; TeamW12-16, 1992). We expect that this visualization contributes to the perception of an expression as an object.

The second part of the teaching materials – 'Changing algebra' – starts with using the programs SHOOT and SLIDE that were installed on the TI-89 calculator[1].
In the game SHOOT, the goal is to hit a target point with a line with equation $y = a \cdot x + 5$. The left-hand screen in Fig. 7.7 shows that the students can set the value of parameter *a*.

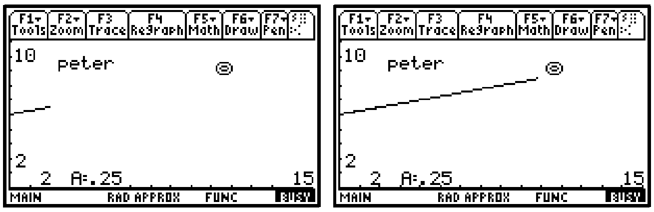
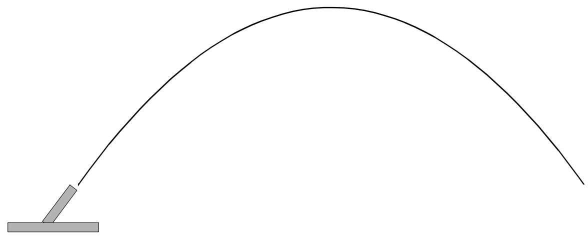



Figure 7.7 The SHOOT game on the TI-89

The right-hand screen shows the shot that is going to hit the point. Two versions are available: one with the coordinates of the target point hidden, and one that shows these coordinates. The aim of the game is that students experience, in a playful manner, that a parameter influences the complete graph, and in this case the slope of the line. The second program, SLIDE, is a more general slider tool, which produces a 'movie' of graphs. The functions, parameter range and steps are user-defined. The questions that accompany the work with SHOOT and SLIDE concern the dynamics, such as: 'What happens with the graph of $y = a \cdot x + 5$ when *a* gets bigger?' and 'What happens with the graph of $y = A \cdot x^2$ when $A$ gets bigger?'
Section 2 of 'Changing algebra' uses contexts of fountains, water pistols and garden sprinklers to experience the dynamics of the sliding parameter. The attention is fo-

1. SHOOT and SLIDE were programmed by Pieter Schadron and can be downloaded from www.fi.uu.nl/~pauld/dissertation.

cused on specific features of the dynamic graph, such as the vertex and the zeros that also change as the graph changes (see Fig. 7.8).



**2.5** This page contains equations of the form $y = x - \frac{1}{a}x^2$

**a** What happens with the graph of these functions when *a* gets bigger?

**b** And what happens with the vertex and the zeros when *a* gets bigger?

Figure 7.8 Properties of the changing graph

Section 3 of ‘Changing algebra’ is an investigation task that asks the students to investigate the changes of the graph of the function *y* with $y = x + a \cdot \sqrt{x^2 + 1}$ while the parameter changes. As a final product, a ‘cartoon sequence’ of graphs is required. Furthermore, the question is to find out which value(s) of parameter *a* is (are) special and why. Whereas the overall task addresses the parameter as changing quantity, the last part anticipates the parameter as unknown.

In Section 4, the graphs that belong to different values of the sliding parameter are drawn together in one figure, so that a sheaf of graphs emerges. The distinct figures in the ‘cartoon sequence’ of graphs are joined to form the ‘family of graphs’. This sheaf of graphs is considered as a whole when the curve that goes through all the vertices of the family is addressed. This is expected to prepare for the parameter as generalizer; the algebraic representation of the sheaf of graphs is a formula that contains a generalising parameter.

**5.4** It is getting boring; it is always the same.
Therefore the following questions.
- How do you solve these tasks all the time? Explain in words.
- Does your method always work?
- Could you write down the result in the form of a formula?

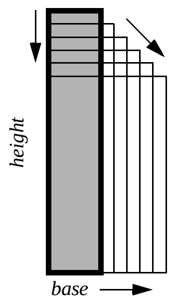



Figure 7.9 Generalization of the sum-difference problem

In Section 5 the central issue is the sum-difference problem, but in a different context than in G9-I, namely the context of a changing rectangle: a narrow, high rectangle with a fixed perimeter of 12 units changes gradually. The base grows, and the height gets smaller. The question is to find the dimensions given the difference between base and height. This question is repeated for several values of the difference and this, we expect, elicits the desire for generalization that is asked for in assignment 5.4 (Fig. 7.9).

Section 6 then treats the general algebraic solution of the sum-difference problem by hand and using the CAS. After that, some exercises are presented, of which assignment 6.7c is the most complex one because of the squared parameters. The squares attract the attention and may ask for actions such as rewriting or calculation. We want to know if students are insensitive to this confusing cue and can still apply the general solution of the sum-difference problem (Fig. 7.10).

**6.7** Solve the following systems of equations:
(...)
**c.** $x + y = a^2$
$x - y = b^2$

Figure 7.10 Abstract sum-difference problem

Section 7 takes the context of the sliding rectangle (Fig. 7.9) as a starting point for the sum-product problem: the perimeter is always 12, but how about the area of the rectangle? Again, the calculation of the dimensions for several values of the area is expected to lead to generalization in a natural way. The method of fair share is an appropriate starting point for finding the general solution (see Fig. 7.11).

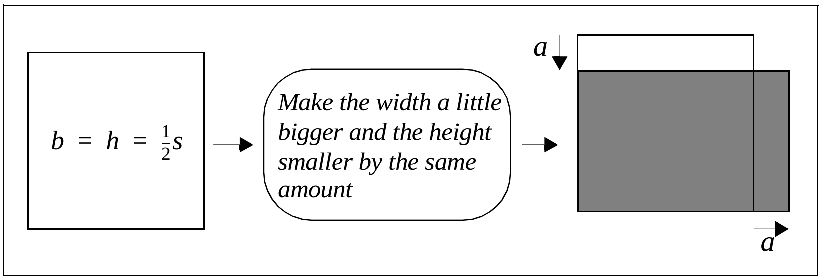



Figure 7.11 Fair share as starting position for solving the sum-product problem

To see the parallel between the approach by hand and the CAS technique, the two methods are both suggested in assignments 7.7 and 7.8 (Fig. 7.12).

**7.7a** Substitute by hand in the product equation:
$b = \frac{1}{2}s + a$ and $h = \frac{1}{2}s - a$ .
**b** Solve the equation that you found in question **a**.
**c** What expressions do you get this way for *b* and *h*?
**d** Rewrite the expressions for *b* and *h* so that $s^2 - 4p$ comes under the square root sign.

**7. 8a** Execute the procedures of the previous assignment once more, but now using the TI-89.
**b** How can you see in the formula for the area after substitution the fair share of the perimeter gives the maximal area?

Figure 7.12 Sum-product by hand and using computer algebra

In Section 8, different ways of solving systems of equations are summarized and compared. The corresponding composed instrumentation schemes are called isolate-substitute-solve, isolate-isolate-equal-solve and the fair-share method.
Section 9 contains a summary and additional assignments, such as the right-angled triangle assignment that was discussed in the previous chapter (Fig. 6.9). Assignment 9.4, on the intersection of a circle and a hyperbola, provides an example of the isolate-isolate-equal-solve scheme. Question 9.4c addresses the parameter as unknown (see Fig. 7.13).

**9.4** Consider the equations
$x^2 + y^2 = 25$ and $x \cdot y = 10$ .
**a** Isolate *y* in both equations and have the graphs drawn.
**b** Solve the system of equations.
**c** What is the biggest value that can be substituted instead of 10 so that the system still has a solution?

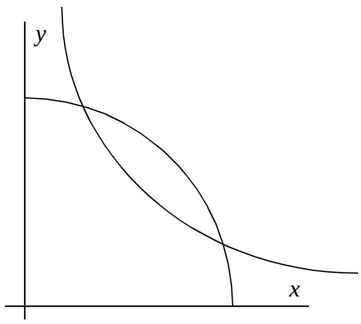



Figure 7.13 The circle-hyperbola task

Section 10 of ‘Changing algebra’ concerns the solution of quadratic equations, a new topic for the students. By expanding the form $(x + p) \cdot (x + q)$ and comparing the result with $x^2 + b \cdot x + c$ , it appears that the general monic quadratic equation $x^2 + b \cdot x + c = 0$ can be solved using the sum-product problem: *p* and *q* have sum *b* and product *c*. Once more a shift of roles: parameters *p* and *q* become the un-

knowns in the new system of equations $p + q = b$ and $p \cdot q = c$. The solution of this system leads to the generic solution of the monic quadratic equation. This is a useful application of the sum-product problem.
The teaching materials finish with a section containing some additional exercises that allow the students to practise and deepen the acquired insights and techniques.

## 7.4 The G9-II teaching experiment

The method of carrying out the teaching experiments in general is described in Section 2.5; information on the school where the experiment was carried out is provided in Section 2.8. In the present section, we first sketch the educational setting of the G9-II teaching experiment (7.4.1). Then we describe the results of the pretest (7.4.2). The global findings of the observations during the experiment are presented in Section 7.4.3. Section 7.4.4 contains the results of the final test. In 7.4.5 and 7.4.6 the data from the evaluation interview with the students and the teachers are presented. Finally, we combine the data from the different sources (7.4.7). The method of selecting and representing protocols is similar to the one described in Section 6.4.3.

### 7.4.1 *Educational setting*

The G9-II teaching experiment was carried out in two ninth-grade classes comprising 54 students. It lasted for 22 lessons in class A and 19 lessons in class B, including a pretest lesson and a posttest lesson. Class A (27 students: 15 females, 12 males) was a normal class with a good working attitude. The average report mark for mathematics on the previous report was 6.6 out of 10. The mathematics teacher was the same as the class A teacher in G9-I, and her teaching style had not changed. As in G9-I, using IT in the mathematics lesson was new to the students of class A.
Class B consisted of 27 students (14 females, 13 males) and had a slightly less good working atmosphere than class A. The average report mark for mathematics on the previous report was 6.5 out of 10. The teacher, the same as the class B teacher in G9-I, managed the class better than she had the previous year. The impression of the observers is, however, that the efficiency of classroom discussions and of the work in pairs was less than in class A. As in class A, the students did not have substantial experience in using IT for doing mathematics.
Before the teaching experiment, the aims and goals of the experiment were discussed with the two teachers. The teachers commented on the teaching materials, and a planning for the teaching experiment was agreed upon. During the experiment, each lesson was discussed by teacher and researcher, and decisions for the next lesson were taken.
Data collection took place in a similar way as in G9-I (Sections 2.5 and 2.8). Additional data were collected by video-recording classroom discussions and one pair of students during the work in pairs. Furthermore, the observers tried to be less guiding during the mini-interviews than they had been in G9-I. Post-experiment interviews

were held with 19 of the 54 students; these 19 students had also been addressed in mini-interviews during the teaching experiment. Section 3.7 gives an indication of the preliminary knowledge of algebra of the students from both classes.

### 7.4.2 *Pretest*

The G9-II teaching experiment started with a pretest taken by all 54 students. The aim was to determine the knowledge of the students concerning parameters, general solutions and dynamics at the start of the experiment. In order to allow a better comparison between the results of pretest and posttest, the G9-I pretest was revised to improve the match with posttest items. Appendix C contains the full text of the pretest.

The results of the pretest are summarized in Table 7.1.

| Item | 1a | 1b | 2a | 2b | 3a | 3b | 4 | 5a | 5b | 5c | 6d |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Correct | 29 | 2 | 39 | 31 | 7 | 23 | 6 | 46 | 44 | 51 | 31 |
| Partial score | 15 | 5 | 0 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0 |
| Incorrect | 10 | 47 | 15 | 19 | 41 | 21 | 48 | 8 | 10 | 3 | 23 |

Table 7.1 Results of the G9-II pretest

We will now briefly address the findings on each of the pretest items.

The results of *item 1* indicated that the students had a good idea of what a variable is: question **a** received 44 good responses, of which the variable as changing quantity was addressed the most (20 times out of the 44). What a parameter is (question **b**) was not clear to 47 of the 54 students; some of the others did have an idea: ‘A parameter is a letter that remains the same number in the formula all the time, for example $y = 2x + b$, $b$ = starting position, starting position = 3, remains the same’ or ‘for example, $y = a - x$, $a$ is a constant and then you get a graph and if it changes then the complete graph goes differently’.

*Item 2* concerned the sum-difference problem, first for the case in which the sum is 120 and the difference is 38, and later in general. Of the 54 students, 39 found the correct answer, using different strategies. Then, question **b** asked for the general description of the problem-solving procedure. This resulted in 31 correct answers, some in natural language, some with exemplary numbers (‘a thinking example’, as it was called by one of the students), and some using formulas. Two examples: ‘If you call the added number $x$ and the difference $y$ then it is $x - y$, divide result by 2’ and ‘An example, together two people are 100 years old. If you subtract one age from the other, you get 26. Add 100 and 26 and divide it by 2. $(100 + 26)/2 = 63$, $63 - 26 = 37$, $63 + 37 = 100$’.

*Item 3* concerned solving the equation $a \cdot x + b = c$, first with respect to $x$ and then with respect to *b*. The students were familiar with solving with respect to *x*, but not when the equation contained a parameter. Many students (41) did not manage to solve this, but it is unclear whether this was because of the parameter or the algebraic manipulation in general. The six partial scores were given to students who did not put parentheses around the numerator: $x = c - b / a$. Solving with respect to *b* was new for the students, but probably because it is algebraically less complicated than solving with respect to *x*, this was done correctly by 23 students. The fact that there is an *x* in the equation, but that it should not be solved with respect to *x*, was apparently not a big obstacle. After the pretest, some students were asked to explain their answer. Let us look at the reactions of Rianne and Sandra.

Rianne got $x = c \cdot a - b$. She was able to solve linear equations such as $3x + 5 = 6$, but on $a \cdot x + b = c$ she commented: 'Then you don't know what you have, I don't know what *a* and *b* and *c* are.'

Sandra, on the other hand, found $x = (c - b) / a$ as a solution for *x*. She commented:

> *Observer:* *Why did you put the parentheses around it?*
> *Sandra:* *Look, you get this answer, c − b, so if you take that as, so to say, an entity, so if you just call it d for example, that you make a d of this, then a times x is d, so d divided by a is x.*
> (G9-II-pretest, item 3)

To Sandra, the idea of a formula as an object (an 'entity') was clear; expressions and formulas seemed to be reified into objects for her.

*Item 4* concerned the substitution of $y = a - x$ into $x^2 + y^2 = 10$. The results were not good: only six of the students gave a correct answer. One of them − Marty − wrote: $(a - x)^2 + x^2 = 10$. After the pretest he explained this as follows:

> *Observer:* *How did you do that, do you remember?*
> *Marty:* *I thought a − x is the same as y, so if I square a − x, then it is the same as y squared, plus x squared is then 10.*
> (G9-II-2/19, item 4)

*Item 5* was on the costs of visiting an attraction park. Deriving the formula for the total costs went quite well (question **5a**). Also, the dynamics of the change of the parameters was described well by most students (question **5b**). To find out the parameter value when the total costs were given was not hard for the students either.

Finally, *item 6* concerned solving equations of the form $x^2 = \ldots$, with $x^2 = a$ as the final one. Out of the 54 students, 31 answered $x = \sqrt{a}$ to the last item. We did not blame them for forgetting the negative solution, for the focus of this item is generalization and the acceptance of $\sqrt{a}$ as a solution. To those students who did not answer this item, the lack of closure seems to be the problem, as is indicated by comments such as '*x* can be anything', 'Here there's nothing to solve' and 'Impossible'. The conclusions of the pretest results were that the students did not know what a parameter is, and that some of them, but not all, suffered from not knowing the value of the parameter. The dynamics of the parameter in the concrete and meaningful context of item 5 was clear to many students. The generalizations in items 2b and 6d sometimes worked out well, although the method in 2d was often informal, and in 6d some students suffered from the lack of closure. The reification of expressions and formulas (items 3 and 4) seemed to be difficult for the students. The instrumentation of solve and substitute, anticipated in items 3 and 4, respectively, will require attention during the teaching experiment.

### 7.4.3 *Experiences during the teaching experiment*

In this section we describe the experiences during the teaching experiment, based on classroom observations and on the written work of the students. First, we address the instrumentation of solve and the reification of expressions and formulas. Then we follow the line of the HLT: the parameter as changing quantity, as generalizer and as unknown.

#### *The instrumentation of solve*

In the first part of the teaching materials – 'Introduction TI-89' – some of the students had difficulties in distinguishing approximated numerical results and exact results (see Section 10.5.2). More attention was paid to algebra than was done in G9-I. We noticed that students wrote down algebraic results in their notebooks (for example, the simplification of *a*b*c*a*b/c* into $a^2*b^2$), but did not seem to wonder why this was an appropriate and correct answer.
Assignments 3.13 - 3.14 (see Fig. 7.5) concerned the instrumentation of the solve command. Two factors played a role: the need to indicate the unknown with respect to which the equation is solved, and the acceptance of an expression as a solution. The instrumentation of solve is addressed in more detail in Section 10.2.
The following fragment indicates that the first issue was not a big problem.

> *Observer:* *And what is the difference between 13 and 14?*
> *Maria:* *That you can calculate here [points at assignment 13] the a [should be x] and here [points at assignment 14] the b.*
> *Observer:* *Yes. And if you were to enter comma-a instead of comma-x or comma-b, what kind of answer would you get then?*
> *Maria:* *With the a at the beginning and then come both the b and the x here [at the right-hand side].*
> (G9-II-1/41, assignment 3.13 - 3.14)

Some of the students wondered that the solution was an expression and not a number. The following dialogue took place after Donald found the solution for *b*. He seems to have a feeling for the expressions as the solution, but is not sure.

*Observer:* *Do you think b is solved?*
*Donald:* *Ah, yes, it is solved as it is, but I don't know what a and b, ehm, and x are, what values you can fill in there, what variables they are.*
*Observer:* *Yes. But for the rest you think it is solved?*
*Donald:* *Yeah, well, I don't know how I might solve it further.*
(G9-II-2/29, assignment 3.13 - 3.14)

### *Reification of expressions and formulas*

The substitution of expressions was expected to contribute to their reification. The cylinder assignment (see Fig. 7.6) was similar to the one in G9-I (see Fig. 6.5) but now the substitution was visualized with ovals and the substitution bar was called the 'wherein bar' or the 'with bar'. This seemed to help the students understand the substitution. Two issues played a role: the substitution mechanism itself, and the idea that an expressions is an object that one can 'pick up' and 'put into a letter'. These issues are addressed in more detail in Section 10.3; here we just present one observation, which indicates that Barbara understood the mechanism and, despite her first reference to numerical values, saw that expressions can be substituted.

*Observer:* *What does that mean, that vertical bar? What does it do?*
*Barbara:* *Then you can one of the letters of the formula … there you can fill in what number has to be there.*
*Observer:* *Fill in what number... but here [in assignment 3.20] there is no number at all.*
*Barbara:* *No, there is a formula to calculate the a with.*
*Observer:* *And what happens to that formula?*
*Barbara:* *Which one of the two do you mean?*
*Observer:* *The one behind the bar.*
*Barbara:* *Well, in fact it is filled in, in the place of a.*
(G9-II-1/45, assignment 3.20 - 3.21)

The overall impression of the introductory part of the teaching sequence is that the instrumentation of algebraic procedures went better than in the previous experiment.

### *The parameter as changing quantity*

The learning trajectory for the concept of parameter started with the parameter as changing quantity. First, the students played the game SHOOT on the TI-89 (see Fig. 7.7). This functioned well in the sense that students noticed the effect of changing the value of the parameter *a* in $y = a \cdot x + 5$ so that the line hits a given point. In the following fragment, Maria found it difficult to explain the dynamics, but in the end she managed. She referred to the SHOOT context by using the word ‘barrel’.

*Observer:* *What do you see happening if that a value changes? What happens in the game?*
*Maria:* *Then you get that x and that y things, they change.*
*Observer:* *But if you shoot with a different value of a, what do you see happening then?*
*Maria:* *Well, then the barrel so to say changes.*
*Observer:* *Yes that barrel changes. So what happens to the line?*
*Maria:* *That graph? It changes.*
*Observer:* *Yes, and in what manner does it change?*
*Maria:* *Yes, it gets higher or lower, I think.*
*Observer:* *On what does that depend?*
*Maria:* *On the value of a.*
(G9-II-3/46, assignment 1.5 - 1.6)

Calculating the appropriate *a* value from the coordinates of the target point was too hard in the context of the SHOOT game. This was not a surprise, as this is an algebraically complicated procedure with ‘clumsy’ decimal numbers. Furthermore, the parameter *a* needed to be considered as unknown, and previous findings indicate that this shift of role is difficult for students to make.
The second TI-89 program that was developed to investigate the parameter as changing quantity was called SLIDE. Unfortunately, during classroom use the program turned out to have some inconveniences. For example, it was not possible to adapt the viewing window within the program, and the parameter change could only take place with constant step size. Therefore, some of the students chose not to use SLIDE; instead, they just entered the function into the function list and substituted the parameter values: $y_1(x) = A \cdot x^2 \,|\, A = \{1, 2, 3, 4, 5\}$. The result is not a sliding graph but a sheaf of graphs that are build up slowly, one by one. Other students stuck to using SLIDE.
Despite the limitations of both strategies for investigating the effect of the sliding parameter, the concept of the dynamics of the sliding parameter was well-perceived by most students. The visualization was attractive to students. For example, one of the students formulated the dynamics of the graph of $y_1(x) = A \cdot x^2$ when *A* changes as follows: ‘If A is negative, the graph gets wider; ah, if A is positive, then the graph narrows.’
Language complicated the description of the dynamics. For example, some students used the expressions ‘getting bigger’ and ‘growing’ in case of negative parameter values for ‘getting further from zero’ rather than ‘sliding to the right’. Also, some students used the word ‘turn’ rather than ‘slide’, whereas for the dynamics from a mathematical point of view the difference between rotation and translation is impor-

tant. However, despite occasionally incorrect formulations, the students perceived the dynamics correctly most of the time.

So far, the concept of the parameter as changing quantity that affects the position or the form of the graph had come across well. Assignment 2.5, which comprises the context of fountains, is more difficult (see Fig. 7.8). The formula under consideration, $y = x - \frac{1}{a}x^2$ is complex and led to problems. For example, some students forgot the multiplication sign * between *a* and $x^2$, which led to $y = x - \frac{1}{ax^2}$. Parentheses did not always help. More important was the difficulty in seeing the relation between the value of the parameter *a* and the right-hand zero of the function. Many students did not see that $x = a$ is a zero in the formula. We did not see students using the TI-89 solve command to find the zeros in general. This suggests that the dynamic effect of the sliding parameter was understood, but that the students did not relate the zeros in the graph with the solutions of the algebraic equation, even with a computer algebra machine at their disposal. The observation of the sliding effect apparently had a visual and superficial character. Only after the coordinates of the zeros and the vertex had been explicitly asked for were the results found, which led to pattern recognition and generalization.

In Section 3 of ‘Changing algebra’ the focus on the parameter as changing quantity ended with the task to make a ‘cartoon sequence’ or a ‘movie’ that represents the dynamics of the graph of the function *y* with $y = x + a \cdot \sqrt{x^2 + 1}$ as the parameter *a* changes. Many students produced reasonable ‘movies’, but observations show that they did not examine the graphs closely. Many did not notice specific features such as asymptotes or asymmetry, and did not look for algebraic explanations of such graphical features. The graphical and algebraic properties were not related. For example, almost all students found it very difficult to see that the formula is reduced to $y = x$ in case $a = 0$, which explains the linear graph for that case (see Section 9.7.3). As a possible explanation for this, we notice that this formula is far more complicated than the students had seen before, and that they therefore were unable to ‘look through’ the formula. Also, the students had had limited experience with graphs, and these were in almost all cases linear or quadratic. Therefore, it is not so surprising that in this assignment many students called the graph a parabola.

### *The parameter as generalizer*

In Section 4 of ‘Changing algebra’ the different graphs from the animation or ‘cartoon sequence’ are integrated into one figure. The students found the sheaf of graphs that emerges this way quite natural and easy to understand. This was the first step towards the parameter as generalizer.

Section 5 addresses the generalization of the sum-difference problem that is presented in the context of the changing rectangle (see Fig. 7.9). The observations suggest three remarks concerning, respectively, the generalization in natural language, the formalization of the generalization and the role of computer algebra in generalization.

First, students showed a sensitivity for generalization on several occasions. Often, the ‘general in the particular’ was expressed in natural language in an imperfect way. The next protocol provides an example for the case of the rectangle with given perimeter and difference of the edges.

| | |
|---|---|
| *Observer:* | *How could you phrase that in general? What you would have to do then?* |
| *Anita:* | *First, yeah, just what is there [on the black board]. First divide the perimeter by the, ehm, what’s that called, the sides, the edges, and then the difference divided by 2 and then ...* |
| *Observer:* | *And how do you get then the values of the base and the height?* |
| *Anita:* | *Plus and minus.* |
| *Observer:* | *Plus and minus. What do you do then?* |
| *Anita:* | *Minus the difference.* |
| *Observer:* | *And did you take the whole difference then?* |
| *Anita:* | *No, half of it.* |

(G9-II-E3/9, assignment 5.4)

The second remark concerns the real difficulty: to compress, to condense the sense of generalization, formulated in natural language, into the compact algebraic language of formulas and expressions. This formalization is an obstacle, even when students understand the generalization informally. The next protocol shows how the observer wanted to help the student to formulate the general solution of the sum-difference problem by means of an example with ‘strange’ numerical values. To avoid this big number, the student used the word ‘perimeter’.

| | |
|---|---|
| *Observer:* | *And if the perimeter is 700 thousand million 231, for example, and the difference is 30501?* |
| *Dean:* | *Then you do divided by 2, ...* |
| *Observer:* | *Yes* |
| *Dean:* | *... and then you do 7, ehm, the perimeter divided by 4 ...* |
| *Observer:* | *Yes* |
| *Dean:* | *... and then you take half of the difference and add it to the one and take it away from the other.* |

(G9-II-6/13, assignment 5.4)

However, the formalization still was the bottleneck. Dean was unable to phrase his procedure as a formula: ‘I really don’t manage because I don’t know how to write down in a formula one quarter of *P.’* By *P* he meant the perimeter.

One of the students, Rianne, managed to express the procedure into a formula. In her notebook she wrote:

$$\frac{perimeter}{numbedges} + \frac{difference}{2}$$

The third remark on the parameter as generalizer concerns the role of computer algebra. In fact, the CAS acts as a generator of examples that form the 'raw data' for the generalization process. It can also help to find general solutions and relations. However, the main step in generalization is a mental one: to distinguish the invariant aspects in the problem-solving procedure, which do not depend on the specific numerical values of the concrete problem situation at hand. While doing so, students jump back and forth between concrete cases, generic situations and exemplary values. This issue is elaborated on in Section 9.4.2.

After the sum-difference problem had been solved in general, the students went on with Section 6, which contained some exercises for practising. The hardest of these was assignment 6.7c, which concerns solving the system of equations $x + y = a^2$, $x - y = b^2$ (see Fig. 7.10). Most students were able to solve this system, and some of them explicitly considered it as a case of the general problem. For example, one student wrote in her notebook $s = a^2$ and $v = b^2$, thus referring to the general problem $x + y = s$, $x - y = v$.

Others did not apply the formula for the general solution in this assignment, but did apply the procedure of dividing by 2 and adding/subtracting the half of the value of *v*. One of the students was asked why he solved with respect to *x* and *y*. His answer 'You have to solve the equations and I think it is meant that you calculate x and y, so that you know the answer' shows that he was aware of the conventional roles of the different literal variables. He considered the formula as a satisfying answer.

The results of this assignment suggest that the students were now so familiar with the sum-difference problem that they could apply the generic approach to a new and complicated situation.

In Section 7 the sum-product problem is generalized. This was difficult for the students, because the formulas are harder to understand than in the case of the sum-difference problem. The by-hand method and the scheme isolate-substitute-solve (ISS) were both discussed in the classrooms, but a complication here was that the CAS solution is represented in a different way than the by-hand solution is:

$$\frac{1}{2}s + \frac{\sqrt{s^2 - 4 \cdot p}}{2} \text{ rather than } \frac{1}{2}s + \sqrt{\frac{1}{4}s^2 - p}$$

Some of the students lost track, and it was necessary to go back to the concrete situations to give meaning to the formulas:

*Observer:* *The height is $\frac{1}{2}$ minus a, well, here one half was 3, that was half of that 6, isn't it?*
*Marg:* *Yes.*
*Observer:* *And if here [in the position of the 6] there had been a 10, what would you get here [in place of the 3]?*
*Marg:* *A 5.*
*Observer:* *And if it were 100?*
*Marg:* *A 50.*
*Observer:* *And if there were an s here?*
*Marg:* $\frac{1}{2}$ *s.*
(G9-II-7/28, assignment 7.7)

Section 8, which aimed at combining different ways of solving systems of equations, was skipped because of time constraints.
Section 9 contains the right-angled triangle assignment described in the previous chapter (see Fig. 6.9). The system of equations $x + y = 35$, $x^2 + y^2 = 25^2$ was supposed to be generalized. One of the students showed a good sense for generalization, but had difficulties in interpreting the algebraic formulas in the 'nasty answer':

*Dean:* *I found something nice, I thought if you put a letter instead of this 35 here and that 25 there, so a and b, there and there, that you get this solution, but then I got a somewhat nasty answer.*
(G9-II-9/29, assignment 9.3)

Rianne, on the other hand, seemed to stick to the placeholder concept of parameter, which led to a lack of closure and prevented her from generalization:

*Observer:* *The 31 first changed into 35, and now it changes again into something else.*
*Rianne:* *Into something.*
*Observer:* *Pardon?*
*Rianne:* *Into simply, that you don't know.*
*Observer:* *Yes. Well let's take a letter for it.*
*Rianne:* *Oh, but then you can't calculate it.*
(G9-II-10/13, assignment 9.3)

### *The parameter as unknown*

In Section 9 the parameter acquired the role of unknown in some of the assignments, such as the circle-hyperbola task (Fig. 7.13). The results of these and similar tasks were not very good for two reasons. First, despite measures taken earlier in the teaching experiment, there were still common instrumentation problems, in particular with the ISS scheme and its components. These reappeared as soon as the formulas and the situations became more complex. Second, the graphical contexts in which the conditions for the parameter were formulated directed the students towards graphical solutions. For example, in assignment 9.2 it is asked for which value of $b$ the line with equation $y = b - x$ 'touches' the half circle with equation $y = \sqrt{625 - x^2}$. Most students solved this graphically, and only started looking for an algebraic method after this had been suggested to them. This indicates that students do not link the graphical and algebraic representations on their own.

Only class A had time enough to work through Section 10, which treats the general solution of the monic quadratic equation $x^2 + b \cdot x + c = 0$. This was successful to the extent that students understood that the problem would be solved by finding a factorization of $x^2 + b \cdot x + c$ in the form of $(x + p) \cdot (x + q)$, and that therefore the sum-product problem re-appeared. This became understood by expanding $(x + p) \cdot (x + q)$ in a multiplication table. The students were able to distinguish the roles of the literal symbols, and noticed that $p$ and $q$ acquired the character of unknowns. The difficult part, once more, was 'looking through' the formulas of the general solutions for $p$ and $q$. In particular, the square root with a square underneath in $p = \frac{1}{2} \cdot b + \frac{1}{2} \cdot \sqrt{b^2 - 4c}$ was overwhelming. It should be noted that this is the first time that solving an equation with respect to the parameter led to expressions rather than formulas.

Rather than working through the exercises of Section 11, the final lessons before the posttest was given were spent on a self-test.

### 7.4.4 *Posttest*

The teaching experiment ended with a written posttest that was taken by all 54 students. The test lasted one lesson (45 minutes) and the students used their TI-89 during the test. The full text of the posttest is presented in Appendix D. Because class A worked through Section 10 during the teaching experiment and class B did not, item 4 of the posttest was different for the two classes.

Table 7.2 shows the results. The column 'Incorrect' includes students who did not answer at all. The A and B for item 4 refer to the differences in the test.

| Item | 1 | 2a | 2b | 3a | 3b | 4a (A) | 4b (A) | 4a (B) | 4b (B) | 5a | 5b | 5c |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Correct | 33 | 46 | 25 | 24 | 11 | 18 | 1 | 21 | 8 | 39 | 31 | 5 |
| Partial score | 11 | 4 | 5 | 4 | 4 | 8 | 1 | 4 | 7 | 4 | 3 | 16 |
| Incorrect | 10 | 4 | 24 | 26 | 39 | 1 | 25 | 2 | 12 | 11 | 20 | 33 |

Table 7.2 Results of the G9-II posttest

The average marks were 5.0 out of 10 for class A, and 5.8 out of 10 for class B. We will now discuss the items of the posttest in relation to the matched pretest items, following the parameter learning trajectory of changing quantity - generalizer - unknown.

Let us first consider the answers to the first posttest question: ‘What is a parameter? Can you give an example?’ Most students (33 out of 54) gave an adequate answer. Some of them referred to the parameter as placeholder: ‘A letter that you use instead of a number.’ Some of the answers suggested some hierarchy, often related to the changes of the graphs: ‘An extra letter in a formula’, ‘A variable that affects the form of the graph.’ Also, the parameter as ‘variable constant’ was often mentioned: ‘A unknown that has a fixed value.’ Compared to the same item in the pretest (item 1b, see Appendix C), where most of the students did not have any idea of what a parameter is, the adequate responses show an improvement in the posttest item.

The parameter as a *changing quantity* played a role in item 5b, which concerned the dynamics of the graph of $y = a \cdot x - 5a + 2$ while the value of $a$ changes. Most of the answers were correct (31 out of 54), including formulations such as ‘the graph gets steeper’. The dynamics of the sliding parameter was clear to most of the students. Compared to the pretest, the improvement was not that spectacular, because the idea of the global effect was already noticed in the pretest (item 5b). A higher level understanding of the parameter as changing quantity therefore cannot really be noticed, although the context of the posttest item was far more abstract than was the problem situation in the pretest.

The parameter as *generalizer* was addressed in items 2b and 3b of the posttest. Item 2b concerned the general solution of the sum-difference problem in the context of calculating ages. Other than in the pretest, the general solution was explicitly asked for. About half the students gave a correct response, which is not many if we take into consideration that such activities had been addressed several times during the teaching sequence. Also, in the corresponding pretest item (2b) many students showed that they already had a sense of generalization. The correct answers to the posttest item 2b reflected different methods: fair share on an empty number line, isolate-substitute-solve (nested or step-wise), isolate-isolate-equal-solve and simultaneous solve. Apparently, standardization of the method took place to a lesser extent than it did in G9-I.

The second posttest item concerning the parameter as generalizer was item 3b. The task was to solve the system of equations $x + y = 10$, $\sqrt{x^2 + y^2} = d$ after the concrete case $d = 8$ had been solved. Of the 54 students, only 11 were able to solve this. Already for the concrete case, only 24 managed to find a correct solution. Students did not know how to deal with this task, or were frightened by the complex-looking formulas. Probably the lack of insight into the formulas in the task and in the answers was the bottleneck here. Furthermore, instrumentation difficulties with the solution scheme played a role. For example, non-isolated forms were substituted (Fig. 7.14

upper part) and equations were solved with respect to the wrong unknown (Fig. 7.14 lower part).
The reification of expressions and formulas played a role in item 4b for class B, and to a lesser extent in the generalization tasks 2b and 3b. The results were not very good; however, we do not know whether it is the reification that caused this, or the complexity of the situation, the generalization or the instrumentation.

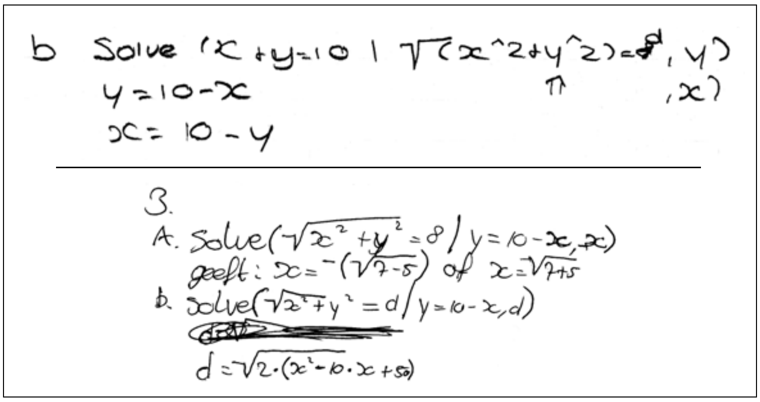


Figure 7.14 Substituting a non-isolated form and solving to the wrong unknown

We conclude that a higher level understanding of the parameter as generalizer was not assessed in the results of the posttest. Possible explanations are the complexity of the problem situations and the formulas, the lack of symbol sense, and the incomplete instrumentation of the schemes involved. This is in agreement with the observations of the students' work on tasks from Section 9.
Finally, we consider the results concerning the parameter as *unknown*. Because this parameter role was not addressed extensively in the teaching sequence, item 4b for the class A posttest was the only item on this issue. In that item, students were asked what conditions hold for $b$ and $c$ so that the general solution of $x^2 + b \cdot x + c = 0$ could be calculated. The results for this item were bad (1 correct answer out of 27). Rather than examining what is under the square root sign, most students saw the square root sign as a reason to argue that $b$ and $c$ should be positive. Clearly, there is no evidence for a higher level understanding of the parameter as unknown; however, once more it can be argued that the hard issue was the complexity of the formula rather than the parameter role.
Looking at the posttest as a whole, the total scores are reasonable and better than was the case in G9-I. The parameter as changing quantity was well understood; however, the results concerning the parameter as generalizer and as unknown are not very satisfying. Possible explanations for this are the complexity of the formulas involved, the lack of symbol sense and the instrumentation problems.

### 7.4.5 *Interview with the teachers*

We interviewed the two teachers after the teaching experiment. In this interview, the teachers reflected on the students' learning process, the role of the computer algebra tool, the results of the posttest and on their own professional learning process. The teachers were satisfied with *the learning process of the students*. The students worked hard and were actively involved. The teaching experiment was an intensive way of doing mathematics for them, and some students were more challenged by the tasks than they were normally. There was time to address issues more in depth; the level of the topics was beyond the regular level for ninth grade. The teachers found that there was much to explore, to find out, to struggle with, and at the end the students were more confident about their mathematical skills.
As a result of the learning process, the students were no longer afraid of using more literal symbols within expressions and formulas. One of the teachers said:

> I think they clearly understand more about all the different letters that one can see, and that they no longer are afraid of that. (...) I think they really understood what the difference is. They do not always formulate it very clearly, but they do see 'these are the letters that belong to the coordinate axes, to state it simply, and the rest are the parameters.

The results on this point were better than in G9-I, according to the teachers. In particular, the students understood the parameter as changing quantity quite well. The learning trajectory changing quantity - generalizer - unknown was better than the G9-I trajectory, because the algebraic work with equations containing parameters is more abstract than watching sliding graphs. The last section on quadratic equations, which was only done in class A, suffered from time constraints and would have needed more time. Another experience of the teachers was that the students were able to see the global line of the learning trajectory without getting lost in the details.
According to the teachers, the use of *the computer algebra tool* enabled the students to keep track of the global line. Leaving the algebraic procedures to the machine allowed them to focus on the main ideas. The capacities to quickly draw a number of graphs and to show the sliding graph were powerful, even if the slider tool was far from perfect. As far as algebraic procedures were concerned, the students had to be very precise in entering expressions and commands; instrumentation difficulties for solve and substitute contributed to the students' insight into what was happening. As one of the teachers put it:

> I think some of the students learned what they are solving, because if you enter an equation such as x+y = something and enter solve, it does not work. You have to think carefully 'what do I have to solve', the comma-x.

Also, the students lost their fear of the machine, which is useful in this technological society.

Concerning *the results of the posttest*, the teachers argued that the posttest was easier for the students than in G9-I because there had been a self-test prior to it, so that they knew better what to expect. This probably contributed to the improved results.
Concerning *the teachers' professional learning process*, both teachers indicated that they felt more confident about what to do, that they were able to 'orchestrate' the use of the computer algebra tool in a more adequate manner than they had been in G9-I and that they enjoyed combining different classroom arrangements such as classroom discussion, demonstration by students, group work and work in pairs. For them, the experiment was less tense than G9-I had been.
Overall, the teachers were positive about the course of the teaching experiment, the relevance for the students and the improvements compared to the G9-I cycle.
Looking back at the teachers' roles during the teaching experiments and their reactions in the interview, we think that they had improved as regards integrating computer algebra use into the teaching and learning, and stressing the key steps in the learning trajectory. Still, we would have appreciated more interactive whole-class discussions for collectively posing the main problems and developing the conceptual thinking.

### 7.4.6 *Interviews with students*

After the teaching experiment, interviews were held with 19 of the students who had been followed during the experiment. During these interviews, which took about 10-15 minutes each, the following three items were discussed. Items 1 and 2 were part of the pretest and the posttest, and item 3 was a general formulation of a pretest task.

**1** What is a parameter? Can you give an example?
**2** Consider the equations

$$y = a - x$$
$$x^2 + y^2 = 10$$

Make one equation out of these that does not contain $y$.
You do not have to solve this new equation!
**3** Consider the equation $a \cdot x + b = c$.
**a** Express $x$ in $a$, $b$ and $c$; in other words, write the equation in the form $x = ....$
**b** Express $b$ in $x$, $a$ and $c$.

During these interviews, the students did not have the symbolic calculator at their disposal, so they could only answer the questions using brains, paper and pencil.
The results of *item 1* concurred with the answers to the corresponding item in the posttest. Of the 19 students, 15 gave a correct answer, although sometimes formulated in a clumsy way. The parameter is seen as an extra letter that determines the form of the graph, and therefore is considered as changing quantity at the level of

graphs. References to the parameter as generalizer or as unknown were not made. Two of the students mixed up the variable and the parameter, while two others described the parameter as determining the value of the solution. For example, if the solution of an equation is $x = a + 3$, then the value of $x$ depends on the value of $a$.
As an example of the reactions, the following fragment shows how Fred thinks about parameters:

> *Fred:* *The parameter indicates, ehm..., how such a formula develops.*
> *Interviewer:* *Yes, that's a nice formulation. What do you mean by that?*
> *Fred:* *Well, then you can see what if you use different values for a slope number or something like that, for a parameter, ehm,... how the formula adapts to that.*
> *Interviewer:* *Yes.*
> *Fred:* *For example, for a parabola, you see it changes completely.*

14 of the 19 students managed to perform *item 2,* in some cases after the interviewer had given some hints. Of the 5 students who did not manage to perform it, 4 made a mistake with the squaring, although they did show insight into the goal of the task and into the process of substitution. One of the students did not know what to do with this task. It is remarkable that all students but one were able to carry out this substitution by hand, because this had not been practised during the teaching experiment. This suggests transfer from the work in the computer algebra environment to the paper-and-pencil environment. This item is discussed in more detail in Sections 10.6.3.
The last part of *item 3* was not discussed in most of the interviews because of time constraints. For item 3a, many students required a concrete example first. The interviewers provided one, and the students were able to solve it. Then, guidance was sometimes needed to advance to the generalization. At the end, all the students found the correct answer, although some of them did not put parentheses around the numerator $(c - b)$. One of the students indicated that $c - b$ should be considered as one thing, as Sandra did in the pretest (Section 7.4.2). Another student took one single numerical example and deduced the general solution, without really understanding the algebra behind it.
The results of the post-experiment interviews indicate that the students understood the parameter as a changing quantity and were able to carry out the substitution of an expression by hand. Solving by hand an equation that contains parameters is difficult if a concrete example is lacking.

### 7.4.7 *Combining different types of data*

Several kinds of data were collected in the G9-II teaching experiment. The main difficulties that students encountered in this teaching experiment concerned the parameter as generalizer, the parameter as unknown and the instrumentation of the ISS scheme. In this section we briefly compare the different types of data on these issues, to get a complete view by data triangulation.

The results of the posttest on the generalization and the application of the isolate-substitute-solve instrumentation scheme were not very positive (items 2b and 3b). For the parameter as unknown, the picture is even more negative (item 4b for class A). Overall, the results of the posttest were better than in G9-I.

The data from observations and from the students' notebooks on these issues showed more nuances. Indeed, the difficulties that appeared in the posttest were also observed during lessons. However, the mini-interviews in the classroom also indicated a growing awareness of generality, a gradual instrumental genesis and instances of understanding the shift of parameter roles. Many good examples of appropriate instrumentation of the ISS scheme were found in the student notebooks.

The interviews with the teachers suggest that the students had learnt a lot about literal symbols and their use, and about using a technological tool for doing mathematics. The post-experiment interviews indicate that the students were able to carry out substitution by hand and to treat expressions as objects; finding a general solution and expressing a variable in terms of others was still hard for them.

If we compare these data, we first notice that the disagreement in the data does not concern the findings in general, but shows gradual difference. In addition to the possible explanations in Section 6.4.7 concerning the phenomenon of differences between data types, we argue that the data differ in character. The data from observations and mini-interviews have a process character: we observed the process of students who were struggling to make sense of phenomena and to formulate this by mathematical means. We noted progress, even if it did not lead to fully correct formulations and procedures. The written final test, on the other hand, has a strong product character: we judged the correctness of the solutions as they were written down, and the student did not have an opportunity to explain. The data from the student notebooks and from the post-experiment interviews are somewhat in between the two extremes: in the notebooks, the students condense their progress in a (usually) neat form, which makes it more product-like. In the post-experiment interviews, students had the opportunity to explain issues that had arisen already in the posttest.

Overall, we see only gradual divergence between the different types of data. We attribute this improved convergence, compared to the G9-I experiment, to an increased attention to practising exercises, so that the benefits of the learning process could be transformed into product performance to a greater extent than was the case for G9-I.

## 7.5 Reflection and feed-forward

In this final section of the chapter we look back at our initial expectations and formulate the feed-forward of the G9-II research cycle for follow-up research cycles.

*Reflecting on the expectations*

We started Section 7.2 by presenting expectations based on the feed-forward of research cycle G9-I concerning the opportunities that computer algebra would offer for the learning of the concept of parameter. Did those expectations come true? We will address them one by one.

1 The first expectation was that the parameter as *changing quantity* would be a more natural start for developing insight into the higher parameter roles than the parameter as generalizer would be. The findings of this teaching experiment confirm this conjecture. Observational data and test results indicate that the students appreciated the parameter as changing quantity that affects the complete graph. This view of parameter is supported by a strong visualization that was explored by means of a sliding tool on the TI-89. However, two remarks can be made. First, it seems that the parameter as changing quantity dominated the other parameter roles too much, probably because of the strong visualization. The parameter roles of generalizer and unknown may have suffered from that. Second, the students' view of the effect of parameter change tended to remain somewhat superficial. Often, graphical effects were not examined carefully, and a link with algebraic features was not established.

2 The second expectation was that a better *slider tool* that allows for gradually changing the parameter value would help students to perceive the parameter as changing quantity. To a certain extent, this conjecture was confirmed in the teaching experiment. Teachers and students referred to the use of the SLIDE program that had been developed for this purpose. However, despite the success of the 'sliding parameter', this slider tool on the TI-89 was too limited, so that the image of a gradual and continuous change of the parameter did not come over very well. For example, students seemed to think that the parameter value could only run through sets of integer values rather than through real intervals.

3 The third expectation was that practising the simple instrumentation schemes for solving and substitution, and the composed ISS instrumentation scheme would lead to a better *instrumentation* that would create room for generalization. The overall impression of the instrumentation in G9-II is that more attention for solve and substitute – with the oval visualization and the name for the vertical bar – at the start of the experiment indeed improved the instrumental genesis of these simple instrumentation schemes. However, at the end of the teaching experiment (Sections 9 and 10 of 'Changing algebra', and the final test) difficulties with the instrumentation showed up again, and in particular with the composed ISS scheme. We explain this by the more complex situations in which this scheme had to be applied, and see this as an indication of an incomplete instrumentation.

4 The fourth expectation was that more attention to generalization as it emerged from the dynamics of the 'sliding parameter' would lead to a better understand-

ing of the *parameter as generalizer*. This conjecture turned out to be false. Although generalization scored better in the posttest than was the case in the G9-I research cycle, the results still were not very good. The transition from the parameter as changing quantity to the parameter as generalizer apparently did not work out very well. Possible explanations for this are the already mentioned dominance of the changing quantity conception of the parameter, and the fact that problem situations that stress the dynamics of the parameter are not necessarily the best contexts to invite for generalization.

### *Feed-forward*

The findings of the G9-II research cycle generated feed-forward for the next research cycles. We distinguished feed-forward that concerns the hypothetical learning trajectory, the instructional activities and the research methodology.

The *feed-forward concerning the HLT* first addresses the global learning trajectory. The parameter learning trajectory from G9-I, which was characterized by the roles generalizer - changing quantity - unknown, was changed for G9-II into changing quantity - generalizer - unknown. This change turned out to be an improvement, because the changing quantity is the most accessible parameter role and offers good opportunities for visualization. Therefore, this line remained unchanged for the next research cycle. However, in order to avoid the parameter as changing quantity being too dominant, we did not want to address this parameter role too extensively, so that it would not hinder the transition to the other parameter roles such as the generalizer and the unknown. The transition from the parameter as changing quantity to the parameter as generalizer required specific attention. The symbol sense – the insight into the structure and the meaning of formulas and expressions – and the instrumentation of the algebraic techniques involved were to be addressed particularly for achieving better understanding of the parameter as generalizer and the parameter as unknown. As far as the role of the teacher is concerned, we regretted that the two G9-I / G9-II teachers do not teach in tenth grade, so that their professional development could not be monitored and used in the next research cycle.

The *feed-forward concerning the instructional activities* follows from the feed-forward for the HLT. First, we aimed at using a better slider tool for the visualization of the parameter as changing quantity than was available in G9-II. To avoid the TI-89 screen resolution limitations, we considered using a PC environment for that purpose. Second, paying more attention to the elementary instrumentation of algebraic techniques by means of more practice at the start of the teaching experiment did improve the instrumentation, so this remained unchanged in the next teaching experiment.

Third, the G9-II teaching material contained more real-life problem situations than was the case in G9-I; however, most of them addressed the parameter as changing

quantity (fountains, sprinklers). For the improvement of the transition to the parameter as generalizer, we aimed at finding real-life problem situations that invite generalization. Rather than directly addressing the general level, we wanted to start with referential models of problem situations that are meaningful to the students and that could develop into models for mathematical reasoning at the general level in terms of the Gravemeijer four-level structure (Section 3.6). This was expected to help the students to give meaning to parameter and to understand the structure of the formulas. In order to diminish the role of the ISS instrumentation scheme, we looked for contexts, which would lead to one function rather than a system of equations.
The *feed-forward concerning the research methodology* addresses the teaching experiments. Despite our intentions after G9-I, there still was a danger that the mini-interviews during the lessons were guiding the students too much. The observer was tempted to influence the learning process. Of course, discussing the students' questions yielded relevant data; however, it could interfere with the methodology of questioning students on specific key items in the teaching material. In the next teaching experiment, the balance between observer-driven interactions and student-driven interactions was to be reconsidered carefully, so that gathering data on key items would not be hindered by long discussions with students, which could acquire the character of instruction.

# 8 The G10-II research cycle: learning trajectory and experience

## 8.1 Introduction

In Chapters 6 and 7 we described the two research cycles in the ninth grade, from the hypothetical learning trajectory through the global experiences in the teaching experiments to the feed-forward for the next research cycle. The present chapter addresses the two research cycles that focused on the tenth grade, viz. G10-I and G10-II. Fig. 8.1 shows the arrangement of the teaching experiments.

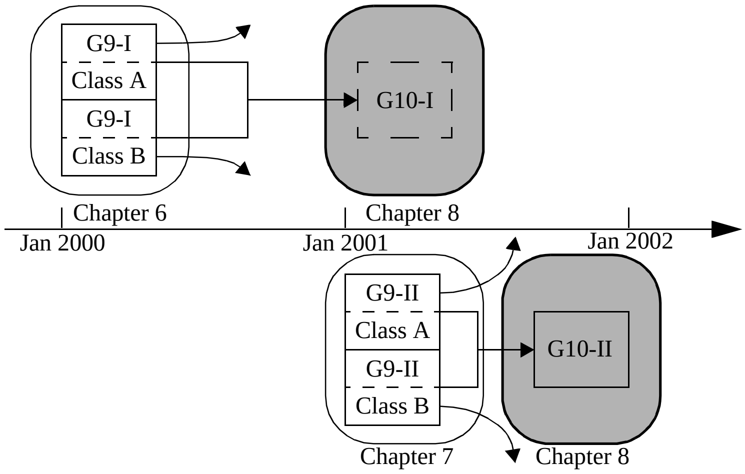



Figure 8.1 Teaching experiment arrangement

The first research cycle for the tenth grade – G10-I – was an intermediate one that included a short teaching experiment of five lessons with about half of the students who had also been involved in G9-I. For several reasons, this research cycle is presented only briefly (Section 8.2).

The G10-II research cycle was the third and last complete research cycle. It is discussed in a similar way as G9-I and G9-II, viz. we first discuss the hypothetical learning trajectory (Section 8.3), then the key activities for the students (8.4) and then the global experiences of the teaching experiment (Section 8.5). The chapter ends with a concluding section (8.6) that evaluates the development of the hypothetical learning trajectory throughout the three research cycles and the global information that emerges from the teaching experiments. In Chapters 9 and 10, the issues of the concept of parameter and the instrumentation of computer algebra are revisited in more detail.

## 8.2 The G10-I intermediate teaching experiment: a global impression

The G10-I teaching experiment was an intermediate experiment carried out in a tenth-grade class comprising 14 students. All but one student who had failed to pass tenth grade the previous school year had participated in the G9-I teaching experiment. After ninth grade, all of these students had chosen the exact stream, so the G10-I population consisted of those students from G9-I who were interested in mathematics and science.

For practical reasons, such as a lack of facilities for additional lessons and the full regular curriculum in tenth grade, only five lessons were available for the teaching experiment. The teacher was an experienced mathematics teacher, who was used to integrating technology into his teaching. Because of the time constraints, he felt the need to stick close to the regular text book. In particular, he recommended the topic of the transformations of graphs. The teacher and the students had been using the TI-83 graphing calculator for eight months by the start of the teaching experiment.

The teaching materials for this short experiment first aimed at refreshing the TI-89 skills learned in G9-I, in particular those concerning algebra. Then, the transformation of graphs was investigated by changing parameter values. The experiment ended with a final investigation task concerning the dynamics of the graph of the function *y* with

$$y(x) = \frac{1}{a^2} \cdot x^3 - \frac{3}{a} \cdot x^2 + 2x$$

Because of practical constraints, the research aims could not be dealt with adequately. A complete HLT could not be developed and tested; rather, we consider G10-I as a short but meaningful experience, as a warm-up for the G10-II teaching experiment that is described hereafter.

On the basis of the analysis of the data from G10-I, the following findings that are relevant to G10-II can be stated. First, the students were able to pick up the TI-89 skills quite easily. Their experience with the TI-83 graphing calculator could be used in the graphical modules of the TI-89, and the algebraic procedures that were used in the G9-I experiment could be refreshed without much effort. Second, the method of changing parameter values to study the dynamics of the graph was adequate, although just replacing parameter values turned out to be an inefficient means of doing so. A slider tool was to be preferred. Third, the algebraic features of the TI-89 were not used very often. For example, only two out of seven pairs used the solve command to find out that (0, 0), (*a*, 0) and (2*a*, 0) were the coordinates of the zeros of the function *y* in the final task. However, some of the students did notice by using the factor command that $y(x) = x \cdot (x - a) \cdot (x - 2a) / a^2$. One pair of students verified that the graph of *y* was obtained by multiplying the graph of *f* with $f(x) = x^3 - 3x^2 + 2x$ by a factor *a* in both the direction of the *x*-axis and the *y*-axis. Expanding $a \cdot f\left(\frac{x}{a}\right)$ gave the expected result.

Theses global conclusions from the G10-I intermediate teaching experiment were useful for the preparation of the G10-II teaching experiment, which is the topic of the rest of this chapter.

## 8.3 The G10-II hypothetical learning trajectory for the parameter concept

In this section we first describe the starting points for the HLT of the G10-II research cycle, and the expectations that are investigated in this final teaching experiment. Then we motivate the global trajectory through the different parameter roles. The activities that are supposed to bring about the transitions between the different parameter roles are described next. This leads to the global HLT. Finally, we address some instrumentation aspects of the learning trajectory.

### *Starting points and expectations*

As with research cycles G9-I and G9-II, the starting point for the development of the HLT was the level structure for defining the higher level parameter roles and the suggestions concerning how to achieve an understanding of these roles by using the opportunities that computer algebra offers (Sections 4.5 and 5.5). Furthermore, we built upon the experiences and the feed-forward from G9-I and G9-II. Essentially, we revisited the approach of the G9-II research cycle. However, the feed-forward of G9-II led to the following intentions and expectations that were intended to improve the results and inform the development of the HLT for G10-II:

1 Insight into *the structure and meaning of formulas and expressions* was one of the bottlenecks in the G9-II research cycle. Specific attention to this would improve the students' symbol sense and facilitate the work in the computer algebra environment.

2 In G9-II the transition from the parameter as changing quantity to the parameter as generalizer was problematic. For the parameter as changing quantity, the use of a seemingly *continuous slider tool* was supposed to support the perception as a parameter that runs through an interval; also, the use of generic functions in the slider tool environment, in which parameters have a graphical meaning, was expected to prepare for the transition towards the parameter as generalizer.

3 In G9-II the students did not see the meaning of the general level, which hindered the conceptual development. Therefore, the use of *real-life problem situations* with an intrinsic hierarchy of literal symbols was supposed to give meaning to parameters at a referential level and foster meaningful generalization at the general level; also, a focus on functions rather than on systems of equations would diminish the instrumentation difficulties.

4 A third means of improving the transition towards the parameter as generalizer was an earlier *intertwinement of the different parameter roles.* This would stimulate the understanding of the parameter as generalizer and the parameter as unknown and prevent the changing quantity role from becoming too dominant, as was observed in G9-II.

*A trajectory towards the higher level understanding of the parameter concept*

One of the conclusions of the previous chapter was that the G9-II global trajectory, which consisted of the line placeholder - changing quantity - generalizer - unknown, was an improvement over the G9-I trajectory. Therefore we followed the same line in G10-II (Fig. 8.2). In order to prevent the parameter as changing quantity from becoming too dominant, the parameter as generalizer was presented as the central issue of the teaching materials by means of an introductory example at the start.

