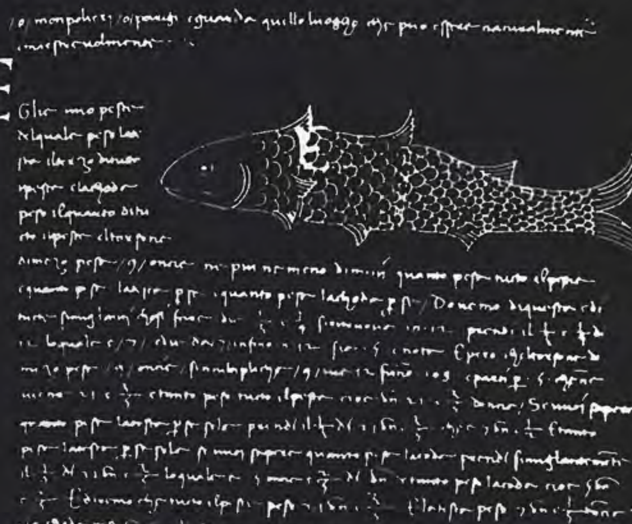
B.A. van Amerom

REINVENTION OF EARLY ALGEBRA

Developmental research on the transition from arithmetic to algebra



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The illustration on the cover is a mathematical riddle from a manuscript by Paolo Dagomari (ca.1281-1370).
Source: Tropfke (1980), p. 577

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REINVENTION OF EARLY ALGEBRA

Developmental research on the transition from arithmetic to algebra

Heruitvinden van aanvankelijke algebra

Ontwikkelingsonderzoek rond de overgang van rekenen naar algebra

(met een samenvatting in het Nederlands)

Proefschrift ter verkrijging van de graad van doctor aan de Universiteit Utrecht
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Preface

To be honest, my appointment to the project 'Reinvention of algebra' was a rather unexpected turn of events. I never had the ambition to pursue an academic career. During the final month of my teacher training I still had every intention to become a secondary school mathematics teacher. However, when I was given the opportunity to combine my two main interests in mathematics – history and didactics – the choice was easily made. Looking back, there have been no regrets. Soon after the project started I took a part-time teaching job on the side, and just as I had hoped, the combination of theory and practice has been inspiring and fulfilling.

At this time I wish to commemorate Leen Streefland, who helped me get started on the project and who supervised my early work. It has been an honor and a privilege to have known him and to have worked with him. He made me feel competent as if I were his equal and not a rookie, but even now there is so much more that I could have learned from him.

Of course I could have never completed this book without the help of others. First of all I want to thank my supervisors Jan van Maanen, Jan de Lange and Koeno Gravemeijer, for their honest and constructive criticism, advice and support. I am especially grateful to them for letting me decide my own course, and on occasions that I seemed to go astray they helped me just in time to find my way back. A special thanks goes to Jan van Maanen, who has guided me since my university years and who brought this project to my attention. He has been more than a supervisor to me, he has also been my friend. It was never too much trouble for him to travel from Groningen early in the morning in order to work together on my project the whole day. Even in rough times he continued to believe in me, sometimes more than I did myself, reminding me not to be too hard on myself.

I also extend my thanks to my colleagues at the Freudenthal Institute, for their expertise, their interest and support – in meetings, during coffee and lunch breaks or while waiting for the xerox machine to finish. In particular I want to thank Hanneke Beemer for designing and testing the student unit *The Fancy Fair*, and Mieke Abels for testing it again with her students and giving suggestions for improvement afterwards. I am very grateful to Marjolein Kool en Julie Menne for their advice and encouragements when the going got rough. Moreover, Marjolein contributed some really good ideas and historical materials for the student activities. Meeting the deadline for this book would not have been possible without the assistence of Betty Heijman, Nathalie Kuipers, Ank van der Heiden-Bergsteijn, Marianne Moonen-Harmsen, Anneleen Post and Ellen Hanepen.

Uiteraard wil ik ook de docenten bedanken die hebben deelgenomen aan de verschillende onderwijsexperimenten. Dit zijn de leerkrachten van groep 7 en/of 8 van RK Basisschool De Kameleon te Utrecht, Koningin Julianaschool te Woudenberg, RK Basisschool ‘t Zonnewiel te Almere, Basisschool Rijnsweerd te Utrecht, CBS Het Anker te Brielle, OBS Lorentz te Hilversum en Basisschool Klaverblad te Amsterdam, en de brugklasdocenten verbonden aan RK Scholengemeenschap College De Klop te Utrecht, Almere College te Dronten en Mendelcollege te Haarlem. In het bijzonder gaat mijn waardering uit naar Jaap Griffioen voor zijn op- en aanmerkingen op het lesmateriaal. Verder wil ik Monique Schipper en Natasja Capelle vermelden vanwege hun inzet bij de lesobservaties en -verslagen.
Niet in de laatste plaats heb ik veel steun gehad van vrienden en familie. Mede dank zij hen is het me gelukt enkele tegenvallers te incasseren en sterker uit de strijd te komen.
Met dit boek sluit ik een drukke, spannende en ook zeer dankbare periode in mijn leven af. De afgelopen maanden heb ik naar een dubbele bevalling toegeleefd – van een kind en van een boek – en nu is het tijd om daarvan te gaan genieten.

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1 Introduction

1.1 Reasons for this study

educational argument

In recent years, many research projects on mathematics education have focused on learning difficulties related to algebra, like the translation of word problems into simple algebraic equations. It is generally agreed that certain obstacles of equation solving can be ascribed to fundamental differences between arithmetic and algebra. For instance, students are required to adopt an algebraic way of reasoning, they have to break away from certain arithmetical conventions and they need to learn to deal with algebraic symbolism. The teaching and learning of early algebra, in particular the transition from arithmetical to algebraic problem solving, form a valid reason for conducting the study ‘Reinvention of algebra’.

curriculum development

Different studies on the discrepancy between algebra and arithmetic have resulted in the acknowledgement of a number of learning obstacles and their possible causes, but it has barely led to the systematic development of a new, coherent learning trajectory for solving algebraic equations, with the exception of the *Mathematics in Context* project – a joint effort of the Wisconsin Center of Educational Research and the Freudenthal Institute – which started in the United States in 1991, and to a certain extent the W12-16 algebra program described in section 2.9. The designers of the algebra strand in the *Mathematics in Context* project were inspired by innovative strategies for solving systems of equations proposed by Van Etten in 1980, but these strategies were not compatible with the traditional view on solving equations that determined Dutch curricula. Rigorous adjustments to the middle school mathematics curriculum in the Netherlands a decade ago have prepared the development of a learning strand on solving (systems of) linear equations, but to this day it has not yet been realized. It is therefore relevant at this time to develop and evaluate an innovative, experimental program on pre-algebraic problem solving.

history of mathematics

Other research indicates that the history of mathematics can make a valuable contribution to mathematics teaching and learning, either through direct integration of historical elements or by providing the curriculum designer with a hypothetical learning trajectory. But very little is known about the effectiveness of history as a didactical tool. This constitutes the third reason for conducting the present study.

1.2 Aims of the study

Given the reasons described in section 1.1, the aims of the study are threefold:

- to produce an experimental (pre-)algebra learning strand for 11- to 14-year-old students, drawing from the *Mathematics in Context* materials as well as the historical development of algebra;
- to conduct developmental research on the teaching-learning process of the teacher(s) as well as the learning process of the students involved;

- and to study the effectiveness of history as a didactical tool for (pre-)algebraic equation solving.

Combining these three elements of the study, the following problem is posed:

In which way and for what possible reason(s) does the experimental algebra strand facilitate a smooth transfer from arithmetical to algebraic problem solving, and in which way and for what possible reason(s) does it not? In particular, what is the role of the history of mathematics in this regard?

The research questions and hypotheses which follow from here are described in chapter 4.

1.3 Duration and partition of the study

The study ‘Reinvention of algebra’ has been conducted at the Freudenthal Institute between september 1995 and september 2001. An intermediate evaluation of the study has led to its partition into two different stages. The *exploratory* phase covers our orientation on the research topic, the initial design of instructional activities and early classroom experiments. The *final* phase of the study consists of a larger classroom experiment – the field test – conducted with revised materials, the analysis of student materials from this experiment and a discourse on the results and implications of the study.

1.4 Reading guide for this book

This book, which has resulted from the study ‘Reinvention of algebra’, comprises seven chapters. In *chapter 1* we describe the origination, aims and goals of the study and the framework of the thesis to which it has led. *Chapter 2* treats the theoretical background of the study: current views on the learning and teaching of early algebra. A brief discourse on the historical development of early algebra and a possible approach to its integration in mathematics education are presented in *chapter 3*, while *chapter 4* describes the research questions, research method and research plan we employed in the study. The design process and learning-teaching experiences with the early instructional materials – the exploratory phase – are described in *chapter 5*. The final phase of the study, which includes an overview of the revised learning strand and the overall results, is elaborated in *chapter 6*. Finally, in *chapter 7* we conclude with a discussion and a number of recommendations for educational designers and teachers. Two sections from the instructional unit *Time Travelers* have been added as an appendix.

We have chosen to include short comments in the margin to emphasize key issues of particular paragraphs. Tables and figures are numbered according to chronological order in a specific chapter; for instance, figure 3.4 is the fourth figure in chapter 3.

2 Learning and teaching of school algebra

2.1 Introduction

The purpose of this chapter is to present the study's theoretical framework and the researcher's perspective on early algebra learning. Considering that the project covers the educational design of (pre-)algebraic activities for students of age 11 to 13, we do not elaborate on the entire domains of algebra and algebra education. Instead we confine ourselves to a discussion of different conceptions, teaching approaches and typical learning difficulties of early school algebra. The emphasis of the second part of the chapter is on the troublesome transition from arithmetic to early algebra, in particular the cognitive obstacles that students encounter when they attempt to symbolize word problems and solve equations, including what is referred to as the *cognitive gap* or *didactical cut*. Since there is no consensus amongst researchers on what algebra is or how it should be taught and learned, section 2.7 describes which standpoint has been taken in this study.

2.2 Traditional school algebra

Algebra is known to be a major stumbling block in school mathematics, both in the past and at present. Historical studies on the developments of algebra education in the twentieth century show that the algebra studied in secondary school has not changed much over the years. Unintentionally algebra has functioned as a means of selecting the more capable learners – the 'happy few' who understand and enjoy the powers of algebra – from the rest, who experience and remember it as an elusive interplay of letters and numbers. Problems with algebra can be ascribed to external factors like the teaching approach and a poor image, but also to intrinsic difficulties of the topic, which will be described in section 2.4.

external factors

Researchers have reported that grown-ups often have a negative image of school algebra, and many students can make no sense of it. There is a plausible explanation for this. Traditional school algebra is primarily a very rigid, abstract branch of mathematics, having few interfaces with the real world. It is often presented to students as a pre-determined and fixed mathematical topic with strict rules, leaving no room for own input. Traditional instruction begins with the syntactic rules of algebra, presenting students with a given symbolic language which they do not relate to. Students are expected to master the skills of symbolic manipulation, before learning about the purpose and the use of algebra. In other words, the mathematical context is taken as the starting-point, while the applications of algebra (like problem solving or generalizing relations) come in second place. Students are given little opportunity to find out the powers and possibilities of algebra for themselves. One can imagine that an average or below-average learner finds little satisfaction in practicing mathematics without a purpose or a meaning. Another characteristic of the traditional approach is the rapid formalization of algebraic syntax. School algebra has always had a highly *structural* character, where algebraic expressions are conceived as objects rather than computations or procedures to be carried out. The *procedural* (or *operational*) aspects of algebra, which are more closely related to the arithmetical background that early algebra learners have, are usually cast aside soon after the introduction. This procedural-operational duality of algebra is discussed in more detail in section 2.4 and section 2.6.

Even though we all have an immediate idea what students learn when they learn school algebra, it is not an easy task to give a cast-iron definition. In an attempt to capture 'school algebra' in one sentence, we might suggest it is the mathematical domain dealing with (general) relationships between quantities on a symbolic level. Still, this description does not do justice to the multiple roles and utilities of algebra. Typical topics of school algebra include simplifying algebraic expressions, the properties of number systems, linear and quadratic equations in one unknown, systems of equations in two unknowns, symbolic representations and graphs of different kinds of functions (linear, quadratic, exponential, logarithmic, trigonometric), and sequences and series. In most of the core activities we find aspects of *algebraic thinking* (mental processes like reasoning with unknowns, generalizing and formalizing relations between magnitudes and developing the concept 'variable') and *algebraic symbolizing* (symbol manipulation on paper). Generally it is agreed that students must acquire both competencies in order to have full algebraic understanding.

2.3 Approaches to algebra

In the last two decades, the growing interest in algebra learning and teaching has instigated an international discussion on what we believe (school) algebra to be and what we believe it *should* be. Contemporary researchers have identified kernal characteristics of algebraic reasoning and algebraic language – such as generalizing, formalizing and symbolizing – which are related to different aspects of algebra (Kieran, 1989, 1990, 1992; Filloy & Rojano, 1989; Sfard, 1991, 1995; Sfard & Linchevski, 1994; Herscovics, 1989; Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996; Bednarz, Kieran & Lee, 1996; Kaput, 1998). A few months ago the Twelfth ICMI study 'The Future of the Teaching and Learning of Algebra' raised issues like 'why algebra?', 'approaches to algebra', 'language aspects of algebra', 'early algebra education', 'technological environments', and more. Meanwhile it has become clear that there is no agreement on what algebra is or what it should be; each classification has its strong and weak points. Therefore, instead of trying to establish what algebra *is*, one might consider algebra in terms of its *roles* in different areas of application instead.

classification of algebra

Bednarz et al. (1996) distinguish four principal trends in current research and curriculum development of school algebra: generalizing, problem solving, modeling and functions. These different roles of algebra can be associated with the various ways in which the authors conceive algebra, and which characteristics of algebraic thinking they believe ought to be developed in order to find algebra meaningful. A fifth perspective presented by Bednarz et al. is the historical one, not as an alternative way to introduce algebra at school but as a valuable pedagogical tool for teachers and educational researchers.

The same researchers recognize that the classification is oversimplified and incomplete, and that various approaches have not yet been adequately researched: "The separation into four approaches to 'beginning algebra' is artificial; all four components are needed in any algebra program. (...) Some other possible approaches have probably been omitted." (ibid., p. 325). Still, Bednarz et al. observe that their classification has helped to structurize their discussion on essential issues of school algebra.

roles of letters

Some years earlier, Usiskin (1988) proposed a slightly different categorization of perceptions of algebra: as generalized arithmetic, as a study of procedures for solving problems, as a study of relationships among quantities (including modeling and functions) and as a study of structures. In each of these approaches to algebra Usiskin identifies different roles of the letter symbols: pattern generalizer, unknown, argument or parameter, or arbitrary object respectively. One might argue that this list is not complete; other meanings of the concept of variable that are mentioned regularly are those of placeholder (a symbol in an arithmetical open sentence such as $3 + \bullet = 5$), letter not evaluated (like π and e) and label (letter to abbreviate an object, or a unit of measurement). A variable that varies (as argument or parameter) is considered to be of a higher level of formality than the variable as generalized number or unknown, which is again more formal than the placeholder; at the top end we find the arbitrary symbol. This subtle variation of meanings of letters has been identified as one of the major obstacles in learning algebra (see also section 2.4).

different characterizations

A number of other characterizations of algebra can be found in the literature. For instance, the National Council of Teachers of Mathematics (1997) identifies four themes for school algebra: functions and relations, modeling, structure, and language and representation. Kaput (1998) has listed five forms of algebraic reasoning: generalizing and formalizing, algebra as syntactically-guided manipulation, algebra as the study of structures, algebra as the study of functions, relations and joint variation, and algebra as a modeling language.

In the present study we do not take an explicit position on what is the best classification of algebra. It is only relevant that we recognize which aspects of algebra are relevant for the proposed learning program. The algebraic activities that we have developed can be described as 'advanced arithmetic', with a large component of problem solving and studying relations (see also section 2.7). We have no clear preference for one classification or the other; it is only for the practical reason of having a framework that we have made a choice. The overviews of perceptions of algebra (below) and its typical learning obstacles (in section 2.4) are based on contributions in Bednarz et al. (1996) and reviews of approximately two decades of research on learning and teaching algebra by Kieran (1989, 1990, 1992).

learning algebra through generalizing

If algebra is construed as a product of generalizing activities, its main purpose is to grasp generality, for instance by expressing the properties of numbers. The Oxford Dictionary exemplifies this perspective by defining algebra as the 'study of the properties of numbers using general numbers'. Algebraic skills are directed at translating and generalizing given relationships among numbers. This approach to algebra stands a better chance if the learner's intuitive base for the structure of algebra is already nourished in arithmetical activities. For example, Booth (1984) has suggested that if a student is to perceive an expression like $a + b$ as an object in algebra, he or she must be able to view the sum 5 + 8 as an object in arithmetic, rather than as a procedure leading to the outcome 13.

algebra as a problem solving tool

Problem solving by constructing and solving equations is not only an historical route into algebra, it has also become a core activity in every algebra curriculum. Translating word problems into equations involves the fundamental issue of transition from arithmetic to algebra, in terms of symbolism as well as reasoning. According to Bell (1996), problem solving seen in a wider sense means 'exploring problems in an open way, extending and developing them in the search for more results and more general ones.' Bell sees algebra not as a separate branch of mathematics but as an integrated strand, wherever its symbolism, concept and methods are appropriate.

learning algebra through modeling

A modeling approach to learning algebra is based on the conception that students need to become flexible in the description and interpretation of phenomena in the world around them. It includes constructing meaning for various representations (graph, table, formula) and transforming one kind of representation into another. Note that these models are not intended to emerge as constructions of students' own mathematical activity, but as given, pre-determined symbolic forms. The modeling approach has at least two factors in common with the functional approach to algebra. First, modeling is based on expressing relations between varying quantities, and second, it contributes to developing the student's sense of what a variable is.

functional approach to algebra

Calculators and computers lead to new possibilities in studying relations between two sets of numbers. Computers may be used, for example, to test whether a certain function is hidden behind the structure of a set of numerical data. Various types of functions and the concept of variable may be investigated in this respect. Bednarz et al. (1996) describe two projects of introducing algebra which are based on this principle.

2.4 Typical learning difficulties

The present study is concerned with students in the age of 11 to 13 years, for which reason we focus on the early learning of algebra. The introduction to algebra usually involves the study of algebraic expressions, equations, equation solving, variables and formulas. According to Kieran (1989, 1992), students' learning difficulties are centered on the meaning of letters, the change from arithmetical to algebraic conventions, and the recognition and use of structure. Some of these problems are amplified by teaching approaches: often the structural character of school algebra is emphasized, whilst procedural interpretations would be more accessible for children (Kieran, 1990, 1992; Sfard & Linchevski, 1994). A more detailed account of the first two categories – meaning of letters and arithmetical versus algebraic conventions – with respect to equation solving is given in section 2.6.1, where we discuss the relation between arithmetic and algebra and the discontinuities between them. In the present section we describe two more general ontological difficulties of algebra: operational (or procedural) and structural (relational) modes of thinking, and problem solving.

processes vs. objects

Sfard and Linchevski (Sfard, 1991; Sfard & Linchevski, 1994) suggest that problems encountered in learning algebra can be partly ascribed to the nature of algebraic concepts. According to Sfard (1991) there are two fundamentally different ways to conceive mathematical notions: *operationally* (as processes) and *structurally* (as objects). Students struggle to acquire a structural conception of algebra, which is fundamentally different from an arithmetical perspective (see also section 2.6.1). To illustrate the operational conception, Sfard and Linchevski (1994) explain that an algebraic expression like $3(x + 5) + 1$ can be seen as a description of a computational *process*. It is a sequence of instructions: add 5 to a certain given number, multiply the result by three and then add 1. From another perspective, the expression can also be viewed as the *product* of the computation, representing a certain number (which at this time cannot be specified). In yet another setting $3(x + 5) + 1$ can behave as a *function*; instead of representing a fixed number, it reflects a change. And at a very simple, superficial level we can even say the expression is a meaningless string of symbols. As an algebraic object, it can be manipulated and combined with other symbolic expressions. The three latter conceptions – as a computational product, a function and a symbolic string – all reflect a structural understanding of algebra. In fact, Sfard and Linchevski argue that these four different notions of an algebraic expression represent different phases in the individual learning of algebra, based on logical, historical, and ontological analyses.

In addition Sfard (1995) has compared discontinuities in student conceptions of algebra with the historical development of algebra. She claims that the *syncopated* stage of algebra (where unknown quantities are represented by abbreviations or letters) is linked to an operational conception of algebra, whereas the *symbolic* stage of algebra (where letters stand for given as well as unknown quantities) corresponds with a structural conception (see section 3.3 in for more information on the terms 'syncopated' and 'symbolic').

process-product dilemma

In Kieran's reviews (1989, 1990) we read that studies on this issue go back a few decades. Matz and Davis, for example, did research in the 1970's on students' interpretation of the expression $x + 3$. Students see it as a procedure of adding 3 to x, whereas in algebra it represents both the procedure of adding 3 to x and the object $x + 3$. In other words, in algebra the distinction between the process and the object is often not clear. Matz and Davis call this difficulty the 'process-product dilemma'. Freudenthal (1983) illustrated the difference between a procedural (in terms of Sfard, 'operational') and a static (corresponding with Sfard's term 'structural') outlook by comparing language use and meaning:

> A powerful device – this formal substitution. It is a pity that it is not as formal as one is inclined to believe, and this is one of the difficulties, perhaps the main difficulty, in learning the language of algebra. On the one hand the learner is made to believe that algebraic transformations take place purely formally, on the other hand if he has to perform them, he is expected to understand their meaning. (...) The learner is expected to read formulae with understanding. He is allowed to pronounce:
>
> $a + b$, $a - b$, ab, a^2
>
> as
>
> *a* plus *b*, *a* minus *b*, *a* times *b*, *a* square.
>
> Yet he has to understand it as
>
> sum of *a* and *b*, difference of *a* and *b*, product of *a* and *b*, square of a.
>
> The action suggested by the plus, minus, times, square and the linear reading order must be disregarded. The algebraic expressions are to be interpreted statically if the formal substitution is to function formally indeed (Freudenthal, 1983, p. 483-484).

Sfard (1987) observes that students are better at writing their solution procedures for solving equations verbally than they are at constructing and manipulating symbolic equations. She therefore proposes to foster students' understanding of processes and algorithms before moving on to the structural perspective. One way of doing this is to incorporate computer programming.

problem solving

After a while learners of algebra can become quite skilled at performing algorithmic procedures (expanding brackets, solving a system of equations), and yet they fail at problem solving. Professional algebraists seem to forget the catch: that rote skills do not help students in getting started. Da Rocha Falcão (1995) suggests that the difficulty is contained in the difference in approach to problem solving. Arithmetical

problems can be solved directly, if necessary with intermediate answers. Algebraic problems, on the other hand, need to be translated and written in formal representations first, after which they can be solved.
To illustrate the direct, arithmetical approach and the indirect, algebraic approach let us consider the following example of the task ‘guess my number’:

Student 1: Think of a number, multiply it by 3 and add 5 to it. What is the outcome?
Student 2: 32.
Student 1: Then the number you thought of is 9.

The most obvious arithmetical solution procedure to follow is to undo the chain of operations: 32 minus 5 gives 27, and 27 divided by three gives 9. It is a direct approach because it works towards the solution right from the start. The algebraic method for solving this problem is to represent the initial number by x, construct the equation $3x + 5 = 32$ and then solve the equation for x. This approach is indirect: the problem is translated first from a dynamic description ‘do this, do that’ to a static, symbolic representation, before moving onto the actual solution procedure. Mason (1996, p. 23) formulates the difference as follows (comments by the researcher between brackets): “Arithmetic proceeds directly from the known (32, in this case) to the unknown (9) using known computations (the inverse operations of ‘times 3’ and ‘plus 5’); algebra proceeds indirectly from the unknown (x in our example), via the known (the operations ‘times 3’ and ‘plus 5’), to equations and inequalities which can then be solved using established techniques.” Learning difficulties related specifically to equation solving are discussed in more detail in section 2.6.1 which deals with the transition from arithmetic to algebra.

purpose and meaning

In order for algebra to be appreciated, its superiority to arithmetic needs to be (made) apparent. It is common to introduce students to equation solving using linear equations in one unknown like the example above, as an alternative to the arithmetical procedure. The algebra expert (teacher, text book author) finds this approach suitable because each step in the solution process can be verified by the arithmetical counterpart (the inverse operation). But in the eye of the learner it is not a logical or natural method; after all, the arithmetical approach is easier and works just as well! Ideally, learners should experience the value and purpose of algebra from the start – for example in situations where arithmetic and common sense no longer comply – but without being forced to a formal level prematurely. In our opinion *purpose* is unmistakably joint with *meaning*. Classroom experiments have shown that algebraic competence depends on the ability to give meaning to equations (Abels, 1994; Van Reeuwijk, 1995, 1996). When equations emerge through a good understanding of the underlying relations – when they make sense to the learner – students have been found to be more successful at solving them as well.

2.5 Symbolizing

As we have already said before, the symbolic language of algebra requires students to learn to look at symbolic expressions in a new way. In traditional teaching approaches algebraic expressions like formulas, equations or arithmetical identities are presented to students as ready-made artifacts. The meanings of the symbols are fixed in a rigid framework of conventions. In reaction to this 'anti-didactical inversion' (Freudenthal, 1973), we advocate a teaching approach which begins with what the learner already knows and does. In other words, algebra learning and teaching should be based on problem situations leading to symbolizing instead of starting with a ready-made symbolic language. In this section we describe some aspects of how symbolizing and meaning may develop in the proposed early algebra program.

current ideas on symbolizing

In recent years, research discussions on symbolizing and modeling show a change in ideas of how symbols and models (also called 'manipulatives') may be used to support the development of mathematical concepts (see, for example, Gravemeijer & Terwel, 2000). Where models and symbols were previously introduced by teachers as ready-made tools with a pre-determined meaning, intended to make abstract mathematics more accessible, they are now seen as products of students' own mathematical activities. The corresponding teaching approach is based on the belief that symbolizing and meaning develop interactively as students engage in reflexive discourse:

> The basic idea is that forms of symbolization (in schemes, diagrams, models or even verbal terms) emerge in the context of activities that require the availability of such symbolic tools, and that the functional requirements of these activities stimulate the improvement of the children's way of symbolizing (Gravemeijer & Terwel, 2000, p. 2).

This dynamic view of symbolizing and modeling has called for another way of speaking about symbolizations. Terms like 'symbols' and 'referents' – connected to the static, representational view of symbolizations – have been replaced by notions like 'sign' and 'inscription'. A sign consists of a pair *signifier-signified*, of which the signified plays a dynamic part in the constitution of new signs. In the so-called 'chain of signification' a certain sign combination becomes the signified of the succeeding sign, so that the meaning of the original sign changes. It is during this dynamic process, where signs and meanings change and produce new signs, that mathematical concepts are developed.

signs and meanings

In an exposition on the interaction between mathematical discourse and mathematical objects, Sfard (2000) also considers the interplay between symbols (*signifiers*) and their objects (*signifieds*). In her conception, signifiers must come before their signifieds, since 'one simply *cannot* speak about the object represented by a symbol before the symbol enters the language and becomes a fully fledged element of the discourse'. Sfard observes an inherent circularity in mathematical discourse: the construction of signifieds relies on talking about their signifiers, while the signifiers themselves obtain their meaning from mathematical discourse. In other words, we have a seemingly paradoxical situation: symbols become meaningful by using them, but how can a symbol be used before it is meaningful? Sfard conjectures that when a new signifier is introduced it does not have a signified yet. It is semantically 'empty', and its meaning develops gradually in mathematical activity. In such a way the apparently vicious circle of mathematical discourse and mathematical objects fuels their simultaneous development. Furthermore, Sfard and Linchevski (1994) argue that symbols seem to be a necessary but not sufficient condition for acquiring a structural mode of thinking:

> It is true that as long as algebraic ideas are dressed in words and in words only, it is difficult to imagine the more advanced structural approach, where the computational processes are considered in their totality from a higher point of view, and where operational and structural slants meet in the same representations. To put it differently, words are not manipulable in the way symbols are. It is this manipulability which makes it possible for algebraic concepts to have the object-like quality (Sfard & Linchevski, 1994, p. 93).

The current perspective on symbolizing as a dynamic process – the interplay between the development of mathematical meaning and symbol use – implies that instructional design should provide opportunities for students to develop their own sense-making symbolism. A teacher-guided mathematical discourse on the meaning, advantages and shortcomings of these symbolic constructs can result in a mutually accepted (pre-)algebraic symbol system as the basis for further algebra learning. Two cases of informal symbolizing and the development of meaning which are relevant for this study are described below, followed by a brief description of some common models for solving equations.

2.5.1 Symbolizing and schematizing

phenomenology

Let us turn for a moment to the realistic instructional theory of Realistic Mathematics Education (see also section 4.3.1). Point of departure is Freudenthal's notion of mathematics as an activity of organizing or mathematizing. In this activity, symbolizing is developed as a personally meaningful and convenient problem solving tool. This notion of symbolizing as a tool for mathematical reasoning serves to explain the use of the terms 'symbolizing' and 'schematizing' in this study.

schematizing

'Symbolizing' and 'schematizing' are sometimes both seen as symbolizing activities. Symbolizing in the narrow sense of the word refers to the construction and use of conventional mathematical symbols: numbers, letters, operators, expressions, and so on. Symbolizing in the wider sense of the word refers to the use of material or visual representations such as drawings, notations, diagrams, tables, or concrete, context-bound marks. In order to make this distinction, the latter conception of symbolizing will be named ‘schematizing’. And when the representation at hand is more than just a calculational tool, we say that a student uses ‘schematizing as a problem solving tool’. For example, figure 2.1 shows a solution to the task: ‘How many quarters and dimes do you get for a coin worth 2.5 guilders?’ The table can inspire the student to use a systematic approach – repeated exchange of 2 quarters for 5 dimes – in order to find all possible combinations. In this way a schematic representation like a table can give meaning to a problem solving strategy.

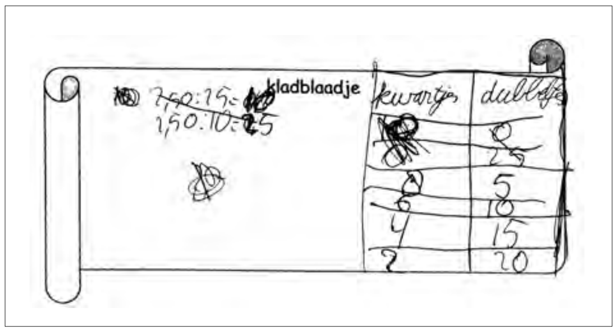


figure 2.1: table as a problem solving tool

2.5.2 Symbolizing equations

letter symbols

The ideas proposed by Sfard (2000) can be linked to the present study by considering the specific case of algebraic symbolizing in the experimental program. In the first few lessons on equation solving students are confronted with different types of symbols: drawings, pictograms, abbreviations (letter combinations) and unknowns, not necessarily in this order.

figure 2.2: two combinations of umbrellas and hats

These symbols suggest a gradual withdrawal from contextual meaning. Let us consider the problem in figure 2.2 taken from the *Mathematics in Context* unit *Compar-*

ing Quantities (Mathematics in Context Development Team, 1998, p. 16) as an example. The drawing represents an embedded system of equations in two unknowns: the price of an umbrella and the price of a hat. Note that the visual representation means that the problem situation is already organized. One can imagine that if the problem had been represented as a description, a student might have chosen to organize the information in a drawing in a similar way. The pictures have a direct reference to the objects they stand for: the (price of a) hat and the (price of an) umbrella. At the informal level we accept that students say '2 umbrellas and 1 hat cost 80 dollars'.

At this stage we do not aim to hear the formal, mathematical expression 'the sum of twice the price of an umbrella and the price of a hat is 80'. The symbols are meaningful, but they are not yet tied to the formal signified. Abbreviations, too, may reflect an informal level of understanding of the signified: in the system of equations

$$2\ um + 1\ ha = 80$$
$$1\ um + 2\ ha = 76$$

the letters *ha* and *um* are used as labels. The link between abbreviations and the context can easily be reconstructed because the abbreviations refer directly to the situational objects: hats and umbrellas. At a formal level, in the system

$$2u + h = 80$$
$$u + 2h = 76$$

the unknowns h and u are signifiers for the mathematical objects (signifieds) 'price of a hat' and 'price of an umbrella'. The letters are no longer labels but magnitudes, in fact they are determinate unknowns. The transition from the conception '2 umbrella's plus 1 hat equal 80 dollars' to the more formal conception will need considerable attention. As we see, early algebraic symbolizing can be meaningful for students from the start and the relation signifier-signified can develop quite naturally over time. The teacher should accommodate a gradual shift from an informal to a more formal conception of an unknown (ending with the concept of variable) in classroom discourse.

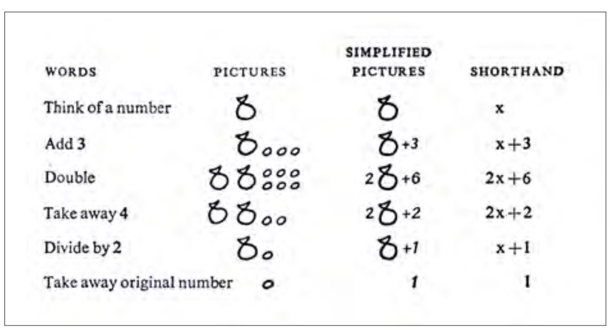


figure 2.3: gradual steps of symbolization

Figure 2.3 shows another example of progressive symbolizing, where informal symbols can have a contextual meaning at first and a more formal (abstract) meaning later on (Sawyer, 1964, p. 73).

2.5.3 Models for equation solving

(common) models for equation solving

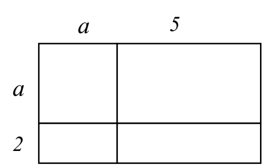
The teaching of equation solving often involves the use of pre-designed models, intended to make the abstract symbolic equation more accessible. Some models serve to visualize the situation (symbolically or schematically), while others take a purely numerical approach. Such models contain a component of *translation*, where objects and operations in abstract situations are given meaning at a more concrete level. It is important that this translation operates in two directions, that students can identify operations and objects at both the concrete and the abstract levels. A second component in modeling concerns the gradual detachment from the context-bound semantics of the model. Filloy and Sutherland (1996) remark that "(...) fixation on the model can delay the construction of an algebraic syntax since this requires breaking away from the semantics of the concrete model" (ibid., p. 150). We describe five of these manipulatives: the balance model, the geometrical model, the arithmetical model, the notebook model and the linear model. Each model has its advantages and disadvantages; the perfect one is yet to be discovered.

balance model

The classical balance model is based on the concept of equal weights on both sides of the scale. For instance, in the equation $3x + 12 = 5x + 8$ the left-hand side of the scale holds 3 elements of weight x and 12 unit weights, while the right hand side holds 5 elements of weight x and 8 unit weights. The weight x can be determined by cancelling equal weights on both sides. The advantage of this model is that it has a meaning in every day life situations, and students can make a mental image of the balance very easily. Moreover, the balance emphasizes the static character of the equation; the concept of equivalence remains in the foreground as the solution procedures are carried out. The major limitation of the balance model – of all physical and visual models for that matter – is the restrictions of its applicability to equations involving negative terms and negative solutions.

geometrical model

The advantage of the geometrical or area model lies in its visualization and concrete meaning (area) for symbolic expressions. This model can be used for linear equations of the form $ax + b = cx$ (a rectangle of length a and width x and a rectangle of area b added together have the same area as a rectangle of length c and width x) but also for quadratic equations. Figure 2.4 shows the geometrical representation of the expression $(a + 5)(a + 2)$ and the corresponding multiplication table as it used in some Dutch mathematics text books. The area model, too, has a few disadvantages. Just like the balance model, the concrete meaning of the area model is limited to positive magnitudes. In addition, it may not be suitable for students who have a weak geometrical foundation. After all, basic geometrical concepts like area and perimeter continue to be an obstacle for many mathematics students. Furthermore, Filloy and Sutherland (1996) observe that automation in both the balance and the area model lead to errors typically associated with algebraic syntax, such as adding and subtracting coefficients of different degree.



×	a	+ 5
a	a^2	$+ 5a$
+ 2	$+ 2a$	+ 10

figure 2.4: area model and multiplication table

arithmetical model

The arithmetical model employs arithmetical identities as precursors of symbolic equations. The identity $3 \times 2 + 12 = 5 \times 2 + 8$, for example, can be used to construct the equations $3 \times ? + 12 = 5 \times ? + 8$, or $3 \times 2 + \bullet = 5 \times 2 + 8$, etcetera. If so desired, the question mark or dot (or any other symbol) can eventually be replaced by a letter symbol to introduce the concept of unknown. After students have seen where an equation might come from and what the solution looks like, one proceeds to teaching the solution method. In order to make the symbolic manipulations meaningful, each step in the solution procedure is demonstrated for the arithmetical identity as well. The solution (the number 2 in the example above) is marked to make it ‘hidden’. Although the arithmetical model does not have the advantage of physical or visual affinity, it makes good use of the arithmetical pre-knowledge that students have and it can be applied to any type of equation.

notebook model

The notebook model supports one of the strategies for solving systems of equations developed for the algebra strand in the *Mathematics in Context* project. Figure 2.5 shows how a realistic context of ordering drinks and food in a restaurant is translated into a mathematical representation of different combinations.