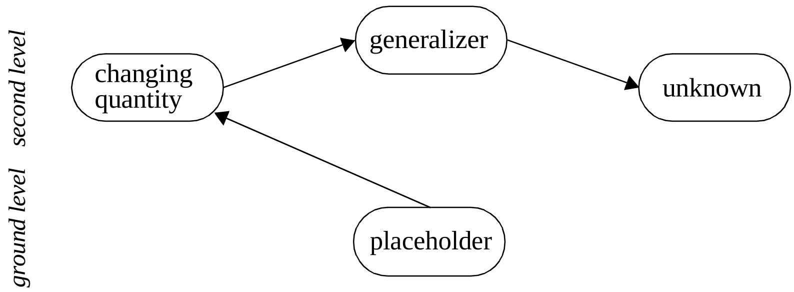



Figure 8.2 Learning trajectory through parameter roles in G10-II

The feed-forward from G9-II suggested some refinements of the learning trajectory within this global outline. First, the development of insight into the structure and meaning of formulas would be stimulated by examining and explaining CAS output, and by horizontal mathematization of problem situations that lead to meaningful formulas. Second, in order to avoid a too limited concept development of the parameter as changing quantity, we intended to address this parameter role not as extensively as in G9-II and to anticipate on the parameter as generalizer and the parameter as unknown from the beginning of the learning trajectory. Third, the referential level, horizontal mathematization and the use of real-life contexts would receive more attention. This should lead to mathematical situations that can be represented by one function rather than a system of equations, and that invite generalization using a meaningful parameter and enable the development of a general level in terms of the four-level structure, and vertical mathematization.

*How to bring about the transitions?*

The character of the student activities that were supposed to foster the transitions from one parameter role to the next have not changed essentially compared to the G9-II research cycle. Fig. 8.3 provides an overview. The learning trajectory runs from top to bottom. The columns subsequently describe the role of the parameter, the algebraic and graphic meaning of this parameter role, the students' overt and mental activity that brings about the transition, and the way computer algebra supports this activity. We do not repeat the motivation for the HLT description that was given already in Section 7.2. Here we just focus on the changes concerning the transition from the parameter as changing quantity to the parameter as generalizer that was one of the hard issues in the G9-II research cycle.

| role of parameter | algebraic meaning | graphic image |
| --- | --- | --- |
| placeholder | one numerical value | one graph |
| changing quantity | a changing numerical value | a changing graph |
| generalizer | a set of numerical values | a sheaf of graphs |
| unknown | subset of numerical values | a subset of graphs in a sheaf |

| student activity | mental activity | role of computer algebra |
| --- | --- | --- |
| slide parameter values | see global dynamics, graph as object | substitute parameter values, animate graph |
| explore similar situations for different values | see similarity, generalize, formula as object | graph, solve equations, general solutions |
| filter specific parameter values | change relation parameter - variable | solve with respect to parameter |

Figure 8.3 HLT scheme for the parameter concept in the G10-II teaching experiment

The transition from the changing quantity role towards the generalizer role was supported by two additional means. First, while working with the slider tool, attention was paid to the invariant properties of the sliding graph, such as zeros that do not move or points that are hit by all graphs. Such graphical invariants can be explained algebraically, and this was supposed to open the horizon to generalization.
Second, more than was the case in G9-II, the starting situations for studying the dynamics of the parameter change in G10-II also involved generic functions that contained parameters, such as $y = a \cdot x^2 + b \cdot x + c$. When two of the three parameters were 'pinned' at a numerical value, the dynamics of the third one could be investigated. This was inspired by Doerr and Zangor, who used a similar approach to match data sets (Doerr & Zangor, 2000).

### *Instrumentation aspects within the HLT*

Computer algebra in the proposed learning trajectory was used for substitution of numerical values and expressions, for solving equations containing parameters, for manipulation of expressions and formulas, and for graphing graphs/sheaves of graphs and for animating graphs.

The instrumentation of the computer algebra schemes was fostered by practising simple instrumentation schemes, as was done in the G9-II teaching experiment. Furthermore, the black box character of work in the computer algebra environment would be countered by asking for explanations of the CAS output. Attention was paid to the congruence of CAS technique and paper-and-pencil technique by comparing them.

Because of the use of problem situations that would lead to one generic function with a parameter rather than systems of equations, we expected that the instrumentation difficulties with the isolate-substitute-solve scheme would be less important than they were in previous teaching experiments.

## 8.4 Key activities in the teaching materials

In this section we describe the elaboration into assignments in the teaching materials of some of the key activities of the HLT for the concept of parameter.

All but one of the students in the G10-II teaching experiment had been involved in G9-II, so they already had some experience in using the TI-89 symbolic calculator. Furthermore, they had used the TI-83 graphing calculator since the start of the school year, so an introduction to the graphical features of the TI-89, which are quite similar to those of the TI-83, was not necessary. Therefore, the teaching materials for G10-II did not contains an introductory part, as was the case in the previous teaching experiments. Fig. 8.4 shows the table of contents of the teaching unit, entitled ‘Fountains and sheaves’; the complete manuscript (in Dutch) is available at www.fi.uu.nl/~pauld/dissertation.

***Fountains and sheaves***

1 Refreshing TI-89 skills
2 Starting problem
3 What happens to the graph?
4 The sliding parabola in TI-Interactive
5 A ‘strip comic’ of graphs
6 Sheaves of graphs
7 Setting formulas containing parameters
8 General solutions
9 Using general solutions

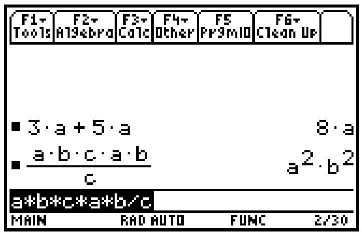
Figure 8.4 Contents of the teaching unit

The first section aims at rehearsing the basic skills of operating the TI-89. It ends with a self-test, which students are supposed to work through without any help.

Two assignments in this section address the development of symbol sense by examining the output of the CAS. The aim of assignment 1.3b is to make students examine the CAS output and realize that it cannot always be trusted blindly (Fig. 8.5). Assignment 1.11c concerns rewriting expressions in other, equivalent forms. Solving equations and substituting expressions are also addressed in the first section.

**1.3 a** Go to the HOME screen and enter: 3a + 5a. You enter the letter a with ALPHA =. You notice that the TI-89 immediately simplifies this.

**b** Enter: a∗b∗c∗a∗b/c
This is immediately simplified, too. Is the answer true for all values of a, b and c?




**1.11** A right-angled iron wire with a total length of 120 cm serves to reinforce a box. At the sides of these boxes advertisement texts will be attached. Therefore $x = 2y$ needs to hold. The volume $I$ of the box is equal to $x \cdot y \cdot z$ .

**a** Enter: I = x*y*z | x=2y.

**b** Use the total length to set a formula that does not contain the height $z$ anymore.

**c** The formula for the volume that you found can be written in different ways. Find at least three different forms.

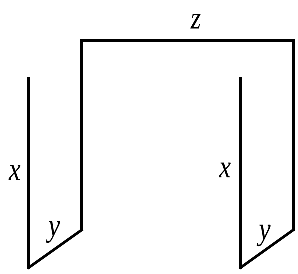



Figure 8.5 First algebra assignments

The second section of ‘Fountains and sheaves’ aims at setting the scene for the teaching sequence by providing an introductory example. Students study the relationship between the skin area and the body weight of different species. This relation can be modelled by the algebraic formula $A = \ldots \cdot W^{2/3}$ . The numbers on the dots – the so-called Meeh coefficients – are given for different species. The central task in this section concerns the generalization: what overarching formula reflects the relation between skin area and body weight for all species?
In Section 3 the parameter as changing quantity is addressed. Parameters in the general quadratic functions are systematically changed using the TI-89. Invariance is also considered: what aspects of the graph do not change? Section 4 elaborates this

using the slider tool of the PC computer algebra package TI-Interactive (Fig. 8.6). Students experience the dynamics of the situation by moving the slider with the mouse.

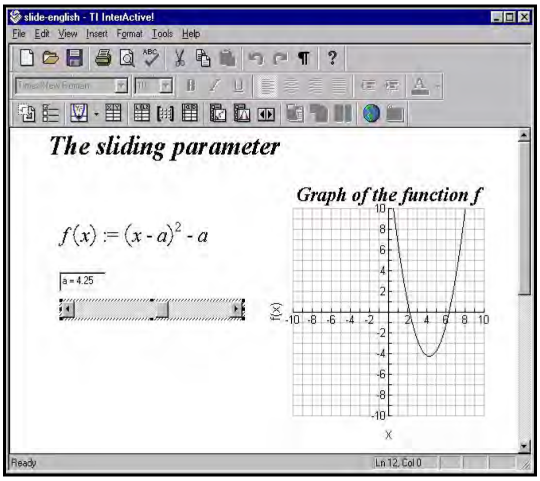


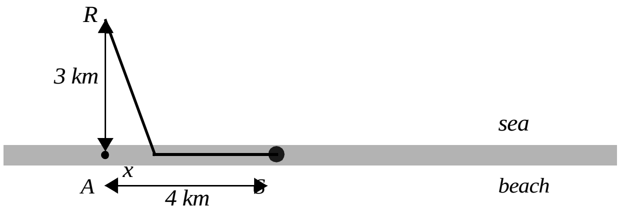

Figure 8.6 TI-Interactive screen for sliding parameter values

**4.4 a** Change the function into $f(x))\ =\ (x-a)^2 + a$ .
**b** Make the graph slide.
What is the effect of an increasing value of a to the graph?
**c** How can you tell the value of *a* from the graph?
Explain this from the formula.
**d** For what value of *a* is $(\frac{5}{2}, \frac{5}{2})$ the vertex of the parabola?

Figure 8.7 Dynamics and meaning of the parameter

Fig. 8.7 shows that the parameter in the assignments has a graphical meaning that can be understood algebraically, such as the slope, the coordinate of an intersection point or a vertex. Assignment 4.4 anticipates on the parameter as generalizer in question **c**, whereas the parameter as unknown is addressed for the first time in item **d**. In Section 5, the slider tool of TI-Interactive is used for the investigation task that ends the part on the parameter as changing quantity. It concerns the context of James Bond, who needs to bring a secret message from position *R* out at sea to point *S* on the beach (see Fig. 8.8). The total time spent on rowing and running has to be minimized. The parameter for the wind velocity leads to a function that is complicated for the students:

$$T = a \cdot \frac{\sqrt{3^2 + x^2}}{6} + \frac{4 - x}{12}$$




**5.3a** Draw in your notebook a ‘comic strip’ that represents the change of the graph of *T* as *a* gets bigger. Please make global sketches and indicate the corresponding values of *a* and the formula.

**b** Also consider values of *a* that are smaller than 1: if the wind blows from the right direction, the rowing time gets shorter!

**c** Describe in words how the graph changes as the parameter value increases.

**d** What values of *a* give a ‘special’ graph? What is special in those cases? How can you explain that using the formula?

Figure 8.8 James Bond’s mission

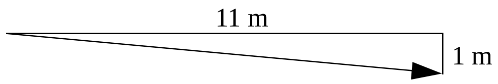
The final question of task 5.3 prepares for the use of the parameter as unknown.
Section 6 concerns the transition from the parameter as changing quantity towards the parameter as generalizer. Initially, this is done by considering sheaves of graphs rather than dynamic graphs. The first context is the traffic flow. A general formula for the circulation flow *f* as function of the speed *v* of the traffic flow is set up, in which the risk factor *r* that drivers take is the parameter:

$$f = \frac{100 \cdot v}{6 \cdot (4 + r \cdot 0,0075 \cdot v^2)}$$

The second context in this section concerns a sheaf of graphs that represent jets of water from a garden sprinkler.
In Section 7, the parameter as generalizer is addressed further by setting up general formulas containing parameters that describe classes of situations. The parameters have a meaning in the problem situations. Fig. 8.9 provides an example.
Section 8 concerns the general solutions of equations with parameters that emerge

from physics or geometry contexts. The aim of the activities is to help students perceive the inefficiency of repeating the same solution procedure for several different parameter values. Therefore, a general solution is useful. Assignments 8.5 - 8.6, which were inspired by Trouche (1998), provide examples of this approach (see Fig. 8.10).

**7.1** A vulture is an excellent glider. While gliding without moving his wings and without turbulence a vulture loses one metre of height for eleven metres of horizontal movement. In a sketch:




**a** A vulture glides away from a rock at a 100 m height. Find a formula for the height and the horizontal distance.

**b** An albatross declines only 1 metre on 20 metres horizontal distance. Adapt the formula for the albatross.

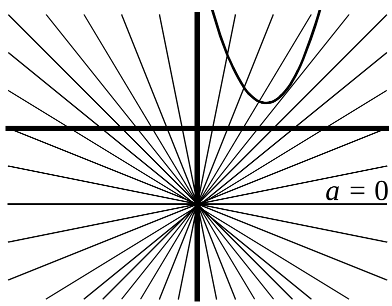
**c** The horizontal distance covered by a bird while losing 1 metre of height is called the finesse *f*. Make up a formula with this for the flight from 100 metre height for every ‘glider’.

Figure 8.9 The glider assignment

**8.5** The parabola with equation $y = x^2 - 4x + 5$ is intersected by a sheaf of lines, represented by $y = a \cdot x - 2$. The central question in this task is: which lines ‘touch’ the parabola?

**a** Calculate the coordinates of the intersection points in case $a = 5$.

**b** Do the same for $a = 6$ and for $a = -10$.




**8.6a** Generalize the problem of the previous assignment: calculate the *x*-coordinates of the intersection points without choosing a value for *a*.

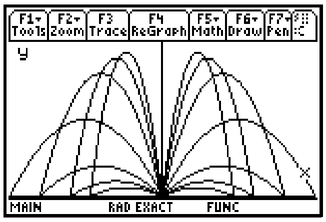
**b** Verify your general solution by substituting a numerical value for *a* in the solution formula and have the graphs drawn.

**c** How does the formula of the general solution show for which values of *a* the lines touch the parabola? Calculate those values of *a*.

Figure 8.10 Which lines touch the parabola?

From now on the parameter acquires the role of the unknown more often. The idea is that selecting one or more parameter values from the complete range reinforces the perception of a parameter representing a reference set and thus the parameter as generalizer. An example of this approach is assignment 9.4 (Fig. 8.11).

**9.4** In assignment 4.5 the jets of water from a fountain were described by the formula $f(x) = a \cdot x - (1 + a^2) \cdot x^2$
It looks as though two jets end at the same point of the ground. You can use algebra to verify that.




**a** Solve the equation $f(x) = 0$ in general.
**b** Verify that this gives the same solutions for $a = 5$ and $a = \frac{1}{5}$, and also for $a = 3$ and $a = \frac{1}{3}$.
**c** Is it true for every value of $a$ that you find the same zeros if you replace $a = p$ by $a = \frac{1}{p}$? Motivate your answer.

Figure 8.11 The garden sprinkler revisited

## 8.5 The G10-II teaching experiment

The way the teaching experiments were carried out is described in general in Section 2.5; information on the school where the experiment was held is provided in Section 2.8. In the present section, we first sketch the educational setting of the G10-II teaching experiment (8.5.1). Then the global findings are sketched (8.5.2). Section 8.5.3 contains the results of the final task, which ended the teaching experiment. In 8.5.4 and 8.5.5 we present the data from the evaluation conducted with the teacher and the students. Finally, we combine the data from the different sources (8.5.6).

### 8.5.1 *Educational setting*

The teaching experiment of research cycle G10-II was carried out in one tenth-grade class of 28 students (11 females, 17 males). It lasted for 15 lessons, including one lesson for evaluation. For practical reasons, the experiment was interrupted for two weeks after the eleventh lesson.

The class was a normal class with a fair working atmosphere. The average report mark for mathematics on the previous report was 7.3 out of 10. With two exceptions (one exchange student and one student who failed to pass tenth grade the previous school year), all the students had participated in the G9-II teaching experiment the previous school year. After ninth grade, these students chose the exact stream; thus the G10-II population consisted of the students from G9-II who were interested in mathematics and science.

The male mathematics teacher was at the beginning of his career and had not been involved in the other teaching experiments. He is qualified to teach mathematics to 12- to 18-year-old students. His style of teaching is informal. He had good contact with and left much responsibility to the students, who were given more freedom than they had in the ninth-grade mathematics classes. Classroom discussions or whole-class instructions were not very frequent. The teacher was not very experienced in using IT in his teaching. He and his students had been using the TI-83 graphing calculator for three months before the start of the teaching experiment.

Pre-experiment meetings were held between the teacher and the researcher to discuss the aims and goals of the experiment. The teacher commented on the teaching materials, and a planning for the teaching experiment was agreed upon. During the experiment, each lesson was discussed by the teacher and the researcher, and decisions for the next lesson were taken.

For the following reasons it was decided that no pretest or posttest would be held. First, for the students doing a difficult pretest can be a discouraging start to the teaching experiment. Second, the preliminary knowledge of the students in this case was known, as they all had participated in the G9-II experiment. Third, the teaching experiment suffered from considerable time constraints, so we preferred to use the time for the teaching experiment itself. As no pretest was held, we thought it would be more informative to end the experiment with a final task rather than with a posttest.

An important difference between the ninth-grade and the tenth-grade experiments was that for the latter experiments there was no extra teaching time available. As a consequence, the regular topics for the G10-II class had to be taught at a higher speed than usual. When the students noticed this, they were less enthousiastic about the teaching experiment.

Data collection took place in a similar way as in G9-II. This is described in Chapter 2. The method of selecting and representing protocols was similar to the one described in Section 6.4.3. After the experiment, the students filled in a short evaluation questionnaire.

### 8.5.2 *Experiences during the teaching experiment*

In this section, we describe the experiences during the teaching experiment, based on the classroom observations and on the written work of the students. First, we address insight into the structure and meaning of formulas. Then we follow the line of the HLT: the parameter as changing quantity, as generalizer and as unknown.

#### *The structure and meaning of formulas*

The teaching experiment got off to a quick start. The students were motivated and soon got used to using the keys on the TI-89 again. In the introductory first section of the teaching unit, the structure and meaning of formulas and expressions were addressed. In assignment 1.3, for example, *a*b*c*a*b/c* is simplified as $a^2*b^2$ by the machine (see Fig. 8.5). The students often did not pay attention to the results that the machine provided. In the following dialogue, a question by the observer was needed to make them consider the CAS output in detail.

| | |
|---|---|
| *Observer:* | *Do you understand what the machine does when it simplifies this [a*b*c*a*b/c] into this [$a^2*b^2$]?* |
| *Ada:* | *No, not really.* |
| *Observer:* | *If you wanted to simplify the left part without using the machine, how would you do it?* |
| *Ada:* | *Well yes, then I would just, in fact I wouldn't use b square.* |
| *Observer:* | *But if you look at this, a*b*c*a*b/c, could you simplify that yourself if you look at it?* |
| *Ada:* | *In the end I would've divided by c, but not that the c disappears. Oh yes, c divided by c disappears; yes it is right.* |
| *Observer:* | *OK, so you would skip the c?* |
| *Ada:* | *And then the rest, a squared times b squared.* |

(G10-II-v1, assignment 1.3)

The question whether this simplification is correct for all values of *a*, *b* and *c* had not been answered. After some explanation, Ada noticed that $c = 0$ is a problem.

Equivalence is an important issue when rewriting expressions and formulas. In assignment 1.11, for example, the substitution *I = x*y*z | x = 2y* and $z = 120 - 4x - 2y$ on the TI-89 resulted in $-20y^2 \cdot (y - 12)$ (see Fig. 8.5). The students were asked to rewrite in equivalent forms. During the classroom discussion, it appeared that that many of the students found this hard. Suggestions such as 'remove brackets' and 'expand it' gave $-20y^3 + 240y^2$. Rewriting this as $20y^2 \cdot (12 - y)$ needed by-hand skills and was only done correctly after some errors. Rewriting the expression in another form by hand revealed a lack of algebraic skills. Meanwhile, the issue of rewriting an expression into a form that is more transparent was addressed; it appeared to be quite new to the students.

During the work on the first section of the teaching materials, the instrumentation of solve and substitute seemed to work well; also, the reification of the formulas and expressions that appeared as solutions or as objects that needed to be substituted was no longer a problem.

### *The parameter as changing quantity*

Sections 3, 4 and 5 of 'Fountains and sheaves' address the parameter as changing quantity. The use of the TI-Interactive slider tool was an improvement compared to the visualization on the TI-89. The students perceived the dynamics, although they did not always find them easy to describe. In the context of the garden sprinkler, for example, students investigated the dynamics of the graph of $f(x) = a \cdot x - (1 + a^2) \cdot x^2$. Maria referred to the context in the description she wrote in her notebook:

> From $\frac{1}{2}$ to 5 -> It is a mountain parabola (below the x-axis) that gets smaller all the time (+higher, a sort of garden sprinkler effect) as a gets bigger.

The context of the garden sprinkler seemed to help some students to imagine the dynamics. However, the complexity of the formula caused difficulties with entering it correctly, and with relating graphical features and algebraic properties.
The intention to also use the sliding parameter to anticipate the parameter as generalizer and the parameter as unknown was realized to a certain extent. For the parameter as generalizer, assignment 4.4 is an example (see Fig. 8.6 and Fig. 8.7). The students explored the dynamics of the graph of $f(x) = (x - a)^2 + a$ as $a$ increased. The question ‘How can you tell the value of $a$ from the graph?’ led to generalization:

*Ada:* *I see it already, the vertex is at a. You see, y=5 here and the vertex is at 5.*
*Maria:* *OK, the vertex of the graph...*
*Ada:* *So the y-coordinate of it is ...*
*Maria:* *No, because, look this is 5 [points at the y-coordinate of the vertex] and this is 5 [the x-coordinate].*
*Ada:* *OK, so let’s try.*
*Ada slides the graph using the slider bar.*
*Maria:* *Yes it is right indeed.*
*Ada:* *2, (2, 2)*
*Maria:* *The vertex of the graph...*
*Ada:* *y- and x-coordinate are the same as a.*
(G10-II-v2, assignment 4.4)

Apparently, the fact that the parameter had acquired a graphical meaning made generalization more accessible to the students .

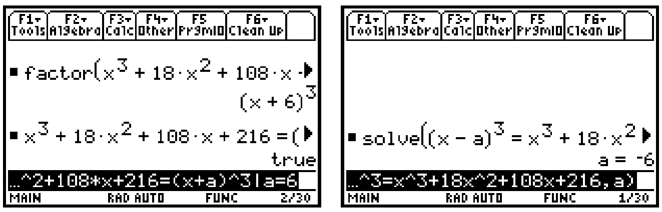



Figure 8.12 Algebraic verification

In assignment 3.3, the graphs of functions of the form $y = (x-a)^3$ were investigated. The perception of the dynamics was no problem. However, it was hard to verify algebraically whether the function $y = x^3 + 18x^2 + 108x + 216$ was a 'member of the family'. No students realized that the expression could be factorised (first line of the left-hand screen of Fig. 8.12). Two students found graphically that $a = -6$, and verified this by checking that $x^3 + 18x^2 + 108x + 216 = (x+a)^3 | a = 6$ yielded 'true' (see last line of the left-hand screen). One student treated the parameter $a$ as the unknown and solved the equation with respect to $a$ (right-hand screen of Fig. 8.12).

The final assignment on the parameter as changing quantity was the James Bond investigation task, which was handed in by the students (see Fig. 8.8). The task was to investigate the dynamics of the graph of

$$T = a \cdot \frac{\sqrt{3^2 + x^2}}{6} + \frac{4-x}{12}$$

as the parameter $a$ (the wind factor) is changing. The product should be a 'cartoon sequence' of graphs. The results of this task show that many students liked the context and referred to it in their answers. For example, translations of values of $a$ into the wind situation were frequently made. To infer the formula from the problem situation was a difficult task, but the students understood the formula, although some needed help.
The use of the slider bar clarified the dynamics of the graph. However, while copying the graphs into their 'cartoon sequence' on paper, some students seemed to believe that the graphs were parabolas. Others did not know what graphical features to pay attention to. This phenomenon is well known from research on the use of the graphing calculator (e.g. Drijvers, 1995b). One of the students integrated the graphs into one picture, commenting: 'I find this easier.' The dynamics were described appropriately in most cases. For example, Martin and Cedric wrote:

> The graph narrows as the parameter increases. For a very small value of a, it's nearly a straight line, and as a grows the graph becomes more narrow.

The final part of the task concerned the values of $a$ that provided a 'special' graph. Most students saw that $a = 0$ was the only special case; it gave a linear graph. However, substituting $a = 0$ into the formula led to calculation errors. For example, Martin and Cedric thought that the graph in that case was the $x$-axis, as they wrote:

> I think the value a = 0 is the only one that gives a special graph. It's a straight line on the x-axis. You do the a times the whole formula, so if you have a = 5 you do 5 times the whole formula. Because 0 times something always remains 0, then you do 0 times the formula. And that is and remains 0.

Apparently, Martin and Cedric did not see the structure of the formula. Others had difficulties with entering the formula correctly. The ‘pretty print’ facility that TI-Interactive offered did not always prevent difficulties. Dealing with the length of the square root sign or with factors that appear in the denominator rather than the numerator, for example, remained difficult.
Despite the difficulties with entering the formula, most students seemed to consider the formula as an object and perceived its parts as entities as well. For example, Rianne and Mandy found that $a = 0$ provided the straight graph, and noticed that this parameter value would cancel the first term of the formula for *T*. However, they did not notice that the remaining part $(4 - x)/12$ was a linear expression.
The work on the James Bond task showed that the problem situation gave meaning to the parameter and the formula, that the parameter as changing quantity was well understood, that copying graphs from the screen was not an obvious thing to do and that substitution in a complex formula can go wrong easily. The reification of the formula and its sub-expressions was not a problem.

### *The parameter as generalizer*

In the second section of ‘Fountains and sheaves’ students studied the Meeh example concerning the relationship between skin area and body weight for different species. The generalization of relations of the form $A = \ldots \cdot W^{2/3}$, with a different number on the dots for each species, into $A = k \cdot W^{2/3}$ was quite natural for the students. The next fragment indicates that Maria understood the power of the general formula:

> *Observer:* *Do you know the advantage of that formula?*
> *Maria:* *Well that you spontaneously ,at once, for example I want to know the bird you can just fill in the bird, the frog.*
> (G10-II-1/24, assignment 2.4)

In the traffic flow problem in Section 6, the students were supposed to generalize the formula of the traffic flow for the risk factor of 0.8 or 0.9 to the general case with *r*:

$$f = \frac{100 \cdot v}{6 \cdot (4 + r \cdot 0,0075 \cdot v^2)}$$

This generalization step was not difficult for the students. The context seemed to support the generalization. Carrying it out, however, sometimes led to errors, due to the complexity of the formula. Maria managed well: she replaced the 0.8 by the parameter *a* and added in the function library: $| \ a = \{0.1, .2, .3, .4, .5, .6, .7, .8, .9, 1\}$. This gave a sheaf of graphs. Jeff, however, had difficulties with the formula and built in the risk factor in a wrong place in the formula.
Section 7 concerns setting up and generalising formulas. The gliders assignment

asks for a formula for the height if the starting position is 100 metres high and the albatross loses 1 metre for every 20 metres of horizontal displacement (see Fig. 8.9). Marty and Sandra had difficulties in finding the formula, but in the end found $h = 100 - x/20$, where $h$ stands for height and $x$ for the horizontal movement. Then the formula was adapted for a real glider, that has a finesse of 30.
The following dialogue took place.

> *Observer:* *So what do you do with that number here, the horizontal movement?*
> *Marty:* *This one, you can make a parameter from it.*
> *Observer:* *Yes. What kind of formula would you get then?*
> *Marty:* *100 – a, no* $100 - x/a$ *, wherein a dot dot dot [referring to the wherein bar of the TI-89].*
> *Sandra says it in fact it is f. (...)*
> *Observer:* *So the general formula?*
> *Marty:* $100 - x/a$ .
> *Sandra:* *Couldn't you better then put here [instead of 100] 'starting height'?*
> (G10-II-2/40, assignment 7.1)

Apparently, the generalization was clear after the concrete cases. Sandra even suggested further generalization. The computer algebra device was of no help in this generalization process.
Once the generalization of the formulas was managed, the general solutions that appeared in Section 8 were no longer problematic. The students accepted the formulas as solutions. For example, solving the lens formula $1/f = 1/v + 1/b$ with respect to $v$ or $b$ was no longer hard. In assignments 8.5-8.6 on the lines that touch the parabola, calculating the general solution was the easiest part of the task (see Fig. 8.10). However, the continuation required changing the roles of the literal symbols.

### *The parameter as unknown*

The perception of the parameter as the unknown requires a change of perspective that many students found difficult to make: the hierarchy between parameter and variable is turned upside down. In the next observation, which concerns the assignment of the lines that touch the parabola (8.5 - 8.6, see Fig. 8.10), Misha determined the $x$-coordinates of the general intersection points of the line with equation $y = a \cdot x - 2$ with the parabola in terms of the parameter $a$. Then he shifted the roles of the literal symbols and solved the equation with respect to the parameter.

> *The observer explained that the two solutions should coincide in order to touch the parabola. Misha had not seen this so far.*
> *Observer:* *When are the two solutions the same?*
> *Misha:* *O, that's very easy: then you do solve once more. But I*

*didn't understand that that was the problem.*

*Observer:* *That's the most difficult thing indeed.*
*Misha:* *No but, that is, you generalize.*
*Misha equals the two general solutions for x and solves this with respect to x rather than a.*
(G10-II-3/25, assignment 8.6)

After Section 8 the experiment was interrupted for two weeks. When we continued with Section 9, we noticed that many students had lost some of their machine skills, as well as the main thread of the teaching sequence. Furthermore, for the students the assignments of Section 9 were very complex.
For example, let us consider assignment 9.4 on the garden sprinkler (see Fig. 8.11). It took some time for Rob and Jack to generalize the idea that two water jets with reciprocal parameter values meet the ground at the same point. They did, however, notice that the graphs for $a = 5$ and $a = \frac{1}{5}$ shared the same zeros:

*Observer:* *Exactly. And then I come by again and ask if it is also right for a=7 and* $a = \frac{1}{7}$ *?*
*Rob:* *You can try that.*
*Observer:* *Yes, but -*
*Jack:* *Then you are busy for hours.*
*Observer:* *Then I would ask: is it also correct for a = 100 and* $a = \frac{1}{100}$*?*
*Rob:* *Yes, one can try that as well, but probably you mean that we reason about that or so.*
*Jack:* *With a, then you have to do a and 1/a.*
(G10-II-4/14, assignment 9.4)

Many students did not do this assignment, but some did reason (often not completely correctly) on the general *x*-coordinate of the zero, that is $x = a/(a^2+1)$. Then they verified that substitution of 1/*a* and *a* provided the same result.

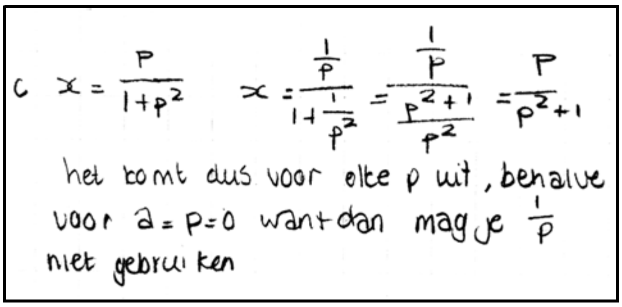



Figure 8.13 Substituting *p* by 1/*p* by hand

This approach shows the ability to deal with the different roles of the literal symbols involved and the perception of the solution expression for $x$ as an object. Helen even did the two substitutions completely by hand (see Fig. 8.13). She added to the calculation: ‘It comes out right for every $p$, except for $a = p = 0$, because then you are not allowed to use $\frac{1}{p}$.’

### 8.5.3 *Final task*

The teaching experiment ended with a task that was performed by 27 students, who worked on it in 12 pairs; 3 students handed in individual work. The task lasted for a double lesson (90 minutes in total). The full text of the assignment is presented in Appendix E.

Table 8.1 shows the results of the 15 groups. The column ‘Incorrect’ includes studnets / pairs of students who did not answer at all.

| Item | 1a | 1b | 2 | 3a | 3b | 4a | 4b | 5a | 5b | 6a | 6b | 6c |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Correct | 15 | 14 | 14 | 13 | 15 | 13 | 10 | 15 | 9 | 12 | 8 | 6 |
| Partial score | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 1 | 0 | 4 | 5 |
| Incorrect | 0 | 1 | 1 | 2 | 0 | 2 | 2 | 0 | 5 | 3 | 3 | 4 |

Table 8.1 Results of the G10-II final task

The assignment revisits the traffic flow context that was addressed in Section 6, and is a follow-up to the assignment on the lines that touch the parabola (see Fig. 8.10).

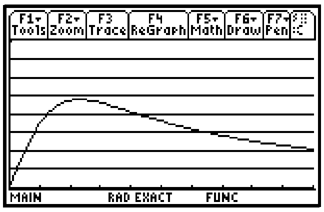



Figure 8.14 Movement of a traffic flow as function of its velocity

The central idea is that the maximum of the movement of the traffic flow as a function of its velocity can be found by intersecting the graph with horizontal lines, and then finding the line that has exactly one intersection point (see Fig. 8.14). Note that differentiation had not been taught to these students.

We will now discuss the items of the final task one by one, and then relate the results to the parameter learning trajectory of changing quantity - generalizer - unknown. Finally we will address the issues of the role of the context, the structure and meaning of the formulas, the reification of the formulas and the instrumentation.

The *first item* concerned reproducing the picture in Fig. 8.14 on the screen of the TI-

89 and approximating the vertex of the graph. Setting up the formula for the flow and drawing the graphs was easy for the students. Most of them read from the graph that the vertex is somewhat below the line at height 50. Some students used trace, while one pair made a table of function values.

The *second item* concerned verifying algebraically that at height 40 there are two intersection points, and at level 50 none. Some students did not understand that the word 'algebraically' refers to solving rather than watching the graph. Of the 14 correct answers, 11 got a numerical result, probably because they entered 0.0075 in the formula for the average distance between two cars. As this is a decimal number, the TI-89 in auto mode only gives approximations. Of course, the problem situation suggests approximated answers.

The *third item* concerned the general formula for intersection points of lines at height *h* and the flow graph. Most students used the solve command here. Five of the correct answers were exact; the others were approximated. Misha and Marty verified their general solution by substituting the known case $h = 40$. In item 3b, all students noticed that there should be only one intersection point in the case of the tangent line.

The *fourth item* was crucial: what is the height of the horizontal line through the vertex? Most students took the expressions below the square root sign in the answer to item 3 and found its zeros by solving with respect to *h*. Only Dirk and Kevin presented the exact expression for *h*; the others gave approximated answers. Some students used a table or a graph. Helen found it difficult to read the formula for the general solution due to the strange representation with the minus signs. Ada combined different approaches, including the graphical intersection method that she preferred because she knew it from the graphing calculator. Two pairs entered solve($x_1 – x_2 = 0, h$), where $x_1$ and $x_2$ were the general solutions from the previous items. In order to find the optimal velocity, most students substituted the optimal flow *h* into the general solution formula. Due to the approximations, this did not always lead to a correct answer.

In the *fifth item* the risk factor of 0.7 had to be incorporated into the formula and the solution procedure was repeated. All students got the right formula. While performing the problem-solving procedure, some errors were made, such as solving with respect to the wrong unknown, or confusing solve and substitute. Some students used the graphical method or a table, and one pair provided the exact results.

The *sixth item*, finally, addressed the generalization over the general risk factor *r*. All but two pairs got the right formula; the two pairs that did not get the right one entered 0.0075^*r* rather than 0.0075**r*. These were probably copying errors, as their answers further on were correct. Some of the students took the question concerning the relation between the risk factor *r* and the optimal flow as a qualitative one, and described the dynamics rather than providing a formula. Only one pair showed an exact formula for the maximum of *f* as function of *r* (see Fig. 8.15). The way in which the change of *r* affected the maximum flow was not clear to all the pairs. Language was complicating this: ‘a higher risk’ meant in this formula that the risk factor *r* was decreasing. This led to mixed answers to question 6c. One of the students formulated his response orally in relation to the context while answering the question what happened with the optimal flow when more risk was taken: ‘The flow gets better, but if only one person brakes, you get an accident.’

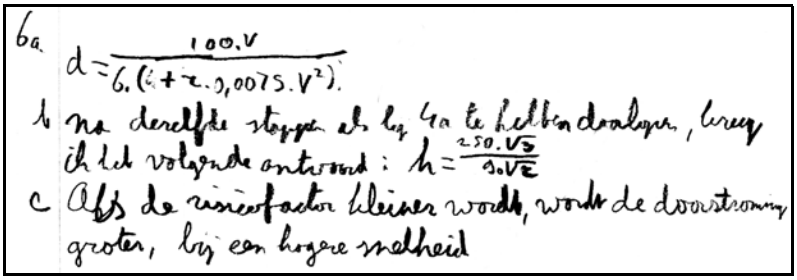



Figure 8.15 Exact solution for the maximum of *f* in terms of *r*

We now briefly review the results of the final task according to the global learning trajectory for the concept of parameter, which followed the line changing quantity - generalizer - unknown. The parameter as changing quantity was involved in items 6c and, to a lesser extent, 6b. The results indicate that the students understood how the changing parameter affected the complete situation, the graph and its vertex, despite some difficulties in explaining this clearly. For the parameter as generalizer, items 3a and 6a-b are of particular interest. The results here show that the idea of generalizing by a parameter was understood by most students, and that they saw the meaning of the general solutions. For the parameter as unknown, the further processing of the general solution in item 4a, and its repetition in 5b and 6b, was informative. The results suggest a better insight into the parameter as unknown than was the case in the previous teaching experiments. Although some confusion concerning the role of the different letters was observed, such as solving with respect to the wrong unknown, the general impression is that the students were flexible enough to deal with the shifts of roles.

What is the role of the problem situation in this final task? The traffic flow context seemed to motivate the students, and allowed for a meaningful parameter in a situation that the students could imagine. A disadvantage of the concrete context is that it possibly encouraged students to calculate numerical approximations and to refrain from algebraic solutions.

The observations concerning the insight into the structure and meaning of the formulas were positive. The students showed insight into the formula, for example when they built in the risk factor. They accepted the formulas as solutions, which suggests the reification of the formulas. Also, they were able to see sub-expressions (such as the expressions under the square root sign) as entities, as objects that were submitted to further processing. A complication was the way in which the TI-89 in some cases represented the formula, as the example in Fig. 8.16 shows.

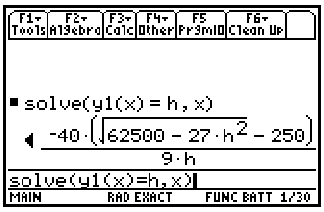



Figure 8.16 TI-89 representation of general solution

As far as the instrumentation was concerned, not many problems were encountered. The solve instrumentation scheme was usually applied correctly, despite incidental difficulties, such as solving with respect to the wrong unknown. For substitution, similar observations were made. Substitution of expressions was no problem and concrete values were used incidentally for verification purposes. The combination of solve and substitute in a more integrated scheme led to confusion in some cases, but this was not the general impression. An important instrumentation issue in this task was the relation between exact and algebraic calculations on the one hand and numerical, approximated calculations on the other hand.

In general, the results of the final task are more positive than those of the posttests in the previous teaching experiments. This suggests a positive conceptual development. We should remember here, though, that the student population of G10-II was a subset of the G9-II population, so this development was to be expected. Furthermore, an important difference between the G10-II final task and the G9 posttest was, of course, that it was done in pairs and that the teacher and the observer were available to give additional explanations if necessary. However, the observational data of the students at work do not differ from the final results. We were surprised by these results as the observational data from lessons 10 - 12 suggested many conceptual difficulties. The items in Sections 8 and 9 were difficult, and the students seemed to suffer from a lack of motivation (because they had become aware of the time consequences of the experiment) and from the interruption of the experiment. Perhaps because the final task was graded, or because the students knew this was the last part of the teaching experiment, their motivation increased and the results showed conceptual progress.

### 8.5.4 *Interview with the teacher*

A post-experiment interview with the teacher was held. During it, the students' learning process was discussed, as were the role of the computer algebra tool and the teacher's professional learning process.

The teacher was critical of the *students' learning process*. He argued that the students did not realize the main idea of the teaching materials, namely that you can affect a function by using a parameter. He considered the level of the teaching materials adequate, but a little too high at the end. The abstraction, the conceptual part and the interpretation of the results were hard. The learning trajectory was built up well, in his opinion. The learning process at the end of the teaching experiment was hindered by the fact that the students realized that they had to go through the regular topics in a shorter time after the teaching experiment. Despite all this, the teacher admitted that the students seemed to know what they were doing and that the results of the final task had surprised him in a positive way.

The availability of the *computer algebra device* changed the work, as the interpretation of the results became more important. However, the teacher argued that working with paper and pencil sometimes seemed to have led to a better understanding than working in the computer algebra environment had. He considered the TI-89 a useful tool for students at a higher level, such as eleventh and twelfth grade of the exact streams.

As far as his *professional learning process* is concerned, the teacher said that he enjoyed participating in an exciting teaching experiment. Looking back, he regretted not having had more time to familiarize himself with the TI-89 before the experiment, because it would have made it easier to foresee student difficulties and to anticipate on them. However, becoming a routine user would have required time that was not available. A second regret concerned the limited time spent on classroom discussions. Were he to do it again, the teacher would organize more classroom discussion and plenary teaching moments, so that the main thread and the essentials of the teaching sequence would be more clear to the students.

In general, the teacher was satisfied with the experience and with the teaching experiment in practice, although he was critical of the conceptual benefits for the students.

Looking back at the role of the teacher and at the interview, we agree that more whole-class discussions to interactively discuss the central issues of the learning line might have stimulated the collective cognitive development in the class.

### 8.5.5 *Student reactions*

During the final lesson of the teaching experiment, 25 of the 28 students filled in a questionnaire that consisted of five questions. This took 10 minutes. We will now summarize the reactions to these questions.

The first question was: what is your opinion of the experiment with the TI-89? Most students replied that they found it 'nice'. Some found it interesting, whereas others stated that it was a waste of time. A few quotations:

- *It took a lot of time that we didn't have, but for the rest it was nice and informative.*
- *Nice. I like working with computers and searching for solutions to problems.*
- *Easy, not much homework.*

The second question was: what did you learn from the experiment with the TI-89? Many students replied that they had learned how to operate the TI-8; not so many, however, referred to the learning of algebra. Perhaps mentioning the TI-89 in the question prompted this. A few quotations:

- *Operating the TI-89 and extra functions.*
- *How to deal with the machine.*
- *To write general formulas.*

The third question was: did you find it useful? Why or why not? Many students replied that it was no use, because they had to return the TI-89 and revert to the graphing calculator after the experiment. Two of them also indicated that they preferred to work with paper and pencil:

- *No, you get lazy from these sort of things. I can't do these kinds of things (solve, expand) on paper.*
- *Yes because you learn to look at mathematics in a different way.*
- *You learn to solve by button pressing and not how to solve it without machine (for example solve).*

The fourth question was: did you enjoy it? Why or why not? Most students enjoyed the experiment, although three students did not. A few quotations:

- *Nice until it appeared that the chapters [from the regular text book] now need to be done extra fast.*
- *Yes, you can do many nice things with it that we did not do in the lessons.*
- *It was nice to work on solutions for a long time, and to find them in the end is a challenge, I think.*

The fifth and last question was: do you have other remarks or comments to make? There were not many reactions to this. A few students commented that they had learned to use the TI-89, but now had to get back to the TI-83 again.
In general, most students had enjoyed working with the TI-89, but were not able to express what they had learned about algebra. They regretted that they had to return the TI-89 and get back to the TI-83; they also regretted that the time for the teaching experiment had been at the expense of the regular teaching. These two points were not relevant to the teaching experiments in the ninth grade, as extra lessons had been

arranged and the students did not have the TI-83, so the differences between the machines was not an issue.

### 8.5.6 *Combining different types of data*

Several kinds of data were collected in the teaching experiment. In this section we briefly compare them. We first discuss the triangulation of observational data and the written work of the students; we then consider the comparison between the observational data and the handed-in final task in particular.

The observational data from mini-interviews during the lessons, from classroom discussions and from video registrations of one pair of students clearly indicated the conceptual and technical difficulties the students encountered while working through the teaching materials. There had been many struggles, conceptual obstacles and misconceptions. The written data from the handed-in task on James Bond and the work from the student notebooks, on the other hand, contained many good solutions and sometimes good explanations as well.

There are several possible explanations for this difference between the data types. First, the observations during the lessons were quite close, so that every temporary wrinkle in the students' thinking was registered, even if it was solved soon afterwards. From this perspective, the observations might have given a too negative impression. Second, students possibly only wrote down in their notebook the things they were really sure about, and perhaps later added results from colleagues or from classroom discussions. In that case, the written data might have provided a somewhat too positive impression. Third, and most important, we would argue that the observational data and the written data are two data types with a different character that complement each other. The observational data give insight into the students' learning process, whereas the written data have a product character. The two types of data stress different aspects of the teaching experiment that are not conflictual but complementary: the observational data help to map the learning process and the conceptual difficulties and developments involved, whereas the written data provide insight into the final products of the learning process.

It is remarkable that the written data from the G10-II teaching experiment are more positive than the observational data, whereas this was the other way around in the previous teaching experiment. We conjecture that this is because the G10-II experiment ended with a task that was done in pairs, and not with a final individual test, as was the case for G9-I and G9-II. The final task provided better results, and linked up better with the style of working during the teaching sequence than was the case for the previous experiments. Partially, this can be attributed to the fact that the G10-II students had been involved in G9-II, and so had had some experience with parameters already.

## 8.6 Reflection and conclusion concerning HLT and teaching experiments

In this final section of the chapter, we first look back to our initial expectations for the G10-II teaching experiments. Then we consider the hypothetical learning trajectory and the teaching experiment experiences throughout the three research cycles.

### *Reflecting on the expectations for G10-II*

Section 8.3 started with expectations concerning the opportunities that computer algebra would offer for the learning of the concept of parameter, based on the feed-forward from research cycle G9-II. Were those expectations justified? We will address them one by one.

1 The first expectation was that specific attention to *the structure and meaning of formulas and expressions* would improve the students' symbol sense and facilitate the work in the computer algebra environment. The findings of this teaching experiment confirm this conjecture. By questioning the meaning of CAS output and by using real-life contexts, the students examined the formulas more carefully and were able to attach meaning to expressions / parts of expressions and formulas. Evidence for this is the way the students dealt with complex formulas, for example in the James Bond context and the traffic flow problem (see Fig. 8.8 and Appendix E). The lack of closure obstacle no longer played an important role; the reification of formulas and expressions seemed to be fostered by the attention to their meaning and structure.

2 The second expectation was that the use of a *continuous slider tool* would support the perception of the parameter as a changing quantity. Also, the use of generic functions in the slider tool environment, in which parameters have a graphical meaning, was expected to prepare the students for generalization. These expectations were justified to a certain extent. The continuous slider tool did stimulate the perception of the parameter as continuously changing quantity. Evidence for this is the students' work on the tasks in Sections 3, 4 and 5 (e.g. see Fig. 8.6 and Fig. 8.7). However, the 'pretty print' facilities of TI-Interactive did not solve all the difficulties with entering complex formulas.
The slider tool and the way its use was didactically structured allowed for addressing the parameter as generalizer and as unknown at an earlier stage than had been the case in the previous teaching experiments. The generalization of the vertex of the graph of $f(x) = (x - a)^2 + a$ in assignment 4.4 is an example of this.

3 The third expectation was that the use of *real-life problem situations* with an intrinsic hierarchy of literal symbols would give meaning to parameters and foster generalization; also, a focus on functions rather than on systems of equations would diminish the instrumentation difficulties. As indicated in point 1 above,

the use of real-life contexts indeed fostered generalization. In this way, the jump from the referential level to the general level was made. Evidence for that is provided by the students' work on the relation between skin area and body weight, and their work in the glider context (Fig. 8.9). The contexts did give meaning to the formulas. Also, the fact that they did not lead to systems of equations made the instrumentation easier; in particular, the difficulties with the isolate-substitute-solve scheme were not as dominant as they had been in previous teaching experiments. Two comments on this issue can be made. First, it seems that the intrinsic generalization step was primarily a mental one, in which the contribution of the CAS was limited. Second, despite the meaningful formulas, in complex situations the students still had difficulties keeping track of the problem-solving strategy and distinguishing the roles of the different literal symbols.

4. The fourth expectation was that, using the appealing visualizations of the parameter as changing quantity, an earlier *intertwinement of the different parameter roles* would stimulate the understanding of the parameter as generalizer and the parameter as unknown. This expectation was justified. The visualizations of the sliding parameter were used to raise questions concerning the parameter as generalizer and the parameter as unknown (e.g. see assignments 4.4 c and d in Fig. 8.7). This earlier intertwinement of the different parameter roles led to a less dominating perception of the parameter as changing quantity and a better understanding of the parameter as generalizer and as unknown. Evidence for this is provided by the students' work on the final task.

While considering these rather positive conclusions, we should stress some important differences between the G10-II teaching experiment and the G9-I and G9-II experiments. First, the students in G10-II were now a year older and had more mathematical experience. Furthermore, the G10-II population consisted of those students in the G9-II population who had chosen the exact stream, so it consisted of students who were better at and more motivated to learn mathematics. Third, many of the topics addressed in the G10-II teaching experiment had been addressed already in G9-II, so they were not completely new to the students. We suppose that the students' experience in G9-II made them more receptive to the notions of generalization and reification. Nevertheless, the findings of G10-II suggest that the new didactic approach with more attention to real-life problem situations, to intertwining the different parameter roles and to limiting the importance of systems of equations was an improvement. We are is tempted to conclude that computer algebra use for the learning of the concept of parameter is more appropriate for tenth grade than for ninth grade.

### *Conclusions drawn from the three research cycles*

Chapters 6, 7 and 8 described the development of the HLT throughout the research cycles. Furthermore, the experiences of the teaching experiments were sketched. In this section we briefly look back at these three chapters and summarize the main issues that will be considered in more detail in Chapters 9 and 10.

The research cycles led to the development of an HLT according to the parameter role line of placeholder - changing quantity - generalizer - unknown. Student activities were developed that were intended to foster the transitions in a natural way. These activities included manipulations in the computer algebra environment and in the paper-and-pencil environment, and, of course, the related mental activities. Depending on the grade and the student population, these activities were changed and emphasized different aspects. In this way, optimization of the HLT was achieved.

This learning trajectory was far from easy for most of the students. We often observed struggles with obstacles and misconceptions concerning the concept of parameter, particularly when the parameter acted as generalizer and as unknown. Bearing in mind the conceptual analysis of the concept of parameter in Chapter 4, this was not a surprise. The computer algebra use was sometimes helpful in this conceptual development; on other occasions, however, it did not help or even confused the students. The last teaching experiment suggests that the role of the problem situations is important, and that real-life contexts may help students to perceive the meaning and structure of the formulas involved. Seeing the structure and meaning of the formulas is part of symbol sense. A related issue is the reification of the formulas,; such reification is required in the computer algebra environment, but was also fostered by the CAS work. The understanding of the different parameter roles, the function of computer algebra therein, and the role of the problem situations and the insight into formulas and expressions are considered in more detail in Chapter 9, where the data on these aspects from the three experiments are combined. In this way, we will address the first research subquestion on the understanding of the concept of parameter.

A second issue is the instrumentation of computer algebra. Many observations from the teaching experiments show how students got bogged down in technical problems while operating in the computer algebra environment. In some cases this lasted so long that it was no longer efficient, but frustrating. Frequently encountered problems concerned the instrumentation of the solve command, of the substitution of expressions and of integrating these two into a composed instrumentation scheme. However, it turns out that these problems are not just technical, but also have a conceptual component. According to the theory of instrumentation, which was discussed in Chapter 5, the conceptual and technical aspects are intertwined in the instrumentation schemes. The instrumentation of computer algebra, the interaction between conceptual and technical aspects, and the obstacles that students encounter while working in the computer algebra environment are considered in more detail in Chapter 10. The data on these aspects from the different teaching experiments will be combined, thus allowing the second research subquestion on the instrumentation of computer algebra to be addressed.

# 9 Results on computer algebra use and the concept of parameter

## 9.1 Introduction

In Chapters 6, 7 and 8 we sketched the development of the hypothetical learning trajectory for the concept of parameter and the global experiences from the teaching experiments for the consecutive research cycles. This led to an HLT for the concept of parameter along the line placeholder - changing quantity - generalizer - unknown.
In the present chapter we discuss the contribution of computer algebra use to the development of the understanding of the concept of parameter throughout the three research cycles. Thereby it treats the first research subquestion:

> *How can the use of computer algebra contribute to a higher level understanding of the concept of parameter?*

The structure of the chapter follows the learning line for the concept of parameter. First, we present the findings concerning the parameter as placeholder (Section 9.2). Then we address the parameter as changing quantity (9.3), as generalizer (9.4) and as unknown (9.5). In Sections 9.6 and 9.7 we discuss two related topics, namely the importance of realistic problem situations (9.6) and of the insight into the meaning and structure of formulas (9.7). Section 9.8 contains an overview and a discussion.
Sections 9.2 - 9.7 share the following structure: they start with the conclusions on the section issue. These conclusions are then motivated in two ways: first we present a global overview of the data by means of representative protocols and (in Sections 9.3 - 9.5) a data table. Second, we illustrate these general findings by more presenting detailed descriptions of the conceptual development of one of the students.

## 9.2 The parameter as placeholder

### 9.2.1 *Conclusions on computer algebra use and the parameter as placeholder*

In Chapter 4 we defined the placeholder view of the parameter as the understanding of the parameter as an ‘empty place’, as a ‘box’ that contains a numerical value. As a consequence, systematic variation of the parameter is not considered and the different parameter values are not related or integrated into a general image. This placeholder role of the parameter is seen as the basic level – in Van Hiele’s terminology, the ground level – of the concept of parameter: the parameter as a non-changing constant, as a pseudo-number. At this level, the students substitute numerical values for the parameter and are reluctant to accept that the parameter value is unknown. Our aim is for the students to develop a more versatile and rich concept of parameter that includes the higher level parameter roles of changing quantity, generalizer and unknown.
We conclude from the data that the students initially did view the parameter as a placeholder for numerical values. This placeholder view can indeed be considered

as the ground level of the concept of parameter and as the starting point for the learning trajectory. The computer algebra environment supports the placeholder view by allowing for substituting numerical values for parameters in a transparent way by means of the 'wherein bar', written as |. For example, $(x-a)^3|a = 4$ yields $(x-4)^3$. Such substitutions are required for using the graphical module within the computer algebra environment: if no numerical value is substituted for the parameter, two-dimensional graphs cannot be drawn and error messages are the result.

Within the students' placeholder conception, we found two main views. The first concerns the parameter as referring to a numerical value, which by preference is known, so that further numerical calculations can be carried out. If the parameter value is unknown, in some cases students choose one themselves. The second view considers the numerical values of the parameter as generic examples, as 'specific cases of the general, without having anything that is specific'. The latter view provides chances for extending the placeholder view towards higher level parameter roles. The computer algebra option to easily change the value of $a$ in $(x-a)^3|a = 4$ can start the preparation for the parameter as changing quantity. The substitution of sets of parameter values opens the horizon for sheaves of graphs and therefore for the parameter as generalizer.

The placeholder conception of the parameter should not be considered as irrelevant or incorrect. In the computer algebra environment, numerical substitutions have to be done in order to produce two-dimensional graphs and to consider concrete cases as verification of general findings. The observations indicate that it was hard for the students to decide when the placeholder conception is appropriate and when it is not. However, an exclusive placeholder view is limited, and combined with the lack of closure obstacle it may hinder the extension of the concept of parameter towards the parameter as changing quantity, generalizer and unknown.

### 9.2.2 *Overview of the data*

Because the placeholder level is the basic level of this study that students are supposed to have already reached at the start, we did not define specific key items for this parameter role in the instructional activities. Instead, all incidences of the students' behaviour that were interpreted as related to the placeholder conception were identified as such and included in the category that our prior coding system had prepared for this parameter role. By means of a constant comparative method the analysis of these observations led to an overview of the data on this issue and to the distinction of different aspects of the placeholder view of parameter.

The results of this analysis first indicate that the parameter as placeholder can indeed be considered at the ground level of the concept of parameter, and that all students considered the parameter as placeholder for numerical values, which links up with the similar perception of 'ordinary' variables. No observations indicated that students had difficulties with the parameter as placeholder. Also, even while considering the higher parameter roles, students regularly came back to the placeholder role. The second result of the analysis is the distinction of five aspects of the placeholder view of parameter that were identified by means of a sub-coding. These are addressed below one by one.

### *a The parameter as a fixed (known or unknown) number*

The students often understood the parameter as a reference to a numerical value, to a particular numerical value or even to more than one number. The parameter was considered as a kind of label that refers to a number. The numerical substitution within the CAS supported this. After substituting a numerical value for a parameter *b*, one of the student expressed this view by saying 'that indicates the points of *b*'. Such a view of the parameter as a 'pseudo-number' can be adequate, but is too limited to deal with other aspects of the concept of parameter.

### *b The need to know the value of the parameter*

It was observed that students sometimes felt that they could not proceed as long as the values of the parameters were not known. A typical dialogue that suggests this took place between Anne and Caroline while they were solving the sum-product system of equations $x + y = a^2$ , $x - y = b^2$ with respect to *x* and *y*:

*Anne:* *OK,* $\frac{1}{2}$ $a^2$ *plus, and the sum is ...*
*Caroline:* *But how can you calculate this then?*
*If you have* $\frac{1}{2}a^2 + \frac{1}{2}b^2$*?*
*Anne:* *You can't calculate it, then you need to enter numbers.(...)*
*Anne:* *But then you cannot fill in x,* $\frac{1}{2}a^2 + \frac{1}{2}b^2$*.*
(G9-II-E4/9, assignment 6.7)

In such observations we see the lack of closure obstacle (Section 3.4): the students perceive $1/2 \cdot a^2 + 1/2 \cdot b^2$ as a non-closed result, probably because the plus operator, and need numerical values for the parameters to be able to carry out the addition. Note that the computer algebra environment does not cause this, as it allows symbolic calculations as well as numerical calculations. This lack of closure obstacle seemed to hinder the progress towards the parameter as generalizer and as changing quantity.

### *c The tendency to choose a parameter value if it is unknown*

Some students solved the lack of closure obstacle by simply choosing a concrete parameter value. In some cases this numerical value had played a role earlier in the context or in a previous assignment; on other occasions it was just an easy numerical value that suited the student.

For example, one of the students wanted to solve the general sum-product problem:

$b + h = s$, $b \cdot h = p$. She entered solve($b + h = s \mid b = s - h, b$), which yielded 'true'. Then she changed her input into solve($b + h = 20 \mid b = s - h, b$), as in the previous assignment the value of $s$ was 20.

*d The use of exemplaric parameter values*

Some observations showed that students substituted a numerical value for a parameter that seemed to have an exemplaric function. The numerical value served to make the situation more concrete, more tangible, but in the meantime the student was aware that the specific value was not important and was no more than a kind of generic example.

In the following observation, Vera tried to explain the general procedure of finding the dimension of a rectangle with a given perimeter and a given difference between base and height. While doing so, she chose a difference of 3, although she did not use that value.

> *Vera:* *Just half of the perimeter and then ... if the difference for instance must be 3, then just the other half added to the half and the other half off.*
>
> (G9-II-E3/11, assignment 5.4)

*e The moving between specific parameter values and a generic parameter*

Along the lines of the previous aspect, some students jumped back and forth between generic parameters and specific, examplaric parameter values. They referred to concrete numbers, but in the meantime formulated in generic terms. This turned out to be a sensible step towards generalization.

The next fragment concerns the same assignment as the previous observation: the task is to find the dimensions of a rectangle with given perimeter and difference between base and height. It shows the use of concrete values to illustrate generic reasoning.