The notebook model resembles a matrix where the entrees in each row represent the number of items for that particular combination. Matrix equation solving in itself is of course not an innovative approach; from the influential text *Nine Chapters on the*

Mathematical Art composed at the time of the Han Dynasty (206 BC to 220 AD) we know that ancient Chinese civilizations used a similar approach.

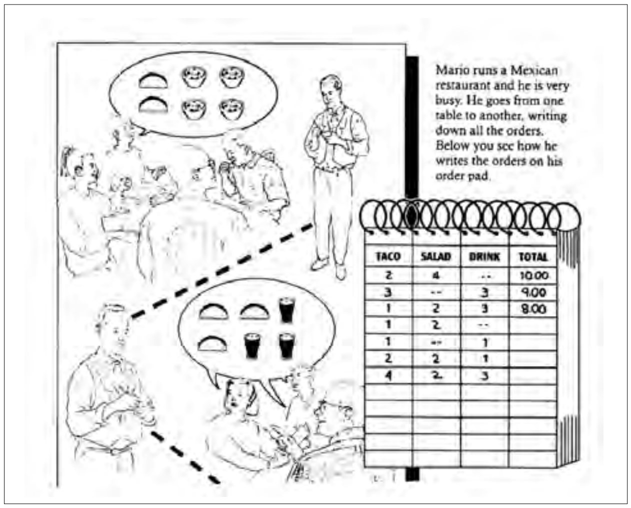


figure 2.5: notebook equations

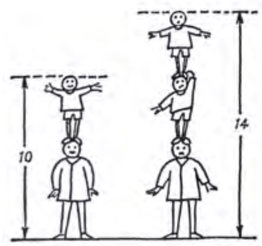
linear model

Another possibility for visualizing linear equations is based on representing an unknown quantity or magnitude by a line, or a strip where only the length of the strip matters. We call this the *linear* model. Let us consider the following problem posed by Sawyer, who already suggested using visual representations of equations decades ago:

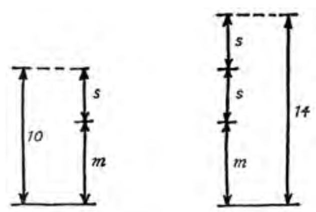
> A man has two sons. The sons are twins; they are the same height. If we add the man's height to the height of 1 son, we get 10 feet. The total height of the man and the 2 sons is 14 feet. What are the heights of the man and his sons? (Sawyer, 1964, p. 40).

The objective of visualization is to organize and thereby clarify the problem situation. Figure 2.6 I illustrates how the description can be transformed into a picture representation, which can then be simplified to a more schematic representation (part II). The visual forms are more accessible for reasoning than the story; at a glance the two human towers are compared to find that each son has to be 4 feet tall. The schematic drawing has the advantages of less work and general applicability, but when it becomes context-free it also loses its meaning. Moreover, students may not accept that an unknown length can be drawn as if it were known. It is therefore important to let students decide their own pace of learning. If at a later stage the student is capable of schematizing a new type of problem in a similar way, we can say that the schematic drawing has become a model. What started as a model *of* a given problem situation has become a model *for* reasoning about a new family of problems.

I



II



III

m	$1s$	10
m	$2s$	14

IV

$1m + 1s = 10$
$1m + 2s = 14$

figure 2.6: drawing - schematizing - symbolizing

Perhaps after a few more problems a student will suggest using a simple table (figure 2.6 III) or even mathematical symbolism closer to a system of two equations (part IV). Otherwise, when the time is ready, the teacher may guide the classroom discussion towards a symbolic representation. In chapter 5 and chapter 6 we describe how the linear model has functioned in the learning strand designed for this project.

2.6 Arithmetic and algebra

In section 2.3 we mentioned four different approaches to the teaching and learning of early algebra: generalizing, problem solving, modeling and functions. It is a classification which compares four kernal activities in mathematics, each leading to algebraic learning. Shortcomings like oversimplification and incompleteness have been recognized. Some readers might suggest classifying algebra in terms of its interfaces with other mathematical terrains instead. For instance, algebra can also arise from mathematical activities in geometry (see section 2.8). In this section we discuss algebra from an arithmetical perspective. The relationship between arithmetic and algebra not only sheds light on some typical learning difficulties of algebra, but it also shows why an approach to early algebra based on arithmetic is a suitable one for this study.

2.6.1 A dual relationship

problematic differences

We have already mentioned in section 2.4 that a number of learning difficulties of early algebra can be ascribed to the different natures of arithmetic and algebra. We can identify differences regarding the interpretation of letters, symbols, expressions and the concept of equivalence. For instance, in arithmetic letters are usually abbreviations or units, whereas algebraic letters are stand-ins for variable or unknown numbers. And in the case of solving linear equations in one unknown there is said to be a discrepancy known as the *cognitive gap* (Herscovics & Linchevski, 1994) or *didactical cut* (Filloy & Rojano, 1989), referring to students' inability to operate with or on the unknown. In this study we use the terms 'cognitive breach', 'cognitive break', 'rupture' and 'gap' to describe the collection of learning difficulties caused by discrepancies between arithmetic and algebra. If in any situation we use the term 'cognitive gap', it is to be understood in this broad sense and not in the specific way Herscovics and Linchevski use it.

reification

According to Sfard and Linchevski (1994) the rupture between arithmetic and algebra is an ontogenetic gap caused by the operational-structural duality of mathematical concepts. In the transfer from an arithmetical to an algebraic conception students need to learn that processes can be seen as objects; they must acquire a dual process-product perception. Sfard (1991) proposes a 'theory of reification' according to which the development of mathematical concepts occurs in 3 phases: interiorisation, condensation, reification. These phases form a hierarchy of perspectives where processes on objects become objects on their own, which can in turn be part of a process at a higher level. It is a theory which resembles Freudenthal's vision on levels of learning (see section 4.3.1). There is evidence that this process of reification is difficult to achieve, not in the least because reification and advanced interiorisation appear to be locked in a 'vicious circle'. On the one hand the ability to perform basic algebraic algorithms is needed to get a feeling for the objects involved, on the other these same objects are needed to gain full technical competence, giving meaning to the algorithms and making it easier to remember them. In section 2.5 we already described how symbolizing and schematizing activities play a role in the reification process. In the case of algebra, Sfard and Linchevski (1994) connect the difficult progression from an operational conception (arithmetic) to a structural one (algebra) with the gap in the process of reification (the 'vicious circle').

dynamic vs. static

We want to remark here that in this book we may use the terms operational' (or procedural) and 'structural' when in fact we mean only to distinguish between their 'dynamic' and 'static' natures respectively. The terms 'static' does not include the notion of 'perception as an object', so whereas the process-product duality of the former qualification indicates a difference in conceptual level, the dynamic-static duality does not. For all the students who participated in the classroom experiments we can say that our reference to an operational or a structural conception is confined to the dynamic-static distinction. The students have not had enough time to actually develop a structural notion of algebra.

procept

Gray and Tall (1994) have suggested the notion of 'procept' and 'proceptual thinking' as an intermediate phase between the operational and the structural level. The procept, intended to build bridges across the 'proceptual divide', consists of three components: a *process* which produces a mathematical *object*, and a *symbol* to represent either of these. Gray and Tall remark that learners who are able to see symbols as objects and use these symbols to produce new mathematical ideas can formalize their thinking, while students who continue to think in terms of processes are likely to remain at an operational level of thinking.

Decades ago Freudenthal already pointed out that inconsistencies between arithmetic and algebra can cause great difficulties in early algebra learning. He observes that the difficulty of algebraic language is often underestimated and certainly not self-explanatory: "Its syntax consists of a large number of rules based on principles which, partially, contradict those of everyday language and of the language of arithmetic, and which are even mutually contradictory" (Freudenthal, 1962, p. 35). He then says:

> The most striking divergence of algebra from arithmetic in linguistic habits is a semantical one with far-reaching syntactic implications. In arithmetic $3 + 4$ means a problem. It has to be interpreted as a command: add 4 to 3. In algebra $3 + 4$ means a number, viz. 7. This is a switch which proves essential as letters occur in the formulae. $a + b$ cannot easily be interpreted as a problem (Freudenthal, 1962, p. 35).

The two interpretations (arithmetical and algebraic) of the sum $3 + 4$ in the citation above correspond with a procedural (operational) and a structural perception respectively.

contrasting characteristics

In spite of the increase in information available from research on the arithmetic-algebra duality, and perhaps also because of it, the demarcation line between arithmetic and algebra is not clear. In this study a magnifying glass was taken to hand, so to speak, to contrast the characteristics of both. We realize that naming the differences in extreme terms might be dangerous. Some descriptions are self-evident, while others are certainly subject to debate. Moreover, the list is probably not complete. However, we feel the end justifies the means because, for two reasons, table 2.1 has been an effective tool. First, the demarcation has provided ideas for constructing bridges between arithmetic and algebra. And second, it has simplified the identification of solution strategies as 'arithmetical', 'pre-algebraic' or 'algebraic' during the analysis of student work. In the next few paragraphs we clarify some of the characteristics of arithmetic and algebra and connect them with typical learning difficulties of algebra. The left-hand column in table 2.1 contains eleven arithmetical characteristics, opposed to eleven algebraic characteristics in the right hand column. The middle column represents the transition zone between arithmetic and algebra, the contents of which – the numbers 1 through 11 – will be elaborated in thematic sections below. In each case we discuss the characteristic at hand and give suggestions for an intermediate, pre-algebraic approach. For example, the characteristics 1, 2, and 3 deal with *generalization,* the theme for the first section.

arithmetic	pre-algebra	algebra
general aim: to find a numerical solution	1	general aim: to generalize and symbolize methods of problem solving
generalization of specific number situations	2	generalization of relations between numbers, reduction to uniformity
table as a calculational tool	3	table as a problem solving tool
manipulation of fixed numbers	4	manipulation of variables
letters are measurement labels or abbreviations of an object	5	letters are variables or unknowns
symbolic expressions represent processes	6	symbolic expressions are seen as products *and* processes
operations refer to actions	7	operations are autonomic objects
equal-sign announces a result	8	equal-sign represents equivalence
reasoning with known quantities	9	reasoning with unknowns
unknowns as end-point	10	unknowns as starting-point
linear problems in one unknown	11	problems with multiple unknowns: systems of equations

table 2.1: characteristics of arithmetic and algebra

generalization (1, 2, 3)

Solving problems in arithmetic is primarily directed at finding numerical solutions in specific situations. The objective of algebra, on the other hand, is usually to discover and express generality of method, looking beyond specificness. Generalization requires the learner to recognize common factors on the one hand and unique characteristics on the other. For example, equation solving is not useful if each new problem requires a new approach. The strength of equation solving is its general applicability: define the unknown(s), describe the relation(s) between the quantities, and solve the problem with algebraic means. Algebra also constitutes the reduction to uniformity; in contemporary mathematics this is done with symbolic language.

At times students carry out activities of generalizing in arithmetic. Generalization of number situations helps students to develop abstract notions of numbers, like the decontextualization of fractions. In doing so students internalize and reify the fraction concept. Algebra, on the other hand, pursues the generalization of relations *between* numbers or methods of manipulating numbers; not the numbers but the *relations* (methods) are the objects of generalization. Generalized relations in turn enable extrapolations and predictions about new situations, broadening the horizon even further. We can illustrate this difference by considering the role of a tabular representation in arithmetic and in algebra. In arithmetic the table is seen as a calculational tool, to support the calculation of ratios or to organize and structurize information. In algebra the table has a purpose in solving problems, for instance to investigate patterns. Table 2.2 can help a student to recognize the relationship between n and A and describe this relationship in general terms.

n	0	1	2	3	4	5	6	*number*
A	0	2	6	12	20	30	42	*number* × (*number* + 1)

table 2.2: from a pattern to a general expression

pre-algebra proposed

In the proposed learning strand the tabular representation is used to solve problems like ‘two numbers added together make 120, while the difference between them is 38’ (see table 2.3). It is very natural for students to use a trial-and-error approach. The table helps students to structurize their attempts, like starting with a difference of zero and then increasing it symmetrically. In both examples the purpose of the tabular representation is to facilitate the acquisition of a general method.

first number	60	70	75	79
second number	60	50	45	41
difference	0	20	30	38

table 2.3: using the pattern to solve the problem

meaning of letters (4, 5)

We have already mentioned that letters can have different meanings and functions in algebra. Early in the process of learning of symbolic algebra – the study of algebraic expressions, equations, equation solving, and formulas – letters usually represent general numbers, unknown numbers, arguments or variables (see also Küchemann, 1978; Usiskin, 1988). According to Kieran (1989, 1990), students have been found to be confused by the different ways that a single letter can be used, leading to incorrect interpretations. Moreover, learners may be reluctant to accept the idea that numbers can be represented by letters or that the expression $x + 3$ can be a final answer (‘cognitive readiness’). Another common difficulty of calculating symbolic expressions in algebra is related to a conflict with the positional system. In algebra the term $6x$ when $x = 3$ is evaluated by calculating 6×3, which clashes with the arithmetical meaning of the digits 6 and 3 in the number 63. When students encounter letters in arithmetic, the role of these letters is very different from algebraic letters (variables). In the Dutch arithmetic curriculum there are only a few instances where letters refer to generalized numbers, for example in the formula for the area of a rectangle $A = l \times w$. In arithmetic a letter usually represent a label for measuring (*m* for meters), counting (*p* for points) or currency (*f* for guilders), or it can be an arbitrary label to abbreviate a word. In each case the letter refers to the measurement unit or object directly. In algebra letters can have a second meaning, namely the *number* of meters, points or guilders. If algebra learners continue to interpret letters as labels instead of variables, they are bound to make what is known as the 'reversal error' in the 'student-professor problem':

> Write an equation using the variables *S* (the number of students) and *P* (the number of professors) to represent the statement 'There are six times as many students as professors at this university' (Rosnick, 1981).

Students who write $6S = P$ instead of $S = 6P$ interpret their expression as '6 students for every professor'. Herscovics (1989) suggests that this error is caused by the interference of natural language and algebra.

pre-algebra proposed

In the experimental learning strand we tackle this problem by confronting students with different meanings of (word) variables in the context of barter trading. For example, students are required to switch from a statement on *value* of goods ('value of a cabbage = 3 × value of an apple') to a table with *numbers* of goods, and vice versa, followed by a task on completing a table with numbers of goods given a barter expression like $1a = 2b$ (1 apple for 2 bananas) where the letters are labels. Subsequently students do an activity with arrow diagrams involving letters where students have to determine the meaning of the letters (representing either the *number* of items or the *value* of an item).

conception of symbolic expressions (6, 7, 8)

As we discussed earlier, in arithmetic students conceive operations as a command to perform an action (addition, multiplication, etcetera). The operation is only the means to an end: finding a numerical outcome. An operation viewed algebraically, on the other hand, is an autonomic object and the outcome is the expression itself; the operation cannot be carried out, so to speak. For example, an expression like '5 + 3' is an open-ended action in arithmetic but in algebra it is a valid, finished product. Along the same lines of reasoning we can say that the arithmetical meaning of the equal-sign is to announce the numerical outcome of a calculation, while the the algebraic, relational conception is to depict a state of equivalence. The former viewpoint agrees with a dynamic, procedural conception of operations and expressions, whereas the latter viewpoint fits a static or – at a formal level – structural perception. With these roles of operations and the equal-sign in mind, symbolic expressions can be viewed as commands for action or as static descriptions.

cognitive gap

Filloy and Rojano (1989) as well as Herscovics and Linchevski (1994) point out a break in the development of operating on the unknown in an equation. Herscovics and Linchevski describe the *cognitive gap* as 'students' inability to spontaneously operate with or on the unknown' (1994, p. 59). They object to the definition of *didactical cut* given by Filloy and Rojano (1989) for restricting the problem to mathematical characteristics, with no eye for the role of solution procedures. In the transfer from a word problem (arithmetic) to an equation (algebraic), the meaning of the equal-sign changes from announcing a result to stating an equivalence. For example, if we symbolize the statement "Jenny is 5 years old, and she is 2 years older than her little brother" in the exact same order, we get the equation $5 = x + 2$. The symbolic expression does not resemble the arithmetical interpretation '5 minus 2 gives the little brother's age' at all. Furthermore, if the unknown appears on both sides of the equal-sign instead of one side, as in $10 - 3x = x + 2$, the equation can no longer be solved arithmetically (i.e. by inverting the operations on the coefficients one by one). Instead the student is required to treat the unknown quantity as if it were a known number. In other words, operating with or on the unknown requires another notion of equivalence as well as the ability to treat an unknown number as if it were known.

cognitive obstacles of symbolizing

Other cognitive obstacles related to manipulating symbolic expressions and solving equations reported by researchers are:

- recognizing equivalence of expressions;
- handling minus signs in an equation: Linchevski and Herscovics (1996) have found that mixed terms in a symbolic expression become detached from the operations: $3x - 5x + 7x$ is interpreted as $3x - 12x$, and $3x + 2 - 8x$ as $11x + 2$;
- combining like terms, i.e. terms of the same dimension;
- misunderstanding the syntax of expressions can cause the so-called 'reversal error' – also referred to as the student-professor problem – where the relation between two quantities is interpreted the wrong way around;
- making formal manipulations meaningful and purposeful;
- most models for solving equations fail to accommodate the transfer from an informal to a formal conception of equations.

pre-algebra proposed

The pre-algebra instructional materials designed for this study deal with just a few of these obstacles because linear equations appear only in context-bound form in the secondary school units. Recognition of equivalent expressions, combining like terms and symbolic manipulations are embedded in situations of fair trade, where purpose and meaning are ensured. In the case of solving iconic systems of equations the informal strategies of repeated exchange (see figure 2.1) and making new combinations (see figure 2.5) provide a natural intermediate transition phase from arithmetic (trial-and-error strategies) to algebra (elimination of one unknown by equalizing coefficients). These early algebra activities can be formalized in a context-free, symbolic environment at a later stage.

problem solving: reasoning with (un)knowns (9, 10, 11)

In arithmetic children reason with and about fixed numbers, mostly in specific, context-bound situations. High ability students may think of numbers as abstract objects, and reason about their properties. Algebraic reasoning involves variables and unknowns instead, and symbolic notation appears to be an additional cognitive obstacle for the novice learner.
Arithmetical unknowns are symbolized by dots, question marks or geometric figures (little squares or circles), or implicitly by 'stains' (drawn to 'hide' the unknown number). The unknown value can be recovered with arithmetical means like calculating in reverse order or trial-and-error strategies; the unknown is not involved in the calculations. In algebra, too, the sole purpose of the unknown is to be revealed. But in algebraic applications the unknown is the starting-point of the solution process, in which the symbol itself is the object of manipulation. We have already mentioned before that these different approaches to problem solving – straight-to-the-point inversion in arithmetic, round-about way of constructing an equation first in algebra – can cause great difficulty. It has been found that children have trouble recognizing the structure of the problem as they try to represent the problem symbolically. They can recognize the solution procedure (for example, inverse calculation) but they cannot reason with the unknowns themselves. Moreover, the informal arithmetical approaches do not go hand in hand with algebraic methods; as they learn algebra, students tend to forget their informal knowledge and with it they lose their framework of meaning.

pre-algebra proposed

The historical development of algebra implies that reasoning with unknowns is found to be more natural to novice learners than symbolizing it. The experimental learning strand therefore aims to stimulate the dual development of symbolizing and reasoning with unknowns using students' free productions wherever possible. This is done by offering students various approaches to symbolizing problem situations (tables, abbreviations, pictures, diagrams) and by letting them switch between these different forms of representation to become familiar with them. Subsequent activities provoke students to use some of these representations as tools for mathematical reasoning. We anticipate that the role of the teacher is very important in this process because students might invent notations which are not compatible with algebraic conventions.

2.6.2 Accesses to algebra in the Dutch arithmetic curriculum

The dual relationship between arithmetic and algebra offers various opportunities for pre-algebraic activities in arithmetic at elementary school. For instance, a solid foundation of number sense is prerequisite for developing an understanding of number properties (algebra as generalized arithmetic). Ratio tables are a suitable setting to study simple number patterns and come to a general formulation. Thirdly, inverting operations in activities like 'guess my number' – where one student has a number in mind, another student names a string of operations to be carried out, the first student gives the outcome and the other student then determines the initial number – helps to prepare students for 'undoing' linear equations of the form $ax + b = c$ in early algebra.

impulses

In the Dutch teaching units *Wis en Reken* (Boswinkel et al., 1997) a first impulse has been given to integrate pre-algebra activities in the elementary school curriculum. Some tasks in grade 6 are based on student materials from the *Mathematics in Context* unit *Comparing Quantities,* like story problems on barter trade (substitution of trade relations) and embedded systems of equations such as in figure 2.8. The objective of these tasks is to develop reasoning strategies with which unknown quantities (in the form of concrete objects) can be manipulated and their values be recovered. Another example of pre-algebraic activity in grade 6 is making the procedures themselves the objects of study. Students are challenged to shorten a string of operations represented by 'calculating machines' by combining additions and subtractions into one. For example, the additions '+ 5' and '+ 0.25' and the subtraction '– 7' can be replaced by the subtraction '– 1.75'.

opportunities not exploited

Two activities for students in grade 4 are based on repeating the same calculational procedures for a range of numbers. The first consists of a list of pies and their prices, and students are asked to determine the price of half a pie. The procedure is as follows: round off the price to the next whole number, divide by 2 and add one guilder. This activity can lead to a general formula for finding the price of half a pie of any kind, and perhaps it can be extended to new situations. However, the teacher guide does not mention generalization as one of the goals of the task, from which we deduce that early algebra is not an explicit part of the curriculum. The second activity we mention here deals with proportions *between* kites and *within* kites. Children measure the dimensions of a series of similar kites drawn on the work sheet. One of the kites does not fit on the page entirely, so its dimensions can only be determined by finding the ratio. The students are then asked to complete a table in which some of the measurements are given. The kite's dimensions are represented by the letters *a* through *e*, as shown in figure 2.7, which lead quite naturally to remarks like '*a* is always twice as much as *b*' (internal proportion). Some values in the table can only be found using the internal proportions. A formulation in general terms could be a suitable extension of the task, although presenting the letters ready-made to the students has already enervated the first step of this process. Again there is no indication of an algebraic intention in the teacher guide.

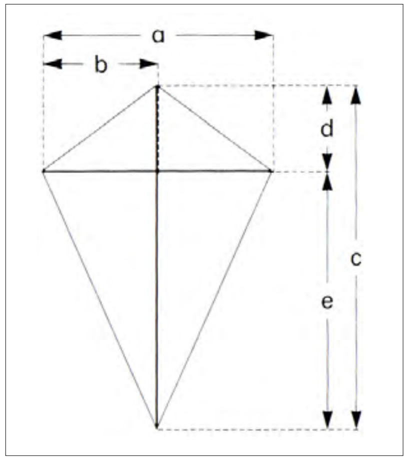


figure 2.7: dimensions of a kite

The examples above illustrate that some curriculum developers have included a few isolated activities which facilitate the development of algebraic reasoning and symbolizing, but not as an explicit learning goal. It is therefore difficult to predict to what extent teachers in elementary school currently make use of accesses to algebraic thinking in arithmetic.

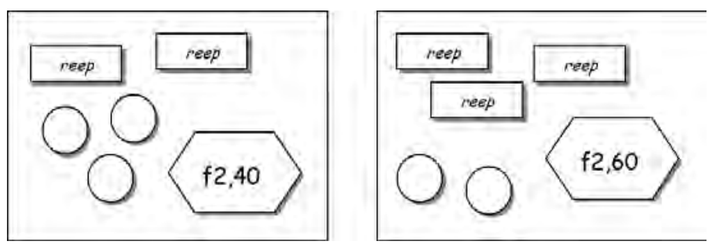
2.7 Pre-algebra: on the way to algebra

In this study we have decided to restrict 'school algebra' to linear relationships and solving equations in one or two unknowns, in particular the transition from descriptions to (semi-)symbolic representations. The proposed learning strand corresponds most with the problem solving perspective of algebra, but it also includes generalizing and modeling activities. Although the learning strand is arithmetically inclined, it does not really fit the definition 'algebra as generalized arithmetic' because it does not aim to generalize number properties (commutativity, distributivity etc.). Instead, algebra and arithmetic are considered to have a dual relationship: algebra has its roots in arithmetic and depends on a strong arithmetical foundation, while arithmetic has ample opportunities for symbolizing, generalizing and algebraic reasoning.

pre-algebra

From this perspective we propose to use the term 'pre-algebra' as the transition zone of informal explorative activity from arithmetic into early algebra. Pre-algebra involves algebraic thinking and informal symbolizing in an arithmetical setting, broadening and strengthening the arithmetical foundations needed for equation solving. For instance, there are indications that poor number sense and little insight in number relations can cause problems in the early learning of algebra with respect to precedence and inversion of operations.

categorization It is therefore important to determine what makes a mathematical problem or activity algebraic, arithmetical or pre-algebraic. The differences between arithmetic and algebra in table 2.1 help to get a clearer view, suggesting opportunities for intermediate, pre-algebraic conceptions for most issues. However, we believe it is not the nature of the *task* but the nature of the *solution method* that matters. The problem in figure 2.8, taken from the first unit in the experimental learning strand, will help to clarify this idea. The picture shows two combinations of candy with two different total prices. There is no algebraic symbolism or other algebraic representation involved. But since the price of a candy bar and that of a magic ball are unknown, the drawings represent an informal system of simultaneous equations. Still, many mathematicians will probably hesitate to call it an algebraic problem.



How much does one candy bar cost? And one magic ball?

figure 2.8: reasoning with iconic unknowns

Let us now consider a few solution methods. A student may solve the task by trial-and-error or trial-and-adjustment, substituting numerical values for each bar and magic ball. This kind of approach is of a primitive, arithmetical level. Another student might compare the two combinations and observe that changing a bar for a magic ball brings down the price by 20 cents. Two more exchanges will result in a combination with only magic balls. This kind of reasoning involves comparing known quantities and continuing the pattern; it certainly has an algebraic tendency. Yet another learner might make new combinations by adding, multiplying and/or subtracting them, until one of the unknowns is eliminated. If these combinations were written in a symbolic form, the solution method would surely be considered algebraic. In other words, the nature of the task – algebraic or otherwise – cannot be seen separate from the solution strategy applied. Similarly we can trace back the beginnings of algebra to ancient Egypt, where an algebraic perception of the unknown (treating it as a known number) accentuates algebraic *method*, while the problems themselves – written in words – are hardly algebraic from the modern perspective (see also section 3.3). Alternatively many problems we would nowadays solve using an algebraic method were tackled successfully with arithmetic for many centuries.

We might even say that algebraic *problems* do not exist; we can only speak of algebraic *methods* or *solutions*.

purpose of the learning strand

Our design process should therefore be directed at creating tasks which facilitate the development of (pre-)algebraic methods, so that the cognitive break between arithmetic and algebra may be (partly) overcome. But towards what kind of algebraic competence do we aim to work with our experimental pre-algebra learning strand? And which algebraic skills and insights do we deem accessible from an arithmetical problem solving perspective? Our overview of differences between arithmetic and algebra has accentuated a number of territories which deserve special care and attention: generalizing, meaning of letters, symbolic expressions, and reasoning with unknowns. In addition to algebraic computational skills and an algebraic way of reasoning, a student should also develop an algebraic attitude. Flexible use of problem solving strategies and feeling confident to reason about unknown quantities are two characteristics of such an attitude. The process of designing the learning strand is described in chapter 5.

notation vs. abstraction

We also wish to find out how algebraic notation and mathematical abstraction are related. Sfard's theory of reification is based on the idea that an operational conception of a notion precedes a structural perception:

> It seems, therefore, that the structural approach should be regarded as the more advanced stage of concept development. In other words, we have good reason to expect that in the process of concept formation, operational conceptions would precede the structural (Sfard, 1991, p. 10).

Sfard conjectures that this is basically true for both the historical development and for the development of the individual learner, and gives the following historical example:

> (...) the science of computation, known today under its relatively new name 'algebra', has retained a distinctly operational character for thousands of years. The so-called 'rhetorical' algebra, which preceded the syncopated and symbolic algebras (the last developed not before the 16th century!) dealt with computational processes as such, while the only kind of abstract objects permitted in the discourse were numbers. Even most complex sequences of numerical operations were presented by help of verbal prescriptions, which bore distinctly sequential character and did not stimulate condensation and reification (Sfard, 1991, p. 23-24).

However, this point of view has also been contradicted. Radford (1997) observes that the categorization rhetoric – syncopated – symbolic is the result of our modern conception of how algebra developed, and that it is often mistaken for a gradation of mathematical abstraction. When the development of algebra is seen from a sociocultural perspective, instead, syncopated algebra was not an intermediate stage of maturation but it was merely a technical matter. As Radford explains, the limitations of writing and lack of book printing quite naturally led to abbreviations and contractions of words. In other words, we would like to determine whether or not a student's progressive formalization of notations is accompanied by a process of abstraction.

2.8 Algebra from a geometrical perspective

geometry and early algebra

Even though generalized arithmetic and problem solving are more frequently taken as the starting-point for early algebra learning – in particular for simplifying symbolic expressions and solving equations – geometrical visualizations form a regular part of many algebra school books. One of the common topics in early algebra is recognizing and continuing a pattern to deduce a general formula. An example of such a task is shown in figure 2.9. Adding up the number of dots in the figures on the top row gives you the *triangular numbers* 1, 3, 6 and 10 (n = 1, 2, 3 and 4 respectively). The learner may be asked to a) continue the sequence by drawing the fifth and the sixth triangular number, and b) give a general formula to describe the n^{th} triangular number.

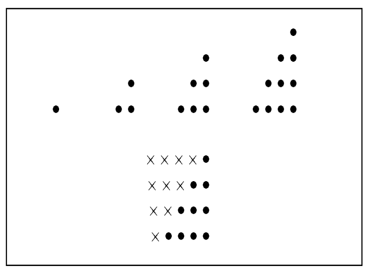


figure 2.9: continuing triangular numbers

Geometrical visualization can help to deduce a general formula, by changing the triangular figure into a rectangular one. In the second row of figure 2.9 we have drawn the number 10 again, adding crosses to complete it to a 5 by 4 rectangle. The number of dots N – which is the triangular number itself – of the 4$^{\text{th}}$ triangular number is found to be $N = \frac{1}{2} \times 5 \times 4 = 10$. If we extend this method to the n^{th} figure, the n^{th} number can be described by the expression $N = \frac{1}{2}\, n\,(n + 1)$. As shown by this example, geometrical figures can support students in shifting from specificness to generality, while at the same time there is the arithmetical component of number relations. In other words, this kind of activity is suitable for a combined arithmetical-geometrical approach to early algebra.

And yet for this project we have chosen not to include geometrical patterns in the student materials, for two reasons. The first reason is related to content. Constructing general expressions involving varying quantities is not a deliberate part of the learning trajectory (which does not mean it cannot occur spontaneously). The core of the program involves a lower level of concept of the letter symbol, such as the letter as unknown (in equations) and the letter as a abbreviation of an object. For instance, a trade expression like $2m = 3a + 4b$ symbolizes the act of trading 2 melons fairly for 3 apples and 4 bananas, and the letters m, a and b act as arbitrary labels for concrete objects. Second, it has been our intention to choose an approach to early algebra which can be considered new and innovative in The Netherlands. From this point of view, recognizing and continuing patterns of geometrical figures is not so appropriate because it is quite a common activity in most Dutch algebra textbooks.

content of the experimental pre-algebra strand

geometry and solving equations

When asked to give an example where geometry can support equation solving, the area model for quadratic equations comes to mind quite readily. However, quadratic equations do not play a role in this study because they are too advanced for our target group. The emphasis of the learning strand lies on developing tools for comparing quantities: a description, a picture, a diagram or symbolism. The contexts which have been chosen do not lead naturally to a geometrical perspective. Early instructional experiments in the study and personal teaching experience have shown that schematic and symbolic representations like a table, a diagram, a pictogram and abbreviations are more accessible and more natural to pre-algebra students than a geometric shape. Beginning algebra books in the Netherlands show the same preference, using ‘calculating machines’, arrow language, pictures and symbols to visualize relations between quantities. The *Mathematics in Context* instructional units also work with tree diagrams.
The empty number line (or any other linear model) can be used as a geometric model for studying linear relations, which is also done in this study. Yet, it is a type of representation that learners do not propose themselves. Teaching practice indicates that students associate geometrical forms such as a rectangle with perimeter and area problems, and the (empty) number line with basic arithmetic.

geometrical models not so suitable

Research results suggest that geometric models may not be appropriate for representing relations and solving equations in one unknown. Kieran (1989) describes that students do not profit from visualizing linear equations of the form $ax + b = cx$ using rectangles. Early results in this study indicate that representing a relation such as $A = 3 + B$ using two rectangular bars is not feasible because younger students are reluctant to draw an indeterminate magnitude. These intermediate results have strengthened our decision to emphasize the arithmetical accesses to early algebra rather than the geometrical ones, although we have maintained the rectangular bar as a model in one of the student units (see also chapter 5). Another drawback of geometrical models is that their generality is limited: sometimes magnitudes and algebraic expressions do not have geometrical representations, and dimensional considerations cause restrictions.

historical development of algebra

The third reason for not taking a geometrical approach to equation solving is found in the history of algebra. From its beginning until the sixteenth century, algebra existed as an advanced form of arithmetical problem solving. With the exception of Diophantus and a few others, algebraic problems were stated and solved in natural language (the phase of rhetoric algebra, see also section 3.3). The solutions to these problems were not accompanied by any kind of explanation, and the rhetorical notation held back the development of a more generalized formulation. There was some visualization in Babylonian, Greek, Indian, and Arabic cultures, but it referred to quadratic and higher order equations, whereas for our present purposes we limit ourselves to linear problems and systems of linear equations. The integration of algebra and geometry came with Descartes in the seventeenth century, but his approach is out of range of this study because the level of symbolic algebra it requires is too high. The historical development of algebra, therefore, does not argue for a geometrical approach of solving linear equations either.

2.9 Early algebra in the Dutch curriculum

The experimental pre-algebra learning strand designed for students in grade 6 and grade 7 is intended as a series of lessons which is complementary to the national curriculum. It does not require specific pre-knowledge, nor does it replace any particular part of the early algebra strand in the regular program. We do foresee that giving more priority to informal methods and symbolism will help to prepare students for their first encounter with algebra as it is currently taught in Dutch schools. Since the national algebra curriculum has not influenced the content of the experimental teaching materials, we assume that a brief description of the early algebra program in the Netherlands will suffice.

algebra program

Two decades ago, the algebra working group of the W12-16 project designed a new algebra program for the first three years in Dutch secondary schools (Algebragroep W12-16, 1990, 1991; W12-16 C.O.W., 1992), although an approach to algebra based on different types of representations is not entirely innovative because it has been suggested previously (Janvier, 1978; Goddijn, 1978). The team deliberately chose to develop algebra from a user's perspective, for which reason important choices were made regarding mathematical content. First, the learning strand concentrates on interpreting rather than manipulating algebraic expressions. As a result algebraic techniques are subservient to studying relations and solving problems; they are not a goal in itself. Second, the problems are situated in realistic contexts as much as possible. Third, the developers emphasize the acquisition of a wide variety of techniques instead of an in depth study of only a few techniques. And finally, different algebraic concepts are developed simultaneously rather than stacking them in a linear order. Students develop algebraic conceptions and skills very gradually from concrete situations by connecting different forms of representation: descriptions of situations, tables, graphs and formulas. Word formulas are used extensively before moving on to formal symbolic expressions because, since word formulas are situated in a context, they enable students to reason and manipulate with understanding. Variables are primarily treated as varying quantities; the concept of ‘unknown’ appears when formulas are transformed into equations or when students are asked to determine the point of intersection of two graphs. All in all the mathematical content of the program shows an integration of early algebra and early analysis based on graphical interpretation and meaning, and symbolic manipulations are kept to a minimum.

solving equations

In the first three years of secondary school, equation solving is not an end goal. Equations are used to study graphical relations or to find a number value in a formula. Equation solving techniques are based on natural strategies like undoing a string of calculations or clenching in the solution by successive bisection of the interval. Formal equation solving, the construction of equations from word problems – which is a very important part of early algebra historically – has been postponed to the higher grades. Meaning and understanding form its foundations: “(...) we think that techniques dealing with ‘manipulating’ succeed directly from ‘interpreting correctly’, and that these techniques will in turn support interpreting” (W12-16 C.O.W., 1992, p. 12, transl.). For more information on the W12-16 algebra program, see also Van Reeuwijk (in press).

The algebra learning strand proposed by the working group has not been implemented nationwide because some essential ideas were not adopted into the national curriculum. Important elements like ‘growth and order of magnitude’, successive bisection and developing general solution strategies instead of specific techniques appear in the national program only sporadically. Also the proposed attention for the structure of formulas has been reduced to a minimum. The W12-16 team developed a learning strand which differentiates between high and average ability students, but since educational authorities decided on one national curriculum for all students aged 12 to 16, what has remained is no more than a diminished version.

2.10 Conclusion

The teaching and learning of school algebra has become a world wide topic of interest over the last few years. An animated discussion on what algebra is and what it should be indicates there is no consensus amongst researchers in the field, resulting in a number of different approaches to how algebra should be learned and taught in school. Still, one matter most people agree on is that students are known to struggle with the structural aspects of algebra. Especially the change from a procedural way of thinking in arithmetic to a structural perspective in algebra causes a rupture in the learner’s development.

The objective of the project is to find ways to overcome the gap between arithmetic and algebra. We attempt to break through the ‘vicious circle’ of interiorisation and reification (Sfard, 1991) by a connected development of skills and concepts. Students will be guided to develop an informal, pre-algebraic concept of problem solving (arithmetical methods) first, followed by pre-algebraic skills (symbolizing, reasoning), to end with formalizing their skills to a level of algebraic conception (equation solving).

3 History in mathematics education

3.1 Introduction

Over the years mathematicians, educators and historians have wondered whether mathematics learning and teaching might profit from integrating elements of history of mathematics. It is clear that mathematics education does not succeed to reach its aims for all students, and that it is therefore worthwhile to investigate whether history can help to improve the situation. The interest in using history and the belief in its value for learning have grown remarkably in recent years – judging from the number of research groups in this field in Italy and France, and the History in Mathematics Education (HIMED) movement founded after a number of successful conferences and if we go back in time we find that acknowledged mathematicians from the eighteenth and nineteenth century held the same point of view. Joseph Louis Lagrange (1736-1813) wrote in one of his lectures for trainee school teachers:

> Since the calculation of logarithms is now a thing of the past, except in isolated instances, it may be thought that the details into which we have entered are devoid of value. We may, however, justly be curious to know the trying and tortuous paths which the great inventors have trodden, the different steps which they have taken to attain their goal, and the extent to which we are indebted to these veritable benefactors of the human race. Such knowledge, moreover, is not a matter of idle curiosity. It can afford us guidance in similar inquiries and sheds an increased light on the subjects with which we are employed (cited in Fauvel & Van Maanen, 2000, p. 35).

And in one of his notebooks Niels Henrik Abel (1802-1829) remarked: "It appears to me that if one wants to make progress in mathematics one should study the masters" (cited in Fauvel & Van Maanen, 2000, p. 35).

Only recently there has been a stronger call for methodological and theoretical foundations for the role of history in mathematics education. The report *History in Mathematics Education: The ICMI Study* (Fauvel & Van Maanen, 2000) has made a valuable contribution in this respect by collecting theories, results, experiences and ideas of implementing history in mathematics education from around the world.

The purpose of this chapter is to explain why and how history of mathematics might play a role in the learning and teaching of early algebra. After a brief account of the historical development of algebra we describe how history has instigated some essential ideas for the experimental learning strand.

3.2 Arguments for using history

We would not plead for the use of history of mathematics in mathematics education if we did not believe that history can make a difference. Incorporating history in mathematics education can be beneficial for students, teachers, curriculum developers and researchers in different ways. We give a number of arguments frequently mentioned to illustrate this (Radford, 1995, 1996ab, 1997; Fauvel, 1991; Fauvel & Van Maanen, 2000).

math as a growing discipline

Students can experience the subject as a human activity, discovered, invented, changed and extended under the influence of people over time. Instead of seeing mathematics as a ready-made product, they can see that mathematics is a continuously changing and growing body of knowledge to which they can contribute themselves. Learners will acquire a notion of processes and progress and learn about social and cultural influences. Moreover, history accentuates the links between mathematical topics and the role of mathematics in other disciplines, which will help to place mathematics in a broader perspective and thus deepen students' understanding.

teacher profits

History of mathematics provides opportunities for getting a better view of what mathematics is. When a teacher's own perception and understanding of mathematics changes, it affects the way mathematics is taught and consequently the way students perceive it. Teachers may find that information on the development of a mathematical topic makes it easier to explain or give an example to students. For instance, heuristic approaches provided by history can be contrasted with more formal, contemporary methods. In addition it is believed that historical knowledge gives the teacher more insight in different stages of learning and typical learning difficulties. On a more personal level, history also helps to sustain the teacher's interest in mathematics.

value for the designer

Not only the mathematics teacher but also the educational developer or researcher can profit from history in studying subject matter and learning processes. It provides teachers and developers with an abundance of interesting mathematical problems, sources and methods which can be used either implicitly or explicitly. Historical developments can help the researcher to think through a suitable learning trajectory prior to a teaching experiment, but it can also bring new perspectives to the analysis of student work. In the context of using history to study learning processes we mention the so-called Biogenetic Law popular at the beginning of the 20th century. The Biogenetic Law states that mathematical learning in the individual (philogenesis) follows the same course as the historical development of mathematics itself (ontogenesis). However, it has become more and more clear since then that such a strong statement cannot be sustained. A short study of mathematical history is sufficient to conclude that its development is not as consistent as this law would require. Freudenthal explains what he understands by 'guided reinvention':

> Urging that ideas are taught genetically does not mean that they should be presented in the order in which they arose, not even with all the deadlocks closed and all the detours cut out. What the blind invented and discovered, the sighted afterwards can tell how it should have been discovered if there had been teachers who had known what we know now. (...) It is not the historical footprints of the inventor we should follow but an improved and better guided course of history (Freudenthal, 1973, p. 101, 103).

learning trajectory

In other words, we can find history helpful in designing a hypothetical learning trajectory and use parts of it as a guideline. For instance, Harper (1987) argues that algebra students pass through different stages of equation solving, using more sophisticated strategies as they become older, in a progression similar to the historical evolvement of equation solving. Harper pleads for more awareness of these levels of algebraic formality in algebra teaching.

3.3 History of algebra: a summary

It is generally accepted to distinguish three periods in the development of algebra – oversimplifying, of course, the complex history in doing so – according to the different forms of notation: rhetorical, syncopated and symbolic, as shown in table 3.1 (see, for example, Boyer & Merzbach, 1989).

	rhetoric	syncopated	symbolic
written form of the problem	only words	words and numbers	words and numbers
written form in the solution method	only words	words and numbers; abbreviations and mathematical symbols for operations and exponents	words and numbers; abbreviations and mathematical symbols for operations and exponents
representation of the unknown	word	symbol or letter	letter
representation of given numbers	specific numbers	specific numbers	letters

table 3.1: characteristics of the 3 types of algebraic notation

rhetorical phase

The *rhetorical* phase lasted from ancient times until around 250 AD, where the problem itself and the solution process were written in only words. In this period early algebra was a more or less sophisticated way of solving word problems. A typical rule used by the Egyptians and Babylonians for solving problems on proportions is the *regula tri* or Rule of Three: given three numbers, find the fourth proportionate number (see also Kool, 1999). In modern notations this means: given the numbers a, b and c, find d such that $a : b = c : d$. The Rule of Three also specifies in which order the numbers in the problem must be written down and then manipulated. An example of such a problem is problem 69 in *Rhind Papyrus* (ca. 1650 BC), which says: “With 3 half-pecks of flour 80 loafs of bread can be made. How much flour is needed for 1 loaf? How many loafs can be made from 1 half-peck of flour?” (note: 1 half-peck ≈ 4.8 liters) (Tropfke, 1980, p. 359). Indian mathematicians (7^{th} and 11^{th} century) extended the Rule of Three to 5, 7, 9 and 11 numbers. Such problems are commonly classified as arithmetical, but in cases where numbers do not represent specific concrete objects and operations are required on unknown quantities, we can speak of algebraic thinking.

unknown

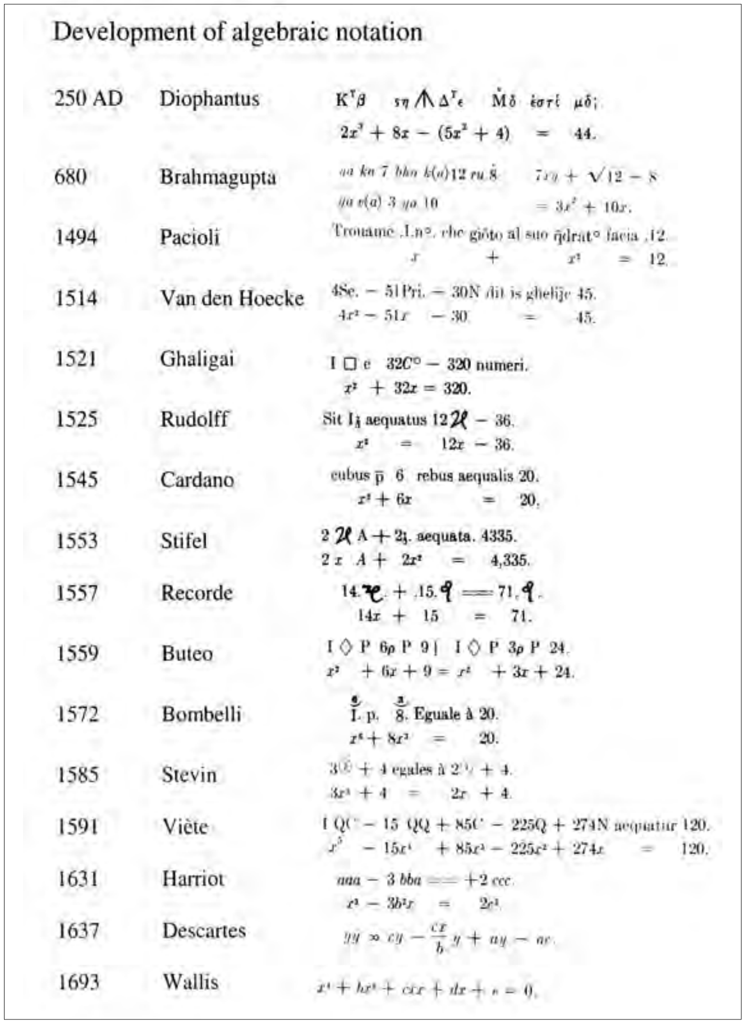
Depending on the number concept of each civilization as well as the mathematical problem, the unknown was a magnitude denoted by words like ‘heap’ (Egyptian), ‘length’ or ‘area’ (Babylonian, Greek), ‘thing’ or ‘root’ (Arabic), ‘cosa’, ‘res’ or ‘ding(k)’ (Western European). The solution was given in terms of instructions and calculations, with no explanation or mention of rules. These calculations indicate that unknowns were treated as if they were known and reasoning about an undetermined quantity apparently did not form a conceptual barrier. For instance, in the case of problems that we would nowadays represent by linear equations of type $x + \frac{x}{n} = a$, the unknown quantity x was conveniently split up into n equal parts.

Rule of False Position

The Babylonians also used linear scaling to solve for the unknown in the equation $ax = b$, much like the *regula falsi* or Rule of False Position first used systematically by Diophantus (Tropfke, 1980). According to this rule one is to assume a certain value for the solution, perform the operations stated in the problem, and depending on the error in the answer, adjust the initial value using proportions. For example, an old Babylonian problem goes: “The width of a rectangle is three quarters its length, the diagonals are 40. What are length and width? Choose 1 as length, 0;45 as width.” (Tropfke, 1980, p. 368, transl.). In the sexagesimal number system, the calculations show the diagonal to be equal to $\sqrt{1^2 + 0;45^2}$; that is, 1;15. Since the diagonals have to be 40 instead of 1;15, the length is then adjusted to $\frac{1}{1;15} \cdot 40$. The Chinese (second century BC) knew a rule based on the same principle that uses two estimates: the Rule of Double False Position. Although the Rule of False Position is generally not said to be an algebraic algorithm, its wide acceptance and perseverance even after the invention of symbolic algebra indicate it was and can still be a very effective problem solving tool.

syncopated algebra

The phase of *syncopated* algebra began around 250 AD when Diophantus introduced shortened notations which enabled him to rewrite a mathematical problem into an ‘equation’ (in curtailed form). He systematically used abbreviations for powers of numbers and for relations and operations. In his equations Diophantus used the symbol ς to denote the unknown and additional unknowns were derived from this symbol (although they were not used in the calculations). Tropfke (1980) explains that this change from representing the unknown by words to symbols really persevered only once the symbols were also used in the calculations. He gives two arguments to indicate that Diophantus appears to have been the first mathematician to do so. First, Diophantus performed arithmetic operations on powers of the unknowns, carrying out additions and subtractions of like terms self-evidently, without explicitly stating any rules. And second, he explained the method and purpose of adding and subtracting like terms on both sides of an equation (Tropfke, 1980, p. 378).

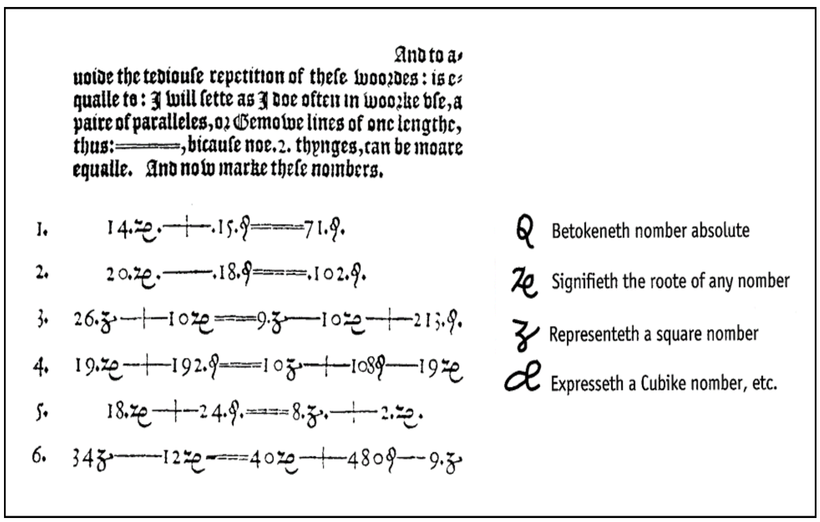


Adapted from an unretrievable source

figure 3.1: progress in symbolizing equations

progressive symbolizing

After Diophantus there were other practitioners of syncopated algebra. Figure 3.1 illustrates a collection of (semi-)symbolic equations throughout time, where each example is followed by the modern representation on the line below. In India (7$^{\text{th}}$ century AD) words for the unknown and its powers – which were extended in a systematic way – were abbreviated to the first or the first two letters of the word. Additional unknowns were named after different colors. In Arabic algebra (9$^{\text{th}}$ century AD) powers of the unknown were also built up consecutively, using the terms for the second and third power of the unknown as base. In abbreviated form, the first letter of these words was written above the coefficient. In Western Europe (13th century) there were minor differences in the technical terms between Italy and Germany, and only in the second half of the 14th century the words 'res' and 'cosa' were shortened to *r* and *s* respectively. In the middle of the 16th century Stifel introduced consecutive letters for unknowns and stated arithmetical rules using these letters. From there Recorde, Buteo, Bombelli, Stevin, and many others each developed a system to symbolize powers of unknowns and formulate equations (Tropfke, 1980, p. 377-378). Recorde introduced the equal-sign in print, saying: "And to avoid the tedious repetition of these words: is equal to: I will set as I do often in work use, a pair of parallels, or Gemowe lines of one length, thus: ═ , because no 2 things, can be more equal." (Recorde, 1557, as cited in Eagle, 1995, p. 82, modernized spelling of the text in figure 3.2).


And to a-
uoide the tediouse repetition of these woordes : is e-
qualle to : I will sette as I doe often in woorke vse, a
paire of paralleles, or Gemowe lines of one lengthe,
thus: ═══, bicause noe .2. thynges, can be moare
equalle. And now marke these nombers.

Date: Recorde (1557), *The Whetstone of Witte*
Source: Eagle (1995), *Exploring Mathematics through History*, p. 82

figure 3.2: algebraic notation of Recorde and introduction of the equal-sign

general methods

In the rhetorical and syncopated periods we see a certain degree of standardization. Problem solving procedures were demonstrated with one numerical example, which the reader could then easily repeat for new problems by simply replacing the numbers. Diophantus, the Arabs and the mathematicians in Western Europe contributed a variety of general methods of solving indeterminate, quadratic and cubic equations. But with the lack of a suitable language to represent the given numbers in the problem, it was still a difficult task to write the procedures down legibly. In a few isolated cases geometrical identities were expressed algebraically (with variables instead of numbers) but nonetheless written in full sentences. Syncopated notation did not (yet) enable mathematicians to take algebra to a higher level: the level of generality. It is important that students experience this limitation themselves in order to appreciate the value and power of modern mathematical notation.

FRANCISCI VIETÆ

ZETETICORVM

LIBER PRIMVS.

ZETETICVM I

DAta differentia duorum laterum, & adgregato eorumdem, invenire latera.

Sit data B differentia duorum laterum, & datum quoque D adgregatum eorumdem. Oportet invenire latera.

Latus minus esto A, majus igitur erit A + B. Adgregatum ideo laterum A 2 + B. At idem datum est D. Quare A 2 + B æquatur D. Et per antithesim, A 2 æquabitur D — B, & omnibus subduplatis, A æquabitur D $\frac{1}{2}$ — B $\frac{1}{2}$.

Vel, latus majus esto E. Minus igitur erit E — B. Adgregatum ideo laterum, E 2 — B. At idem datum est D. Quare E 2 — B æquabitur D. & per antithesim, E 2 æquabitur D + B, & omnibus subduplatis E æquabitur D $\frac{1}{2}$ + B $\frac{1}{2}$.

Data igitur differentia duorum laterum & adgregato eorumdem, inveniuntur latera. Enimvero

Adgregatum dimidium laterum minus dimidia differentia æquale est lateri minori, plus eadem, majori.

Quod ipsum est quod arguit Zetesis.

Sit B 40. *D* 100 *A sit* 30. E 70.

FIRST BOOK

Zetetic I[2]

Given the difference between two roots and their sum, to find the roots.

Let B be the difference between two roots and let D be their sum. The roots are to be found.

Let the smaller root be A. The greater will then be $A + B$. So the sum of the roots is $2A + B$. But this has been given as D. Hence

$$2A + B = D$$

and, by transposition,

$$2A = D - B.$$

Having divided through by 2,

$$A = \tfrac{1}{2}D - \tfrac{1}{2}B.$$

Or let the greater root be E. The smaller will then be $E - B$. Therefore the sum of the roots is $2E - B$. But this has been given as D. Hence

$$2E - B = D$$

and, by transposition,

$$2E = D + B.$$

Dividing through by 2,

$$E = \tfrac{1}{2}D + \tfrac{1}{2}B.$$

Given, therefore, the difference between two roots and their sum, the roots can be found, for

Half the sum of the roots minus half their difference is equal to the smaller root, and [half the sum of the roots] plus [half their difference is equal] to the greater.

It is this that zetetics makes clear.
Let B be 40 and D 100. A is then 30 and E is 70.

Date: Viète (1593), *Zeteticorum Liri Quinque*
Source, left: Latin text from F. van Schooten's edition, p.42 (Leiden, 1646, reprint in Hofman, 1970); source, right: English translation in Witmer (1983), p. 83-84

figure 3.3: symbolic algebra

symbolic period

The development of algebraic notation in the 16th century was a process still instigated by problem solving (see also Radford, 1995, 1996a). In 1591 Viète introduced a system for denoting the unknown as well as given numbers by capital letters, resulting in a new number concept, the 'algebraic number concept' (Harper, 1987). The signs and symbols became separated from that what they represent (a context-bound number) and *symbolic* algebra became a mathematical object in its own right. For a solution to a typical Diophantine problem in the style of Viète, see figure 3.3. A few decades later Descartes proposed the use of lower case letters as we do nowadays: letters early in the alphabet for given numbers, and letters at the end of the alphabet for unknowns. With the creation of this new language system, earlier notions of the 'unknown' had to be adjusted. The first objective had always been to un-cover the value of the unknown but in the new symbolic algebra the unknown served a higher purpose, namely to express generality, since x could stand for an arbitrary number. Algebra as generalized arithmetic was a fact, and in its new role algebra detached itself from arithmetic.

simultaneous linear equations

The historical development of algebra shows that, no matter how revolutionary symbolic algebra was, it was not a necessary condition for the existence of equations. As a matter of fact, linear equations were very common in Egypt and the Babylonians knew how to solve linear, quadratic and specific cases of cubic equations. Babylonian problems in two unknowns concerning sum and difference were frequently solved using the rule (in modern notation): if $\frac{x+y}{2} = S$ and $\frac{x-y}{2} = D$ then $x = S + D$ and $y = S - D$ (Tropfke, 1980, p. 389). The Babylonians were also familiar with the subtraction (elimination) method. In order to solve the system of equations (in modern notation)

$$\begin{aligned} x + \tfrac{1}{4}y &= 7 \\ x + y &= 10 \end{aligned}$$

the first equation was multiplied by 4 and the second equation was then subtracted from the first, which gave $3x = 18$. Hence $x = 6$, and from the second equation it followed that $y = 4$.

general method in China

In a very different part of the world a systematic treatment of solving equations developed in ancient China. Just like the ancient civilizations, the Chinese lacked a notational system of writing problems down in terms of the unknowns, but the computational facilities of the rod numeral system enabled them to surpass the rest of the world in equation solving. The *Jiu zhang suanshu* or *Nine Chapters on the Mathematical Art* was a very influential text, composed around the beginning of the Han dynasty (206 BC to 220 AD). It is an anonymous collection of 246 problems on socio-economic life, including numerical answers and algorithmic rules. The book discusses the Rule of Three, which originated in the barter trade, the Rule of Double False Position and several other methods of solving linear equations (Lam Lay-Yong & Shen Kangshen, 1989). More significantly, it is the oldest book known until now that contains a method of solving any system of n simultaneous linear equations with n unknowns, with worked-out examples for $n = 2$, 3, 4 and 5. This was done using the method *fang cheng* (calculation by tabulation), writing the coefficients down or organizing them on the counting board in a tabular form and then performing column operations on it (much like the Gauss elimination method of a matrix). Figure 3.4 shows an example of such a problem and two representations in tabular form. The general application of the *fang cheng* method led quite naturally to negative numbers and some rules on how to deal with them, which is in great contrast with the late acceptance of negative numbers in other parts of the world.

Three bundles of high quality rice, two bundles of medium quality and one bundle of low quality rice yield 39 *dou*; two bundles of high quality, three bundles of medium and one bundle of low quality yield 34 *dou*; one bundle of high quality, two bundles of medium quality and three bundles of low quality rice make 26 *dou*. How many *dou* are there in a bundle of high quality, medium and low quality rice respectively?[34]

Using modern notation to express the information of this problem, a system of three linear equations in three unknows emerges;

$$3x + 2y + z = 39$$
$$2x + 3y + z = 34$$
$$x + 2y + 3z = 26$$

Numerical data is entered onto the board by use of rods working from right to left and from top to bottom. Resulting rod configurations for the set of equations are shown in Figure 3(a). Following rod algorithm, elementary column operations reduce a column to two entries: a variable's coefficient and an absolute term. See first column in Figure 3(b). In this reduced matrix form, a solution for one variable is obtained ($36z = 99$) and back substitution supplies the remaining required values. For the given problem, $z = 2\frac{3}{4}$ *dou*, $y = 4\frac{1}{4}$ *dou* and $x = 9\frac{1}{4}$ *dou*.

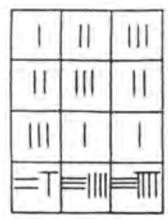


FIGURE 3a

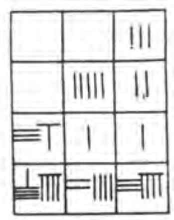


FIGURE 3b

Date: *Nine Chapters*, 200 BC - 200 AD
Source: Calinger (Ed.) (1996), p. 92-93

figure 3.4: *fang cheng* method

Diophantus

Diophantus certainly demonstrated a pursuit of generality of method, but his first concern was to find a (single) solution for each problem. His major work, the *Arithmetica* (ca. 250 AD), is a collection of about 150 specific numerical problems that exemplify a variety of techniques for problem solving. Diophantus distinguished different categories and systematically worked through all the possibilities, reducing each problem to a standard form. Negative solutions were not accepted, and if there was more than one solution, only one was stated. He solved linear equations in one unknown by expressing the unknown and the given numbers in terms of their sum, difference and proportion. If a problem contained several unknowns, he expressed all the unknowns in terms of only one of them, thereby dealing with successive instead of simultaneous conditions. Consider for example the type: “To split up a given number in two parts that have a given difference” (Tropfke, 1980). The Babylonians solved such a problem with unknowns using a standard rule involving two unknowns, as mentioned previously in this section. Diophantus, on the other hand, defined one unknown in the problem, represented it by a symbol and combined the two conditions into one single equation. For example, he assumed the smaller of the two numbers to be ς; the other number then had to be $\varsigma + d$ and the sum $2\varsigma + d$ had to be equal to n.

Diophantus is also known for his treatment of indeterminate equations: equations of the second degree and higher with an unlimited amount of rational solutions. Once again the general method involved reducing the problem to one unknown and finding a single solution. Often the method he used for determining this single solution was displayed in such a manner that the reader is able to repeat it and find an arbitrary number of other solutions.

Arabic standard equations

Much later the Arabs also played an important role in the historical development of equation solving. Although the boundaries of my research have been set at (systems of) linear equations, their achievements on quadratic and cubic equations deserve to be mentioned. A well-known book on Arabic algebra is al-Khwarizmi's *Hisab al-jabr w'al-muqabalah*, written around 825 AD. It contains a clear exposition of the solutions of six standard equations (in modern notation, where a, b and c are positive numbers): $bx = c$, $ax^2 = bx$, $ax^2 = c$, $ax^2 = bx + c$, $ax^2 + c = bx$ and $ax^2 + bx = c$, followed by a collection of problems to illustrate how all linear and quadratic equations could be reduced to these standard forms. Al-Khwarizmi also gave geometric proofs and rules for operations on expressions, including those for signed numbers, even though negative solutions were not accepted at that time. But as far as the difficulty of the problems and the notations are concerned, the book remained behind compared to the work of Diophantus; everything was written in words, even the numbers. The Arabs did not succeed at solving cubic equations algebraically, but in the 11^{th} century AD. Omar Khayyam presented a well-known yet incomplete treatise on solving cubic equations with geometric means.

Arabic algebra became known in the Western world in the twelfth century, when al-Khwarizmi's work was translated by Robert of Chester. By the fourteenth century, mathematical textbooks on arithmetic and algebra were very common in certain parts of Europe, and equation solving (even of the third and fourth degree) had become a regular subject in the Italian abbacus schools. In 1545 Cardano presented the solution of the general cubic equations by means of radicals. After the invention of symbolic algebra, equation solving developed very rapidly and soon found new applications in other areas of mathematics.

3.4 Implementing historical elements of algebra

integrating history in teaching

There are different ways of implementing history in educational design. First, it can be used as a designer guide. Milestones in the development of mathematics are indications of conceptual obstacles. We can learn from the ways in which these obstacles were conquered, sometimes by attempting to travel the same course but at other times by deliberately using a different approach. 'Reinvention' does not mean following this path blindly. On the contrary, it means that developers need to be selective and should attempt to set out a learning trajectory in which learning obstacles and smooth progress are in balance. History can set an example but also a non-example. And second, we can choose between a direct and an indirect approach, bringing history into the open or not. Learning material can be greatly enriched by integrating historical solution methods and pictures and fragments taken from original sources, but in some situations it may be more appropriate that only the teacher knows the historical information.

research objectives

Having decided to use history of mathematics as a source of inspiration and an educational tool for both the researcher and the students, it is one of the main objectives in this project to determine the role of history in the experimental lesson series:

- How does the historical development of algebra compare with the individual learning process of the student following the proposed learning program?
- Do historical problems and texts indeed help students to learn algebraic problem solving skills?
- How do students react to historical elements in their mathematics lessons?

3.4.1 Reinvention of early algebra

Prior to the start of the design process, we turned to the history of algebra for possible signs of appropriate and inappropriate activities. Rojano extracted some lessons from history, two of which agree with Realistic Mathematics Education theory, which we discuss in section 4.3.1 (Rojano, 1996). First, Rojano observes that since problem solving has always proven to bestow meaning on new knowledge in the past, we must be aware of the risk of teaching symbolic manipulation in advance of situations where symbolism is meaningful. And second, Rojano points out that we should not deny students' informal knowledge and methods but use them to give meaning to the new area of knowledge, in particular symbolic algebra. After all, when Viète introduced symbolic algebra – which formed a rupture with previous algebraic thinking – he used the skills and knowledge of classical Greek mathematics to achieve his aims (ibid.).

historical accesses to early algebra

In order to facilitate the 'reinvention of early algebra' in the classroom, we need to investigate where the historical development of algebra indicates accesses from arithmetic into algebra. First, word problems and informal algebraic methods form an obvious link between arithmetic and algebra. In its early days, algebra was considered to be 'advanced arithmetic'. For many centuries algebra was intended for fluent arithmeticians as a problem solving tool; only in recent centuries algebra evolved as an axiomatic study of relations and structures. Although algebra has made it much simpler to solve word problems in general, it is remarkable how well specific cases of such mathematical problems were dealt with before the invention of algebra, using arithmetical procedures. Some types of problems are even more easily solved without algebra! As the 'science of restoration and opposition', which is the literal translation of the title of al-Khwarizmi's algebra treatise *Hisab al-jabr w'al-muqabalah*, algebra was founded on arithmetical techniques like the Rule of Three and the Rule of False Position. Both these techniques provide opportunities for reasoning with unknown or variable quantities within an arithmetical context (see also section 5.3.3). Another possible access is based on the use of notation, for instance by comparing the historical progress in symbolization and schematization with that of modern students. And third, in order to anticipate possible objectives that students might have to the algebraic approach, educational developers and teachers could study old textbooks on early algebra in order to learn more about how algebra was understood and applied just after it became accepted.

3.4.2 Historical influences on the experimental learning strand

The historical development of algebra indicates certain courses of evolution that the individual learner can reinvent. Ideally, the student will acquire a new attitude towards problem solving by developing certain (pre-)algebraic abilities: a good understanding of the basic operations and their inverses, an open mind to what letters and symbols mean in different situations, and the ability to reason about (un)known quantities. In this section we discuss only a few general themes and topics selected for the program; a more detailed description of how certain historical problems and methods have been integrated in the experimental teaching units is elaborated in section 5.3.3.

word problems

Word or story-problems seem to constitute a suitable foundation for a learning strand on early algebra. This type of problem offers ample opportunity for mathematizing activities. Babylonian, Egyptian, Chinese and early Western algebra was primarily concerned with solving problems from every day life, although one also showed interest in mathematical riddles and recreational problems. Fair exchange, money, mathematical riddles and recreational puzzles have shown to be rich contexts for developing convenient solution methods and notation systems and are also appealing and meaningful for students.

rhetorical algebra

The early rhetorical phase of algebra finds itself in between arithmetic and algebra, so to speak: an algebraic way of thinking about unknowns combined with an arithmetic conception of numbers and operations. The natural preference and aptitude for solving word problems arithmetically form the basis for the first half of the learning strand, where students' own informal strategies will be adequately fit in. The transfer to a more algebraic approach is instigated by the guided development of algebraic notation – in particular the change from rhetorical to syncopated notation –as well as a more algebraic way of thinking. One of the study's aims is to establish whether or not the evolvement of intuitive notations used by the learner shows similarities with the historical development of algebraic notation.

barter equations

The barter context in particular appears to be a natural, suitable setting to develop (pre-)algebraic notations and tools such as a good understanding of the basic operations and their inverses, an open mind to what letters and symbols mean in different situations, and the ability to reason about (un)known quantities. The following Chinese barter problem from *Nine Chapters on the Mathematical Art* taken from Vredenduin (1991) has inspired us to use the context of barter as a natural and historically-founded starting-point for the teaching of linear equations:

> By selling 2 buffaloes and 5 wethers and buying 13 pigs, 1000 qian remains. One can buy 9 wethers by selling 3 buffaloes and 3 pigs. By selling 6 wethers and 8 pigs one can buy 5 buffaloes and is short of 600 qian. How much do a buffalo, a wether and a pig cost?

In modern notation we can write the following system:

$$2b + 5w = 13p + 1000 \qquad (1)$$

$$3b + 3p = 9w \qquad (2)$$

$$6w + 8p + 600 = 5b \qquad (3)$$

where the unknowns b, w and p stand for the price of a buffalo, a wether and a pig respectively. From the perspective of mathematical phenomenology, Streefland and Van Amerom posed a number of questions regarding the origin, meaning and purpose of (systems of) equations (1996, p. 140). The example given above is interesting especially when looking at the second equation, where no number of 'qian' is present. In this 'barter' equation the unknowns b, w and p can also represent the animals themselves, instead of their money value. The introduction of an isolated number in the equations (1) and (3) therefore changes not only the medium of the equation (from number of animals to money) but also the meaning of the unknowns (from object-related to quality-of-object-related).

Several historical texts have been integrated in the instructional materials to illustrate the inconvenience of syncopated notations and the value of modern symbolism. Other authentic sources are used to let students compare ancient solution methods like the Rule of False Position and the Rule of Three with contemporary techniques (see also the appendix).

3.4.3 Expectations

It is our belief that history of mathematics can play a positive role in mathematics education and educational design, as mentioned in section 3.2. The historical development of algebra indicates sufficient possibilities of integrating old problems and methods in a pre-algebra program of problem solving. Another aspect concerns student reactions to learning history in a mathematics lesson. In this regard we anticipate that most students will enjoy the change of scenery. The historical contexts are different from the common ones in contemporary mathematics text books and yet they show a clear relation with everyday life events. For instance, some problems involve old currencies, exchange rates for trading goods, or proportions of ingredients for baking bread. Tasks that involve acting out a story, such as 'the hermit' problem (see section 6.7.6), provide opportunities for a creative classroom activity like drama. We assume a number of students at secondary school level – having a higher intellectual capacity on average than the primary school pupils – to be interested in learning about the history of mathematics for its own sake: to see where mathematics comes from, how people learned mathematics centuries ago, and the types of problems people encountered then. They can experience mathematics as a human activity: mathematics produced, changed, and improved by man.

4 Research plan and methodology

4.1 Introduction

Combining our intentions of implementing history (see chapter 3) and bridging the gap between arithmetical and algebraic conceptions (chapter 2), we now rephrase the study's prime aim to fit these objectives:

> This study aims to determine in which way an informal approach towards algebra – starting from arithmetical abilities that students already have – containing historical elements helps to reduce the discrepancy between arithmetic and algebra. In particular we shall investigate which early algebra activities in the learning strand help students to proceed more naturally from the arithmetic they are familiar with to new algebraic territories, and how procedural and structural properties in both algebra and arithmetic can become integrated.

After presenting *what* we wish to know – in terms of main research questions and sub-questions – we turn to *how* we intend to attain this knowledge. This is done by describing the research method we have employed and the research plan according to which we have conducted the study.

4.2 Research questions

The research questions which have directed the decisive phase of the study are related to two main theoretical issues – the cognitive breach between arithmetic and algebra (see chapter 2) and the didactical value of history (see chapter 3). In the course of the study these questions have been altered, sharpened and extended as a result of intermediate findings and the learning process of the researcher. In particular the pilot experiment (see section 5.4) and the peer review (described in section 5.6) have aided in the formulation of sub-questions, having two important purposes. First, the sub-questions make the main research questions more concrete and operational, as if we have taken a magnifying glass to look more closely. Second, together with the peer review recommendations on mathematical content the sub-questions have determined the adjustments to the teaching materials as well as the spearheads for the field test analysis. The research results – including the answers to the research questions – are presented in chapter 6.

4.2.1 Main research questions and hypotheses

The two main research questions are:

1 When and how do students begin to overcome the discrepancy between arithmetic and algebra, and if they are hampered, what obstacles do they encounter and why?
2 What is the effect of integrating the history of algebra in the experimental learning strand on the teaching and learning of early algebra?

We have formulated two hypotheses to announce our expectations with regard to these questions.

Hypotheses

1 With regard to the first question, we hypothesize on account of an anticipated learning trajectory that students can succeed in overcoming part of the gap between arithmetic and algebra.
A few recent studies on the learning and teaching of early algebra – the *Mathematics in Context* project, the candy experiments conducted by Streefland – as well as the pilot experiment described in chapter 5 indicate that we can expect students to be able to reason about quantities algebraically without teacher interference. The discourse on characteristics of arithmetic and algebra in section 2.6 has resulted in an overview of pre-algebra activities which can help students breach some of the typical discrepancies between arithmetic and algebra. These activities have been linked to the most relevant cognitive obstacles associated with equations and problem solving, and analyzed for potential leaps of progression from arithmetical to (pre-)algebraic behavior. In section 6.2 we describe how these activities have been structured into a global learning trajectory which is intended to construct a bridge from students' arithmetical knowledge to informal algebra.

2 On the second question, we hypothesize that history of mathematics can have a positive effect on the learning and teaching of pre-algebra.
First, the historical problems selected for the learning strand are well suited for the arithmetical, informal approach to learning algebra. We anticipate also that studying historical developments contributes to the learning process on a more reflective level. For example, the historical problems and methods included in the learning strand illustrate that algebra has a long and complex history and that mathematicians struggled very hard at developing a suitable notation system (see section 5.3.3). Second, it is expected that the historical tasks will fascinate the students and thus provide a positive contribution to the math lesson. And third, the historical context provides opportunities for integrated learning, constructing links between mathematics and other subjects. In addition we hypothesize that integrating history successfully in the mathematics classroom depends significantly on the teacher. We expect the teachers involved in the classroom experiments to be enthusiastic about using history in their lessons, but we also suppose that with their lack of experience it will be difficult to exploit all the potentials.

4.2.2 Sub-questions

The sub-questions formulated below bear upon the most relevant issues addressed by the main research questions. We do not consider it compulsory to find exact answers to all these questions, because their main purpose is to make the main research questions accessible. The research data we have gathered – using the sub-questions as points for attention – should produce enough information collectively for answering the main research questions.