> *Donald:* *OK. First you do the perimeter that you need to have, in most cases that is 12 or 15, divided by 4, and that number that you get then plus the difference that you should have, yes, well there are differences that you need to have, for example 5, yeah, for example 3, and then you do 3 + ...*
>
> (G9-II-6/5, assignment 5.4)

Looking back at these five issues, we notice that the issues *a*, *b* and *c* reflect the limited view of a parameter as referring to a numerical value, which by preference should be known. This view is related to the lack of closure obstacle (Section 3.4) and links up with the tendency to prefer numerical answers to algebraic results, especially in concrete problem situations. However, it hindered the acceptance of algebraic expressions as results as long as the parameter values are not known, and therefore was an obstacle for the transition to the higher parameter roles of changing quantity and generalizer.

The views described under points *d* and *e*, on the other hand, reflect the idea that a parameter may have a certain numerical value but also could have another numerical value, and that the concrete value may not be important. This view opens the way to a higher level understanding of the parameter as generalizer and, to a lesser extent, as changing quantity.

The substitution of several different parameter values, and the discussion of the parameter as placeholder versus its variation, are activities that foster the transition from the limited placeholder view explained in *a - c* to the generic view addressed in *d* - *e*, and to the parameter as generalizer.

Throughout the teaching experiments, sometimes the conception of the parameter as placeholder for numerical values appeared, possibly also because it is indispensable for graphing. We conjecture that language also plays a role in the students' tendency to revert to the placeholder view. For example, words like 'solve', 'calculate' and 'determine' were related to numerical solutions for the students, whereas sometimes the answer was a formula. It might have been better to use formulations as 'express ... in ...' or 'find a formula for ...' in order to avoid the placeholder association.

### 9.2.3 *Exemplary development of one of the students*

In addition to the global findings concerning the parameter as placeholder that were presented in the previous section, we now describe some episodes on this issue from the work of Maria, who sometimes collaborated with Ada. In the G10-II teaching experiment, this pair was constantly observed by means of video registration.

Maria is a student with a good working attitude, who likes to express herself. Her mark for mathematics on the previous report was 8 out of 10. She has an efficient and practical approach to mathematics, directed towards 'knowing how' and not so much to 'knowing why'. Her partner in the video pair, Ada, scored 7 out of 10 for mathematics on her previous report. Dutch is not her native language. She speaks less than Maria, and operates quite independently. She seems to have a more investigative and exact attitude than Maria, although the report marks do not express this. Ada thinks about what is behind the tasks and enjoys understanding why, but is not very concerned about the effect on her marks.

The following episodes illustrate how Maria struggled with the lack of closure obstacle. She found it hard to accept not knowing the parameter value, which made it hard to understand the parameter as changing quantity or as generalizer.

At the start of G10-II, Maria and Ada were using the continuous slider tool in TI-Interactive to investigate the dynamics of the graph of $f(x) = x^2 + 2x + c$ as $c$ changed (see Fig. 8.6). However, they did not change parameter values using the slider bar, but substituted each concrete parameter value in the formula by hand. As a consequence, they did not perceive the dynamics in the situation. This indicates a placeholder view of the parameter. In the next assignment they saw that the graph intersects the vertical axis at height *c*, which is a form of generalization, and then used a concrete value to verify this. Ada commented: 'Try another one, then we're sure, take *c*= 5.' For verification, the placeholder view is adequate.

Later in the teaching experiment, Maria got used to formulas that contain parameters, but sometimes forgot that the parameter needed to have a value before the graph could be drawn. For example, when asked to make a sheaf of graphs, Maria entered $y_1(x) = a \cdot (1 + a^2) \cdot x^2$, but did not substitute a value for *a*, so she did not get any graphs.

Later, in the context of the lens formula, Maria and Ada suffered from the lack of closure obstacle when asked to isolate *b* in $1/5 = 1/b + 1/v$. They felt unable to proceed as long as they did not have values for the parameters. Ada commented: 'I mean you can't find it. It's just $1/5 = 1/b + 1/v$, you can't find anything else.'

Similarly, the task of finding the general coordinates of the intersection points of the graph of $f(x) = a \cdot x^2 + b \cdot x + c$ with the horizontal axis presented the two students with problems:

| | |
|---|---|
| *Maria:* | *This is an extremely difficult question. Find the general coordinates. How can you find coordinates of something if it has no numbers?* |
| *Ada:* | *[reads the assignment]* |
| *Maria:* | *That's impossible, isn't it?* |
| *Ada:* | *$ax^2$+bx+c ... last year ...* |
| *Maria:* | *But how can you know in general where it intersects? That's different for each...* |
| *Ada:* | *Intersection points with the x-axis is just filling in zero, that's what they want, isn't it?* |
| *Maria:* | *But you can't fill in anything in a formula with a's, b's and c's?* |

(G10-II-v6, assignment 9.1)

These observations show that is it difficult for Maria and Ada to go beyond the placeholder view and see the parameter as generalizer, despite the earlier work with the slider tool and with generalizations in easier examples.

In the final example of this section, the task was to find the general coordinates of the vertex of the function $f(x) = x^2 + b \cdot x + 1$. In the assignment it was suggested to first calculate the zeros in general, because the *x*-coordinate of the vertex would be the average of the two zeros. Maria once more was reluctant to work with the zeros in general:

| | |
|---|---|
| *Maria:* | *But how do you get the zeros then?* |
| *Observer:* | *Then you say* $x^2 + b \cdot x + 1 = 0$. |

*Maria:* *But it's no use to know that it [the vertex] lies between the two zeros if you can't calculate them.*
*Observer:* *You get something that contains the b, and you can continue with that.*
(G10-II-4/16, assignment 9.2)

## 9.3 The parameter as changing quantity

### 9.3.1 *Conclusions on computer algebra use and the parameter as changing quantity*

In Chapter 4 we described how viewing the parameter as changing quantity concerns systematically changing the parameter value, so that the parameter acquires the character of a really *changing* constant or, in other words, a sliding parameter. The parameter runs through a reference set in a continuous, dynamic way. This parameter change affects the problem situation at a higher level than variable change does: it changes the formula as a whole, and moves the complete graph.

We conclude from the data that the designed instructional activities indeed brought about the transition from the placeholder view of parameter towards the changing quantity view. Students experienced how parameter change affected both the graph as a whole and the entire formula. In terms of the level structure for the concept of parameter (Section 4.5), this is seen as a higher level understanding. Throughout the consecutive teaching experiments, a modest improvement of insight into the parameter as changing quantity was observed, which we attribute to the improved learning trajectory and the use of a slider tool.
The computer algebra environment supported this development by allowing for easy changes of the parameter value, substitution of sets of parameter values, and dynamic change of the parameter value using a slider tool. The latter is not exclusive to computer algebra environments: graphical and numerical software (e.g. Excel) offers similar options. The additional power of computer algebra came into play as soon as the relations between graphical features and algebraic properties were considered, so that the algebraic facilities of the CAS could be integrated into the graphical approach. During the teaching experiments this integration was not obvious. We noticed that students examined the dynamics of the graphs only superficially, without linking graphical phenomena to algebraic properties: they saw *what* happened, but did not know *why*. A careful design of the assignments and an alert teacher were needed to link the graphics with the algebra and to capitalise on the affordances of computer algebra. Only when the teacher suggested that the students should use algebra to explain what they saw, as was done in G10-II, did computer algebra use have a surplus value compared to graphical and numerical software. If the parameter had a graphical meaning for the students, asking for an algebraic explication of

graphical features created the need for algebraic verification and generalization. This way, the parameter as changing quantity anticipated on the parameter as generalizer and as unknown.

Besides this risk of superficial examination of the effects of the changing parameter, we also noticed the danger of the parameter as changing quantity becoming too exclusively the students' parameter view, so that the development of the other parameter roles (generalizer, unknown) was hindered. Therefore, instructional activities in which the parameter as changing quantity invited generalization had the best effect. Furthermore, the observations show that some students tended to see the parameter as running through integer values rather than continuously changing. The use of the continuous slider tool helped them to overcome this. As a final remark, the students had difficulties with adequately formulating the perceived dynamics in the moving graph.

### 9.3.2 *Overview of the data*

Before the teaching experiments, we identified key instructional activities that were supposed to foster the understanding of the parameter as changing quantity and that would be appropriate to monitor the students' development on this issue. We formulated our expectations on these key items and tested them by means of mini-interviews with students during the teaching experiment.

While analysing the data, the collection of key items was slightly changed, for example when a key item had been skipped because of time constraints, or when observations were not done for practical reasons. Then the observations were analysed and coded. To each observation one of the labels '+', 'o' or '–' was attributed.

The + scores refer to observations that reveal the expected insight into the parameter as changing quantity. The next protocol provides an example of a '+' score. Josef showed an adequate grasp of the dynamics of the graph of $y = a \cdot x - (1 + a^2) \cdot x^2$ when $a$ increased.

| | |
|---|---|
| *Josef:* | *Then you see that the vertex gets higher and the zero goes to the y-axis.(...)* |
| *Student:* | *Josef, the vertex gets higher?* |
| *Josef:* | *Ehm, yes.* |
| *Student:* | *What do you mean 'higher'?* |
| *Josef:* | *The a gets bigger and then you see that it gets, ehm, higher all the time.* |
| *Teacher:* | *And what happens to the shape of the graph?* |
| *Josef:* | *Well yes, it gets more narrow.* |

(G9-II-5/3, assignment 2.8)

The 'o' scores refer to insights into the parameter as changing quantity that are not completely correct but do contain valuable elements. For example, the next protocol was on the James Bond assignment. The description of the effect of an increasing value of *a* on the graph of $a \cdot \sqrt{3^2 + x^2}/6 + (4 - x)/12$ by Robert and Jack is valued with an 'o':

> *Observer:* *Do you know what happens to the vertex if the value of a changes?*
> *Rob:* *The vertex?*
> *Jack:* *It gets higher or lower.*
> *Rob:* *At a certain moment somewhat more to the left.*
> (G10-II-2/11, assignment 5.3)

The '–' scores refer to conceptions concerning the parameter as changing quantity that are not adequate. For example, in the following observation the students do not know what happens to the line with equation $y = s - x$ when the sum *s* increases:

> *Teacher:* *Minke, what happens to the graph of* $y = s - x$ *when the sum gets bigger?*
> *Minke:* *It gets steeper.*
> *John:* *They get lower.*
> (G9-I-A14-15, assignment 6.5)

Table 9.1 provides the scores for the key items concerning the parameter as changing quantity for the three teaching experiments.
As a first comment on this table, it should be remembered that the learning trajectories were not the same for all teaching experiments. In G9-I, the parameter as changing quantity was addressed after the parameter as generalizer, whereas this order was inverted in G9-II and G10-II.
A second difference between the learning trajectories concerns the use of a slider tool: this was not done in G9-I, a simple tool was used in G9-II and a more sophisticated one in G10-II.
The number of observations on the key items is smaller for G10-II than for G9-II because only one class was involved. For G9-I it is smaller than for G9-II because of a less efficient way of carrying out the observations.
Table 9.1 confirms the conclusions from Chapters 6, 7 and 8: most observations on the key items indicate an understanding of the parameter as changing quantity. Students noticed the effect of parameter change and were able to formulate it. This reveals insight into the dynamics of the sliding parameter. By means of the proposed activities, the students developed the image of a sliding parameter that affects the graph and the formula as a whole. This view of the parameter seemed to be quite natural.

| **G9-I** key item | Changing quantity scores | **G9-II** key item | Changing quantity scores | **G10-II** key item | Changing quantity scores |
|---|---|---|---|---|---|
| 6.5bc | +++oo––– | 1.5 - 1.6 | +++++++++ | 3.1c | + |
| 6.8 | ++o | 1.11 | ++++oo | 3.3b | + |
| 7.1b | +o– | 1.12 | ++++o | 4.2c | ++ |
| 7.2c | ++++– | 2.5 | ++++++++++oo– | 4.4c | +++++ |
| 7.4c | ++ | 2.8a | +oo | 4.6b | + |
| | | 2.9b | +++++++++o–– | 4.7a | ++++++– |
| | | 3.1a | ++++++++o– | 5.3abc | +++++++o |
| | | | | 6.5b | o |

Table 9.1 Key item scores for the parameter as changing quantity

The positive results that appear in Table 9.1 link up with the findings of the final tests of G9-I and G9-II and the final task of G10-II (6.4.4, 7.4.4 and 8.5.3, respectively). These results also revealed a good insight into the parameter as changing quantity. In the interviews with some of the students after G9-II, this parameter conception turned out to be the most dominant one (Section 7.4.6).

The conclusions from classroom observations in Chapters 6, 7 and 8 were that the change in the HLT and the use of a better slider tool improved understanding of the parameter as changing quantity. The slider tool apparently offers an appropriate model for the notion of the sliding parameter. However, these positive results need to be considered with care. A further analysis of the observations and of the written materials revealed four issues that complicated the understanding of the parameter as changing quantity. The following are the four issues.

*a The superficial examination of the effect of the sliding parameter*

Although the image of the dynamically moving graph is a powerful one, students often examined the effect of parameter change only superficially and found somewhat phenomenological results, such as ‘when the parameter slides to the right, the graph does ....’. They often did not wonder why this happened, and did not examine characteristic features of the graph, such as zeros or vertices. Relations between the graphical change and the algebraic properties of the corresponding formula were hard to make. The subtlety of the dynamics of the graph was not always discovered. For example, after drawing parabolas with equation $y = x^2 - a \cdot x + a + 3$ for different values of the parameter *a*, one of the students thought that the parabola simply shifted to the right as the parameter *a* increases, whereas the movement is more complex than that. The complexity of the formula seems to play a role here: the results on the dynamics of the graph of $y = (x - a)^2 + a$ (G10-II task 4.4) were better.

*b The dominance of the changing quantity view of parameter*

As was indicated by the results of the G9-II posttest and post-experiment interviews, the conception of a parameter as a changing quantity that affects the graph tended to dominate the concept of parameter (7.4.4 and 7.4.6). One of the conclusions of G9-II was that this dominance hindered the development of insight into the parameter as generalizer and as unknown. In order to reduce this dominance in G10-II we tried to intertwine the different parameter roles and to link the parameter as changing quantity to the parameter as generalizer or unknown at an early stage. This turned out to be an improvement, and the conclusion is that the possible dominance of the changing quantity view of the parameter has to be dealt with carefully in the learning process.

*c The perception of the parameter as a discrete changing quantity*

Often students seemed to consider the parameter as an integer value rather than a continuous changing quantity. They often chose values 1, 2, 3, 4, ... for the parameter, and rarely fractions or negative values. The use of the continuous slider tool helped in perceiving the parameter running through a continuum. We also made students investigate the dynamics of parameter changes with steps smaller than 1. The following dialogue shows how Michael reacted to change with a step size of $\frac{1}{2}$.

*Observer:* *What's the difference if the step size is $\frac{1}{2}$, because in other assignments it was 1? Do you know what the difference is?*
*Michael:* *Here it's more exact, for $\frac{1}{2}$.*
*Observer:* *What do you mean 'more exact'?*
*Michael:* *Then you see better, ehm, (...), then you better see the changes.*
(G9-II-3/44, assignment 1.12)

*d The formulation of the dynamics in natural language*

It was observed regularly that students did perceive the dynamics caused by the sliding parameter correctly, but were not able to express it clearly in natural language. For example, sometimes the phrase 'the graph goes upwards' was used to indicate that the graph got steeper. Also, the words 'rotate' and 'slide' were sometimes confused. We noticed a difference between the informal natural language and the precise use of some terms within mathematics. An additional complication while formulating the dynamics was the meaning of 'getting bigger', 'growing' or 'increasing'. Students had difficulties with the meaning of these expressions. For negative parameter values they indicate that the parameter values slide towards 0, but students sometimes interpreted them as increasing absolute values, thus a movement in the opposite direction. In one of the observations, one of the student expressed this while discussing the dynamics of the graph of $y = A \cdot x^2$. He put it as follows: 'For negative it is in fact: if it [the parameter value] gets bigger in the negative, it gets smaller, but if it gets more positive, it [the graph] gets bigger.'

### 9.3.3 *Exemplary development of one of the students*

In addition to the overall picture of the previous section, we now describe the behaviour of Maria, who was sometimes assisted by Ada. Generally speaking, Maria understood the parameter as changing quantity quite well. The only issue that really caused her difficulties was the relation between 'special' graphs that show up during the process of sliding the parameter, and the corresponding 'special' parameter values. Substitution of parameter values in a complex formula was the bottleneck.

One of the observations in Section 7.4.3 concerned the SHOOT game on the TI-89, in which the parameter $a$ in $y = a \cdot x + 5$ determines the direction of shooting (Fig. 7.7). It showed that Maria gave meaning to the parameter in the context, saw the dynamics and realized that the graph depended on the value of the parameter. She did not notice from the formula that the graph was a straight line for each parameter value. Later she explained the dynamics to the class, but had difficulties phrasing it. The teacher might have pointed out that the parameter was the coefficient of the variable, thus therefore the slope of the line:

*Teacher:* *And what's the influence on the graph itself of that slope number?*
*Maria:* *Well that it changes, so to say.*
*Teacher:* *If the a gets bigger ...*
*Maria:* *It changes the direction, then it changes here, it gets bigger or ...*
*Teacher:* *Yes. How does it change? Show it with your hand.*
*Maria:* *Yeah, then it changes like this, I think, more upwards.*
*[Maria makes a gesture of getting steeper]*
*Teacher:* *Then it gets steeper.*
*Maria:* *Steeper, yes, that's what I mean.*
(G9-II-3/47, assignment 1.5-1.6)

The G9-II investigation task concerns the 'cartoon sequence' of the graphs of $y = x + a \cdot \sqrt{x^2 + 1}$ as $a$ changes (Section 7.3). Maria produced a fair cartoon sequence on paper. After that, she commented on the difference between drawing one graph by hand and considering a sliding graph in the computer algebra environment. In fact, she seems to be referring to the fact that the graph is no longer the result of a process of drawing point after point, but is an object that is submitted to the process of sliding.

*Maria:* *Yeah, but I think it's a bit strange because normally you have a graph and you draw from point to point, but here*

*you suddenly have for each a a different graph.*
*Observer:* *Yes.*
*Maria:* *Whereas if you draw yourself, that never happens.*
(G9-II-5/11, assignment 3.1)

In the G9-II post-experiment interview the changing quantity appears to be Maria's main perception of the parameter, although her reference to the set of graphs also suggests the generalizer role. The next fragment starts with Maria reading her answer to the posttest question of what a parameter is:

*Maria:* *A parameter is an extra letter, so not x or y, in a formula or equation. The graph changes if you change the value of the parameter. For example, in* $y = ax + 1$ *, a is the parameter. The graph looks different when you fill in different values for a. a thus indicates a set of graphs.*
*Interviewer:* *Well done. That's a very good answer, too. Now what's the function, what can it be useful for, such a parameter?*
*Maria:* *Well, then, ehm, you can see what kind of graph it is, a little bit its shape. And yes, if you didn't have the parameter, then you couldn't say anything about it. Well, at least if you have no parameter, you can't indicate the set of graphs, so to say.*
(G9-II-12/11, final interview)

In G10-II a continuous slider tool was used. As we indicated (9.2.3), Maria and Ada first substituted different parameter values by hand. After starting to really use the slider tool, the sliding graphs soon led to them noticing that the vertex of the moving graph of $y = (x-a)^2 + a$ had coordinates (*a, a*). Ada commented that 'the vertex is at *a*' and '*y*- and *x*-coordinate are the same as *a*'. Later, during the whole-class discussion, Maria explained this generalization by referring to an exemplaric parameter value.

*Maria:* *Yes, I think that vertex, that were the x and y coordinates, they were, so to say, equal to the a. The vertex was (1, 1), then a was 1 too.*
(G10-II-2/18, assignment 4.4)

These observations show how the parameter as placeholder may raise the issue of generalization in a natural way.

## 9.4 The parameter as generalizer

### 9.4.1 *Conclusions on computer algebra use and the parameter as generalizer*

In Chapter 4 we argued that viewing the parameter as generalizer concerns generalising over similar concrete cases and unifying them by means of one parametric formula. In this way, the 'family parameter' represents a sheaf of graphs and a class of problems that can be solved in a generic algebraic way. The formulas and the solutions share a meta-character, in that they organize all concrete instances of the problem or the solution. This involves the reification of the formulas that represent the generic problem and the general solution.

We conclude from the data that the designed instructional activities indeed fostered the development of the generalizer view of the parameter, but that this development was successful only to a limited extent. Students experienced how the parameter as generalizer unified a class of situations, formulas and solutions, but in the meantime the generalizer view often remained abstract and not very meaningful to them. If the generalization started with a meaningful context, the jump from the referential to the general level was more likely to be successful. In terms of the level structure for the concept of parameter (Section 4.5), this is seen as a modest form of higher level understanding. Comparison of the HLTs for G9-I and G9-II suggests that the order changing quantity - generalizer is better than the inverse order.

The computer algebra environment supported this development by allowing for recalculating problem situations by substituting different parameter values. In this way the CAS generated the examples that formed the basis for the generalization, and freed attention for the investigation of regularities and patterns. However, in some cases such activities led to superficial pattern recognition rather than meaningful generalization. In those cases we observed empirical abstraction rather than reflexive abstraction. The black box of computer algebra did not automatically foster the understanding of the pattern. This phenomenon is not new, although examples of the successful use of CAS as a generator of examples can be found in the literature as well (Berry et al., 1994; Drijvers, 1999b; Trouche, 1998). Factors behind these successes are an intriguing problem situation that students can imagine, a valuable contribution of the CAS for generating examples, and the need to reason with the examples.

As a second means of supporting the development of the generalizer view of parameter, the computer algebra environment allowed for carrying out procedures without substituting a parameter value. This enabled the students to formulate generic equations and to find generic solutions that consisted of algebraic expressions rather than of numerical values. This required that the students were able to perceive expressions as entities, in other words, were able to reify expressions and formulas. On the other hand, this kind of CAS use, which for the students had a somewhat top-down character, stimulated the reification of formulas and helped them to overcome the placeholder conception and the lack of closure obstacle. The need for and the power of general solutions were not always felt by the students, who found it difficult to attach meaning to the resulting formulas. Switching between concrete and generic sometimes promoted the generalization.

Third, the graphics module within the CAS allowed for the fast drawing of a sheaf of graphs that represented the family of functions. This representation emerged quite naturally from the sliding graph that resulted from the changing parameter. The adjective ‘fast’ refers to a comparison with by-hand drawing: compared to the PC environment, graphing sheaves on the symbolic calculator is quite slow. The TI-89 notation $y_1(x) = x \cdot (s-x)|s = \{0, 1, 2, 3, 4, 5\}$ helped the students to realize that the parameter represents a set of values rather than one individual value (Section 10.3.3). Of course, the affordance of graphing sheaves is also provided by graphical software and is not exclusive to computer algebra environments.

Finally, an essential step in the generalization process – the setting up of the general relation or formula that represents the generic problem – is not done by the computer algebra environment. The power of computer algebra lies in solving general problems, not in posing them. This is probably the hardest mental task involved in the process.

### 9.4.2 *Overview of the data*

Before the teaching experiments we identified key instructional activities that were supposed to foster understanding of the parameter as generalizer and that would be appropriate to monitor the students’ development on this issue. We formulated our expectations on these key items and tested them by means of mini-interviews with students during the teaching experiment.

While analysing the data, the collection of key items was slightly changed, for example when a key item had been skipped because of time constraints, or when observations were not done for practical reasons. Then the observations were analysed and coded. To each observation one of the labels ‘+’, ‘o’ or ‘–’ was attached.

The ‘+’ scores refer to observations that reveal the expected insight into the parameter as generalizer. For example, the following dialogue shows how Aisha was able to generalize her solution procedure after solving the sum-difference problem $x + y = 700$, $x - y = 600$.

*Observer:* *And suppose there was another number instead of the 600, how would you do it?*
*Aisha:* *The same way.*
*Observer:* *And if you got x+y is a number s and x-y is another number v, how would you tackle it?*
*Aisha:* *Well then you have 2x is s+v, so you have x equals one half s plus one half v, and y is then s-v, 2y=s-v...*
*Observer:* *How do you do that then?*
*Aisha:* *Well, x +y=s, x-y=v, if you do then 2y, then that s is v bigger [than 2y] and then you have one half s minus one half v.*

(G9-I-B11, assignment 2.1)

The ‘o’ scores refer to insights into the parameter as generalizer that are not completely correct but do contain valuable elements. For example, after solving the above system of equations, $x + y = 700$ and $x - y = 600$, Bert was able to apply the procedure to other concrete cases, but he did not formulate the generalization.

> *Observer:* *And now suppose there was a 800 instead of that 700?*
> *Bert:* *With a difference of 600? Then you simply do 400, then this [the x] becomes 700 and this [the y] 100.*
> *Observer:* *Yes. And if there had been 10000?*
> *Bert:* *50000, ehm, 5000 and then the difference has to be 600 again, so then it gets 5300 and 4700.*
> (G9-I-A9, assignment 2.1)

The '–' scores refer to conceptions concerning the parameter as generalizer that are not adequate. For example, Mia was unable to generalize the solution procedure for the sum-difference problem and stuck to the trial-and-improve method:

> *Mia:* *In fact I just try a little bit. But I don't know a formula or something for it. That would be useful, I think, because for some numbers it [the trial-and-improve method] is quite complicated.*
> (G9-II-E3/9, assignment 5.4)

Table 9.2 provides the scores for the key items concerning the parameter as generalizer for the three teaching experiments. As a first comment on the table, it should be remembered that the learning trajectories were not the same for all teaching experiments. In G9-I, the parameter as generalizer was addressed before the parameter as changing quantity, whereas this order was inverted in G9-II and G10-II.
Table 9.2 shows many positive scores. Indeed, the students managed to see the parameter as a means for generalization over concrete situations and relations, and for formulating general solutions. What the table does not show, however, are the difficulties the students had in achieving the target generalizations. Often, help from other students or indicative questions from the teacher or the observer were needed before progress was made. Frequently, students reverted to the placeholder conception of the parameter, were hindered by the lack of closure, or got lost in the problem-solving procedure and in the roles of the different literal symbols. The results shown in Table 9.2 should be considered in this perspective.
The results from Table 9.2 were also put into perspective by the modest results of the posttests of G9-I and G9-II on the items that concerned generalization (Sections 6.4.4 and 7.4.4). On the other hand, the results of the generalization items of the final task of G10-II were satisfying (8.5.3). Apparently, the generalizer concept of the parameter was not mature enough to be dealt with adequately in the stress of the test situation and was hindered by instrumentation problems, which during the lessons could be overcome with the help of others.

| **G9-I** key item | Generalizer scores | **G9-II** key item | Generalizer scores | **G10-II** key item | Generalizer scores |
|---|---|---|---|---|---|
| Intro 5.2-5.3 | +++++++++++++ ++++oooooo––– – | 5.4 | +++++++++++++ +++++++++o––– | 2.4b | ++++ |
| 2.1b | ++++++oo– | 5.8 | ++++ | 6.4a | ++o |
| 2.6 | ++ – | 5.9-6.1 | ++++++++++oo– | 7.1c | ++++o |
| 4.6-4.7 | ++ | 6.2 | ++++++– | 7.2b | ++ |
| 5.4c | ++o | 6.7c | ++++++++++o– | 7.3abc | +++++ |
| 5.5b | ++o | 7.7-7.8a | +++++++oo–– | 7.4 | + |
| 11.1 | ++++oo | 9.2c | +++oo– | 7.5 | + |
| | | 9.3c | ++++ | 8.3b | ++ |
| | | | | 8.6ac | ++o |
| | | | | 8.8 | + |
| | | | | 9.1a | + |
| | | | | 9.2bc | +++++o |
| | | | | 9.3bcd | +– |
| | | | | 9.4ac | +++ |

Table 9.2 Key item scores for the parameter as generalizer

Table 9.2 does not suggest an improvement on the issue of generalization within each of the experiments, or between the different experiments. We attribute this to the increasing complexity of the problem situations and the corresponding formulas, and to the fact that later in the teaching experiments the parameter as unknown interfered.

Further analysis of the observations and of the written materials showed four issues that were relevant to the process of acquiring understanding into the parameter as generalizer. These issues are addressed below.

### *a The need for generalization*

Students did not always feel the need for generalization and did not perceive its power. As soon as they understood the problem-solving procedure and were able to apply it in concrete cases, the need for a general formula was no longer felt. Especially when the 'price' of generalization was high, because of the complexity of the general solution formulas, some students wondered what it was good for. The availability of computer algebra did not change this. For example, when the system of equations $x + y = 40$, $x \cdot y = p$ was solved, Martin reacted: 'But why do so much work to

express $x$ in a formula, if you can do the same as above?' By 'the same as above' he was referring to the previous assignment in which $p$ had a numerical value.
Apparently, the assignments did not always elicit the need for generalization. The argument that generalization saves work if you have to apply a procedure many times is somewhat artificial. More relevant is the argument that the generalized form summarizes and integrates all specific cases, and reduces them to one. Jack formulated this after the teacher asked for the relevance of the general solution like this: 'You you can fill in every number there, in fact it is all the others together.'
If the generalization is made for application to concrete cases, it may be sufficient to understand the general problem-solving procedure; a condensation into general formulas may not be needed. Therefore, we recommend the use of meaningful problem contexts in which generalization also contributes to insight into the situation, so that there is more than just applying the general to the specific.

### *b Shifts between concrete and general*

As indicated in Section 9.2.2 on the parameter as placeholder (issues *d* and *e*), students sometimes shifted between the concrete level and the general one. The concrete parameter value then had an exemplaric character as 'a specific number without anything specific'. Sometimes in the generalization process students reverted to concrete cases, possibly to avoid the lack of closure obstacle or because it was more easy to imagine. An example of this is the observation of Donald that is presented in Section 9.2.2. The other way around, namely replacing numbers by names of variables or parameters, was also observed. This happened when the student realized that the specific value of the number was not important, or when he or she wanted to avoid 'clumsy' numbers. The observation of Dean, which was presented in Section 7.4.3, is an example of the latter category.
Apparently, switching between concrete and general seems to be a natural intermediate phase in the process towards generalization. The use of meaningful contexts and parameters that play a recognizable role in that is recommended, to allow for this jumping back and forth between concrete and general.

### *c Generalization and pattern recognition*

In Section 4.5 we distinguished empirical and theoretical generalization, or, otherwise stated, empirical and reflective abstraction (Dubinsky, 1991; Mitchelmore & White, 2000). This distinction can be applied to generalization in the computer algebra environment. The theoretical generalization that we aimed for concerns insight into what happens in general and why this is the case, thus a generalization that is meaningful to the students. Beside this kind of generalizations, we also observed instances of a superficial, phenomenological generalization that had the character of pattern recognition without understanding. In that case the students saw 'what' but did not know 'why', as was the case for the superficial perception of the dynamics of the sliding parameter in Section 9.3. It is therefore recommended to require explanations of the regularities and that students have the means to understand the patterns that were found.

An example of empirical generalization or pattern recognition within the computer algebra environment that originally occurred without intrinsic understanding was observed in the expansions of $(x + c \cdot y)^2$ (see Fig. 6.6). The results of this assignment show that most students found an adequate description of the pattern in the expanded expression. However, 24 of the 47 students did not provide an explanation of the pattern, although they had means to do so, such as a multiplication matrix for polynomials (see Fig. 9.1). Pattern recognition with a CAS did not automatically generate the question of the intrinsic meaning and explanation of the pattern. We admit that in the task an explanation of the pattern was not explicitly asked for. In the observation below, only one of the students, Cindy, raises the 'why' question.

*Observer:* *What is the result of $(x + 5 * y)^2$ – without using the calculator?*
*Carly:* $x^2 + 10 * x * y + y^2$.
*Cindy:* *That's not logical. Why that 10?*
*Carly:* *That's the same everywhere, so it's the double here as well.*
(G9-I-6A, assignment 5.1)

In the handed-in work, these students did provide an explanation for the pattern using a multiplication matrix in which several errors occur (see Fig. 9.1). We conjecture that Cindy's doubt led to a further inspection of the problem.

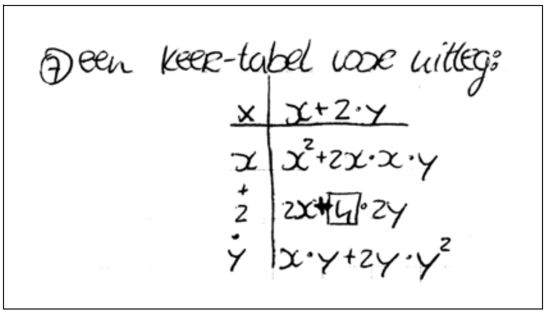


Figure 9.1 Multiplication matrix for $(x + 2 \cdot y)^2$ with errors

*d Natural language and formulas*

The description of the solution procedure in natural language was a common intermediate phase towards generalization for many students. These descriptions often made use of action language: they indicated what needs to be done to get the answer. The step from such recipe descriptions to formulas was not evident. We observed that students often formulated their generalizations in natural 'rhetoric' action language: 'Take that number, do this with it, then you get...'

The next protocol shows how Shirley tended towards generalization of the sum-difference problem $x + y = s$, $x - y = v$. In her notebook she called the sum the 'totalnumber' and the difference the '*x-y* number'.

> *Divide the totalnumber by 2 and do the same with the x-y number.*
> *Subtract x-y:2 number from totalnumber/2 and then you have y.*
> *y + x-y = x [Here x-y once more refers to the difference]*
> (G9-I-11B, assignment 2.1)

The next fragment shows how Sandra explained the general solution of the sum-difference problem, as it was presented in the textbook, to her neighbour by translating it back into action language.

> *Sandra:* *Yes, but here they let you know that if you take the half and then on both sides half of the difference added and subtracted, that you get the corrects answer immediately, or something.*
> (G9-II-vb11, assignment 5.8)

### 9.4.3 *Exemplary development of one of the students*

In this section we follow the development of Maria's insight into the parameter as generalizer.

In the following fragment Maria generalized the sum-difference problem from the concrete case that $b + h = 700$ and $b - h = 400$. The variables *b* and *h* refer to the base and the height of a rectangle. She replaced 400 by *difference* and substituted $b = h + difference$ into the first equation.

> *Observer:* *Now, could you make a general formula for that without using the 700 and the 400?*
> *Maria:* *Well, I think b + h = h + h + difference, yes or the other way around.*
> *Observer:* *Yes, OK, but let's take one of the two cases. So then you get b + h = h + h + difference, together it was 700 in the example, so in general let's say total or sum or whatever you like to call it. What kind of a formula do you get then?*
> *Maria:* *If you call this sum? Well, sum = h + h + difference.*
> (G9-II-6/20, assignment 5.4-5.6)

Simplifying this to *sum* = *2h + difference* was not easy for Maria, but then she solved the problem despite the presence of parameters. She did not suffer from the lack of closure obstacle:

> *Observer:* *Can you now calculate h from the equation* $sum = 2 \cdot h + difference$ *?*
> *Maria:* *Well I think here you've got the sum, you take the difference off and you divide that by 2.*
> *Observer:* *Yes. How do you express that in a formula?*
> *Maria:* *Well, sum – difference divided by 2.*
> (G9-II-6/20, assignment 5.4-5.6)

Maria also applied the general formulas for solving concrete cases of the sum-difference problem. After finding the general solution of the sum-product problem, she explained the importance of the general formulas: 'That's very easy. Once you got these formulas, you can do all these assignments quite easily.'
In the G10-II teaching experiment Maria was able to formulate the relevance of the generalization in the Meeh context on the relation between skin area and body weight. As a reply to the question what the advantage of the general formulas would be, she said: 'Well, that you spontaneously at once, for example, I want to know the bird, you can just fill in the bird, the frog.'
In Section 9.3.3 we described how Maria and Ada's use of the slider tool led them to find the general coordinated of the vertex of a class of parabolas.
At the end of G10-II one of the assignments was to find the vertex of the graph of $y = x^2 + b \cdot x + 1$ by first finding its zeros. In Section 9.2.3 we indicated that the concept of 'general coordinates' was difficult for Maria. However, shortly after that she solved the parametric equation and she seemed to understand the result as well, as she commented: 'So now you can calculate the zeros for every arbitrary value of *b*.'
These observations indicate that Maria had developed a sense for generalization and for the purpose of the general solutions. Meanwhile, the examples presented in this section provide a somewhat too positive image, as the observations on the parameter as placeholder showed Maria's difficulties with dealing with solutions as long as the parameter value is unknown (see Section 9.2.3). Later, the interaction between the parameter roles of generalizer and unknown confused her, as we will describe in Section 9.5.3.

## 9.5 The parameter as unknown

### 9.5.1 *Conclusions on computer algebra use and the parameter as unknown*

In Chapter 4 we described the parameter role of the unknown. Whereas the parameter as generalizer concerns unifying a set of situations or solutions, the parameter as unknown selects cases that fulfil a specific condition of this set. This involves the mental shift of the relation between parameter and variable: so far the variable usually has been the unknown. In this sense, the hierarchy between variable and parameter is turned upside down. In particular, students need to notice that equations can be solved with respect to the parameter. In most cases, the result of such a solving procedure will be a set of numerical values of the parameter; incidentally, the result is a formula that needs to be accepted as an object that represents the answer.

We conclude from the data that the designed instructional activities indeed fostered the development of the unknown view of the parameter but that this development was successful only to a limited extent. The students experienced how the CAS allowed for solving with respect to the parameter and how the roles of the literal symbols could change. They showed an increasing flexibility in dealing with literal symbols. However, particularly in complex situations they often lost track of the roles of variable, parameter as generalizer and parameter as unknown, and were not able to bring the solution process to an adequate end. In terms of the level structure for the concept of parameter (Section 4.5), this is seen as a modest form of higher level understanding. The instructional activities on setting extra conditions to general solutions and 'filtering out' parameter values of the total set were perceived as natural. This supports the line of the HLT that addressed the parameter as unknown after the parameter as generalizer and as changing quantity.
The computer algebra environment supported this development by allowing for solving equations with respect to the parameter. By doing so, the students became aware that an equation is always solved with respect to an unknown (see Section 10.2). Furthermore, solving with respect to the parameter required a mental shift of roles which students found difficult to make, probably because it affects the hierarchical relation between variable and parameter that they began to understand for the parameter as generalizer. This was particularly hard when the meaning of the parameter or the problem situation was not completely clear to them. Meanwhile, the need to explicitly identify the parameter as unknown in the syntax of solve made the parameter as unknown more accessible to the students.

As a second means of supporting the development of the view of the parameter as unknown, the computer algebra environment allowed for the flexible use of literal symbols in general. In the computer algebra environment, 'all variables are equal'. This meant that the relativity of the roles of the literal symbols was made tangible to the students. The different meanings and roles of the variables and parameters, which are 'in the eyes of the beholder', were no longer visible in the CAS. The CAS flexibility concerning the roles of literal symbols stimulated the students' flexibility while working with variables and parameters, which is a first step towards a versatile use. Evidence for this is the way the students dealt with renaming variables and parameters. However, distinguishing the roles of the different literal symbols, which may change throughout the problem-solving process, was often not successful, especially when the students found the formulas too complex to understand their structure and meaning.

A third way in which the computer algebra environment supported the development of insight into the parameter as unknown was by graphing sheaves of graphs. Questions on particular graphical features emerged in a natural way and could be approached in an algebraic way using the algebraic facilities of the CAS.

### 9.5.2 *Overview of the data*

As for the parameter as changing quantity and as generalizer, we identified key instructional activities that were supposed to foster the understanding of the parameter as unknown and that would be appropriate to monitor the students' development on this issue. We formulated our expectations on these key items and tested them by means of mini-interviews with students during the teaching experiment. Because of the expected difficulties with the parameter as unknown and its place in the learning trajectory, the number of key items was smaller than was the case for the changing quantity and the generalizer, and most key items were in the last parts of the teaching materials.

While analysing the data, the collection of key items was slightly changed, for example when a key item had been skipped because of time constraints, or if observations were not done for practical reasons. For example, assignment E3a from the final task of G10-II is added to the key items, but was not designed at the start of the teaching experiment.

The observations were analysed and coded. Because shifts of roles between literal symbols were considered to be important for the understanding of the parameter as unknown, incidences of shifting roles that did not particularly involve the parameter as unknown were also included in the analysis, such as students renaming variables and parameters, or students correctly or incorrectly distinguishing the different roles. After that, we attached one of the labels '+', 'o' or '–' to each observation.

The '+' scores refer to observations that reveal the expected insight into the parameter as unknown. For example, one of the students drew the function $f$ with $f(x) = a \cdot x - (1 + a^2) \cdot x^2$ for several values of $a$ using the view screen. In the classroom discussion that followed, Helen was able to distinguish the roles of the different letters, despite the confusing question posed by the observer:

*Observer:* *If you look at the formula, what kind of graph will you get, what form?*
*Helen:* *A parabola.*
*Observer:* *Why?*
*Cedric:* *Because there's a square in it.*
*Helen:* *Because it's to the power 2, so the highest power is 2.*

*Observer:* *Because it's a to the power 2?*
*Helen:* *Because it's x to the power 2.*
(G10-II-2/19, assignment 4.7)

The 'o' scores refer to insights into the parameter as unknown that are not completely correct but do contain valuable elements. An 'o' score was also attributed to observations in which we doubted whether the students really understood it or whether the teacher or observer had helped too much.
The '–' scores refer to conceptions concerning the parameter as unknown that are not adequate. For example, Misha wanted to find the coordinates of the intersection points of a parabola with equation $y = x^2 - 4x + 5$ and lines with equations $y = a \cdot x + 2$, but he lost track of the meaning of the different letters and solved with respect to the parameter. He entered solve($x^2$-4$x$+5=$a$*$x$-2, $a$), which yielded $a = (x^2 - 4x + 7)/x$.
Table 9.3 provides the scores for the key items concerning the parameter as unknown for the three teaching experiments.

| **G9-I** key item | Unknown scores | **G9-II** key item | Unknown scores | **G10-II** key item | Unknown scores |
|---|---|---|---|---|---|
| Intro 4.12-13 | ++ | Intro 3.13-14 | ++++++++++++++ | 1.8bc | ++ |
| 6.6c | ++++o | 1.5 - 1.6 | +++– | 4.2e | +o |
| 6.8b | o | 6.7c | ++++++++++ | 7.6 | ++– |
| 10.9b | ++–– | 7.7-7.8a | ++++++– | 8.6ac | +o––– |
| 10.10 | +––– | 9.2c | – | 9.2bc | +––––– |
| 10.11 | ++o– | 10.7 | o | 9.3bcd | – |
| 11.1 | ++ | 10.8 | + | 9.4ac | + |
| | | | | E3a | +++–– |
| | | | | E4a | + |

Table 9.3 Key item scores for the parameter as unknown

As a first comment on Table 9.3, we notice that the results for the key items at the start of the teaching experiments were quite good. Assignments 4.12-4.13 (G9-I), 3.13-3.14 (G9-II) and 1.8 (G10-II) concern solving the equation $a \cdot x + b = 5$ with respect to different variables. The students experienced that such an equation can be solved with respect to different letters and that it is reasonable that the CAS requires its specification.

*Observer:* *Now, what's the difference between the two results, that of 13 [solve* $a \cdot x + b = 5$ *with respect to x] and 14 [solve* $a \cdot x + b = 5$ *with respect to y]?*
*Michael:* *The one, then it is x= and the other b=.*
*Observer:* *Yes. And if you would say solve that equation and then comma-a, what kind of thing would you get then?*
*Michael:* *Then it solves a.*
*Josef:* *a equals and then x and b.*
(G9-II-1/43, assignment 3.13-3.14)

These positive results should be interpreted with care: although the students discovered the idea of expressing one variable in terms of others, in these situations there was no meaningful hierarchy between variable and parameter.
Later in the teaching experiments, when there was a hierarchical relation between parameter and variable, the results shown in Table 9.3 were less positive. The students had more difficulties in distinguishing the changing roles of the literal symbols during the solution process. This is the case for all three columns, indicating that these difficulties were not solved during the subsequent teaching experiments.
Table 9.3 shows a decreasing number of observations towards the end of the G9-II teaching experiment. This is due to the difficulty of the assignments; students had so many questions that the possibilities for mini-interviews were limited. In addition to this, students often were happy with a numerical-graphical approach to the sliding parameter that was probably suggested by the graphical problem situation.
The data of Table 9.3 suggest that students were able to deal with the parameter as unknown in simple situations, but that in more complex situation they often got confused. These findings link up with the data from the final tests of G9-I and G9-II (see Sections 6.4.4 and 7.4.4), where the results concerning the parameter as unknown in complex situations were poor. The results of the final task of G10-II, on the other hand, were better (see Section 8.5.3): many students managed to switch to the parameter as unknown when required by the context, though the formulas in the context were complicated too. Once more it appears that data from interactive classroom work and from individual tests can lead to different findings. Furthermore, the traffic flow problem situation in the G10-II final task was meaningful to the students.
Further analysis of the observations and the written materials revealed three issues that were relevant to the process of acquiring understanding into the parameter as unknown. These issues are addressed below.

*a Renaming variables and parameters*

While working in the computer algebra environment, the students sometimes preferred to use letters other than those suggested by the context or the assignment. The first reason was that for drawing graphs on the TI-89, the dependent variable needs

to be called $y$ and the independent $x$. For example, the context of the relation between skin area and body weight led to the formula $a = 11 \cdot w^{2/3}$, where $a$ stands for area and $w$ for weight. In order to graph this relation, one of the students dealt with this renaming by means of substitution and entered: $y_1(x) = 11 \cdot w^{2/3} | w = x$.
A second reason to rename a variable or parameter was that the TI-89 keyboard allows for the direct entering of some letters ($x$, $y$, $z$ and $t$), whereas other letters can be entered only after pressing the alpha key first. Therefore, students called a parameter $z$, for instance, so that it could be entered more efficiently. It was interesting to notice that students who did so never got confused about the two names for the parameter; apparently, they were fully aware of what they were doing.
Such renaming activities stimulated the students' flexibility with literal symbols and anticipated on the shifts of letter roles and on the use of the parameter as unknown.

#### *b The parameter as unknown with respect to which an equation is solved*

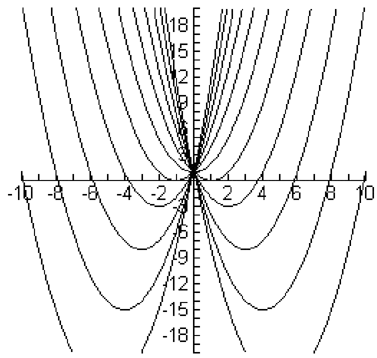
We mentioned earlier in this section that the perception of the parameter as unknown with respect to which an equation can be solved required a mental shift that affects the hierarchy between variables and parameters. Students found this shift hard to make. The following fragment is taken from a classroom discussion after playing the SHOOT game on the TI-89. The aim of the game was to set $a$ so that the line with equation $y = a \cdot x + 5$ would hit a point with given coordinates. The teacher asked whether you could calculate the appropriate value of $a$ from the coordinates of the target point. Kevin explained this in partially general terms and perceived the parameter $a$ correctly as the unknown.

> *Kevin has an idea of how to achieve this and writes this on the blackboard for the target point (11.6 , 3.84). He explains:*
>
> *Kevin:* *The formula was $y = a \cdot x + 5$. So then you have to, ehm, the second coordinate is y, 3.84, then you subtract 5 from that, then you have a times x left, that is 11.6, you know x already, that is 11.6. So then you have to take the result from 3.84 - 5.*
>
> *(...)*
>
> *Kevin:* *You have to divide that by x and then you have a.*
>
> (G9-II-3/47, assignment 1.5-1.6)

Fragments like this were exceptional. We regularly observed students who solved equations with respect to a variable when the parameter got the role of the unknown. If they forgot the letter specification, the CAS provided an error message; solving with respect to a wrong letter, however, led to results that were useless for the intended purpose. This issue is related to the instrumentation of the solve instrumentation scheme addressed in Section 10.2.

*c The distinction of the roles of the different letters and of the shifts of roles*

In complicated situations, the students found it difficult to distinguish the roles of the different literal symbols, particularly when the role of the parameter shifted towards the role of the unknown. An observation that illustrates this concerns assignment 9.2 from the G10-II teaching experiment (see Fig. 9.2).




**9.2** Above you see a sheaf of graphs of the family $y = x^2 + b \cdot x + 1$ .
We pay special attention to the vertices of the parabolas.

**a** Mark and connect the vertices of the parabolas in the figure.
What kind of curve do you seem to get?

**b** Express the coordinates of the vertex of a ‘family member’ in *b*.
Hint: the *x*-coordinate of the vertex is the middle of the *x*-coordinates of the two zeros, if these exist.

**c** Find the equation of the curve through the vertices of the sheaf and draw some graphs for verification.

Figure 9.2 Assignment 9.2 from G10-II

First, the equation $x^2 + b \cdot x + 1 = 0$ should be solved with respect to *x*, but Sandra solved it with respect to *b*, which yielded $b = -(x^2 + 1)/x$ . After she corrected this and found that the *x*-coordinate of the vertex was -*b/*2, she wanted to substitute this into the function equation $y = x^2 + b \cdot x + 1$ . This way, she got $y = 1 - b^2/4$ whereas a relation between *y* and *x* was asked for.
To improve the perception of the roles of the literal symbols, we recommend reviewing the complete solution strategy after it is completed, and then considering the changes of the roles of the variables and parameters involved.

### 9.5.3 *Exemplary development of one of the students*

In this section we follow the development of Maria’s insight into the parameter as unknown throughout the G9-II and G10-II teaching experiments.

Her results on the introductory assignments in G9-II and G10-II on solving the equation $y = a \cdot x + 5$ with respect to different unknowns indicated that Maria understood the role of the unknown after the comma in the solve command.

In G10-II, the question was to rewrite the lens formula $1/f = 1/v + 1/b$ for the case $f = 5$ so that the graph of $b$ as function of $v$ can be drawn. Distinguishing the roles of the letters was not easy for Maria. She did not know with respect to what unknown the equation $\frac{1}{5} = 1/b + 1/v$ needed to be solved in order to graph $b$ as function of $v$. She got help from Ada: '*y* is the function of *x*, so *v* is the *x*.' Maria solved with respect to $b$ and got $b = 5v/(v-5)$. She replaced the *v*'s by *x*'s, entered this in the function library and obtained the graph. Maria clearly was able to rename the variables, so that the graph could be drawn. Later the question was to express $b$ in $f$ and $v$ in general. Maria obtained this by entering solve(1/*f* = (1/*b*)+1/*v*, *b*). She seemed to understand the different roles of the letters.

At the end of the G10-II teaching experiment, the situations became more complex and Maria seemed to lose track of the problem-solving strategy and the roles of the different letters. Errors with the solve command occurred. For example, in assignment 8.7 she solved the equation $x = 2/(y-2) + 2$ with respect to $x$, although $x$ was already isolated.

We will look at Maria's work on assignment 9.2 on the family of functions $y = x^2 + b \cdot x + 1$ in somewhat more detail, because it shows how different kind of problems can interfere and may lead to long struggles (see Fig. 9.2).

Maria thought that question **a** concerned reproducing the picture with the TI-89. With some technical problems (*a* rather than *b* as parameter, wrong minus key, window settings) she got the correct picture (see Fig. 9.3) and commented: 'It works! This is a revolution in my math calculator thing!'

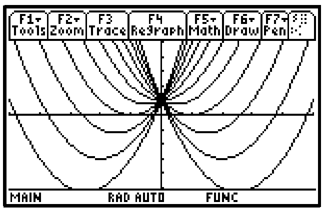



Figure 9.3 Maria's sheaf of graphs on the TI-89

For question **b** Maria first tried to find the zeros with solve($x^2 + b \cdot x + 1$ , $x$). This gave an error message, as what is to be solved is not an equation but an expression. Maria's comment was: 'It hates me, that calculator!' She then changed this into solve($x^2 + b \cdot x + 1$ , $b$) and got the same error message again. Then she reverted to her graph, to show it to the rest of the class by means of the view screen projection to which her calculator was connected.

In the next lesson, Maria started by entering the formula again. This time, however, she forgot the multiplication sign between *b* and *x*: $y_1 = x^2 + bx + 1$ . No graph appeared, not even after adding the parameter values. After a hint about the missing multiplication sign, she corrected it and got the same picture as in the earlier lesson. Maria's reaction was: 'Oh, that has to be times. Yes, that's always me, I never do it completely right.'

To find the zeros, Maria entered solve($x^2 + b \cdot x + 1 = y$, *b*). She now realized that the argument should be an equation rather than an expression, but she solved with respect to the wrong unknown. The roles of the different letters were not clear to her. Then Maria wanted to substitute a value for *b*, but this did not work and led to her comment: 'I just don't understand it; I'm simply too stupid.'

After a discussion with the observer Maria noticed that the value 0 should be substituted for *y* in order to find the intersection points with the *x*-axis. She entered solve($b = (-x^2 - 0 + 1)/x$ , *x*) and this gave the *x*-coordinates of the zeros in terms of *b* (see Fig. 9.4), a result that could have been obtained in a more direct way.

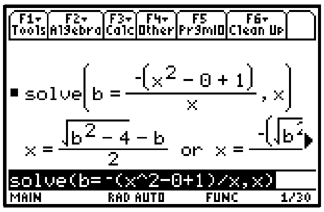



Figure 9.4 Zeros in terms of parameter *b*

She copied it into her notebook, but in the second solution she made the square root sign too long (see Fig. 9.5).

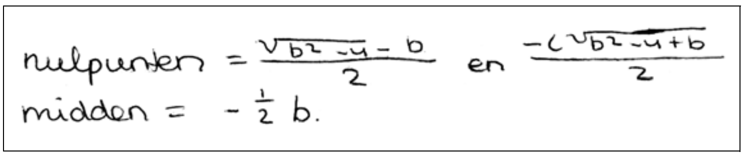



Figure 9.5 Copying solutions into the notebook

With some help from the observer she accepted the symbolic answers. She calculated the average of the two general solutions for *x* and found that the *x*-coordinate of the vertex equals –*b*/2. Maria was happy with that: 'Well, I did find this out quite nicely.'

Maria then substituted $x = -b/2$ in the formula of the function using the solve command: solve($y$ = -0.5*$b^2$+$b$*-0.5*$b$+1, $y$). She forgot the parentheses around –*b*/2, so the result $y = 1 - b^2$ was not correct. Also, using solve to carry out a substitution is not an efficient way of working; $x^2 + b \cdot x + 1 | x = b / 2$ would have been more direct.

For question **c**, Maria wrote down in her notebook that the extreme points have coordinates (– ½*b*, 1–$b^2$) and concluded that the equation of the curve through the extreme points was $y = (1 - b^2) \cdot x - \frac{1}{2}b$. Again, this indicates a mixing up of the meaning of the different literal symbols involved. However, probably with the help of her neighbour, she corrected her error and ended up with the right formula for the curve through the vertices: $y = 1 - x^2$.

This report on Maria's work on assignment 9.2 shows that despite the initial understanding of solving equations with respect to the parameter, Maria now had difficulties with that. She tried to solve expressions rather than equations, she solved with respect to the wrong unknown or forgot to indicate the letter, and she seemed to confuse solving and substituting. This prevented her from following her original plans and led to erroneous behaviour. A clear understanding of the problem situation and of the role of the parameter as unknown in it was lacking. Maria's comments on what was happening, her feelings of victory when she managed and her frustration when things did not seem to work indicate strong emotions.

## 9.6 Realistic contexts and the understanding of the parameter concept

In the previous sections we followed the line of the hypothetical learning trajectory through the parameter roles of placeholder, changing quantity, generalizer and unknown. On several occasions it was argued that the meaning that students attached to the parameter was relevant to the understanding, and that insight into formulas and expressions, the symbol sense, was an obstacle. In this section we address the meaning of the parameter as it emerges from a realistic context; in the subsequent one we discuss the insight into formulas. For that we leave the line of the HLT and the method of key items that were identified in the teaching materials and scored in tables.

### 9.6.1 *Conclusions on the use of realistic contexts and the parameter concept*

We identified the tension between concrete, context-bound meaning and insight that goes beyond the specific context and refers to a framework of mathematical relations as one of the difficulties of the learning of algebra (see Sections 3.2, 3.6 and 4.5). A concrete, realistic problem situation that students can imagine can give meaning to a parameter and to the formula that it appears in. This links up with the *model-of* concept at the referential level of the four-level structure of Gravemeijer (Section 3.6). On the other hand, the learning process aims at developing a framework of mathematical objects, relations and properties. This involves the development of a general *model-for* mathematical reasoning. This section concerns the development from referential model to general model in the learning of the concept of parameter in a computer algebra environment.

At the start of this study, we conjectured that students in the Netherlands might have had sufficient experience with graphs, formulas and functions in realistic contexts in seventh and eighth grade (Section 3.7). Therefore, the primary focus of our work was on the general model-for and on vertical mathematization towards the framework of mathematical relations and objects. We considered not paying much attention to real-life contexts and horizontal mathematization, as we assumed that the computer algebra environment – which does not refer to real-life contexts but acts as a 'microworld' on its own – would facilitate making real-life contexts redundant and would elicit the development of a mathematical reality.

We conclude from the data that this view was too simplistic. Insight into the concept of parameter seemed to develop the best when there was a meaningful problem situation to start with, and when this concrete context gradually became redundant because of the development of a framework of meaningful mathematical objects and relations; in fact, this is the model-of/model-for development that is described in Section 3.6. Students appreciated starting with concrete problem situations that they could imagine, that they found motivating and in which the formulas and the parameter had meaning to them. Just using a CAS did not replace this need for meaningful contexts. Using computer algebra to address the mathematical level directly therefore was not a fruitful approach for students of these grades and levels.

On the other hand, using the computer algebra device to work on the realistic problems that form the starting point stimulated generalization of the model by means of a process of vertical mathematization. A mathematical reality was developed from the concrete situation, as CAS use allowed for 'taking distance' from the concrete context. The realistic context did not hinder generalization and abstraction beyond the specific situation.

In short, we needed concrete, realistic starting points and referential models-of for developing the general model-for. This involved horizontal mathematization in the concrete contexts, that could not be separated from the vertical mathematization towards a model-for. Examples of contexts that invited generalization and abstraction were the Meeh context on the relation between skin area and body weight, fountains and garden sprinklers, the James Bond situation and the traffic flow problem. Although the vertical mathematization by means of setting up formulas and generalising relations and procedures was difficult for most students, the meaning of these situations supported this process.

The data suggest that graphs in the graphical module of the CAS could mediate between realistic contexts and mathematical abstraction. Graphical features were easily understood by the students, while evoking entry to a mathematical world. For example, the parameter as changing quantity was meaningful in the context of the SHOOT game, where it represented the slope of the line. The use of a slider tool was

appropriate to redirect the focus towards the dynamics of the graph.
These findings link up well with the ideas on emergent modelling and horizontal and vertical mathematization within the RME theory and with theories on symbolizing that argue that students need concrete starting points that provide opportunities for developing a mathematical reality. Although CAS use may support that process, the meaningful realistic starting points cannot be missed. As long as the mathematical framework has not been developed, it cannot be addressed directly in the way we originally intended.

### 9.6.2 *Overview of the data*

In the analysis of the data the role of realistic problem situations in the learning of the concept of parameter was a specific focus. As distinct from the method for the parameter roles that was described in the previous sections, we did not identify prior key items. Instead, a free coding approach was followed that aimed at identifying observations that involved the influence of the problem situation on the learning process.
The results on this issue do not support the original idea that the role of realistic problem situations could be reduced in favour of problem situations within the world of mathematics. The teaching materials of the first teaching experiment, G9-I, did not contain many realistic problem situations. However, the data suggest that students missed a concrete frame of reference that would have given meaning to the mathematical objects and procedures and to the parameter in particular. On the other hand, when students were given the opportunity to develop a story that would lead to the sum-difference problem, good situations were described:

> *Dean:* *You bought two new games, you've got three hours and you're going to figure out how long you play, if for example you like one game most, you play that one for, for instance, half an hour longer.*
>
> (G9-I-B10, assignment 1.11)

Therefore, the student activities for the second teaching experiment, G9-II, included more problem situations that were meaningful to the students. For example, in the SHOOT game on the TI-89 the parameter was seen as the slope of the barrel of the 'gun' by many students. The context of the rectangle with constant perimeter and changing base and height was a meaningful introduction to the sum-difference and the sum-product problem for many students. However, when the generalization of the sum-product problem was approached, things got too far away from the meaningful context for some of them. One of them commented, while pointing at the general formulas: 'Then something stops in my mind.' Re-establishing the relation with the meaningful rectangle often helped in such cases.

In other assignments, the possible benefits of a real-life context were hindered by instrumentation problems or by too complicated formulas. Also, some of the abstract situations proved to be meaningless to the students.
The teaching materials of the third teaching experiment, G10-II, contained more realistic problem situations, and the students used them to attribute meaning to the mathematical situations and symbols. The next fragment shows how the teacher wanted to reach for a higher abstraction level, whereas the student answered in terms of the problem situation, in this case James Bond rowing and running towards a spot on the beach (see Fig. 8.8).

*Teacher:* *Why is zero a special case?*
*Student:* *There's no distance any more, then he doesn't do anything any more; in fact he stands still.*
*Teacher:* *And what does that mean in the formula, if you have a equal to zero?*
*Student:* *It means that he's only running.*
(G10-II-2/20, assignment 5.3)

More than was the case in the earlier teaching experiments, the contexts in G10-II such as the garden sprinkler, the James Bond task and the traffic flow problem contributed to a meaningful perception of the formulas and the parameters, whereas generalization or abstraction did not seem to be hindered by this.
Overall, we see a development throughout this study from the original idea of focusing on the general level without paying much attention to realistic problem situations at the referential level, towards using realistic contexts as meaningful starting points for the targeted generalization by means of parameters.
Further analysis of the observations and the written materials revealed three issues that are addressed below.

*a Realistic contexts as meaningful starting points for generalization*

Students appreciated starting with a realistic problem situation because it allowed them to imagine the problem, and to attach meaning to the formulas and to the parameter. While working they often referred to this meaning and interpreted results in the framework of the context. This way, the context provided an adequate starting point for generalization by means of the parameter. The protocol above illustrated this. Another example was observed when a student explained the dynamics of the graph of the function $y = x - (1/a^2) \cdot x^2$. He referred to the context of the garden sprinkler when he explained what happened when $a$ was increasing: 'So then if you had a garden sprinkler, then it gets further.' An other student in a similar situation: 'It looks rather like the power that is behind it, with which you open the tap.'
The results described in the previous sections indicated that a graph – or a dynamic

graph or a sheaf of graphs – can take over the meaning from the original context and may act as a context itself. A graph provides a powerful visualization that may replace the original context and may be a first step towards a general model.

*b Difficult formalization departing from realistic problem situations*

Within a meaningful problem context, students often were able to find informal strategies and to describe them adequately. However, condensing these procedures, capturing them in formulas, was difficult for many of the students. This issue was mentioned when we discussed the parameter as generalizer (Section 9.4.2). The following fragment shows that Bob thought of a strategy for the sum-difference problem in the context of a rectangle with given perimeter and difference between base and height, but was not able to formalize it into a general, context-independent mathematical formula.

*Observer:* *Did you manage?*
*Bob:* *No.*
*Observer:* *[reads from Bob's notebook] Perimeter divided by 4 plus difference divided by 2 is the length of two edges. 12 minus the length of two edges is something, and something divided by 2 is the length of the other two edges.*
*Bob:* *Yes I cannot write it down shorter.*
(G9-II-6/16, assignment 5.4)

*c A meaningful mathematical reality built up from realistic problem situations*

Some of the observations show that students replaced the reference to the problem situation by a more abstract and general reference to a mathematical reality, whereas others were reluctant to do so. For example, assignment 6.7c from G9-II concerned solving the system $x + y = a^2$, $x - y = b^2$ as a context-free general application of the sum-difference problem (see Fig. 7.10). Although other students had difficulties with this assignment and argued that it made no sense, for Donald the sum-difference problem seemed to be an abstract class of problems that he could deal with.

*Donald:* *I divided $a^2 - b^2$ by 2 because that's the sum equation minus the difference equation divided by 2, so (?) [inaudible] is something, and then plus the difference is something else. So I don't know a-two [a squared], what you have to fill in there. If you knew what $a^2$ and $b^2$ is, then you could calculate it.*
(G9-II-vb12, assignment 6.7c)

### 9.6.3 *Exemplary development of one of the students*

In this section we again follow Maria, this time while she is working on the sum-difference problem in the context of the rectangle with given perimeter and difference between base and height (Fig. 7.9). Her work on this assignment was addressed in Section 9.4.3, but now we focus on the process of gradually taking more distance from the problem situation and developing a more formalized method. The first observation below was made before the observation described in Section 9.4.3. She was working on the concrete case with a perimeter of 1400 and a difference between base and height of 400: $P = 2 \cdot (b + h)$ and $b - h = 400$ .

> *Maria wrote in her notebook:*
> *You look for two numbers that form together half of the perimeter, with eventually a specific difference. I then find out by trying which numbers it has to be.*
> *P = (b+h)*2 and h = a + b = difference.*
> *[The latter '=difference' seems to concern only the a]*
> (G9-II-6/20, assignments 5.4 and 5.6)

This reveals a trial and error like strategy, related to the problem situation. However, the next observation shows formalization.

> *Observer:* *And how do you know, for example, if the perimeter is 12 and the difference is 3, what do you do then? Try?*
> *Maria:* *Yes, but this one [assessment 5.6, perimeter is 1400 and difference is 400] I did by just using formulas and stuff.*
> *Observer:* *[Reads in her notebook] There you have P = 1400,* $P = (b + h) \cdot 2$ , $1400 = (b + h) \cdot 2$ , $700 = b + h$ , $b - h = 400$ , $b = 400 + h$ , $700 = h + (h + 400)$ , $b = 550$ , $h = 150$ . *Oh, that's nice what you did!*
> *Maria:* *Yes. And then I combined them.*
> (G9-II-6/20, assignments 5.4 and 5.6)

This fragment shows how Maria derived meaning to the problem from the context, for example by using the word perimeter and the letters *P*, *b*, and *h*. On the other hand, she manipulated the equations in an algebraic manner that was meaningful to her without reference to the concrete context.
Later Maria was confused by the symmetry of the problem: an exchange of the values of *h* and *b* leads to the same perimeter and the same (absolute) difference. Her final remark reveals an adequate distinction of the non-symmetric mathematical situation and the symmetric real-life context.

> *Maria:* *Because you don't know what h is or b. I think in principle you can say 2 times b + difference too [rather than 2 times h + difference]*
> *The observer explains that the equation* $b - h = 400$ *leads to* $b = 400 + h$ .

*Maria:* *But that's by chance in this assignment, but not in general. In general the h and b are basically the same, because you never know what h or b is.*
(G9-II-6/23, assignment 5.4)

While giving meaning to *h* and *b* Maria seemed to refer to the context of the rectangle. She jumped back and forth between realistic context and mathematical reality in an appropriate way.

## 9.7 The role of symbol sense in understanding the parameter concept

### 9.7.1 *Conclusions on computer algebra use, symbol sense and the parameter concept*

We described symbol sense as a multi-faceted ability that includes seeing the structure of an algebraic formula or expression, giving meaning to an expression, a part of an expression or a formula, and conceiving it as a mathematical object (Section 3.5). The learning of the concept of parameter in a computer algebra environment was supposed to require a certain degree of symbol sense, as well as to further develop it. We also conjectured that manipulating parametric expressions and formulas in the computer algebra environment would foster their reification (Section 5.5).
So far, the results on acquiring insight into the higher level parameter roles by using computer algebra suggest that the lack of symbol sense was one of the main reasons why students failed. Therefore, we now consider the development of the symbol sense while working on the concept of parameter in the computer algebra environment. This brings us to the interface between the two research subquestions: on the one hand, symbol sense is involved in the learning of the concept of parameter, while on the other hand it plays a role in using a CAS in general, and is a more general instrumentation issue. Therefore, formulas and expressions will be considered in Section 10.5 as well.
We conclude from the data analysis that the introduction of parameters made the formulas more complex, simply because they contained more literal symbols than in non-parametric situations. The data indicate that the development of insight into the concept of parameter in some cases was hindered by a lack of symbol sense; on the other hand, by studying the concept of parameter, the students' insight into the meaning and structure of formulas was stimulated as well. In short, the development of insight into the concept of parameter and of the symbol sense were intertwined. Although the development of symbol sense was stimulated in the teaching experiments, this was not always successful: students kept having difficulties with insight into the meaning and structure of formulas. As we argued in the previous section, a meaningful problem situation contributed to insight into the structure of the formulas that emerged from it.
The computer algebra environment supported the simultaneous development of

symbol sense in three ways. First, the CAS use stimulated the students' insight into the meaning and structure of formulas and expressions. It provided means of visualizing algebraic expressions by graphing. Entering complicated formulas with square root signs, fractions and parentheses forced the students to consider the scope of the operators. The interpretation of the CAS output stimulated the students to examine the structure of the formulas and expressions more closely. Furthermore, the computer algebra environment allowed for dealing with bigger and more complex formulas than would have been possible by hand. In order to have an overview, the students needed to split these bigger formulas up into meaningful parts. However, the CAS use was only partially successful: although it helped the students to focus on symbol sense, errors showing lack of insight into the structure and meaning of formulas and expressions were observed frequently.

As a second means of supporting the development of symbol sense, the CAS work on parameters stimulated the reification of formulas and expressions. While solving parametric equations in the computer algebra environment, the students were confronted with solutions that are expressions. These were treated as objects that were processed further by substitution or other procedures. This reinforced the perception of formulas as 'things'. In this way, the students moved away from the formula as a process, and developed an object view of formulas and expressions.

Third, we found that the use of symbols and notations in the computer algebra environment was a complicated matter. The data show that students found it hard to formulate findings and procedures in the form of algebraic formulas. The use of computer algebra did not offer much on this issue. Furthermore, the differences between CAS conventions/notations and paper-and-pencil use caused obstacles. However, this lack of congruence was also an occasion to discuss paper-and-pencil symbolizations and notational conventions that so far had remained implicit. These opportunities of CAS use were not fully exploited in the teaching experiments.

### 9.7.2 *Overview of the data*

The development of symbol sense while learning the concept of parameter was a specific focus in the analysis of the data. As distinct from the method for the parameter roles described in Sections 9.2 - 9.5, we did not identify key items beforehand. Instead, a free coding approach was followed.

The results of this data analysis support the idea that symbol sense is an important factor in the learning of the concept of parameter in a computer algebra environment and of using a CAS in general. The students had difficulties with understanding the structure of formulas, which resulted in the inability to deal with the input and output of the CAS. Notational and syntactical difficulties exacerbated these problems. On the other hand, we observed incidents of understanding the meaning of the formula and of perceiving it as an object that could represent a solution. Using the CAS stim-

ulated the reification of the formulas. An appropriate problem situation that gave meaning to the formula and its components was an important help.
A further analysis led to the distinction of three aspects of symbol sense that were important in the learning of the concept of parameter in a computer algebra environment. These are addressed below.

*a Insight into the structure and meaning of formulas*

A lack of insight into the structure and meaning of algebraic formulas was manifest in the observation of errors with algebraic manipulation by hand. Many of these errors are described in the literature and are not specific to this study. What is noteworthy here, is that the students could have avoided them by using the CAS, but apparently felt so confident that they found no need to use the machine. Most of these errors concern incorrect simplifications. Table 9.4 provides some examples.

| Original expression, formula or equation | Students' simplification made by hand |
|---|---|
| $h + h$ | $h^2$ |
| $\frac{1}{2}a^2 + \frac{1}{2}b^2$ | $a + b$ |
| $(x - 2)^3$ | $x^3 - 8$ |
| $\sqrt{3^2 + x^2}$ | $3 + x$ |
| $x^2 + y^2 = 25$ | $x + y = 5$ |
| $y = \sqrt{625 - x^2}$ | $y + x^2 = \sqrt{625}$ |
| $1/f = 1/v + 1/b$ | $f = v + b$ |
| $x^2 - 4x + 5 = a \cdot x - 2$ | $x - 4x + 7 = a$ |

Table 9.4 Students' errors in simplification by hand

Many of these algebraic errors can be seen as manifestations of the linearity illusion (de Bock et al., 1998; de Bock et al., 2002). For example, in the simplifications of $x^2 + y^2 = 25$ and $1/f = 1/v + 1/b$, the students seemed to think that the square function and the reciprocal function are linear, which is a well-known misconception.
In some cases, the CAS output looks different from the expected outcome or from the results obtained by hand. The question whether the two outcomes are equivalent requires a closer inspection of the formulas. For example, solving the general sum-difference problem $b + h = s$, $b - h = v$ by hand gave $b = 1/2 \cdot s + 1/2 \cdot v$, whereas CAS use led to $b = (s + v)/2$. One of the students explained this by saying:

'Well, because it is in fact, because you divide by 2 it is in fact already one half, so that comes down to the same.' The issue of equivalent expressions will be addressed further in Section 10.5.4.

The dynamics of the sliding parameter combined with questions on the properties of the graph in some cases invited a detailed look at the formula. For example, when students investigated the dynamics of the graph of $y = (x-a)^2 + a$ as the value of *a* changed, Jeff noticed that the point (*a*, *a*) was the vertex of the parabola. After a question from the observer, he looked at the formula and explained: 'That [*y* when *x*=*a*] is always just *a* alone, because these [*x* and *a* in $(x\text{-}a)^2$] cancel out.'

Using a problem context that students can imagine can help to see the structure of the formula. Mattijs and Thomas worked on the James Bond assignment, where the rowing and the running were reflected in separate parts of the formula for the travel time *T*: $T = a \cdot (\sqrt{3^2 + x^2})/6 + (4-x)/12$. The context helped them to understand the parts of the formula:

> *Observer:* *Do you understand where that formula comes from?*
> *Mattijs:* *Yes, this [the first term] is the part over the sea, and this [the second term] is over the land.*
> *Observer:* *And why does that part over the sea look like that?*
> *Mattijs:* *Yeah, I don't understand that.*
> *Thomas:* *The velocity of 6, that is below.*
> (G10-II-2/12, assignment 5.1)

The importance of the meaning that students can attach to formulas and their components also explained the results of the traffic flow final task of the G10-II teaching experiment. Dirk and Kevin, for example, were able to interpret the symbolic relation $f = (250\sqrt{3})/(9\sqrt{r})$ that stood for the optimal movement of the traffic flow and the risk factor that models the drivers' behaviour. In their report they wrote: 'If the risk factor gets smaller, the optimal flow gets bigger, at a higher velocity.'

The assumption that the complexity of the formulas can be an obstacle to seeing its structure and meaning was supported by observations. For example, while solving the sum-product problem, Misha solved the equation $1/4 \cdot s^2 - a^2 = p$ with respect to *a*. The TI-89 provided two solutions for *a* and added the condition that $s^2 - 4p \geq 0$. The situation was too complex for Misha to understand this condition.

In general, we first conclude that the availability of computer algebra does not prevent student from making errors in by-hand calculations. Second, algebraic and graphical affordances of the computer algebra environment may stimulate looking at formulas in more detail. Third, the data provide further evidence for the statement that a meaningful context helps to see the structure of a formula. Finally, the analysis supports the idea that the complexity of the formulas may hinder insight into their structure.

As stated in Section 9.4.2, setting up formulas was difficult for the students and computer algebra was not of much help in that.

*b The reification of formulas*

One manifestation of the perception of a formula as an object is its acceptance as a solution of an equation. Observations, in particular at the start of the ninth-grade experiments, indicate that some students suffered from the lack of closure obstacle that hinders this acceptance (see Sections 3.4 and 9.2.2). These students thought the formula or the expression had no sense as long as it contained operators that could not be worked out as no numerical values were substituted.
However, as the experiments proceeded, the students got used to seeing a formula or expression as a meaningful object that could represent the solution of an equation or that could be processed further by other procedures such as substitution. For example, when the teacher asked the students to rewrite the lens formula $1/f = 1/v + 1/b$ into something that started with *b* =, one of them replied: 'Solve gives $b = f \cdot v/(v - f)$ '. He perceived this formula as a full-fledged solution.
In the next fragment, Misha anticipated the fact that the result of a solve procedure would be a formula that could be processed further. He showed an object view of the formula as well as a sense of the overall problem-solving strategy:

> *Misha:* *You can solve these to each other, with solve, and then have x calculated, and if you then solve the formula that you get this way to zero, x, (...) then you get a formula with an a in it, and if you solve that ...*
>
> (G10-II-3/25, assignment 8.6)

Substitution of expressions was a means to stress their object character and seemed to support reification. For example, Misha solved the equation $1/4 \cdot s^2 - a^2 = p$ with respect to *a*. The TI-89 provided two solutions for *a*, $a = (\sqrt{s^2 - 4p})/2$ and $a = -(\sqrt{s^2 - 4p})/2$ , and added the condition $s^2 - 4p \geq 0$ . Although Misha did not understand this condition, he felt confident to use the object for substitution: 'It may look threatening, but if you substitute this in that, in the formula, then it should give a solution, shouldn't it?' Substitution is addressed in more detail in Section 10.3.
Many observations show a progressive reification of formulas and expressions by means of the language the students used. In particularly, nouns and descriptions such as 'that thing' were regularly used and interpreted by us as indicators of reification. For example, the square root part of the expression $b = 1/2 \cdot s + \frac{1}{2}\sqrt{s^2 - 4p}$ was described by one of the students as 'that square root thing'.
Contrary to the previous section, in which we stated that the complexity of the formula might inhibit the understanding, the data suggest that the complexity of the formulas may stimulate reification: in a complex formula there is more need to see parts

as entities than there is in a simple one, as the complete formula cannot be overviewed.
All together, the data provide evidence that the work in the computer algebra environment fostered the perception of formulas and expressions as objects that can represent solutions of equations, that can be substituted into other formulas or processed further in another way. The lack of closure obstacle, although sometimes persistent, in most cases was overcome. The impression is that complicated (though not too complicated) expressions and formulas invite an object view, because they are otherwise too difficult to overview.

### *c Symbols and notations*

The third aspect of understanding the structure and meaning of formulas that emerged from the data has a technical and linguistic character. It concerns the use of symbols and notations. The conventions of the TI-89 machine do not always agree with the paper-and-pencil conventions. This caused difficulties that became manifest when students entered formulas in the CAS or copied CAS results into their notebook.
As a first comment we note that informal notations cannot be used as easily in the CAS as in the paper-and-pencil environment. For example, to describe the pattern in the expansion of $(x + c \cdot y)^2$ (see Fig. 6.6) some students suggested using '?' to denote the parameter. This cannot be done in the CAS environment.
A number of observations on the issues of notation and symbols concern the entering of formulas and expressions. Students had difficulties with some of the TI-89 settings or conventions. For instance, the machine in default setting requires multiplication signs to be entered between two literal symbols. Two students entered $x - \frac{1}{a}x^2$ as $x - 1/ax^2$, which was interpreted by the machine as $x - \frac{1}{(ax)^2}$, because $ax$ was seen as one variable.
The unidimensional line editor for entering formulas into the TI-89 might explain these difficulties. However, while working in TI-Interactive – a software environment that provides bidimensional 'pretty print' entering and editing of formulas – the problem was not solved. For example, one of the students wanted to enter $\sqrt{a^2 - x^2}$, but instead got $\sqrt{a}^{2 - x^2}$. Referring to the square root sign she commented: 'Look, now it does make that thing longer.' Apparently, the linear entering that the TI-89 requires was not the only cause of the difficulty; also symbol sense was involved. We will revisit the issue of entering formulas in Section 10.5.3.
Students encountered difficulties with symbols and notations not only while entering expressions into the TI-89, but also while copying CAS results into the hand-written notebook. As an example of this we look at the equation $31 - x = \sqrt{625 - x^2}$, which appeared in one of the assignments. To enter the equation, the square root argument needs to be entered inside parentheses. The output screen of the TI-89, how-

ever, shows a long square root sign (see upper part of Fig. 9.6). In her notebook, Barbara first wrote a short square root sign, which raises the question whether she understood the structure of the formula (see lower part of Fig. 9.6). While referring to the solve command, on the other hand, she gave the square root sign the appropriate length.

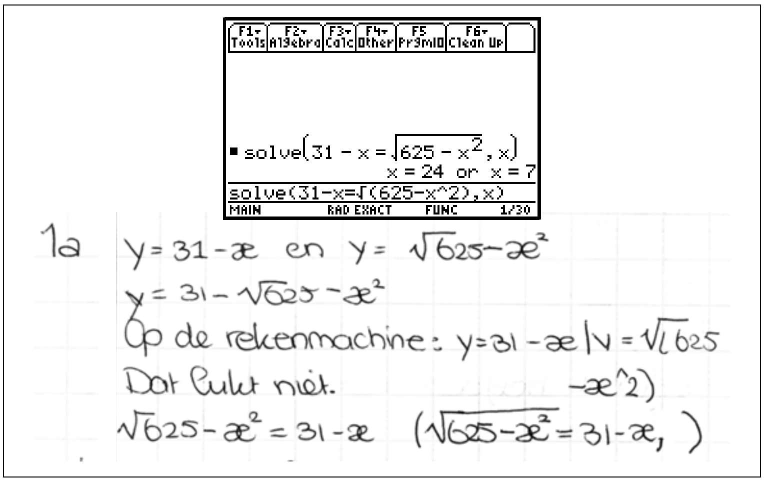



Figure 9.6 The length of the square root sign

The different notations may confuse to the students. However, some of them understood the CAS conventions quite well. For example, Rob entered $y_1(x) = \sin(x) \cdot a | a = \{1, 3, 5, 9, -1, -3, -5, -9\}$ in order to obtain a sheaf of graphs. The teacher addressed the parentheses: ‘Does it matter if you put the bracket behind the *x* or behind the *a*?’ Rob replied: ‘Yes, I think so, otherwise it takes the sine of *x* times *a*.’

All together, the data show that students had difficulties with the symbols and notations of the CAS. Evidence for this appeared when entering formulas and expressions into the CAS and when writing down results from the CAS by hand. The unidimensional editor of the TI-89 is not the only cause of this; also a general lack of experience with formulas and symbol sense plays a role, as do the differences between the CAS environment and the traditional paper-and-pencil use. These issues are revisited in Sections 10.5 and 10.6 on obstacles for instrumentation of CAS and the relation between CAS use and paper-and-pencil use.

### 9.7.3 *Exemplary development of one of the students*

In this section we confine ourselves to one long observation (protocol G9-II-5/11) of Maria as she tried to substitute a parameter value in a formula by hand. The observation reveals how she struggled to understand the structure of the formula, and to perceive parts of it as entities. Moments of insight alternated with moments of frustration. We will interrupt the fragment for some comments.

The dialogue took place after Maria had made a 'cartoon sequence' of graphs of $y = x + a \cdot \sqrt{x^2 + 1}$ for various values of $a$ (assignment 3.1, Section 7.3). The 'cartoon sequence' showed curved graphs, but one of them was linear, namely the case $a = 0$. The observer then came in and asked for an explanation of the straight line.

| | |
|---|---|
| *Observer:* | *For a=0 you have a straight line. Can you see this in the formula, too?* |
| *Maria:* | *Ehm, no.* |
| *Observer:* | *That's a pity.* |
| *Maria:* | *Yeah, but with the calculator, I think it is much more clumsy, because normally I understand it very well, but such a formula, I don't see much in it if I just enter it into the calculator and it draws the graph.* |
| *Observer:* | *And if you just look at it, without calculator, you take x, add a times the square root of* $x^2$ *plus 1, what happens then if a = 0?* |
| *Maria:* | *Well then it gets straight but I really don't know why, no idea.* |

Apparently, Maria had not thought about understanding the straight line from the formula. Furthermore, she indicated that the use of the CAS calculator did not help her to understand.

| | |
|---|---|
| *Observer:* | *What happens with a times the square root of a equals zero?* |
| *Maria:* | *Ehm, well then the square root will be zero as well?* |
| *Observer:* | *Yeah, so what will be left of the formula in fact?* |
| *Maria:* | *x plus a times* $x^2$ *plus 1, isn't it?* |

The observer initially thought Maria's explanation was right, but the last line indicates that she is not.

| | |
|---|---|
| *Observer:* | *But a was zero, remember?* |
| *Maria:* | *Yes.* |
| *Observer:* | *And in this case...* |
| *Maria:* | *Let's look, well then … Well the square root is then zero and the square, yes zero squares is also zero, so in fact, then I think this complete part is skipped, or isn't it?* |
| *Observer:* | *And what will remain?* |
| *Maria:* | *Ehm, x plus a times … plus 1 or something?* |
| *Observer:* | *No x isn't zero but a equals zero, doesn't it?* |
| *Maria:* | *… O yeah … Well then, then I think the square root is dropped.* |

| | |
|---|---|
| *Observer:* | *Yes.* |
| *Maria:* | *And the rest remains.* |
| *Observer:* | *Yes, and what is the rest then?* |
| *Maria:* | *Well x + a times* $x^2$ *+1, .. , or not?* |

Surprisingly, when Maria said that the square root is dropped, she thought the expression below would remain. The square root seems to be a separated sign to her that is independent from its arguments. Or was she getting confused by the observer, who was guiding the discussion too much?

| | |
|---|---|
| *Observer:* | *But a was zero?* |
| *Maria:* | *Oh, then it is, ehm, x +* $x^2$ *+1* |
| *Observer:* | *No, because, ehm, it says, for this a you should read a zero in this case.* |
| *Maria:* | *Mmm...* |
| *Observer:* | *If a = 0, then you get x + 0 times, a whole part.* |
| *Maria:* | *Yes.* |
| *Observer:* | *But how much is zero times a whole part?* |
| *Maria:* | *Zero.* |
| *Observer:* | *Yes. So what will be dropped?* |
| *Maria:* | *In fact the complete last part?* |
| *Observer:* | *Yes* |
| *Maria:* | *Oh.* |
| *Observer:* | *So what will remain?* |
| *Maria:* | *x + a?* |

Her answer indicated that she did not cancel the *a*, although strictly speaking she is right, as $a = 0$.

| | |
|---|---|
| *Observer:* | *No, because a = 0, yes, so* |
| *Maria:* | *x.* |
| *Observer:* | *Yes. Are you guessing now or ...?* |
| *Maria:* | *No, I really think so.* |
| *Observer:* | *OK, I also really think so.* |
| *Maria:* | *Then it is only x.* |

Here we are finally. This fragment shows the relevance of symbol sense and of the ability to view part of the formula as an entity, as was described by Arcavi and Wenger (Arcavi, 1994; Wenger, 1987).

## 9.8 Conclusions on computer algebra use and the concept of parameter

In the final section of this chapter we first summarize and discuss the results concerning the contribution of computer algebra use to the understanding of the concept of parameter. Then we reflect on the role of the theoretical framework that was described in Chapters 3 and 4. We conclude with the formulation of some consequences for teaching.

### 9.8.1 *Summary and discussion*

The central question in this chapter is the first research subquestion:

> *How can the use of computer algebra contribute to a higher level understanding of the concept of parameter?*

This higher level understanding was defined as the extension of the placeholder view of the parameter towards insight into the higher parameter roles of changing quantity, generalizer and unknown. We now summarize the results of the study by following the line placeholder - changing quantity - generalizer - unknown. Then we reflect on these findings by considering the role of realistic contexts, symbol sense and the character of computer algebra tools.

For the parameter as *placeholder*, the results from the study indicate that the students did initially indeed perceive the parameter as a placeholder for numerical values. In some cases this view was persistent and, in combination with the lack of closure obstacle, hindered the transition to the parameter roles of changing quantity and generalizer. The use of computer algebra allowed for substituting different numerical values for the parameter. This offered opportunities for extending the concept of parameter with the higher parameter roles. Understanding the exemplaric character of the numerical parameter value was a first step in this direction. Graphing two-dimensional graphs required substituting numerical values for the parameter, so in that case the placeholder view was appropriate.

For the parameter as *changing quantity*, the results indicate that varying and replacing parameter values in the computer algebra environment and using a slider tool indeed fostered the transition from the parameter as placeholder to the parameter as changing quantity. Students understood that the changing parameter affected the complete graph. In that sense, computer algebra use contributed to a higher level understanding of the concept of parameter, although the availability of the slider tool is not specifically a computer algebra feature. Subject to adequate problem situations, examining the sliding graph led to the discovery of graphical properties that invited algebraic verification and generalization. In this way, the parameter as changing quantity anticipated on the parameter as generalizer and as unknown. Therefore, addressing the changing quantity before the generalizer, as we did in G9-II and G10-II, seemed to be the best learning trajectory. As a comment on these positive results, we observed the risk of superficial examination of the effect of the sliding parameter on graph and formula. Furthermore, the changing quantity view of parameter became so dominating for some students that the development of other parameter roles seemed to be hindered.

For the parameter as *generalizer*, the results were mixed. The computer algebra environment allowed for easy repetition of procedures for different values of the parameter and thereby generated examples that were the source for generalization. Furthermore, it allowed for a general algebraic approach to a class of problem situations. In this way the students experienced how the parameter as generalizer unified a class of situations, formulas and solutions. This is seen as the start of a higher level understanding of the concept of parameter. Manipulating parametric expressions and formulas in the computer algebra environment also fostered their reification: it helped the students to accept formulas as solutions and to overcome the lack of closure obstacle. However, there were important limitations. First, the students sometimes did not feel the need for generalization. Second, the starting point for generalising by means of parameters is primarily a mental step that was a difficult one for the students. Third, students found it difficult to understand the general results. In some cases, the examples generated by the CAS led to superficial pattern recognition and not to meaningful generalization. Fourth, the generalization led to formulas that were sometimes too complex for the students to see through. Finally, the generalization was hindered by the too complex instrumentation scheme that we proposed in G9-I and G9-II; this was improved in G10-II. All together, we noticed CAS affordances for the insight into the parameter as generalizer, but for different reasons they fostered the improvement of the understanding only to a limited extent.

For the parameter as *unknown*, the results were once more mixed. The flexibility of the CAS concerning the meaning and the roles of literal symbols forced the students to be more conscious of the letter roles and allowed for a flexible use of literal symbols, such as renaming variables and parameters for practical purposes. The CAS possibility to solve equations with respect to any unknown stimulated the students' flexibility and fostered the change of roles and hierarchy when the parameter acted as unknown. The approach of selecting parameter values from the general case that fulfilled an extra condition provided a natural transition from the parameter as generalizer towards the parameter as unknown. Addressing the parameter as unknown after the parameter as generalizer, therefore, seems to be the appropriate order. In simple cases that students could grasp, the role of the parameter as unknown was often understood well. However, in more complex situations students had difficulties keeping track of the overall problem-solving strategy and distinguishing the different roles and meanings of the variables and parameters, especially when these changed during the solution process. We argue that this is caused by the change in the hierarchy between variable and parameter. The higher level understanding therefore primarily concerned the flexibility with literal symbols and the awareness of the roles of the variables and parameters involved. In complex situations the tasks were often not brought to a satisfying end.

We conclude that computer algebra use contributed to insight into the parameter as placeholder and as changing quantity, but that the higher level understanding of the parameter as generalizer and unknown was only achieved to a limited extent. What factors inhibited better results for the parameter roles of generalizer and unknown? The observations suggest that the following factors played a role: the limited importance of realistic problem situations in the instructional activities, the students' lack of symbol sense, and the character of the CAS. Furthermore, we conjecture that the role of the teacher, the student population and the separation of the parameter roles may also have played a role. We will now address each of these factors.
The limited importance of *realistic problem situations* in the instructional activities seemed to hinder meaningful generalization. For students of this age and level, a mathematical framework that allowed for reasoning with formulas and expressions had not yet been developed. Therefore, the students appreciated starting with concrete problem situations that they could imagine, that they found motivating and in which the formulas and the parameter were meaningful to them. Just using a CAS did not replace this need for meaningful contexts. In terms of Gravemeijer's four-level structure (Section 3.6), we conclude that a referential model is needed before the general can be addressed. We attribute the improved results of the G10-II teaching experiment to the increased attention to realistic problem situations. Meanwhile, the concrete starting points did not hinder the generalization and abstraction beyond the specific situation. Using the CAS fostered the development of the targeted mathematical framework. For example, the work in the computer algebra environment stimulated the object view of formulas as mathematical objects that could represent solutions. These findings link up well with the ideas on emergent modelling and mathematization within RME and with theories on symbolizing.
A second factor that explains the modest results on the understanding of the parameter as generalizer and unknown, is the students' lack of *symbol sense*. We defined symbol sense as insight into the meaning and structure of formulas and expressions. The use of parameters led to more complicated formulas than those the students were used to, and the CAS capacities did not limit the complexity of formulas in the way that paper and pencil does. However, we recommend a carefully designed increase of this complexity, to prevent students from working with meaningless formulas in which they do not see the structure. Working with parametric formulas in the computer algebra environment was hindered in some cases by a lack of symbol sense; on the other hand, finding general solutions and substituting expressions fostered the development of symbol sense as well. For example, the CAS output, algebraic or

graphical, invited a closer inspection of the formulas and expressions involved. Furthermore, we already mentioned that the reification of formulas and expressions was stimulated by the use of parameters in the computer algebra environment. The use of symbols and notations turned out to be complicated by incongruence between notations of the computer algebra tool in use and paper-and-pencil conventions.

The third factor that explains the modest results on the understanding of the parameter as generalizer and unknown is the *character of the CAS*. Although the affordances of computer algebra use for the parameter learning trajectory met the expectations that were stated in Section 5.5, the character of the CAS was not completely appropriate for the students of the ninth and tenth grade classes in our study. The top-down and formal character of computer algebra with its sometimes idiosyncratic language and notation did not fit very well with the informal mathematical background of the students, who had not developed an abstract framework of mathematical relations. Therefore, computer algebra use did not replace the realistic and concrete problem situations for them. Our ambitions to address such a difficult topic as generalization directly without embedding it in the referential level were simply too high with students of this age and level in such a short period.

Fourth, although we speak about the affordances of computer algebra, we should notice that the CAS environment on its own does not constitute the educational setting. Factors such as the hypothetical learning trajectory, the didactic embedding, the instructional activities and the orchestration of the CAS use by the teacher strongly influenced the results of the study, which might have been different had other factors been involved in a different way. Particularly, we conjecture that more and deeper whole-class discussions, directed by a teacher who raises questions and stresses central issues, might have led to better results, as well as instructional activities that showed a more gradual increase in complexity.

Fifth, it must be remembered that the two cohorts of students that took part in this study were considered as weak cohorts by their teachers. Evidence for this is the low student participation in the exact stream in eleventh and twelfth grade.

Finally, although we structured the HLT along the lines of the parameter roles that we distinguished in the conceptual analysis, we should note that our experiences in G10-II suggest that it is good to mix up the different roles more than we did. For example, addressing the parameter as generalizer and as unknown after investigating the sliding parameter can be natural, and may help the student to acquire a more integrated view of the concept of parameter.

### 9.8.2 *Reflection on the theoretical framework*

The elements of the theoretical framework that are relevant to the first research subquestion are the level theories, the concept of the hypothetical learning trajectory, the instruction theory of RME, the process-object duality and the ideas on symbol sense. How do the findings of this chapter link up with these theoretical elements?

We will briefly address them one by one.

The *level theories* that were discussed in Section 4.5 played a role in the definition of the higher level understanding in the research subquestion. We identified the placeholder view of the parameter as the lower level, and the other parameter roles as the higher levels of understanding the concept of parameter. The Van Hiele idea of somewhat discrete levels was useful for designing the learning trajectory. Gravemeijer's four-level structure as well as the dichotomy model-of and model-for were helpful for making explicit our goal of going from the referential to the general level, and for understanding why the original approach of directly aiming at the general level did not work out well.

The concept of the *hypothetical learning trajectory* was used as a means to capture the development within each research cycle and throughout the consecutive cycles. The method of identifying key items and their expected outcomes beforehand was essential for making explicit the goals of the teaching experiment and for keeping track of the links between theory and practice, between thought experiment, teaching experiment and reflection. The HLT approach in fact was a concrete means to work out the design research methodology.

The instruction theory of *Realistic Mathematics Education* was used particularly while reflecting on the relation between concrete and abstract and on using real-life contexts versus using mathematics and computer algebra as a context. The notions of model-of/model-for, of levels of referential and general activity, and of horizontal and vertical mathematization provided insight into the processes of generalization and abstraction, and the intertwinement of the horizontal and the vertical components.

The *process-object duality* was embodied in the need to reify algebraic formulas and expressions, and to overcome the lack of closure obstacle generated by the mathematical topic – the concept of parameter – and by the computer algebra tool. The duality proved to be useful for analysing students' difficulties and for identifying the process view that students often had of 'doing mathematics'. The reification of formulas and expressions could be analysed thanks to the framework.

The notions on *symbol sense* had to be specified for the purpose of this study. Besides the aforementioned reification of expressions and formulas, it includes the insight into the meaning and the structure of formulas and expressions and the use of symbols and notations within the computer algebra environment. This notion was helpful to focus the attention on insight into formulas and its relation with the development of the concept of parameter.

### 9.8.3 *The role of the teacher and consequences for teaching*

What was the role of the teacher in the development of the concept of parameter? The teacher plays an important role in establishing the socio-mathematical norms and the didactical contract (Brousseau, 1997). He or she is the students' main source of feedback and can stimulate reflection on the work by raising issues, making suggestions and instigating whole-class discussions. For instance, such suggestions as to try different values for the parameter, to explain graphical features by algebraic means, and to solve problems in general turned out to be essential in the learning process. Whole-class discussions on different views of the parameter, which ensured collective conceptual development, might have been fruitful, but were held effectively only to a limited extent. Stressing the key issues in the learning trajectory was another aspect of the role of the teacher that was not fully realized, probably because of a lack of insight into the researcher's goals and aims. Therefore, we recommend paying more attention to communication between the researcher and the teachers, so that the teachers have more clear guidelines on what questions to ask and what issues to raise in discussions with their students.

The results presented in this chapter suggest some pedagogic consequences for the teaching of the concept of parameter. From various places throughout the chapter, we distil the following suggestions:

- For the parameter as *changing quantity*, activities should be designed that elicit a closer look at specific graphical features that can be understood algebraically, and in which the parameter is meaningful to the students. These activities should also address the parameter as generalizer and as unknown, to prevent the sliding parameter from becoming the dominant parameter view. In order to avoid the perception of a sliding parameter having integer values, we recommend including fractional parameter values and using a continuous slider tool.
- For the parameter as *generalizer*, meaningful problem contexts should be used in which generalization contributes to insight into the situation, so that just summarizing the specific cases is not the only motive. To foster the process of generalization, the use of meaningful contexts with parameters that play a recognizable role is recommended. These contexts should allow for jumping back and forth between concrete and general. In order to avoid empirical abstraction that comes down to pattern recognition, it is suggested that explanations of the regularities be required and that students have the means to understand the patterns they find.
- For the parameter as *unknown*, the problem situation should naturally generate the question which parameter value fulfils a condition, and bring about the shift of roles that this parameter role requires. In general, we recommend stimulating a flexible use of variable and parameter roles. The computer algebra environment can reinforce this. Reviewing the solution strategy after it is completed, and considering the changes of the roles of the variables and parameters involved, can be an effective means to improve this flexibility.
- Insight into the concept of parameter is related to *insight into the meaning and structure of formulas*. Therefore, we recommend using formulas that emerge in

a meaningful way from problem situations. For the development of symbol sense, entering formulas into the CAS and interpreting the output may be a means of studying the structure of the formula. Manipulating formulas can be used to develop a feeling for equivalent forms. For the reification of formulas and expressions, we suggest having expressions appear as solutions of parametric equations that need to be processed further, for example by substitution.

# 10 Results concerning the instrumentation of computer algebra

## 10.1 Introduction

In Chapters 6, 7 and 8 the development of the hypothetical learning trajectory for the concept of parameter and the global experiences in the teaching experiments were sketched throughout the research cycles. The observations in these chapters indicated that students often encounter difficulties while working with the computer algebra machine, and that these difficulties can be considered as instrumentation issues that include conceptual and technical aspects.

Now that the first research subquestion on the concept of parameter has been dealt (Chapter 9), the present chapter addresses the instrumentation of computer algebra, and deals with the second research subquestion:

> *What is the relation between instrumented techniques in the computer algebra environment and mathematical concepts, as it is established during the instrumentation process?*

In Section 5.6 we described how the process of instrumental genesis leads to the development of simple instrumentation schemes that are the building blocks for composed instrumentation schemes. By developing such schemes, the tool gradually becomes part of an instrument that integrates the physical tool and the accompanying mental schemes for types of tasks. In this study, we restrict the instrumentation to algebra-related techniques, so to guarantee the coherence with the research question on the concept of parameter.

The theory of instrumentation guides the content of this chapter. First, we discuss two simple instrumentation schemes: the solve scheme, which influences the conceptual understanding of the parameter as unknown (10.2), and the scheme for substitution of algebraic expressions, which affects the reification of algebraic expressions and formulas (10.3). Then we present a composed instrumentation scheme that integrates both simple schemes: the isolate-substitute-solve (ISS) scheme (10.4). In the teaching experiments, this scheme was an obstacle to developing insight into the parameter as generalizer.

Section 10.5 describes obstacles that hindered the instrumental genesis, but at the same time offered opportunities for teaching and learning. Then we consider the relation between instrumented computer algebra techniques and the 'traditional' paper-and-pencil techniques the student is familiar with (10.6). Because the collective instrumentation in the classroom is guided by the teacher, we consider the orchestration of instrumentation by the teacher in Section 10.7. Finally, Section 10.8 is the synthesis of the chapter.

## 10.2 The solve instrumentation scheme

### 10.2.1 *Description of the solve instrumentation scheme*

The solve scheme is an example of a simple instrumentation scheme. We chose to analyse this scheme in depth because of its relation with the development of the concept of parameter – in particular with the parameter as unknown – and because of the difficulties that were encountered in the instrumental genesis.

The syntax of the solve command on the TI-89 is *solve*(*equation, unknown*). Fig. 10.1 shows how the equation $a \cdot x + b = 5$ can be solved with respect to different unknowns.

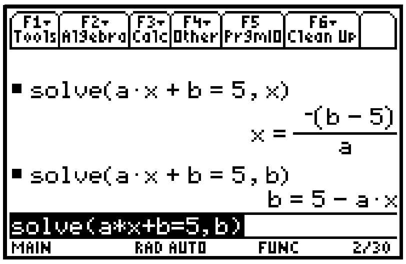



Figure 10.1 Solving on the TI-89

What does instrumentation of the solve command mean? The instrumentation scheme for solve contains the following elements:

1 Knowing that solving an equation is a step towards solving the problem at stake, and that it can be used to 'isolate' one of the variables in particular;
2 Remembering the TI-89 syntax of the solve command;
3 Knowing the difference between an expression and an equation;
4 Realizing that an equation is solved *with respect to* an unknown;
5 Being able to type in the solve command correctly on the TI-89;
6 Being able to interpret the result, particularly when it is an expression, and to relate it to graphical representations.

This list of elements is not intended to suggest a logical order; the order in which the student needs each element may vary. We do notice that some of the elements have a primarily technical character (elements 2 and 5) whereas the others have a mainly conceptual character.

### 10.2.2 *Conclusions on the solve instrumentation scheme*

We conclude from this study that despite the elementary character of the solve scheme, its application generated many errors that turned out to be quite persistent throughout the teaching experiments. Even if the instrumentation of the scheme on its own seemed to be in order, the integration of the scheme into a composed scheme

and its application in situations with complex formulas remained difficult.
The three main aspects of the instrumentation that we identified concern the syntax of solve, the extension of the meaning of solve, and the graphical meaning of an algebraic solution.
Concerning the *syntax of the solve command* (elements 2 and 5 in the list), students often forgot to specify the unknown with respect to which the equation is solved. Overcoming this instrumentation problem requires the insight that an equation always is solved with respect to an unknown, and that the CAS cannot know which literal symbol is the unknown in the case of more variables. A second syntactic difficulty concerned entering an expression rather than an equation, especially when the equation ended with '=0'. The third syntactic mistake was to forget the word 'solve'. This may have to do with a lack of insight into the difference between solve and simplify. Although the syntactic difficulties are primarily technical and were overcome in simple applications, they have a conceptual side as well (elements 3 and 4).
The second category concerns the need to *extend the meaning* of solve in two ways (elements 1 and 6). First, students needed to realize that expressing one variable in terms of others is also a way to solve an equation. As long as they did not understand solve in this way, they did not think of using it for this purpose. The result of such an isolation is an expression. Therefore, the second conceptual extension is the notion that a solution of an equation can be an expression and not only a numerical value. This requires reification of expressions and overcoming the lack of closure obstacle. This aspect of the instrumentation of solve was successful.
The third category concerns the *graphical meaning of an algebraic solution* (elements 1 and 6). The data suggest that students appreciated the graphical-numerical approach more than the algebraic, and did not connect the two approaches by themselves. When the teacher stressed that relationship, graphical properties could encourage solving algebraically and giving meaning to algebraic solutions. This was difficult for the students.

### 10.2.3 *Overview of the results*

The instrumentation of the solve command was a specific focus in the analysis of the data. As distinct from the method for the parameter roles that was described in Sections 9.3 - 9.5, we did not identify key items beforehand because it is hard to foresee the appearance of instrumentation matters. Instead, a free coding approach was followed to identify episodes that involved the instrumentation of solve. By doing so, all observations concerning this issue were gathered.
The results of this first round of analysis indicate that the instrumentation of the solve command required much effort. Furthermore, the difficulties that the students encountered while using solve were quite persistent throughout the teaching experiments. Finally, the instrumentation of the solve command interfered with the insight into the parameter as unknown.

Further analysis and categorization of these data revealed three issues that were relevant in the instrumental genesis of the solve scheme:

*a* The syntax of the solve command;
*b* The extension of the meaning of solve;
*c* The graphical meaning of an algebraic solution.

Point *a* stresses the technical part of the scheme, whereas points *b* and *c* have a more conceptual character. A second round of coding categorized the instances of solve into one of these three. There were also observations that did not fit into one of these three aspects, but their number was limited. Below, we elaborate on each of these issues.

### *a The syntax of the solve command*

The mastering of the apparently simple syntax of the solve command on the TI-89 was an aspect of the instrumentation that required the students' attention (elements 2 and 5 in the list). Errors were made and reappeared later in the teaching sequences. The most frequent errors were: no specification of the unknown, the solving of an expression rather than an equation, and no solve command.

– *No specification of the unknown*

  The most frequent syntactic error with the solve command was forgetting to specify the unknown with respect to which the equation should be solved.
  For example, students entered solve($a \cdot x + b = 5$) rather than solve($a \cdot x + b = 5$, $x$). They were not used to being explicit about the unknown, and often did not realize that in the case of more variables or parameters in the equation, the CAS could not know which one was meant to be solved.
  However, as some of the students noticed correctly, the specification of the unknown that the TI-89 requires is superfluous if the equation contains only one variable, such as $x^2 = 3$. For practical reasons, the computer algebra algorithm does not distinguish between cases with one or more variables, so the specification of the unknown needs to be done in both cases. This was confusing for some students, who felt it was not necessary to add 'comma *x*' for $x^2 = 3$. One of them did not add the 'comma *x*', got the error message 'too few arguments' and commented: 'But here you say already that it should do the *x*, by putting the *x* between the parentheses.'
  On the issue of adding 'comma unknown', we see that technical and conceptual aspects come together. The need to specify the unknown in the syntax of solve led to the students' growing awareness that equations are always solved *with respect to* an unknown (element 4). On the other hand, as soon as a student realized that, the need for this specification was understood, particularly in cases with more than one literal symbol, and this specification was no longer forgotten so

often.

– *Solving an expression rather than an equation*
Sometimes students entered an expression rather than an equation to be solved (element 3). This seems to be a mistake with no serious conceptual component; rather, students simply forgot one side of the equation, in particular when the right-hand side was '=0'.
The next protocol concerns the traffic flow problem (Appendix E). Ada wanted to solve the equation $y_1(x) = h$ to calculate the coordinates of the intersection points of the graph of $y_1$ and the horizontal line at height *h*. She first entered an expression rather than the equation and then solved with respect to the wrong unknown. This happened at the end of the G10-II teaching experiment, so it was not due to unfamiliarity with the machine.

*Ada first tries solve(y1(x),x) which gives an error message as the first argument is not an equation.*
*Ada:* *Then it is just like this [solve(y1(x), x)]*
*Observer:* *No, because you're going to intersect it with a horizontal line at height h.*
*Ada:* *So = y?*
*Observer:* *Well, y, I would prefer to use h.*
*Ada:* *So this [y1(x)], comma h.*
*Observer:* *No, comma x.*
*Ada:* *Oh yes, intersection point.*
(G10-II-5/15, assignment E3)

– *No solve command*
A third common mistake, though less frequent than the two previous ones, was the omission of the word 'solve'. The conceptual aspect of this mistake probably is the confusion between solving and simplifying. The automatic simplification of the TI-89 – for example, entering $3x + 5x$ immediately yields $8x$ – may have led the students to also expect automatic solving after entering an equation. The next protocol shows an example of this confusion. Fred wanted to simplify $0.5* a^2 – 0.5* b^2$. He used solve but did not enter an equation, or specify an unknown.

*Fred enters solve(0.5* $a^2$ – 0.5* $b^2$), which gives an error message.*
*Observer:* *What do you want to do really?*
*Fred:* *I want to solve this one and look what...*
*Observer:* *What do you mean by solving?*
*Fred:* *Yeah, simplifying or so, but that's not possible.*
(G9-II-6/34, assignment 6.7)

Apart from these three most important errors, other syntactic difficulties with the solve command were observed incidentally, such as the simultaneous solving of equations or using more than one unknown. Overall, the syntactic difficulties with the solve command became less frequent as the teaching experiments advanced, which suggests a progressive instrumental genesis. However, when situations became more complex or when the solve scheme was integrated into a composed scheme, the 'old' errors showed up again.

*b The extension of the meaning of solve*

An appropriate application of the solve instrumentation scheme required extending the meaning of solve, which at that stage meant to the students calculating the numerical value of the unknown. This extension concerned two aspects that appeared while solving equations with more than one variable. The first is the notion that solving an equation with more than one variable meant 'expressing one variable in terms of the other' or, as it is called in the teaching materials, 'isolating one variable'. The second is the notion that the solution of an equation could be an expression rather than a numerical value. The lack of insight into these issues could result in syntactic errors such as the ones mentioned above. We elaborate on these two issues below.

– *Solve to express one variable in the others*

  Many students were unfamiliar with the idea that solve can be used to express one variable in terms of the others, viz. to 'isolate' one variable by means of solving (element 1 in the list). As a result, students sometimes solved with respect to the wrong unknown. A language aspect is involved here: some students misunderstood 'express $x$ in $y$' as 'solve with respect to $y$.' The following observation illustrates this. The task was to express the zeros of the parabola with equation $y = x^2 + b \cdot x + 1$ in $b$.

  *Maria:* *So you do =0 so to say, and then 'comma b', because you have to solve it with respect to b.*
  *Observer:* *Well, no.*
  *Maria:* *You had to express in b?*
  (G9-II-4/16, assignment 9.2)

  Not all students had difficulties with the isolation view of solving. An adequate explanation was given by Jerome, who described 'isolating a letter' as 'taking apart a letter that you are going to use next to find out what that letter is', thus as an intermediate step in the problem-solving process. Apparently, some further steps are required after solving.

  The last line of the next fragment shows how one of the students, Mart, reacted to a question from the teacher by realizing that in an equation with more than one variable, one can choose which variable is isolated. This indicates an extended concept of the solve procedure. As a result, he did not forget to specify the unknown in the syntax of solve.

> *The teacher writes down on the blackboard:* $ap + q = 5$
> *Teacher:* *Can you solve this equation mentally?*
> *Student:* $ap + q - 5 = 0$
> *Teacher:* *And then?*
> *Mart:* *Which things do you have to know?*
> (G10-II-1/15, assignment 1.6 - 1.8)

– *Solutions can be expressions*
  The students are used to equations having numerical solutions. The solving of parametric equations yields algebraic expressions as solutions, and the students had to include that in their concept of solving (element 6).
  The next fragment shows how solving a simple equation such as $a \cdot x + b = 5$ with respect to different variables prepared for this extension. Marg successfully solved this equation with respect to *x* (assignment 3.13, Fig. 7.5) and *b* (3.14, Fig. 7.5). Despite the somewhat artificial situation, in which there is no obvious need to solve the equation, Marg accepted expressions as solutions that indicate 'how you have to calculate' the unknown. Her interpretation of the formula, therefore, has a process character.

> *Observer:* *You had that formula,* $a \cdot x + b = 5$ *and now I suddenly get x equals b minus 5 over a.*
> *Marg:* *Yes, because, look, because you had to calculate x, so you had to know what x was.*
> *Observer:* *And what is then the difference, how is that for 14 [solve with respect to b]?*
> *Marg:* *Well, in 14 you had to know what the b is, there you had to know how you have to calculate b.*
> *Observer:* *Yes. And if you would enter solve that 'comma a'?*
> *Marg:* *Well then you get a= .. a story.*
> (G9-II-1/39, assignment 3.13 - 3.14)

In more complex situations students accepted expressions as solutions that could be processed further by substitution or other procedures. This issue is related to overcoming the lack of closure obstacle and to the reification of expressions and formulas (see Section 9.7). Even students who initially suffered from the lack of closure obstacle, after some time found it normal that the result of a parametric equation was an expression. This aspect of the instrumentation was successful.

### *c The graphical meaning of an algebraic solution*

This point concerns perceiving the relation between algebra and graphical visualization (elements 1 and 6). When assignments addressed the relation between algebraic solutions and graphical properties, the students often were satisfied with the graphical-numerical approach and only considered algebra after questions from the teacher. Apparently, most problem situations did not bring about the need to apply an algebraic approach.

In the following dialogue the task was to find the value for the parameter *c* so that the distance between the zeros of the function *f* with $f(x) = x^2 + 2x + c$ is 6. The students did not look for an algebraic procedure, but found that for $c \approx -7, 25$ the zeros are (approximately) 2 and -4. After a question from the teacher, Thomas found an algebraic way of verification. With the *abc*-formula he referred to the solution formula for the general quadratic equation $a \cdot x^2 + b \cdot x + c = 0$.

*Kevin and Thomas found c=-7.25 by using the slider tool in the graph.*

| | |
|---|---|
| *Observer:* | *Are you sure about that? Is it exactly 7.25? Does it exactly hit the points 2 and -4?* |
| *Kevin:* | *You can check that, but I don't know exactly how.* |
| *Thomas:* | *You can do that with the abc formula.* |
| *Observer:* | *How?* |
| *Thomas:* | *If you use the abc formula, then a=1, b=2 and c=-7.25 and if that then gives the values -4 and 2 then it really is it.* |

(G10-II-1/40, assignment 4.2)

## 10.2.4 *Exemplary development of one of the students*

In this section we follow the instrumental genesis of the solve instrumentation scheme by Maria throughout the G9-II and G10-II teaching experiments. The overall impression gained from these observations is that the aforementioned instrumentation problems did indeed play a role, and were persistent: we observed an interplay of conceptual and technical difficulties that reappeared in complex situations, after successful use in easy cases had suggested an already completed instrumentation.

The first observation concerns solving the equation $a \cdot x + b = 5$ with respect to *x* (assignment 3.13) and *b* (3.14). Maria indicated that the isolated form $x = -(b-5)/a$ was easier than the original equation and the protocol in Section 7.4.3 shows that she understood that the unknown indicated after the comma in the solve command was isolated.

Later in G9-II Maria isolated a variable mentally, but the next observation indicates that she knew that this could be done with solve as well. The extension of the meaning of solve towards isolation, therefore, seemed to be in order.

| | |
|---|---|
| *Observer:* | *Would you be able to do that step with the machine? Would you be able to get from b*h=540 to h=?* |

> *Maria: Yes, just 'comma h', in solve.*
> (G9-II-9/2, assignment 9.4 - 9.5)

During the interview after G9-II Maria showed she was able to solve the equation $a \cdot x + b = c$ with respect to *x* with paper-and-pencil. She first asked if the task was 'to get the *x* in front of the =' and then wrote down the formula (Fig. 10.2). The parentheses around $c - b$ indicate that the structure of the formula was clear to her: an observation of what Wenger called the global substitution principle (Wenger, 1987). An expression as a solution no longer surprised Maria.

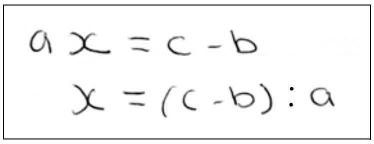



Figure 10.2 Maria's solution by hand

In G10-II the situations became more complex and Maria had difficulties in dealing with that. For example, in the assignment on the traffic flow, the task was to make one formula out of the following three: $f = 100 \cdot v/(6 \cdot (l + a))$, $a = 0{,}0075 \cdot v^2$, $l = 4$. Maria exhibited erroneous, unsuccessful behaviour. She first entered solve((100/4)/(6*(4+*a*)) without specification of the unknown. Then she entered solve((100/4)/(6*(4+*a*), *a*), but the first argument was not an equation. She changed the input into solve((100/4)/(6*(4+*a*)=*y*),*a*) after a helpful error message. She then seemed to lose track of her problem-solving strategy. Other students, who could follow her work by means of the view screen, made comments and Maria pulled the view screen cable from her calculator.
In the assignment on the lens formula $1/f = 1/v + 1/b$ Maria calculated *b* correctly for the case *f* = 5 and *v* = 15: solve(1/5 = (1/*b*)+(1/15), *b*). In the case that *v* was not given, she was unsure which unknown she would solve to, but help from Ada solved this problem. Maria accepted the solution to be an expression, and later applied solve to similar cases: solve(1/8 = (1/*b*)+1/*v*, *b*). She used some redundant parentheses, probably for 'safety reasons'.
At the end of the G10-II teaching experiment Maria became frustrated by the errors in the solve command, which she did not understand immediately. She also seemed to confuse solve and substitute, which indicates a mixing up of procedures.

> *Maria enters solve(a*x$^2$+b*x+c, x), which results in an error message because she forgets the '=0'.*
> *Maria: Why doesn't it give an answer? I don't understand it.*
> *She changes her input to: solve(a*x$^2$+b*x+c | x=x*
> *That doesn't work either, but gives 'error: memory'.*
> (G10-II-v6-59, assignment 9.1 - 9.2)

Finally, the earlier description in Section 9.5.3 indicates that Maria encountered many difficulties in applying the solve command and in keeping track of the roles of the different letters and of the problem-solving strategy while working on the rest of the assignment.