1 With respect to the discrepancy between arithmetic and algebra:
   - How do students conceive symbolic notations as a mathematical language, which type of shortened notations do children use naturally, and how do we obtain an acceptable compromise between intuitive, inconsistent symbolizations and formal algebraic notations?
   - To what extent and in what way can students become aware of different meanings of letters and symbols?
   - Is there a correlation between the form of notation students use (rhetoric, syncopated, symbolic) and their level of algebraic thinking; in particular, are there signs of progress from a procedural to a structural conception of algebra?
   - How can students actively take part in the process of fine-tuning notations and establishing (pre-)algebraic conventions?

2 With respect to the didactical value of history of mathematics:
   - What is the effect of integrating history in the mathematical classroom on the students, in particular their motivation and their learning process, and what is the possible influence of age, gender, intellectual level and the teacher?
   - How does the learner's symbolizing process compare with the historical development of algebraic notations?
   - Which parallels, if any, do we observe between the development of algebraic thinking amongst individuals and the epistemological theory?

4.3 Research method

The present study is conducted according to the method of developmental research and the educational theory of Realistic Mathematics Education. In this section the background will be sketched of both of these, using two representative publications: *Developmental Research: fostering a Dialectic Relation between Theory and Practice* (Gravemeijer, 2001) and *Hans Freudenthal: a mathematician on didactics and curriculum theory* (Gravemeijer & Terwel, 2000). The section ends with a brief elaboration on explorative research.

4.3.1 Curriculum development and Realistic Mathematics Education

educational development

Hans Freudenthal (1905-1990) was an outspoken critic of the customary model of educational research and educational theory 'research, development and diffusion' of his time. He referred to this method as being theory-driven and top-down, far removed from the classroom practice of education and teaching. According to Freudenthal, curriculum theory is not a fixed set of theories, contents and method, but a by-product of the practical enterprise of curriculum development. He proposed a new approach to curriculum development which he called 'educational development', in which curricula are seen as processes subject to developments and change. Educational development was intended by Freudenthal to involve more than the development of curriculum materials; it should encourage and integrate actual change in on-going classroom teaching. Driven by the method of developmental research (see section 4.3.2), it is an "... all-embracing innovation strategy, based, on the one hand, on an explicit educational philosophy and, on the other hand, incorporates developments in all sorts of educational materials as part of its strategy" (Gravemeijer & Terwel, 2000, p. 780).

RME

Freudenthal's views on curriculum development are connected with his ideas on mathematics education, which has become known as Realistic Mathematics Education (RME). For instance, he firmly believed that mathematics should be taught *in order to be useful*. If mathematics education is intended for the majority of students, its main objective should be developing a mathematical attitude towards problems in the learner's every-day life. This can be achieved when mathematics is taught as an activity, a *human* activity, instead of transmitting mathematics as a pre-determined system constructed by others. His phrase 'anti-didactic inversion' implies that if the starting-point for teaching is the result of an activity instead of the activity itself, the situation is upside down.

mathematizing

Freudenthal pleaded for 'mathematizing', meaning 'organizing a subject matter, either mathematical or taken from reality'. This image of mathematical activity has been taken as a paradigm for RME, where the emphasis should not be on the form of the activity but the activity itself and its effect. For example, mathematizing should not be seen as merely a translation into conventional symbolism, but as a process of organizing from which a way of symbolizing might emerge. More concretely, mathematizing can be understood as 'making more mathematical', involving typical characteristics of mathematics such as generality, exactness, certainty and brevity. One can think of the following kind of competencies (Gravemeijer & Terwel, 2000, p. 781):

- for generality: generalizing (looking for analogies, classifying, structuring);
- for exactness: modeling, symbolizing, defining (limiting interpretations and validity);
- for certainty: reflecting, justifying, proving (using a systematic approach, elaborating and testing conjectures, etc.);
- for brevity: symbolizing and schematizing (developing standard procedures and notations.

horizontal and vertical mathematization

Treffers (1987) extended Freudenthal's ideas on mathematizing by making a distinction between *horizontal* and *vertical* mathematization. Horizontal mathematization concerns the conversion from a contextual problem into a mathematical one, whereas vertical mathematization refers to the act of taking mathematical matter to a higher level. The latter can be induced by facilitating problem solving on different levels of mathematics. Freudenthal later (1991) described horizontal mathematizing as leading 'from the world of life to the world of symbols', where the world of life should be understood as 'what is experienced as reality', and vertical mathematizing comprises the adjustments, manipulations and reflections in the world of symbols. He emphasized that the distinction between the two kinds of mathematizing is not very clear and depends on what is understood by 'reality'. In his definition reality means what is experientially real from the actor's point of view, which grows and changes during the individual's learning process.

connected learning

Freudenthal (1991) is also known for his plea for 'connected' learning. Here we mean the way learning is organized. For instance, Freudenthal pointed out the advantages and importance of *prospective* learning, also known as 'anticipatory learning', where students solve problems in informal situations before they learn a systematic method. He considered it to be a natural and common sense way to prepare students for more formal mathematics. Its counterpart, *retrospective* learning – which means recalling and reviewing matter learned previously – strengthens the old roots and forms the foundation for the new matter. In other words, prospective and retrospective learning pursue the integration of past and future learning processes. From another perspective learning processes can become more connected by intertwining learning strands. According to Freudenthal it is neither sensible nor desirable to organize learning on separate tracks which are largely independent of each other. Instead he favored a long and strong mutual intertwinement of learning strands, perhaps even involving moments of prospective and retrospective learning for this purpose. This principle of connected learning was pursued by Streefland (1996ab), among others.

In summary, from Freudenthal's perspective mathematics must above all be seen as a human activity, a process which at the same time has to result in mathematics as its product. The question arises how a curriculum can be designed to achieve both goals. Freudenthal thought through a few design principles referred to as 'guided reinvention', 'levels in the learning process' and 'didactical phenomenology'.

guided reinvention

The reinvention principle (Freudenthal, 1973) – later renamed by Freudenthal as 'guided reinvention' – specifies that students should have the opportunity to experience the development of a mathematical matter similar to its original development. To this purpose a learning route needs to mapped out along which students can find the intended mathematics for themselves. The term 'guided' accentuates the acquisition of one's own knowledge – under the supervision of the teacher – instead of actually inventing it as such. The designer envisions in a so-called thought-experiment a possible trajectory by which a student may arrive at the solution. Possibly the history of mathematics can be used as heuristic device and a source of inspiration, for instance as indicator of intermediate steps and possible learning obstacles. On the one hand the thought-experiment involves anticipating student reactions, on the other it requires a suitable course of action in response to student reactions. The interaction between students (mutually) and the teacher is what effects the re-invention of mathematical matter.

levels in the learning process

The learning process of reinvention from an observer's point of view can be complemented with the learner's own perspective. The learner should experience the learning process as 'progressive mathematization' (Treffers, 1987). Informal solution strategies often anticipate more formal procedures; if the mathematical problems selected by the researcher allow for a range of strategies on different levels, it is not unlikely that the compilation of these will indicate a possible learning route. At the start a student mathematizes a given subject from reality, and then he or she should experience a moment of vertical mathematization, by analyzing and reflecting on one's own mathematical activity. As Freudenthal (1971, p. 417) put it: 'The activity on one level is subjected to analysis on the next, the operational matter on one level becomes subject matter on the next level'. This change in conception from operational to subject matter agrees with what Sfard (1991) describes as the development from an operational to a structural perspective (see also section 2.4). The level-theory proposed by Freudenthal has formed the basis for another principle of RME, namely the use of emergent models.

didactical phenomenology

According to the principle of didactical phenomenology (Freudenthal, 1983), the designer is required to study situations where a given mathematical topic is applied. Freudenthal emphasized the selection of 'phenomenologically rich' situations: situations that can be organized by the mathematical objects which the students are intended to construct. Moreover, the situations need to be assessed on to what extent they are suitable components in the process of progressive mathematizing. In other words, the phenomenological investigation is directed at finding situations which can be mathematized in a situation-specific way first, after which a process of generalizing and formalizing can form a basis for vertical mathematization. In section 5.2.1 we give an example of a mathematical-didactical analysis of (systems of) linear equations.

emergent models

Emergent models act as a fourth design heuristic in supporting the progression from informal to more formal mathematics (Streefland, 1985; Treffers, 1987; Gravemeijer 1994, 2001; Van den Heuvel-Panhuizen, 1995). These models are constructed by students themselves as referents to a given situation or activity. At first the models derive their meaning from situations which are familiar and real to the student, allowing for informal strategies which correspond to the context of the problem. After a while the character of the model changes, when through a process of generalizing and formalizing the student begins to focus on the strategies themselves. The model itself becomes the object of study, and now its meaning is found in the mathematical framework around it. After this switch the *model of* acting in a situation has become a *model for* mathematical reasoning.

RME: a dynamic reform movement

The Dutch reform movement now known as Realistic Mathematics Education and its underlying educational theory are presently still under development. New developmental research studies produce new impulses for theoretical ideas, and implementation in classroom practice has not yet been completed. In the last two decades different researchers have continued and refined Freudenthal's initial ideas on RME. It is impossible to pay due respect to everyone, so we confine ourselves to some of the most influential contributions.

RME in development

It was Streefland (1988) who carried out an extensive study on guided reinvention of fractions, while Treffers (1979) and Dekker, Ter Heege and Treffers (1982) have reported on self-discovery of multiplication and division algorithms. As we have already mentioned, Treffers (1987) extended Freudenthal's ideas on mathematizing by making a distinction between horizontal and vertical mathematization. Freudenthal's emphasis on the link to reality and focus on application has been extended beyond real life situations. The key issue is to choose contexts which can be organized mathematically and which stimulate students to imagine themselves being in the situation. This aspect of 'imagining themselves' (Van den Brink, 1973) has instigated the name Realistic Mathematics Education. Several research studies (Streefland, 1985; Treffers, 1987; Gravemeijer, 1994, 2001; Van den Heuvel-Panhuizen, 1995) show that students themselves indicate a path of progressive schematizing, which has resulted in the theoretical idea of emergent models described earlier in this section. Finally, we can say that the characteristics of RME with respect to views on mathematics and the learning and teaching mathematics have also determined developments of RME assessment. For further reading on this topic we refer to publications by De Lange (1987) and Van den Heuvel-Panhuizen (1996).

4.3.2 Developmental research

Developmental research is a combination of educational research and curriculum development, resulting in the formation of educational theory. This occurs locally at the level of the teaching experiment and at a more general level (per topic). The development of instructional activities is employed to elaborate, test and refine what Gravemeijer refers to as a general, *domain-specific* instruction theory. In a cyclic process of theory and practice, the researcher alternatively conducts thought experiments and teaching experiments. Reflections on the outcome of the teaching experiment results in the development of an educational theory, which is then fed back into a new thought experiment and a new teaching experiment (see figure 4.1, Gravemeijer, 2001). By doing so, the researcher aims to establish a well-considered *local* instruction theory founded on empirical grounds.

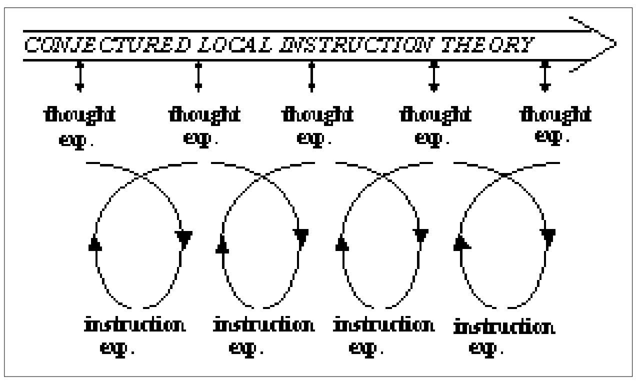


figure 4.1: reflexive relation between theory and experiments

To Freudenthal the aim of developmental research was to enable outsiders to retrace the researcher's learning process, also called 'trackability'. It is the learning process that justifies the instruction theory which is developed, in order for it to be accepted by a community of practitioners and researchers. This principle of 'trackability' is important, first of all, from a methodological point of view, but also for the teachers who wish to use the instructional sequence. They can profit from the researcher's experiences and considerations to decide which learning trajectory might be appropriate in their own teaching situation. Another criterion used for establishing the validity of developmental research is based on the observed range of solution strategies. If at a particular moment in the instruction experiment the researcher observes a cross section of the learning trajectory – lower ability learners who remain at an informal level, average students who are at the intended level, and 'prospective learning': fast learners who anticipate more formal methods to come – the instructional sequence is said to have a strong vertical component. Such a range in learning levels is taken as a concrete indication that the process of progressive mathematization can effect the envisioned learning trajectory (Treffers, 1992).

4.3.3 Explorative research

The present study can best be described as an *explorative research* project. The emphasis lies on the early experimental phase of developmental research, involving the design of new teaching materials and small scale instruction experiments. The cyclic process of practice and theory formation is expected to result in a very tentative local instruction theory based on noteworthy findings, and an experimental instructional sequence on pre-algebra. Our objective is to gain insight and look for 'trends' and indications regarding the research topic, which can direct future research. The study does not intend to include a comparative element where the results of the experimental group of students are compared with a control group. Nor does the duration of the project enable a longitudinal study of individual learning processes. It is therefore not expected that we will produce a fully tested learning strand ready to be implemented, nor a well-considered and empirically grounded instruction theory. The present theory and instructional design should be seen as intermediate products which need to be refined in the future.

4.4 Research plan

The research plan of the present study is founded on the developmental research method (see section 4.3.2). It consists of consecutive cycles of thinking through a hypothetical teaching-learning trajectory, designing instructional materials, testing these materials in classroom experiments, analyzing and reflecting on the learning trajectory observed in the experiment, adjusting the teaching materials and carrying out a new teaching experiment.

Table 4.1 shows an overview of these cycles, including a reference to the section where the various parts are described. Cycle 1* is not considered a full-fledged cycle because it did not involve adjusting the instructional materials. Since the research results are based on our findings in the three teaching experiments *case studies, pilot experiment* and *field test,* we have decided not to describe issues of method for the other small-scale try-outs. The process of adjusting and refining the instructional activities is intended to be conducted in the spirit of developmental research, where cycles of mental and instructional experiments result in the formation of a (tentative) local instructional theory (section 4.3.2).

Note that we distinguish between instructional *activities,* which result in *educational development*, and instructional *materials* which is the medium in which the activities are carried out (*curriculum* development). Reasons for adjusting the materials, activities or even the hypothetical learning trajectory might be an inappropriate order of activities or problems, a misjudgment of the schematizing skills that students have available, insufficient attunement to the learners' informal knowledge, or otherwise.

Due to the explorative character of the study the developmental process may be quite coarse, with fundamental adjustments to the instructional sequence and theoretical considerations. A more detailed account of the design process, including the researcher's reflections and theoretical considerations of consecutive teaching experiments, is given in chapter 5.

cycle	**theoretical foundation**	**school level**	**instructional materials**	**name teaching experiment + section**	**evaluation and reflection**
1	empirical studies	primary	activity sheets	*case studies* section 5.2.3	section 5.2.4
	empirical studies	secondary	teaching unit	*systems of equations* section 5.3.2	section 5.3.2
	history of algebra	secondary	teaching unit	*pre-algebraic strategies from the past* section 5.3.3	section 5.3.3
1*	evaluation *case studies*	primary	collection of open problems	*mathematical starting level* section 5.3.1	section 5.3.1
2	evaluation *case studies + mathematical starting level*	primary	teaching units	*pilot experiment* section 5.4	section 5.5
	evaluation *systems of equations*	secondary	teaching units	not reported	not reported
3	evaluation *pilot study* + peer review	primary	teaching units	*field test* section 6.5	section 6.6 + chapter 7

table 4.1: overview of research plan

Figure 4.2 illustrates the composition of the *field test* design cycle. Reading the diagram from left to right, the cycle starts with the two main theoretical issues which instigate the design of RME instructional units satisfying a number of learning targets, followed by a teaching experiment.

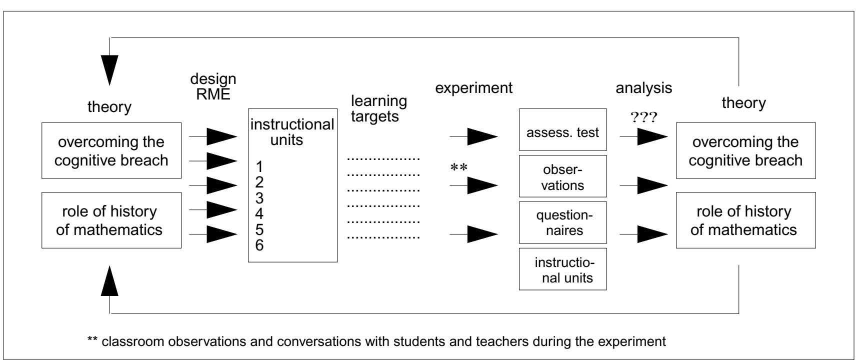


figure 4.2: diagram of iterative design process

During (indicated with **) and after the experiment the researcher analyzed the data: written tests, classroom observations and protocols, student questionnaires and students' written work in the instructional units. Evaluation and reflection led back to the theory.

4.4.1 Important changes of focus in the study

set-up of the study

Initially the intention of the study was to investigate the transfer from pre-algebra to algebra for two topics: solving equations and negative numbers. Promising results from a few small developmental research projects conducted in 1993-1995 by Streefland on solving equations (Streefland, 1995ab) and on negative numbers (Streefland, 1996a), as well as the *Mathematics in Context* instructional unit *Comparing Quantities* (*Mathematics in Context* Development Team, 1998), formed the starting-point for the study. The scientific product should be a local instruction theory brought forth by the iterative process of mental and instructional experiments. The practical output of the study was intended to be twofold: first, a provisional instructional sequence, and second, a systematic survey of recent literature on the didactical value of history of mathematics. However, due to an early re-evaluation of the project's main goal, the literature survey has been withdrawn. Moreover, early results in the study indicated that a progression from arithmetic to pre-algebra based largely on free productions is not self-evident (see chapter 5).

change of focus

Consequently the focus of the study shifted to pre-algebraic problem solving, in particular to how it might connect the worlds of arithmetic and algebra. As time passed, we realized that focusing only on the pre-algebraic approach to solving equations and the role of history in it would be a considerable study by itself, and so it was decided that the topic 'negative numbers' would be dropped. In other words, what started as a study on a pre-algebra learning strand based on free productions of equations and negative numbers, changed into a study on a pre-algebra strand for solving equations aimed at bridging the gap between arithmetic and algebra. In addition, we took more care of integrating history of mathematics in the final design cycle.

three streams

The learning process of the researcher constituted three separate components: becoming acquainted with the theoretical background (the learning and teaching of school algebra, RME and developmental research), gaining experience in educational design, and learning about the historical development of algebra. Right from the beginning the three streams were developed simultaneously as much as possible, so that each would attune to the others. Theoretical and historical issues not only influence the researcher's thought experiment and design heuristics but they can also complement each other to deepen the researcher's understanding, while the design process can in turn instigate a new perspective on the theory. As the study progressed, the historical component developed independently of the theory but continued to direct the design process. The theoretical stream and the design stream continued to influence each other mutually until the last design phase was completed. In the analysis phase of the study the three components were reunited to interpret student results.

4.4.2 Audience

The present study is expected to be of interest to educational researchers, curriculum developers, teachers and teacher trainers. School algebra and history of mathematics are currently (high) priority issues. International study groups like PME and ICMI have special algebra working groups, while the study groups HPM and HIMED have helped to promote the integration of history and mathematics education. Teachers and teacher trainers may be interested to use (a part of) the instructional sequence, or just learn about the teaching-learning processes observed in the study. In order to make the researcher's learning process and design choices accessible to the outsider, a thorough account of each cycle of thought experiment and instructional experiment is given in chapter 5.

4.4.3 Educational design

orientation

Prior to the first design activities of the study, a number of sources were studied for orientation purposes. The units *Patterns and Symbols, Dry and Wet Numbers, Expressions and Formulas, Comparing Quantities* and *Decision Making* (*Mathematics in Context* development team, 1998) constitute the algebra strand in the mathematics text books *Mathematics in Context,* a recently developed RME curriculum. These instructional units set an example as an informal introduction to algebra in grades 5 through 8. The unit *Comparing Quantities* in particular inspired us to elaborate an approach to equation solving in the context of barter and other forms of fair trade. The historical development of algebra, too, shows that barter is a natural and suitable context for developing algebraic reasoning and symbolizing (see also section 3.4.2). Some of the mathematical problems and solution methods selected for *Comparing Quantities* can be traced back to early Indian and Chinese algebra texts. For more reading on the *MiC* algebra instructional material we refer to Van Reeuwijk (1995, 1996).

Two important sources of inspiration for using history were several classroom experiences of integrating sixteenth century arithmetic problems in contemporary mathematics lessons (Kool 1993, 1994ab, 1995) and the *Mathematical Gazette* special (1992) on implementing history in the mathematics classroom. For more general information on the historical development of algebra several reputed secondary sources (Struik, 1987; Tropfke, 1980) were consulted. Finally, ideas were drawn from the work of Bednarz and Janvier (1996, on different types of algebraic problems and strategies), Streefland (1995abc, on constructing and solving systems of equations) and Harper (1987, on parallels between the historical development of algebra and contemporary algebra learning of solving linear equations). Streefland's experiments showed that the children's activities led to algebraic abilities like constructing and transforming symbolic equations, substituting numerical values, recognizing equivalent equations, grouping like terms in an equation and solving simple equations (ibid., 1995abc).

early ideas

We remark here that in the present study our objective of integrating history cannot be combined with the use of computer-supported interventions. Right from the start we have given first priority to pencil and paper work, and of course the instructional materials should be designed in accordance with the principles and theories of RME (see section 4.3.1): opportunities for mathematizing (algebraizing, in this case), emergent models to support algebraic reasoning, and guided reinvention of algebraic notations and methods. Some key issues in the *Mathematics in Context* unit *Comparing Quantities* (*Mathematics in Context* Development Team, 1998) were adopted for the study 'Reinvention of algebra': making quantities comparable, mathematizing trade situations, flexible switching between different forms of representation (description, pictures, symbols, tabular form) and solving (embedded) system of equations by combining and exchanging iconic or symbolic equations (see also section 5.2).

In section 5.2.1 we describe how the mathematical analysis of the topic of equations raised questions about the meaning, purpose and origin of one or more linear equations (Streefland & Van Amerom, 1996). An example of an embedded system of equations is given below to illustrate the design heuristic of progressive mathematization as it has been applied in the *MiC* unit *Comparing Quantities*.

progressive mathematization of a system of equations

The following problem constitutes a system of two equations with two unknowns:

> 3 soft drinks and 4 pieces of pie together cost 25 guilders,
> 4 soft drinks and 3 pieces of pie together cost 24 guilders.
> How much does one soft drink cost, and how much does one piece of pie cost?

This type of problem can be solved at different levels, creating opportunities for progressive mathematization. The current system of equations can result from a preliminary activity of generating expressions in situations where the prices are known at first, like in Streefland's candy experiment (1995) mentioned in section 5.2. From here the switch to unknown prices is quite natural. The student may organize the problem situation using representations like pictures, abbreviations, symbols or letters. The strategies can include trial-and-adjustment, systematic exchange (exchanging 1 soft drink for 1 piece of pie brings down the price by 1 guilder), operating on the equations and substitution. The process of progressive mathematization concerns both the use of notation and strategy. Such a gradual build-up of strategy levels and notation can result in the end in formal, algebraic representations and standard procedures for solving them. Another important factor of formalization concerns the different roles of letters: letters as *unknowns*, where students can reason in terms of objects – soft drinks and pieces of pie – and turn to prices only at the end, or letters as *variables*, standing for variable numbers. In both cases context problems are eventually replaced by context-free situations, in order to generalize situation-specific strategies and symbolism to a more formal level (see also Van Reeuwijk, 1995, 1996).

4.4.4 Teaching experiments

For the teaching experiments *case studies, pilot experiment* and *field test* we first give a global description of the research group, the researcher's position and the instruments used to collect data. A more precise description of the groups of students participating in the respective experiments is given integrally with the results in chapter 5 and chapter 6.

experimental group

The instructional sequence is intended for learners of grade 6 (primary school) and grade 7 (secondary school), roughly in the age of 11 to 14. Since the lesson sequence is written from an RME point of view, the participants – teachers and students alike – were required to satisfy certain conditions. First, we made sure to select a research group that worked with an RME curriculum. Both students and teachers should be familiar with the pedagogical principles of this approach to education, having an appropriate attitude towards (learning) mathematics. Here we may think of investigative students who are not reluctant to explain their thinking, and a teacher who can stimulate and guide informal strategies to a higher level. Second, the emphasis on free productions and envisioned variety of solution strategies asked for a wide distribution of capacities within the group. Third, the teacher preferably takes an interest in the historical background of mathematics, sending out a positive signal to the students in this respect.

researcher's position

different roles

In the first experiment we worked with several pairs of students individually, outside the classroom. Due to the explorative character of this try-out we specifically chose to give the instructions ourselves instead of the regular teacher. In such a situation the researchers take on different roles. In the role as teacher we gave instructions and had conversations with the learners. As designer, the researchers are responsible for writing and adjusting the instructional materials. In the role as researcher we observed and analysed the learners' activities and their learning process. In this way the researchers were actively engaged in the teaching practice and influenced the course of the experiment quite directly, sometimes at very short notice. In the pilot experiment the researcher only occasionally played an active role in the classroom, in order to allow the instructional sequence to take place almost naturally. The field test was deliberately kept as objective as possible; the researcher did not participate in the lessons but merely observed the classroom interaction going on.

instruments

Developmental research is primarily qualitative in nature, and relies largely on the observation and registration of mathematical activity by the observer, for example in protocols. The prime source of data for the project was the written work done by the students (activity sheets, instructional units, tests). The tests and – to some extent – student work in the instructional units enabled us to assess the learning effect of the lesson series for *individual* students. We emphasize that not only failure or success but also the strategies students used indicate their level of understanding and competence. In fact, we are more interested in correct *strategies* than correct *answers*. Classroom observations informed the researcher on learning processes in the group, the role of the teacher and special circumstances and events that matter for the analysis. The lessons were recorded on tape or video to reduce the risk of (unintended) subjectivity on behalf of the observer. These recordings enabled us to analyze significant parts of the protocols in more detail, for example where students came to new insights. Usually a global account of what was said was sufficient to reconstruct a student's train of thought or a group discussion, especially with the students' written work at hand. In each of the three teaching experiments we collected written observations and student work, but there were also a few differences between the three.

case studies

In the first try-out, alternating pairs of students completed a series of activity sheets in a two-on-two situation. The researcher and supervisor alternately took the role of interviewer-teacher and observer. One asked the students questions, gave instructions and guided the interactive learning process, the other wrote down what happened as precisely and objectively as possible. Usually the conversations were recorded on tape, and sometimes on video. The students did not take a test at the end of the sequence.

pilot experiment

The pilot experiment was carried out in two mixed classes grade 5 and 6, each with a full-time teacher. The data were collected through the observation of lessons, participation of classroom discussions, the analysis of video recordings and the evaluation of two written tests. For a number of reasons the students' written work in the instructional units was not included in the analysis. Students often worked in groups, and some students corrected their answers after class discussions, which made it difficult to determine what a student really did on his own. We also found that interesting personal notations and strategies were often used for classroom discussions and reflection anyway. Due to unforeseen circumstances it was not possible to test the second half of the program in a classroom situation. Instead, three students – one of relatively high, moderate and low ability – were selected from each class in order to test the second half of the learning strand. During these work sessions, which were all recorded on video, the researcher took the role of teacher/individual tutor.

field test
The field test was conducted in four primary school classes (grade 6) and two secondary school classes (grade 7). Three classes completed the entire program, while two schools tried out only the first half of the program. These schools dropped out of the experiment, but the data have been included in the analysis. There is one class which has been left out of the analysis altogether (see bottom row in table 4.2). As the experiment progressed, it became clear that this school did not satisfy the RME conditions for the teaching experiment.
Audio-visual classroom observations, protocols and short conversations with the students were used to supplement the analysis of the written work – tests and instructional units – which was taken as the prime source of information. At the end of the experiment the students were asked to fill out a questionnaire. Their answers have informed us on their attitudes and opinions on school mathematics and the experiment in particular.

school	# boys	# girls
A	12	11
B	4	14
C	12	20
D	16	14
E	14	16
*	10	12

table 4.2: number of students in target group (* data not included)

4.4.5 Analysis of student work

In this section we describe how the analysis of student work was carried out. Due to the gradual decrease of classroom participation on the part of the researcher and a notable difference in method, we have decided to distinguish between the final teaching experiment and the two main teaching experiments which proceeded it.

case studies

In the first teaching experiment – the 'case studies' – the analysis was conducted in an informal, unstructured way. We did not use a categorization of strategies because we could not foresee the strategies students might use and also because we wanted to keep an open mind. Moreover, the target group changed all the time which means we did not have enough data to validate a systematic, quantitative analysis. Instead, we took a qualitative approach to look for trends and gather ideas for the next design cycle. We also used our observations to prepare the next session with the students.

pilot experiment

For the pilot experiment our analysis was guided by our experiences from the case studies results. The researcher's observations of the lessons were documented in reports where the emphasis was placed on classroom discussions, reflections and interesting individual contributions by students. Comments on individual written work were included when relevant. As explained before, the instructional units were not studied systematically, because the students' solutions were influenced too much by group work and classroom discussions. Individual written tests for the first part of the experiment were categorized according to use of strategy and symbolizing. A first round of analysis enabled us to draw up a categorization of strategies and notations, after which the data were recorded. These data were then sorted according to score, strategy level, explanation of the answer, gender, age and peculiarities in order to find trends. For more details we refer to section 5.4.1 in chapter 5. The assessment of the second part of the experiment involved only 6 students, so we integrated these results with the classroom observations on the various aspects of the learning strand.

field test

Most lessons in the final teaching experiment were observed and recorded on audio and/or video tape by the researcher and two assistants. The observation reports focus on various points of attention which were identified for each lesson in advance: symbolizing, advanced reasoning, reflections, sudden moments of insight, expected obstacles, matters of attitude, reactions to historical elements etc. This categorization has facilitated the sorting and listing of classroom activities for each of these points of action, enabling us to distill trends which cannot be detected from written work alone.

first impression of results

In the remainder of this section we explain how our method of analyzing the students' written work in the field test evolved. Theoretical reflection prior to the field test had turned our attention to strategy use and symbolizing at three levels: arithmetic, pre-algebra and algebra. The results of the first primary school test showed signs of a few trends and peculiarities with respect to reasoning strategies, symbolizing and frequent errors. At first glance students' levels of reasoning and symbolizing seemed to be independent of each other, and symbolizing and schematizing seemed hardly effective as a problem solving tool. Frequently observed errors appear to be related to differences between an arithmetical and an algebraic outlook on the test problems.

developing a method

Since the primary level test task *Number Cards*, which is discussed in detail in section 4.5, revealed the largest variety of problem solving strategies and symbolizations, it was chosen as the starting-point of a second, more thorough analysis. We compared our findings with classroom work on similar problems in the unit *Exchange*, which led to the formulation of three conjectures. Instead of repeating such an unstructured, open-minded analysis for each test task separately, we opted for a more pragmatic and coherent approach. Eight more test tasks (at primary and secondary school level) and isomorphic tasks in the student units were selected to put our conjectures to the test and reformulate them when necessary. By including corresponding problems in the student units we are able to evaluate to a certain extent the individual student's development, comparing his or her understanding during classroom work with the test results. We decided it was not necessary for each task to be relevant for all three conjectures, as long as there was sufficient argumentation for the final conclusions.

analysis and theory formation

Figure 4.3 illustrates the interactive and developmental nature of this method, where analysis of student work leads to local theory formation (conjectures), which in turn is followed by a new analysis and theoretical reflection. In each cycle the conjectures acquire more substance and take on a more definite character. This iterative method demonstrates the same dynamic, evolving character of developmental research itself.

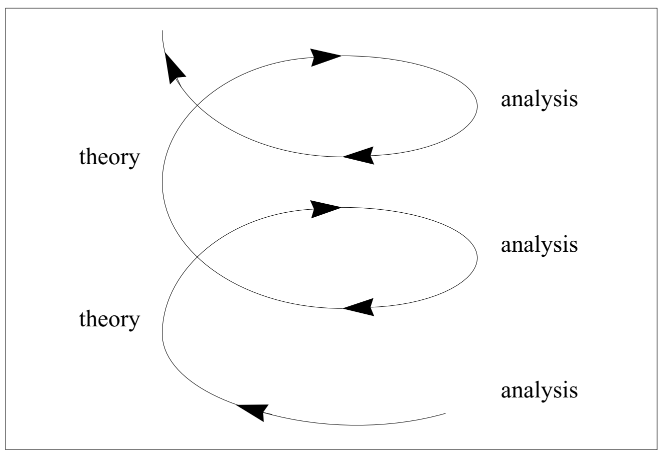


figure 4.3: iterative method of analysis and reflection

model for analysis

Whilst trying to get a better grip on what we mean by a pre-algebraic solution – how it can be distinguished from arithmetic and from algebra – we developed a simplistic but purposeful model:

algebraic reasoning + (pre-)algebraic symbolizing = (early) algebra
algebraic reasoning + arithmetical notations = pre-algebra
arithmetical reasoning + (pre-)algebraic notations = pre-algebra
arithmetical reasoning + arithmetical notations = arithmetic

'Reasoning' here refers to students' mental processes, either for specific problem situations and using only known, fixed quantities (arithmetical reasoning) or on a more formal (general) level using unknown or variable quantities (algebraic reasoning). By 'notations/symbolizing' we mean the written notes *during the solution process*, instead of the answer itself or the initial problem representation. Using this model, we can say that some students have reached the highest of the four levels in the secondary school instructional unit *Fancy Fair.* In the unit *Time Travelers* students encounter linear problems with symbolic representations, but we have found no evidence that students have succeeded in combining algebraic reasoning *and* symbolizing.

4.4.6 Peer review

Half-way during the study we consulted a panel of experts from the field to evaluate the results of the study up to that point. This peer review shows certain similarities with the qualitative research technique known as the Delphi method, which belongs to the category of 'interactive survey structuring', i.e. research as public dialogue (Bastiaensen & Robbroeckx, 1994). It is a method of structuring a group communication process by consulting a group of individual experts on a research question or a complex issue. The reason for using this procedure can be that the researcher or the group wants to explore solutions, to obtain advice or legitimation for the next research phase, to design and implement a certain instrument, or to bring out different points of view. Two distinct features are the exchange of knowledge between the researcher(s) and a diverse panel of frank experts, and that this exchange is an iterative process. The peer review deviates from the Delphi method since it comprised just one round of communication, but nevertheless a certain group consensus was reached on how the study should be pursued. A discussion of the outcome of the peer review session is given in section 5.6 in chapter 5.

4.5 Analysis of test tasks: a paradigmatic example

This section describes the classification of arithmetical, pre-algebraic and algebraic solution strategies for the primary level test task *Number Cards*. The purpose of this detailed description is to give the reader the opportunity to look through the eyes and mind of the researcher. In addition we describe a hypothetical learning path from one strategy to the next which the student might follow. Concise expositions of observations and interpretations like these are meant to enhance the study's credibility, as mentioned in section 4.2. The actual results of the task, the formation of a tentative local theory and the rest of the results can be found in chapter 6.

4.5.1 Test task *Number Cards*

Mathematically the *Number Cards* task (see figure 4.4) is 'a visual system of equations in two unknowns', and it belongs to a class of problems – restriction problems – that play a key role in both primary school units. The task is similar to the number riddles in the unit *Exchange,* section 3, but in a new kind of representation. The number riddles are based on a typical problem in Diophantus' *Arithmetica* (ca. 250 AD) on the sum of two numbers and the difference between them, and they are stated verbally (see also section 5.3.3). Students are meant to recognize *Number Cards* as a restriction problem with two conditions, which have been studied in the lessons in different situations and variations in order to develop abilities of systematizing. Or, if the problem recognition is complete, the student can consider the test task as a number riddle on sum and difference like the ones in the unit *Exchange*. In other words, the task can be understood and solved at different levels.