## 10.3 The substitute instrumentation scheme

### 10.3.1 *Description of the substitute instrumentation scheme*

The second simple instrumentation scheme that we consider is the substitution scheme. It involves the substitution of numerical values for variables and parameters as well as the substitution of algebraic expressions. Like the solve scheme, the substitute scheme can be integrated into a composed instrumentation scheme. We chose to analyse this simple instrumentation scheme in depth, because of its relation with the development of the concept of parameter, and with the parameter as placeholder and as generalizer in particular. The substitution of expressions is important for the generalization of problem-solving schemes. The reification of expressions is a prerequisite for such operations; on the other hand, these operations may reinforce the object character of expressions and formulas and thereby complete their reification. The syntax of the substitution of algebraic expressions on the TI-89 is:

$$\textit{expression1} \mid \textit{variable} = \textit{expression2}$$

The first expression, *expression1,* can be any algebraic form. The second, *expression2,* has to be an expression in the limited sense of the word, viz. it cannot contain the = sign. The *variable* can also be a parameter, and of course the substitution only makes sense if this variable appears in *expression1*. Fig. 10.3 shows an example, together with a visualization by means of arrows.

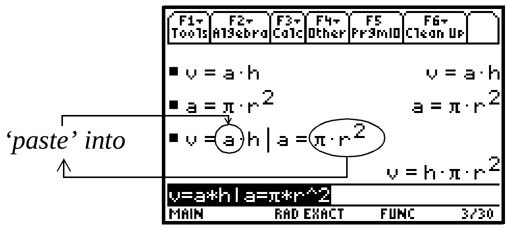



Figure 10.3 Substituting on the TI-89

What does instrumentation of the solve command mean? The instrumentation scheme for substitute contains the following elements:

1 Knowing that substituting is a step towards solving the problem at stake, and imagining the substitution as it is visualized in Fig. 10.3 in particular;

2 Remembering the TI-89 syntax of the solve command and the meaning of the vertical bar symbol in it;
3 Realizing which expressions play the roles of *expression1* and *expression2*, and considering *expression2* as an object rather than a process in particular;
4 Being able to type in the substitute command correctly on the TI-89;
5 Being able to interpret the result, and particularly to accept the lack of closure when the result is an expression.

As for the solve instrumentation scheme, this list of elements does not suggest a logical order; the order in which the student needs each element may vary. We notice that some of the elements have a primarily technical character (elements 2 and 4) whereas the others have a mainly conceptual character.

### 10.3.2 *Conclusions on the substitute instrumentation scheme*

We conclude from our data that the instrumental genesis of the substitution scheme advanced during the teaching experiments. Three aspects of instrumentation emerged from the data analysis: the substitution of numerical values, the syntax of the substitution of expressions and the insight into the substitution of expressions.

For the *substitution of numerical values,* instrumentation was easy. The students understood what happened when a value was substituted. This is in line with the findings on the parameter as placeholder (9.2). By-hand substitution of numerical values in some cases was hard. This was a matter of algebraic skill and symbol sense.

For the *syntax of the substitution of expressions* we conclude that the notation seemed to be natural (elements 2 and 4 in the list). The procedure was transparent in the sense that the students were able to verify the machine's output in simple cases. However, a frequent error was the substitution of non-isolated forms (elements 1 and 3). This mistake was persistent and seems to be related to a limited understanding of the substitution of expressions or to an overestimation of the CAS potential.

For the *insight into the substitution of expressions* we conclude that the extension of the concept of substitution towards the substitution of expressions was indeed made. Substitution seemed to foster the reification of the expressions (elements 3 and 5). The meaning of the substitution bar was clear to most students, and they were able to explain this. Giving the substitution bar a name (the 'wherein bar') was helpful. The visualization by means of ovals supported the concept of substitution (element 1). What creates doubt, however, is that the problem of the substitution of non-isolated forms was not easily solved, and also appeared later in the teaching experiments, particularly in combination with the solve command in a composed scheme.

### 10.3.3 *Overview of the results*

In the analysis of the data we followed a similar method as for the solve scheme. A free coding approach was used to identify episodes that involved the instrumentation of substitute. By doing so, all data concerning this issue were gathered and coded.

The results of this first round of analysis indicate that the instrumentation of the substitution involved unexpected aspects that the students found hard. Overall, however, the obstacles seemed to be overcome as long as substitution was not embedded into a composed scheme; in more complex situations, the 'old' errors showed up again.
Further analysis and categorization of these data revealed three issues relevant to the instrumental genesis of the substitution scheme:

*a* The substitution of numerical values;

*b* The syntax of the substitution of expressions;

*c* The insight into the substitution of expressions.

Points *b* and *c* are related, as they stress the technical and the conceptual part, respectively, of the instrumentation scheme. A second round of coding categorized the incidences on substitution into these categories. A small number of observations did not fit into any of these three aspects. We now elaborate on each of these issues.

### *a The substitution of numerical values*

The substitution of numerical values concerns replacing variables or parameters with numbers. The students were familiar with this – at least for variables – and this point provided hardly any conceptual difficulties. The substitution linked up with the students' concept of variables and parameters as placeholders. On the TI-89, substitution is done with the 'wherein bar'. For example, $3x + 5|x = 2$ substitutes 2 for *x*. The following fragment contains an explanation by one of the students.

> *Observer:* *Do you know what that bar means?*
> *Barbara:* *Ehm, yes, that you have to fill in behind what the letter then stands for.*
> *Observer:* *And if you press enter then, what happens then with what's behind the bar? For example if you substituted this [points at 3x + 5 | x=2], what does it have to do?*
> *Barbara:* *Fill in the number for the letter.*
> (G9-II-E2/3, assignment 3.22)

For the calculation of function values and for verification of a solution, usually the students applied numerical substitution correctly. However, observations from the G9-II teaching experiment show that by-hand substitution in complicated formulas was difficult. Maria's struggle to substitute $a = 0$ into $x + a \cdot \sqrt{x^2 + 1}$ is an example of this (Section 9.7.3). This difficulty seemed to be caused by a lack of symbol sense rather than by a lack of insight into numerical substitution. A more conceptual error was made by a student who wanted to solve the equation $(x-5) \cdot (x+1) = 7$ by hand by substituting 12 for the first instance of *x* and 0 for the second.
For drawing two-dimensional graphs of parametric functions, the substitution of nu-

merical values for the parameter was indispensable. This corresponds with the placeholder conception of parameter. In some cases, students forgot the substitution and tried to graph the general function. An error message usually revealed this omission. To draw a sheaf of graphs, which prepared for the parameter as generalizer, more than one numerical value needed to be substituted into the formula of the parametric function. The teaching material suggested the following notation for drawing the graph of $y_1(x) = x \cdot (s - x)$ for the values 0, 1, 2, 3, 4, 5 of *s*:

$$y_1(x) = x \cdot (\{0, 1, 2, 3, 4, 5\} - x)$$

However, this notation caused the students difficulties, until one of them, Arno, found a more natural notation:

$$y_1(x) = x \cdot (s - x) | s = \{0, 1, 2, 3, 4, 5\}$$

This notation reflects the hierarchic relation between the variable *x* and the parameter *s*: a change of *s* clearly affects the whole function, whereas this is not the case for *x*. Furthermore, the notation indicates that *s* stands for a set of values rather than a single value, and thus prepares for the concept of parameter as generalizer. After this notation was introduced, the substitution of values in parametric functions was no longer a problem.
The more complex case of two parameters required a double substitution. That presented some technical obstacles, such as using two vertical bars rather than one joint substitution. For example, $y_1(x) = a \cdot x + b | a = 5$ and $b = 3$ is syntactically correct, whereas $y_1(x) = a \cdot x + b | a = 5 | b = 3$ is not. Also, to the surprise of the students, the TI-89 accepted $y_1(x) = a \cdot x + b | a = 5$ and $b = \{1, 2, 3\}$, but not $y_1(x) = a \cdot x + b | a = \{5\}$ and $b = \{1, 2, 3\}$, because the sets that appear in the function definition need to have an equal number of elements.
Altogether, the instrumentation of the substitution of numerical values went well and did not cause any lasting problems.

### *b The syntax of the substitution of expressions*

The syntax of substitution was an aspect of the instrumentation, which generally was mastered quite well (elements 2 and 4). Only one error was persistent and troubled students on many occasions: the substitution of non-isolated forms. For example, the right-angled triangle assignment (see Fig. 10.4 in Section 10.4) led to the system of equations $x^2 + y^2 = 25^2$, $x + y = 31$. In such situations, substitution of a non-isolated form meant entering $x^2 + y^2 = 25^2 | x + y = 31$. This gave an error message, as the CAS did not know which variable in the quadratic equation was to be replaced. There are several explanations for this error. First, the two equations in the system have equal roles, whereas in $x^2 + y^2 = 25^2 | x + y = 31$ they have distinct roles, and are treated in a different manner. Second, students may have the idea that while entering the two equations 'the CAS will find a way to combine them'. The analogy with the combined solve command, solve($x^2 + y^2 = 25^2$ and $x + y = 31$, *x*) that does work on the TI-89, may play a role as well. The students see the 'wherein bar', annotated by |, probably as a more symmetric 'with', close to the set theory notation $\{x^2+y^2= 25^2 \mid x + y = 31\}$. A mature insight into the process of substitution would prevent this error from occurring, so once more we see the relation between the syntactic mistake and the conceptual understanding of the process of substituting expressions (element 1). The difficulty of non-isolated substitution was persistent, particularly when substitution was combined with solve.

The following remarkable incident shows how notations can be confusing. In the cylinder assignment (see Fig. 6.5), Ramon substituted the formula for the area *a*, $a = \pi \cdot r^2$, into the volume function $a \cdot h$. He thought the result $\pi \cdot h \cdot r^2$ was logical, because he confused the vertical wherein bar '|' with the slash '/' for division.

> *Ramon:* *So here you have an a [before the bar], and there one [after the bar], and the one a cancels out the other, and then you get h times pi times r squared, the two a's cancel out so to speak.*
> *Teacher:* *Instead of a you get pi r squared.*
> (G9-I-B6, assignment 4.17)

In later teaching experiments, a more conscious use of the substitution bar was stimulated by calling it the 'wherein bar' and by phrasing $3x + 5|x = 2$ as '3*x*+5 wherein *x* = 2' or '3*x*+5 whereby *x* = 2'.

The final observation of this section provides an example of using substitution for renaming the variables, which was presented in Section 9.5.2. The variable *w* (which stands for body weight) was replaced by *x* to allow for drawing the graph. One of the students used substitution to rename *w* as *x*:

$y1(x) = 11*w^{2/3} \mid w = x$

Overall, the substitution of non-isolated forms was the main syntactic obstacle of the substitute scheme. The next subsection deals with the conceptual aspects of substitution.

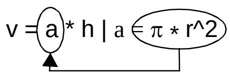
### *c The insight into the substitution of expressions*

The instrumentation of the substitute scheme involves the conceptual extension of the substitution of numerical values towards the substitution of expressions.

Introductory assignments on the substitution of expressions showed good results. The meaning of the substitution was often explained adequately. For the assignment on the rectangular cylinder (see Fig. 7.6), Fred carried out the following substitution: $v = a \cdot h|a = \pi \cdot (d/2)^2$ and $h = 2r$. The *v* stands for the volume, *a* for the area of the bottom, *h* for the height, *d* for the diameter of the bottom and *r* for the radius. He explained the substitution as follows and got help from Cedric:

*Teacher:* *Now what exactly does that vertical bar mean?*
*Fred:* *Well, it means that here [after the bar] an explanation is given for that variable.*
*Teacher:* *You mean the variable is explained, so to say?*
*Fred:* *Yes. Anyway, that's what I thought and that came out more or less.*
*Teacher:* *But what do you mean exactly, the variable that is before the bar, or...*
*Fred:* *Yes the ones before the bar, they are explained here.*
*Teacher:* *Yeah, that's true, but... Who can explain what exactly the effect of that 'whereby' is?*
*Cedric:* *Well it, that is in fact a manual for the formula that stands beside [in front] because it says, well, v equals a times h and besides it says yes and there a, that is pi and then r squared.*
(G9-II-1/46, assignment 3.20-3.21)

Similar explanations were observed more often. Students understood the substitution of an expression as 'explaining a variable', as a 'manual' or an 'instruction'. Some students came up with more vague terms such as 'combining', 'simplifying' or 'make one formula out of it'. The word 'substitution' itself was difficult for some students to use, even if they understood its meaning.
The students who used the more vague explanations seemed to have missed the idea of substitution that was visualized by means of ovals (element 1):




This visualization supported the image of 'cutting an expression' and 'pasting it into' the right position(s) in the expression before the wherein bar. Our impression is that this visualization contributed to the development of an appropriate conceptual understanding of substitution and helped in overcoming the syntactic obstacle of the non-isolated substitution that dominated the previous section.
The substitution of expressions stressed the object character of the expressions that were substituted and thus both required and contributed to the reification of formulas and expressions (see Section 9.7 and elements 3 and 5). In the interviews after G9-II we observed transfer on the issue of substituting expressions from the computer algebra environment to the paper-and-pencil environment (Section 7.4.6 and Section 10.6).
To conclude this section we will mention an incident in which the student substituted the expression $-b/2$ by hand into the $x$ in $x^2 + b \cdot x + 1$ , but did not realize that this should be done for all instances of $x$. She replaced the $x$ only in the second term, and

not the $x$ in $x^2$. This is similar to the mistake reported in point *a* on numerical substitution. Still, such errors were rare.

### 10.3.4 *Exemplary development of one of the students*

In this section we follow the instrumental genesis of the substitute scheme by Maria throughout the G9-II and G10-II teaching experiments. The overall impression of these observations is that initially instrumentation advanced well; later on, however, the combination of substitute and solve in more complex situations confused Maria.
The first fragment from the start of the G9-II experiment shows that Maria developed a sense for substitution. Numerical substitution was clear to her, but she described substitution of expressions in somewhat vague terms:

> *Observer:* *And what happens now for example in assignment 20,* $v = a \cdot h$ *where* $a = \pi \cdot r^2$ ?
> *Maria:* *Well, that it just makes one formula out of it.*
> *Observer:* *Yes, and how does it do that then?*
> *Maria:* *By joining them or something?*
> (G9-II-1/42, assignment 3.20 - 3.21)

Later Maria was able to explain the substitution of $a = \pi \cdot r^2$ in $v = a \cdot h$ during the classroom discussion. She noticed that the 'value' for *a* now is an expression:

> *Teacher:* *Who can explain what the exact effect is of that wherein?*
> *Maria:* *I think it's like this. You have a formula, and then you fill in a in the formula, so that the results is, so to say, the same formula, but you gave a a value. That value is a formula once more. And then you give that a value, then you fill it in.*
> (G9-II-1/51, assignment 3.21)

Shortly after that, Maria managed the syntax of the substitution, although she did not feel confident about it. She entered $y = x + 1 | y = \sqrt{25 - x^2}$ and asked: 'Did I do it well like this? I wasn't sure if it was correct now.'
Maria's long struggle to substitute $a = 0$ into $x + a \cdot \sqrt{x^2 + 1}$ shows that she had difficulties with substituting numerical values by hand (see Section 9.7.3). However, this seemed to be caused by a lack of symbol sense rather than by a lack of insight into the substitution process.
The sum-difference problem in the context of the rectangle with perimeter *P* and difference $v$ between base and height led to the equations $P = 2 \cdot (b + h)$ and $b - h = v$. By hand, Maria rewrote the second equation as $b = h + v$ and substituted this result into the first equation: $sum = 2 \cdot h + difference$. This indicates a good insight into the process of substituting expressions (see Section 9.4.3).

One of the next assignments concerned solving the system $b \cdot h = 540$, $b^2 + h^2 = 39^2$. Once more, Maria started to work by hand. After an error, she managed to rewrite the first equation as $h = 540 / b$, but the substitution in this case was harder to carry out. The observer felt the need to guide her quite a lot. At the end, she showed she was able to translate her substitution into machine language.

| | |
|---|---|
| *Observer:* | *And could you use this now in the first equation?* |
| *Maria:* | *I don't know, because of course it is a square.* |
| *Observer:* | *Yes, but what does it matter?* |
| *Maria:* | *Maybe I can fill it in for the h, but then I still have that square.* |
| *Observer:* | *But you got rid of the h.* |
| *Maria:* | *But can I square this [540 / b]?* |
| *Observer:* | *You can.* |
| *Maria:* | *That comes down to the same, in fact?* |
| *Observer:* | *Yes. If h = 540 / b then you can fill in here for* $h^2$ *the square of 540/b.* |
| *Maria:* | *Oh yes.* |
| *Observer:* | *What would you get then?* |
| *Maria:* | *Well* $b^2 + (540/b)^2 = 1521$. |
| *(...)* | |
| *Observer:* | *And how would you fill in that h = 540/b in that first equation without doing it yourself?* |
| *Maria:* | *Well, I think, then you have to do this [*$b^2 + h^2 = 39^2$*] and then such a wherein and then wherein h=, and then you fill in this [h = 540/b].* |

(G9-II-9/2, assignment 9.5)

In the next teaching experiment, G10-II, Maria was still able to carry out substitutions with the CAS. She also substituted numerical values by hand. However, later on in G10-II she started to confuse solving and substitution. For example, she used solve in combination with substitution, where substitution alone would have sufficed.

One of the last observations of Maria concerning substitution shows how she substituted the expression $-b/2$ by hand into $x$ in $y = x^2 + b \cdot x + 1$. Then she solved the result with respect to $y$, although $y$ was already isolated. She seemed to have the idea that a substitution always had to be followed by a solve command.

## 10.4 The isolate-substitute-solve (ISS) instrumentation scheme

### 10.4.1 *Description of the ISS scheme*

While describing the instrumentation of solve and substitute in Sections 10.2 and 10.3, we made the remark that the instrumentation was shown to be incomplete when these simple schemes were integrated into a composed instrumentation scheme. We now illustrate this by discussing such a composed scheme, the ISS scheme.

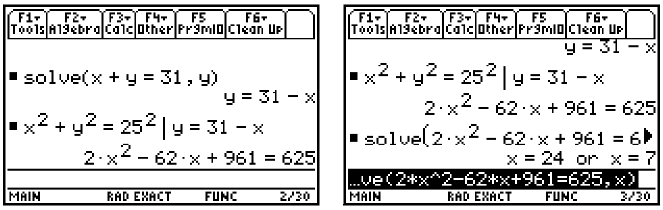



Figure 10.4 Assignment inviting application of the ISS scheme

**5.4** The two right-angled edges of a right-angled triangle together have a length of 31 units. The hypotenuse is 25 units long.

**a** How long is each of the right-angled edges?

**b** Solve the problem also in case the total length of the two edges is 35 instead of 31.

**c** Solve the problem in general, that is without the values 31 and 25 are given.

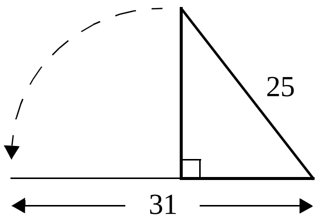



Figure 10.5 The ISS scheme on the TI-89

We explain the ISS scheme for the right-angled triangle assignment from G9-I (see Fig. 10.4). Question **a** leads to a system of two equations with two unknown, $x + y = 31$ and $x^2 + y^2 = 25^2$, where *x* and *y* stand for the lengths of the two perpendicular edges. Fig. 10.5 shows the proposed solution strategy for this case on the TI-89. First, one of the variables, in this case *y*, is chosen to be isolated in one of the equations, here $x + y = 31$. The result is then substituted into the other equation using the wherein operator, symbolized by the vertical bar |. In the resulting equation, there is only one variable left (here *x*) and a second solve command provides the answer, in this case $x = 7$ or $x = 24$. The calculation of the corresponding values of the other variable, *y*, is a matter of substituting the values of *x* in one of the equations. In question **b** this scheme is repeated for other values, and in **c** it is generalized for the case with parameters.

It is important to notice that many students did not carry out the ISS scheme as proposed; they developed variations, such as condensing the scheme into a one-line nested form that in this case comes down to solve($x^2+y^2=25^2 \mid y = 31 - x, x$). The next section will show that this form in particular led to instrumentation problems.

What does the instrumental genesis of the composed ISS scheme mean? The scheme contains the following elements:

1 Knowing that the ISS approach is a way to solve the problem, and being able to keep track of the global problem-solving strategy in particular;
2 Being able to apply the solve scheme for the isolation of one of the variables in one of the equations;
3 Being able to apply the substitute scheme for substituting the result from the previous step into the other equation;
4 Being able to apply the solve scheme once more for calculating the solution;
5 Being able to interpret the result, and particularly to accept the lack of closure when the solution is an expression.

This list of elements does suggest an order, as the sequence of steps 2 - 4 cannot be changed. We notice that steps 2, 3 and 4 are simple instrumentation schemes, which were described in the previous sections. They include technical and conceptual aspects and are embedded here in a composed scheme.

### 10.4.2 *Conclusions on the ISS instrumentation scheme*

We conclude from the data analysis that the instrumental genesis of the ISS scheme was very difficult, even though the instrumentation of the components schemes of solve and substitute seemed to be adequate. The difficulties that the students encountered were similar to those described in Sections 10.2 and 10.3. The use of the nested form of the ISS scheme, however, made it harder to overcome them and to overview the problem-solving strategy. The following aspects of the instrumental genesis of the ISS scheme emerged.

First, many errors concerned skipping the isolation step and directly substituting a non-isolated form. This mistake, which turned out to be persistent, is related to the understanding of the substitution procedure and to its TI-89 notation with the where-in bar. When the substitution stood on its own, as was the case in Section 10.3, this difficulty was overcome, but in the more complex context of the ISS scheme this turned out to be harder, whether or not parameters were involved.

Second, the use of the nested form of the ISS scheme, in which the scheme is condensed into one command, led to solving with respect to an unknown that no longer appeared in the equation. Students found it difficult to foresee what variable would remain in the equation after substitution. The step-by-step method shown in Fig. 10.5 is easier than the nested form in this respect.

Third, strategic errors were observed that were related to a lack of overview of the whole problem-solving process. These errors included the ‘circular’ substitution in the same equation, and doing ‘double work’ such as substituting twice rather than once, so that none of the variables is eliminated.

Finally, variations to the ISS scheme appeared. One of these variations, which we called isolate-isolate-equal-solve, seemed to be more accessible to the students because of its relation with the paper-and-pencil approach and with the graphical interpretation of solving equations.

All together, we conclude that the integration of simple instrumentation schemes into a composed scheme requires a higher degree of instrumentation of the components than does the application of the separated simple instrumentation schemes, and an overview of the problem-solving procedure as a whole. For the ISS scheme, we question whether it is the most natural approach to solving systems of equations.

### 10.4.3 *Overview of the results*

In the analysis of the data concerning the ISS scheme we followed a similar method as for the solve and substitute schemes. A free coding approach was used to identify episodes that involved the instrumentation of the ISS scheme. By doing so, all data concerning this issue were gathered and coded.

The results of this first round of analysis indicate that the instrumentation of the ISS led to many unexpected errors, if we take into account the relatively smooth instrumentation of the component schemes. These difficulties hindered carrying out the ISS scheme throughout the teaching sequences. Even in the final tests of G9-I and G9-II, many students were not able to apply the scheme adequately. Overall, the instrumentation of the ISS scheme was more successful in G9-II than in G9-I. Possible factors that caused this are the increased amount of practice with the simple instrumentation schemes and a better orchestration by the teachers, who were more experienced in G9-II. In G10-II the ISS scheme played a less important role, because of a decreased importance of systems of equations in the instructional activities.

Correct ISS procedures were observed as well. For example, Carolien adequately solved the system of equations $a + v = 62$, $4a - v = 13$ that emerged from a context about the ages of mother and daughter. She isolated $v$ in the second equation and said that the result should be substituted into the first equation. She combined this with solving the equation with respect to the correct unknown by entering: solve($a$+$v$ = 62 | $v$ = 4$a$–13, $a$).

In simple cases, some students carried out the first step of the ISS scheme – the isolation – in their head. For example, the equation $x + y = 31$ was immediately substituted as $y = 31 - x$ in the other equation. This, of course, is an efficient strategy.

Further analysis and categorization of these data led to the following categories:

*a* Errors in the isolation and substitution phase;

*b* Errors in the solution phase;

*c* Strategic errors;

*d* Variations on the ISS scheme.

A second round of coding categorized the incidences of the ISS scheme into one of these categories. Only a limited number of observations did not fit into any of these four categories, which are elaborated on below.

*a Errors in the isolation and substitution phase*

The first and second step in the ISS scheme are solving one of the equations for isolation one of the literal symbols and substituting the result into the other equation (steps 2 and 3 in the list). The frequently observed error here was that the first step was skipped and the non-isolated equation was substituted directly into the other. This problem was described in Section 10.3 on the instrumentation of substitution. As an example, the next protocol shows how John and Rob worked on the right-angled triangle problem (see Fig. 10.4). While solving the system $o + a = 31$, $o^2 + a^2 = 625$, Rob substituted the non-isolated form, but John helped him.

| | |
|---|---|
| *Observer:* | *Rob, you wrote down solve($o^2 + a^2 = 625$ \| $o + a = 31, a$). How about that bar?* |
| *John:* | *You have to put one apart: you have to put either the o or the a apart.* |
| *Rob:* | *You can only work on one letter?* |
| *Observer:* | *John, could you explain this to Rob?* |
| *John:* | *I think if you have got the vertical bar, you are allowed to explain only one letter,* so *o* =. |
| *Rob:* | *Oh, $o = 31 - a$.* |

(G9-I-A13, assignment 5.4)

As we explained in Section 10.3, the substitution of non-isolated forms shows a limited understanding of the substitution process. We also described some possible reasons for this error. Apparently, embedding the substitute scheme into the more comprehensive ISS scheme causes this error to occur more frequently.

*b Errors in the solution phase*

The most frequent error in the last solution step of the scheme was to solve the equation with respect to the wrong unknown (step 4 in the list). This was observed particularly when the students used the nested form of the ISS scheme, in which the substitution and the solving are combined in one command.
For example, one of the students used the nested form to solve the system of equations $x^2 + y^2 = 25$, $x \cdot y = 10$. He entered: solve($x^2$+$y^2$=25 | $x$*$y$=10, $y$). This is an example of non-isolated substitution. When the observer pointed this out to him, he changed the input into solve($x^2$+$y^2$=25 | $y$=10/$x$, $y$). The result of this was a complicated equation in *x* rather than the intended solution for *y*: $1 = 25x^2/(x^4 + 100)$. To be able to choose the unknown correctly, the students needed to mentally divide the nested command into two subprocesses, and to realize that if the substitution part starts with *y*=, then the resulting equation will no longer contain *y*, so it has to be solved with respect to *x*. One of the students avoided this mistake by symbolizing the two subprocesses in the nested command with an extra pair of parentheses:

$$\text{solve}((x^2 + y^2 = 25^2 \mid y = 31 - x), x)$$

A second way to avoid this difficulty is to use the step-by-step method proposed in Fig. 10.5. Many students preferred the nested form, and this preference was related to the classroom culture and the orchestration by the teacher (see Section 10.7). In a paper-and-pencil environment this difficulty probably does not occur as the step-by-step approach is a natural one. The power of computer algebra to integrate two steps requires an increased awareness of what is happening.

*c Strategic errors*

Besides errors in carrying out components of the composed ISS scheme, we observed errors that go beyond the level of the simple instrumentation schemes and concern the overall problem-solving strategy (step 1 in the list). Two kinds of strategic errors were observed, one showing a ‘circular approach’ and the other showing ‘double work’.

The circular approach concerns the substitution of an isolated form into the equation from which it was derived. For example, one of the students isolated $y$ in $x + y = 5$ and then continued by substituting: $x + y = 5 | y = 5 - x$. The result was the message ‘true’. A second example of this was described in Section 9.2.2 where one of the students tried to solve the general system of equations $b + h = s$, $b \cdot h = p$ by entering solve($b + h = s \mid b = s - h$, $b$).

As a last example of such circular substitution, the following dialogue shows that Donald did not know how to solve the system of equations $x \cdot y = 540$, $x^2 + y^2 = 39^2$.

| | |
|---|---|
| *Observer:* | *What would you do now?* |
| *Donald:* | *Ehm, well, I calculated that x is 540 divided by y.* |
| *Observer:* | *Exactly.* |
| *Donald:* | *Well, and if ... you then can replace this [x in x*y = 540] by that [540/y].* |
| *Observer:* | *Well, yes, but that is not sensible, because x is 540 divided by y but you shouldn’t fill that in in the equation where you got that from, where you derived it from.* |
| *Donald:* | *Oh.* |
| *Observer:* | *So you’d better substitute it into the other equation.* |
| *Donald:* | *In that one [the quadratic equation].* |
| *Observer:* | *Right. So if you fill in this [540/y] here [x in* $x^2+y^2=39^2$*] ..* |
| *Donald:* | *... and that one below [y = 540/x] there [the y in the quadratic equation].* |

(G9-II-10/7, assignment 9.5)

The last line of the above dialogue reveals the strategic error of the ‘double work’: by means of two ‘isolations’, both $x$ and $y$ are isolated from $x \cdot y = 540$. Then $x$ in

$x^2 + y^2 = 39^2$ is replaced by 540/*y* and simultaneously *y* is replaced by 540/*x*. That clearly is double work with no progress towards the result.
Both the circular approach and the double-work error show a lack of overview of the solution strategy and of the composed scheme as a whole. These errors are not new: Arcavi described how these problems emerged in paper-and-pencil work (Arcavi, 1994). This illustrates that ‘traditional’ algebraic difficulties in the paper-and-pencil environment can show up in comparable forms in the computer algebra environment.

*d Variations of the ISS scheme*

The ISS scheme was used in a step-wise form (Fig. 10.5) and a nested form. Besides these two forms the students developed other variations that could be applied to the same type of problems. The following alternatives were observed:

– The method of simultaneous solving
  The system $x + y = 31$, $x^2 + y^2 = 25^2$ can be solved simultaneously on the TI-89 by the command solve($x^2 + y^2 = 25^2$ and $x + y = 31$, *x*). This option was not foreseen during the preparation of the G9-I teaching experiment but became quite popular. In G9-II it was indicated to the students that this combined solve command works only for a very limited number of systems.
– The method of isolate-isolate-equal-solve
  In the above system this approach starts with isolating the same variable, for example *y*, in both equations: $y = 31 - x$ and $y = \sqrt{25^2 - x^2}$. Often, students forgot the negative square root. The next step is to equal both isolated forms and solve the resulting equation: solve($31 - x = \sqrt{25^2 - x^2}$, *x*). This approach links up with the interpretation of solutions of equations as coordinates of intersection points of graphs, and with the method that the students used by hand. Incidentally, the last solve command was entered by means of substitution: solve($y = \sqrt{25^2 - x^2} \mid y = 31 - x$, *x*).

These variants sometimes developed individually; their popularity was related to classroom practice and the orchestration by the teacher (see Section 10.7).

### 10.4.4 *Exemplary development of one of the students*

In this section we follow the instrumental genesis of the ISS composed scheme by Maria in the G9-II teaching experiment.
The sum-difference problem in the context of the rectangle with perimeter *P* and difference of *v* between base *b* and height *h* led to the equations $P = 2 \cdot (b + h)$ and $b - h = v$ (see Fig. 7.9). In Section 9.4.3 we described how Maria isolated *b* in the second equation by hand and substituted the result manually into the first equation. The natural way in which she developed this method suggests that is an intuitive ap-

proach for students. The next fragment shows how Maria explained the step-wise ISS method to her fellow students in a classroom discussion for the case that $P$ = 1400 and $v$ = 400.

> *Maria:* *So that P was 1400. And you know that b-h=400. Well this [P] divided by 2 is 700. And then it is this, so to say: 700 = h+ (h+v). And so you do next, if you 700, well, minus 400 [the value of v] is already 300, divided by 2 is 150. Then you have 150 + 150 + 400 is 700. So then you know already that this one [700] minus 150 is 550, so then you know that b is 550 and h 150.*
>
> (G9-II-E3/20, assignment 5.7)

In assignment 9.4 it was suggested to isolate $y$ in both equations of the system $x^2 + y^2 = 25$, $x \cdot y = 10$ and to graph the two resulting functions. After that, Maria tended to solve the system graphically, as though the graphs led to neglecting the algebraic method. In the next assignment, 9.5, substituting $h = 540/b$ by hand into $b^2 + h^2 = 39^2$ was difficult for her, as was shown in Section 10.3.4.
In the final test of G9-II, Maria showed that she was able to carry out the nested ISS scheme correctly for the system $x + y = 10$, $\sqrt{x^2 + y^2} = 8$ (Fig. 10.6).
All together, the instrumentation of the ISS scheme during the G9-II teaching experiment was accomplished well by Maria.

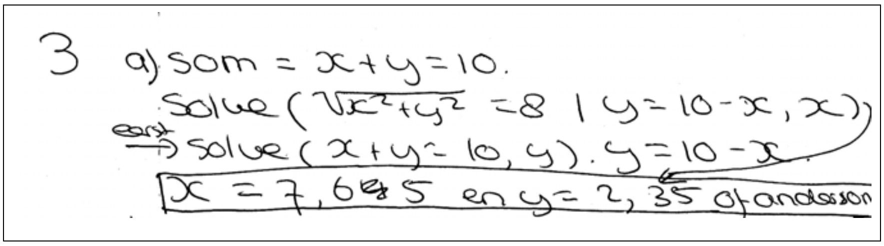



Figure 10.6 Maria's ISS scheme at the final task of G9-II

## 10.5 Obstacles to the instrumentation of computer algebra

### 10.5.1 *Conclusions on obstacles to the instrumentation of computer algebra*

In Chapter 5 we presented evidence from previous research indicating that the instrumentation of computer algebra often does not go smoothly. The previous sections of the present chapter confirm this as far as the instrumentation schemes for solve, substitute and ISS are concerned. In Chapter 5 we also argued that it can be important to pay attention to the obstacles that students encounter during the instrumentation, because an inventory of such obstacles might reveal the students' technical and conceptual difficulties, which is a first step towards understanding them.

Therefore, in this section we take a seemingly negative perspective by focusing on the obstacles that we observed. We define obstacles here as technical and/or conceptual barriers encountered in the CAS environment that prevent the student from carrying out the instrumentation scheme he/she had in mind. We confine ourselves to obstacles that are related to algebraic insight, which is the main topic of this study. Because such obstacles were encountered on various occasions, we no longer focus on the development of one particular instrumentation scheme.
We conclude from the data analysis that obstacles inhibited the instrumental genesis on many occasions, and that similar obstacles showed up in different situations. For example, the difference between the exact calculations that the CAS carries out and the approximated numerical results that the students appreciate was one of the obstacles that played a role in different kinds of problems. Furthermore, insight into the structure of formulas and expressions – symbol sense – was an important obstacle to correctly entering expressions and dealing with equivalent formulas.
In our analysis of the data we followed a method similar to that applied to the instrumentation schemes in the previous sections. As the occurrence of obstacles is difficult to predict, a free coding approach was used to identify episodes that involved obstacles to the instrumentation. By doing so, all data concerning this issue were gathered and coded. Further analysis of these episodes revealed three categories:

a Numerical approximation versus exact calculation;
b Entering expressions;
c Equivalence of expressions.

A second coding round categorized all instances of obstacles into one of these three categories. Only a limited number of observations did not fit into one of these categories. In the next subsections we elaborate on each of these categories. The relation between the computer algebra environment and the paper-and-pencil environment, which caused most of the equivalence issues, is elaborated in Section 10.6. A broader inventory of obstacles that students encounter while working in a computer algebra environment can be found in Drijvers (2000, 2002b).

### 10.5.2 *Numerical approximation versus exact calculation*

The first category of obstacles that students encountered while working in the computer algebra environment concerns the difference between numerical approximations and exact calculations, and the implicit way the CAS often deals with this. Three kinds of observations are presented on this issue: the students' preference for decimal approximations, their understanding of exactness, and the way the TI-89 deals with approximations and exact calculations.
One of the capacities of computer algebra tools is the ability to deal with exact numerical values such as fractions and square roots. However, the students often perceived such numerical results as not informative; instead, they had a strong preference for approximated, decimal results, which is also what their regular calculator

provides. For example, one of the students complained when the result of entering 100/6 was 50/3. He commented: 'But I want the result.'
Such observations are related to the expected-answer obstacle and the lack of closure obstacle that were described in Section 3.4: the students did not see a fraction as an outcome, but as an instruction to carry out a division process. They expected a decimal answer that provides information on the order of magnitude. In the next two observations, the task was to enter 8/15 and to calculate the decimal approximation, which is 0.53.

| | |
|---|---|
| *Observer:* | *Probably you find 0.53 a more real answer?* |
| *Marg:* | *Yes, that is simply more clear, this [8/15] is like if it is still a task. (...) That is a little bit nothing-like.* |

(G9-II-1/5, assignment ZT1.1)

| | |
|---|---|
| *Observer:* | *Do you understand the difference between that 8/15 and that 0.53333?* |
| *Nick:* | *Yes, that [8/15] is a fraction, so to say 8 divided by 15.* |
| *Observer:* | *Yes, and the other one?* |
| *Nick:* | *That is the results of 8 divided by 15.* |

(G9-II-1/4, assignment ZT1.1)

These observation also suggest a relation with the process-object issue: for the students, exact answers signify a process or calculation. In contrast, they perceive an approximation as a result. The fraction is not seen as an object on its own. The same holds for results that contain square root signs. In the next protocol, the results of factor ($x^2$–2, $x$) and factor($x^2$–2.0, $x$) were compared.

| | |
|---|---|
| *Teacher:* | *What's the difference between that 2 and the 2.0?* |
| *Student:* | *For the first it gives a square root function, and for the other the outcome of that square root.* |

(G9-I-A4, assignment 4.7)

On another assignment, one of the students got $\sqrt{5} + 1$ as a result. She found this a strange results, because she 'did not see much in it'. In fact, she is right in that $\sqrt{5} + 1$ still 'contains some algebra' and is harder to imagine than a decimal approximation that indicates the order of magnitude. That explains why students are reluctant to process the exact answer further. The advantage of working with exact results is that the original input can be better recognized in the results. On the other hand, decimal approximations are meaningful in concrete problem situations and in graphs.
A second issue concerns the misconceptions that students have about exactness. In particular, some students confused exactness with precision: they considered an approximated answer with many decimals as 'more exact' than the exact value. In the next protocol, the results of solve($x^2$=2, $x$) and solve($x^2$=2.0, $x$) were compared.

*Teacher:* *And then that 2 point 0, who remembers the difference?*
*Dirk:* *Then you can calculate it exactly.*
*Teacher:* *For the 2 point 0 or the 2?*
*Dirk:* *2 point 0.*
*Teacher:* *Why, (...) What do you think is more exact?*
*Dirk:* *2 point 0.*
(G9-II-va5, assignment 3.6)

An important factor in the confusion between exactness and precision might be the different approaches in mathematics on the one hand and in physics and science on the other hand. In mathematics exactness is stressed, whereas in science and physics the number of significant digits and the precision of the results are the primary focus.
The third issue concerns the way the TI-89 deals with exact answers and decimal approximations. The TI-89 has three settings for numerical calculations: the exact, the approximate and the auto mode. In exact mode, fractions such as $3/2$ are not approximated. However, the approximate key of the TI-89, [≈], in this case gives 1.5, which is not an approximation but an exact value in the eyes of one of the students: 'Why does it say approximated and not just equal?' In exact mode, 1.5 is simplified as 3/2, and 3.14 as 157/50. The fractions that emerge this way can be confusing to the students.
In approximate mode, the TI-89 approximates all values. As a result, the formulas in some cases become harder to overview. For example, in the traffic flow problem, $l = 4$ and $a = 0{,}0075 \cdot v^2$ are substituted into the flow formula $f = 100 \cdot v/(6 \cdot (l + a))$ In approximate mode, the decimal value in the expression for $a$ results in $f = 2222,22222222 \cdot v/(v^2 + 533,333333333)$, which gives no clue as to how these numbers are related to the input values.
In auto mode, the TI-89 calculates with exact values such as 3/2 or $\sqrt{2}$ if they are entered as such, whereas decimal numbers remain unchanged. Because of the demands of science and physics, many students set their machine to approximate mode, thereby losing the exact calculations that we appreciate so much in mathematics.
The CAS settings offer opportunities for paying attention to this issue: whole-class discussions focused on the effects of the different settings of the TI-89 may encourage the development of an insight into the different kinds of calculations. Such a collective instrumentation was observed in the teaching experiments only to a limited extent; we conjecture that more attention to this would have been worthwhile.

### 10.5.3 *Entering expressions*

The second category of obstacles that students encounter while working in the computer algebra environment concerns entering formulas and expressions. We distinguish two subcategories, the first dealing with the structure of the formulas that are entered, and the second with the notations and the conventions of the CAS.

Entering formulas and expressions in the computer algebra environment requires insight into their structure. In Section 9.7.2 examples were given of how the TI-89 unidimensional line editor requires entering formulas using parentheses in order to adequately deal with rational expressions, square roots and exponents. In the case of the traffic flow problem, for example, some of the students entered the flow function without the appropriate pairs of parentheses, and got:

$$f = \frac{100 \cdot v}{6} \cdot (l + a) \text{ rather than } f = \frac{100 \cdot v}{6 \cdot (l + a)}.$$

Our observations with students working with TI-Interactive, which provides bidimensional entering of formulas, show that this did not solve all problems. Students still had to realize the scope of the square root signs and of the exponents. Observed errors included entering $\sqrt{x^2} - 3$ rather than $\sqrt{x^2 - 3}$ and $\sqrt{a}^{2-x^2}$ rather than $\sqrt{a^2 - x^2}$.

Another example of difficulties with entering formulas concerns the James Bond assignment (see Fig. 8.8). The context led to a sub-expression that was entered as 3*√(3^2+x^2)/6 and was displayed in the output window as:

$$\frac{3 \cdot \sqrt{(3^2 + x^2)}}{6}, \text{ where the student expected } 3 \cdot \frac{\sqrt{(3^2 + x^2)}}{6}.$$

Recognizing the equivalence of such expressions is the topic of the next section.

Entering expressions and formulas using parentheses, and adequately dealing with rational expressions, square roots and exponents, required insight into the structure of the formulas, which was identified as an important aspect of symbol sense. This work in the computer algebra environment, however, also supported the development of this insight.

The notations and the conventions of the CAS, which often differ from the paper-and-pencil notations that the students are used to, can be obstacles as well. Such obstacles are specific to the CAS; however, similar obstacles were observed in other computer algebra environments as well.

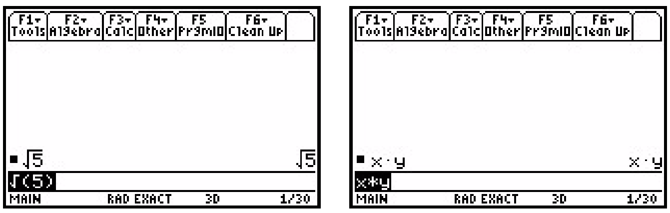



Figure 10.7 Entering square roots and products on the TI-89

A first notational issue concerns parentheses once more. The TI-89 in some cases

generated parentheses that the students did not expect. For example, pressing the square root key √ leads to square root and an opening bracket √( in the editor line. After adding, for example, the value 5, a closing bracket is required, but the output shows √5 without parentheses (see left-hand screen of Fig. 10.7).
A second notational difficulty is shown in the right-hand screen of Fig. 10.7 and concerns multiplication. In default mode, the product of $x$ and $y$ has to be entered into the TI-89 as $x$×$y$ rather than $xy$, because the latter is interpreted as a single two-digit variable name. Furthermore, the $x$ and the × key should not be confused; $x*y$ appears in the input line, whereas the output screen shows $x \cdot y$ . Such notational inconveniences were obstacles to productive CAS use for the students.
Third, in the Netherlands the comma is used for decimal digits, whereas the TI-89 uses the decimal point. On some occasions this confused the students. For example, in the solve command some students wanted to solve the equation $x^2 = 3{,}0$ (Dutch notation) by entering solve($x^2 = 3{,}0$), which is interpreted by the machine as a request to solve with respect to 0.
Fourth, the notation of the vertical bar for substitution was mixed up with the slash / for division. An example of this confusion can be found in Section 10.3.3.
Fifth and final, the meaning of the equal sign = is different. While working with paper and pencil, the equal sign is used for equations, definitions, identities and equivalences. Like most computer algebra environments, the TI-89 and TI-Interactive use := rather than = for definitions. As a result, the students in some instances entered definitions as equations and got unexpected results.
In the described notational obstacles we recognize the phenomenon of pseudo-transparency of the CAS (see Chapter 5): the notations and conventions were close to what the students know from their paper-and-pencil experience, but not exactly the same. Minor differences may have important consequences, and become serious obstacles to the progress of the work. Students' insight often lacks the subtlety to understand these differences and to deal with them adequately. Furthermore, the notational obstacles show a combination of technical and conceptual aspects. For example, if one does not understand the difference between an equation and a definition, it is hard to deal with the technical difficulty of the difference between the = sign and the := sign. The relation between the work in the computer algebra environment and the paper-and-pencil background of the students is elaborated on in Section 10.6.

### 10.5.4 *Equivalence of expressions*

The representation of the CAS output $\frac{3 \cdot \sqrt{(3^2 + x^2)}}{6}$ in the previous section already raised the issue of interpreting results, and of dealing with equivalent expressions in particular. Recognizing equivalence is an important skill when working in a computer algebra environment, as the CAS often represents the result in another way than the user would expect or would perceive as the simplest form. The inability to recognize equivalence in the experiments was an obstacle in many instances.

The first fragment below shows how Rob, while working on the general sum-difference problem, needed a numerical example to see that $s/2$ and $\frac{1}{2}s$ are equivalent:

*Rob:* *s, oh no 1/2 s, s divided by 2, is that the same?*
*Observer:* *Is s/2 the same as 1/2 s?*
*Rob:* *Well anyway here it is the same.*
*Observer:* *Yes. But you can understand that, I think?*
*Rob:* *s/2 is 1/2 s.*
*Observer:* *Yes. Why then?*
*Rob:* *10/2 = 5 and 5 is one half, is half of 10.*
(G9-II-7/16, task 6.2)

| **Entered in the TI-89** | **(part of) TI-89 result** | **Expected form** | **Task** |
|---|---|---|---|
| solve($b$*$h$=7 and $b$+$h$=6, $b$) | $-(\sqrt{2}-3)$ | $3-\sqrt{2}$ | Task 7.2 |
| factor($x$^2+3$x$+1, $x$) | $\frac{(2x+\sqrt{5}+3)\cdot(2x-\sqrt{5}+3)}{4}$ | $\left(\frac{1}{2}x+\frac{1}{4}\sqrt{5}+\frac{3}{4}\right)\cdot\left(\frac{1}{2}x-\frac{1}{4}\sqrt{5}+\frac{3}{4}\right)$ | Task 3.1 |
| solve($x$+$y$=$s$ \| $y$=$x$+$v$, $x$) | $x = \frac{s+v}{2}$ | $x = \frac{1}{2}s+\frac{1}{2}v$ | Task 5.4 |
| (1-$x$)^2 | $(x-1)^2$ | $(1-x)^2$ | Task 8.7 |
| factor($x$^2-13/2*$x$-7/2, $x$) | $\frac{(x-7)\cdot(2x+1)}{2}$ | $(x-7)\cdot(x+\frac{1}{2})$ | Task 10.5 |
| $x$*(40 – $x$) | $-x\cdot(x-40)$ | $x\cdot(40-x)$ | Task ZT2 |

Table 10.1 Equivalent expressions?

Table 10.1 summarizes some more cases in which the equivalence of expressions was an obstacle. The following observation, corresponding to the third line of Table 10.1, shows how Maria understood algebraically that $b = (s+v)/2$ and $1/2s+1/2v$ are equivalent:

*Maria:* *In fact that comes down to the same thing.*
*Observer:* *Could you explain that?*
*Maria:* *Well, because this is in fact, because you divide by 2 it is in fact already one half, so that comes down to the same. This one [(s+v)/2] in fact is easier than that one.*
*Observer:* *Yeah, that depends.*
*Maria:* *So in fact I found already another formula for the same problem.*
*Observer:* *Yes, another formula but one that...*
*Maria:* *Comes down to the same thing.*
(G9-II-6/34, task 5.4)

The next fragment shows how the output from the CAS can lead to misconceptions.

It concerns the fourth line of Table 10.1, in which the TI-89 represents $(1-x)^2$ as $(x-1)^2$. The formula emerged as the area of two equilateral right-angled triangles with edges $1-x$. As a result, Maria thought that $1-x$ equals $x-1$:

*Maria:* *The one edge is x-1*
*Teacher:* *The one is 1-x.*
*Maria:* *Yes, then you can also say x-1?*
*Teacher:* *Is that so?*
*Maria:* *Well that's what my calculator did, so I though it was possible.*
*Student:* *That's something very special.*
*Maria:* *Yes I thought so, too, but my calculator did it.*
*The teacher fills in a numerical value to show that it is not correct.*
*Maria:* *Yes I do understand it, but my calculator solved it this way, yes, I have a very strange calculator.*
(G10-II-4/1, task 8.7)

This observation and similar ones indicate that students take the output from the machine very seriously and find it difficult to verify whether it is equivalent to what they expected or obtained in another way. These difficulties are caused by the expression ordering algorithm of the CAS. Checking the equivalence requires insight into the structure of the formulas; on the other hand, the differences between obtained and expected results may encourage a closer inspection of the formulas to try to see whether 'they come down to the same thing', as Maria put it. This can contribute to the development of symbol sense. Some students found an efficient, technical way to use the CAS for this verification: they set the obtained result equal to the expected one, and hoped that the TI-89 reaction would be 'true' (see Fig. 10.8).

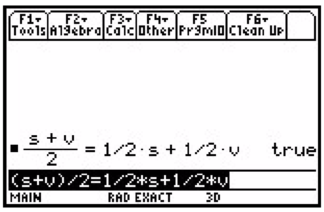



Figure 10.8 TI-89 verification of equivalence

The issue of recognizing equivalent forms is not new or specific for technology use, but also a well-known difficulty from the paper-and-pencil work that shows up when students found different but equivalent solutions. This was the case when the equation of lines through the point (3, 0) was asked for. Mike found the formula $y = a \cdot (x-3)$, but Sandra said she had a different one, $(x-3)/a$. Mike understood that these two formulas represented the same family of lines, although his explanation was not perfect: 'If you do $a$ times 0.1 for example then you get the same as when you do divided by 10.' What he seemed to be saying is that the value $a = 0,1$ in the first formula corresponds to $a = 10$ in the latter.

## 10.6 Instrumented CAS techniques and paper-and-pencil techniques

### 10.6.1 *Conclusions on instrumented CAS techniques and paper-and-pencil techniques*

An important aspect of instrumentation is the intertwining of technical and conceptual aspects within the instrumentation schemes. In the obstacles that students encountered during the instrumentation process, both aspects could often be recognized as well. The relation between techniques in a computer algebra environment and the students' mental conceptions was considered as crucial for appropriate instrumentation. However, the students in the teaching experiments already had a lot of experience with techniques in another medium: 'by-hand' techniques using paper and pencil. This was their main frame of reference, so it seemed appropriate to consider the relation between the new computer algebra techniques and the old paper-and-pencil techniques. The aim of this section, therefore, is to report on observations concerning the relationship between computer algebra use and traditional paper-and-pencil work in relation to the students' thinking. In short, to consider algebra on the screen, on paper and in the mind (Drijvers, 2002a, in press).

We conclude from the data that the paper-and-pencil techniques the students knew already did indeed interfere with the instrumentation of computer algebra. While considering the screen - paper - mind triangle, we observed the transfer of notations, techniques and language from the paper-and-pencil tradition towards the computer algebra environment. Obstacles to this transfer were the lack of congruence between the CAS techniques and the paper-and-pencil techniques, and the black box character of the CAS. Stressing the similarities and the differences between the work in the two environments and comparing the two approaches helped the students to translate the paper-and-pencil techniques into computer algebra techniques.

The transfer of computer algebra techniques to the paper-and-pencil environment was observed for the substitution of a parametric expression in the post-experiment interviews of the G9-II teaching experiment. These data support the idea that working in the computer algebra environment does not necessarily imply that by-hand skills will not develop or are neglected. Such a transfer did not take place for solving equations. We interpret this as support for the transparency condition: the substitution procedure on the TI-89 was far more transparent for the students than was the black box solve procedure, in that they felt able to 'look through' the way the CAS carried out the substitution and could relate this to their own experience with paper-and-pencil substitution.

The students differed in their preference for CAS work or paper-and-pencil work.

Sometimes they preferred the latter because it was more efficient, or because they felt they had more control over and understanding of what they were doing. The confidence in their skills in both media also played a role. The fact that the students had to hand in the machines after the experiment meant that it seemed somewhat risky to neglect by-hand skills and to rely too much on the machine. Finally, an important factor in the students' preference was the teacher's attitude to by-hand skills and machine skills, which is the topic of Section 10.7.

In the analysis of the data concerning the relation between computer algebra techniques and paper-and-pencil techniques, we followed a similar method as for the instrumentation schemes and the obstacles in the previous sections. All data concerning this issue were gathered and coded. Further analysis of these episodes revealed three categories concerning the relation between computer algebra technique and paper-and-pencil techniques:

a Transfer of paper-and-pencil techniques to the computer algebra environment;
b Transfer of computer algebra techniques to the paper-end-pencil environment;
c Student preferences for one of the two media.

A second round of coding categorized the relevant instances into one of these categories. Only a limited number of observations did not fit into one of them. In the next sections we elaborate on each of these categories.

### 10.6.2 *Transfer of paper-and-pencil techniques to the computer algebra environment*

In this section we describe how students translated a problem-solving method that they found natural from their paper-and-pencil work into the computer algebra environment.

One of the most salient examples of this concerns the ISS scheme. In Section 10.4.3 we described the variation on this scheme that we called isolate-isolate-equal-solve (IIES). For example, while solving the system of equations $x + y = 31$, $x^2 + y^2 = 25^2$ some of the students preferred to first isolate $y$ in both equations. That gave $y = 31 - x$ and $y = \sqrt{25^2 - x^2}$. The negative square root often was ignored. Then they set both right-hand sides equal and solved with respect to $x$: solve($31 - x = \sqrt{25^2 - x^2}$, $x$). Marty explained this scheme to the class by saying: '31 – $x$ is equal to the square root of 625 – $x^2$, and if you put these against another and then compare, you get the answer.' We conjecture that this IIES scheme comes closer to the paper-and-pencil technique than the ISS scheme, which in the nested form here would be solve($x^2 + y^2 = 25^2 \mid y = 31 - x$, $x$). In the paper-and-pencil work and in the textbooks, functions and graphs were often the points of departure. In these contexts, the graphing requires that one of the variables, usually $y$, is isolated. In that sense, the IIES scheme fits to the paper-and-pencil practice better than the ISS scheme does.

Transfer from paper-and-pencil technique to computer algebra technique is not evident. In Section 9.4.2 we described how Aisha was able to generalize the solution of the sum-difference problem $x + y = s$, $x - y = v$ mentally and by using paper-and-pencil. In her own words, her method consisted of 'adding the difference' to the two equations, which led to $2x = s + v$. However, she was unable to translate this approach into a TI-89 technique.

When a student did not understand the output of the CAS, and there seemed to be a lack of transfer between paper-and-pencil technique and CAS work, the question how it would be done with paper-and-pencil served as a cue. The next fragment, which was presented in Chapter 8, illustrates this. Ada entered $a \cdot b \cdot c \cdot a \cdot b / c$ and the results was $a^2 \cdot b^2$.

| | |
|---|---|
| *Observer:* | *Do you understand what the machine does when it simplifies this [a*b*c*a*b/c] into this [$a^2$*$b^2$]?* |
| *Ada:* | *No, not really.* |
| *Observer:* | *What would you do if you wrote down the left part and tried to simplify it without that machine?* |
| *Ada:* | *Well yes, then I would just, in fact I wouldn't use b square.* |
| *Observer:* | *But if you look at this, a*b*c*a*b/c, could you simplify that yourself if you look at it?* |
| *Ada:* | *In the end I would have divided by c, but not that the c disappears, oh yes, c divided by c so disappears, yes it is right.* |
| *Observer:* | *OK so you would skip the c?* |
| *Ada:* | *And then the rest, a squared times b squared.* |

(G10-II-v1, assignment 1.3)

While translating paper-and-pen procedures into the computer algebra environment, the students encountered the notational obstacles we described in Section 10.5.3, such as incongruence concerning entering fractions, square roots, exponents, using parentheses, and differences between the two minus signs. This lack of congruence complicated experiencing the similarities of the corresponding techniques in both media.

In one case, one of the students tried to make the TI-89 notation more congruent with the paper-and-pencil notation. She entered the square root sign √ and got it with an opening bracket √(. Using the backspace key, she deleted the bracket. When asked why she did that, she replied: 'That bracket is not in the book either.'

### 10.6.3 *Transfer of computer algebra techniques to the paper-end-pencil environment*

Observations concerning the transfer of computer algebra techniques to the paper-and-pencil environment are relevant, because they may help to answer the question whether the use of computer algebra comes at the expense of diminishing by-hand skills. From this perspective, the results of the post-experiment interviews of the G9-

II teaching experiments were interesting.
As described in Section 7.4.6, after the teaching experiment we interviewed nineteen students, who had been followed during the experiment. One of the questions in these final interviews was a context-free substitution task (see Fig. 10.9). All observations in this section refer to these interviews on this assignment.
This assignment was also part of the pretest, and the pretest results were not good: hardly any of the students managed to deal with the assignment. Because the students had already handed in the TI-89 by the time of the post-experiment interview, they were forced to use the traditional paper-and-pencil approach.

| | |
|---|---|
| **2** | Consider the equations $y = a - x$<br>$x^2 + y^2 = 10$ .<br>Make one new equation from these in which $y$ does not appear.<br>You do not need to solve this new equation! |

Figure 10.9 Second assignment from G9-II post-experiment interviews

Of these nineteen students, fourteen found the correct answer $x^2 + (a - x)^2 = 10$ or, in one instance, $a - x = \sqrt{10 - x^2}$ . Some of these fourteen students first made an error that they corrected after getting a hint from the interviewer. Four other students performed the substitution but made an error that was not corrected by the interviewer. The answers these four students gave were:

(1) $x^2 + a^2 - x^2 = 10$

(2) $x^2 + a - x = 10$

(3) $a - x = 10 - x^2$

(4) $a - x = \sqrt{}(10 * x)$

In answer (1) the student squared both $a$ and $x$ rather than squaring their difference. In answer (2) the student forgot the squaring. In answer (3), the student missed the square root on the right-hand side, and in (4) the student multiplied 10 and $x$ while isolating $y$ rather than subtracting $x^2$ from 10. However, despite these errors, all four students showed some understanding of the principle of substitution. Only one student out of nineteen did not arrive at a sensible answer.
The following fragment of the interview with Donald is illustrative. After some struggling with the task, Donald carried out the substitution correctly and gave an adequate explanation.