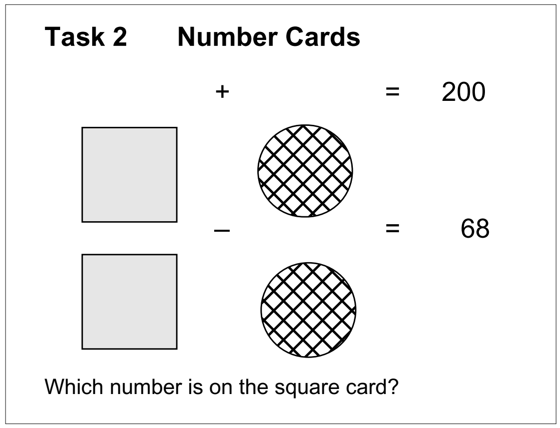


figure 4.4: *Number Cards* task

4.5.2 Solution strategies for *Number Cards*

We can distinguish 6 types of correct solution strategies in the students' work, of which we characterize two as algebraic, two as pre-algebraic and two as arithmetical (see table 4.3). Solutions which could not be classified as any of the above, have been categorized as 'answer only' (no notes, only a correct answer), and 'incorrect' (incorrect or unclear strategy with an incorrect answer, or with no answer). Students who have not made any attempt at all, are labelled 'none'. The strategies are described and sometimes illustrated below, starting with the lowest arithmetical level and ending with the highest algebraic level.

strategy level	algebraic	pre-algebraic	arithmetical
1	elimination of one unknown		
2	algorithm of halving the difference		
3		adjusting the difference symmetrically	
4		reason-and-trial	
5			trial-and-adjustment
6			trial-and-error

table 4.3: list of strategies for *Number Cards*, test 1 primary level

arithmetical strategies

trial-and-error

A student using this strategy tries to find the two unknown numbers by performing calculations at random, without reflecting on the error. The next trial is equally as random as the previous one, and the error in the solution does not necessarily become smaller in each step. The calculations do indicate understanding of the problem: the sum of the numbers is always 200 and the difference between them is 68. However, the student does not rise above the level of specific cases of calculations.

trial-and-adjustment

A superior level of solving by trying is shown when a student does consider the error before trying again, for instance by making the error smaller. For instance, a student tries the numbers 90 and 110 and realizes the difference is too small. The next attempt is directed at obtaining a larger difference: a number lower than 90 and the other number higher than 110. In some cases consecutive attempts will converge 'directly' to the correct answer, the difference becoming larger in each step but never too large. In other cases the convergence may be indirect, so to speak, when the difference between the numbers is alternatively too large or too small. The adjustments are of a qualitative kind; there is no sign of how much the correction should be, only 'higher' or 'lower'.

pre- algebraic strategies

reason and trial

To this category we allocate strategies consisting of reasoning followed by numerical attempt or vice versa. For example, a student may use the trial-and-adjustment approach for a while and then in the final stage discover that the error in the difference found (for example, if the attempt gives a difference of 60 instead of 68, the error is 8) must be divided in half and then be distributed to find the solution. If we recognize this process of reasoning early on in the solution process, the strategy is considered to be of the kind 'adjusting the difference symmetrically'.
One student in the field test reasoned very differently, namely about the structure of the numbers involved (see figure 4.5). The last digits of the numbers 200 and 68 also determine the last digits of the two numbers which form the solution. It is quite hard to try and reconstruct the student's thought process, but let us start with the most natural way to read the draft notes: in vertical columns from left to right. Doing so, the student appears to have written down the two problem conditions first . Next she reasons that the final digits must be 4 and 6 – quite an abstraction! – and tries a few possible combinations in the table underneath. However, in this way the remaining column of calculations, in which she investigates how to obtain a difference ending with 8, is not relevant in the solution process and actually seems a step backwards; indeed, she has already decided that 4 and 6 are the correct digits. It is also puzzling why she does not consider uneven numbers at all, which might have generated 9 and

1 as another valid combination. A more probable order of events, therefore, is the following. The student copies down the two conditions, leaving the right hand side for draft calculations. She first considers the condition of 68 difference, generating a column of possible differences ending with 8, in an upward direction starting with 10 – 2.

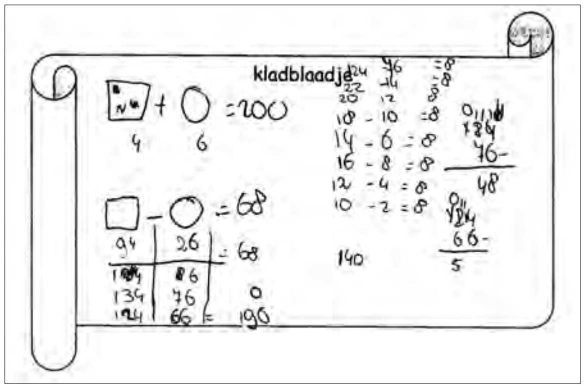


figure 4.5: reason and trial

However, how can we explain the combination of 124 and 76 at the top? It is a rather large step after 22 – 14. It may even be possible that it said 24 – 16 first, since the 7 is slightly untidy. On the other hand, if we link the calculation of 124 – 76 to the table of values in the bottom left corner, as a follow-up to the attempt 124 – 66, then the list of differences must have been constructed in a downward direction. This makes 10 – 2 a rather simple last attempt. In other words, the left-hand side and the right hand side of the draft notes are hard to relate.

Nevertheless, in spite of the inaccuracy in the vertical calculation, 124 and 76 is the best attempt so far: a combination that satisfies the first condition and that also gives 8 as the last digit for the difference. It appears that by now she has realized that the sum has to end with a 0, giving the combination 4 and 6 as the only correct one (below the square and the circle in the left-hand corner). She then tries a few number combinations with final digits 4 and 6, starting with 94 and 26 (in the table). The difference is 68, but the sum is not right. Her next attempt, 134 and 76, with a calculation error, because the difference is not 68 but 58, gives a sum of 210, which is too much. Note how she applies shortcuts to her calculations, bothering only with the minimum. She then lowers the values to 124 and 66, which means she brings down the sum by 20 instead of 10. Perhaps she only interpreted the error qualitatively, not quantitatively, or perhaps she unthinkingly adjusted it twice. Nonetheless, this attempt is not correct either, as she realizes half-way through the column calculations. Her strategy brings her very close to the solution. The remaining calculations are not

written down and are probably done mentally, because her answer is correct. There are two uncertain factors in this reconstruction. Why did she not start closer to 124 and 76? And when did she decide to reason about the last digits, before or after the attempt 124 and 76? These questions cannot be answered.
This strategy is classified as algebraic on account of the reasoning involved: the student displays an abstract notion of number – with understanding of number properties – a notation for the unknown number (an empty space with two dots) and reasoning about an unknown number. In other words, the strategy illustrates the capacity to think algebraically. However, the method of trial-and-adjustment in the second part of the solution process lacks the generality of method found in the other algebraic strategies.

adjusting the difference symmetrically
Contrary to the qualitative character of the trial-and-adjustment strategy, this strategy is based on handling the difference between the unknown numbers quantitatively. The superiority of the approach lies in the series of deliberate steps to correct the difference symmetrically, i.e. the increase and decrease are the same. The strategy takes on a more general character when the student determines the starting value as half the sum, which works for every problem of this type. In the test we see only numerical representations amongst the correct answers (with calculations or in a table), but in the unit *Exchange* students also use a visual representation such as the number line, as shown in figure 4.6. The student on the left first determines half the sum, 100, which is the first attempt for the two unknowns, and then uses a direct approach of increasing the difference from 0 to 68. He tries 132 and 68 as the second attempt, enlarging the difference from 0 to 64, and then proceeds in the right hand column. He has seen that the difference is 4 short, which he divides in half (132 – 2 and 68 + 2), but when he checks the conditions he discovers that it should be the other way around (132 + 2 and 68 – 2). Not only does the student check his solution (the column calculations at the bottom) but he also demonstrates the correctness on the left again by substituting the numbers for the icons in the horizontal expressions.

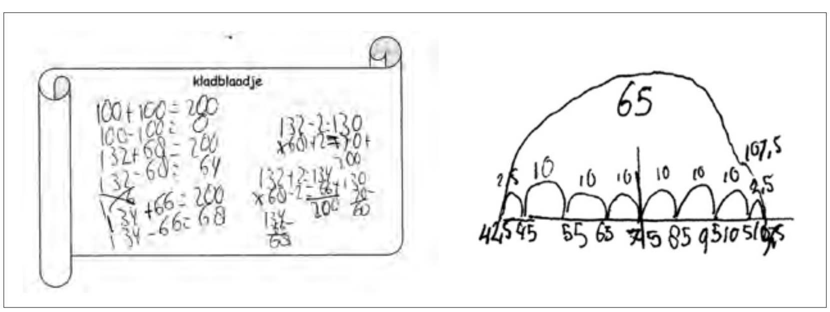


figure 4.6: adjusting the difference symmetrically

The other example in figure 4.6 is the solution to one of the tasks in *Exchange*; it illustrates a more visual approach to the strategy of adjusting the difference. This student has first tried to find the solution by way of trial-and-error, as she has done in all the preceding tasks; there is reason to suspect that the teacher may have given her some help. The jumps on the number line are very small and indicate that the student is not yet confident with this method. Since there are no similar tasks in the unit after this one, we cannot tell whether this student might have experienced some progress, but she did not succeed at solving the *Number Cards* task.
The strategy 'adjusting the difference symmetrically' seems not to belong indisputably to arithmetic, nor to algebra. The solution method involves the arithmetical handling of two given numbers – the sum and the difference – but it also requires thinking about unknown numbers. We can say that this strategy finds itself on the verge of algebra. The number of steps are not strictly defined; a student can take as many as needed. But we can say that fewer, larger steps indicate a higher level of algebraic thinking than many small ones. It is in fact reasonable to expect a student who has solved various problems of this kind to discover the general algorithm discussed next.

algebraic strategies

algorithm of halving the difference

The most advanced performance of adjusting the difference from the middle – in just one step – is an algorithm applicable to any problem of this type. The general character of this strategy, on top of the presence of unknowns, legitimizes us to call it algebraic. The outward representation of algebraic thinking can be entirely arithmetical; symbolic notation is not a prerequisite for algebraic thinking. This strategy has been characterized by Harper as rhetorical, corresponding with the pre-Diophantine period of algebraic history (Harper, 1987).

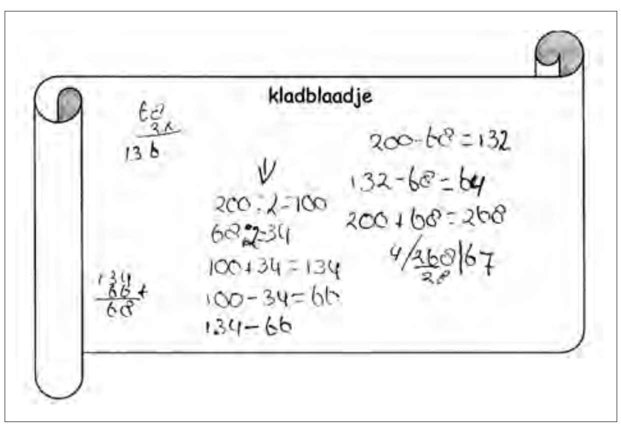


figure 4.7: algorithmic solution method

Figure 4.7 shows how a student begins with a trial-and-adjust approach and then switches to the algorithm of halving the difference. It is plausible that she suddenly recognizes the isomorphism of the task.

reasoning with unknowns

One student – with no more than an average score on the test – used a strategy in which he eliminates one of the unknowns. His calculations are so neat and efficient (see figure 4.8) that we can assume he knows and understands what he is doing and that it is not a coincidence. It is not possible to explain how he discovered this method; the classroom observations and his solutions to similar problems in the instructional unit *Exchange* do not disclose any progressive formalization of method. In fact, we do not even see an algebraic solution strategy in his classroom responses; he has answered the most complex problems with no calculations at all. In addition there is a small chance that he was shown this strategy by another person: teacher, parent or sibling. Since we have no sign of external influence, we must base our conclusions on the pen and paper work present.

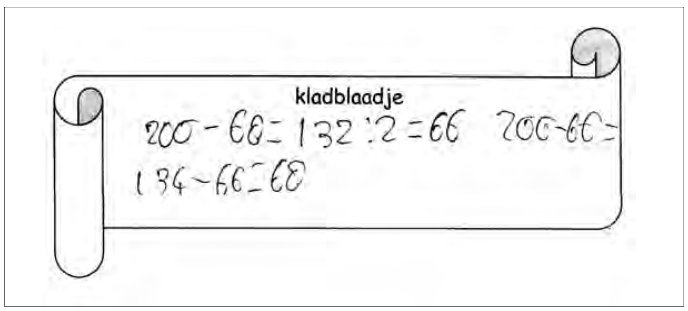


figure 4.8: elimination of one unknown

We can compare his calculations to the method of elimination in a system of equations. The subtraction 200 – 68 corresponds with subtracting the second equation from the first, which gives him 132 for twice the value of a circle. He then divides 132 to get 66, the value of the circle, and subtracting 66 from 200 gives him the value of the square. In writing down the answer he makes a small mistake, but his verification on the bottom line shows his understanding of the problem. We cannot conclude that he substitutes the value 66 back into the first equation, to get 'square + 66 = 200', because there is no physical proof; we can only see that he subtracts 66 from 200 to find the value of the square.

The limitations of arithmetical notations cause the student to operate only on the numbers, all the while keeping track mentally of the unknown he is dealing with. Such handling of the unknown shows strong similarities with the way secondary school students worked on the unit *Time Travelers* (see also section 6.7.7) as well as the solution in figure 5.17 which is explained in section 5.3.2. This solution strategy reflects a breach between arithmetic and algebra: algebraic thinking but arithmetical notations. Apparently the student sees no reason to draw the unknowns as part of his calculations; either he does not need them for support or clarity, or he feels unsure about it, or perhaps it has not even crossed his mind. If the system is solved formally, subtracting the second equation from the first involves manipulating unknowns on the left-hand side of the equality-sign: square minus square leaves nothing, and circle minus negative circle equals two circles. We assume this formal approach to eliminating one unknown is beyond a primary school student's understanding. The target group has not yet encountered the properties of negative numbers, and the notion that subtracting a deficit is equivalent to adding is rather abstract. This means that if we rule out the possibility of consciously subtracting the circles, we cannot explain how this student decides to divide by two.

historical perspective

It is also interesting to compare the student's strategy to an old Babylonian method mentioned already in chapter 3. The Babylonian method states (in modern notation): if $\frac{x+y}{2} = S$ and $\frac{x-y}{2} = D$ then $x = S+D$ and $y = S-D$. In other words, if half the sum is S and half the difference is D (where D must be positive), then the larger number is equal to $S+D$ and the smaller number is equal to $S-D$. In the case of the *Number Cards* task, the 'square' is x and the 'circle' is y, which makes the calculations $\frac{x-y}{2} = D$ and $y = S-D$ equivalent to the student's method for finding the value of the circle. However, this is where the similarity ends, for the student does not determine the value of half the sum.

other strategies

answer only

Students who write down the correct solution without any calculations or explanation can do so for a number of reasons. A student can find it trivial to give an explanation, even though elaboration is specifically asked for. Perhaps the student deliberately objects to writing down the calculations; draft notes are sometimes considered a sign of weakness. It might also be a matter of laziness or lack of interest. And lastly, classroom observations have proven that some students have great difficulty in explaining their thinking and might therefore involuntarily opt for writing just the answer. In spite of the inappropriateness of giving an answer without explanation, and the lack of proof, we feel that these students must have a better mathematical understanding of the task than those using an incorrect strategy. Hence we consider this category to be of a higher level than incorrect strategies.

incorrect strategies

common errors

This category includes solution methods which reflect a poor understanding of the nature of the task. A small number of students have no clue at all and probably write down an answer because they are expected to, but most students referred to here make a serious attempt. For example, a very common thinking error is to link up the relations stated in the problem with the given values 200 and 68, like 200 + 68 (‘sum of the numbers’) or 200 – 68 (‘difference between the numbers’). Another type of error is related to a misunderstanding of the principle of simultaneous conditions, as shown in figure 4.9. Looking at the answer on the lines below the draft work, the left ‘system of equations’ shows two values for the circular card, namely 100 for the first ‘equation’, 32 for the second, and just to the right we see the correct combination of values. She also tried 200 – 68 and 200 + 68, the errors discussed before. So although this student solved the problem correctly – with the algorithmic strategy even! – she obviously was not able to determine which answer is correct.

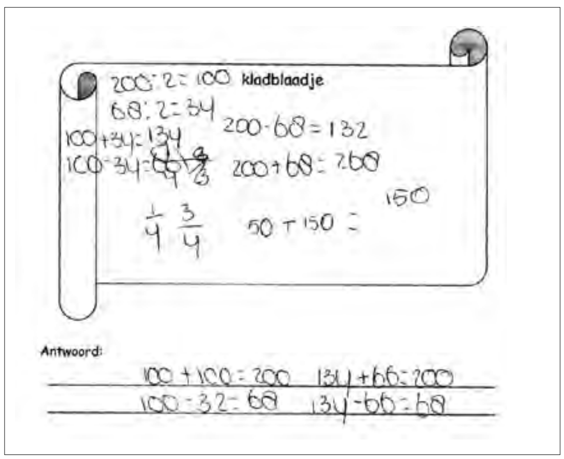


figure 4.9: incorrect strategy

none

Students who write down nothing at all, who appear not to have made any attempt at dealing with the problem, form the final group. Although we are inclined to say that these students therefore do not understand what the task means, there is also the possibility of extreme lack of interest or deliberate sabotage, in particular in the case of school *C* where a combination of circumstances eventually resulted in a premature termination of the experiment.

4.5.3 Progressive formalization

The overview of strategies for the *Number Cards* task reflects a gradual progression in mathematical thinking, starting at an arithmetical level and ending at an algebraic level. These strategies have been observed amongst different students for one task, illustrating how students can solve the problem at different levels. Such a cross section of learning levels at one given moment in the instructional sequence can be seen as a validity criterion, as we explained before in section 4.3.2. If we can identify the same progressive development of learning in an individual student, we speak of *vertical mathematization* (i.e. formalization of mathematical activity). According to RME theory, mathematical learning proceeds most effectively by way of vertical mathematization. This trajectory of learning can take place in due time as a student works through a lesson series, assuming that the mathematical activities enable and – when necessary – induce the invention of a more advanced strategy at each level. But, as we will see for the present case, the path is not always smooth, linear and unequivocal.

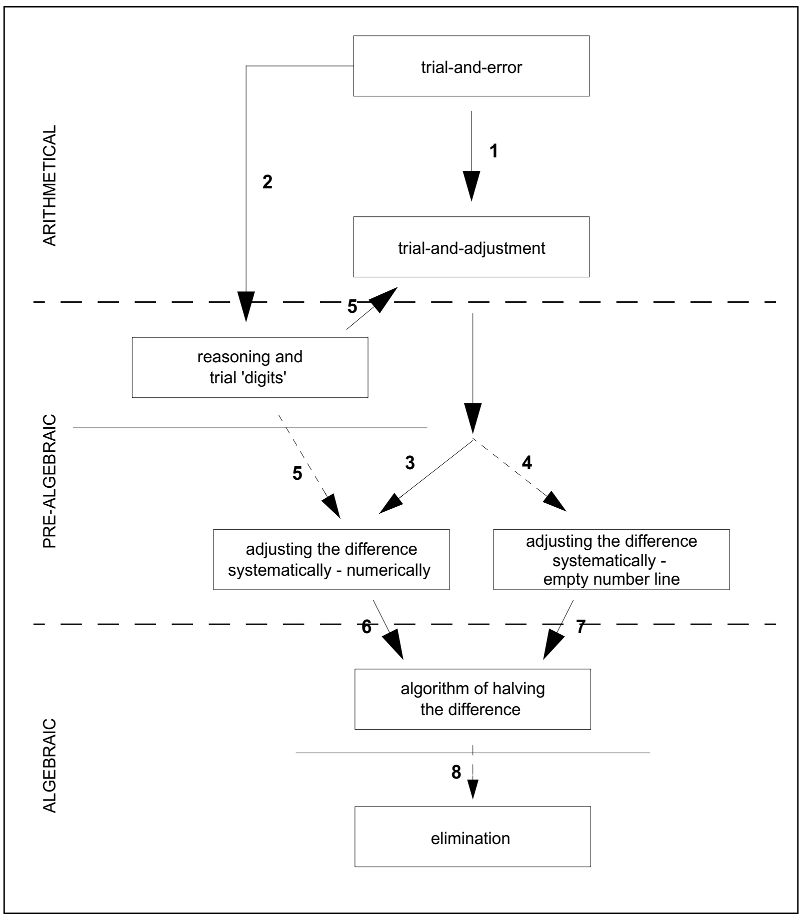


figure 4.10: vertical mathematization for problems like *Number Cards*

Figure 4.10 illustrates some feasible paths of progressive formalization, for example

1-3-6-8, 1-4-7-8 and 2-5-3-6-8. Individual students will not pass through every stage of learning, because some arrows are parallel developments. We describe how an imaginary student named Nicky – boy or girl; we alternate 'he' and 'she' per section – might come to new insights and move from one strategy to the next (arrows 1 through 8). It will become clear that Nicky sometimes needs to take a step back to be able to proceed. One or two strategies seem to be isolated from the rest so accessible. The strategies themselves (described in section 4.5.2) are presumed to be clear to the reader; we concentrate on what might happen inbetween in Nicky's head. Note that we have chosen to depart from the first condition in the task (two numbers adding up to 200) and work towards satisfying the second, but the same path of development is applicable for students who depart from the other restriction.

1 Assume that Nicky has solved two or three problems with the trial-and-error method. It has taken him a while, and he has noticed that other students around him are faster. Besides, the work is tedious; he wants to find a more efficient way. Although he is getting more feeling for the numbers, which makes it possible for him to do a good first estimate, each next attempt is always another guess. In the next problem he compares his second try with the first: am I getting closer? He can reason as follows: I need to get a difference of 68. Is the difference I found closer to 68 or not? If not, I adjusted the wrong way! Nicky can now see to it that each new attempt is an improvement. After a while he can even decide in which direction to adjust the numbers, instead of comparing consecutive errors. He knows now, that if the difference is larger than 68 that the numbers must be closer together, and if the error is smaller than 68 they must be further apart. He still calculates the second value by subtracting the first from 200, though, because he does not realize at this level how adjusting one number automatically determines the other by symmetry.

2 An alternative insight: is it possible to reduce the number of possibilities from the start? Nicky's estimates are improving, but that is not good enough. What can she say about the numbers on the cards? Her experience with add-end problems and dot-problems enables her to do a little number theory. Am I looking for odd or even numbers? Thinking about what happens when she adds two numbers, perhaps trying a few, she can discover that only certain combinations of digits, when added, will give a 0 as the last digit. If she does the same for the difference, she can reduce the number of possibilities even further! Only numbers ending with 4 and 6 or 9 and 1 satisfy the conditions.

3 Nicky has successfully used the strategy of trial-and-adjustment. He can now tell by the size of the error whether to try a higher or a lower number for each card, and also whether or not he is still far removed from the solution. For example, he takes the following steps: first attempt 60 and 140, difference is 80 but should be 68, too much so I have to bring the numbers closer together. How much closer?

How much do I need to adjust? The next step depends on a moment of insight, of recognition. Nicky needs to notice that if he adds 5 to the smallest number, then he must subtract 5 from the other number. In other words, he needs to identify the symmetry. Once the symmetry is established, the qualitative interpretation of the adjustment can be replaced by a quantitative interpretation. So now he can reason about the size of the adjustment: if he adds 5 to 60, the other value will be 140 – 5 by symmetry, and the error will decrease with 10. It is no longer necessary for him to calculate 200 – 65, which saves him a lot of time. The more general level of this strategy can follow if Nicky realizes that no matter how large the error is after the first attempt and no matter how many steps it takes, you can be certain to find the solution because the method always works. In other words, it requires Nicky to switch his perspective from 'the most accurate attempt' to 'the easiest first attempt'. Taking into account that students have a natural inclination to take the average, it is quite reasonable that Nicky decides that the easiest first attempt is half the sum.

4 This part of the vertical mathematization process is difficult to make plausible, which is why the arrow is a dotted line. Nicky is clearly taking a numerical approach to the problem, so it is not a logical step for him to switch to a visual representation. We assume that she is familiar with the empty number line, but in a very different role: to support the learning of basic skills, and not for problem solving. So the chances are small that the empty number line will emerge as a model through the numerical trial-and-adjustment approach. It may be a more likely continuation in a trajectory that involves a visual representation from the start. Initially the learning program introduced the rectangular bar to represent a known quantity, but in the previous try-out we found that the step from known to unknown quantity requires more attention than we had foreseen (see section 5.4.2). Perhaps a geometrical approach to algebra can bring out opportunities of a natural implementation of such a model (see also section 7.2).

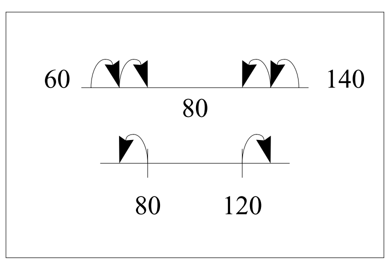


figure 4.11: symmetrical jumps on a line segment

But – going back to our line of thought – if Nicky is 'visually inclined' and confident with positioning numbers on a line, she might draw how far apart the num-

bers lie that he has found. Let us say she does it as follows: she draws a horizontal line (segment), the smallest number on the left, the largest on the right. The difference between them is written below the line. It is not unthinkable that Nicky visualizes the numbers coming closer together by drawing jumps – much like the addition and subtraction exercises in grade two (see top of figure 4.11).
Or, if the difference she finds is smaller than 68, she can locate the two numerical values and increase the difference between them by drawing symmetrical jumps (bottom of figure 4.11). Either way Nicky draws the difference, a positive number, as a line segment. It requires a switch of perspective to think of drawing a difference of zero as the first attempt – which you cannot draw! – and then increase it with symmetrical jumps. And so the position of the empty number line strategy in the trajectory of progressive formalization is not so solid.

5 Nicky has been reasoning about the digits of the numbers he wants, but it still requires a lot work for each new problem. In fact, he will not be able to continue along these lines. Further formalization of the number theory strategy would require very abstract thinking or even symbolic algebra, which is not desirable at this stage. And it is not logical to suddenly reason quantitatively about consecutive attempts if the emphasis has been on the numbers themselves; the two types of reasoning are not compatible. And so Nicky will need to take a step backwards (to a more concrete thinking level) to get back on track.

6 Nicky can use the symmetry of adjustment to find the solution in just a few steps. She just divides the number in the first expression by two and then makes symmetrical adjustments until the second expression also applies. Nicky thinks, ‘It is really a good method: it is faster, it always works and it is easier to calculate’. She also realizes that the larger the steps, the sooner she finds the solution. As she combines steps of 10 into steps of 20 or more, she discovers that taking away the largest ‘nice’ number (for example, the largest multiple of ten) works even faster. Her attention will therefore shift from the size of the *adjustment* (the step) to the size of the *error*.
This renewed attention can result in another moment of insight: the total of symmetrical adjustments is equivalent to dividing the error by two. So all she needs to do is divide the given difference by 2 right from the start. Her own active participation in developing this strategy should ensure that it will not become a rote skill without understanding; if necessary, she can go back one step to the pre-algebraic level because she knows where the algorithm comes from.
Nicky can now perform the algebraic strategy of halving the sum and the difference. One can imagine that the Babylonian method for this type of problem might have originated in a similar way, driven by a need for efficiency and generality. The strategy has two strong points: it does not require a notation for the unknown, and it can be extended to problems involving fractions, irrational num-

bers or even variables for the more advanced algebra learner. Of course the most obvious shortcoming is its limited domain of application: it only works for a very specific type of system of equations.

7 Nicky knows how to solve number riddles using symmetric adjustments on the empty number line. If the given difference is a simple number, he uses a few big steps, but sometimes he prefers to be safe and take smaller steps. The act of drawing a bow for each jump – one left of the middle, one right – emphasizes that each adjustment involves a division into two equal portions, and so the insight of halving the difference at once is just a matter of time. The act of 'halving the sum' is already a part of his strategy. So we can expect Nicky to move onto the algorithm of halving the sum and the difference, perhaps accompanied at first by a drawing of the empty number line, but after a while the visual aid will probably become superfluous.

8 Nicky is now confident to use the strategy of dividing the sum and the difference. The emphasis of this method lies on the two given numbers in the problem situation. It is not relevant for her to compare the iconic unknowns on the left-hand side of both equations. How, then, can Nicky suddenly be inspired to begin to reason about the icons? She needs to change her perspective first. The path from the algorithm to a strategy of elimination is not plausible, and is not a natural continuation of the learning trajectory. It would seem more reasonable to expect a strategy like elimination to emerge from a situation where comparing known quantities precede the comparison of unknowns. But, disregarding the learning trajectory for a minute, is it not possible that Nicky can reason with her common sense? The task is not a typical every day life situation, but the mathematical context of 'secret numbers' is experientially real. In arithmetic class Nicky always does well at problem solving, and this problem is really more like a puzzle with pictures. In other words, the strategy of elimination may not be a logical constituent of the process sketched above, but it is certainly accessible to students who have a feel for thinking logically.

shortcuts and non-linearity

The learning trajectory described is merely a theoretical learning environment. In the actual classroom only a few students, if any at all, might experience this development, but the various stages have been identified amongst different individuals. It is therefore more likely that a student will pass through certain levels of the learning process. While some learners might not start at the arithmetical level, others might not reach the algebraic level at all. It is also probable that faster students will skip a level somewhere. In each of these situations, measured steps of formalization enable a student to fall back on a less formal approach whenever he or she wants. In other words, gradual formalization at the learner's own pace is important for mathematics learning to be meaningful. For the same reason we might expect didactical tension in learning situations where a sudden, new insight is required, for instance where the transition from one level to another is not so ‘smooth’ (like the broken arrows in figure 4.10). Premature formalization encouraged by the teacher or fellow students can lead to artificial, meaningless understanding, making it more difficult for the student to recall and fall back on a less advanced method. These cognitive jumps in the learning trajectory may help to explain why the pre-algebraic strategies ‘adjusting the difference systematically’ did not function as expected (see section 6.6).

5 Design process

5.1 Introduction

hypothetical learning trajectory

This chapter deals with the design process of the experimental pre-algebra strand, from the first design ideas until the field tested version. In most educational design and curriculum development projects the learning process of the designer is left unmentioned; people take less interest in the underlying process than in the final product. One of the principles of developmental research, however, is to make the design process reconstructible. This is how the designer can strengthen the validity of the result (the principle of 'trackability' is discussed in section 4.3.2). The cyclic process of envisioning and trying out has not only resulted in a provisional early algebra learning strand, but it has also produced theoretical ideas on early algebra learning and teaching. In order to make the research process explicit we describe the evolution of and reflection on instructional materials, including the application of various Realistic Mathematics Education design heuristics (guided reinvention, didactical phenomenology, levels of learning and emergent models). The study consists of three cycles of design and trying out: an orientation phase, the pilot experiment and the field test. For each cycle the teaching and learning processes observed in the classroom are compared with the conjectured trajectory, and recommendations are made for revision. In the current chapter we report on the first two research cycles, the third and final one is evaluated in chapter 6.

5.2 The orientation phase

theoretical inducement for the first design phase

Starting-point for the first tentative thought experiment were arguments of a theoretical and an empirical nature. In chapter 2 we elaborate that studies of the last two decades have reported on difficulties of teaching and learning algebra. Students struggle with the semantics of algebraic language, they make so-called reversal errors when transforming a verbal description into a formula, they have trouble to obtain both a dynamic (procedural) and a static conception of algebraic notions, for example in order to translate a word problem into an equation, and so forth (see section 2.4). These problems ask for an approach to algebra that departs from the familiar terrain of arithmetic, that gives meaning to algebraic notations and that pays attention to static and dynamic aspects of algebra.

empirical inducement

On the other hand, empirical studies in the *Mathematics in Context* project (Van Reeuwijk, 1995, 1996) have revealed that young learners can reason algebraically in realistic problem situations, using their common knowledge and informal strategies. For instance, in the *MiC* student instructional unit *Comparing Quantities* combinations of quantities are represented by pictures and stories – like the tug-of-war task in figure 5.1 where students are asked to figure out which team is the strongest – challenging students to create and develop suitable notations to describe their actions. Reasoning with quantities is seen as a suitable predecessor to equation solving because students develop a notion of equivalence, they can solve problems at their own level and they symbolize their thinking without losing sight of the meaning. Other instructional units in the *MiC* algebra strand – *Patterns and symbols, Expressions and Formulas, Operations* and *Building Formulas* (*Mathematics in Context* Development Team, 1998) – display a progressive use of informal letter notation and arrow language as precursors of variables and formulas.

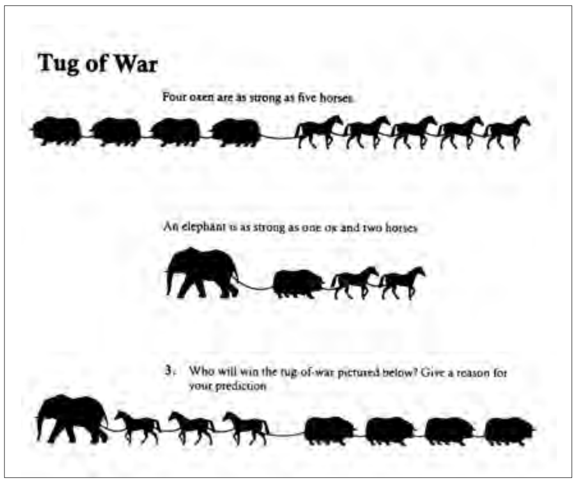


figure 5.1: 'tug of war' problem

Classroom experiments by Streefland (1995abc) show how young students create their own system of notations to describe combinations of candy and their combined price.

1. 4 = 0,40
2. 1 zakje autodrop + 2 gevulde koek = 2,25
   1 autodrop + 6 spekkies = 1,85
3. 6 l_2 = 1,50
4. 4 zuurballen = 50 c
   4 fruitballen = 50 c
5. 1 l_1 + 2 gr r = 4,05
6. 3 dv + 2 m_1 + 2s = 2,20
7. 4 dropv + mars2 + 5 spekjes = 2,05
8. 2 lol 1 + 1 gr. reep = 2,25
9. 2 spekkies + 1 nuts = f 1,05
10. 2 grote repen = 3,90
11. 1 autodrop + 1 gev.k = 1,75
12. 8 dropv + 2 mars2 + 8 sp = 3,90
13. 1 sp + 2 n = 1,80
14. 2 autod + 2 sp = 2,70
15. 3 dv + 1 m_1 + 2s = 1,35

figure 5.2: anthology of equations

Student work demonstrates a transition from notations in columns (in which each price is known) to notations in horizontal expressions – some including abbreviations – where the prices have become unknown. In his experiments Streefland presented an anthology of student work to the classroom community to negotiate the meanings of symbolic expressions, and he asked the students to decide which equations belong together and according to which criteria (see figure 5.2). In subsequent activities the students combined and manipulated equations to determine unknown prices.

5.2.1 RME design heuristics

reinvention

On the basis of these theoretical and empirical findings, we designed a series of activity sheets for students aged 10 or 11 in grade 5 of primary school to get started. The design heuristic 'guided reinvention' instigated us to look at the historical development of algebra for potential barriers and plausible learning moments (see section 3.4). First, we found indications for the development of algebraic notations and symbol use, namely that verbal descriptions precede syncopated and symbolic notations and that the invention of symbolic notation was a long and difficult process. And second, it became clear that practical problems naturally led to the existence of systems of simultaneous equations in Babylonian and Chinese mathematics, perhaps earlier than but certainly at the same time as single equations in one unknown. This is not the only reason why we decided to begin with activities on systems of two simultaneous equations, ahead of activities of single equations in one unknown. Another reason is the fact that simple linear equations in one unknown can be solved quite easily arithmetically – as was done in the past – which makes a switch to algebra neither meaningful nor practical.

mathematical analysis

The next step in the design process is to conduct both a mathematical and a mathematical-didactical analysis of the content. The term 'phenomenology' (as described in section 4.3.1) cannot yet be justified in the orientation phase. Here we give an example of such an analysis for the case of (a system of) linear equations. We start with the system of equations:

$$\begin{aligned} 2x + 3y &= 8 \qquad &(1)\\ 4x - 9y &= -14 \qquad &(2) \end{aligned}$$

Traditionally the unknowns x and y are determined through one of the following procedures:

a *eliminating one of the unknowns*: multiplying equation (1) by 2 and subtracting equation (2) from it gives

$$\begin{array}{rcl} 4x + 6y &=& 16 \\ 4x - 9y &=& -14 \quad - \\ \hline 15y &=& 30 \Rightarrow y = 2 \end{array}$$

Substitution of $y = 2$ into equation (1) will result in

$$2x + 6 = 8 \Rightarrow 2x = 2 \Rightarrow x = 1$$

Therefore the solution of the given system is $x = 1$ and $y = 2$.

This procedure is as curtailed as possible. The question is: what purpose is served by it? The equations are compared to make the unknowns known, and this is done by making them more easily comparable first.

b *substituting one of the equations into the other:* rewriting equation (1) such that x is written in terms of y, we can replace x in equation (2) as follows

$$x = 4 - 1.5y$$

$$4(4 - 1.5y) - 9y = -14 \Rightarrow 16 - 6y - 9y = -14 \Rightarrow -15y = -30 \Rightarrow y = 2$$

Substitution of $y = 2$ into equation (1) or (2) will again give $x = 1$.

The purpose of this procedure is also to make the unknowns known, this time not by making the equations comparable but by expressing one of the unknowns in terms of the other.

A third method of elimination, which is not so common, goes as follows:

c multiply equation (1) by 7 and equation (2) by 4, and add the second equation to the first to give

$$\begin{array}{l} 14x + 21y = 56 \\ \underline{16x - 36y = -56} + \\ 30x - 15y = 0 \Rightarrow 30x = 15y \Rightarrow 2x = y \end{array}$$

This means that x is to y as 1 to 2. If we choose $x = 5$ and $y = 10$ as a possible solution, then substitution into equation (1) or (2) shows that these values are 5 times as much as they should be, hence $x = 1$ and $y = 2$.

We see that elimination of the known terms in the equations leads to a ratio expression of the form $ax + by = 0$, that is, one equation in two unknowns.

The mathematical analysis of simple algebraic expressions and equations raises questions about their nature, origin, structure and meaning from a mathematical and an historical point of view, such as

What is the purpose of such a system of equations?
Where does it come from?
What phenomena is it supposed to organize?
What is the meaning of the unknowns?
Why are they unknown?