*Interviewer:* *You understand the problem, don't you?*
*Donald:* *Yes, I don't know how to do it without calculator.*

*Interviewer:* *How would you do it then with your calculator?*
*Donald:* *Well, ehm, solve, and then, let me think, no, ehm, yes, I think solve, doesn't contain y,*
*Interviewer:* *Ehm…*
*Donald:* *Gosh, ehm, maybe no, wait a moment, let me have a look.*
*Donald writes down:* $x^2 + (a – x)^2 = 10$
*Interviewer:* *Wonderful, how do you get this?*
*Donald:* *I was thinking too hard. Here is already a y and here it says what y is, so you just have to replace this with that. I was just thinking too hard.*
*Interviewer:* *Right. And do you now know how you would do it with the calculator? Of course, you could type it in like this.*
*Donald:* *Then I would do this… with y = a – x.*
*Donald adds to* $x^2 + y^2 = 10$ *on the sheet of paper:* $| y = a – x$.

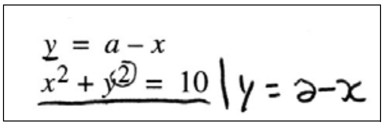



Figure 10.10 Donald's transfer of notation

Fig. 10.10 shows that Donald wrote down the vertical bar notation for substitution, which is a transfer of notation from the CAS to paper and pencil. Another example of transfer from computer algebra technique to paper-and-pencil work is the way Kevin looked at the equation $y = a – x$. In his reasoning he made use of TI-89 language, but he also related the machine technique to his understanding. What is remarkable is the fact that he converted the first isolated equation into the symmetrical, non-isolated form, like he encountered them often in the TI-89 environment.

*Kevin:* *Well, I think x + y = a, doesn't it?*
*Interviewer:* *Yes.*
*Kevin:* *But in fact this is, this is already solved. Because if you do solve, x+y = a, comma y with solve, then you get this.*
*[points at y = a – x on the paper]*

As we described in Section 10.6.2, Marty preferred the IIES scheme of double isolation. By mistake, during the post-experiment interview he isolated $y$ in the second equation by hand as follows: $y = \sqrt{}(x*10)$. His equation then was $a – x = \sqrt{}(x*10)$, but he did not feel sure about the right-hand side. The interviewer asked him how he would check this if he had the calculator in his hands. He said he would use solve. On his sheet of paper he changed $x^2 + y^2 = 10$ into solve($x^2 + y^2 = 10, y$)

and $a - x = \sqrt{(x*10)}$ into solve($a - x = \sqrt{(x*10)}$, $x$).
As in the previous section, we saw that asking students how they would do it by hand could help them to understand the CAS technique, the inverse question 'how can you do that with the CAS' helped Marty to clarify his paper-and-pencil strategy.
To summarize this section, the data suggest that the transfer of computer algebra techniques to the paper-and-pencil environment does take place, including the corresponding notations and language. Although it could not be observed frequently, because the CAS was permanently available during the teaching experiments, the data from the post-experiment interview of G9-II in particular provide evidence for the conclusion that such a transfer can be effective. We interpret this outcome as modest evidence that working on substitution in the computer algebra environment improved the students' understanding so that transfer to the paper-and-pencil environment could take place.

### 10.6.4 *Student preferences for one of the two media*

The last issue concerning the relation between computer algebra techniques and paper-and-pencil techniques is the students' preference for one of the two media.

In some cases students preferred to do things mentally or to use paper-and-pencil methods rather than use CAS techniques. For example, in Section 10.4.3 we noticed that in simple cases students carried out the isolation step of the ISS scheme mentally. The next fragment shows how Cindy did that for $b \cdot h = p$ and immediately substituted the isolated form into the other equation, $b + h = s$. Robert needed an explanation.

> *Cindy enters: solve(b+h=s | b = p/h, h)*
> *Robert:* *Why divided by h?*
> *Cindy:* *We found the formula of b times h is p, and then if you want to isolate b then you have to divide p by h.*
> *Teacher:* *So she isolated b mentally.*
> (G9-I-A12, assignment 4.6)

This is a matter of efficiency: if you do it faster mentally, then why use the machine? Some students also preferred to work mentally or with paper-and-pencil in more complex cases, such as solving the equation $(x - 5) \cdot (x + 1) = 7$. Probably they felt that they knew better what they were doing in this way. For example, Maria commented: 'I can always see it easier mentally than with the calculator, because then I always forget what I'm doing'. This links up with another remark that she made earlier concerning graphing with the TI-89:

> *Maria:* *Yes but it is with that calculator, yes I think it is much more clumsy, because normally I understand that very well but such a formula, yeah, I don't see so much in it*

*if I just enter it into the calculator and it draws the graph.*
(G9-II-5/11, assignment 3.1)

We conjecture that the preference for paper-and-pencil and mental work is caused by the black box character of the computer algebra environment that elicits the students' feeling of not understanding what is happening. Working in the 'traditional' manner gives them a firmer grip on the process. This links up with findings from previous studies. For example, the following fragment reveals a similar attitude:

*Esther solved an optimization problem graphically.*
*Observer:* *It can also be done with differentiation.*
*Esther:* *But I cannot differentiate this function yet.*
*[She means she cannot differentiate it manually]*
*Observer:* *But the machine can.*
*Esther:* *Yeah, but then you don't know what you're doing!*
(Drijvers, 2000, p. 206)

The preference for the paper-and-pencil method because otherwise 'you don't know what you're doing' shows a striving to understand what is happening and to control the problem-solving process. The above quotations show that students sometimes miss this in the CAS work. Apparently they want the 'black boxes to become white'. This tendency indicates a critical mathematical attitude that we appreciate and not like to discourage.
In addition, students in some cases seemed to use the by-hand method because they did not realize that the CAS could do the work for them. For example, they solved an equation with paper and pencil because they still were not used to the idea that the CAS could take over this task. The risk of this preference for paper-and-pencil work is the occurrence of elementary algebraic errors that hinder the progress, such as the mistakes summed up in table 9.4.
Other students, or the same students in other situations, preferred using the CAS to the paper-and-pencil method. For example, Maria used the TI-89 to solve the lens equation $1/5 = 1/v + 1/b$ with respect to *b*. After the observer asked if she could do it by hand as well, she answered: 'Well, I have the calculator for that. I always struggle with the machine so I'm happy I can use it for something.'
All together, the students had different preferences on the CAS / by hand issue. As was shown in Fig. 8.13, Helen was one of the students who preferred to work with paper and pencil. The following dialogue took place after she had solved the general lens equation $1/f = 1/v + 1/b$ on the blackboard by hand. Linda supported Helen's approach, whereas Maria and Ada preferred the CAS method.

*The teacher explains Helen's solution and says it can also be done with solve on the TI-89.*

| | |
|---|---|
| *Linda:* | *But that's far too easy, isn't it? Then you have to know the result, or don't you? [She means that you should be able to solve it by hand]* |
| *Teacher:* | *No, that doesn't matter, you can do it with the solve command. It should be done by hand as well, but you're now working with the machine.* |
| *Maria:* | *Yes indeed.* |
| *Ada:* | *Solve, very easy.* |

(G10-II-3/34, assignment 8.3)

The students knew that they were using the TI-89 for a temporary project, and that after that they would have to be able to do the algebra by hand. Of course, this influenced their attitude concerning CAS use versus paper-and-pencil work. Furthermore, the demands of the teacher concerning the by-hand skills to a great extent determined this attitude. The role of the teacher in using the CAS or working in the paper-and-pencil environment is addressed in Section 10.7.3.

## 10.7 Orchestration of instrumentation by the teacher

### 10.7.1 *Conclusions on the orchestration of instrumentation by the teacher*

Many research studies stress the role of the teacher in the instrumental genesis of technological tools (Doerr & Zangor, 2000; Guin & Trouche, 1999, 2002; Kendall & Stacey, 1999, 2001). The teacher is the exemplary user of the tool, and orchestrates the instrumentation of the technology by means of individual interactions, classroom discussions and demonstrations.

Although the role of the teacher in the instrumentation was not a primary focus of this study, the data suggest that the way in which the teacher orchestrates the instrumentation influences the practice that is developed by the students. This links up with the findings of the studies mentioned above. Furthermore, the integration of computer algebra requires a rethinking of the relation between carrying out algebraic operations by hand and leaving this to the computer algebra environment. This results in a new didactical contract (Brousseau, 1997). Reflecting on the teaching experiments of this study, we reconstruct this new contract as one that stressed the combination of the CAS approach and the paper-and-pencil approach, so as to consider the differences and similarities of the techniques in the two media. This was instructive for the students, but using the computer algebra environment in addition to the paper-and-pencil work is time-consuming and does not relieve the students. Looking back we regret that we did not communicate this new didactical contract more clearly to the teachers and the students.

Two issues are worth discussing in more detail. As an example of the teachers' orchestration of computer algebra use, we will first describe how the instrumentation of the ISS scheme was influenced by the teachers of the ninth grade classes (10.7.2). Second, we will address the didactical contract that was not clear to the teacher and the students in the tenth-grade G10-II teaching experiment (10.7.3).

### 10.7.2 *Orchestration of the instrumentation of the ISS scheme*

We described the isolate-substitute-solve scheme (ISS, see Fig. 10.5) in Section 10.4. The students appeared to develop several variations of the proposed scheme. The popularity of these variations was largely influenced by the way the teacher dealt with them. As an example of teachers influencing the instrumentation of CAS techniques, we will now compare the orchestration of the ISS scheme in classes A and B of the G9-I teaching experiment.

In class A, the first context of the ISS scheme was the sum-difference problem, in which two numbers with given sum and difference had to be found. The teacher immediately used the nested form that integrates the substitution and the solution into one combined command. She wrote on the blackboard:

*$x + y = s, \rightarrow y = s - x$, isolate*
*$x - y = s$, substitute*
*TI-89 solve: solve($x - y = v \mid y = s - x, x$)*
*fair share:*

$$x = \frac{1}{2}s + \frac{1}{2}v \quad \frac{s+v}{2}$$

$$y = \frac{1}{2}s - \frac{1}{2}v \quad \frac{s-v}{2}$$

In this approach we appreciate the attention paid to the equivalent forms of the solution formulas. However, the nested form is complex because of its compactness, and led to errors such as substituting non-isolated forms or solving with respect to the wrong unknown (see Section 10.4.3.).

In class B things went quite differently. First, while working for themselves some of the students discovered the option to solve systems of equations using 'and'. For example, solve($x + y = 700$ and $x - y = 600$, $x$) solved the system directly. One of the students called this a 'double solve'. This simultaneous method soon spread throughout the class. In a classroom discussion, the observer acting as a guest teacher demonstrated the step-wise method. The teacher followed this approach in the subsequent lessons.

For example, while solving the system of equations $x^2 + y^2 = 625$, $x + y = 31$ she demonstrated the step-wise method on the TI-89 view screen as follows:

*Entered:* $x^2 + y^2 = 625 \mid y = 31 - x$
*This yields:* $2x^2 - 62 \cdot x + 961 = 625$

*Entered:* *Solve*$(2x^2 - 62 \cdot x + 961 = 625, x)$
*This yields:* $x = 24$ *or* $x = 7$
(G9-I-B15, assignment 5.4)

The different approaches of the teachers led to a different instrumentation of the ISS scheme, as is reflected in the handed-in results of the right-angled triangle assignment (see Fig. 10.4). The distribution of methods in Table 10.2 shows differences between the two classes. The nested form is popular in class A, whereas the step-wise method and the simultaneous method are preferred in class B. This indicates that the teacher's approach led to standardization and collective instrumentation. This is in line with the findings of Doerr and Zangor (2000), who argued that the orchestration by the teacher is very influential, and of Kendall and Stacey (1999, 2001), who stated that teachers privilege specific techniques and schemes, and dis-privilege others.

| Solution method | Class A | Class B |
|---|---|---|
| ISS step-wise | | 6 |
| ISS nested form correct | 15 | 2 |
| ISS nested form incorrect | 1 | 1 |
| ISS simultaneous form correct | | 8 |
| Divers methods, partially correct | 4 | 2 |

Table 10.2 Different methods for carrying out the ISS scheme in task 5.4

In the next teaching experiment in ninth grade, G9-II, the differences between the two classes were not so obvious for several reasons. First, the class A teacher was more careful in introducing the nested method than she had been the year before, because of the difficulties that had shown up. Second, the teaching materials contained more assignments that could not be solved with the simultaneous method. As a result, the proposed step-wise method was used more often than in G9-I and the classes did not differ so much. When students changed to the nested form after mastering the step-wise method, errors were less frequent than when they started with the nested form, probably due to an increased insight acquired by the step-wise method.

After solving the system of equations $x^2 + y^2 = 625$, $x + y = 31$ with the ISS scheme using the TI-89, the teacher encouraged the use of the machine and told the students that after the experiment they would have to do it with paper-and-pencil again. When one of the students asked 'Why do we need to do it by hand then?' the teacher replied 'Because by then you will have handed in the calculator.'

This fragment raises the issue of the didactical contract concerning computer algebra use and the paper-and-pencil work, which is the issue of the next section.

### 10.7.3 *Teachers' influence on the relation between computer algebra use and paper-and-pencil work*

The use of a computer algebra machine in a mathematics course raises the question what operations and procedures should be mastered by hand. As the teacher sets the norms, the students watched the teacher's attitude very closely: what is the new didactical contract concerning by-hand skills? In fact this was not so clear in the teaching experiments. On the one hand, the teachers felt that the students should use the CAS as it was an important aspect of the experimental setting. On the other hand, after the experiment the students would be thrown back on their by-hand algebraic skills. Therefore, it was sometimes recommended to carry out procedures with paper and pencil as well.

For example, in the G10-II teaching experiment one of the tasks consisted of finding the coordinates of the intersection points of the line with equation $y = a \cdot x - 2$ and the parabola with equation $y = x^2 - 4x + 5$. On that occasion the teacher said it could be done by hand, but that it was better to do it with the machine now, as this was a calculator experiment. On the other hand, teacher and observer often asked the students who found a result in the computer algebra environment whether they would be able to find it by hand as well. The reason for this question was to avoid the risk of getting too dependent on a machine that would be gone within a few weeks, as well as the idea that comparing the two methods and experiencing their congruence/incongruence would increase the insight into the procedure.

All in all, the attitude of teachers and observers concerning carrying out the operations by hand in the paper-and-pencil environment versus leaving it to the CAS was ambiguous; as a result, the didactical contract was not clear to the students, who were aware of the temporary character of the teaching experiments and the availability of the technology. The final observation of Section 10.6.4 shows this ambiguity in the teacher' reaction, and the mixed feelings of the students afterwards.

## 10.8 Conclusions on the instrumentation of computer algebra

In the final section of this chapter, we first summarize and discuss the results concerning the instrumentation of computer algebra. Then we reflect on the role of the theoretical framework that was described in Chapters 3 and 4. Finally, some pedagogic consequences are formulated in the form of recommendations to teachers.

### 10.8.1 *Summary and discussion*

The central question in this chapter is the second research subquestion:

> *What is the relation between instrumented techniques in the computer algebra environment and mathematical concepts, as it is established during the instrumentation process?*

In Chapter 5, we defined instrumented techniques and schemes of instrumented action as mental schemes for solving specific types of task by means of sequences of actions in a technological environment. Characteristic of such schemes is the interplay between mathematical conceptual understanding and technical skills for carrying out procedures in the technological environment.
The conclusion of this chapter is that there does indeed exist a close and reciprocal relation between CAS techniques and conceptual understanding. Furthermore, the paper-and-pencil techniques were identified as a third element involved in this relationship. We found differences between the instrumental genesis of simple schemes and composed schemes, and identified conceptual obstacles that hindered the instrumentation as well as technical obstacles that sometimes were related to the limitations of the CAS interface. The teacher seemed to play an important role in the orchestration of the instrumentation.
While the students worked with the TI-89 they developed instrumentation schemes of different complexity. For the schemes that were investigated in particular – the solve scheme, the substitute scheme and the ISS scheme – the data showed that technical and conceptual aspects were integrated and developed interactively. Limited conceptual insight hindered the instrumentation, and the instrumental genesis fostered the conceptual development. For example, the solve instrumentation scheme required the extension of the idea of solving for 'expressing one variable in terms of others' and the identification of the unknown with respect to which the equation was solved. For the substitute scheme, the conceptual development included the understanding that only isolated forms could be substituted.

We found a difference between simple instrumentation schemes such as the solve scheme and the substitute scheme, and composed instrumentation schemes such as the ISS scheme. The instrumental genesis of the last-mentioned scheme showed the above-mentioned difficulties for the component schemes of solving and substituting to a greater extent than was observed when the simple schemes were carried out on their own. The integration of the partial schemes into a composed scheme led to an increasing number of errors. We conclude that such an integration sets high demands on the instrumentation of the component schemes. Furthermore, in the comprehensive scheme the students sometimes lost track of the global problem-solving strategy, especially when the problem situation, the formulas and the expressions became more complex.

A close relation between the instrumented techniques in the CAS and the understanding of the mathematical concepts was assessed. A third element involved in this relation was the paper-and-pencil practice that the students were familiar with. Incongruence between computer algebra technique and paper-and-pencil technique caused instrumentation problems. When the techniques in the two media were congruent, transfer of notation, strategy and technique took place. A second condition for transfer is the transparency of the computer algebra technique: if it is not transparent, but rather a black box, the student would be unable to link it with the paper-and-pencil technique and with the mathematical concepts he has in mind. We recommend to regularly ask the students to compare the computer algebra technique with the paper-and-pencil method. Whole-class discussions of both kinds of techniques may clarify instrumentation difficulties. In short, we plea for the integration of algebra on the screen, on paper and in the mind.

Often the instrumentation in the teaching experiments did not take place in a smooth way. We identified a number of obstacles that hindered the instrumentation of algebraic techniques that were related to the primary focus of the study, viz. the understanding of the concept of parameter. In addition to this, some more general obstacles were defined that limited the effectiveness of the work in the computer algebra environment, such as the understanding of the difference between numerical and exact calculations and the symbol sense needed to enter expressions and to recognize equivalent formulas. There are several reasons to pay attention to these obstacles. First, obstacles can discourage students. Second, obstacles can strongly delay the instrumentation and make the lessons ineffective. We have observed students spending twenty minutes on one minor obstacle. The third and main reason to pay attention to obstacles is that they offer opportunities for learning (Drijvers, 2002b). Many of the obstacles are existing cognitive obstacles that are simply made more manifest in the computer algebra environment, and not created by it. Working on understanding and overcoming an obstacle also means making computer algebra work more effective, and improving the conceptual development of the mathematical concepts behind the obstacle. Such activities as investigating what the problem really is, finding out what the 'logic' of a specific syntax is, discovering the meaning of the output and inventing a new strategy that is more feasible in the computer algebra environment offer opportunities for a better understanding, an improved conceptual development and a good mathematical attitude. In that sense, obstacles are opportunities for mathematics education.
The students' instrumentation and their attitudes to working in the computer algebra environment are influenced by the teacher. Different orchestration of techniques led to different collective instrumentation and classroom practices. Classroom demonstrations and whole-class discussions are recommended in order to achieve collective instrumentation. Furthermore, the didactical contract concerning the relation between by-hand work and machine work was affected, but it was not clearly stated how; teachers, observers and students struggled with the question what part of the procedural work could be left to the CAS and what had to be done by hand as well.

### 10.8.2 *Reflection on the theoretical framework*

The main elements of the theoretical framework of the study for this chapter are the theory of instrumentation, the instruction theory of Realistic Mathematics Education and theories on symbol sense and symbolizing. In this section we reflect on the relation between these theoretical elements and the findings of this chapter.

The central issue in the *theory of instrumentation* is the instrumental genesis, which consists of building up simple and composed instrumentation schemes. Such schemes have technical and conceptual aspects that are closely related.

The findings of this chapter show that the perspective of instrumentation was valuable for interpreting the observations. Indeed, this chapter describes the building up of some instrumentation schemes in detail. A technical and a conceptual aspect could be distinguished in many of the problems that the students encountered while doing so. Conceptual limitations could hinder the instrumental genesis, but the CAS requirements for building up the schemes could elicit a further conceptual development as well, such as the extension of the understanding of solution and substitution. These findings do not suggest modifications or extensions of the instrumentation theory; rather, they show its usefulness for understanding the student - machine relation, and they translate the theory to the concrete practise of classroom observations. For a successful instrumentation, the transparency of the computer algebra tool turned out to be crucial, as did the congruence of the computer algebra technique and the corresponding paper-and-pencil technique. If congruence is lacking, the absence of a link with the paper-and-pencil tradition may hinder instrumentation. If the computer algebra technique is not transparent, CAS use acquires a black box character and is no longer meaningful to the students because they are unable to 'look through' the way the CAS finds its results, and cannot relate these results to their own experience with paper-and-pencil techniques

The idea that procedures and concepts have to be meaningful for the students is one of the key issues in the *instruction theory of Realistic Mathematics Education*. As soon as the students are not able to understand what is happening within the computer algebra environment, or to give meaning to results and to the formulas that appear, the work acquires a black box character and no longer leads to 'a growing common sense', to quote Freudenthal (1991). This was observed for the simple solve instrumentation scheme. Students are confronted in a top-down manner with results that, to them, do not refer to a kind of reality. The connection with reality can be established by using real-life contexts, as was shown in Section 9.6. Relating the computer algebra work to paper-and-pencil techniques that the students are familiar with can also help. Computer algebra as a means for vertical mathematization only seems to work if the situation is experienced as meaningful by the student; the general level of the mathematical framework can only be built up from the referential level. The

opportunity to use computer algebra for addressing more complex and more realistic problem situations, as was suggested in Chapter 5, was exploited only to a limited extent in this study.
Like the theory of instrumentation, the notion of *symbol sense* was useful for explaining student behaviour in the computer algebra environment as well. In particular, entering formulas correctly into the CAS and interpreting the CAS output requires the ability to 'see through' the formulas and expressions, to understand their meaning and structure. On the other hand, the work in the computer algebra environment can contribute to the development of this kind of symbol sense, if the entering and interpretation of expressions and formulas is taken as an opportunity to analyse their structure. In this sense, the CAS is an appropriate environment for experimenting with algebraic expressions and formulas. It is up to the teacher to stimulate such a closer look into the formulas. The sometimes subtle differences in notation between the computer algebra environment and the paper-and-pencil work may generate obstacles to the development of this kind of symbol sense.

An important element in the *theories on symbolizing* is the signification process in which the student develops his/her own symbolizations, and improves and uses them in relation to the meaning he/she attaches to them. The observations presented in this chapter did not show much of this bottom-up process, because the computer algebra environment already offers a symbol system and is not very flexible concerning notation and syntax. This inhibits a free development of symbolizations at the level of ninth and tenth grade. For supporting the progressive development of notations and informal strategies the CAS is not the ideal medium for students at this level. However, the confrontation with a ready-made symbol system also involves the process of developing meaning while using. This links up with the technical and conceptual aspect of instrumentation. In that sense, theories on symbolization provided a background for understanding the growing sense for the symbols within the computer algebra environment.

### 10.8.3 *Consequences for teaching*

As a consequence for teaching, Section 10.8.1 suggests a pedagogic strategy of considering obstacles seriously, paying attention to them and taking advantage of the opportunities they offer. As Simon (1995) stated in a more general sense than our specific case of working in a computer algebra environment that:

> Conceptual difficulties that I have previously observed in students are not to be avoided; rather, they provide particular challenges, which if surmounted by the students, result in conceptual growth.
> (Simon, 1995, p. 139)

Rather than trying to smooth over the obstacles, we suggest making them the subject of classroom discussion in which the meaning of the techniques and the concepts is developed. The mathematical ideas behind the obstacles should be considered explicitly, and the computer algebra environment should be used as an inspiring object of study rather than an 'oracle'. Such an approach turns the obstacles to computer algebra use into opportunities for learning, and enriches mathematical discourse in the classroom.
If we aim at mathematics education that focuses on the development of a meaningful network of mathematical objects by using computer algebra, the following suggestions for teachers can be derived from the teaching experiments and may help to orchestrate the instrumentation.

- *The didactical contract*
  When starting with CAS use in the classroom, we recommend establishing a didactical contract concerning by-hand skills and computer algebra use and to discuss it with the students, so that they are aware of the new situation and of what is expected of them. Furthermore, new socio-math norms will need to be developed. For example, one of the norms might be that numerical or graphical answers should be explained by algebraic means. In that case, the teacher should use problems that can be solved and explained algebraically.
- *Technical and conceptual difficulties*
  We recommend paying attention to the difficulties students encounter while working in the computer algebra environment, and trying to identify technical and conceptual factors therein. This could be done during interactions with individual students. The main observations could be reviewed and the main problems discussed in classroom teaching. Obstacles should be addressed and made explicit rather than 'smoothed over'.
- *Comparing paper-and-pencil and CAS techniques*
  We suggest carrying out selected tasks by hand on the blackboard and also in the computer algebra environment using a data projector. While doing so, the students should be asked to compare both methods, so that they become aware of the transparency and congruence issues. However, the use of technology in this approach does not save time. To develop new CAS techniques, initial examples should not be too complex; on the other hand, they should not be so simple that they can be solved mentally at once.
- *Discussing the entering of commands and interpreting output*
  It is recommended to discuss with your students the logical and illogical aspects of entering commands into the CAS and interpreting the CAS output. Show them that perceived strange aspects may have to do with their paper-and-pencil habits, with peculiarities of the CAS or with limited conceptual understanding. Possibly address the way the CAS works internally. For example, ask students to develop

their own ‘machine language’: how could you explain unambiguously to a machine what you want it to do?

- *Discussing problem-solving strategies*
  In order to prevent students from losing track of the overall problem-solving strategy, review the line of the strategy and its rationale after it has been finished.
- *Demonstrations*
  Have students demonstrate their work in the computer algebra environment using a data projector, and discuss the methods with the class, while also considering alternatives. An other option is to have a ‘Sherpa student’ (Guin & Trouche, 1999) manipulating the computer algebra environment while the teacher explains. Use these demonstrations to develop collective instrumentation and machine routine.

# 11 Conclusions and discussion

## 11.1 Introduction

In this final chapter we discuss the detailed results that were presented in Chapters 9 and 10 from a more global perspective. The aim is to make a synthesis of these results and to reflect on the findings. First, we summarize the conclusions on the two research subquestions and derive the answer to the main research question (11.2).
Then, we compare these conclusions with our expectations at the start of this study. We reflect on the influence of the theoretical framework and discuss the methodology, as well as the generalizability of the results (11.3). The last Section, 11.4, contains recommendations for teaching mathematics using computer algebra, for designing computer algebra software for educational purpose and for further research on the use of technology in mathematics education.

## 11.2 Conclusions

In this section we first summarize the findings on the two research subquestions, the first one concerning the concept of parameter, the second the instrumentation of computer algebra. These results have been presented in more detail in Chapters 9 and 10. Then we infer an answer to the more general main research question on the contribution of computer algebra use to the learning of algebra.

*Conclusions concerning the concept of parameter*

The first research subquestion was: how can the use of computer algebra contribute to a higher level understanding of the concept of parameter? The results described in Chapter 9 led to a hypothetical learning trajectory for the concept of parameter that followed the line placeholder - changing quantity - generalizer - unknown. The students did acquire a higher level understanding of the parameter as changing quantity; insight into the parameter as generalizer and as unknown, however, was achieved only to a limited extent.
Computer algebra use supported the understanding of each of these parameter roles and the transitions to the higher parameter roles in various ways. Work in the computer algebra environment allowed for varying parameter values, for repeating problem-solving procedures, for calculating generic algebraic solutions and for shifting roles of the parameter. The last is relevant to the parameter as unknown. The use of the slider bar illustrated the parameter as changing quantity, and the link between graphical features and algebraic properties (Section 9.8).
In the computer algebra environment, the students encountered expressions as solutions to equations that were processed further by other procedures such as substitution. This stimulated the reification of algebraic formulas and expressions as well as overcoming the lack of closure obstacle. In some cases the work in the computer

algebra environment improved the students' insight into the structure of the – in some cases parametric – formulas and expressions and thus the students' symbol sense. A precondition for this to happen seemed to be that the formulas and expressions had to be meaningful to the students, for instance by means of a connection with a problem situation that was perceived as realistic. This shows that the original focus of this study – namely, to use computer algebra as a means of entering the general level of generalization – could not be achieved without paying attention to the referential level and to realistic problems as starting points.

The data also suggest limitations of the positive effects of computer algebra use. Instances of long periods of non-productive work in the computer algebra environment were observed, misconceptions in some cases turned out to be persistent, and the test results revealed a lack of technical skills and conceptual understanding. Although these modest results should be interpreted in the context of the relatively weak student cohorts involved and the teachers' lack of experience in the integration of technology, we conjecture that the following reasons explain the limited conceptual benefits:

- The type of tasks was new to the students. Their previous education had hardly prepared them for generalization and symbol sense.
- The teaching experiments were too short to allow the students to really get used to 'doing mathematics with a machine'.
- The ninth-grade level is early to start working with computer algebra, as the by-hand experience of the students is still limited.
- The hypothetical learning trajectory may have treated the different parameter roles too much as separated concepts. Integrating and mixing the parameter roles might have led to a more comprehensive understanding.
- The targeted general level was addressed too directly, at the expense of the referential level that provided the meaningful starting points. Some of the instructional activities were too general and too complex.
- The teachers had reservations about having whole-class reflective discussions and about being explicit about the didactical contract. A comparison of G9-II with G9-I shows that as the teacher gained experience, the teaching improved.
- Instrumentation and interface problems sometimes frustrated the learning process. This issue is addressed below.

### *Conclusions concerning the instrumentation of computer algebra*

The second research subquestion was: what is the relation between instrumented techniques in the computer algebra environment and mathematical concepts, as it is established during the instrumentation process? The overall conclusion from Chapter 10 is that there is a close relation between instrumented techniques, which are condensed into simple and composed instrumentation schemes, and conceptual understanding. The students can only understand the logic of a technical procedure from a conceptual background. What seemed to be technical difficulties were often observed to have a conceptual background, and the relation between technical and conceptual aspects made the instrumental genesis (the development of instrumentation schemes) a complex process. The examples of the solve scheme and the substitution scheme show in detail how the mixture of technical and conceptual aspects may have led to difficulties that were not easy for the students to overcome. The instrumentation of the scheme isolate-substitute-solve indicates that the integration of elementary instrumentation schemes into a composed scheme requires a high-level mastering of the component schemes. Evidence from the data suggests that congruence between instrumentation schemes and schemes that the students had developed within the paper-and-pencil environment is an important factor in the instrumental genesis. This concerned the overall problem-solving strategy as well as the notations, symbols and syntax used. The students encountered many obstacles during the instrumentation process that could be explained in terms of the technique - concept interaction and the incongruence between the CAS and the paper-and-pencil environment. We argue that such obstacles offer opportunities for learning that can be capitalized on by reflecting on their conceptual aspects and the relation with the corresponding paper-and-pencil technique. This requires orchestration of the instrumentation process by the teacher, attention to instrumentation issues in instructional activities, and time.

Two comments should be made on these conclusions. First, one can ask whether computer algebra is an appropriate tool for algebra education if the instrumentation requires so much effort. Indeed, the CAS has a top-down character in the sense that ‘everything is already there’, that no symbolizations have to be developed, and that it is not flexible with respect to notational or syntactic differences. The students’ work may suffer from a perceived lack of congruence with paper-and-pencil techniques or a lack of transparency, so that the CAS is seen as a black box. Although the flexibility of any IT tool is limited, this issue is striking for CAS use at this level of education. In Section 10.8 we argued that overcoming obstacles is a process that involves conceptual development, and that instrumentation includes a signification process of giving meaning to algebraic objects and procedures. Once the students have dealt with most obstacles, the CAS offers a wide scope of algebraic experimentation room. For the ninth- and tenth-grade students in our study we conclude that the balance between the costs of instrumentation and the benefits of algebraic insight was not positive for these relatively short teaching experiments; instrumentation was time-consuming. Although interesting issues came up during the instrumental genesis, the opportunities were not easy to capitalise on. We conjecture that integrating CAS into algebra education is better suited for longer periods, so that students would

be able to really get used to the tool, or in higher grades, when they have more algebraic experience.
The second comment concerns the embedding of CAS use in the educational setting. As for the first research subquestion, one can ask whether the results would have been better if the orchestration by the teacher had been different, if other activities had been designed, or if another computer algebra tool had been used. Of course, the educational context affected the instrumentation difficulties. For example, a better orchestration of instrumentation might have led to more reflection on the instrumented techniques. In the meantime, we argue that the educational setting does not change either the nature of the instrumentation process that students have to go through while working in a computer algebra environment, or the importance of this process.

### *Conclusions concerning algebra and computer algebra*

The main research question that was phrased in Chapter 1 was:

> *How can the use of computer algebra promote the understanding of algebraic concepts and operations?*

Through extrapolating answers to the two research subquestions to a more general level, we arrive at the following conclusions concerning the main research question. First, the findings concerning the concept of parameter suggest that the use of computer algebra can promote algebraic understanding, but that the effect depends on the didactic and educational settings. Computer algebra use can generate opportunities for learning, for gathering experiences, for making explicit algebraic issues and for developing conceptual insights. The contribution of the computer algebra work to learning, however, is small if it is embedded in education in which the computer algebra work is not related to paper-and-pencil work and in which whole-class discussion on the interpretation of CAS results and reflection on the consequences and the meaning of the findings are lacking.
Second, several of the benefits of the work in the computer algebra environment that were observed in this study go beyond the concept of parameter and address algebra in general. For example, the CAS work allows for combining different representations, and in particular for linking graphical and algebraic representations. In the computer algebra environment, operations and procedures can be easily repeated under slightly different or more general conditions, which stimulates the construction of a mathematical framework of algebraic relations and objects.
Symbol sense, insight into the structure of algebraic expressions and formulas, and the reification of formulas and expressions can benefit from computer algebra work as well: the confrontation with equivalent expressions stimulates a closer look at algebraic structures, and the submission of expressions to subsequent processes fos-

ters the perception of formulas as objects. On the other hand, a productive use of computer algebra already requires a certain level of symbol sense.

Third, the findings concerning the instrumentation exceed learning the concept of parameter as well. The instrumentation of computer algebra is an important issue. An incomplete instrumentation may hinder learning; on the other hand, the instrumental genesis often includes a conceptual development, provided the individual and collective instrumentation is orchestrated adequately. Therefore, the instrumentation of computer algebra offers opportunities for the learning of algebra.

Fourth, we distinguish some prerequisites for productive computer algebra use in algebra education. The findings suggest that the algebraic expressions and formulas that the students work with in the CAS have to be meaningful to them. Realistic problem situations play an important role here. In the first teaching experiments we focused on the general level of mathematical objects, while neglecting the referential level of problem situations that are realistic to the students. Because the framework of mathematical relations had not yet been developed by the students, this turned out to be an unbalanced approach that led to results that were meaningless to the students. Instead, taking realistic problem situations at a concrete referential level as starting points for building up the mathematical objects at the general level was more successful, as we saw in the last teaching experiment.

Conditions for meaningful work in the computer algebra environment include the students' perception of congruence between instrumented technique and mathematical concept, and between notations and symbols within the CAS and in the paper-and-pencil environment. The CAS procedures should not be perceived as black boxes. In short, the work in the CAS should connect with the mental algebraic concepts of the students and with the paper-and-pencil tradition.

This relation between screen, paper and mind implies transfer: if the students perceive transparency and congruence, their work in the computer algebra environment influences their paper-and-pencil techniques and their conceptual understanding. On the other hand, difficulties with the instrumentation of computer algebra techniques are often related to a lack of conceptual understanding. Taking these difficulties seriously may foster an extension or adaptation of the algebraic concept.

## 11.3 Reflection and discussion

In this section we consider the results of this study from different perspectives.

First, we compare the conclusions with our initial expectations, and reflect on changes in these issues if we were to start again. Second, we reflect on the merits of the theoretical framework and the role it played in the study. Third, we discuss the appropriateness of the research methodology and design. Finally, we address the generalizability of the conclusions.

*Reflecting on the initial expectations and changes in the case of a restart*

Looking back on our initial expectations (Sections 1.3, 5.4 and 5.5), some came true and others did not. What came true is that computer algebra does indeed offer opportunities for the learning of algebra: we already mentioned the findings concerning generalization, reification of formulas and expressions, insight into the structure of formulas and expressions and exploration of algebraic relations. In the meantime, computer algebra environments are difficult tools for students of the age and level of the subjects of our study. The computer algebra environment was more an explorative tool than an expressive tool, to use the terminology of Doerr (2001): students can use the tool to explore the conventional symbol system, but not to express themselves by means of their own symbols.

The results show that the instrumentation difficulties were more important than we had expected beforehand. For example, the difficulties with the ISS instrumentation scheme were quite dominant. We had not foreseen the obstacle that the integration of simple instrumentation schemes into composed schemes could present.

An outcome of the study that we did not expect beforehand concerns the importance of meaningful, realistic problem situations as starting points for instructional activities. The CAS did not act as a microworld that could replace the realistic contexts, at least not for students of this age and level.

Finally, our initial belief that CAS use would help to focus on the overall problem-solving process without being distracted by calculational details is not supported by the data. One possible explanation is the interfering ‘noise’ of instrumentation problems; a second reason might be a lack of attention to classroom discussions that focus on this aspect.

If we were to start this study again, what would we do differently? We would start at a referential level with concrete realistic problem that students could meaningfully engage in, so that a framework of mathematical relations and objects would be construed by the students, leading to entering the general level. This would involve both horizontal and vertical mathematization. Furthermore, we would pay more attention to the instrumental genesis, individually by means of instructional activities and collectively by means of frequent and deepening whole-class discussions. The new didactical contract would require more discussion as well. Rather than focusing so much on the development of instructional activities, we would pay more attention to clarifying the main ideas to the teachers and to offering them more guidelines for establishing a classroom culture in which this new didactical contract and the accompanying social and mathematical norms would be realized (Gravemeijer, 1995). We would try to encourage the teachers to have whole-class discussions in which reflection on the tasks is stimulated by appropriate questions, thus fostering a collective cognitive development. As far as the hypothetical learning trajectory for the concept of parameter concept is concerned, we would start mixing the different parameter roles somewhat earlier and more extensively than we did, in particular in teaching experiments G9-I and G9-II.

### *The role of the theoretical framework*

In Section 1.5 we introduced the theoretical framework of the study, which was composed of elements with different backgrounds, such as the domain-specific instruction theory of Realistic Mathematics Education, level theories, theories on symbol sense and symbolizing, the process-object duality and the theory of instrumentation. We now briefly address each of these theoretical notions and reflect on their role in this study and on their contribution to the results. This will indicate whether the situation of the study is within the scope of the theoretical notion. We also wonder what our study will contribute to the further development of the theoretical element.

In this study the domain-specific instruction theory of *Realistic Mathematics Education* played a role in three related ways. First, the four-level structure of mathematical activity helped to understand why the original idea of using mathematics and computer algebra as contexts at the general level did not work out well: the students failed to attribute meaning to the algebraic objects and procedures, because an appropriate network of mathematical relations had not yet been built up. Therefore, concrete and realistic problem situations that were meaningful to the students at the referential level provided better starting points. The notion of referential and general level provided insight into the processes of generalization and abstraction, and the intertwinement of horizontal and vertical mathematization. Second, the RME view was helpful in identifying the problem of the black box character of using the CAS in cases where the students were not comfortable with their inability to understand what was happening within the computer algebra environment, and to give meaning to the algebraic results that appeared. These contributions of RME to this study do not affect the theory; rather, they support the general character because application to the specific situation of using IT in mathematics education proved to be both appropriate and valuable.

The *level theories* (Section 4.5) were of limited importance in this study. They acted as background for defining the higher level understanding of the concept of parameter. The Van Hiele level theory clarified the need to develop a framework of mathematical relations for the concept of parameter. In addition to what was said in the previous paragraph, the four-level structure of Gravemeijer and the distinction between model-of and model-for stressed the relevance of a longitudinal perspective for considering the transition from action language to formal language, the development of the object view of formulas and expressions/sub-expressions.

The notions on *symbol sense* had to be specified for the purpose of this study. We confined ourselves to insight into the structure and the meaning of formulas. After this definition, the notion of symbol sense was one of our primary means of interpreting the students' behaviour in the computer algebra environment with formulas and expressions. Both the insight into the concept of parameter and the instrumentation of computer algebra were related to this kind of symbol sense: complex formulas appeared while working with parameters. Using the CAS required insight into the structure of expressions, such as $\sqrt{a^2 - x^2}$, to be able to enter them correctly. Recognizing equivalence, even in such simple cases as $-x \cdot (x - 40)$ and $x \cdot (40 - x)$, also required symbol sense. On the other hand, CAS use seemed to reinforce the development of this kind of symbol sense. As feedback to the theoretical notion of symbol sense, we suggest that the notion as such lacks a more precise definition.

*Theories on symbolizing* stress the signification process in which the student develops his/her own symbolizations. The character of the computer algebra environment and the initial choices of this study did not offer much opportunity for such a bottom-up approach. However, symbolizing also involves reasoning with conventional mathematical symbols (Gravemeijer et al., 2000). We conclude that the instrumentation of computer algebra in the context of the concept of parameter may foster the development of meaningful mathematical objects. In that sense, the instrumental approach can be seen as a symbolization perspective. The feedback to the theories of symbolizing, therefore, is the question how symbolization takes place in a case where the environment offers only limited freedom of notation and strategy.

The *process-object duality* helped us to state one of the goals of the learning trajectory, viz. the reification of algebraic formulas and expressions. Furthermore, the duality proved to be useful for identifying the influence that computer algebra use had on the development of an object view of formulas and expressions by the students. Together with the notion of symbol sense, reification was the main frame of reference for monitoring the progress of algebraic insight into formulas and expressions.

The *theory of instrumentation* was an important framework for interpreting student behaviour in the computer algebra environment. The close relation of technical and conceptual components within instrumentation schemes was observed and helped us in understanding the instrumentation process. The instrumental genesis sometimes was hindered by conceptual barriers, but also fostered further conceptual development, as was suggested by the theory. This study worked out the instrumentation theory for the concrete case of some detailed schemes. Furthermore, we stressed the relevance of the congruence and transparency criteria and suggested to pay more attention to the link between the computer algebra technique and the corresponding paper-and-pencil technique.

All together, most of the elements of the theoretical framework contributed to the results of this study, which was not obvious beforehand: only the instrumentation theory had been customized especially for application to learning algebra in a computer algebra environment. Apart from some suggestions for further development, the main value of this study for the theoretical framework is its perceived contribution in a setting for which it was not developed. This suggest a wide applicability.

### *The methodology of the study*

In this study we used the design research methodology. We now reflect on the main characteristics of this methodology that were described in Chapter 2.

Overall, we felt that the *design research methodology* was appropriate for investigating the research questions of this study, given the state of knowledge. In particular, the study aimed at understanding how and why computer algebra affected learning, and not only whether it would. The fact that a dedicated theoretical framework and condensed hypotheses were not yet available linked up well with the cyclic character of design research, which allows for changing the learning trajectory in the next cycle according to new insights from the previous cycle. Furthermore, no instructional activities were available, and their design was an important means of capturing the development of hypotheses and insights.

Looking back at our methodology, we notice that at the micro-level the cyclic character of 'on the run' adaptations for 'tomorrow's lesson' was only used to a limited extent; the macro-cycle design, however, was very important. We now reflect on each of the phases of a macro-cycle: the preliminary phase (consisting of the development of the hypothetical learning trajectory and the design of instructional activities), the phase of the teaching experiment, and the phase of the retrospective analysis.

The preliminary phase of the design research cycle includes the *development of the hypothetical learning trajectory*. The HLT proved to be an adequate research instrument for monitoring the development of our hypotheses and expectations within design research within and throughout the research cycles. The advance identification of key items in the instructional activities and their expected outcomes made explicit the goals of the teaching experiment and guaranteed links between theory and practice, between thought experiment, teaching experiment and reflection. The HLT approach was a concrete means to deal with the design research phasing.

The other main component of the preliminary phase of the design research cycles was the *design phase*. In Chapter 2 we described some design principles that might be helpful, namely guided reinvention, didactical phenomenology and mediating models. If we look back at the design process, these design principles acted only in the background. The guided reinvention approach best reflects the design process. The intrinsic tension between 'guided' and 'reinvention' was reinforced by the tension between bottom-up ideas on learning and the top-down means offered by computer algebra. For didactical phenomenology, the conceptual analysis of the parameter concept that was presented in Chapter 4 guided the design, so that probably it would be better to speak of mathematical phenomenology instead. Didactical phenomenology mainly was a means to find appropriate problem situations. The mediating models were only considered while designing activities that would lead to generalization. Altogether, the way the design was carried out remained intuitive. Peer review of instructional activities by fellow researchers and teachers was helpful in making more explicit our goals and expectations.

The second phase of the design research cycle is the *teaching experiment phase*. The HLT that resulted from the preliminary phase included the identification of key instructional activities. This set of key items and their expected effects guided classroom observations: For each lesson, the key items were studied and questions to ask pairs of students in short mini-interviews were formulated. These mini-interviews assessed the results on the key items for a subset of the students, which guaranteed a link between HLT, expectations and observational data. This method also contributed to structuring the observations. As we noticed that whole-class discussion were not always held with the targeted effect, we concentrated our observations on these mini-interviews. The data from one selected pair of students and from the class as a whole provided information additional to that obtained during the mini-interviews. Overall, this method worked out in a satisfying way, as it allowed for a structured means of gathering data that corresponded to the HLT and the aims of the teaching experiment. In practice, though, we had to deal with two kinds of difficulties. First, students sometimes considered the observers as ‘extra teachers’ and wanted to ask them questions. Although these questions led to interesting observations, they also led to the observers having less time for mini-interviews on the key items. Second, in some cases data on key items could not be assessed as the items had been part of the homework or had already been discussed in the classroom. Despite these limitations, the method of identifying key items and observing by means of mini-interviews was a valuable one.

The final phase of the design research phase is the *retrospective analysis*. For the data analysis, we used a bi-directional method: we made a coding system related to the HLT and the key instructional activities before the teaching experiment, and we started the analysis with an ‘open’ way of coding. Confrontation of the preliminary coding system with the ‘bottom-up’ impressions led to adaptations of the coding system. We were satisfied with this method, because it ensured the link with our preliminary ideas and was also open to changes according to new insights. The coding by a second researcher was worthwhile because it led to a higher quality of data analysis and forced the first researcher to be explicit about the coding system. After the analysis, the design research idea that each cycle leads to feed-forward to the next cycle was helpful in drawing up the balances at the end of a research cycle.

*The generalizability of the conclusions*

The study focused on the concept of parameter and on the instrumentation of computer algebra. Choices had to be made concerning the computer algebra tool, as well as the grades and the school where the teaching experiments were to be held. How about the generalizability of the conclusions of this study? Can they be extrapolated to other situations, or are they specific to the particular settings and choices? We now address generalizability to algebra in general, to other computer algebra tools and IT tools in general, to other grades and levels and to Dutch school settings other than the school that was involved.

Many of the conclusions of this study go beyond the concept of parameter, and concern *learning algebra in general*. For example, computer algebra use contributed to the reification of formulas and expressions as well as to the development of symbol sense and insight into the structure of formulas and expressions. The conclusion that while using computer algebra realistic problem situations at a referential level are still necessary starting points, is not specific to understanding the concept of parameter either. These conclusions do not seem to be related to the particular topic of the concept of parameter. Therefore, we generalize them and expect them to be valid for learning algebra in general.

How about the generalization of the conclusions to *computer algebra environments* other than the TI-89? Of course, differences between computer algebra environments affect instrumentation at a detailed level. For example, in some other computer algebra systems it is not necessary to specify the unknown of an equation if it contains only one variable, as this is self-evident. However, this instrumentation issue will then be encountered later, when equations with more variables are solved, which will make no essential difference in the instrumentation process. Other examples are the reification of formulas and expressions, and the acceptance of expressions as solutions: even if the syntax of solve is different in another computer algebra environment, this conceptual development will be required. In general, we feel that similar instrumentation issues will be encountered while using another CAS, and similar affordances are offered, though perhaps in a slightly different form. The experiences with the Studyworks and TI-Interactive software packages (see Section 2.6 and Chapter 8) are in line with this claim of generalizability. This also applies to the findings of related projects that use other computer algebra environments (e.g. van de Giessen, 2002; Pierce & Stacey, 2002). Something that might have affected the findings is the decision to use a handheld CAS. However, as the students mostly worked in pairs in the classroom, we do not feel that the collective instrumentation suffered from the individual character of the handheld machine use. Therefore, the findings can be extrapolated to computer algebra environments in general. Less clear is the extrapolation to other IT tools. We conjecture that the instrumentation perspec-

tive is a valuable one that could be useful in understanding student - machine interactions for other IT tools; as yet, not many examples of this approach are available (Mariotti, 2002).
Concerning the generalizability of the findings to *other grades and levels*, the study suggests that overcoming the obstacles that students encounter while working with the CAS requires a conceptual development that was hard for ninth- and tenth-grade students to achieve in the short periods of the teaching experiments. Therefore, we would expect even larger instrumentation difficulties in grades lower than the ninth. For higher grades, we expect similar difficulties as reported in this study, but assume that students will have more mathematical experience with which to overcome them. Data from previous studies confirm this (Drijvers, 1999a, 1999b, 2000).

We do not think that the special pedagogic philosophy of the school where the teaching experiments were carried out (see Section 2.8) influenced the conclusions. In fact, classroom practice was quite close to that in regular Dutch school settings. Because the student cohorts involved in this study were considered as weak cohorts, and the teachers were not experienced in integrating technology into their teaching, we expect that the results in another Dutch school setting would not be worse than they were in our study. On the other hand, the setting in the study was artificial in that the researcher and observers supported the students and the teachers to a certain extent. All together, we think that the conclusions can be generalized to *Dutch school settings* other than the school that was involved in this study.
Two additional remarks about the general character of the conclusions can be made. First, the fact that some of the algebraic difficulties we observed were already described in the research literature on algebra in general (e.g. the lack of closure obstacle, the reification of expressions, the equivalence of expressions) suggests that the research context is not so particular and thus enables the finding of general results. Second, the findings result from the complete educational setting and not from the CAS use alone. Therefore, the conclusions can be extrapolated only if the specific didactical embedding is taken into account.

## 11.4 Recommendations

This final section contains recommendations concerning teaching mathematics using computer algebra and the design of computer algebra software for educational purposes, as well as recommendations for further research on the use of technology in mathematics education. These recommendations are induced from the conclusions of this study and serve two purposes: they condense the findings into concrete suggestions (this is the case for the recommendations for teaching and for software design), and they highlight the issues that have not been addressed of fully addressed in this study and therefore require further research.

*Recommendations for teaching*

In Section 9.8.3 the answer to the research subquestion on the learning of the concept of parameter using computer algebra led to identifying consequences for teaching. Similarly, the answer to the research subquestion on instrumentation suggested consequences for the orchestration of the instrumentation of computer algebra use (10.8.4). We now synthesize these recommendations into more general suggestions concerning teaching mathematics using computer algebra that go beyond the learning of the concept of parameter.

- *Anticipating on computer algebra use*
  Teaching can anticipate on the use of computer algebra for learning algebra. The prerequisite symbol sense can be addressed in several ways, such as:
  - Having students realize which variable plays the role of unknown with respect to which an equation is solved, and having them solve equations with more than one variable, so that solving also comes to signify expressing one variable in terms of others;
  - Having students reflect on the different roles of variables and stressing the flexibility of changing these roles;
  - Having students develop an insight into the structure of expressions and formulas by highlighting sub-expressions with tiles, boxes or ovals;
  - Making students experience the object character of expressions and formulas by submitting these algebraic forms to procedures such as substitution;
  - Reflecting with students on the global solution strategy after a problem has been solved;
  - Discussing with students the difference between numerical, approximated answers and exact, algebraic results;
  - Having students reflect on the transition from natural action language to static algebraic language.
- *The didactical contract*
  When starting with CAS use in the classroom, we recommend establishing a didactical contract concerning by-hand skills and computer algebra use and to discuss it with the students, so that they are aware of the new situation and of what is expected of them. Furthermore, new socio-math norms will need to be developed. For example, one of the norms might be that numerical or graphical answers should be explained by algebraic means. In that case, the teacher should introduce problems that can be solved and explained algebraically.
- *Individual instrumentation*
  While integrating computer algebra into mathematics education, the instrumental genesis by individual students can be addressed by using simple mathematical problem situations to develop instrumentation schemes. If difficulties are en-

countered, the teacher should be aware of the relation between the technical and the conceptual aspects of instrumentation schemes.

- *Collective instrumentation*
  Collective instrumentation can be orchestrated by means of classroom discussions and demonstrations of computer algebra techniques by the teacher or by students using data projection facilities. The discussions can focus on the obstacles that were encountered or on the concepts behind a computer algebra technique, or on the relation with paper-and-pencil techniques. The demonstrations can show the different approaches the students have developed and may lead to the convergence of the instrumentation schemes.
- *Computer algebra technique and paper-and-pencil technique*
  The new computer algebra techniques, which the students develop, can be compared with the paper-and-pencil techniques they are familiar with. We recommend solving relatively easy problems by hand as well as in the computer algebra environment, so that difficulties and similarities of a technique in the two media can be discussed. This will raise the issues of the transparency of the computer algebra technique and of its congruence with the corresponding paper-and-pencil technique. Once the computer algebra technique and its relation to the paper-and-pencil technique are clear, the by-hand method is no longer needed in more complicated cases.
- *The CAS as object of study*
  It is recommended to occasionally make students consider the computer algebra environment as an object of study. For example, the logic – or lack of such – behind representations and reactions of the CAS may be a subject of whole-class discussion. Peculiarities of the CAS may be addressed, as may the way the CAS works internally. Often, interesting mathematical elements will soon come up in such discussions.
- *Reflection on CAS use*
  As is the case with many IT tools – and with work with traditional media – students often do not reflect on their results, which in this case is the output of the CAS. Therefore, we recommend the teacher to ask for the meaning of the findings and encourage reflection on the CAS use. For example, questions concerning the graphical interpretation of algebraic results may lead to making them concrete and visual and to integrating multiple representations. Questions about other ways of writing the algebraic results and about comparing them to by-hand outcomes may work as well.

#### *Recommendations for software design*

What do the results of this study imply for the design of computer algebra software? Although software design is not one of the focuses of this study, the findings do suggest some criteria and recommendations for the future development of computer algebra environments.

The first criterion concerns the black box character of the CAS. A lack of transparency leads to the perception of CAS as a black box oracle that students feel unable to 'look through' and cannot relate to their own experience with paper-and-pencil techniques. In order to avoid this, we recommend that computer algebra tools provide means of carrying out procedures step-wise, so that the student determines the problem-solving process at a more detailed level and is able to compare CAS technique with by-hand technique. Pseudo-transparency – which occurs when the process seems to be transparent but is carried out in a different manner within the CAS – should be avoided by enforcing a correspondence between CAS methods and paper-and-pencil methods. There will be a risk of pseudo-transparency as long as the CAS makes use of very sophisticated methods that the students do not understand.
Second, the lack of congruence between CAS technique and paper-and-pencil technique caused difficulties. Therefore, we recommend a closer relation between CAS notation, symbols and syntax on the one hand and the paper-and-pencil techniques on the other. The confusion between the TI-89 wherein bar and the mathematical meaning of the same symbol | in set theory is an example of a subtle notational misfit.
Third, the CAS proved to be not very flexible concerning syntax and notations, and it did not offer the students many opportunities for developing symbolizations. Therefore, we would appreciate a computer algebra environment that provides room for constructing procedures in a more intuitive manner. However, we realize that this will be difficult to achieve. In fact, freedom is often limited in an IT environment. The difficulty with computer algebra is that the freedom that it offers, for example for defining procedures and programming problem-solving strategies, is useful at a level higher than that of the nine- and tenth-grade students in our study. Perhaps such dedicated IT tools as Java applets would exhibit these limitations to a lesser extent.
An improved user-friendliness of the interface of the computer algebra tool we used in this study, the TI-89, might have reduced the instrumentation difficulties. For example, the machine might scan an equation before solving it and, when only one variable was detected, accept the lacking specification of the unknown. Similarly, for substitution the machine might draw up a list of variables that appear in the expression before the substitution bar, and provide a choice by means of a small menu. In this way, some of the instrumentation problems could be avoided. However, in complex cases, such as equations with more than one variable, some of them would still occur. Also, the learning effect of the instrumental genesis would be reduced if all procedures were menu-driven and the user's choice was further limited. Therefore, although improvements of the user interface would be welcomed, we do not plea for

a computer algebra environment that offers a limited set of 'cooked up' options and choices. We do appreciate that other technological tools such as dedicated Java applets offer room for reinventing symbolizations and models. Such environments might serve to foster the development of symbol sense that using a CAS requires.

### *Recommendations for further research*

The recommendations for the teaching of mathematics using computer algebra and for the development of computer algebra software originated in the findings of this study. The recommendations for further research concern ideas that emerged in this study, but that were not worked out yet.

First, our study considered computer algebra use in a rather isolated way. Although we noticed that it is embedded in a didactic and educational setting, the relation between this setting and the effects of computer algebra use was not considered in detail. A recommendation for further research, therefore, is to investigate the influence of computer algebra use in a more holistic way with a closer *relation to the didactical setting*. In particular, the roles of the teacher, the didactical contract and the classroom norms deserve further investigation.

Second, we noticed that the theories of symbolizing, which stress the joint development of symbols and meaning, were not used extensively in this study because of the limited room for developing symbols in the computer algebra environment. Instead, we focused on the instrumental genesis of schemes that integrate technical and conceptual aspects. As symbolizing also involves reasoning with conventional mathematical symbols (Gravemeijer et al., 2000), the question is whether *the symbolization perspective and the instrumentation perspective* can be integrated into a more comprehensive framework if the use of IT tools is concerned. This issue could be used as a starting point for further theoretical development. The perspective of cultural-historical activity theory might be considered for inclusion in this framework as well because of its stress on the social and cultural-historical context of tool use, which is related to the 'community of practice' (e.g. Engeström et al., 1999; Wenger, 1998). This would allow for a coordination of the social and the psychological perspective (Yackel & Cobb, 1996). A longitudinal study on symbolizing in a computer algebra environment that includes preparatory teaching is recommended.

Third, the *instrumental approach* to tool use was helpful in this study to interpret interactions between student and computer algebra environment and to understand the difficulties that were encountered. A question that requires further research concerns the scope of the instrumentation perspective. Does this view lead to insights into the affordances of other IT tools for the learning of mathematics, such as software for dynamic geometry or interactive Java applets?

Fourth, our study focused on the concept of parameter. The issue of the reification of formulas and expressions played an important role. We noticed that the reification of the *concept of function* was involved as well; in fact, in most cases the formulas and expressions were related to functional relations. However, we decided that the concept of function was too multi-faceted to address in this study (Section 3.4). A question for further research therefore concerns the contribution of computer algebra to an understanding of the concept of function that includes its object character. One might expect that the reification of expressions and formulas that we found in this study influences the development of the concept of function.

The fifth and final recommendation for further research concerns *assessment*. The results of the final tests of the teaching experiments G9-I and G9-II were not very convincing. The integration of technology into assessment seems to be even harder than its integration into teaching. The use of technology in assessment requires re-balancing the relevance of concepts and skills, and rethinking the methods and formats for assessment. Taking into consideration the importance of assessment to educational practice, this issue requires clear research conclusions to allow for a real integration of technology into mathematics education.

# Summary

## 1 Research questions

The project ‘Learning algebra in a computer algebra environment’[1] reported upon here is a case of research on the integration of information technology (IT) into education, and on its influence on the learning of mathematics in particular. In this study we focus on two issues that are relevant to mathematics education today: the learning of algebra and the integration of technology – and particularly of computer algebra – into it.

Algebra has always been an important topic in school mathematics curricula, and often is a stumbling-block to students. The difficulties of learning algebra include its formal and algorithmic character, the abstract level at which problems are addressed, the object character of algebraic expressions and formulas, and the compact algebraic language with its specific conventions and symbols. Because of these difficulties, students often do not perceive algebra as a natural and meaningful means for solving problems (Bednarz et al., 1996; Chick et al., 2001).

The integration of IT is one of the ways in which solutions for these problems of learning algebra are sought. IT use is expected to contribute to the visualization of concepts, and can free students from carrying out operations by hand, thus directing their attention towards concept development and problem-solving strategies. In this way, IT use might lighten the traditional algebra curriculum for them. In the meantime, the integration of technology raises questions concerning the goals of algebra education and the relevance of paper-and-pencil techniques, now that they can be left to a technological device. For algebraic skills, the use of a computer algebra system (CAS) is of particular interest, as it provides a complete repertoire of algebraic procedures and operations (Heid, 1988; O’Callaghan, 1998).

The issue of integrating computer algebra use into the learning of algebra brings us to the main research question of this study:

> *How can the use of computer agebra promote the understanding of algebraic concepts and operations?*

This question needs further specification. As algebra as a whole is too big a topic to address in this study, we confine ourselves to the concept of parameter. Parameters may emerge quite naturally from concrete contexts, and may also be means of generalization and abstraction. Therefore, addressing the concept of parameter may stimulate students to enter the algebraic world of formulas, expressions and general solutions. Also, the concept of parameter allows for revisiting the different roles of ‘ordinary’ variables that the students have met before. The use of parameters may

1. Granted by the Netherlands Organization for Scientific Research, project number 575-36-003E.

improve the students' insight into the meaning and structure of algebraic formulas and expressions (Bills, 2001; Furinghetti & Paola, 1994). Therefore, the general research question is specificated in the first research subquestion:

> 1. *How can the use of computer algebra contribute to a higher level understanding of the concept of parameter?*

Previous research on the integration of computer algebra into mathematics education shows that the idea of technology carrying out the elementary operations, so that the students can concentrate on conceptual understanding, is too simplistic (Artigue, 1997b; Drijvers, 2000; Guin & Trouche, 1999; Lagrange, 2000; Trouche, 2000). The technical skills that the students need to carry out procedures in the CAS environment both require some conceptual understanding and affect that understanding. As a second focus of this study, therefore, we consider the intertwined development of techniques in the computer algebra environment and of mathematical understanding in terms of the students' mental schemes. This instrumental approach to IT use – which concerns the instrumentation process of establishing the dual relation between conceptual understanding and techniques in the IT environment – is the topic of the second research subquestion:

> 2. *What is the relation between instrumented techniques in the computer algebra environment and mathematical concepts, as it is established during the instrumentation process?*

## 2 Theoretical framework

While phrasing the research question, we wondered what theoretical perspectives could help us to investigate these issues. A ready-to-use theoretical framework for the study of learning algebra in a computer algebra environment was not available. Therefore, we selected those theoretical elements from research on learning algebra and mathematics in general that seemed promising for application in and adaptation to the focus of this study. As a consequence, most of these elements were taken out of their usual scope and localized for the aim of this study. This eclectic approach to theory is called 'theory-guided bricolage' (Gravemeijer, 1994). The following elements were included in our theoretical framework.

- *The domain-specific instruction theory of Realistic Mathematics Education*
  Key elements in the domain-specific instruction theory of Realistic Mathematics Education (RME) are guided reinvention and progressive mathematization, horizontal and vertical mathematization, didactical phenomenology and emergent modelling (Freudenthal, 1983; Gravemeijer, 1994; de Lange, 1987; Treffers, 1987a, 1987b). Guided reinvention, progressive mathematization and didactical phenomenology were supposed to be useful for developing a hypothetical learn-

ing trajectory for the concept of parameter, and for designing instructional activities. The issue of horizontal and vertical mathematization might clarify the relation between concrete problem situations and the abstract 'microworld' of the computer algebra environment, which tends to have a top-down character (Drijvers, 2000). The notion of emergent modelling was supposed to be useful for distinguishing levels of activity. As is shown in Fig. 1, Gravemeijer (1994, 1999) distinguishes four levels of mathematical activity. The idea of emergent modelling is that models, which initially refer to a concrete context that is meaningful to the students, gradually develop into general models for mathematical reasoning within a mathematical framework.

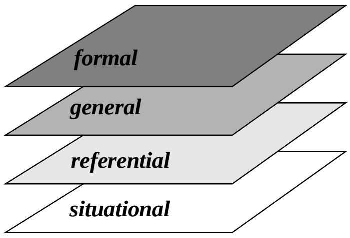



Figure 1 Four levels of mathematical activity (Gravemeijer, 1994, 1999)

- *Level theories*
  The first research subquestion speaks of 'a higher level understanding of the concept of parameter'. To make explicit what is meant by that, several perspectives were considered. First, Van Hiele's level theory distinguishes between a ground level, a second level and a third level of insight (Van Hiele, 1973, 1986). This distinction might be useful for defining levels of understanding of the concept of parameter. A second approach to the levels involves the concepts of emergent models and levels of activity (see Fig. 1). In fact, in this study we aim at the transition from referential to general level of activity.
- *Theories on symbol sense and symbolizing*
  The notion of symbol sense concerns the sense for algebraic entities in general and the insight into formulas in particular (Arcavi, 1994). In this study, we defined symbol sense as the insight into the meaning and the structure of algebraic expressions and formula. Working in a computer algebra environment with parametric formulas was supposed both to require this kind of symbol sense and to foster it.
  How does one acquire symbol sense? Theories of symbolizing stress the parallel development of symbols and meaning by means of a signification process (Gravemeijer et al., 2000). Because giving meaning to algebraic techniques, formulas and expressions as they appear in the computer algebra environment

seemed to be an important aspect of the instrumentation process, a symbolization perspective was supposed to be relevant in this study.

- *The process-object duality*

  The process-object duality concerns the idea that a mathematical concept can be considered both as a process and as an object. Often, students first experience the process aspect; on the basis of this they may develop the object aspect, which is a prerequisite for conceptual progress. This development is called reification (Sfard, 1991) or encapsulation (Dubinsky, 1991) and results in proceptual understanding (Tall & Thomas, 1991). These theoretical notions concern mathematics in general, but can also be applied specifically to the learning of algebra.

  In this study we considered the reification of expressions and formulas, because perceiving formulas and expressions as objects is important for developing the concept of parameter. Furthermore, computer algebra use might affect the relation process-object. Reifying formulas and expressions involves overcoming the lack of closure obstacle, viz. the students' inability to see expressions and formulas as results if these contain operators; in $a+b$ or $x+3$ for example, the students would prefer to carry out the addition (Collis, 1975; Küchemann, 1981; Tall & Thomas, 1991). The reification of formulas and expressions does not imply the reification of function, which involves more aspects.

An important theoretical element in our study is the instrumental approach to IT use. We address that issue separately in Section 4.

## 3 Analysis of the concept of parameter

This study included a conceptual analysis of the concept of parameter.

We started this conceptual analysis with an investigation into its historical development. Essential in this development was the transition from syncopated to symbolic algebra that is characterized by the work of Diophantus and Viète (Boyer, 1968; Harper, 1987). Whereas Diophantus (ca. 250 A.D.) used literal symbols to denote unknown variables but not parameters, Viète (1540 - 1603) provided general, parametric solutions. He distinguished unknowns from parameters by using vowels and consonants. Apparently, he accepted expressions as solutions to general, parametric equations and thus considered them as objects. In our study, we aimed at this jump 'from Diophantus to Viète', or from the referential to the general level (Fig. 1). The time that this step took in history evinces its complexity.

The conceptual analysis of the notion of parameter led to the idea that a parametric formula or expression represents a second-order function. For example, the short arrow in Fig. 2 indicates that $x \to a \cdot x + 5$ for a fixed value of the parameter $a$ represents a – linear – function in $x$. As soon as the value of $a$ changes, the long arrow in Fig. 2 indicates a second-order function, with the parameter as argument and linear expressions as function values (Bloedy-Vinner, 2001). As the main difficulties of the concept of parameter we identified this hierarchical relation with ordinary variables, represented in Fig. 2 and in the expression ‘variable constant’, and the distinction of the different roles of the parameter, which may change during the problem-solving process.

$$a \longrightarrow (x \to a \cdot x + 5)$$

Figure 2 The parameter as argument of a second-order function

The conceptual analysis led to the identification of four parameter roles, which were similar to the roles of ordinary variables:

- The parameter as *placeholder* represents a position, an empty place, where a numerical value can be filled in or retrieved from. The value in the ‘empty box’ is a fixed value, known or unknown, that does not change. This is the ground level of understanding the concept of parameter.
- The parameter as *changing quantity* concerns the systematic variation of the parameter value. The parameter acquires the dynamic character of a ‘sliding parameter’ that smoothly runs through a reference set (van de Giessen, 2002). This variation affects the complete situation – that is, the formula as an object, and the global graph – whereas variation of an ordinary variable only acts locally.
- The parameter as *generalizer* generalizes over a class of situations. By doing so, this ‘family parameter’ (van de Giessen, 2002) unifies such a class, and represents it. This generic representation allows for ‘seeing the general in the particular’, for the generic algebraic solution of categories of problems, and for formulations and solutions at a general level. This general solution of all concrete cases at once by means of a parametric general solution requires the reification of the expressions and formulas that are involved in the generic problem-solving process.
- The parameter plays the role of *unknown* when the task is to select particular cases from the general parametric representation on the basis of an extra condition or criterion. This often requires a shift of roles and of hierarchy (Bills, 2001).

The conceptual analysis and the theoretical framework provided means to define the higher level understanding of the concept of parameter. Fig. 3 visualizes this level structure. In terms of the Van Hiele levels, we considered the placeholder level as the ground level of understanding the concept of parameter that is the basis for the second level. The three roles of changing quantity, generalizer and unknown share the property that formulas that contains them are considered as objects. Therefore, they are part of the second level understanding. The most important of the three was supposed to be the generalizer, as generalising is a key activity in algebra. The targeted higher level understanding, therefore, involved the jump from placeholder to the other parameter roles, ‘from Diophantus to Viète’, or in terms of the four-level structure (Fig. 1), the transition from the referential level to the general level.

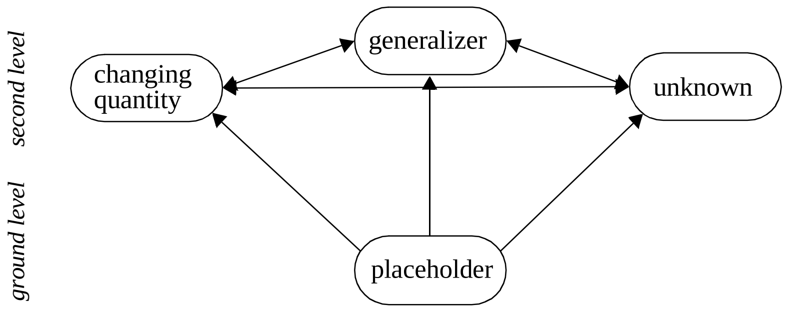



Figure 3 Levels of understanding the concept of parameter

The understanding of the higher parameter roles was supposed to require the reification of parametric expressions and formulas (Gravemeijer et al., 2000).

## 4 The instrumental approach to using computer algebra

The question now is how computer algebra use can contribute to the higher level understanding of the concept of parameter.

Generally speaking, computer algebra environments were supposed to offer opportunities for algebra education. Unlike other IT tools, the CAS has a full repertoire of algebraic procedures and representations. By freeing the students from the algebraic calculations, computer algebra would offer opportunities to concentrate on concept development and on problem-solving strategies. CAS use might help students to distinguish between concepts and skills (Monaghan, 1993) and to readjust the balance between them (Heid, 1988; O’Callaghan, 1998).

The integration of CAS into algebra education also might introduce some pitfalls. Because the CAS already ‘contains all the algebra’, the computer algebra tool might have a somewhat abstract and formal top-down character, and might turn out to be inflexible with respect to informal notation and syntax. Furthermore, the CAS might be a black box for students, as it carries out complex procedures in a way that is not transparent to them. Finally, the CAS might seem to be a microworld to the students that is not connected to the world of real-life problems or that of paper-and-pencil and mental mathematics (Drijvers, 2000).

For the purpose of understanding the concept of parameter, we identified the following CAS affordances. First, getting algebraic expressions as solutions to parametric equations and processing algebraic expressions further by means of substitution was expected to support the reification of algebraic expressions and formulas. Second, algebraic explorations that generated examples were supposed to foster generalization over these situations, thus opening the way for the parameter as generalizer. Third, the flexibility concerning literal symbols and their roles that the CAS offers could be used to change the parameter role into that of unknown. Finally, although not exclusive to the computer algebra environment, the availability of a slider tool was supposed to support the view of parameter as changing quantity.

For the teaching experiments, we had to choose a specific computer algebra tool. For practical reasons we decided to use the handheld TI-89 symbolic calculator: the students would be able to use it permanently at school as well as at home, and it would not require a reorganization of the lessons. The limitations of the screen resolution were mitigated by having some lessons in the computer lab, using the TI-Interactive software package. We hoped that the individual character of the handheld calculator would not hinder collaboration between students, and decided to have them work in pairs in order to obviate that problem.

As we announced in the description of the theoretical framework, we used the instrumental approach to computer algebra use as a framework for understanding and interpreting the student-machine interactions. The central idea of the instrumental approach to using IT tools is that a 'bare' tool or artifact is not automatically a useful instrument. While using such a tool, the user has to develop mental 'schemes of instrumented action', or – to put it somewhat shorter – instrumentation schemes (Artigue, 1997b, 2002; Guin & Trouche, 1999, 2002; Lagrange, 2000; Trouche, 2000). An instrument, according to this view, consists of the artifact (or a part of it), the mental scheme and the type of tasks for which it is appropriate (see Fig. 4).

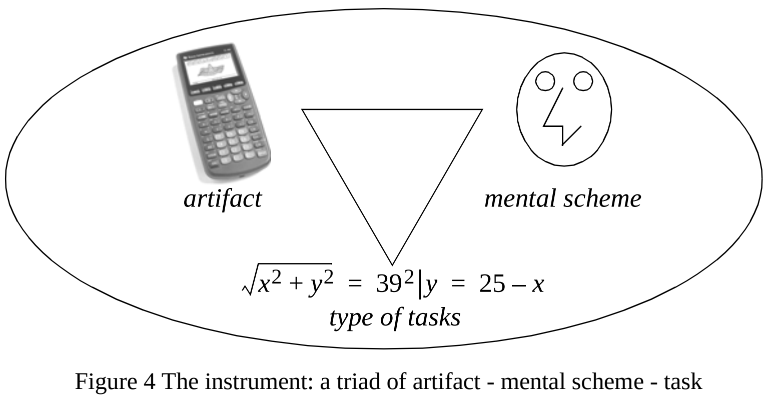



Figure 4 The instrument: a triad of artifact - mental scheme - task

The instrumentation schemes integrate technical skills and conceptual insights; difficulties in the development of the schemes – the instrumental genesis – often involve both aspects. As we cannot look insight the heads of the students to observe the mental schemes, we focused on the techniques, which can be considered as the observable parts of the instrumentation scheme, viz. the set of procedures in the en-

vironment that the students use for solving a specific type of problems.
In this study, the instrumental approach offered a framework to investigate the interaction between student and the computer algebra environment. The combination of technical and conceptual aspects within the instrumentation scheme made the theory promising for this purpose, because it both goes beyond the somewhat naive idea of reducing skills and reinforcing concepts, and takes seriously the difficulties of the instrumental genesis.

## 5 Methodology

We used a design research methodology in this study. Design research – which is also known as developmental research or development research – aims at developing theories and an empirically grounded understanding of 'how learning works' (Research Advisory Committee, 1996). Its main objective is to understand the students' learning process. This connects with the character of our research questions, which start with 'How can ... ' and not with 'Can ... '. One characteristic of design research is the importance attributed to the design of instructional activities, which is seen as a meaningful part of the research methodology as it forces the researcher to be explicit about choices, hypotheses and expectations (Edelson, 2002). Another important feature of design research is the adaptation of the learning trajectory throughout the research: instructional sequences and teaching experiment conditions are adjusted according to previous experience. Therefore, design research was particularly suitable for this study, because a full theoretical framework was not yet available and hypotheses were to be developed. Adapting the experimental situation is possible by means of the cyclic design that is used in design research. A macro research cycle consists of a preliminary phase (which in our case included the development of a hypothetical learning trajectory and the design of instructional activities), a teaching experiment phase and a retrospective phase, which includes data analysis and leads to feed-forward for the next research cycle (Gravemeijer, 1994).This study consisted of three main research cycles – G9-I, G9-II and G10-II – and one intermediate cycle, G10-I. The 9 and 10 stand for the grades in which the teaching experiments were conducted. The I and II refer to the first and second cohort of students involved. The tenth-grade population was a subset of the ninth-grade population, with the exception of a few students who entered the teaching experiment in tenth grade (see Fig. 5).
We will now briefly explain each of the phases within one research cycle. In the *preliminary phase*, a hypothetical learning trajectory (HLT) was developed (Simon, 1995). This involved assessing the starting level of understanding, formulating the end goal and developing a chain of mental steps towards that goal, as well as instructional activities that were expected to bring about this mental development. This was accompanied by the designer's description of why the activities were supposed to work and what kind of mental developments were expected to be elicited. Because

of its stress on the mental activities of the students and on the designer's motivation of the expected results, the HLT concept was a useful research instrument for monitoring the development of the hypotheses and for capturing the researcher's thinking.

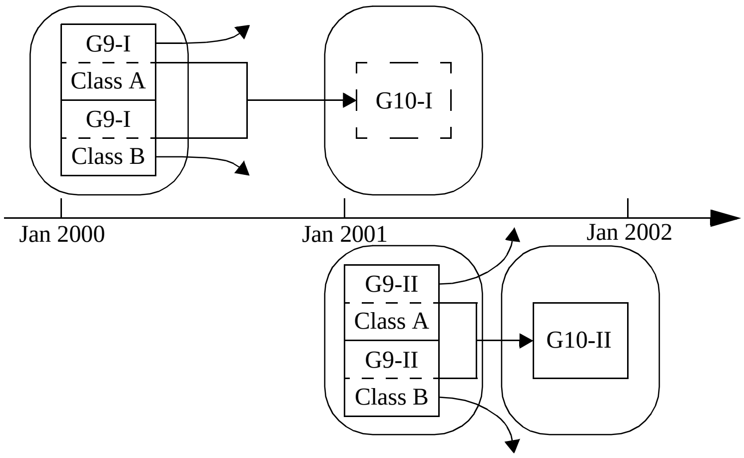



Figure 5 Arrangement of research cycles and teaching experiments

The development of the HLT was closely related to the design of instructional activities, which was done by using design heuristics such as guided reinvention, didactical phenomenology and mediating models. Key items in these instructional activities that would serve to monitor students' mental development in relation to the HLT were identified; also, questions for interviewing students on these key items were formulated and a prior global coding system for observations was developed.

The second phase of the design research cycle is the *teaching experiment phase* (Steffe, 1983; Steffe & Thompson, 2000). During the teaching experiments, we focused on data that reflected the learning process and provided insight into the students' thinking. The main sources of data, therefore, were observations of student behaviour and interviews with students. Most lessons were observed by two observers, who made field notes. The key instructional activities were assessed by means of mini-interviews with a selection of the students; video registrations were made of classroom discussions and of a selected pair of students. Written materials (notebooks, pretests and posttests) were gathered from all students. Altogether, 110 students were involved, and over 100 lessons were observed.

The final phase of the research cycle is the phase of *retrospective analysis*. A first step in this phase was the selection and analysis of data. The initial method of analysis was inspired by the constant comparative method (Glaser & Strauss, 1967; Strauss, 1987; Strauss & Corbin, 1998). The first findings led to adaptation of the prior coding system. Then the data were coded. The coding process was partially done by two researchers, in order to achieve an intersubjective agreement. The conclusions of the data analysis were translated into feed-forward for the next research cycle. The feed-forward comprised changes in the HLT and in the instructional activities, and even slight changes in the focus of the subsequent teaching experiment.

## 6 Through the research cycles

Fig. 5 shows the arrangement of the design research cycles in this study. We will now briefly review each of these cycles, and the teaching experiments in particular. These teaching experiments were carried out at a school in Bilthoven, a town not far from the city of Utrecht.

The first research cycle (G9-I) included a teaching experiment that was carried out in two ninth-grade classes of the pre-university stream (in Dutch: vwo, 14- to 15-year-old students). The experiments ran for five weeks with four mathematics lessons each week. The HLT for these experiments followed the line parameter as placeholder-generalizer-changing quantity-unknown. The instructional activities were presented in two booklets: ‘Introduction TI-89’, which was intended to introduce the students to the use of the TI-89 symbolic calculator, and ‘Changing algebra’, which was intended to help them to develop an insight into the concept of parameter.

The results showed that generalization was hindered by instrumentation problems, in particular with a scheme for solving systems of equations. Fig. 6 shows this scheme of isolate-substitute-solve (ISS) for the system $x + y = 31$ , $x^2 + y^2 = 25^2$ .

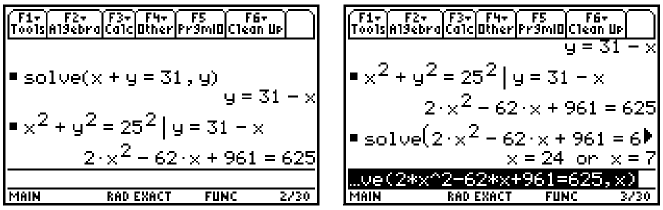


Figure 6 The isolate-substitute-solve scheme on the TI-89

In some cases students felt no need for generalization, or the generalization had a superficial character of pattern recognition without intrinsic understanding. The parameter as changing quantity was understood better, although we missed features for visualizing the dynamics on the TI-89. The parameter as unknown did not receive much attention. The shift of roles of the parameter that was involved presented the students with difficulties. The substitution of expressions and the appearance of expressions as solutions of parametric equations supported reifying expressions and overcoming the lack of closure obstacle. The results of the posttest confirmed that both the generalization and the application of the ISS scheme were bottle-necks for many students. As feed-forward from the G9-I research cycle, we decided to reorder the global HLT line into placeholder-changing quantity-generalizer-unknown, in order to delay generalization until a more dynamic and hierarchic view of parameter was established. For this purpose, we aimed at finding a better means to represent the sliding parameter on the TI-89. To overcome instrumentation problems, we included practising simple instrumentation schemes for solving and substituting more extensively before integrating them into the composed ISS scheme.

The teaching experiment of the second research cycle (G9-II) was carried out in a second cohort of two ninth-grade classes of the pre-university stream (14- to 15-year-old students). The experiment was similar to the G9-I teaching experiment and ran for five weeks with four mathematics lessons each week. The HLT followed the line that emerged from the feed-forward from G9-I. The concept of parameter as changing quantity was supported by two TI-89 programmes called SHOOT and SLIDE. The student text included a part that introduced the students to the use of the TI-89 and a part that focused on developing an insight into the concept of parameter.

The results showed that the concept of parameter as changing quantity was supported by the SHOOT and SLIDE applications, although some students tended to see the parameter as running only through integer values. Starting the HLT with this parameter role was an improvement compared to the G9-I HLT. For generalization, we encountered similar difficulties as in G9-I, though to a lesser extent. Despite the attention to simple instrumentation schemes for solving and substituting, the composed ISS scheme still caused difficulties. Also, the students often did not feel the need for generalization, perhaps because of the abstract and complex problem situations. The parameter as unknown received more attention than it did in G9-I, but the complexity of the problem situations prevented the students from keeping track of the problem-solving strategy and the changing roles of the parameter therein.

The results of the posttest confirmed these findings, although the performance of the ISS instrumentation scheme and the generalizations with parameters slightly improved compared to the G9-I results. After the posttest, final interviews were held with 19 students. During these interviews the students did not have a symbolic calculator at their disposal. The results of these interviews suggested a transfer of substitution as a computer algebra technique to substitution as a paper-and-pencil technique, and an improved reification of expressions. For example, the students were able to substitute $y = a - x$ into $x^2 + y^2 = 10$ by hand, and one of them used the TI-89 notation for substitution while doing so (Fig. 7).

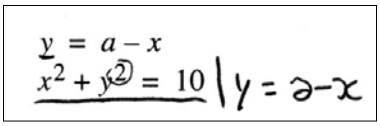



Figure 7 Transfer of notation

As feed-forward from the G9-II research cycle, we first noted that the global HLT along the line placeholder-changing quantity-generalizer-unknown was adequate and needed no further change. For the parameter as changing quantity, the use of a continuous slider tool might be used to avoid the notion of a discrete reference set. The findings suggested that the parameter as generalizer might be better addressed without the complication of the instrumentation of the ISS scheme. This required looking for problem situations that could be solved with one equation rather than with a system of two equations. Furthermore, the problem situations were targeting the general level too directly. The use of realistic contexts in which the parameter would have a meaning to the students might lead to easier and more meaningful generalizations, and would address the referential rather than the general level. Graphs might be used as contexts as well, and could be useful for the transition from the parameter as changing quantity to the parameter as generalizer. The use of realistic problem situations might also address the parameter as unknown in a more natural way.

The G10-I intermediate cycle included a five-lesson teaching experiment in tenth grade. These students had opted for the exact stream and all but one had participated in G9-I as well. Because of a lack of time and the full regular curriculum in tenth grade, this intermediate cycle rather was more of a short 'finger exercise' for G10-II than a full research cycle. The results showed that the students easily recalled the TI-89 skills they had learned a year earlier. Once more, using a slider tool for the parameter as changing quantity was preferred to replacing parameter values by substitution.

The third and last complete research cycle (G10-II) included a 15-lesson teaching experiment in one tenth-grade class of the exact stream of pre-university stream. All but two students had participated in the G9-II teaching experiment. In accordance with the feed-forward from G9-II, we stuck to the global trajectory of placeholder-changing quantity-generalizer-unknown. For the parameter as changing quantity, the continuous slider tool of the TI-Interactive software package was used. To address meaningful parameters and formulas and natural generalizations, realistic problem situations were more prominent than in the instructional activities of the previous teaching experiments. We expected this would also improve the understanding of the parameter as unknown.

The results showed that the use of the continuous slider tool improved the perception of the changing parameter running through a continuum. Contexts in which the parameter had an evident meaning in the graph allowed for meaningful generalizations. Realistic contexts also improved insight into the meaning and structure of formulas and expressions. Meanwhile, entering complex expressions with fractions and powers and recognizing equivalent expressions provided difficulties. The fact that the problem situations led to one rather than to two equations averted the difficulties of the ISS instrumentation scheme. The parameter as unknown was addressed with more success than in previous experiments, probably also due to the more meaningful problem contexts. However, the complexity of the formulas involved seemed to be a crucial factor.

The results of the final task that ended the G10-II teaching experiment were positive. Many appropriate generalizations were found and the students dealt adequately with complex formulas. The realistic problem situation on the speed of a traffic flow helped them to give meaning to the results.

## 7 Results on computer algebra use and the concept of parameter

The first research subquestion of this study concerns the contribution of computer algebra use to a higher level understanding of the concept of parameter. This higher level understanding was defined as insight into the higher parameter roles of changing quantity, generalizer and unknown. We will now summarize the results on this issue.

As expected from our conceptual analysis, the placeholder view of the parameter was the starting level for almost all students. To extend this concept to the role of *changing quantity*, the computer algebra environment allowed for changing the parameter values one by one. Investigating the dynamic effects of the 'sliding parameter' by means of a slider tool – which is not an exclusive CAS feature – proved to be more efficient. This led to the perception that parameter change affected the complete situation and the graph as a whole, and was a second-order change. If the parameter had a meaning to the student in the context or as a graphical property, examining the sliding graph invited algebraic verification and generalization. However, there was a risk of only superficial examination without reflection or verification. Furthermore, the parameter as changing quantity tended to dominate the students' view of parameter, so that they sometimes seemed to forget the other parameter roles. Altogether, a higher level understanding of the parameter as changing quantity was achieved.

The work in the computer algebra environment supported the transition to the parameter as *generalizer* by allowing the repetition of similar procedures for different parameter values. This generated examples that were the basis for generalization and for solving parametric equations. The fact that parametric equations led to expressions rather than numerical solutions fostered the reification of expressions and formulas, and the overcoming of the lack of closure obstacle. The dynamic graphs that appeared from the work with the sliding parameter in some cases elicited generalization, for example in the case of the graph of $y = (x-a)^2 + a$. One pair of students noticed that the vertex of the graph had coordinates ($a$, $a$) for all values of $a$. One of them commented that 'the vertex is at $a$' and that 'the $y$- and $x$-coordinates are the same as $a$'. Later, during the whole-class discussion, the other student explained this generalization by referring to an exemplary parameter value.

*Maria: Yes, I think that vertex, that were the x and y coordinates, they were, so to say, equal to the a. The vertex was (1, 1), then a was 1 too.*

Many students achieved a higher level understanding of the parameter as generalizer only to a limited extent. In some cases, the students did not feel the need for generalization. In other cases, generalization came down to phenomenological pattern recognition without intrinsic understanding. The instructional activities were aimed too directly at the general level, and the formulas in some cases were too complex. Finally, instrumentation problems hindered generalization. The essential step of generalising relations – in which parameters are used to unify a class of situations – seemed to be primarily a mental one, upon which computer algebra had hardly any influence.

The possibility to solve equations in the computer algebra environment with respect to any unknown improved the students' flexibility concerning the roles of the literal symbols. The results on parameter as *unknown* were mixed: although this parameter role was understood in simple cases, in more complex ones the students failed to distinguish the changing roles and meanings of the variables and parameters. The change in the hierarchy between variable and parameter in complex cases was more difficult to keep track of.

Two issues seemed to influence the development of the higher level understanding of the concept of parameter: the use of realistic problem situations and the insight into the meaning and structure of expressions and formulas. We have already mentioned the importance of *realistic problem situations*. Despite our initial ideas, using the CAS as a mathematical environment that would make references to reality redundant, did not work out for the students of this age and level. Addressing the general level directly did not work out either, because the students had not yet developed an appropriate mathematical framework. Instead, they needed meaning for parameters, expressions and formulas from the realistic problem situation at a referential level. Meanwhile, realistic problem situations did not hinder generalization and abstraction beyond these specific contexts.

The *expressions and formulas* that the students encountered were more complex than usual because of the presence of parameters. The CAS output, whether algebraic or graphical, invited a closer inspection of the formulas and expressions involved, particularly when this was suggested by the task or by the teacher. Entering complex formulas required insight into the algebraic structure of the formulas and expressions. Therefore, although insight into the meaning and structure of expressions and formulas was an obstacle during the CAS work, such work seemed to improve the insight. The work in the computer algebra environment also supported the development of symbol sense, the reification of formulas and expressions, and the overcoming of the lack of closure obstacle. As an overall conclusion on the contribution of computer algebra use to a higher level understanding of the concept of parameter, we did indeed find that the CAS activities contributed to the development of insight into the different parameter roles. This contribution was more evident for the parameter as changing quantity than for the generalizer and the unknown. Furthermore, the reification of formulas and expressions was supported. In the meantime, we found that realistic starting points were indispensable for meaningful work in the computer algebra environment.

Several factors might optimise the targeted learning process. First, the data from the final teaching experiment suggest that it might be fruitful to integrate rather than separate the different parameter roles, so that the students develop a more integrated view of the parameter. Second, we noticed that starting with models that refer to concrete problem situations leads to better results with students of this age and level. Third, we conjecture that the contribution of computer algebra to the conceptual development would improve if it were used for a longer period, so that instrumentation difficulties were less dominant. Finally, we think that whole-class discussions, guided by the teacher, might have stimulated the collective conceptual development to a greater extent. We notice that the teachers considered the student cohorts involved in this study as weak cohorts. They might have needed more guidance.

The difficulties the students encountered while working with the CAS suggest that the instrumentation process is important. We will discuss this issue in the next section.

## 8 Results concerning the instrumentation of computer algebra

The second research subquestion concerned the instrumentation of computer algebra and the relation between computer algebra techniques and conceptual development. We will now summarize the results of this study on this issue.

As simple instrumentation schemes, the solve scheme and the substitute scheme were investigated. For both schemes, which are related to the concept of parameter, the instrumental genesis involved the extension of conceptual understanding. For the *solve* scheme, the syntax and the application to parametric equations led the students to realize that an equation is always solved with respect to an unknown, that solving a parametric equation leads to expressing one variable in term of others, and that the solution in that case is an expression. For *substitution*, the substitution of expressions within the computer algebra environment led the students to extend their conceptual understanding with the notion that only isolated forms can be substituted, and that expressions can be considered as ‘things’ that can be ‘pasted’ into a variable. In line with Sfard’s ideas, the operation on the expression enhances its object character (Sfard, 1991).

Although the instrumental genesis of the solve and substitute schemes seemed to progress smoothly, combining them into the composed *isolate-substitute-solve* (ISS) scheme caused persistent difficulties. This indicated that the integration of simple schemes into a more comprehensive scheme requires a high level mastering of the component schemes. The instrumental genesis apparently needed more time than was given to the students.

We identified a number of obstacles that hindered the instrumental genesis, such as the way in which the CAS deals with the difference between numerical approximations and exact algebraic results, the difficulties with entering expressions containing parentheses, square root signs and powers, and interpreting results. Recognizing the equivalence of CAS output and expected results was a difficult issue. Although many of these obstacles were related to a lack of symbol sense, they also fostered its development. And although the students' lack of insight into the structure of expressions and formulas led to errors in the CAS work, these problems stimulated a closer look at these structures. In that sense, obstacles also provided opportunities for learning, when managed appropriately by the teacher.

Some of the observed difficulties with entering expressions and interpreting algebraic results were related to the paper-and-pencil work that the students were used to. Incongruence between computer algebra technique and paper-and-pencil technique explained some of the instrumentation problems. When the techniques in the two media were congruent, the transfer of notation, strategy and technique between the paper-and-pencil and computer algebra environments was observed. An explicit comparison between computer algebra technique and paper-and-pencil method was effective for making students record the degree of congruence. A second condition for transfer was the transparency of the computer algebra technique: when the students were able to understand the way in which the CAS arrived at its results rather than perceiving it as a black box, transfer to the paper-and-pencil work took place. The data indicate that the students developed different preferences concerning paper-and-pencil work versus work in the computer algebra environment.

The teacher played an important role in the instrumental genesis. Different teacher behaviour led to different instrumented techniques. Whole-class demonstrations and discussions proved to be important for collective instrumentation; this aspect of instrumentation should have received more attention in the teaching experiments. Furthermore, a new didactical contract had to be established concerning the relation between by-hand work and machine work, and between numerical-graphical methods and algebraic methods. The fact that for the teachers the integration of computer algebra technology into teaching was new and lasted for a short period made it hard to establish a new didactical contract and to foster collective instrumental genesis.

Overall, the results of this study suggest a close and reciprocal relation between CAS techniques and conceptual understanding. Students had to build up instrumentation schemes that combined technical and conceptual aspects. This instrumental genesis required time and effort, and obstacles had to be overcome. Seemingly technical difficulties were often related to a lack of conceptual understanding. Paper-and-pencil routines were also involved in this relationship. As consequences for teaching, we recommend developing a new didactical contract, fostering collective instrumentation by means of whole-class discussions and demonstrations, and discussing the congruence or incongruence between paper-and-pencil and CAS techniques.

## 9 CAS use and the learning of algebra in general

In this section we first discuss the conclusions concerning the main research question. Then follows a reflection on the study. We conclude with some recommendations.

### *Conclusions*

The main research question concerns the contribution of computer algebra use to understanding algebraic concepts and operations in general. By means of extrapolating the answers to the two research subquestions to this more general level, we conclude the following.

First, some of the findings not only concern the understanding of the concept of parameter, but also show that computer algebra use can contribute to the understanding of algebraic concepts and operations in general. Symbol sense could be developed, as could insight into the meaning and structure of formulas and expressions. Algebraic notions were extended. Computer algebra offered affordances for combining different representations, repeating procedures as a preparation for generalization, and dealing with expressions as objects. Critical for capitalising on these affordances were the didactic embedding and the use of meaningful formulas that referred to realistic problem situations, rather than to the general level directly.

Second, the observed instrumental genesis of CAS instrumentation schemes, provided that it was orchestrated adequately, included conceptual development concerning algebra in general. The development of the solve scheme and the substitute scheme led to new insights into solving and substituting. Although incomplete instrumentation hindered learning, instrumentation obstacles provided learning opportunities. Conditions that fostered the instrumentation of computer algebra were the congruence between CAS technique, mental conception and paper-and-pencil technique, and the transparency of CAS procedures. These turned out to be hard issues, as some CAS procedures were not transparent to the students, and CAS notations and syntax differed from what the students were used to from paper-and-pencil experience.

When we compare these finding with our initial expectations, we notice that the instrumentation difficulties were more persistent than we had expected, and that the role of the teacher was more important than we had foreseen.

Furthermore, our ideas on focusing on the general rather than on the referential level did not come true, and realistic problems could not be missed as starting points. What is realistic to students depends on their age and level: we conjecture that students in advanced mathematics can more easily experience the CAS as a realistic environment on its own.

### *Reflection*

When we look back at this study, what can we say about the role of theory in it, about the methodology that we used and about the generalizability of the results? We will first look back at the elements of the *theoretical framework*.

The domain-specific instruction theory of *Realistic Mathematics Education* helped us to understand why the students needed realistic problem situations to start with: the algebraic objects and procedures were not yet part of an appropriate network of mathematical relations. The distinction of the referential level and the general level (see Fig. 1) clarified our goals as well as the difficulties that the students encountered. The notions of horizontal and vertical mathematization made it clear why generalization and abstraction did not work out on many occasions. Finally, the RME view was helpful in understanding the problem of the black box character of using the CAS.

The *level theories* acted as background for defining the higher level understanding of the concept of parameter and helped in developing the learning trajectory and for designing instructional activities.

We defined *symbol sense* as insight into the structure and meaning of formulas and expressions. Looked at in this way, it was a means of interpreting the students' behaviour with formulas and expressions in the computer algebra environment: symbol sense was a prerequisite to productive CAS use, but could also develop from it. As feedback to the theoretical notion of symbol sense, we would suggest that it needs a more precise definition.

*Theories on symbolizing* made it clear that the character of the computer algebra environment and the initial choices of this study did not offer much opportunity for a bottom-up signification process. However, the instrumentation of computer algebra fostered the development of meaningful mathematical objects, and of a framework of mathematical relations, which was connected with symbols and expressions that the students encountered while working in the computer algebra environment. Therefore, the relation between the instrumentation approach and theories on symbolizing deserves further investigation.

The *process-object duality* was useful for identifying the influence that computer algebra use had on the development of an object view of formulas and expressions by the students. Together with the notion of symbol sense, reification was a frame of reference for monitoring the progress of algebraic insight into formulas and expressions.

The *theory of instrumentation* helped us to interpret student behaviour in the computer algebra environment and to observe the relation between technical and conceptual components within instrumentation schemes. Sometimes the instrumental genesis was hindered by conceptual barriers, but at other times it fostered further conceptual development, as was suggested by the theory. This study worked out some concrete schemes and stressed the relevance of the congruence and transparency criteria. We suggest paying more attention to the link between the computer algebra technique and the corresponding paper-and-pencil technique.

Altogether, we see that most of the elements of the theoretical framework contributed to this study, despite the fact that the elements of the framework – except for the instrumental approach – were not dedicated for use in the context of this study. One of the main results of this study for the theoretical framework, therefore, is the theories' perceived contribution in a setting for which they were not developed, which suggests a wide applicability.

How do we look back at the *methodology*? Reflecting on this, we feel that the design research paradigm was appropriate for the purpose of this study. The cyclic character allowed for adapting experimental settings, and the design of the learning trajectory and instructional activities served for making explicit our intentions. In the preliminary phase, the hypothetical learning trajectory was a productive way of monitoring the development of the hypotheses and expectations throughout the research. Prior identification of key items and their expected outcomes in the design phases guided classroom observations during the teaching experiments and data analysis in the retrospective phases. The mini-interviews on these key items provided relevant data, although this data gathering technique required a good attunement with the teacher and refrainment from playing the role of assistant teacher. The whole-class videos did not contribute much to the data, as the whole-class discussions in most cases did not address the key items or concepts that we wanted to observe. In the retrospective phase, the data analysis method – which combined a prior coding system with an approach similar to the constant comparative method – was effective. Working together with a second researcher who coded the data independently improved both the quality of the data analysis and the coding system. The method of formulating feed-forward for the next research cycle was helpful for capturing research progress.

How about the *generalizability* of the findings? Do the findings of this study depend on the specific choices that were made, or can they be generalized to other situations? Many issues that showed up in this study – such as the reification of formulas and expressions, and the development of symbol sense – are not specific to the concept of parameter. Therefore, we think that the scope of these findings exceeds the understanding of the concept of parameter and can be generalized to learning algebra in general. We also think that the conclusions can be generalized as regards instrumentation. Despite differences between computer algebra environments at a detailed level, these systems are based on common principles and similar instrumentation issues play a role. The question whether the instrumental approach can be applied to other IT environments deserves further research.

Generalization to other Dutch schools can be made as well, as long as the specific educational setting is taken into account. Generalization to CAS use in other grades should be considered with care. We expect more instrumentation problems to appear in lower grades, and we recommend longer periods of CAS use in ninth and tenth grades. For higher grades, we expect similar instrumentation issues and affordances to play a role, and we conjecture that in those grades the students will have more mathematical means for overcoming obstacles and benefiting from opportunities.

### *Recommendations*

The conclusions of this study led to recommendations for teaching, for software design and for further research. As regards *teaching algebra using computer algebra*, the results suggest that it is important to anticipate on computer algebra use, to be explicit about the changing didactical contract, to orchestrate individual and collective instrumentation, to have students compare CAS techniques with paper-and-pencil techniques and to have students reflect on the way CAS works.

As for *designing algebra software* for educational purposes, we recommend taking care of the criteria of the transparency and congruence of the technological environment. The black box character and idiosyncratic features inhibit instrumentation. Flexibility with regard to notations, syntax and strategy is a third criterion that should be met.

As regards *further research* on learning mathematics in a technological environment, we recommend taking into account the complete pedagogic situation rather than investigating the use of the technological environment as an isolated phenomenon, so that the social and the psychological perspective can be coordinated (Yackel & Cobb, 1996). Here, the cultural-historical activity theory – which considers tool use as embedded in a social practice – could provide a fruitful additional perspective. Furthermore, the relation between the instrumental approach and theories on symbolizing would be an interesting starting point for further theoretical development. A longitudinal study on symbolizing in a computer algebra environment, including adequate teaching that prepares for computer algebra use, is recommended. The extension of the scope of this study to other mathematical topics – such as the concept of function – and to other IT tools – such as software for dynamic geometry or Java applets – is also recommended. Finally, the role of technology in assessment requires further study.

# Samenvatting

## 1 Onderzoeksvragen

Het project 'Algebra leren in een computeralgebra omgeving'[1] waarover we hier rapporteren, maakt deel uit van onderzoek naar de integratie van informatie- en communicatietechnologie (ICT) in het onderwijs in het algemeen en naar de invloed daarvan op het leren van wiskunde in het bijzonder. In deze studie richten we ons op twee kwesties die op dit moment actueel zijn voor het wiskundeonderwijs: het algebraonderwijs en de integratie van technologie – in het bijzonder computeralgebra – daarin. Algebra is sinds jaar en dag een belangrijk onderwerp in de schoolwiskunde, dat een struikelblok is voor veel leerlingen. De moeilijkheden van het leren van algebra zijn gelegen in het formele en algoritmische karakter, het abstracte niveau waarop problemen worden benaderd, het object karakter dat algebraïsche expressies en formules hebben en de compacte algebraïsche taal met haar specifieke conventies en symbolen. Door deze moeilijkheden ervaren leerlingen algebra vaak niet als een natuurlijk en betekenisvol middel om problemen mee op te lossen (Bednarz et al., 1996; Chick et al., 2001).

Integratie van ICT is een van de manieren waarop de moeilijkheden met het leren van algebra zouden kunnen worden aangepakt. Het gebruik van ICT kan naar verwachting bijdragen aan visualisatie van begrippen en kan de leerling het uitvoeren van bewerkingen met de hand besparen. Daarmee kan hij/zij zich concentreren op begripsontwikkeling en probleemaanpak. Op deze manier zou het gebruik van ICT het traditionele algebracurriculum kunnen verlichten. Tegelijkertijd leidt de integratie van ICT tot vragen met betrekking tot de doelen van het algebraonderwijs en de relevantie van pen-en-papier methoden, nu die immers kunnen worden overgelaten aan het technologisch gereedschap. Wat betreft algebraïsche vaardigheden is het gebruik van een computeralgebra systeem (CAS) bijzonder interessant, omdat het een compleet repertoire aan algebraïsche procedures en operaties biedt (Heid, 1988; O'Callaghan, 1998).

De mogelijke integratie van computeralgebra in het algebraonderwijs leidt tot de volgende centrale onderzoeksvraag van deze studie:

> *Hoe kan het gebruik van computeralgebra het inzicht bevorderen in algebraïsche concepten en operaties?*

Deze vraag moet nader worden gespecificeerd. Omdat algebra als geheel een te veelomvattend onderwerp is, beperken we ons hier tot het parameterbegrip. Parameters komen op natuurlijke wijze naar voren in concrete probleemsituaties en zijn tevens middelen voor generalisatie en abstractie. Daarom kan het onderwerp parameter de

1. Dit onderzoek is mogelijk gemaakt door de Nederlandse organisatie voor Wetenschappenlijk Onderzoek NWO, projectnummer 575-36-003E.

leerling uitnodigen om de ‘algebraïsche wereld van formules, expressies en algemene oplossingen’ binnen te gaan. Verder omvat het parameterbegrip ook de verschillende rollen die bij ‘gewone’ variabelen een rol spelen en die de leerlingen eerder hebben leren kennen. Het gebruik van parameters kan het inzicht van de leerlingen in de betekenis en de structuur van algebraïsche formules en expressies verbeteren (Bills, 2001; Furinghetti & Paola, 1994). Om deze redenen is de onderzoeksvraag gespecificeerd in de volgende deelvraag:

1. *Hoe kan het gebruik van computeralgebra bijdragen aan een hoger niveau van inzicht in het parameterbegrip?*

Eerder onderzoek naar de integratie van computeralgebra in het wiskundeonderwijs toont aan dat het idee dat technologie de elementaire bewerkingen uitvoert zodat de leerling zich kan concentreren op begripsontwikkeling te simplistisch is (Artigue, 1997b; Drijvers, 2000; Guin & Trouche, 1999; Lagrange, 2000; Trouche, 2000). De technische vaardigheden die de leerling nodig heeft om procedures in de CAS omgeving uit te voeren vereisen conceptueel inzicht en tegelijkertijd beïnvloeden ze dat inzicht. Het tweede aandachtspunt in dit onderzoek is dan ook de verweven ontwikkeling van technieken in de computeralgebra omgeving en wiskundig inzicht in termen van mentale schema’s die de leerlingen ontwikkelen. Deze zogeheten instrumentele benadering van ICT-gebruik, die betrekking heeft op het instrumentatieproces waarin de duale relatie tussen begripsontwikkeling en techniek in de ICT-omgeving zich ontwikkelt, is het onderwerp van de tweede onderzoeksdeelvraag:

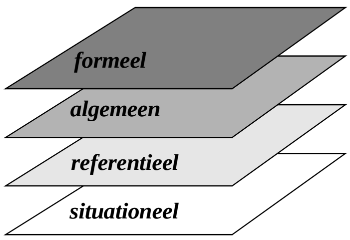
2. *Wat is de relatie tussen geïnstrumenteerde technieken in de computeralgebra omgeving en wiskundige begrippen, zoals die tot stand komt in het instrumentatieproces?*

## 2 Theoretisch kader

Bij het formuleren van de onderzoeksvragen vroegen we ons af welke theoretische invalshoeken van pas zouden komen bij het onderzoeken van deze kwesties. Een kant-en-klaar theoretisch kader voor een onderzoek naar het leren van algebra in een computeralgebra omgeving was niet beschikbaar. Daarom hebben we die theoretische elementen uit onderzoek naar het leren van algebra en wiskunde in het algemeen geselecteerd die kansrijk leken voor toepassing op het onderwerp van deze studie. Dat betekende dat deze elementen uit hun gebruikelijke context zijn gehaald en zijn aangepast aan het doel van dit onderzoek. Deze eclectische benadering van theorie is theorie-geleide bricolage genoemd (Gravemeijer, 1994). De volgende elementen zijn opgenomen in het theoretisch kader:

- *De domein-specifieke instructietheorie van Realistisch Wiskundeonderwijs*
  Kernbegrippen in de domein-specifieke instructietheorie van Realistisch Wis-

kundeonderwijs zijn geleide heruitvinding, progressief mathematiseren, horizontaal en verticaal mathematiseren, didactische fenomenologie en zich ontwikkellende modellen (Freudenthal, 1983; Gravemeijer, 1994; de Lange, 1987; Treffers, 1987a, 1987b). Geleide heruitvinding, progressief mathematiseren en didactische fenomenologie zouden naar verwachting bruikbaar zijn bij het ontwikkelen van een hypothetisch leertraject voor het parameterbegrip en van onderwijsactiviteiten. Het onderscheid horizontaal-verticaal mathematiseren zou de verhouding duidelijk kunnen maken tussen contexten en de abstracte microwereld van de computeralgebra omgeving, die een top-down karakter heeft (Drijvers, 2000). Het idee van zich ontwikkelende modellen zou bruikbaar kunnen zijn voor het onderscheiden van niveaus van activiteiten. Zoals aangegeven in Fig. 1 onderscheidt Gravemeijer (1994, 1999) vier niveaus van wiskundige activiteit. Het idee van zich ontwikkelende modellen is dat modellen die in eerste instantie verwijzen naar een concrete context met betekenis voor de leerlingen, zich geleidelijk aan ontwikkelen tot algemene modellen voor redeneren binnen een wiskundig kader.




Figuur 1 Vier niveaus van wiskundige activiteit (Gravemeijer, 1994, 1999)

- *Niveautheorieën*

  In de eerste onderzoeksdeelvraag is sprake van ‘een hoger niveau van inzicht in het parameterbegrip’. Verschillende perspectieven kunnen duidelijk maken wat hieronder wordt verstaan. Ten eerste onderscheidt de niveautheorie van Van Hiele een nulde, eerste en tweede niveau van inzicht (Van Hiele, 1973, 1986). Dit onderscheid zou bruikbaar kunnen zijn om niveaus van inzicht in het parameterbegrip te definiëren. Een tweede benadering van niveaus is het hierboven beschreven idee van zich ontwikkellende modellen en niveaus van activiteit (zie Fig. 1). In wezen is het streven in dit onderzoek gericht op de overgang van het referentiële naar het generieke of algemene niveau.
- *Theorieën over ‘symbol sense’ en symboliseren*

  De notie van symbol sense behelst het ‘gevoel’ voor algebraïsche entiteiten in het algemeen en het inzicht in formules in het bijzonder (Arcavi, 1994). In dit onderzoek is symbol sense gedefinieerd als het inzicht in de betekenis en de structuur van algebraïsche expressies en formules. Het werken met parametrische formules in een computeralgebra omgeving vereist naar verwachting symbol sense, maar bevordert het ook.

Hoe wordt symbol sense verworven? Theorieën over symboliseren benadrukken de parallelle ontwikkeling van symbolen en betekenis door een significatieproces (Gravemeijer et al., 2000). Omdat het geven van betekenis aan algebraïsche technieken, formules en expressies zoals ze naar voren komen in de computeralgebra omgeving naar verwachting een belangrijk aspect van de instrumentatie is, leek het symboliseren een relevant perspectief voor dit onderzoek.

- *De proces-object dualiteit*

  De proces-object dualiteit betreft het idee dat een wiskundig begrip als proces en als object beschouwd kan worden. Vaak ervaren leerlingen eerst de proceskant; op basis daarvan kunnen ze de objectkant ontwikkelen, die verdere conceptuele ontwikkeling vereist. Deze ontwikkeling heet reïficatie (Sfard, 1991) of inkapseling (Dubinsky, 1991) en resulteert in 'proceptueel' inzicht (Tall & Thomas, 1991). Deze theoretische inzichten hebben betrekking op het leren van wiskunde in het algemeen, maar kunnen worden toegepast op het leren van algebra.

  In dit onderzoek was de reïficatie van expressies en formules van belang omdat het bij het ontwikkelen van het parameterbegrip nodig is om formules en expressies als objecten te beschouwen. Verder zou het werk in de computeralgebra omgeving de verhouding tussen proces en object kunnen beïnvloeden. De reïficatie van expressies en formules veronderstelt dat leerlingen het zogenaamde 'lack of closure' obstakel overwinnen. Dit obstakel houdt in dat leerlingen niet in staat zijn om expressies en formules als resultaat te beschouwen zolang deze nog operatoren bevatten; leerlingen willen dan bijvoorbeeld in $a+b$ of $x+3$ de optelling uitvoeren die wordt gesymboliseerd door de + (Collis, 1975; Küchemann, 1981; Tall & Thomas, 1991). Reïficatie van formules en expressies betekent niet vanzelf reïficatie van functies; dat laatste is veelomvattender.

Een belangrijk theoretisch element in dit onderzoek is de instrumentele benadering van ICT-gebruik. Dit punt komt aan de orde in sectie 4.

## 3 Conceptuele analyse van het parameterbegrip

De conceptuele analyse van het parameterbegrip omvatte allereerst een onderzoek naar de historische ontwikkeling ervan. Essentieel in deze ontwikkeling was de overgang van syncopische naar symbolische algebra, die gekarakteriseerd wordt door het werk van Diophantus en Viète (Boyer, 1968; Harper, 1987). Waar Diophantus (rond het jaar 250) letters gebruikte om onbekenden aan te duiden, maar niet voor parameters, beschreef Viète (1540 - 1603) algemene, geparametriseerde oplossin-

gen. Viète onderscheidde onbekenden en parameters door het gebruik van klinkers en medeklinkers. Hij accepteerde expressies als oplossingen van algemene, parametrische vergelijkingen en beschouwde die als objecten. In dit onderzoek streefden we naar deze 'sprong van Diophantus naar Viète' ofwel van het verwijzende naar het algemene niveau (zie Fig. 1). Dat deze ontwikkeling veel tijd kostte in de geschiedenis kan als waarschuwing voor de complexiteit van deze stap worden opgevat.
De conceptuele analyse van het parameterbegrip leidde tot het beeld van de parametrische functie als een functie van de tweede orde. De korte pijl in Fig. 2 bijvoorbeeld geeft aan dat $x \rightarrow a \cdot x + 5$ voor een vaste waarde van de parameter $a$ een (lineaire) functie in $x$ is. Wanneer echter de waarde van $a$ verandert, dan geeft de lange pijl in Fig. 2 een tweede orde functie aan, met de parameter als argument en de lineaire uitdrukking als functiewaarde (Bloedy-Vinner, 2001).

$$a \longrightarrow (x \rightarrow a \cdot x + 5)$$

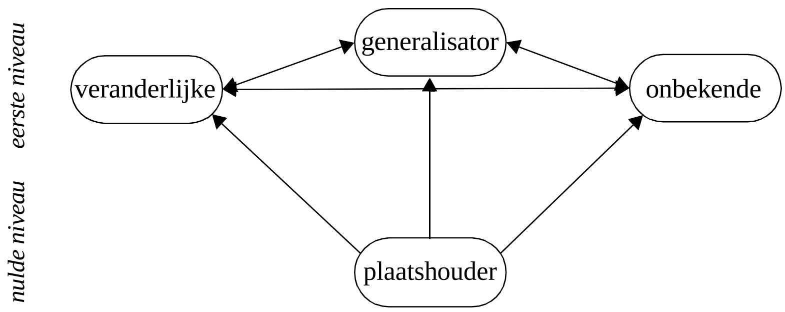
Figuur 2 De parameter als argument van een functie van de tweede orde

Als moeilijke aspecten van het parameterbegrip identificeerden we ten eerste deze hiërarchische relatie met 'gewone' variabelen, zoals weergegeven in Fig. 2 en in de uitdrukking 'veranderlijke constante'. Een tweede moeilijkheid is het onderscheiden van de verschillende rollen die de parameter kan spelen, die bovendien kunnen veranderen tijdens het oplossingsproces.
De conceptuele analyse leidde tot het onderscheiden van vier parameterrollen, die vergelijkbaar zijn met overeenkomstige rollen van de gewone variabele:

- De parameter als *plaatshouder* staat voor een positie, een lege plaats, waar een numerieke waarde kan worden ingevuld of uitgehaald. De waarde in de 'lege doos' is vast – bekend of onbekend – en verandert niet. Dit beschouwen we als het nulde niveau van het parameterbegrip.
- Bij de parameter als *veranderlijke* wordt de parameterwaarde systematisch gevarieerd. De parameter krijgt het dynamische karakter van een 'schuifparameter' die vloeiend een referentieverzameling doorloopt (Van de Giessen, 2002). Deze variatie beïnvloedt de hele situatie, de formule als object en de globale grafiek, terwijl variatie van de gewone variabele slechts lokaal doorwerkt.
- De parameter als *generalisator* generaliseert over een klasse van situaties. Daardoor verenigt deze 'familieparameter' (Van de Giessen, 2002) zo'n klasse en representeert die. Deze generieke representatie maakt het mogelijk om 'het algemene in het bijzondere' te zien en om categorieën van problemen op een generiek niveau te formuleren en op te lossen. Deze algemene oplossing van alle concrete gevallen ineens door middel van een parameter veronderstelt de reïficatie van de expressies en formules die in het generieke oplossingsproces voorkomen.

- De parameter speelt de rol van *onbekende* wanneer de vraag is om specifieke gevallen uit de algemene parametrische representatie te selecteren op basis van een extra conditie of criterium. Dit veronderstelt vaak een wisseling van rol en hiërarchie (Bills, 2001).

De conceptuele analyse en het theoretisch kader leverden een manier om het hoger niveau van inzicht in het parameterbegrip te definiëren. Fig. 3 visualiseert deze niveaustructuur.




Figuur 3 Niveaus van inzicht in het parameterbegrip

In termen van de Van Hiele niveaus beschouwen we de plaatshouder als het nulde niveau van het parameter inzicht, dat de basis vormt voor het eerste niveau. De drie rollen van veranderlijke, generalisator en onbekende hebben als gemeenschappelijk kenmerk dat de formules die de parameter bevatten beschouwd moeten worden als objecten. Daarom zijn die onderdeel van het inzicht het eerste niveau. De belangrijkste van deze drie 'hogere' parameterrollen is de generalisator, aangezien generaliseren een kernactiviteit is in de algebra. Het beoogde hoger niveau van inzicht in het parameterbegrip bestaat dus uit de sprong van de plaatshouder naar de andere parameterrollen, 'van Diophantus naar Viète' of, in termen van het vierlagen model (Fig. 1) de overgang van het referentiële naar het algemene niveau. Het inzicht in de hogere parameterrollen vereist naar verwachting de reïficatie van parametrische expressies en formules (Gravemeijer et al., 2000).