> Were they already unknown before the equations were composed?
> ...
> Does one equation in two unknowns also have a meaning?
> (Streefland & Van Amerom, 1996, p. 140)

The last question can be linked back to solution procedure c as discussed above. Moving from the mathematical to the mathematical-didactical analysis, the following questions on organizing linear relationships become relevant: “Are there situations from which the combining of objects evolves in a natural manner (...) situations that can be organized by novice learners by means of algebraic tools such as letters for (un)known objects, (linear) equations, (systems of) linear equations, solving procedures for these, and so on?” (ibid., p.142.) And what kind of competencies are required to facilitate the development of such algebraic tools? In order to come to grips with these questions we drew up a diagram of subsidiary abilities as shown in figure 5.3. This diagram afterwards led to a list of (pre-)algebraic competencies and a directed map of consecutive abilities, which are both given in section 5.2.2.

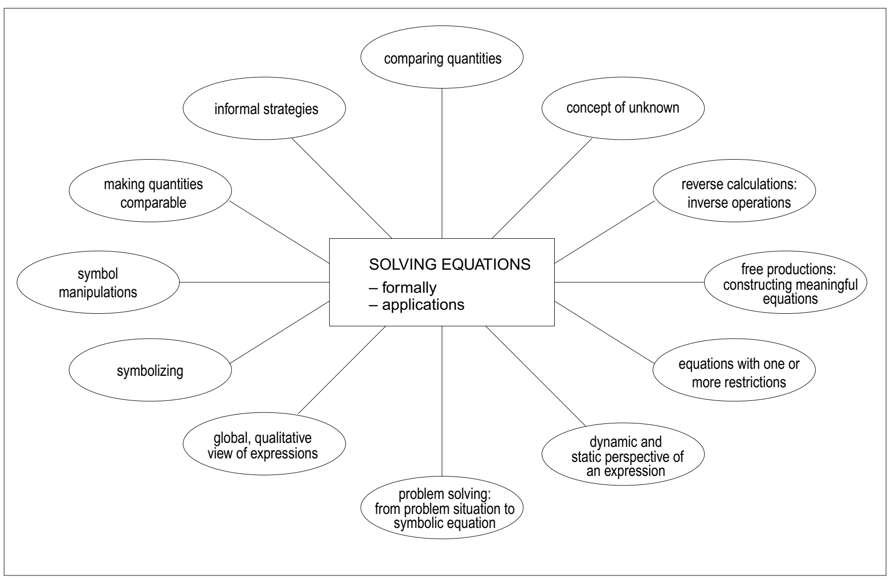


figure 5.3: subsidiary abilities when solving equations

integrating historical elements

The mathematical didactical analysis also showed possibilities for a meaningful application of the history of algebra in the classroom, as we have described in chapter 3. In our quest for appropriate activities and problems with an historical background we discovered that early algebra originated in word problems dealing with everyday life situations, for instance barter, money exchange, fair distribution of property, etcetera. Mathematical activities in such contexts are expected to be closely tied to students' own experiential world and we therefore anticipate that students will engage in discussions with each other and the interviewer.

model

With regard to yet another design heuristic, it is our intention that the designed activities constitute settings from which a model might emerge. During the mathematical didactical analysis we sought a diagram that might develop naturally as a visual model for the students. This should preferably be an item that students already know and use in their arithmetical activities, and which can be easily extended to an algebraic problem solving tool. The rectangular bar or strip – which is used to teach fractions and percentage in primary school – seemed suitable: it can emerge as an abstraction (model) of a drawn series of marbles or coins, while in later activities it can support mathematical reasoning by representing an unknown quantity or number. However, at this stage the bar does not (yet) qualify as an overarching model.

5.2.2 Mathematical content

It must be said that the learning trajectory envisioned for the activity sheets was incomplete. The design heuristics led to the formulation of certain pedagogical ideas on early algebra abilities, but we did not foresee the entire learning process in advance. In other words, when we started our work sessions with the students we had worked out only part of the mathematical content and a rough map of the mathematical goals. The case studies were intended to give information on how the activities might facilitate development of algebraic thinking on the way to (systems of) equations. During and just after the case studies, ideas on the remainder of the mathematical content were worked out and tested in separate classroom experiments (see section 5.3). Hence the orientation phase of the research project should not be seen as a full-fledged hypothetical learning trajectory, but as a feasibility test for four separate parts: the mathematical content, and on a more practical level the chosen contexts, the activities and tasks, and the build-up of the learning strand.

The mathematical content of the learning strand is aimed at creating opportunities for students to develop algebraic reasoning and symbolizing competence. Theoretical reflection on the mathematical content, and arithmetic and algebra in general (see also section 2.6), has resulted in a list of pre-algebraic competencies.

pre-algebraic competencies

1 *comparing quantities*

Recognizing and describing how quantities relate is a mathematical activity that establishes a link between arithmetic and early algebra. Starting from given, numerical quantities that can be treated arithmetically, it is possible to help students develop their reasoning on different levels. The term 'reasoning' used here is limited to thinking logically about constant number values, whereas the more abstract kind of reasoning discussed further on refers to pre-algebraic abilities like looking at quantities qualitatively, generalizing relations, and reasoning with variable magnitudes and unknowns. Activities on comparing quantities allow different solution strategies and give students opportunities for developing notations to support their reasoning (for example, writing a symbolic expression for two quantities of equal value). Since equations originate from situations where quantities or values are compared, we conjecture that making two quantities comparable – expressing them both in terms of the same unit of measure, or in terms of each other – is a skill that needs be mastered in order to give meaning to solving equations.

2 *representing a relation: descriptively, numerically, visually or symbolically*
Understanding the concept 'relation' incorporates the ability to perceive it in different forms. This is not only a matter of representation, but also a matter of how the relation is conceived: as a procedure or as a static object. For example, 'Peter gets twice as much pocket money as John' is a verbal, static perception, whereas 'double the amount John gets and then you have what Peter gets' is a verbal, procedural conception. The relation 'twice as much' can also be expressed numerically: in a table or with numbered pairs (2,4) (3,6) (4,8) etcetera. A symbolic representation could be $P = 2 \times J$, which we imagine will be interpreted statically by most students, whereas a dynamic, symbolic form could be an arrow diagram: $J \xrightarrow{\times 2} P$. From the types of representations named here, we assume that only the symbolic ones will be new to the experimental group of students.
The historical development of algebraic notation shows how symbolic algebra emerged only after a long, difficult process of inventions, adjustments and recesses. Prior to the sixteenth century, mathematical problems were described in words or with abbreviations, and then solved algebraically (calculating with an unknown). The principles of the Biogenetic Law and guided reinvention suggest that we may expect students to globally follow the phases rhetorical – syncopated – symbolic notation, especially the first two. Compared to syncopated and symbolic notation, rhetorical notation is closer to the context and more appropriate to explain procedures of calculation. It might be more natural for children to start with word problems first and while dealing with them, develop short hand notations. If students can experience the practicality and power of generalized notations, they can learn algebra in a sense-making and purposeful way. But, to be realistic, this cannot be expected from the average student! In the syncopated phase we can imagine that students will use abbreviations and mathematical symbols like +, –, ×, : and = . Even though some classroom studies are promising (Streefland, 1995abc; Van Reeuwijk, 1995, 1996), we are doubtful that students will develop symbolic algebra – where the letters are actual variables – on their own, because there is not enough time for students to formalize (or reify) their syncopated notations. Moreover, there is little opportunity in the current teaching program for students to practice developing their own notation; they will probably feel insecure about shortcuts other than abbreviations. But this is not our intention either at a pre-algebraic level.

Considering the procedural nature of arithmetic, it is quite likely that the static conception of a relation between two quantities (like ‘twice as much as’) will be difficult for the students. In particular we expect the symbolic, static representation to be the most difficult, because of its double complexity: conception as a product instead of a process, and an unfamiliar abstract appearance. Perhaps students can profit from the visual support of the rectangular bar in conceiving the relation as a given state between two quantities, i.e. to see the quantities simultaneously instead of consecutively.

3 *meaning of notations*

A major stumbling block for learning algebra is the way in which letters can have different roles and meanings (see section 2.4). Letters can represent constant numbers, they can be substituted by a number and evaluated, or they can refer to an object or a quality of an object. The proposed learning strand deals with different meanings of letters: in symbolic expressions that behave dynamically, but also in static trade terms where the letters represent trade goods. In our opinion it can be very useful to confront students with this phenomenon even at an informal level, when they construct their own symbolic language. Streefland (1995ac) has found in his teaching experiment on candy that the meaning of literal symbols changes during the production process of the students, namely from object (candy) to quality of an object (price of the candy). This development agrees with the historical development of algebra, where a Chinese barter problem in *Nine Chapters on the Mathematical Art* (ca. 200 BC) shows that the unknowns are both objects as well as trade values of objects (see also section 3.4.2). Streefland warns against international trends of algebra education to skip the (pre-algebraic) phase of constant values for unknowns and proceed directly to the more formal conception of variables, since “(...) this would be a jump into the deep end which, from an historical perspective, cannot be justified” (Streefland, 1995a, p. 35, transl.). In fact, according to Streefland the change in meaning of literal symbols is an important constituent of the vertical mathematizing process (progressive formalization) of the pupils. “The changes of meaning that letters undergo, need to be observed and made aware very carefully during the learning process. In this way the children’s level of mathematical thinking evolves” (Streefland, 1995a, p. 36, transl.). He also mentions the example of asking for the meaning of the equal-sign: indicating ‘costs so much’ as well as the usual ‘equal to’. Students’ own productions can be used to raise issues like clarity, efficiency, and consistency. In this way it is possible to reflect on algebraic language in a meaningful way, without pursuing a formal level.

4 *looking at a relation globally*

One of the properties of algebraic expertise is the ability to study the expression as an object, to make qualitative comments about it and to look for general characteristics (Gravemeijer, 1990). 'Looking globally' means being able to change perspective, dismiss the details, see something in a new light. For example, at an arithmetical level a student may observe that the numbers 5 and 10 are related in different ways: '5 more', '5 less', 'half' or 'double'. Changing to yet another perspective: have the student think up new number pairs that satisfy this relationship, which makes the relation dynamic with two variables. We conjecture that a playful, global approach towards arithmetic relations will not only strengthen the foundation of basic competencies, but will also deepen the student's conceptual understanding, for instance of how inverse operations are interrelated. This should form a basis for studying and manipulating simple symbolic expressions, remembering to stay close to the context and to always keep the verbal description of the relation at hand. In a broader sense, 'global view' can also be connected with problem solving. A skilled problem solver is able to study a problem from different points of view without losing track of information, for instance in solving restriction problems (see point 8).

5 *investigating and interpreting equivalent expressions*
The issue of equivalence plays an important role in algebra learning and teaching. By confronting students with different interpretations of 'equivalence' we want students to develop a better understanding of the concept. Recognizing that two expressions (trade terms, linear equations, formulas) are equivalent is a specific case of a global conception of expressions as described in point 4. If an expression is viewed as a process instead of an object, two equivalent expressions will still be seen as different actions. For example, to find x in $x = 4 + y - 2$, you can follow the instructions 'add y to 4' and 'subtract 2 from the result', whereas $x = 2 + y$ might be read as 'add y to 2' to get x. If the two expressions are perceived as objects describing a state of being, they are the same.

At this time we distinguish two types of equivalence: dependent expressions (multiples) and inverse expressions (with the independent – dependent variables interchanged). Let us consider a simple linear expression like the trade term $1p = 8b$ (1 pineapple is worth 8 bananas); it is a simple task to generate multiples like $2p = 16b$, $4p = 32b$, etcetera, falling back on the students' knowledge of ratio. We conjecture that students will be capable of manipulating and substituting syncopated and symbolic expressions as long as the expressions are meaningful to them. Even trade terms involving three items should not be a problem. However, the equivalence of two forms of the same formula will probably be less obvious. In this case students need to check a general statement instead of arithmetical expressions. This can be done at two levels: either making the step directly by describing the formula in words, or substituting numerical values for the variables to determine the equivalence. We expect the latter strategy to be used more frequently because it reduces the abstractness of the formula.

A static perception of expressions (as products) gives the equal-sign a different meaning: two sides of the expression have equal value, rather than the procedural interpretation of announcing a result. A very common error made by students that illustrates this difference is writing down in action language a string of calculations including intermediate outcomes like $2 + 5 = 7 \times 2 = 14$. In fact, we expect to encounter this error in the working sessions and when we do, we will emphasize the interpretation of 'equal value' by going back to fair trade terms.

6 *reasoning about varying and unknown quantities*
Given the fact that we are dealing with a *pre*-algebra learning strand, we pursue an approach that enables students to solve problems informally using the tools they already have at their command. Common sense and an investigative mind are tools that support the advancement of mathematical ideas, both in the course of history and in the individual learner. In situations where formal problem solving techniques are not (yet) at hand, informal strategies like logic thinking and trial-and-adjust are a good alternative (take for instance the historical Rule of False Position discussed in section 5.3.3). One obstacle that students must overcome is the paradox of predicting or concluding something about a variable or an unknown number, based on the information given. One can compare it with asking a student to close his eyes and describe an object just by smelling and touching it. Even though the object is not determined or unknown, it is possible to say something about it. It is typical of algebraic thinking to make assumptions about unknown quantities, like appointing a numerical value (see also table 2.1 in chapter 2). Students need to warm to the idea that an unknown number can be treated as though it is known, with the important difference that you don't get a numerical result. It is plausible that accepting the idea of reasoning about variables and unknowns precedes matters like understanding, constructing and manipulating symbolic expressions.

7 *schematizing as a problem solving tool*
The selection of the target group is based on the assumption that these students have been taught mathematics according to RME principles. In other words, we may expect them to be familiar with tables, visual diagrams and notes to organize mathematical information. Problem solving in the proposed learning strand is based on developing, improving and possibly generalizing informal strategies. These strategies depend on a logical, systematic way of working: reasoning in clear steps, trial-and-adjust according to a plan, or performing a series of consecutive calculations. Some of the activities will be too long or too complex to be solved mentally, without pen and paper; they are intended to instigate students to produce their own schematic notations to support and explain their reasoning.

8 *interpreting problems in two unknowns with one or more restrictions*
A system of two equations in two unknowns gives two conditions which the vari-

ables have to satisfy. Considering the restrictions separately, the variables are indeed varying and there will be more than one solution (sometimes even an infinite number of solutions). As soon as a second condition applies, the variables become unknowns which can be determined – provided that the equations are not dependent, which is usually the case – and the number of solutions is reduced (in this learning strand, to one). In other words, a second piece of information about how the variables are related often enables you to find the solution. Since it concerns the idiosyncrasy of a system of equations, some may consider this kind of knowledge to be meta-cognition and as such too advanced. On the other hand, our point of view is that restriction problems bring out the essence of a system of equations and we believe they help students to understand what a system of equations means and how and why it works. In fact, we conjecture that students can cope with these problems because reasoning is a natural activity.

connection between competencies

The following map illustrates how the pre-algebraic elements described above are related and which of them are seen as prerequisite to other, higher order activities (figure 5.4). The left half of the map is concerned with symbolizations, notations and diagrams, i.e. with mathematizing on paper. The right half deals with mental work: looking, comparing, reasoning. These two streams come together at the highest level of reasoning, where symbolizations facilitate problem solving. It is intended that students will develop both types of competencies simultaneously.

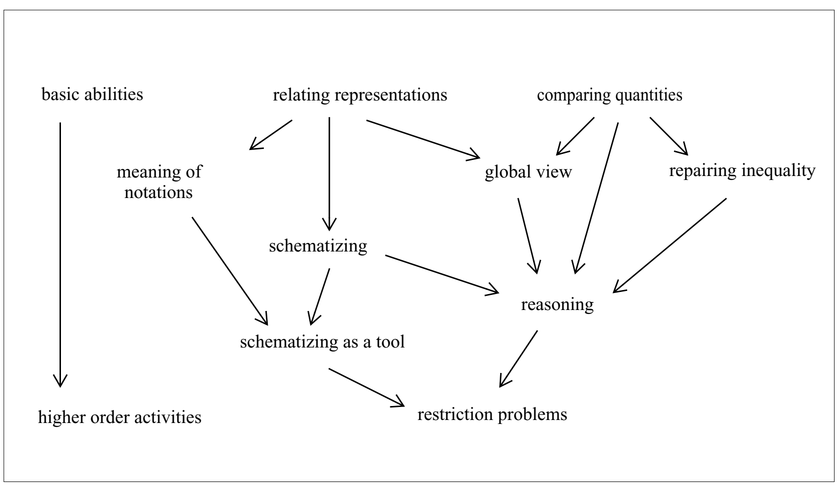


figure 5.4: map of pre-algebraic elements and activities

The basic abilities at the top of the map form the starting-point of the learning trajectory; they function at an arithmetical level. Relating representations and comparing quantities are strongly bound to context situations – realistic everyday settings but also arithmetical settings. These abilities – including understanding of the concepts 'relation' and 'equivalence' – develop progressively as they are evaluated, applied in other settings or seen from a new perspective. As students evaluate and reflect on their notations and diagrams, their schematizing improves. The comparison of quantities is then extended to include taking on different perspectives and identifying multiple ways to repair inequality. Each of these components relies on reasoning, but also contributes to its development. We foresee that in this dual relationship students can formalize their mathematizing activities to become tools for mathematical reasoning, which can then be applied in complex restriction problems.

The hypothetical path of learning illustrated by the map does not represent a sequence of mathematical activities; it merely reflects connections within the mathematical content. The progression from lower to higher level abilities is by no means intended as a chronological, one-directional process. Most abilities are practiced throughout the program, usually progressively and within different contexts (as indicated by the titles of the instructional units: *Marbles, Pocket Money, Playing Cards* and *Barter*). At the start of a new series of activities we step in at ground level, but we expect to move ahead faster each time. However, we also foresee that students might take another path or need to take a step back in order to move forwards.

activity sheets

In summary, at this stage of the research project it was essential to determine whether the hypothetical construction of algebraic abilities as shown in figure 5.4 is consistent with the actual learning process demonstrated by the children. At a more practical level we wished to test the activity sheets we designed on feasibility of:

- the *contexts*: are the students familiar and/or motivated by them? do they understand the unwritten rules within each context?
- the *activities* (especially those on higher order reasoning): are the activities too difficult? or too easy? are they sufficiently open and challenging? do they instigate active, constructive learning? do they allow strategies on different levels? are the activities balanced out where (pre-)algebraic competencies are concerned?
- the *order* of the activities: are the activities increasingly complex? how well do the activities set out a learning trajectory? do the contexts support continuity of the learning process?

In the following section we look at some activities and at how the students performed in more detail.

5.2.3 Case studies

The activity sheets were tried out with nine pairs of grade 5 students from an urban primary school. We intended to work four or five sessions with each pair of students in order to watch their developments without taking too much of their regular class time. We soon realized there would not be enough time to do all the activity sheets unless we let some students step in halfway. Table 5.1 shows which students did which tasks (X = all the tasks, * = a few tasks). There was no assessment at the end. The step-in level of mathematics for the activity sheets is an average mathematical competence of basic arithmetic operations, ratio tables and problem solving. The activities are based primarily on money and equal value, which are daily life topics that students are used to and that facilitate rich problem situations. The students were asked to compare values or quantities in four different settings: marbles, pocket money, points won in a game of playing cards, and barter. In order to vary the levels of abstraction we integrated concrete materials (marbles, plastic coins), students' own experiences, and written tasks.

number of students	marbles	pocket money part 1	playing cards	pocket money part 2	barter
4	*				X
4	X				
1	X				X
2		X			
2		X	X	X	
2		*	X	X	
1					X
2					*

table 5.1: numbers of students doing tasks (X = all the tasks, * = a few tasks)

We discuss the activities in chronological order in order to clarify certain decisions, and at the start of each new setting we have listed the mathematical abilities that will be practiced.

marbles

comparing quantities, shortening notations, reasoning

Four students were asked to tell about how they trade marbles, and write down their terms for fair trade. By choosing an informal approach, problems of equal-sign or meaningless notations are avoided and students can decide their own representation (words, drawings, abbreviations). These trade rules would form the starting-point of the mathematizing activities: shortening notations, combining rules to make more complex trades, determining which combination of marbles are worth the most, etcetera. Special attention was paid to *intuitive notations* (abbreviations, mathematical symbols, schematic diagrams); *solution strategies* (for example, comparing quantities by grouping, by calculating the numerical values, or by cancelling) and *conflicts between mathematizing and reality* (indifference to damaged or partial marbles, preference of taste, equal number of items versus equal value of items).

inconsistent rules

Contrary to our expectations, the starting activity turned out to be inappropriate. Students were not actively playing at the time, and in the school there was not one unambiguous, consistent system of fair trading. The rules that the students constructed were inconsistent and based on prettiness and amount of damage. Consequently the remaining pairs of students were given a set of marble trade agreements to work with.

notations

As far as notations are concerned, students showed a clear preference for rhetoric descriptions – sometimes even for numbers! – no matter how much time it consumed. Not once did a student decide to write it down in a shorter, more efficient way, and so we explicitly asked students to do so. Seven students suggested using abbreviations, but encouragements to use mathematical symbols (+, =, ×) in their trade rules usually met with surprise and confusion. This suggests that the students did not conceive the activities as being mathematical, which hindered the process of mathematization.

The activities included filling in a table with two types of marbles and deducing a rule of fair trade, comparing two combinations of drawn marbles to see if the trade is fair, and dividing a given amount of marbles into equal portions. We found that students performed best at solving problems in context situations, but they needed much more incitement and instruction to write down their strategy than had been foreseen. Especially explaining the method of reasoning proved to be a problem.

The most difficult task turned out to be working out the correct numbers of marbles if the relation involved fractions (figure 5.5).

aantal spikkels	1	2	X ~~2~~ $\frac{1}{2}$	10	1000	2	100	1000000
aantal uppies	2	4	5	~~5~~ 20	2000	4	200	2000000
aantal supers	X	X	1	~~2~~ 5	500	X	40	400,000

figure 5.5: three types of marbles

The aim was to stimulate students to multiply the table entries until the numbers were compatible. On a higher level of abstraction this is equivalent to operating on symbolic expressions, for example doubling $1s = \frac{1}{2}b$ to $2s = 1b$, which is one of the anticipated recurring activities in the barter activity sheets.

pocket money part 1

comparing quantities, shortening notations, adjusting inequalities, recognizing patterns

context

The choice of a pocket money context is based on expectations that every student is familiar with it, both in and outside school. Money is a phenomenon in real life that can be easily mathematized, and at different levels of abstraction: from students' own experiences to reasoning about patterns and relations between numbers. Of course there will be differences in the amounts of money that students get, but it will be the teacher's task to make this acceptable.

The first activity sheet shows two amounts of pocket money: 4 and 8 guilders belonging to two boys, Mark and Eelco, and the question was: 'What do you notice?' The purpose of the activity was to determine how students describe additive and multiplicative relations between two quantities (using words like more/less, times as much as) and the reverse descriptions, and how they shorten their notations.

reversal error

Quite surprisingly all six students gave general statements on the amounts of money from the start; we had expected to get numerical statements involving the numbers 4 and 8. The relation 'times two' and its reverse 'half of' were easily established, but writing it down in a short way was not natural. For instance, Hamza wrote 'Mark × 2 as Eelco' and 'Eelco gets : 2 as Mark'. Let us consider the second expression. From Hamza's verbal explanation it is clear that he intended this expression to be of the same nature as the first, i.e. he meant to say 'what Eelco gets should be divided by 2 to get Mark's allowance'. The words 'gets as', on the other hand, suggest a static situation like 'Eelco gets half as much as Mark', in which case the expression is exactly the wrong way around! So here the infamous reversal error comes to the fore in an attempt to translate action language into static symbolism (see also section 2.4). When asked to read it aloud, Hamza did not notice the controversy; he merely suggested adding the words 'pocket money'. Indeed, since he has constructed the expression in a way that is understandable to himself, he will not be able to imagine how others might find it unclear.

notations

Terms like 'more' or 'less' were also written down fully, perhaps because students could not translate it to mathematical symbols. From the six students who filled out the first activity sheet, it appears that students quite naturally choose abbreviations for names but less so for nouns, and they also prefer to include the measuring units (figure 5.6).

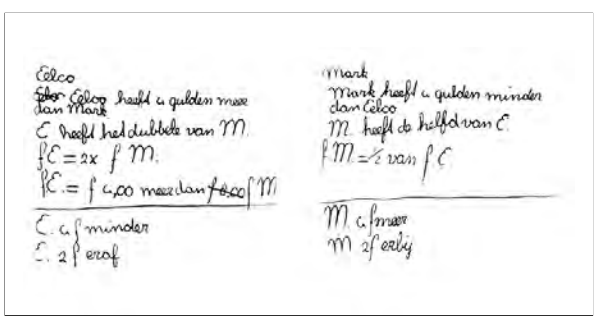


figure 5.6: abbreviations

Perhaps these letters help to give the symbols more meaning by referring to the context, but they are not used consistently. Figure 5.6 shows how Karin placed the letter *f* (which is the symbol for Dutch money, like the dollar sign $) in front of the capital letter E to express 'money belonging to Eelco', but the *f* also appears behind the number 2 where it means '2 guilders'. Moreover, the conventional way to write 2 guilders is '*f* 2'! It seems that shortening notations is not natural for this student, and she writes the abbreviations in the same order that the words are spoken aloud in.
The second activity is on recognizing a pattern in the following series of pocket money amounts: *f* 3,- and *f* 7,-; *f* 5,- and *f* 10,-; *f* 4,50 and *f* 8,50; *f* 10,- and *f* 14,-; ..., namely that every pair of numbers but one satisfies the relation 'four more' (or 'four less', depending on how you look at it). Although recognizing and formulating a relation is a key issue in this project, this is the only occasion that students are asked to do so through pattern spotting. The first two students did not understand the objective, or perhaps they did not see the regularity. Maybe a tabular representation might be better than a horizontal list of numbers? Another possibility might be turning the order around: given a certain relationship, which numbers can you think of that comply? This activity did not catch on and clearly needed to be revised for the next try-out, so it was left out of the remaining work sessions.

adjusting inequality

On the third activity sheet the amounts of 4 and 8 guilders are compared again, this time to adjust the inequality. Six students were asked to suggest ways of making the amounts equal, and to write these actions down. They used strategies of finding the midrange, and of adding and taking away amounts of money. Once again the inverse procedures came to mind very easily. The notations are mostly rhetoric, sometimes syncopated, and two students gave a calculation instead of action language. All in all the students dealt well with the change of perspective and the procedural approach in this activity.

interpreting symbolic expressions

In the next session the order from-description-to-symbolic-representation was turned around. Four of the same students were given a list of descriptions with both an operational and a static conception of the relation between 4 and 8, in symbolic and rhetoric notation. They were asked to fill in the blanks (figure 5.7).

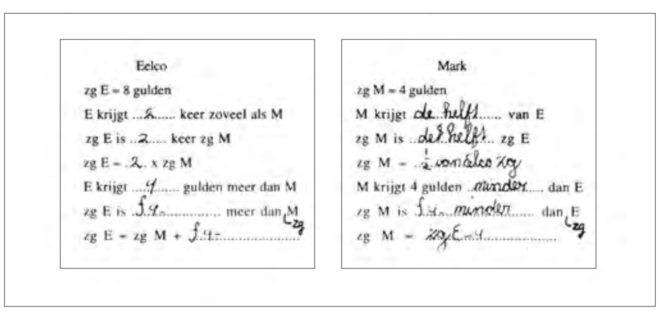


figure 5.7: list of expressions (rhetoric – syncopated – symbolic notation)

This activity did not go well with the first pair of students. They needed a lot of guidance to read the expressions aloud. The notations seemed meaningless to them, even the syncopated ones, in spite of their own constructions the previous time.

numerical support

The next pair of students, Hamza and Fransien, were first asked to give numerical values that agree with the description ‘twice as much’; this translation from a verbal description to a numerical representation went much better. To change their perspective, the students were then asked to write down an expression for the description ‘Mark and Eelco have 15 guilders altogether’, which they did without hesitation. The activity in figure 5.7 was also done better by these students, perhaps due to the numerical foundation. On the last line Hamza wrote *zg M* = *zg E* – *zg M* (as a stand-in for 4 = 8 – 4), where one of the letter combinations appears on both sides of the equal-sign. By removing the last numerical value from the expression, he has turned the expression into an abstract, static entity and really let go of the arithmetical point of view. Seen in this light, it could be a step towards algebra.
The different forms of representation served as a good warm-up for the next activity on playing a game of cards.

playing cards

interpreting and applying formulas, equivalent expressions

The third series of activities was based on the idea of finding out how many points were won in a game of cards. We expected to get more response from students in a problem solving environment, where the shortened notations are more secretive and the aim is to find out how they work. The emphasis was placed on the transfer from a symbolic to a numerical representation: to determine how well students could interpret, calculate and reason with the symbolic representations.

two-letter variable

Based on abbreviations that students themselves used with the marbles, we wrote the formulas as follows: $pA = 3 \times pJ$, meaning that Annelies has three times as many points as Jeroen. The two letters *pA* are to be treated as one variable. The lower case letter *p* provides the connection to the context, but since the students naturally treat the letter *p* as a unit for counting points too, we realize the meaning of the letter *p* is not consistent. This matter must be studied in the next try-out. The expressions describe a relation between two quantities at a set moment in time, namely at the end of a round of play, which means the letters cannot vary with true meaning. The values are predetermined by the situation. For this reason some readers may object to the author’s use of the word ‘formula’, and might prefer something like ‘symbolic expression’. But since sequential activities are based on numerical substitutions and calculations in any round of play, no longer referring to a static situation but dynamic like ‘true’ formulas, we have decided to use the formal term ‘formula’, also for reasons of convenience.

static interpretation

In the interviews we found that Karin and Michel – two students who had also done the pocket money activities – had no trouble to express the symbolic representation in words, but they did so very literally (i.e. in a static way): “the points of Annelies are 3 times the points of Jeroen”. This observation agrees with the earlier analysis of the syncopated expressions in figure 5.7, where Karin tried to symbolize action language. Apparently it is hard to translate a static symbolic formulation into a normal sentence, and vice versa, even in this context situation. The theoretical discussion on the process-product controversy and the cognitive difficulties it causes (see section 2.4) seems to be legitimate and calls for further investigation.

calculating with a formula

Direct application of the formulas in the context gave no unexpected results; students very easily made up numerical examples that satisfy the relation, starting with either variable. We asked one student to say which person (variable) was easier to start with. She replied: “With Jeroen, because then you can do times 3 immediately. With Annelies you have to divide and that is not always possible.” Her answer suggests that the students probably worked with the procedural verbal description rather than the static formula. One of the students had more difficulty to interpret the first two expressions and to produce numerical values; he made the reversal error more frequently than the other students, even repeatedly with the same expression.

reversal error

The last expression, $pH = pA - 10$, gave more problems than the first two. Three students initially interpreted the expression the wrong way around. We asked how the number of points of both players could be made equal; one student suggested that 10 points be added to those of Annelies, to revert the operation ‘– 10’. This implies that the term ‘– 10’ was seen by some students as an action: if 10 points are subtracted from Annelies, Annelies will have 10 less, instead of describing the state ‘10 less than Annelies’. And the advice to test the expression numerically did not catch on, perhaps because students tend to be satisfied with a solution too quickly. Another explanation could be that numerical substitution as a testing device is oversimplified. Not only does it assume the student to understand the concept of variable – that it can be replaced by a range of number values –, it also plays down the complexity of dependent and independent variable, i.e. that the number value of one of the variables follows immediately from the substitution of the other variable. If numerical checking is to be a mathematical asset in the learning program, then students need to have more opportunities for developing a conceptual understanding of variables.

reverse formulas

The second problem on the work sheet also deals with reverse relations; not in words or with numbers but with inverse formulas, where the independent variable has become dependent and vice versa. Students have to decide whether the formula $pA = pH + 10$ is the same as $pH = pA - 10$. This activity also links back to the first pocket money problem, where students did the opposite, namely construct their own inverse expressions to compare two given amounts of pocket money. However, this time the students lacked the numerical support. They now have to check a general, symbolic statement. They can do this at two levels: either by reasoning with the formula directly (which means explaining the formula in words) or by substituting numerical

values for the variables in both formulas to determine the equivalence.
We had hypothesized that students would prefer a numerical method, finding support in concrete numbers. However, the interviews showed that students naturally choose to interpret the relation directly, which in fact concurs with what we saw in the previous activities. Three used the word ‘opposite’ and all four noticed that the other variable was now ‘up front’. These two facts seem to have made it a simple task for the students to produce the inverse formulas for the other two expressions themselves. They did this quite literally, writing $pJ = : 3\ pA$ with the operation ‘: 3’ in the same position as its reverse ‘× 3’. The notations give the impression that these students don’t get much practice in own productions, despite progressive teaching.

pocket money part 2

model use, reverse calculations, restriction problems
In the next phase of the learning trajectory we integrate two approaches to pre-algebraic problem solving strategies with numerical and visual representations in the pocket money context. Concrete materials in the form of plastic coins will enable the students to manipulate the amounts of money, and will also facilitate the wish for a more efficient alternative to drawing circles on paper. The introduction of the rectangular bar is planned as a logical representative for – or model *of* – the plastic coins: familiar, unsophisticated, and much quicker. Moreover, visual representations often have a stronger effect than numbers and words. In a later stage the bar can be a model *for* an unknown quantity, and two bars can express how quantities are related.
The work session started with a simple shopping problem in one unknown which can be solved arithmetically. Undoing a chain of calculations is commonly used as an informal strategy for simple algebra problems. The initial amount of money can be recovered by reverting a series of expenditures. The next work sheet begins with a simple restriction problem in two variables: demonstrate the statement ‘two girls have 16 guilders altogether’ using plastic coins and draw how much each girl has. Students gave 3 or 4 random solutions; the question did not give cause for more, or for a systematic way of finding them. Perhaps the problem can be adapted in such a way that more sophisticated strategies are evoked.

emergent model

The second question was ‘how much do they each have if one girl has 4 guilders more than the other?’ Hamza formed two equal portions by allocating twelve chips one by one, and then allocated the remaining four. He laid them down neatly in two lines to show that the difference was indeed four. He also drew the circles neatly in two horizontal lines on the work sheet. We can imagine that the model can be introduced to this student naturally, giving an argument like ‘drawing a rectangular bar is easier and quicker than separate circles’. Fransien clearly indicated that the coins were silly and that she didn’t need them; she drew the circles randomly and to her they were not functional at all. However, in the next session when the activity was reviewed, she also drew the circles in a systematic way (figure 5.8). In both interviews the students understood well what the effect is of giving a second restriction, for example ‘one girl has three times as much as the other’. The two restrictions combined resemble an informal system of two linear ‘equations’, i.e. in words. But drawing 12 and 4 circles did not bother or bore the students; they were rather surprised by the suggestion to use a more efficient shape. And when they were asked to show how much money the bars represent, two students suggested making unit boxes (see figure 5.8).

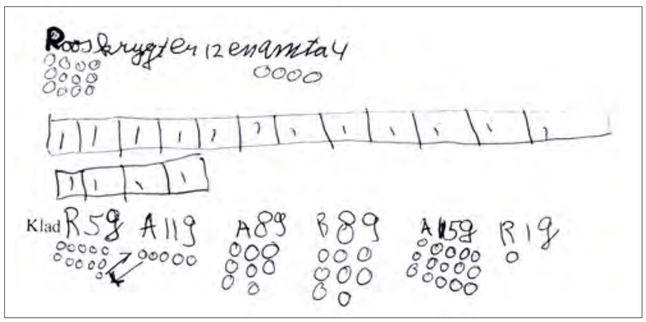


figure 5.8: drawn coins

This development turned out to be a hindrance for the next step of drawing a bar to represent an unknown quantity: if the total value of the bar is not known, it is not possible to draw the boxes. In other words, the hypothetical introduction of the model was not as suitable as expected. Trying an alternative approach, we introduced the rectangular bar to Karin and Michel sooner, to give them visual support working with formulas in the context of playing cards. Not only did it mean skipping the preparatory phase of working with concrete materials, but it also accelerated the application of the bar as a problem solving tool. Translating the multiplicative formula $pA = 3 \times pJ$ to a visual representation using two rectangular bars (one three times as long as the other) posed no problems, but the additive formula $pR = pJ + 5$ was slightly confusing for both students, especially the translation of ‘+ 5’ to ‘5 more’. Initially Karin interpreted the relation exactly the wrong way:

Researcher: Who has more points?
Karin: Rosemarie, because you have to add 5 points to make it equal
R: Add 5 points to whom?
K: To Rosemarie.
R: Really?
Karin looks again and shakes her head.
S: No, to Jeroen.

reversal error

Is this the reversal error playing up again? It seems to be the same process-product controversy that we encountered before in the first pocket money activity. And although both students subsequently drew the bars for the second expression correctly – even the additional part worth 5 points – they misinterpreted the last expression again! (see figure 5.9). In the next session they immediately noticed their mistake, but they could not explain why they had done it wrong before.
Karin and Michel also solved the pocket money restriction problems (on linear relation in two variables), starting with ‘two girls have 16 guilders altogether’. The bars had a double role: to bring out the relation between the quantities at first and to facilitate the solution process later. The students were asked to draw two bars, knowing the total value is 16 guilders but not knowing which bar corresponds to which girl. They easily generated a table of possible values. The second restriction followed on the next activity sheet, and Karin and Michel immediately identified which bar should correspond with which girl. We asked them to show in their drawings the second relation ‘6 guilders more’; they both drew a vertical in the largest of the two bars and wrote f6 in it. Finding the exact amounts of pocket money was now an easy task.