## 4 De instrumentele benadering van het gebruik van computeralgebra

De vraag is nu hoe het gebruik van computeralgebra kan bijdragen aan het verwerven van dit hogere niveau van inzicht in het parameterbegrip.
Om verschillende redenen veronderstelden we dat een computeralgebra omgeving kansen biedt voor het algebraonderwijs. In vergelijking met ander ICT-gereedschap voegt computeralgebra een compleet repertoire aan algebraïsche procedures en representaties toe. Doordat leerlingen bevrijd worden van algebraïsche berekeningen,

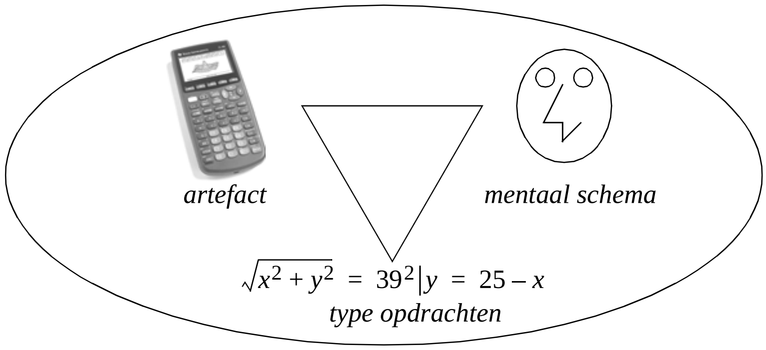
zou computeralgebra concentratie op begripsontwikkeling en probleemoplossen mogelijk maken. Het gebruik van een CAS zou leerlingen kunnen helpen concepten en vaardigheden te onderscheiden (Monaghan, 1993) en om de balans tussen die twee opnieuw vast te stellen (Heid, 1988; O'Callaghan, 1998).

De integratie van computeralgebra in het algebraonderwijs zou ook risico's met zich mee kunnen brengen. Omdat het CAS 'alle algebra al bevat' zou het een wat abstract en formeel 'top-down' karakter kunnen hebben en mogelijk inflexibel zijn bij het hanteren van informele notatie en syntax. Verder zou het CAS voor leerlingen een 'black box' kunnen lijken, omdat het complexe procedures uitvoert op een manier die voor hen ondoorzichtig is. Een laatste risico is dat de computeralgebra omgeving een microwereld voor de leerlingen schijnt, die niet is verbonden met de wereld van realistische problemen of van wiskunde met pen-en-papier en het hoofd (Drijvers, 2000).

Met het oog op het inzicht in het parameterbegrip zijn de volgende mogelijkheden van computeralgebra onderkend. Ten eerste verwachtten we dat het gebruik van computeralgebra de reïficatie van algebraïsche expressies en formules bevordert, doordat expressies optreden als oplossingen van parametrische vergelijkingen die vervolgens gesubstitueerd worden. Ten tweede veronderstelden we dat het genereren van voorbeelden door algebraïsche exploratie aanleiding zou zijn voor generalisatie over de situatie om zo een opening te bieden voor de parameter als generalisator. Ten derde zou de flexibiliteit van computeralgebra ten aanzien van lettersymbolen en hun rollen van pas kunnen komen bij de blikwisseling die optreedt bij de parameter als onbekende. Als laatste punt, hoewel niet exclusief voor computeralgebra omgevingen, werd verondersteld dat de beschikbaarheid van een schuifbalk het inzicht in de parameter als veranderlijke zou ondersteunen.

Voor de uitvoering van de onderwijsexperimenten moest een specifieke computeralgebra omgeving worden gekozen. Om praktische redenen hebben we de 'hand-held' TI-89 symbolische rekenmachine gekozen. Dit apparaat zouden de leerlingen permanent – zowel op school als thuis – ter beschikking hebben en zou niet nopen tot verandering van de organisatie van de lessen. De beperkte schermresolutie zou kunnen worden ondervangen door enkele lessen in het computerlokaal te houden, waarbij het softwarepakket TI-Interactive gebruikt zou kunnen worden. We hoopten dat het individuele karakter van de rekenmachine de samenwerking tussen leerlingen niet zou verhinderen en dachten dat te ondervangen door de leerlingen in tweetallen te laten werken.

Zoals al aangekondigd in de beschrijving van het theoretisch kader, functioneerde de instrumentele benadering van het gebruik van computeralgebra als kader voor het begrijpen en interpreteren van de interactie tussen leerling en machine. De kerngedachte van instrumentele benadering van het gebruik van ICT-gereedschap is dat een 'kaal' stuk gereedschap, een artefact, niet vanzelf een bruikbaar instrument is. Bij het hanteren van zulk gereedschap ontwikkelt de gebruiker mentale schema's, schema’s van geïnstrumenteerde actie, of kortweg instrumentatieschema’s (Artigue, 1997b, 2002; Guin & Trouche, 1999, 2002; Lagrange, 2000; Trouche, 2000). Een instrument, in deze optiek, bestaat uit (een deel van) het artefact, het mentale schema en het type taken waar de schema’s betrekking op hebben (Fig. 4).

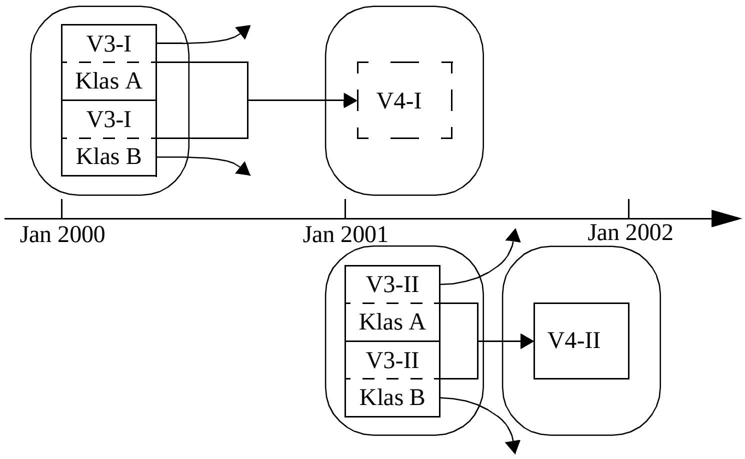



Figuur 4 Het instrument als driehoek artefact - mentaal schema - taak

In deze instrumentatieschema’s zijn technische vaardigheden en conceptuele inzichten geïntegreerd; moeilijkheden met het ontwikkelen van zulke schema’s, de zogenaamde instrumentele genese, hebben vaak betrekking op beide aspecten. Omdat we niet ‘in de hoofden van leerlingen kunnen kijken’ om de mentale schema’s te observeren, hebben we ons gericht op de technieken die we beschouwen als de observeerbare kant van de instrumentatieschema’s: de collectie procedures in de computeralgebra omgeving die de leerling gebruikt om een bepaald type problemen op te lossen.
In dit onderzoek is de instrumentele benadering gebruikt als kader om de interactie tussen leerling en computeralgebra omgeving te onderzoeken. Door de combinatie van technische en conceptuele aspecten binnen het instrumentatieschema leek deze theorie veelbelovend, omdat ze uitstijgt boven het wat naïeve idee van ‘vaardigheden terugdringen, begripsvorming benadrukken’ en omdat ze de moeilijkheden van de instrumentele genese serieus neemt.

## 5 Methodologie

In dit onderzoek is de methode van ontwikkelingsonderzoek gebruikt. Ontwikkelingsonderzoek richt zich op het ontwikkelen van theorieën over en empirisch onderbouwd inzicht in ‘hoe leren werkt’ (Research Advisory Committee, 1996). Het belangrijkste doel is het begrijpen van het leren van de leerling. Dit komt overeen met het karakter van onze onderzoeksvragen, die immers beginnen met ‘Hoe kan...’en niet met ‘Kan...’. Een karakteristiek van ontwikkelingsonderzoek is het belang dat wordt gehecht aan het ontwerpen van onderwijsactiviteiten, dat gezien wordt als betekenisvol onderdeel van de onderzoeksmethode omdat het de onderzoeker dwingt zijn keuzes, hypotheses en verwachtingen te expliciteren (Edelson, 2002). Een tweede belangrijk kenmerk van ontwikkelingsonderzoek is het bijstellen van het leertraject in de loop van het onderzoek. Op basis van eerdere ervaringen worden de onderwijssequentie en de experimentele condities aangepast. Dit maakt ontwikkelingsonderzoek bijzonder geschikt voor dit project, aangezien een volledig theoretisch kader nog niet beschikbaar was en hypotheses ontwikkeld dienden te worden. Het aanpassen van de experimentele situatie is mogelijk door het cyclische karakter van ontwikkelingsonderzoek. Een macrocyclus bestaat uit een voorbereidende fase, die in dit onderzoek de ontwikkeling van een hypothetisch leertraject omvat en het ontwerp van onderwijsactiviteiten), een onderwijsexperiment en een retrospectieve fase waarin de data-analyse plaatsvindt en die leidt tot ‘feed-forward’ voor de volgende onderzoekscyclus (Gravemeijer, 1994).




Figuur 5 Arrangement van onderzoekscycli en onderwijsexperimenten

In dit onderzoek zijn drie onderzoekscycli uitgevoerd, aangegeven met V3-I, V3-II en V4-II, en een tussencyclus, V4-I. De onderwijsexperimenten van de V3-cycli vonden plaats in vwo-3 klassen en die van de V4-cycli in vwo-4. De I en II slaan op de cohorten leerlingen die in de cyclus betrokken waren. De V4-I populatie was (op een enkele uitzondering na) een deelverzameling van de V3-I populatie en hetzelfde gold voor het tweede cohort leerlingen (zie Fig. 5).

We lichten nu kort de verschillende fasen binnen een onderzoekscyclus toe. In de *voorbereidende fase* werd een hypothetisch leertraject (HLT) ontwikkeld (Simon, 1995). Dit houdt in dat het aanvangsniveau werd vastgesteld, dat het einddoel werd geformuleerd en dat een sequentie van mentale stappen die tot dat doel leiden is onderscheiden, samen met onderwijsactiviteiten die naar verwachting die stappen teweeg brengen. Hierbij is beschreven waarom deze activiteiten geacht worden te werken en welk soort mentale ontwikkeling daarmee wordt beoogd. Vanwege de nadruk op mentale ontwikkeling en op het motiveren van de verwachte effecten door de onderzoeker was het HLT een bruikbaar instrument om de ontwikkeling van hypothese bij te houden en het denken van de onderzoeker in kaart te brengen.

De ontwikkeling van een HLT was nauw verweven met het ontwerp van onderwijsactiviteiten, dat plaatsvond aan de hand van ontwerpheuristieken zoals geleide heruitvinding, didactische fenomenologie en bemiddelende modellen. Daarbij zijn kernopgaven geselecteerd, die zouden kunnen dienen als ijkpunten voor de mentale ontwikkeling van de leerlingen in relatie tot het HLT; ook zijn vragen opgesteld om leerlingen te stellen bij deze ijkpuntopgaven en is een a priori coderingssysteem voor observatie ontwikkeld.

De tweede fase van een cyclus in het ontwikkelingsonderzoek is die van het *onderwijsexperiment* (Steffe, 1983; Steffe & Thompson, 2000). Tijdens de onderwijsexperimenten lag de nadruk op data die het leerproces weergeven en inzicht konden geven in het denken van de leerlingen. De belangrijkste bronnen daarvoor waren observaties van leerlingengedrag en korte interviews met leerlingen. De meeste lessen zijn geobserveerd door twee observatoren die aantekeningen maakten op observatieformulieren. De ijkpuntopgaven werden onderzocht door middel van ‘mini-interviews’ met een deel van de leerlingen; videoregistraties zijn gemaakt van klassengesprekken en van enkele geselecteerde tweetallen leerlingen terwijl ze aan het werk waren. Schriftelijk materiaal (schriften, voortoets en natoets) is verzameld van alle leerlingen. In totaal zijn 110 leerlingen bij de onderwijsexperimenten betrokken geweest en zijn meer dan 100 lessen geobserveerd.

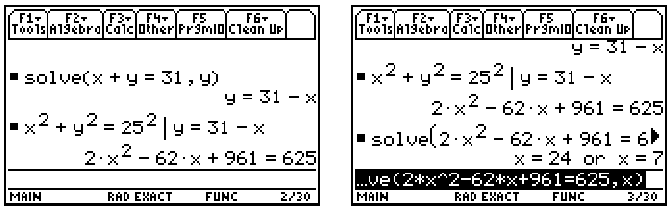
Een onderzoekscyclus eindigt met de *retrospectieve* fase. Een eerste stap hierin was de selectie en analyse van de gegevens. De aanvankelijke analysemethode was geïnspireerd op de constante comparatieve methode (Glaser & Strauss, 1967; Strauss, 1987; Strauss & Corbin, 1998). De eerste bevindingen hieruit leidden tot aanpassing van het a priori ontworpen coderingssysteem. Vervolgens werden de gegevens gecodeerd. Dit gebeurde voor een deel door twee onderzoekers, om zo een intersubjectieve overeenstemming te bereiken. De conclusies van de data-analyse werden vertaald in feed-forward voor de volgende onderzoekscyclus. Deze feed-forward betrof aanpassingen van het HLT, van de onderwijsactiviteiten of veranderende aandachtspunten voor het volgende onderwijsexperiment.

## 6 De drie onderzoekscycli

Fig. 5 geeft het arrangement van de onderzoekscycli weer. Deze sectie beschrijft kort elk van de cycli en de onderwijsexperimenten daarbinnen in het bijzonder. Deze onderwijsexperimenten vonden plaats op een school in Bilthoven.

De eerste onderzoekscyclus, V3-I, omvatte een onderwijsexperiment in twee vwo-3 klassen met leerlingen van 14-15 jaar oud. De experimenten duurden vijf weken met

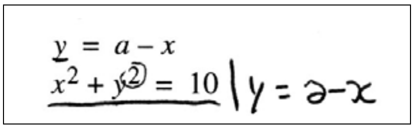
vier lessen wiskunde per week. In het HLT voor deze experimenten kwamen de parameterrollen voor in de volgorde plaatshouder-generalisator-veranderlijke-onbekende. De onderwijsactiviteiten waren beschreven in twee pakketten, ‘Introductie TI-89’ dat beoogde de leerlingen vertrouwd te maken met de TI-89 symbolische rekenmachine en ‘Veranderlijke algebra’ dat gericht was op de ontwikkeling van inzicht in het parameterbegrip.
De resultaten toonden aan dat generalisatie werd gehinderd door instrumentatieproblemen, in het bijzonder bij het schema voor het oplossen van stelsels vergelijkingen. Fig. 6 toont dit schema van isoleren-substitueren-oplossen, afgekort tot ISO, voor het stelsel $x + y = 31$ , $x^2 + y^2 = 25^2$ .




Figuur 6 Het schema isoleren-substitueren-oplossen op de TI-89

In sommige gevallen hadden de leerlingen geen behoefte aan generalisatie, of had de generalisatie het oppervlakkige karakter van patroonherkenning zonder intrinsieke betekenis. De parameter als veranderlijke werd beter begrepen, hoewel een optie voor het visualiseren van de dynamiek op de TI-89 werd gemist. Aan de parameter als onbekende is niet veel aandacht besteed. De rolwisseling die dit vroeg was moeilijk voor de leerlingen. Het substitueren van expressies en het voorkomen van expressies als oplossingen van geparametriseerde vergelijkingen bevorderde de reïficatie van expressies en het overwinnen van het ‘lack of closure’ obstakel. De resultaten van de eindtoets bevestigden dat de generalisatie en het toepassen van het ISO-schema de twee grootste hindernissen waren voor de leerlingen.
De feed-forward van de V3-I onderzoekscyclus hield onder meer een herordening in van de globale leerlijn door de parameterrollen; die werd gewijzigd in de volgorde plaatshouder-veranderlijke-generalisator-onbekende, om de generalisatie uit te stellen tot een dynamische en hiërarchische kijk op het parameterbegrip was ontwikkeld. Daarom is ook gezocht naar een betere manier om de dynamiek van de schuifparameter op het scherm van de TI-89 in beeld te brengen. Om de instrumentatieproblemen te beperken, zijn in het lesmateriaal meer oefeningen met elementaire schema’s voor oplossen en substitueren opgenomen door deze te integreren in het samengestelde ISO-schema.
Het onderwijsexperiment van de tweede onderzoekscyclus, V3-II, is uitgevoerd in het tweede cohort van twee vwo-3 klassen. De opzet was vergelijkbaar met die van

V3-I. Het experiment duurde vijf weken van vier lessen per week. Het HLT volgde de lijn die uit de feed-forward van V3-I naar voren was gekomen. Het beeld van de parameter als veranderlijke werd ondersteund door twee TI-89 programma's die voor dit doel ontwikkeld waren, SCHIET en SCHUIF. De leerlingentekst bestond weer uit een deel gericht op de kennismaking met de TI-89 en een deel gericht op het inzicht in het parameterbegrip.
De resultaten toonden aan dat het begrip van de parameter als veranderlijke inderdaad werd bevorderd door de programma's SCHIET en SCHUIF, hoewel sommige leerlingen de parameter beschouwden als lopend door gehele waarden. De nieuwe volgorde in het leertraject, waarin de parameter als veranderlijke eerder aan de orde komt, was een verbetering ten opzichte van de lijn van V3-I. Voor wat betreft generalisatie kwamen vergelijkbare moeilijkheden aan het licht als in V3-I, zij het in mindere mate. Ondanks de aandacht besteed aan elementaire instrumentatieschema's voor oplossen en substitueren veroorzaakte het samengestelde ISO-schema nog steeds moeilijkheden. Ook hadden de leerlingen regelmatig geen behoefte aan generalisatie, misschien vanwege de abstracte en complexe probleemsituaties. De parameter als onbekende is meer aan de orde geweest dan in V3-I, maar de complexiteit van de probleemsituaties maakte dat de leerlingen niet in staat waren de grote lijn van de oplossingsstrategie te doorzien en de veranderende rol van de parameter daarin.
De resultaten van de eindtoets bevestigden deze bevindingen, hoewel de uitvoering van het ISO-schema en de generalisaties met parameters iets verbeterd waren ten opzichte van de resultaten van V3-I. Na de eindtoets zijn eindinterviews gehouden met negentien leerlingen. De resultaten van deze interviews, waarbij de leerlingen geen technologie gebruikten, suggereerden transfer van substitutie van expressies als computeralgebra techniek naar substitutie als pen-en-papier techniek en een verbeterde reïficatie van expressies. De leerlingen waren bijvoorbeeld in staat om $y = a - x$ met de hand in $x^2 + y^2 = 10$ te substitueren. Een van hen gebruikte daarbij de TI-89 notatie voor substitutie (Fig. 7).




Figuur 7 Transfer van notatie

Als feed-forward van de V3-II onderzoekscyclus concludeerden we dat het globale HLT langs de lijn plaatshouder-veranderlijke-generalisator-onbekende adequaat was en geen verdere wijziging behoefde. Voor de parameter als veranderlijke zou het gebruik van een continue schuifbalk wellicht het idee van een discrete referentieverzameling kunnen vermijden. De bevindingen suggereerden dat de parameter als generalisator beter aan de orde kan komen zonder de complicatie van het ISO-schema. Dat betekende dat probleemsituaties gezocht moesten worden die met één vergelijking in plaats van een stelsel van twee vergelijkingen konden worden opgelost. Verder richtten de probleemsituaties in V3-II zich te direct op het generale niveau. Het gebruik van realistische contexten, waarin de parameter op het referentiële niveau betekenis heeft voor de leerlingen, zou wellicht kunnen leiden tot eenvoudiger en meer betekenisvolle generalisaties. Grafieken zouden ook als context kunnen functioneren en zouden bruikbaar kunnen zijn bij de overgang van de parameter als veranderlijke naar de generalisator. Het gebruik van realistische contexten zou de parameter als onbekende ook op een meer natuurlijke manier kunnen benaderen.

In de V4-I tussencyclus vond een kort onderwijsexperiment van vijf lesuren plaats in vwo-4 wiskunde B. Alle leerlingen op één na hadden ook deelgenomen in V3-I. Vanwege een gebrek aan lestijd en een te vol regulier programma was deze tussencyclus eerder een vingeroefening voor V4-II dan een volledige onderzoekscyclus. De bevindingen waren dat de leerlingen de TI-89 vaardigheden die ze een jaar tevoren hadden geleerd snel weer oppakten. Eens te meer werd duidelijk dat een schuifbalk voor de parameter als veranderlijke te prefereren zou zijn boven het vervangen van parameterwaarden door substitutie.

In de derde en laatste volledige onderzoekscyclus, V4-II, vond een onderwijsexperiment plaats van vijftien lesuren in vwo-4 wiskunde B. Alle leerlingen met uitzondering van twee doubleurs hadden ook aan het V3-II onderwijsexperiment deelgenomen.

Overeenkomstig de feed-forward van V3-II is in V4-II de globale leerlijn van plaatshouder-veranderlijke-generalisator-onbekende aangehouden. Voor de parameter als veranderlijke is de continue schuifbalk van het softwarepakket TI-Interactive gebruikt. Meer dan in voorgaande experimenten kwamen realistische contexten aan de orde waarin parameters en formules betekenis hadden en die op een meer natuurlijke manier tot generalisatie uitnodigden. De verwachting was dat dit ook het inzicht in de parameter als onbekende ten goede zou komen.

De resultaten toonden aan dat het gebruik van de continue schuifbalk inderdaad het beeld van de parameter als veranderlijke door een continuüm verbeterde. Contexten waarin de parameter een evidente betekenis had in de grafiek maakten generalisaties betekenisvol. Realistische contexten leidden ook tot een beter inzicht in de betekenis en structuur van formules en expressies. Toch was het invoeren van gebroken expressies en uitdrukkingen met exponenten moeilijk, evenals het herkennen van equivalente expressies. Doordat de probleemsituaties leidden tot één in plaats van twee vergelijkingen werden de moeilijkheden met het ISO-instrumentatieschema vermeden. Wellicht ook als gevolg van de betekenisvolle probleemsituaties kwam de parameter als onbekende met meer succes aan de orde dan in de voorafgaande experimenten, al leek de complexiteit van de formules hierbij een cruciale factor.

De resultaten van de eindopdracht van V4-II waren positief. Veel goede generalisaties werden gemaakt en de leerlingen konden in het algemeen goed overweg met (delen van) de complexe formules. De realistische probleemsituatie van de snelheid van een file hielp de leerlingen om betekenis aan de uitkomsten te geven.

## 7 Het gebruik van computeralgebra en het parameterbegrip

De eerste onderzoeksdeelvraag betreft de bijdrage van het gebruik van computeralgebra aan het hoger niveau van inzicht in het parameterbegrip. Dit hoger niveau van inzicht was gedefinieerd als inzicht in de hogere parameterrollen van veranderlijke, generalisator en onbekende. We vatten de resultaten op dit punt samen.

Zoals verwacht op basis van de conceptuele analyse, was het plaatshouder-beeld van de parameter voor vrijwel alle leerlingen het startniveau. De computeralgebra omgeving droeg bij aan het uitbreiden van het parameterbegrip met de rol van *veranderlijke* door de mogelijkheid om de parameter steeds een andere waarde te geven. Het onderzoeken van het dynamische effect van verandering van de 'schuifparameter' met behulp van een schuifbalk – niet exclusief voor computeralgebra – bleek echter meer effect te sorteren. Dit leidde tot het inzicht dat verandering van de parameterwaarde verandering van de grafiek als geheel tot gevolg had, een verandering 'van de tweede orde'. Als de parameter voor de leerling een betekenis had in de context of als grafische eigenschap, dan nodigde het onderzoek van de dynamische grafiek uit tot algebraïsche verificatie en generalisatie. Het gevaar van oppervlakkige waarneming zonder reflectie of verificatie was echter aanwezig. Verder domineerde de parameter als veranderlijke het parameter-beeld van de leerlingen, zodat ze de andere parameterrollen soms leken te vergeten. Al met al was een hoger niveau van inzicht in de parameter als veranderlijke bereikt.

Het werk in de computeralgebra omgeving bevorderde de overgang naar de parameter als *generalisator* door herhaling van dezelfde procedure voor verschillende parameterwaarden mogelijk te maken. Dit genereerde voorbeelden die de basis vormden voor generalisatie en voor het oplossen van parametrische vergelijkingen.

Het feit dat de parametrische vergelijkingen leidden tot expressies in plaats van numerieke oplossingen, bevorderde de reïficatie van formules en expressies en het overwinnen van het 'lack of closure' obstakel. De dynamische grafieken die voortkwamen uit het werk met de schuifparameter leidde in sommige gevallen tot generalisaties, bijvoorbeeld in het geval van de grafiek van $y = (x-a)^2 + a$. Een tweetal leerlingen zag dat de top van de grafiek coördinaten $(a, a)$ had voor alle waarden van $a$. Een van hen zei 'de top is in $a$' en '$y$- en $x$-coördinaat zijn hetzelfde als $a$'. Later, in een klassengesprek, legde de andere leerling van het tweetal de generalisatie uit door naar een voorbeeldmatige parameterwaarde te verwijzen.

> *Maria:* *Ja volgens mij die top dat waren de x- en y-coördinaten, die waren zeg maar gelijk aan de a. De top was (1, 1) en dan was a ook 1.*

Veel leerlingen bereikten het hoger niveau van inzicht in de parameter als generalisator slechts in beperkte mate. In sommige gevallen hadden de leerlingen geen behoefte aan generalisatie. In andere gevallen kwam generalisatie neer op fenomenologische patroonherkenning zonder intrinsiek begrip. De onderwijsactiviteiten waren te direct gericht op het algemene niveau en de formules waren in sommige opgaven te complex. Instrumentatieproblemen bemoeilijkten de generalisatie. De essentiële stap van het generaliseren van relaties, waarbij parameters gebruikt worden om een klasse van situaties te verenigen, bleek primair een mentale te zijn waaraan computeralgebra niet zo veel bijdroeg.

De mogelijkheid om in de computeralgebra omgeving vergelijkingen op te lossen naar elke onbekende, vergrootte de flexibiliteit van de leerlingen ten aanzien van de rollen van de lettervariabelen. De resultaten betreffende de parameter als *onbekende* waren wisselend: in eenvoudige gevallen werd deze parameterrol begrepen, terwijl de leerlingen in ingewikkelder gevallen de veranderende rollen en de betekenissen van variabelen en parameters niet meer uit elkaar konden houden. De verandering in hiërarchie tussen variabele en parameter was in complexe situaties moeilijker te doorzien.

Twee kwesties leken de ontwikkeling van een hoger niveau van inzicht in het parameterbegrip te beïnvloeden: het gebruik van realistische probleemsituaties en het inzicht in de betekenis en structuur van expressies en formules. Het belang van *realistische probleemsituaties* is al eerder genoemd. In tegenstelling tot onze oorspronkelijke ideeën maakte het gebruik van een computeralgebra omgeving referenties naar de werkelijkheid buiten die omgeving voor leerlingen van deze leeftijd en dit niveau niet overbodig. De directe gerichtheid op het generale niveau werkte niet, omdat de leerlingen nog geen adequaat wiskundig relatienet hadden ontwikkeld. In plaats daarvan hadden ze een realistische probleemsituatie nodig waaraan de parameters, expressies en formules betekenis ontleenden op een referentieel niveau. Deze realistische probleemsituaties stonden generalisatie en abstractie die uitsteeg boven de context niet in de weg.

Door de aanwezigheid van parameters waren de *expressies en formules* die de leerlingen in het lesmateriaal tegenkwamen complexer dan gewoonlijk. De CAS output, algebraïsch of grafisch, nodigde uit tot een nadere beschouwing van de formules en expressies die daarin voorkwamen, vooral als dit werd gesuggereerd door het lesmateriaal of door de docent. Het invoeren van complexe formules vereiste inzicht in de algebraïsche structuur. Daarom was inzicht in de betekenis en structuur van expressies en formules aan de ene kant een obstakel tijdens het werk met het CAS, maar werd het er aan de andere kant door bevorderd. Het werk in de computeralgebra omgeving ondersteunde de ontwikkeling van symbol sense, de reïficatie van formules en expressies en het overwinnen van het lack of closure obstakel.

De globale conclusie is dan ook dat het gebruik van computeralgebra inderdaad kan bijdragen aan het ontwikkelen van het hoger niveau van inzicht in het parameterbegrip. Voor de parameter als veranderlijke was deze bijdrage evidenter dan voor de parameter als generalisator en als onbekende. Verder werd de reïficatie van expressies en formules bevorderd en waren realistische startpunten onmisbaar voor betekenisvol werk in de computeralgebra omgeving.

Verschillende factoren zouden het beoogde leerproces verder kunnen optimaliseren. Ten eerste suggereren de data dat het vruchtbaar zou kunnen zijn om de verschillende parameterrollen te integreren in plaats van ze sterk te scheiden, zodat de leerlingen een meer geïntegreerd beeld van de parameter ontwikkelen. Ten tweede is al opgemerkt dat modellen die verwijzen naar concrete probleemsituaties bij leerlingen van deze leeftijd en niveau tot betere resultaten leiden dan abstractere en algemenere modellen. Ten derde zou computeralgebra naar verwachting meer bijdragen aan de conceptuele ontwikkeling als het voor een langere periode wordt gebruikt, zodat instrumentatieproblemen minder dominant zijn. Ten slotte zouden naar verwachting klassengesprekken, geleid door de docent, de collectieve conceptuele ontwikkeling meer hebben kunnen bevorderen. De docenten beschouwden de cohorten in de onderwijsexperimenten als zwakke cohorten. Mogelijk zouden deze groepen gebaat zijn geweest bij meer sturing.

De moeilijkheden waar leerlingen bij het werken met het CAS tegenaan liepen suggereren dat het instrumentatieproces van belang is. Dat is het onderwerp van de volgende sectie.

## 8 De instrumentatie van computeralgebra

De tweede onderzoeksdeelvraag heeft betrekking op de instrumentatie van computeralgebra en op de relatie tussen computeralgebra technieken en conceptuele ontwikkeling. Deze sectie vat de resultaten op dit punt samen.

Als enkelvoudige instrumentatieschema's zijn het oplossingsschema en het substitutieschema onderzocht. Voor beide schema's, die samenhangen met het parameterbegrip, hield de instrumentele genese een uitbreiding in van het inzicht. Voor het *oplossingsschema* leidden de syntax en de toepassing op parametrische vergelijkingen ertoe dat leerlingen zich realiseerden dat een vergelijking altijd wordt opgelost naar een onbekende, dat het oplossen van een parametrische vergelijking betekent dat een variabele wordt uitgedrukt in de andere en dat de oplossing in dat geval een expressie is. Bij het *substitutieschema* leidde het substitueren van expressies in de computeralgebra omgeving ertoe dat leerlingen zich realiseerden dat alleen geïsoleerde vormen kunnen worden gesubstitueerd en dat expressies als 'dingen' kunnen worden beschouwd die 'in een variabele kunnen worden geplakt'. Overeenkomstig de ideeën van Sfard bevorderde de operatie op de expressies het objectkarakter (Sfard, 1991).

Hoewel de instrumentele genese van het oplossingsschema en het substitutieschema vlot leek te verlopen, veroorzaakte het combineren van deze elementaire schema's in het samengestelde schema *isoleren-substitueren-oplossen* hardnekkige moeilijk-

heden. Dit geeft aan dat de integratie van enkelvoudige schema's in meeromvattende schema's een beheersing op hoog niveau van die elementaire schema's vraagt. De instrumentele genese had kennelijk meer tijd nodig dan de leerlingen gegeven werd.
Een aantal obstakels stond de instrumentele genese in de weg, zoals de omgang van het CAS met numerieke benaderingen en exacte algebraïsche resultaten, de moeilijkheden met het invoeren van expressies die parameters, wortels en machten bevatten, en het interpreteren van de resultaten. Het herkennen van de equivalentie van CAS output en verwachte resultaten was een bijzonder lastige kwestie. Veel van deze obstakels hielden verband met een gebrek aan symbol sense, maar konden ook leiden tot de ontwikkeling daarvan. Door een gebrek aan inzicht in de structuur van expressies en formules maakten de leerlingen fouten bij het werk in de computeralgebra omgeving, maar deze problemen leidden ook tot een nadere blik op deze structuren. In die zin brachten de obstakels ook kansen voor het leren met zich mee, als de docent daar adequaat mee omging.
Sommige van de geobserveerde moeilijkheden met het invoeren van expressies en het interpreteren van algebraïsche uitkomsten hingen samen met het werken met pen-en-papier zoals leerlingen dat gewend waren. Incongruentie tussen computeralgebra techniek en pen-en-papier techniek verklaarde een deel van de instrumentatieproblemen. Wanneer de technieken in beide media wel congruent waren, vond transfer van notatie, strategie en techniek plaats. Een expliciete vergelijking van computeralgebra techniek en pen-en-papier techniek was effectief om leerlingen de mate van congruentie vast te laten stellen. Een tweede voorwaarde voor transfer was de transparantie van de computeralgebra technieken; als leerlingen konden begrijpen hoe het CAS aan de uitkomsten was gekomen en het niet als een 'black box' beschouwden, kon transfer naar het werk met pen-en-papier plaatsvinden. Uit de gegevens bleek verder dat leerlingen verschillende voorkeuren ontwikkelden ten aanzien van werk met pen-en-papier versus werk in de computeralgebra omgeving.
De docent speelde een belangrijke rol in de instrumentele genese. Verschillend gedrag van de docent leidde tot verschillende instrumentatietechnieken. Demonstraties in de klas en klassengesprekken bepaalden de collectieve instrumentatie; dit aspect van instrumentatie had meer aandacht verdiend in de onderwijsexperimenten. Verder moest een nieuw didactisch contract worden vastgesteld ten aanzien van de verhouding tussen werken met-de-hand en werken met de machine en ten aanzien van grafisch-numerieke methoden en algebraïsche methoden. Het feit dat de integratie van computeralgebra nieuw was voor de docenten – en ook van tijdelijke aard – maakte het moeilijk om zo'n nieuw didactisch contract vast te stellen en om een collectieve instrumentele genese te bewerkstelligen.
Al met al suggereren de resultaten van dit onderzoek een nauwe wederzijdse relatie tussen CAS techniek en conceptueel inzicht. Leerlingen moesten instrumentatieschema's opbouwen die technische en conceptuele aspecten combineerden. Deze instrumentele genese kostte tijd en moeite, waarbij obstakels overwonnen moesten wor-

den. Op het oog technische moeilijkheden bleken vaak samen te hangen met hiaten in het inzicht. Pen-en-papier methodes speelden hierin ook een rol. Voor het onderwijs is het dan ook aan te bevelen om een nieuw didactisch contract te ontwikkelen, om collectieve instrumentatie teweeg te brengen door middel van klassengesprekken en demonstraties en om congruentie en incongruentie tussen pen-en-papier en CAS techniek tot onderwerp van gesprek te maken.

## 9 Algebra leren en het gebruik van computeralgebra in het algemeen

In deze sectie bespreken we eerst de conclusies ten aanzien van de onderzoekshoofdvraag. Dan volgt een reflectie op het onderzoek en we besluiten met enkele aanbevelingen.

### *Conclusies*

De centrale onderzoeksvraag van dit onderzoek betreft de bijdrage van het gebruik van computeralgebra aan het inzicht in algebraïsche concepten en operaties in het algemeen. Door de antwoorden op de onderzoeksdeelvragen te extrapoleren naar dit algemenere niveau zijn de volgende conclusies getrokken.

Ten eerste betreffen sommige bevindingen niet het inzicht in het parameterbegrip maar blijkt dat het gebruik van computeralgebra kan bijdragen aan het inzicht in algebraïsche begrippen en operaties in het algemeen. De leerlingen ontwikkelden symbol sense en inzicht in de structuur en betekenis van formules en expressies. Algebraïsche notaties werden uitgebreid. Computeralgebra bood mogelijkheden om verschillende representaties te combineren, procedures te herhalen als voorbereiding op generalisatie, en expressies als objecten te behandelen. Bepalend voor het effect van deze kansen waren de didactische inbedding en het gebruik van betekenisvolle formules die voortkwamen uit realistische probleemsituaties, in plaats van direct te verwijzen naar het algemene niveau.

Ten tweede is geconstateerd dat de instrumentele genese van CAS instrumentatieschema's, mits adequaat georkestreerd, ook conceptuele ontwikkeling met betrekking tot algebra in het algemeen met zich meebracht. De ontwikkeling van het oplos- en substitutieschema leidde tot nieuwe inzichten in het oplossen en substitueren. Incomplete instrumentatie stond het leren in de weg, maar instrumentatieobstakels boden ook kansen voor het leren. Condities die de instrumentatie bevorderden waren de congruentie tussen CAS techniek, mentaal beeld en pen-en-papier techniek en de transparantie van de CAS procedures. Dit bleken lastige voorwaarden te zijn, aangezien een aantal CAS procedures niet transparent was voor de leerlingen en omdat CAS notaties en syntax afweken van wat de leerlingen gewend waren vanuit hun pen-en-papier ervaring.

Als we deze bevindingen vergelijken met de aanvankelijke verwachtingen, blijkt dat de instrumentatiemoeilijkheden hardnekkiger waren dan verwacht en dat de rol van de docent belangrijker was dan voorzien. Verder bleken onze ideeën om direct naar het algemene in plaats van het referentiële niveau te streven niet haalbaar en konden realistische probleemsituaties als beginpunt niet worden gemist. Wat leerlingen als realistisch ervaren is afhankelijk van hun leeftijd en niveau: we nemen dan ook aan dat de computeralgebra omgeving voor oudere leerlingen, die op een hoger niveau wiskunde leren, eerder als realistische omgeving op zichzelf wordt ervaren.

### *Reflectie*

Wat kunnen we terugkijkend op dit onderzoek zeggen over de rol van theorie, over de gebruikte methodologie en over de generaliseerbaarheid van de resultaten? We lopen eerst de elementen van het theoretisch kader langs.

De *domein-specifieke onderwijstheorie van Realistisch Wiskundeonderwijs* hielp ons om te begrijpen waarom de leerlingen realistische probleemsituaties nodig hadden als startpunt: de leerlingen hadden nog geen geschikt wiskundig netwerk van algebraïsche objecten en procedures ontwikkeld. Het onderscheid tussen het referentiële en het algemene niveau (zie Fig. 1) verhelderde onze doelen en tevens de moeilijkheden die leerlingen hadden. De noties van horizontaal en verticaal mathematiseren maakten duidelijk waarom generalisatie en abstractie in veel gevallen niet uit de verf kwamen. Ten slotte kwam de RME van pas bij het begrijpen van het probleem van het ‘black box’ karakter van het gebruik van een CAS.

De *niveautheorieën* fungeerden op de achtergrond bij het definiëren van het hoger niveau van inzicht in het parameterbegrip en hielpen bij het ontwerp van het leertraject en de onderwijsactiviteiten.

We hebben *symbol sense* gedefinieerd als inzicht in de structuur en betekenis van formules en expressies. In die interpretatie vormde symbol sense een manier om de omgang van leerlingen met formules en expressies in de computeralgebra omgeving te interpreteren: symbol sense was een voorwaarde voor productief gebruik van een CAS, maar kon zich ook van daaruit ontwikkelen. Wel zouden we willen suggereren dat het begrip symbol sense nauwkeuriger wordt gedefinieerd.

*Theorieën over symboliseren* maakten duidelijk dat het karakter van de computeralgebra omgeving en de aanvankelijke keuzes van dit onderzoek niet veel ruimte boden voor een ‘bottom-up’ significatieproces. Anderzijds leidde de instrumentatie van computeralgebra tot het ontwikkelen van betekenisvolle wiskundige objecten en een netwerk van wiskundige relaties dat verbonden was met de symbolen en expressies die de leerlingen in de computeralgebra omgeving tegenkwamen. De relatie tussen de instrumentele benadering en theorieën over symboliseren verdient nader onderzoek.

De *proces-object dualiteit* was nuttig voor het onderkennen van de invloed die het gebruik van computeralgebra had op de kijk van leerlingen op formules en expressies als objecten. Samen met symbol sense was reïficatie een referentiekader om de vooruitgang van het algebraïsch inzicht van de leerlingen te volgen.

De *theorie van instrumentatie* hielp bij het interpreteren van leerlingengedrag in de computeralgebra omgeving en bij het in kaart brengen van de relatie tussen technische en conceptuele aspecten binnen een instrumentatieschema. De instrumentele genese werd soms gehinderd door conceptuele obstakels, maar kon ook de conceptuele ontwikkeling bevorderen, zoals de theorie suggereert. In dit onderzoek is een aantal concrete instrumentatieschema's uitgewerkt en zijn de criteria van congruentie en transparantie benadrukt. Het verband tussen computeralgebra techniek en overeenkomstige pen-en-papier techniek verdient meer aandacht.
Samengevat zien we dat de elementen van het theoretisch kader aan het onderzoek hebben bijgedragen, ondanks het feit dat ze – met uitzondering van de instrumentele benadering – niet voor gebruik in deze context ontwikkeld zijn. Een belangrijke opbrengst van dit onderzoek voor het theoretisch kader is dan ook de waargenomen bijdrage van de theorieën, wat een bredere toepasbaarheid suggereert.
Hoe kijken we terug op de gevolgde *methodologie*? Het onderzoeksparadigma van ontwikkelingsonderzoek was geschikt voor het doel van dit onderzoek. Het cyclische karakter maakte het mogelijk om de experimentele omstandigheden aan te passen en het ontwerp van leertraject en onderwijsactiviteiten dwong ons tot explicitering van de intenties. In de voorbereidende fase was het hypothetisch leertraject een productief middel om de ontwikkeling van de hypotheses en verwachtingen vast te leggen. De vooraf geïdentificeerde ijkpuntopgaven en hun verwachte resultaten stuurden de observaties tijdens de onderwijsexperimenten en de data-analyse in de retrospectieve fase. De mini-interviews over deze ijkpuntopgaven leverden relevante gegevens op, hoewel deze methode van dataverzameling een goede afstemming met de docent vraagt en de mogelijkheden voor de observator om als 'assistent docent' op te treden verkleint. De video-opnames van klassikale delen van de lessen waren niet zo waardevol, aangezien de klassengesprekken niet altijd de ijkpuntopgaven of centrale concepten van het onderzoek betroffen. In de retrospectieve fase was de methode van data-analyse effectief, die codering volgens een vooraf opgezet coderingssysteem combineerde met een aanpak geïnspireerd op de constante comparatieve methode. De samenwerking met een tweede onderzoeker die de data onafhankelijk van de eerste codeerde, verbeterde de kwaliteit van de data-analyse en het coderingssysteem. Het formuleren van feed-forward voor de volgende onderzoekscyclus was bevorderlijk voor het vastleggen van de voortgang van het onderzoek.
Hoe generaliseerbaar zijn de bevindingen van dit onderzoek? Zijn de resultaten afhankelijk van de specifieke keuzes die zijn gemaakt, of kunnen ze worden gegeneraliseerd naar andere situaties? Veel zaken die in dit onderzoek naar voren kwamen, zoals de reïficatie van formules en expressies en de ontwikkeling van symbol sense, zijn niet beperkt tot het parameterbegrip. We nemen dan ook aan dat deze bevindingen verder strekken en gegeneraliseerd kunnen worden naar algebra leren in het algemeen. Voor wat betreft instrumentatie zijn we van mening dat de conclusies eveneens gegeneraliseerd kunnen worden. Ondanks de verschillen tussen computeralge-

bra omgevingen op detailniveau, zijn deze systemen gebaseerd op vergelijkbare principes en zullen vergelijkbare instrumentatiekwesties een rol spelen. De vraag in hoeverre de instrumentele benadering ook van toepassing is op andere ICT omgevingen verdient nader onderzoek. De bevindingen kunnen ook worden gegeneraliseerd naar andere Nederlandse scholen, zolang rekening gehouden wordt met de specifieke educatieve omstandigheden. Bij generalisatie naar het gebruik van computeralgebra in andere klassen en op andere niveaus moet voorzichtigheid worden betracht. In lagere klassen zullen de instrumentatieproblemen naar verwachting groter zijn. In de derde en vierde klas is gebruik van computeralgebra voor een langere periode aan te raden. In hogere klassen verwachten we vergelijkbare instrumentatieproblemen en mogelijkheden, maar we nemen aan dat leerlingen dan meer wiskundige middelen hebben om de moeilijkheden te overwinnen en te profiteren van de mogelijkheden.

### *Aanbevelingen*

De conclusies van dit onderzoek leiden tot aanbevelingen voor het onderwijs, voor software ontwerp en voor verder onderzoek. Voor het *onderwijs in algebra met computeralgebra* suggereren de resultaten dat het belangrijk is om te anticiperen op het gebruik van computeralgebra, om expliciet te zijn over het veranderende didactisch contract, om de individuele en collectieve instrumentatie goed te orkestreren en om leerlingen CAS technieken en pen-en-papier technieken te laten vergelijken, zodat ze nadenken over de manier waarop computeralgebra werkt.

Voor het *ontwerpen van software* voor educatieve doeleinden bevelen we aan om de criteria van transparantie en congruentie van de technologische omgeving serieus te nemen. Het ‘black-box’ karakter van het ICT gereedschap en idiosyncratische trekjes bemoeilijken de instrumentatie. Flexibiliteit ten aanzien van notatie, syntax en oplossingsstrategie vormt een derde criterium waaraan bij voorkeur voldaan dient te worden.

Voor *vervolgonderzoek* naar het leren van wiskunde in een technologische omgeving raden we aan om de gehele pedagogische situatie in ogenschouw te nemen en niet het gebruik van ICT als geïsoleerd verschijnsel te onderzoeken, zodat het sociale en het technologische aspect met elkaar in verband worden gebracht (Yackel & Cobb, 1996). De cultureel-historische handelingstheorie, die gereedschap ziet als onderdeel van een sociale praktijk, zou hierbij een vruchtbaar perspectief kunnen zijn. Verder zou de relatie tussen de instrumentele benadering en theorieën over symboliseren een interessant startpunt kunnen zijn voor verdere theorievorming. Een longitudinaal onderzoek naar symboliseren in een computeralgebra omgeving, inclusief adequaat onderwijs dat daarop voorbereidt, verdient aanbeveling. Daarnaast is uitbreiding van dit onderzoek wenselijk naar andere wiskundige onderwerpen – denk aan het functiebegrip – en ander ICT gereedschap zoals software voor dynamische meetkunde en Java applets. Ten slotte vraagt de rol van technologie bij toetsing nader onderzoek.

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## Curriculum Vitae

Paul Drijvers was born on 16 April 1958 in Breda, the Netherlands. In 1983 the Katholieke Universiteit Nijmegen awarded him a Masters in Mathematics for his thesis on the teaching and learning of differential equations. He minored in Psychology.

Between 1984 and 2000, Paul was a pre-service teacher trainer at the Hogeschool van Arnhem en Nijmegen, where he taught courses in statistics, calculus and computer use. Since 1997, he has been involved in in-service teacher training courses at the Algemeen Pedagogisch Studiecentrum in Utrecht, where he is programme director of the 'Teachers Teaching with Technology' project.

Since 1990, Paul has also been working at the Freudenthal Institute. After performing some short pilot studies on computer algebra use in secondary education, he was involved in research on the integration of the graphing calculator in upper-secondary mathematics and in a secondary curriculum development project (Profi).

In 1999 he started his PhD research study. He has published in journals on mathematics education - mainly on the integration of technology into the learning and teaching of mathematics - and is assistant editor of the International Journal for Computer Algebra in Mathematics Education.

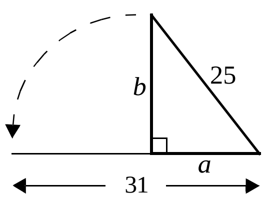
## Appendices

### A Pretest G9-I

**1** Between $x$ and $y$ the relation $y = 5 - 2 \cdot x$ holds.
What happens to $y$ when $x$ gets larger?

**2** If I add my father's age to my own age, I get 120.
If I subtract my age from that of my father, the result is 38.
How old am I? How did you find it?

**3** The two right-angled edges of a right-angled triangle, $a$ and $b$, together have a length of 31 units.
The hypotenuse is 25 units long.
So: $a + b = 31$ and $a^2 + b^2 = 25^2$.
How long are the right-angled edges?
How did you approach this problem?




**4** What do you think of when you look at the formula $y = 15 + p \cdot a$?

**5** Have another look at the formula $y = 15 + p \cdot a$.
What could be the meaning of the letters $y$, $p$ and $a$?

**6** The entrance tickets to an amusement park cost 17.50 guilders.
Furthermore, one can buy tokens that give access to the different attractions.
These tokens cost 1 guilder, each.

**a** Translate this situation into mathematics.

**b** What changes if the entrance price gets higher?

**c** And if the price of the attractions gets higher?

## B Posttest G9-I

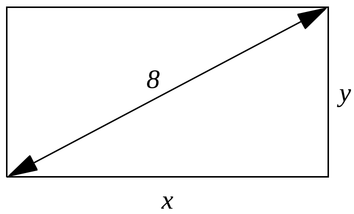
Please write down for each assignment what you do and why you do it, and which TI-89 commands you use. Good luck!

**1** Lines have equations of the form $y = a \cdot x + b$ .

**a** What happens with such a line when the value of $a$ changes?
You may sketch what you mean.

**b** And what happens with such a line when $b$ changes?

**2** A rectangle with a perimeter of 20 has a diagonal of length 8.



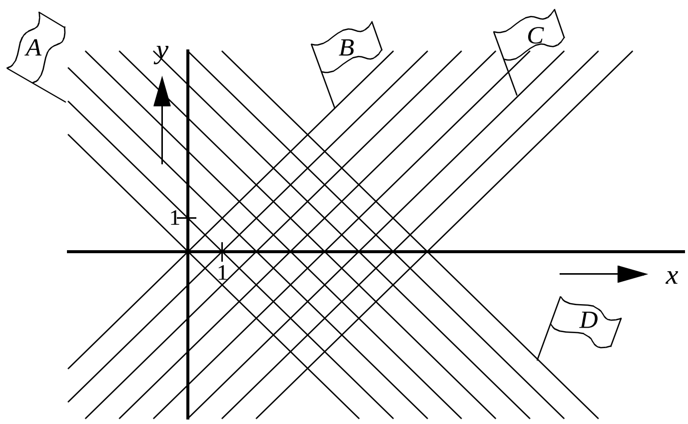

For the length and the width of the rectangle, $x$ and $y$, holds that $x + y = 10$ and that $\sqrt{x^2 + y^2} = 8$ .

**a** Calculate the length and the width of the rectangle.

**b** Solve this problem also in general, when the diagonal has length $d$:
for which $x$ and $y$ holds that $x + y = 10$ and $\sqrt{x^2 + y^2} = d$ ?

**c** How can you see in the answer to question b that $d$ can never be equal to 6?

**3** Below, the graphs of $y = s - x$ and $y = x - v$ are drawn for several values of $s$ and $v$. With 'flags' you can indicate which value of $s$ or $v$ corresponds to a given line. Four of these flags are sketched.

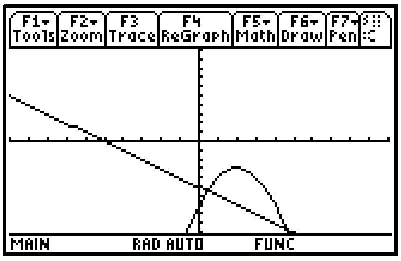


**a** What is in flag *A*? And in *B*, *C* and *D*?

**b** What happens to the lines with equation $y = x - v$ when *v* grows?

**c** In the picture, choose an intersection point of an ‘*s*-graph’ and a ‘*v*-graph’. What happens to this intersection point when *s* and *v* both become larger by 1?

**4** The figure below shows the parabola with equation $y = -x^2 + 4 \cdot x - 7$ and the line with equation $y = -x - 5$ .




**a** The line intersects the parabola in two points.
Calculate the *x*-coordinates of these two points.

Now a parameter is included in the equation of the parabola: $y = -x^2 + c \cdot x - 7$

**b** Have the parabolas drawn for $c = 1$, $c = 2$, ...., $c = 5$ and copy the graphs onto your sheet of paper.

**c** Calculate the values of *c* for which the line and the parabola share only one point.

**5** Only if you have time left:
Write down briefly your opinion of this experiment with the calculator, and what you learned from it. Of course, this will not determine your test mark!

## C Pretest G9-II

**1a** What is a *variable*? Can you give an example?

**b** What is a *parameter*? Can you give an example?

**2** If I add my father's age to my own age, I get 120.
If I subtract my age from that of my father, the result is 38.

**a** How old am I? How did you find it?

**b** Suppose you know the sum of the ages of two people, and the difference between their ages. Could you explain how to calculate their ages?

**3** Consider the equation $a \cdot x + b = c$.

**a** Express $x$ in $a$, $b$ and $c$; in other words, write the equation in the form $x = ....$

**b** Express $b$ in $x$, $a$ and $c$.

**4** Consider the equations

$$y = a - x$$
$$x^2 + y^2 = 10$$

From these two equations, make one equation that does not contain $y$.
You do not have to solve this new equation!

**5** The entrance tickets to an amusement park cost 17.50 guilders.
Once you are inside, you can buy tokens that give access to the different attractions. These tokens cost 1 guilder each.

**a** Draw a graph with on the horizontal axis the number of attractions you enter and on the vertical axis the total amount of money this will cost.

**b** What happens to the graph if the entrance price gets higher, for example 20 guilders ?

**c** Suppose the entrance fee is still 17.50 guilders, but the price of the tokens has increased. You visit 8 attractions and the total cost is 29.50 guilders.
How much did the tokens cost?

**6** Solve the following equations:

**a** $x^2 = 4$

**b** $x^2 = 81$

**c** $x^2 = 10$

**d** $x^2 = a$

## D Posttest G9-II

Please write down for each assignment:

– what you do and why you do it;
– which TI-89 commands you use.

Good luck!

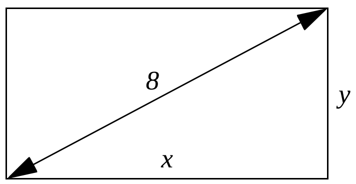
**1** What is a *parameter*? Can you give an example?

**2** Once more the age problem.
If I add my father's age to my own age, I get 120.
If I subtract my age from that of my father, the result is 38.

**a** How old am I? How did you find it?

**b** Suppose you know the sum of the ages of two people, and the difference between their ages: the sum is $s$ and the difference is $v$.
Make two formulas that you could use to calculate the two ages (so express the ages in $s$ and $v$).

**3** A rectangle with a perimeter of 20 has a diagonal of length 8.




For the length and the width of the rectangle, $x$ and $y$, it holds that $x + y = 10$ and that $\sqrt{x^2 + y^2} = 8$ .

**a** Calculate the length and the width of the rectangle.

**b** Suppose you do not know the length of the diagonal.
Call this length $d$ and solve this problem in this general case as well.

Assignment **4** for Class A:

**4** The solutions of the equation $x^2 + b \cdot x + c = 0$ are

$$x = -\tfrac{1}{2}b + \tfrac{1}{2}\sqrt{b^2 - 4c}$$

and

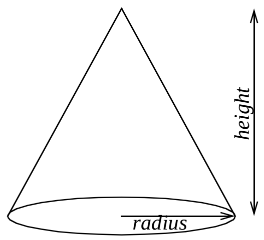
$$x = -\tfrac{1}{2}b - \tfrac{1}{2}\sqrt{b^2 - 4c}$$

**a** What solutions for $x$ do you find when $b = 4$ and $c = 3$?

**b** What conditions do $b$ and $c$ have to fulfil, if you want to be able to calculate solutions using this formula?

Assignment **4** for Class B:

**4** The volume of the cone is equal to the area of the bottom multiplied by the height divided by 3, so $volume = \frac{1}{3} \cdot area \cdot height$.
The area of the bottom equals π times the square of the radius, so $area = \pi \cdot radius^2$.




**a** Calculate the volume of the cone if the radius is 5 and the height is 9.

**b** Suppose you do not know the height and the radius, but you do know that the height is double the radius. What formula for the volume can you infer in that case?

**5** For each value of $a$ $y = a \cdot x - 5a + 2$ is the equation of a straight line.

**a** Have the line drawn for at least 4 values of *a* and sketch a copy of the screen on your paper. Please indicate your WINDOW settings.

**b** What happens to the line as *a* gets larger?

**c** Which point lies on *all* of the lines? How can you verify in the equation that eve ry line does indeed hit this point?

## E Final task for G10-II

*The optimal velocity of the traffic flow*

### *Working arrangements*

Work in pairs and pay attention to the following issues.

– Cooperate and hand in a neat report, one per pair;
– Include in your report more than just answers: explain how you found them and what they mean for the problem situation;
– Also describe your reasoning and motivate your problem-solving method.

### *The traffic flow problem*

The traffic flow problem was addressed in section 6 of the booklet. The central question is: what velocity of a slowly driving traffic flow will lead to an optimal circulation of cars?

The following formula represents the circulation flow:

$$f = \frac{100 \cdot v}{6 \cdot (l + a)}$$

The letters have the following meanings:

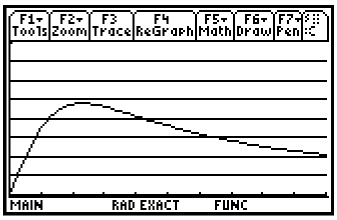
*f*: the flow in number of cars passing in a minute
*v*: the velocity of the traffic flow in km/hour
*l*: the average length of the cars in the traffic flow in metres
*a*: the average distance between two cars in the traffic flow in metres.

We conjecture that $l$ is about 4 metre and that the drivers stick to the prescribed safe inter-car distance, that is:

$$a = 0{,}0075 \cdot v^2$$

This task concerns a new approach to find the optimal circulation flow and the corresponding velocity using these formulas.

**1** Below you see the graph of $f$ as function of $v$. The figure also shows horizontal lines at heights 10, 20, 30, 40, 50, 60 and 70.



**a** Make this figure on the screen of your TI-89. In the viewing window, the horizontal axis goes from 0 to 100 and the vertical one from 0 to 80.

**b** How big is the optimal flow approximately, according to this graph? How do you see that using the horizontal lines?

**2** Verify algebraically that the line at height 40 has two intersection points with the graph of *f*, and the line at height 50 none.

To find the optimal flow we use the following idea. We intersect the graph of *f* with horizontal lines. If we find such a line that has one rather than two intersection points with the graph, then we have a tangent line and therefore the maximum value of the graph. So we look for the height of the horizontal line that is tangent to the graph of *f*.

**3a** Calculate in general the coordinates of the intersection points of the line at height *h* and the graph of *f*.

**b** How many solutions should you find if the line touches the graph?

**4a** What is the height of the horizontal line through the top?

**b** How big is the optimal velocity of the traffic flow then?

In reality drivers often keep less distance to the car in front than is recommended. They take a certain risk with that. Suppose for example that most drivers keep a distance that is only 70% of the safe one.

**5a** If necessary look back to section 6 and build in the risk factor of 0.70 into the formula for *f*.

**b** Use the method of the previous assignments to find the optimal flow and the corresponding velocity of the traffic flow.

**6** Suppose the risk factor in general is called *r*.

**a** What formula for *f* do you get then?

**b** What is the relation between the risk factor and the optimal movement of the traffic flow?

**c** What happens to the optimal flow if more risk is taken by the drivers? How can you infer this from the relation of question **b**?