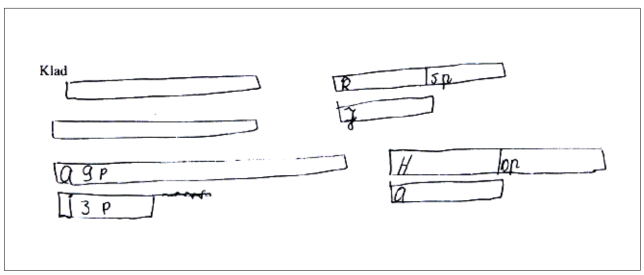


figure 5.9: reversal error demonstrated by rectangular bars representing $pH = pA - 10$

In the following task Michel and Karin were asked to find the solution if instead of ‘6 more’ the second restriction was ‘3 times as much’. Karin divided the larger bar into 3 equal parts and shortened the other bar to one such part, and she then announced that she could solve it with of the table of 4: 4, 8, 12, 16 (see figure 5.10).

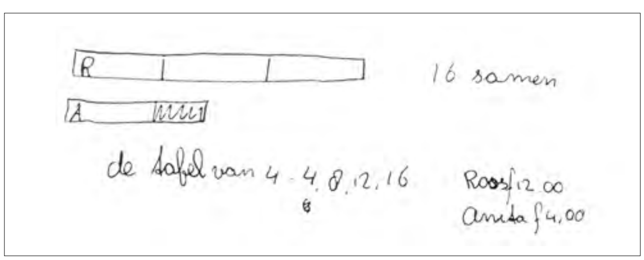


figure 5.10: solving a restriction problem

problem solving obstacles

The third shopping problem was another restriction problem, consisting of two parts:

1. Michael and Rose have equal amounts of pocket money, and then Michael spends 3 guilders; how much do the children have?
2. Rose now has twice as much money as Michael; how much did they have at first?

An infinite number of solutions is possible in part 1, but in part 2 only one combination satisfies both conditions.
Hamza and Fransien wrote down a few solutions to part one, including rectangular bars fitted with vertical lines. The bars did not function as a model for the unknown quantities, since they were drawn to correspond with the exact quantities. However, the bars in part 1 were no longer applicable for the second restriction unless the scale was adjusted. For example, Hamza had drawn bars with 12 and 9 boxes respectively. These bars stood for 12 and 9 guilders in part 1, but they had to represent 6 and 3 guilders in part 2. Both students tried in vain to solve the problem mentally, constantly deterred by their drawings. Moreover, they usually did not even remember the question they were answering! Unfortunately there was not enough time for another problem in a different context.
Apparently a system of equations represented in this way is hard to contain mentally, and so is translating the verbal information into mentally represented mathematics. In the next design more attention should be given to mathematizing simple problems first in order to stimulate the use of tables and diagrams, followed by systems of restrictions which the students have already symbolized, and only then context problems.
Karin and Michel drew correct bars for the first restriction, but then they got stuck. Karin had forgotten that the answer to part one is a possible solution but not necessarily the final answer. Michel could not change perspective to comply with the second restriction; he could not perceive that his answer was correct for part 1 but not for part 2. Perhaps if they had written down more solutions for part one, and had thought about there being infinitely many, this confusion would not have occurred.

drawing unknown quantities

Let us consider for a moment how students dealt with drawing unknown quantities. We expected students to argue that you can only draw a bar if you know how long it is, but we encountered no reluctance in this respect. Only Fransien explained that it is not always convenient to use bars: “Sometimes they are hard to draw, because you don’t have enough information.” It also depends what the first relation is: a multiplicative relation is easier to draw than an additive one. The relation ‘3 times as much’ enables you to draw the bars in the correct ratio; only the scale is still unknown. If you only know ‘they have 16 altogether’, the lengths of the bars are indeterminate. Not surprisingly, the bars drawn by the students were hardly ever good representations of the solution, and the students were quite annoyed with this. Perhaps the multiplicative relations are more suitable to start with, and additive restrictions may be useful to start a discussion with the students why drawing the bars is sometimes difficult. We also expected that students might assume a certain value represented by the bar to avoid thinking about an unknown quantity, and then later adjust it. Two students had the habit of drawing vertical lines in the bars to create unit boxes, and they worked only with known numbers. For instance, they first drew bars that corresponded with the first relation, and then corrected their drawings to comply with the second restriction. It was clear that they wanted the bars to be an accurate model of the quantities rather than a rough sketch used for problem solving. In fact, we found that the students solved the problem in their mind (usually with a lot of help) and then they validated the solution by drawing the correct bars for it. The development from ‘model of a situation’ to ‘model for reasoning’ was not realized with these activities, and has been reconsidered in preparation of the final design cycle.

barter

comparing quantities, shortened notations, changing meanings of letters, representing a relation, reasoning, equivalent expressions, organizing information schematically

The activities on barter form the final phase of the learning trajectory. They integrate nearly all the pre-algebraic abilities elaborated in section 5.2.2. Although these activities require no specific knowledge from previous ones, we expected students to profit from the exploratory activities on marbles, in particular on conflicts with fractions and reasoning. However, due to lack of time we could not work with those particular students again except for one, Rosemarie, which means that we cannot test the effect of prior knowledge. Sidney, Wendy, Angelique and Jeffrey participated in the very first constructive activity on marbles, and Linda was entirely new. Michael and Christa caught on to the activities so slowly that we decided to stop with them after two sessions.

situation

The activities are embedded in a story on barter in a fishing village in New Guinea, where Marcus, the principal character, encounters different types of trading conflicts. The story and an additional sheet with trade terms in picture form – for example, 1 fish on the left and 3 apples on the right – provide students with a consistent system of trade relations. In addition the students had access to cardboard chips of all the goods, with which they could manipulate the trade terms physically. Most students worked only with the picture sheet, though; they had no need for concrete materials. In the remainder of this section we describe the mathematical contents of the activities and report on the noteworthy findings of agreement and disagreement between the conjectured and the actual learning process.

notation

The first written assignment consists of writing down the trade terms in a convenient

way. All eight students chose a rhetoric description, as we had expected, but with no mathematical symbols at all. The instruction to write it down shorter resulted in the use of letters or abbreviations – sometimes clumsily or inconsistently – but the symbols + and = were still not applied. In most of the interviews we tested the students' understanding of the relations orally by asking for multiples (larger or smaller trades with the same goods). Just like the previous sessions we expected students to organize their answers in a table or something similar, but again we found that they needed a nudge in this direction. The students also constructed some of their own trade terms by multiplying, adding and substituting the given ones; this activity enabled the children to work at their own speed and level, and gave us the opportunity to observe each student's understanding.
After a few sessions Sidney and Jeffrey demonstrated growing courage and confidence where notation is concerned: a freer use of symbols (for example the symbol ⇒ for 'is more than'), explanations in terms of consecutive trade terms, and a broader application of tables. However, at other times they chose for a less sophisticated or even non-mathematical notation. For instance, Sidney once suggested inverted comma's would be more efficient than writing multiples of the trade term. In other words, more confidence does not necessarily mean better understanding. The girls admitted that they preferred to explain their reasoning in words rather than with the trade formulas 'because it is more clear that way'.

meaning of letters

For the purpose of explicitation and reflection, we made a detour in the second session to discuss the meaning of letters. We confronted students with a new interpretation. The trade terms students constructed express how to trade goods fairly. For example, $1f = 3a$ represents the trade '1 fish for 3 apples'. The *f* stands for 'fish' and the *a* for 'apple', and the students encountered little trouble making a table with the number of goods. Then we asked the students what the price of a fish would be if an apple were to cost a quarter, and to make a second table with possible prices for one fish and one apple each.

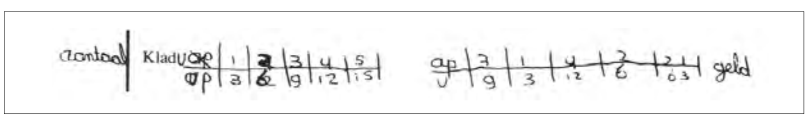
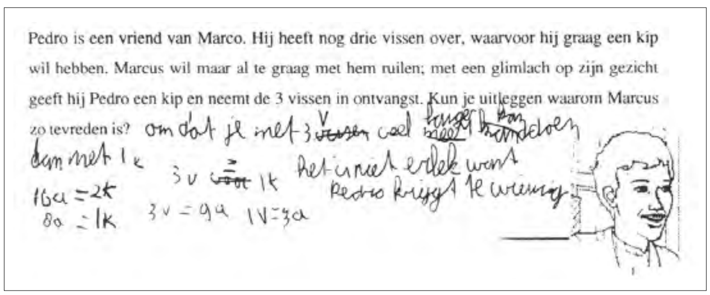


figure 5.11: tables for number of goods (left) and for prices (right)

Comparing the two tables, students quickly saw that the numbers are just the other way around, but explaining it was more cumbersome than we had expected. Moreover, in cases where we talked about the two situations first, and only afterwards asked students to make the tables, students sometimes still got confused. Figure 5.11 shows how Linda got it wrong at first. Angelique was not able to generalize the phenomena to a new situation; she needed to check another example numerically. And unlike we conjectured, we found that some students showed better reasoning about this change of perspective in general terms, without the numerical 'evidence'.

reality conflict

One of the essential issues of fair trade is that the value of the goods is determined by the system of trade terms, and not by bargains, taste or damage. In other words, every day reality will sometimes interfere with the mathematical interpretation of the context. As a result of this conflict, some of the activities on reasoning did not prompt students to make goods comparable by way of consecutive substitutions. Jeffrey, for example, insisted on giving a personal point of view: 'Marcus is happy because 3 fish last you longer than 1 chicken' (see figure 5.12). Although students are encouraged to use their common sense as practitioners of mathematics, it is of course essential that they learn when to overrule subjective arguments.


Pedro is een vriend van Marco. Hij heeft nog drie vissen over, waarvoor hij graag een kip wil hebben. Marcus wil maar al te graag met hem ruilen; met een glimlach op zijn gezicht geeft hij Pedro een kip en neemt de 3 vissen in ontvangst. Kun je uitleggen waarom Marcus zo tevreden is? om dat je met 3 v veel langer kan doen dan met 1 k

3 v = 1 k

het is niet eerlijk want Pedro krijgt te weinig

16a = 2k
8a = 1k

3v = 9a
1V = 3a

figure 5.12: personal opinion versus mathematical interpretation

symbol for the unknown

The next group of activities consists of finding combinations of goods that amount to a given value, and developing an efficient strategy for it. The first problem asks for possible trades of both apples and bags of flour for a total of 10 fish. We instructed the students to write down the trade term first, to summarize the problem. In the current situation it appeared natural to investigate students' own proposals for symbolizing an unknown number. We asked the children to make up a symbol to represent the yet unknown numbers of apples and bags of flour. One student suggested a horizontal line, the others proposed a dot; they did not respond to the idea of a question mark or a box.

systematic exchange

We conjectured that the most frequent strategy for finding the combinations would be trying at random. Other strategies could be starting with a maximum number of one item and then systematically exchanging one for the other, or by converting all the goods to apples (the smallest value) and grouping them. We found that three of the six students started at the end of the range, and worked systematically in a table to find the other solutions.

In figure 5.13 we see how Angelique starts with the case of only apples – 10 *f(ish)* = 30 *ap(ples)* – followed by the situation of only bags of flour – 30 *ap* = 5 *fl(our)* – and then all possible combinations of both items. (The Dutch abbreviation for 'flour' is unequivocal (the letter *m*), but the abbreviation for 'apple' and 'potato' in Dutch is the letter *a*; the choice for 2 letters instead is made by the pupils themselves.) Asking her how she found the combinations, she explained that for every 6 apples you get an extra bag of flour. The table on the right is the result of the discussion afterwards, where we asked students for a more efficient notation. Linda had the right idea of taking one more bag of flour each time, but instead of subtracting 6 apples immediately – using the previous combination – she calculated the total number of apples over and over again.

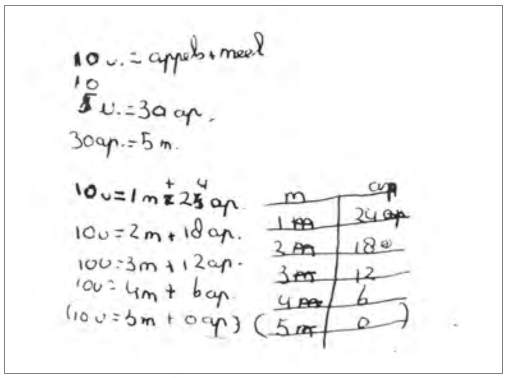


figure 5.13: systematic way of finding combinations

ratio tables vs. combination tables

At this point the table has become a common representation for the students, but the variety and the interpretation is worth mentioning. Some tables included only the numbers of apples and bags of flour (like in figure 5.13), others also mentioned the number of fish. It is important to realize that unlike the ratio-tables (representing ratio of trade) dealt with until now, the tables for this problem contain numbers of goods that are to be combined rather than traded! This did not deter the students, perhaps it was not even noticed. But we found that when it was revised in the next session, some students were puzzled at first. Judging two examples of Linda's notes, she must have been confused. Asked to write an expression for one of the combinations in her table, she wrote $f = 10 = 3m + 12\ a$ at first, and then changed it to $f = 10 + 3m + 12\ a$. She could not account for her train of thought, but we can attempt a hypothetical explanation. The part $f = 10$ might state the number of fish if the letter f were to refer to the number of objects instead of the object itself. This would point to a spontaneous change in perception of the letter variable. The decision to replace the equal-sign by a plus may be the result of a faint recollection that the items in the table were to be combined. There are no clues for an interpretation of the expression as it stands. The second indication of Linda's misconception is illustrated by two tables which she made for the next activity. Both tables were intended to contain combinations amounting to 10 fish, but Linda made the mistake of treating the first table as a ratio-table! She doubled her combination of 1 chicken, 8 apples and 2 bags of flour, without realizing that this would not be a valid trade.

negative terms

It had not been anticipated that the situation of systematic exchange of goods might be a good opportunity for investigating trade formulas with negative terms. It was a moment of inspiration that we asked students to continue exchanging apples for a bag of flour, even though the value of 10 fish had been reached. This means continuing the pattern from the trade terms $10f = 4fl + 6a$ and $10f = 5fl + 0a$ to the term $10f = 6fl - 6a$ and so on. The children explained that this means the customer has to pay the 6 apples back later, which was rephrased as ‘debt’. Linda constructed the expression $10f = 7m - 12a$ by calculating how many apples the bags of flour are worth (42) and then subtracting 30 (which is what the 10 fish are worth) to find a debt of 12 apples. She is able to solve this problem free of the context. Only Jeffrey and Angelique have trouble to understand the idea of negative possession; they suggest taking one bag of flour less to make the trade fair. This topic was concluded with an interactive reflection on the minus sign and its relation to the concept of debt.

equivalent expressions

The expression $10f = 6fl - 6a$ also enabled us to discuss the equivalent expression $10f + 6a = 6fl$ representing the situation where 6 additional apples are paid from the start. Although Jeffrey immediately noticed the expression was reversed, he seemed confused by the change of perspective; talking about an analogous situation with small change at a cashier did not seem to help. We also added in a short activity on halving and doubling the expression, which went well with all the students as expected. In general the students’ work at this activity supports our conjecture that students are successful at manipulating and substituting symbolic expressions when it has meaning for them.

restriction problem

The next activity sheet deals with a problem on restrictions:

> Having traded the 10 fish for bags of flour and apples, Marcus decides to go in search of chickens. He wants to take home at least two bags of flour, though. If he chooses one chicken, how many bags of flour and apples will he have left? And how many of each if he chooses two chickens? (Be aware which solutions from the previous problem you may use).

The children need to combine a number of facts: at least two bags of flour, 1 chicken is worth 8 apples, 1 or more chicken(s), number of apples and bags of flour remaining. We found that the students got stuck trying to remember everything in their head, so we suggested that they write down the combinations of flour and apples from the previous work sheet as well as the desired trade term that includes chickens. The clearest signal of students’ shortcomings at complex exchange problems like this one is their trouble to remember the actual question.The students needed considerable support to get started and to check their course along the way. If the problem demands a complex series of substitutions, the students lose themselves in the arithmetic, often making mistakes and even losing confidence in their approach to the problem. This was certainly never intended. In fact, the purpose of such complex reasoning problems is to instigate mathematization and the application of models or diagrams to facilitate the solution process. The results in this phase of the research project suggest that students will need to practice and develop these abilities before they can succeed at solving complex exchange problems more independently.

generalizing

The last few tasks on the barter activity sheets are concerned with reasoning more generally about the trade terms to make predictions. First of all we used a debatable answer of four and a half apples in one of the problems to raise the issue of why barter may be inconvenient. There was a discussion of alternative means of payment, and their advantages and drawbacks. The written activities thereafter are based on the change-over to bags of salt, starting with the question what should be the value – expressed in bags of salt – of an apple so that all the values would be whole numbers. It was expected that a few students might notice immediately that the fraction can be avoided if the scale is doubled, in other words, if an apple were worth two bags of salt. However, this did not happen. Some students tried one or two values and then waited for an instruction; Wendy and Sidney were more inventive and seemed to realize somehow what the limitations were. For example, they found out that an odd number will always give fractions. In all cases the students were guided towards the answer because it took much longer than we expected and again the students had trouble remembering all the restrictions. The next task was to write down the value of each item in terms of bags of salt. The aim of course was to get students to first express the values in terms of apples (which they had all discovered doing the different activities), and then just double the values. This kind of higher order reasoning was indeed the aim of the task. We were a little surprised to see that a couple of students failed to see this shortcut, even after a while; they returned to the loose sheet of trade terms and performed many redundant calculations! This type of activity will certainly need to get more attention in the learning strand.

unforeseen complications

In the activity that followed it became clear that the switch to bags of salt caused more pain than pleasure. Children are asked to make a shopping list of items that amount to exactly 50 bags of salt, and it satisfies many conditions of RME: it is a challenging, open problem that instigates mathematization and which allows for many solution strategies of different levels. However, the two nouns in 'bag of salt' in Dutch start with the letter 'z', and therefore students abbreviate to *z* or *zz*, which is hard to distinguish from numbers involving the digit 2. Another problem which we had not foreseen was the confusion caused by the 'new' values. By this time the children knew the trade terms by heart, but all the numbers are doubled if the smallest value is represented by a bag of salt. In fact, the children continued to think in terms of apples and the extra step of transforming to bags of salt turned out to be unpractical. In the next design we must make sure that the switch to a new currency is the final exercise, because the purpose is not to calculate at a concrete level but to instigate higher order reasoning.

The final task on barter asks students to predict the influence of a change in prices.

The problem goes as follows:

> In another village 100 miles away a heavy storm has destroyed part of the harvest. The prices of all the goods in that village are twice as high as in Marcus' village. Explain whether a chicken is also worth a bag of flour and two apples there.

We hypothesized that most students should be able to explain the consistency of the trade terms, and students in doubt can check their presumption by calculating the actual values. The interviews point out that some children have trouble distinguishing between absolute value and relative value; when they say that the value stays the same, they meaning the relative value, that is, the trade term. Apparently the concept of value is more complex for these children than we anticipated.

mathematizing a cartoon

As finishing act we gave the students a cartoon on barter in the stone age (see figure 5.14). Due to lack of time and earlier experiences of overestimating the students' capacity of keeping a helicopter view, we decided to structure the activity by instructing the students to write down the trade and how much each character possesses afterwards, and to decide if it is fair.

figure 5.14: stone age trading

But first we read the cartoon together in a role-play. Judging from the reactions of the students, some of them already had a hunch. We expected students to be able to mathematize the story using convenient notations, and we anticipated that the students, supported by their own notes, would find the clue on their own. Sidney showed the most assertive attitude: he clarified his choice of abbreviations, he differentiated between the equal-sign (for fair trading) and a colon (to announce the possessions), and he was very confident that he could explain the cartoon. Two of the other students were deterred by the context for a moment; they commented on the value of the club and the wheel, instead of on the trades. Generally speaking the children had developed enough awareness of fair trading and a suitable mathematical approach for it to complete this final task.

5.2.4 Evaluation and reflection

During and just after the work sessions an evaluation of the hypothetical and actual learning trajectories informs the researcher how to continue the study. The most significant products of the researcher's reflections are the formation of local theories (conjectures) and the consequent adaptation of instructional activities and/or design materials to initiate the next research cycle (see section 4.2). In this light the most relevant findings from the orientation phase are summarized according to topic – mathematical content, mathematical activities, target group – in random order of importance. These conclusions are based on a qualitative analysis of student work and interview protocols.

mathematical content

It is difficult to conclude one-mindedly how the students performed; some parts of the mathematical content were more successful than others. For instance, comparing quantities and adjusting inequality are activities that students did well. They had no trouble inverting operations and changing their perspective in context situations, which precede the more complex skill of inverting formulas. Tasks of doubling and halving trade expressions were also done very well.

switch between representations

Evaluating the activities on switching between representations, we observe that verbal descriptions and (partially) symbolic representations are dominant in the activity sheets. To bring a better balance into the program, more practice is needed on transforming a verbal description into a tabular or numerical representation and vice versa. One of the activities which might become more accessible in this way is pattern recognition. The transition from a series of number pairs to a table might enable more students to recognize and continue the pattern. The only setback to giving or suggesting a table is, that students are no longer required to organize the information themselves. Giving the table in advance means taking away an opportunity of self-induced schematization.

static-dynamic

One result of the case studies which stands out is the tension between a static and a dynamic, procedural perspective of relations between quantities. It appears that students naturally think in terms of procedures rather than static situations. First, the translation from a static, symbolic expression to a dynamic statement in the vernacular was problematic, and vice versa. In fact, the switch from a procedural description to a static, symbolic expression caused students to interpret the expression the wrong way around (the reversal error). For example, formulas which involve a negative term, such as $pH = pA - 10$, tended to confuse students. When they were asked to interpret the expression globally (‘which person has more points, Henk or Annelies’), they were misled by the term ‘– 10’. The student answer ‘*pA* is 10 less than *pH* because you subtract 10 from it’ implies that ‘– 10’ is seen as an action instead of a state. And third, most students succeeded at using a formula as a calculating device but they were reluctant to check their solution by substituting both values back into the formula. For instance, our hint to substitute values for *pH* and *pA* to determine which must be more did not catch on.

reversal error

The reversal error – where a relation between two quantities is conceived exactly in the opposite way – in an expression is not uncommon in the learning of algebra. It is, however, our impression that this phenomenon occurs in specific situations. We have therefore decided to investigate in the next experiment whether or not reversal errors are especially common in situations of premature (mechanic, meaningless) use of symbolic notations, where the learner struggles with the procedural-static duality of expressions. This struggle can, for instance, be recognized by a misconception of the equal-sign, an unnatural, static verbal description of an expression or a poor global interpretation of the expression.

global reasoning

Reasoning globally about relations and unknown quantities ran aground not only on account of the product-process dilemma, but also due to students’ arithmetical shortcomings, conflicts between mathematical and every-day-life interpretations, or misjudgement on behalf of the designers. From the barter activity sheets, for instance, the task on reasoning about numbers of goods opposed to values of goods and explaining why the relation is inverted was more difficult for students than we had foreseen. Numerical examples did not bring relief, either. Also the prediction whether doubling the values would influence the equivalence relations turned out to be too abstract for students.Even though they have not been productive, we still consider activities for developing higher order reasoning and the concept of variable sufficiently important to give them higher priority in the next design.

shortened notations

The case studies point out that students are naturally inclined to use verbal representations. When students were asked to write it down in a shorter way, they abbreviated nouns and names but they did not translate operations like ‘more than’ or ‘times as much’ into mathematical symbols. The abbreviations suggested by the children are not always efficient, consistent or even desirable from an algebraic point of view. For instance, symbolic expressions often included a literal label (unit). In some cases students did not understand their own constructions in the next session. Some students openly showed reluctance towards shortened notations. Generally speaking we can say that for these students, experience with constructing one’s own symbolizations did not lead to a better understanding of (partially) symbolic expressions later on. The activities on pocket money, in particular, made clear that if shortened notations are not logical nor necessary, students will not respond. These results emphasize the value of sense-making activities that allow students to develop their notations in a natural, purposeful way and at their own pace. However, this ideal case may not always be possible; if different roles and meanings of letters is a mathematical goal, it may be necessary to limit the students' freedom and provide more structure. The next experiment is expected to bring more clarity on this issue.

schematizing

At first the students were not inclined to use efficient notations like tables and other organizational tools. As the interviews progressed, the students became more skilled at symbolizing and schematizing their solution (trade terms, tables, bars), but not to a level of problem solving. Schematic problem solving proved to be a major obstacle for students. Although the table as an efficient representation – a calculational tool – became more and more standardized, neither the table nor the rectangular bar developed into a tool for problem solving. For example, a task on finding all combinations totalling a certain value can be done by random trial-and-error, or by using a structured approach: repeated exchange of goods. None of the students thought of such an advanced method, or recognized its value when they heard of it. After a while students adopted the strategy but only with artificial understanding. We found that one session later, students were confused by the fact that the numbers did not comply with the trade term. They had apparently forgotten that the numbers represent a combination, satisfying a given total value, rather than a fair trade. There is one positive result: continuing the exchange beyond the boundaries (all items of one type) was a natural way to introduce students to expressions with negative terms ('debt').

rectangular bar

The next content item we evaluate is the conjectured development of the rectangular bar as an emergent model. According to RME theory, mathematical activities should ideally facilitate the emergence of schematic representations and models. It has already been said that the bar functioned sufficiently as a symbol for the relation between two variable or constant quantities. We observed that symbolizing a multiplicative relations (like 'three times as much') was easier for students than an additive one (such as '6 more than'), judging by the reversal errors that were made. However, at a higher level, the visual representation was not used by students as a tool for mathematical reasoning. First of all, the activities which dealt primarily with using the bar for this purpose, were found to be solved mentally just as easily, which is what students did. Secondly, although students were not opposed to drawing a bar of indeterminate length, it is still the most likely reason why it did not satisfy as a model for mathematical reasoning. We found that students were displeased to draw a picture which they knew was probably not 'correct' where exact length is concerned. These students were unable to shift from the concrete level – precise, scaled drawings to the abstract level – global, indeterminate sketches. We believe that the hypothetical development from *model of* (acting in a specific situation) to *model for* (mathematical reasoning) was too short; students were not given the opportunity to grow accustomed to the idea of indeterminate magnitude and then formalize their understanding progressively. It is essential that in future designs the rectangular bar develops naturally (not enforced) as a problem solving tool, or otherwise it must be considered inappropriate.

restriction problems

Students' higher order reasoning ability is a direct consequence of their (limited) competence at schematizing and recognizing patterns. Complex trade problems involving restrictions, where quantities have to be made comparable, failed because of simple calculating mistakes and low order strategies. Many students continued to use trial-and-error approaches because they had not acquired schematizing as a tool for mathematical reasoning. Moreover, they tried to remember all the conditions in the problem and usually resorted to mental arithmetic. Perhaps in the eye of the student it is not acceptable to make draft notes to support your thinking process. This result suggests that a more suitable approach may be to delay complex problem solving and give priority to subservient, partial abilities like working with simple patterns and restrictions first, as well as learning to write down intermediate steps of the solution procedure. Both issues have been incorporated in the design adjustments.

classroom norms

The difficulties with complex problem solving described above can be partially ascribed to classroom norms and to students' arithmetical background. The students were found to be very reserved about making their thinking process accessible – verbally, on paper, to the interviewer and to each other. It seems these students were not as accustomed to communicating their strategy as we had expected. As a result the more investigative problems hardly stood a chance, relying first of all on the student's attitude to attempt mathematization, and secondly on classroom interaction to propel the solution process. We recommend, therefore, that mathematical discourse and an active problem solving attitude – including courage and stamina – be given more attention in the arithmetic curriculum. In some cases the problem situations were inappropriate; for example, if a problem had already been solved, students were not motivated to try again using a different approach.

map of pre-algebraic elements and abilities

hypothetical vs. actual learning process

Let us consider once more the map of pre-algebraic elements and activities that constitutes the hypothetical learning trajectory. Figure 5.15 shows the 'hypothetical' map first, and the 'actual' map as observed in the case studies underneath. The 'successful' parts of the hypothetical learning trajectory have been put in a frame. Since 'global view' and 'reasoning' have only been partially achieved, their frames are dotted. It appears that students perform better in the right hand stream than the left, i.e. that pre-algebraic reasoning is more accessible for students than symbolizing and schematizing activities. The two streams have therefore not developed simultaneously; in fact, the slow development of schematizing competence has impeded the development of higher order reasoning. On the other hand, some students succeeded

to solve restriction problems mentally, which means that advanced schematizing is not a prerequisite. In other words, these competencies can be independent endpoints of the schematizing and reasoning streams respectively, where schematizing as a tool can support the reasoning process if necessary.
In spite of the limited interaction of the symbolizing and reasoning streams and the disappointing results of some of the parts of the mathematical content, we have chosen to maintain the core of the mathematical content for further research. Furthermore, in our perception a dynamic development of reasoning and symbolizing competence, where each spurs on the other, is still the best option. The second map shows how the results of the first try-out have been incorporated to improve the hypothetical learning trajectory.

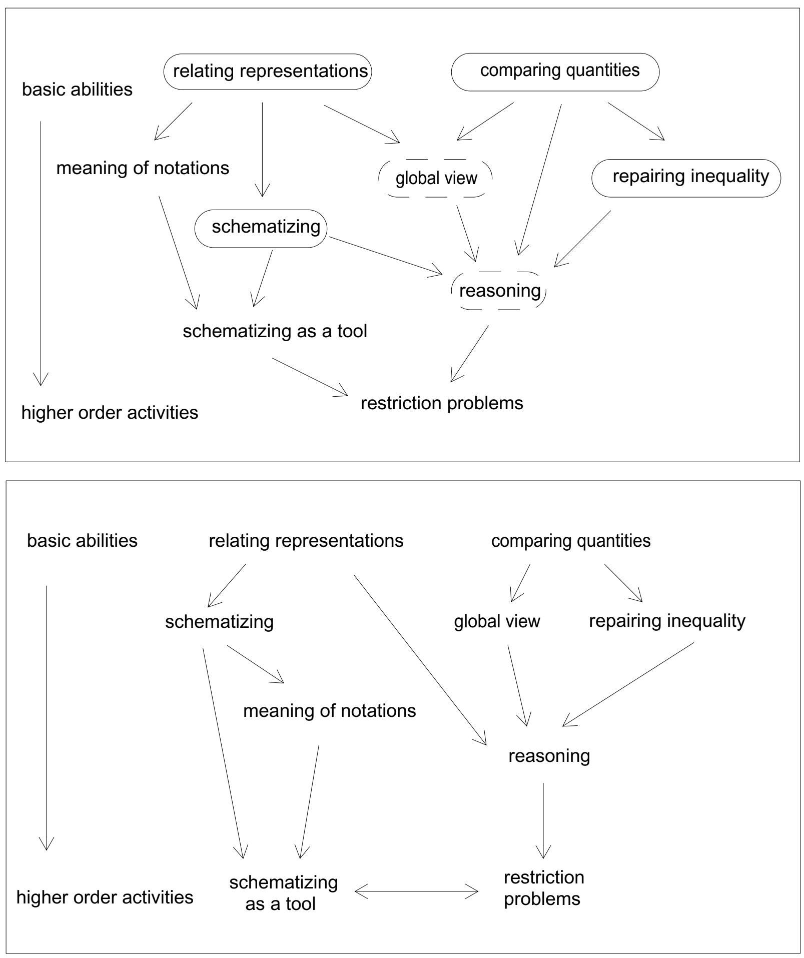


figure 5.15: original map (top) adjusted (below)

mathematical activities and contexts

Having reconsidered the mathematical content of the learning strand, we proceed to reflect on feasibility of the contexts, situations and problems. In other words, have the mathematical goals been translated appropriately into mathematical activities, and which changes are needed?

pocket money: meaning of letters

The contexts of barter and playing cards were received well by the students. They are especially suited for problem solving: comparing quantities, complex substitution problems, finding unknown values, and higher order reasoning. The problems on pocket money need to be adapted to become more meaningful, especially the ones where shortened notations constitute the mathematical goal. It is not a mathematical goal in the program to formalize students' letter use, but to improve their awareness. One part of this awareness is confronting students with different meanings of letters: letter as an unknown number, as a referent to an object (abbreviation), or as a referent to a property of an object (magnitude). In order to make symbolizing a more natural and intuitive activity, the instructional materials have been revised to create problems in which shortened notations provide a challenge (on the basis of 'secrecy') or an obvious advantage for solving problems. In students' own productions the teacher is responsible for making students aware of arithmetical conventions, efficiency and consistency.

equivalence and fair trade

The introduction to the marble context was already adjusted during the experiment. The marble problems based on comparing values serve well as an introduction to fair trading, but the complex ratio tables (on compatibility of numbers of marbles) might better be left out, unless the children are given more time to investigate the problem. More attention should be given to the concept of consistency of trade terms, especially in situations where a conflict arises with reality. The interviews have shown that students are not always capable of distinguishing between mathematical and every-day-life norms, and more awareness may be helpful in these cases. The complex reasoning problems towards the end of the learning trajectory need more structure to guide students towards reasoning strategies, because in principle we continue to believe in their mathematical value.

target group

The mathematical attitude and arithmetical capacities of the students are discussed as well because we feel that these have been an influential factor. The students in the target group were a grade 5 class at primary school. The pre-algebra activities are based on the assumption that students have mastered the basic arithmetic skills and are familiar with the RME attitude towards learning mathematics. Generally speaking their arithmetical competence was disappointing; their work was untidy and full of mistakes. Most students were reluctant to do rough work on paper even when mental calculations failed. We were also unpleasantly surprised to see that the majority of students did not live up to the RME standard. They were not used to constructing their own mathematics, working together or presenting and explaining their solution process to each other. We had expected a more assertive, responsive and venturous target group, but perhaps the learning environment made the students shy and insecure. Certainly we assume to get more response and a greater variety of solutions in a real classroom situation, where the students can interact. As a consequence two important points for revision are: to create more opportunities for weaker students to practice basic skills, and to make mathematics communication (written and verbal) a key issue. In addition we consider choosing slightly older children for the next try-out, to minimize arithmetical set-backs.

5.3 Additional try-outs for orientation

In the previous section we described the first try-out results of the pre-algebra learning trajectory for primary school students. The learning materials cover only part of the mathematical content; the remaining part is intended for students in their first year of secondary school. The continuity of the learning trajectory is conjectured in two different ways. Firstly, a progressive formalization of the pre-algebraic abilities that have been built up so far has been combined with a new substrand on systems of linear equations in the instructional unit *The Fancy Fair* (see section 5.3.2). Secondly, in accordance with the research intentions, a prototype substrand has been developed around historical problems on early algebra and linear equations (see section 5.3.3). In addition the case studies suggest a rather optimistic conception of the target group's mathematical capacities, and so we decided to conduct a small investigation into algebraic susceptibility, which we discuss first.

5.3.1 Mathematical starting level

It was considered useful and important to collect information on the actual pre-algebraic possibilities and limitations of the target group before revising the student materials. One of the reasons for this experiment was to get feedback on students' own knowledge and capacities in the following domains: organizing and visualizing mathematical information, making assumptions when information is missing, reasoning and calculating with undetermined values, and dealing with multiple restrictions. This information should enable us to:

- determine the conceptual starting-point of learners in grade 6, primary school;
- gather student productions for the next thought experiment;
- try contexts and situations for new mathematical activities;
- test activities for feasibility of progressive mathematization;
- investigate the learners' boundaries of algebraic reasoning.

Making up your own problems! is a collection of open and open-ended problems, some of which have to be completed by the students.

The problems can be solved at different levels; the weaker students get support from the closed questions in the beginning, whereas the better students have the opportunity to reason at a more formal level. Figure 5.16 illustrates how such problems can be compiled. We expected that comparing students of different ages would give a wider range of results and thereby a more complete view of the situation. For this reason the activities were tried out in an average grade 5 class and an above average grade 7 class. The tasks were done individually, as a written test. There are no observations or recordings, merely the paper work. Considering the fact that the results are of marginal interest for the research project, we confine ourselves to a brief summary.

How far away from school?
Danny lives 4 times as far from school as Michael.
1 Can they ride to school together?
2 Draw a map to show how they could ride from home to school.
3 If Michael lives 1 km away from school, how far away from school does Danny live? And how far away from Michael does Danny live in that case?
4 One afternoon Danny says to Michael: 'It takes me 5 minutes longer to ride to your home than it takes to ride to school.' Can you say anything now about how far apart their homes lie? And about their location?
5 Now make up some questions about how far away you and your biking friend live from school. Write down the answers, too.

Family riddle
Father is 5 times as old as his daughter Pam, and 4 times as old as his son Robert.
1 Who is older, Pam or Robert?
2 How old can Robert, Pam and Father be?
3 How old will Robert be at least? And how old can he be at most?
4 Now make your own family riddle and solve it.

figure 5.16: open-ended problems

evaluation

Symbolizing: In both classes the students barely used visual means to investigate the problem situation, unless the problem specifically asked for it. They used mostly rhetorical notation, in spite of having encountered other forms of representations in their mathematics lessons. The primary school students sometimes used tables.
Dealing with the unknown: The students were quite willing to make assumptions, the girls more so than the boys. Another significant observation on gender differences is the fact that boys are reluctant to answer indeterminate questions, but they show more courage and imagination in their own productions. The girls tend to produce questions similar to the given ones.
Algebraic thinking: We observed a reasonable difference in algebraic susceptibility between the two classes. Not surprisingly the older students performed more venturously and better at abstract reasoning, changing perspective and solving restriction problems. In terms of the problems in figure 5.16, the grade 7 students have a wider scope of the possible solutions (although not one student suggested that initially all possible locations of one home form a circle with the other home at the center).

conclusion

All in all the primary school students have shown enough (pre-)algebraic potential for us to continue with the same composition of the target group. Disappointing results in the case studies cannot, as it seems, be ascribed to an unattainable mathematical content. Unfortunately it has been difficult to determine the upper boundary of the students' abilities. Not only did the students have trouble to communicate their reasoning, they also failed to see the possibilities of some of the open problems. If the tasks had been slightly more structured, the students might have had access to higher level interpretations of these problems.

5.3.2 Systems of equations

Towards the end of the first year an educational science student joined the project to conduct a small study on teaching and learning systems of equations. Encouraged by the ideas and results of a classroom experiment by Streefland (1995abc), Beemer designed a series of lessons based on the mathematization of fancy fair attractions into equations (Beemer, 1997). She envisioned a hypothetical learning trajectory starting from informal symbolic notations and ending with the construction and solving of systems of linear equations. The starting-point of this learning trajectory was chosen to comply with the primary school trajectory, but in effect the instructional unit *De Kermis* (*The Fancy Fair*) was written and tested as a small, independent learning strand.

The mathematical content of *The Fancy Fair* is structured as follows:

1 construction of trade terms for trading chips, using a given price list of the fancy fair attractions; in words, pictures, symbols or a combination of these;
2 arithmetical problems on the trade terms and the price list;
3 making combinations of prices amounting to a given total price;
4 development of suitable notations to express trade terms, resulting in the use of one letter to represent the value of a chip;
5 reverse calculations with one unknown;
6 comparing two combinations (in letter notation) of chips with a given total price – a system of equations – to determine the price of each chip;
7 formalization of the concept 'unknown': solving symbolic systems of equations in a game situation, disconnecting the letter from the object it refers to;
8 problem solving: translating word problems into a system of equations;
9 awareness of solving equations: writing your own student guide.

Beemer also listed several hypotheses on the expected learning process and student preferences, some of which will be mentioned as we present the results. The materials were tested in two different situations: first in a group of 4 students with Beemer herself as teacher, and then – after some minor adjustments in the material – in a regular classroom situation.

notations

Just like earlier findings, students are not inclined to represent information visually or shorten their (mostly descriptive) notations. Students used a variety of mathemat-

ical symbols but less than expected, and some of these are inappropriate in a mathematical setting (e.g. a dash and a comma for addition, an arrow for equivalence).
Figure 5.17 illustrates test work by Christian, the best student in the group of four. The problem is about writing combinations of narrow and wide cabinets to fit along two walls, and then finding the width of both types of cabinets. In the top left corner the student has constructed a system of equations, using the letter *w* for 'wall' to remember the meaning (a wall of length 4.80 meter and one of 1.60 meter).

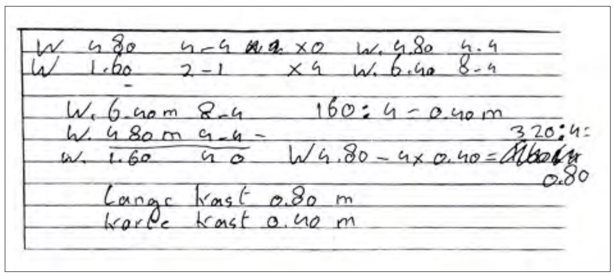


figure 5.17: constructing a system of equations

In formal notations the system would be:

$$4x + 4y = 4.80$$
$$2x + y = 1.60$$

To express addition, Christian has used a dash. His letter use is also unconventional; he has left out the unknowns in the equations, but he has written down letters that refer to the context (*w* for wall, measurement unit *m* for meters). This phenomenon of not writing down the unknown – which also appears in traditional algebra classes – was observed in the main research experiment as well, and will be discussed in more detail in chapter 6. The calculations only involve numbers, the position of the unknowns are constantly remembered; the column notation of the coefficients even savors matrix notation. This student has apparently taken shortened notations even a step further! His error of multiplying with 0 instead of 1 is quite common with students and not important at this stage.
The students intuitively perceived the equal-sign arithmetically, to announce a result, which can explain why students did not use it much (like in figure 5.17). Spontaneous constructions of horizontal expressions like Streefland encountered in his candy experiments did not occur. Students rarely used a letter for the unknown, probably because it is not necessary and not natural in situations where reverse calculations can be done arithmetically. This means that stage 5 in the hypothetical learning trajectory does not connect with stages 4 and 6. Contrary to Beemer's expectations, students continued to prefer letter combinations (one letter for each syl-

lable), even in the course of the experiment. And if a single letter was used, it was usually the first letter of the object which, as students explained, 'makes it easy to remember the meaning'. In the classroom situation students adopted each other's strategies; gradually more students came to realize that letters can make calculations easier. A small number of students achieved the formal level of letter use.

reasoning

Solving a system of equations through comparing and reasoning is a skill that needs more practice than had been anticipated. Students used mostly guess-and-check or trial-and-improve strategies, even after the strategy of reasoning had been discussed. However, Christian intuitively applied a strategy of repeated exchange, which supports the belief that reasoning strategies can be organized didactically to be reinvented by the students (see also Van Reeuwijk, 1995, 1996). Beemer admits this strategy has unfortunately not been anticipated well in the learning strand. The game activity contains a number of didactically strong points (own productions, good practice, context-free use of letters, element of competition), but needs a few practical adjustments for classroom situations.

learning trajectory

Instead of thinking in terms of chips and prices, most students thought in terms of actions: the attractions themselves. Consequently, within the context, letters continued to refer to the attraction (action) and did not formalize progressively to being a referent to the quality of the attraction (price). In fact, the hypothetical development of conceptual understanding of the unknown (left: informal, right: formal):

chip representing an action $\longrightarrow$ value of the chip $\longrightarrow$ letter representing a value

differed from the actual development:

action $\longrightarrow$ abbreviation of the action $\longrightarrow$ letter representing a value

Beemer observed that the transition from abbreviations to a single letter representing a value takes place too quickly in the learning strand. She advises to keep pace with the students, combining the two streams and postponing the introduction of chips until there is a practical need for them: when combinations of chips form a system of equations in two unknowns. In our opinion the two streams need not be seen as consecutive processes, especially since some students think along the top stream and others along the bottom stream. The different meanings of the letters can coexist, remaining at an informal level as long as necessary. Vertical mathematization will happen in due time; the context continues to make the letters meaningful until the students are ready to proceed to a more formal conception of the unknown.
In addition to raising problems, Beemer pleads for more practice throughout the lesson series on solving equations in one unknown (especially to develop symbolic notations, i.e. to strengthen stage 5 in the learning trajectory), gradually increasing in complexity, ending with conventional manipulations on systems of equations. In other words, she suggests integrating symbolic algebra into the learning strand wher-

ever suitable. This suggestion was taken into consideration when the materials were revised, as well as other practical suggestions concerning classroom organization not mentioned here.

teacher and student norms

According to Beemer, students need to learn to reflect on situations where several answers and strategies are sought for, and the teacher has a task as guide to create opportunities for students to activate these reflections. The learning strand is based on the presence of an open learning environment, which sometimes confused the students as well as the teacher. The pedagogical demands of a constructive learning environment should not be underestimated. Beemer feels that unfamiliarity with this approach to teaching and learning mathematics has had a negative effect on the outcome of the experiment. This is something to remember in the classroom experiments still to come.

5.3.3 Pre-algebraic strategies from the past

sources of inspiration

From the history of algebra we know that methods of advanced arithmetic played an important role throughout the rhetorical and syncopated stages of algebra (see also Kool, 1999). Particularly the Rule of Three and the Rule of False Position seemed appropriate for a lesson series on linear equations. Inspired by Calandri's fish problem (figure 5.18) we designed a series of activities on body proportions of different types of fish, followed by a dissection of Calandri's own solution based on the Rule of False Position. Students are then asked to apply this rule themselves and comment on its appropriateness as an alternative strategy to their own. The final paragraph of the resulting instructional unit, *Guessing and Fishing*, requires students to apply the Rule of False Postion to a few number riddles found in *Rhind Papyrus* (ca. 1650 BC). Some of the tasks are based on materials by Ofir and Arcavi (1992). (The current version of these activities as they appear in the eventual instructional unit *Time Travelers* is included in the appendix).

The head of a fish weighs $\frac{1}{3}$ of the whole fish.

The tail weighs $\frac{1}{4}$ and the body weighs 300 grams.

How much does the whole fish weigh?

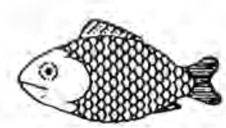


figure 5.18: Calandri's fish problem

Another idea came to mind when we learned that Brazilian fishermen use proportions when they are at sea (Schliemann & Nunes, 1990). Apparently it happens that fishermen who calculate swiftly weight ratios of fresh and cleaned fish in a real life situation, struggle with similar ratio problems from a school book. This observation instigated us to use the fishing context as a realistic and natural situation for exploring different kinds of proportion problems in the opening section of the unit.

try-out

The instructional unit was tried by the researcher's own seventh grade students and then revised. The main objective of this first experiment was to get some feedback on the mathematical content and the formulation of the tasks; an analysis of the students' learning process was left to the next try-out. The activities which were effective were retained, those which were not were adjusted or removed. The resulting instructional unit *Time Travelers* is different from *Guessing and Fishing* in the following ways:

1. The historical component of the unit has been intensified. The Rule of Three is no longer just the researcher's source of inspiration but it is now also an explicit and quite prominent part of the student activities;
2. The final paragraph in the unit on the application of the Rule of False Position includes some problems on sum and difference and is now deliberately aimed at bringing together different mathematical topics: the Rule of False Position and (semi-)symbolic equation solving as developed in the instructional unit *Fancy Fair*;
3. In order to bring the history to life and get students more involved, the activities are now situated in a story on two seventh grade students who meet people from other cultures and eras and learn about their mathematics.

Rule of Three

Proportions constitute a large part of the mathematical content of *Time Travelers*, starting with simple ratios (assumed to be common knowledge) which are then extended in two more unit sections. In the section on applications and variations of the Rule of Three, students solve authentic word problems from different cultures and eras: Egypt (2000 BC), China (400 BC), India (12th century AD) and Western Europe (1568 AD). Comparing the various representations of the rule, students can experience how common features of specific cases can result in a generalized formulation, which is a valuable activity of algebra. Another section, on more complex proportions of part–remainder–whole, is intended to facilitate a smooth switch to linear problems in one unknown solved by reasoning or by the Rule of False Position.

Some methods and problems which constitute the historical component of the instructional unit *Exchange* are discussed integrally with the student results in paragraph 6.7.6.

5.4 Pilot experiment

Revision of the activities has resulted in four instructional units for students in grade 5 or 6: *Pocket Money, Playing Cards, Marbles* and *Barter.* The activities on trading marbles now serve, in compressed form, as introduction in the *Barter* unit. In this second design cycle we have adjusted the learning trajectory and worked it out into a mathematical program (see table 5.2) to show how abilities are developed from one unit into the next (reading from left to right). Reading in a column, row numbers 2 to 6 (from ‘comparing quantities’ to ‘restriction problems’*)* illustrate the *reasoning* stream, and rows 7 to 10 (from ‘relating representations’ to ‘schematizing as a tool’) represent the *schematizing* stream. Other aspects of the instructional sequence, such as the integration of historical elements and the algebraic aspects of the tasks are described in section 5.3.3 and in chapter 6 (integrally) respectively.

The main purpose of this second research cycle is two-fold:

1 to investigate whether the proposed learning strand is suitable for teaching and learning pre-algebra; in particular, to compare the anticipated learning trajectory as sketched in figure 5.15 with the learning trajectory observed in the classroom;

2 to determine how to carry on with the next phase of developmental research as sketched in section 4.3.2: development of local theories, further adjustments to the learning trajectory or instructional materials, changes of style or organization, and extending the learning trajectory to include the secondary school activities.

		Pocket Money	**Playing Cards**	**Marbles**	**Barter**
reasoning stream	**comparing quantities**	numbers		value of marbles	value of goods
	global view	inverse operations; pattern spotting; generalizing	reversed calculations; inverse formulas		predictions
	repairing inequality	numbers		value of marbles	value of goods
	reasoning		inverse formulas	inverse formulas; compatibility of trades	substitutions; exchange; compatibility of trades; predictions
	restriction problems	2 simultaneous relations	calculating with formulas	compatibility of trades	trade restrictions
symbolizing stream	**relating representations**	number pairs; description; syncopated notation; visual/drawing	descriptions; syncopated notation; word formulas	table; syncopated notation; visual/drawing	table; description; syncopated notation; visual/drawing
	schematizing	rectangular bar; table	rectangular bar; table	trade terms; table	trade terms; table
	meaning of notations	shortening notations; role of letters	shortening notations; role of letters	shortening notations;	shortening notations; role of letters
	schematizing as a tool	rectangular bar			table

table 5.2: pre-algebra content in primary school

target groups

The pilot experiment (1997-1998) consisted of testing the four instructional units (approximately 25 lessons) in a regular classroom situation, at a rate of two or three lessons per week. The school is a Roman Catholic school located in the suburbs of a middle size town, neither white nor black. The school staff is in favor of the principles of RME (see section 4.3.1). It was agreed that two mixed classes grade 5 and 6 would participate, each with a full-time teacher. Both teachers were guided intensively prior to and during the experiment. We had agreed to try the lessons with the entire class, but also to do justice to developmental differences between grade 5 and 6 whenever needed. One class consisted of nineteen girls and twelve boys, the other of seventeen girls and fourteen boys, between ten and twelve years of age. In each class one girl and one boy did not take the written test at the end and were therefore left out of the analysis.

methodological issues

Most lessons were observed and evaluated immediately, in order to decide the next move. The experiences in the first group (14 lessons) were used to adjust the student and teacher materials for the second group (12 lessons). In this way a miniature round of design was realized. Due to unforeseen circumstances it was not possible to test the second half of the program (the instructional units *Marbles* and *Barter*) in a real classroom situation. Instead, three students – one of relatively high, moderate and low ability – were selected from each class in order to test the second half of the learning strand. During these work sessions, which were all recorded on video, the researcher took the role of teacher/individual tutor. An account of the second half of the experiment is given in section 5.4.2.

The results of the pilot experiment are based on data collected through the observation of lessons, participation of classroom discussions, the analysis of video recordings and the evaluation of written tests 1 (for *Pocket Money* and *Playing Cards*) and 2 (for *Marbles* and *Barter*). The instructional units were not included in the analysis for a number of reasons. Students often worked in groups, and some students corrected their answers after class discussions, which makes it difficult to determine what a student really did on his own. We also found that interesting personal notations and strategies were often used for classroom discussions and reflection. In the next section we describe classroom experiences most appropriate for 1) illustrating new elements in the learning strands 2) making a comparison between the actual and the hypothetical learning trajectories and 3) explaining how the results have affected the continuation of the research project. Since we do not discuss the whole learning strand, we present the classroom events in order of mathematical content, and summarize the results of the test 1 in a separate section.

5.4.1 Classroom experiences and results

comparing quantities and switching between representations

orientation

The orientation task in the student unit *Pocket Money* is like an advanced organizer: it addresses most of the mathematical abilities that will be dealt with in the unit. The problem is about comparing amounts of pocket money between pairs of children (figure 5.19). Mathematically this means looking for, describing and generalizing relations between two numbers. Students are asked to investigate how the amounts of pocket money of two imaginary students sitting at a table are related (for example, twice as much, 4 more, a difference of 3, etc.). In order to get students to compare number pairs rather than looking at each pair of numbers independently, we gave the condition that a relation is accepted only if two or more tables satisfy it. Obviously some pairs of numbers cohere in more than one way, for example with a multiplicative and an additive relation. Students thus seek out relations by looking at numbers from various points of view.

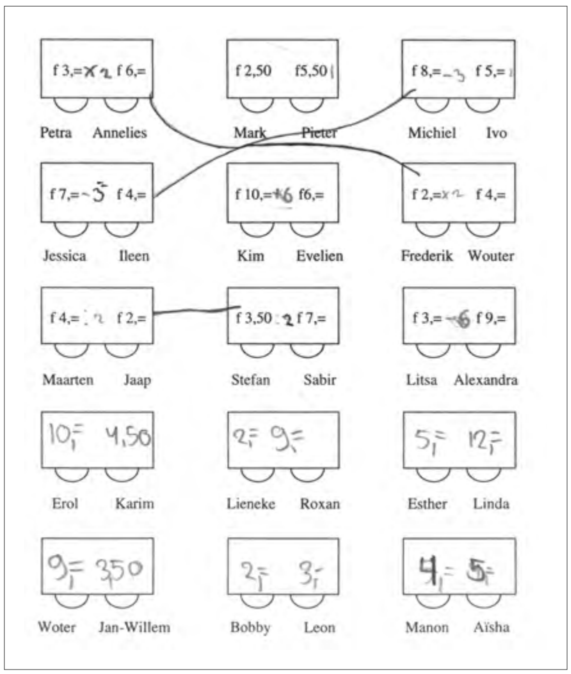


figure 5.19: preliminary activity in *Pocket Money*

symbolizations

We were of course very interested to see how students describe these relations and how they indicate which tables belong to each relation. Working in groups of three or four, students discussed how to express the relations efficiently on paper. Most students used symbolic notations to identify the relations, either directly in the drawing (as in figure 5.19) or in combination with a legenda. Some students made a separate overview of all the relations they found, together with the amounts of pocket money.

different perspectives

There was a good class discussion whether inverse operations like ‘twice as much’ and ‘half as much’ should be considered as different, and how ‘a difference of 4’ compares with ‘4 more’. After that, the students had to change perspective again to check their relations with the rest of the tables (number pairs) in the drawing, until all possible relations had been identified. To involve students more actively and reverse their perspective, this activity was adjusted slightly for the second class: we had students construct new pairs of pocket money for each relation they had found (like the two lowest rows of tables in figure 5.19).

making sums

The final task in this setting asks students to construct different sums for each pair of numbers. The idea of using arithmetic sums occurred to the designer as a suitable way of putting more emphasis on representing a relation numerically. The activity has a double function: it lets students discover how a relation can be checked or demonstrated in an arithmetical setting – which we anticipated would also support the understanding of formulas in the instructional unit *Playing Cards* – and it enables weaker students to practice basic operations and their inverses. In a class discussion (see figure 5.20) three different sums were given for the relation (called 'table group' by the students) 'a difference of 3' using the amounts 5.50 and 2.50:

> Teacher: Who can give an example, that belongs with another table group? Then we can do that one together.
> Bas: Minus 3.
> T: 3 guilders difference. Which names belong with this one?
> B: Marc and Pieter.
> T: Can you make a sum with it?
> B: 5.50 minus 3.
> T: Who can do another one? We have seen before, that it doesn't always have to be minus.
> Hyacinth: 5.50 minus 2.50 is 3.
> Jeffrey B.: I know another sum ... 2.50 add 3 gives 5.50.

figure 5.20: making sums

inverse operations

Activities on recapitulating and practicing operations and inverse operations return throughout the proposed learning program. Whenever a student made up a sum to illustrate the coherence between two numbers, the teacher asked for its so-called 'inverse sum'. The sum $4 \times 3 = 12$ and the inverse sum $12 : 3 = 4$, for instance, show that '3 times as much' and 'divided by 3' are inverses. After all, the relation between two quantities can always be interpreted in at least two ways. Judging the reaction of a 5th grade student, Renske, it seems that inverting operations appealed to children: "Because times and divided by belong together, and plus and minus belong together! If you can make a times-sum, you can also make a division-sum. It is the opposite." In both classroom evaluations in the final lesson, students mentioned 'inverse sums' as a very recognizable and useful topic that they had learned.

Similar to the orientation task, most (clusters of) activities in the learning trajectory relate to more than one mathematical topic (as listed in the left column of table 5.2). For this reason the activities will be classified based on the most representative or dominant mathematical aim, and as much as possible according to the order in the learning trajectory.

schematizing as a tool for organizing and problem solving

rectangular bar

Early on in *Pocket Money* the students are introduced to the rectangular bar as a way of visualizing how two quantities are related. It was anticipated that the teacher would guide a discussion on different ways to write down amounts of money, including drawing. We conjectured that drawing separate coins for large amounts of money would easily propel students to suggest drawing paper money (a bill), which places the rectangular bar within reach. In both classes the model emerged naturally, and students had little trouble to draw and interpret the bars correctly. At first the bars represented fixed amounts of money. Students were free to choose the value of one grid, which led to a nice variety of solutions and a discussion why this is. The remaining tasks were based on unknown amounts of money that satisfy one relation and then two relations, with different representations (description → bars, bars → description, bars → numerical solution). Although some students were put off by the open character of the tasks, the results were satisfactory; several children even formulated relations with fractions ('$1\frac{1}{2}$ times as much', '$\frac{3}{4}$ of', etcetera). But just like in the case studies, the model did not develop into a schematizing tool for solving restriction problems (at the end of *Pocket Money*) nor for students' self-made problems (*Pocket Money* and *Playing Cards*). The hypothetical trajectory from *model of* a situation to *model for* mathematical reasoning (see section 4.3.1) did again not take place in the primary school part of the program.

use of tables

Prior to the start of the experiment, the teachers were instructed to use efficient and schematic notations that students are already familiar with, to stimulate a natural development of organizing competence from student experience whenever possible. Both teachers used tables to organize information on the black board, leaving it up to the students to do the same in their booklets or not. We anticipated that students would show an increasing tendency to use tables as a convenient way of writing down their answer (ratios, or combinations of numbers that satisfy a formula) but we assumed that perhaps only a few students would discover that a table can be a tool for mathematical reasoning (for instance, in trying to discover a pattern). We felt very strongly, though, that the teacher should not teach this application top-down (anti-didactic inversion, see section 4.3.1). We envisioned that the activities on recognizing a pattern or the exchange problems in *Barter* might instigate a student to use a table and discover its convenience. In that case it is the teacher's task to make such a discovery known by inviting the student to tell or show the strategy to the class and have it evaluated.

In the first half of the experiment we did not observe advanced use of the table as an aid for mathematical reasoning, but the ratio table was commonly used by students and teachers as an efficient way of reorganizing information or arranging the answer.

meaning of notations and symbols

The second series of activities in *Pocket Money* is based on the case studies' activity sheets dealing with shortening notations and adjusting inequality. Unlike before, the students first encounter shortened notations in a non-committal way in a slightly exploratory setting, and they have to reflect on the different types of notations. In time students become more and more acquainted with syncopated expressions and formulas, where the process of shortening should become more purposeful. The teacher has the task of guiding in the right direction at the right moment, like the use of abbreviations and symbols for operations (perhaps as late as the last unit, *Barter*), in order to secure a natural learning trajectory towards actual algebraic letter manipulations to be attained in the secondary school strand.

syncopated notation

The first problem is as follows: two boys receive pocket money – one 4 guilders, the other 8 guilders – and there is a list of expressions that describe how the amounts are related (in words, in word formulas using operation symbols and the equal-sign or in syncopated notation). Shortening notations works like before: one letter (initial or unit) or two letters (one for each syllable), so letters only refer to objects or constant numbers. Most statements in the list are correct, but in some of them the relation is reversed. The students have to determine which statements are correct, and afterwards they can decide what they think of the shortened notations (convenient/difficult/clear, or not).

The purpose of this problem is threefold. Firstly, since the shortened notations make the operation in the expression more apparent, it was expected that some students will think it is easier and quicker. Secondly, since the activity gives an indication of the students' initial preferences, it will be possible to get more information on how accessible these shortened notations really are and also whether students change their minds about using abbreviations. And thirdly, it was intended that if the teacher has the opportunity, students can be made aware of the different roles of the letters in the expressions. (In a later stage students are required to decide themselves what the letters refer to, making up their own letters and expressions, even where the same letter can refer to an object or a quality of that object.) Based on previous experiences we expected students to be able to interpret syncopated and symbolic notations, but not to use them voluntarily. The majority will probably prefer to use rhetoric notations, sometimes including mathematical symbols and abbreviations.

student preferences

Classroom discussions reveal that student opinions varied (see figure 5.21). The teacher respected that and gave students a wide berth to use abbreviations or other shortened notations in the next problem and onwards if they wanted to.

However, in time we discovered that this freedom to use informal notations conflicted with the aim of having students develop understanding of symbolic expressions. Student notations were so diverse that classroom discussions remained at a level of outer looks. The teacher never got round to talking about the different roles of the letters.

The next task asks for own productions: statements to compare your classmate's and your own allowance. The aim of this problem is to allow students to use shorthand notations, and make the situation more personal and appealing. However, the teacher was not comfortable about making the students' allowances common knowledge, and so she designed paper cards instead, on which she wrote imaginary but realistic amounts of money. This act of improvisation actually had two advantages: she could make sure that the number pairs fit at least two descriptions (relations), and it gave her the opportunity to choose 'easy' amounts for weaker students and challenging numbers for the better students.

Teacher: What does pm mean?
Class: Pocket money.
T: And the M?
Class: Mark.
T: And the E?
Class: Eelco.
T: Is it convenient, to use abbreviations?
Sanne: No, because pm can also be something else.
Ryan: Yes.
T: Why then?
R: It is shorter.
T: Yes, writing it down will be quicker.
Esther: But g can also be gram.
T: So we have to decide together what the letters mean. Is it clear to all of you what the letters mean here?
Class: Yes.
T: Who thinks he will do it in the same way?
Thomas and Sander react first, then about half the class reacts.
T: And who will certainly not do it?
Four students react, others say that they don't know yet.

figure 5.21: meaning of notations

To have students reflect on the clarity of their notations and the likeliness of inverse expressions, we included questions like "Did you make up the same statements? How many?" and "Did you understand each other's statements? Why or why not?" As it turned out, questions like these are best posed by the teacher in a class discussion because the students read them at a very shallow level.

repairing inequality

One of the activities maintained in the hypothetical learning trajectory deals with repairing inequality. Students are asked how two amounts of pocket money can be made equal: first by checking a list of rhetoric statements concerning 4 and 8 guilders like before, and then by constructing statements about their own amounts. Students who attempted to syncopate their notations generally had trouble to symbolize the procedural character. For this reason the combination of shortening notations and inequality is not appropriate. Symbolization of procedures or manipulations are premature at this stage because it causes improper notations and unnecessary confusion.

orientation

Shortened notations are a recurring theme throughout the program, but in a varying appearance. The introduction in the second instructional unit, *Playing Cards*, is

about an imaginary group of children playing a game of cards. In this activity the students have to interpret descriptions of what happens in each round of play. There are four types of descriptions printed on card board cards: an account in words of what happens in a round, the resulting scores of all the players, a description in words saying how the scores are related (for example, ‘Petra has 5 points more than Jacqueline’) and a word formula for that relation (‘points Petra = points Jacqueline + 5’). The children have to match the cards with each round of play, and then design their own cards for two more rounds. Sometimes there are several cards to describe the scores, when the scores are related in two different ways (for example, ‘twice as much’ and ‘5 more’), which means that students are required to vary their perspective much like the preliminary activity in *Pocket Money* in figure 5.19. We hoped to find out if syncopating notations increases the chance of misinterpreting the relation (reversal error), and indeed we found that students frequently chose the wrong word formulas while interpreting the verbal relation correctly. This issue will be discussed in more detail further on.
In figure 5.22 below, Robert reads to the class a statement he made up that could describe how the next round of play might go. The fragment shows how unspoken rules obvious to professional algebraists need to be made explicit to novice learners.The teacher tries to explain to the class that if abbreviations are used, it must be clear to everyone what they stand for, but it is unsure whether the message was understood. It illustrates again the issue of whether we should encourage informal notations if it means accepting the untidy side-effects. Although most students understood what the letters mean in each context (even in the test), we believe they did not develop awareness that being precise and consistent might be important.

Robert: Points Petra plus Anton is Jacqueline.
Teacher: Something is not right there .... Well, maybe the calculation is.
Tim: Points Petra plus points Anton.
Teacher: Indeed! You have shortened too much. It is all about the points of these people. You can't just add people to the points that they get, that's impossible!
The class has to laugh.
Teacher: It is actually about the number of points: the number of points that Petra has plus the number of points that Anton has is the number points that Jacqueline has.
Renske: That's what I had!
Teacher: Yes, we talked about that for a little bit yesterday. It is not wrong what you say, Robert, but it is not clear when it is too short. It is important that you say it clearly.

figure 5.22: constructing statements

conflicts of notations

Just like the previous experiment, we observe that students choose mathematically unfit symbols more often than was foreseen. During one of the lessons on *Playing Cards*, students were asked what could be the meaning of the expression

$pA = 3 \times pJ$. Based on intuitive notations students had used in previous try-outs, we chose for letter combinations to maintain a link with the context: the letter *p* stands for 'number of points belonging to' and the capital letter stands for the person in question, in this case Annelies and Jeroen. In the expression, such a letter combination behaves like a variable for which numbers can be substituted. One girl suggested the numerical values 3 and 9, which she wrote on the black board as follows:

$$A - 9\,p \qquad j - 3\,p$$

Another girl in the class made a comment on the dashes, and said she would write an equal-sign instead. Situations like these offer the opportunity to have students reflect on mathematical symbols, like why they were invented and why we find them useful. We also intended the teacher to discuss the student's choice to write a capital letter *A* and then a lower case letter *j*, and the fact that she uses the letter *p* as a unit even though it is already part of the variable. She did not do that on the first occasion, but only towards the end of the lesson series. It was not a problem to students that letters mean different things at the same time, or perhaps some did not even notice. The next example of a conflict of meaning involves the equal-sign (see figure 5.23). The task is to calculate how many points Henk has, using the expression $pH = pA - 10$, in the case that Annelies has 45 points.

Hans writes on the blackboard:	*pA* = 45 – 10 = 35	
Teacher: Remember, you must put Henk's points down.		
Hans writes *pH* behind the number 35:	*pA* = 45 – 10 = 35	*pH*
Researcher: I read, Annelies has 45 minus 10 and therefore 35 points. Huh? Didn't Annelies have 45 points? What should it be? I am confused. Does anyone agree, or is it just me?		
Some children are nodding.		
Teacher: It looks like Henk and Annelies have the same number of points.		
Hans then changes it to:	*pA* = 45	
	45 – 10 = 35	*pH* 35

figure 5.23: meaning of the equal-sign

static vs. procedural

Hans starts with the number of points belonging to Annelies, then he performs the calculation, and the equal-sign announces the result. He strings the different parts of the calculation (writing down the intermediary results) and thereby violates the symmetry and transitivity properties of the equal-sign. On top of that, if the string of calculations is conceived statically, he seems to have made a reversal error (placing the operator beside the wrong variable). According to the student's first calculation (read as a static algebraic entity), 10 points are taken from Annelies instead of Henk. But if the calculation is read as a procedure, it is a correct representation of the steps involved: Annelies has 45 points, take away 10 to get 35 points for Henk. Since Hans has not put the operator '– 10' immediately behind the variable *pA* but on the right hand side of the equation (having written the value of *pA* first), his notations suggest that he might have understood the relation incorrectly.

Prior to the experiment, we reasoned that as long as the students and the teacher are aware of the limitations and conditions of symbolic notation, the development and refinement would be a natural process. But unwanted complications like inconsistency, premature formality and unnatural choice of symbols in both rounds of try-out show that the proposed approach to developing symbolic notations is not appropriate. It was not our intention to formalize the conception of letters, but the double role of the letter *p* conflicts so much with algebraic beliefs that this matter should not be ignored. However, a discussion of this kind does not fit in an informal, pre-algebra class! If children have a natural tendency to use the letter *p* as a label for the unit ‘point’, we believe it should not be included in the expressions and formulas. But, this automatically means that in symbolic expressions, the most obvious link with the context might be lost for students.
At this point of the project a compromise between precise and unambiguous notations on one hand, and intuitive, often inconsistent productions of children on the other is not at hand. The current approach to development of symbolic notations in the learning strand will have to be adjusted. And since the second half of the experiment was different in set-up, we have decided not to analyze student work for a significant change in notation behavior, which we initially set out to do.

global view

The mathematical skill referred to in the learning trajectory as ‘global view’ is a collective term for a variety of qualitative reasoning abilities: comparing quantities qualitatively, spotting patterns, generalizing relations or mathematical characteristics, interpreting information from different perspectives (inverse operations, repairing inequalities), and so on. Some of these thinking skills have already played a role in the activities described so far, and others will be introduced here.

spotting a pattern

Compared with the activity sheets, a few activities on recognizing patterns and generalizing have been added to the program. The *Pocket Money* orientation task has already been discussed in this respect, but in addition a task was developed on looking for regularities and change. The problem is based on the idea that when children get older, they get more pocket money, but on which terms? One option could be to raise the amount each year by adding half of it. The first series of questions asks students to investigate the situation of the boys Mark and Eelco from before, who received 4 and 8 guilders respectively:

> How much will they receive next year?
> And how much in 3 years' time?
> Compare the amounts over the years; what happens with the relation ‘twice as much'?
> and with the relation 'a difference of 4'?
> Do the same for your and your class mates' amount of pocket money.

During the first classroom try-out we found that this activity caused unexpected

complications. The calculations involved in the second question withheld the weaker students from attaining the level of generalizing. In spite of the structured build-up of questions, several students did not understand the essence of the task and were clearly unmotivated. Unlike we expected, most students did not write down the intermediate answers to question 2, which automatically meant that they could not answer question 3. As a result the teacher had no choice but to give the students a clue. In the other class the question was changed to avoid this problem. The fourth question, too, was not clear to students, because their own situation usually meant other descriptions than 'twice as much' and 'a difference of 4'. Students did not see the isomorphism of the tasks. And even though the amounts were chosen carefully by the teacher in advance, it frequently occurred that the numbers became very inconvenient after two consecutive halvings. Some students resorted to rounding off, but this of course interfered with number patterns.
The next part of the problem contains the same questions but now with relation to the rule of raising the amounts of money each year with 4 guilders. Only the last question is different: 'Predict, without calculating, if the same will happen with your and your classmates' amounts of money, and explain why. Then check your prediction.'
Also in this sequence of questions errors in calculations and loss of interest turned out to be the main reasons for disappointing results. The last question was meant to instigate students to reason at an abstract level, but the struggles with the previous series of questions really put this goal out of reach. In fact, the limited response to these two tasks on recognizing patterns, generalizing and reasoning make it impossible to conclude anything on how students performed. Evidently this activity does not appear in this form in the next try-out.

investigating and interpreting equivalent expressions

Comparing and rewriting symbolic expressions is an integrated activity that combines reasoning with symbolizing competence. We conjectured that sums and inverse sums would provide the students with supportive insight because they make the formulas tangible.

interpreting formulas

In the instructional unit *Playing Cards* the expression $pH = pA - 10$ means "Henk has 10 points less than Annelies", although again we found that students tend to say "the points belonging to Henk are the points belonging to Annelies minus 10". Some mathematical problems require students to calculate the number of points belonging to the players in a forward direction (when pA is known), and others in a reverse order (with inverse sums, when pH is given). A classroom discussion of the answers indicated that most students experienced no trouble calculating with the formulas, although the calculations were not always written down correctly, as we saw in figure 5.23. In another problem the student has to decide whether or not the formula

$pA = pH + 10$ is equivalent to $pH = pA - 10$, for example with the help of substituting numbers for the variables. Some students decided the two formulas are different even though they describe the same relation, but in a class discussion they reverted to the general opinion that these two expressions are really the same. After that, there are some activities about rewriting (inverting) formulas. We expected students to fall back on the meaningful context situation when necessary, checking their answers by substituting the players' points into the expressions.

reversal error

The greatest difficulty that students encountered when they tried to rewrite a formula was to put the operator in the right place. Consider the formula $pR = pJ + 5$, for example. Children frequently made the mistake of only inverting the operation ($pR = pJ - 5$) or only exchanging the positions of the variables ($pJ = pR + 5$). The difficulty is contained in the fact that the operator '+ 5' is not written next to the variable having a surplus of 5, but beside the other one. And not only students make this mistake! During the last lesson of the first series of lessons, there is an opportunity for students to make up their own story problem with two restrictions. Figure 5.24 shows how such a problem was discussed in the class. The teacher wrote the relations read aloud by Sander on the blackboard (the letters *S* and *R* stand for the unknown amounts of pocket money belonging to Sander and Robert, and the relations Sander gave are between brackets).

S	*R*	
$S \times 2$		(Sander has twice as much)
$S + 500$		(Sander has 500 more)
S – 1/4 own money		(Sander spends $^{1}/_{4}$ of his money)

figure 5.24: incorrect teacher notations

At least two incorrect notations come to the fore immediately. The teacher uses the letter S as a substitute for 'Sander has', and tries to symbolize the important information 'twice as much', '500 more' and 'one fourth less' in a compact way. The first two operations describe a state, but a professional algebraist will interpret the expressions $S \times 2$ as the procedure 'multiply Sander's amount by 2'. The notation in columns would lead naturally to the formula $S \times 2 = R$ instead of the correct formula $S = R \times 2$. The operation '$\times 2$' will automatically appear beside the wrong unknown. And the same observation goes for the additive relation in the third line. In the last line the term $S - \frac{1}{4}$ represents "Sander has one fourth of his money less", which should formally be written as $S - \frac{1}{4}S$ or $\frac{3}{4}S$. The teacher's choice to write the relations like this is understandable, since it is closest to the intuitive way students use abbreviations, as we have also seen in the case studies. Interpreting the expression correctly requires a formal conception of letters, and seeing the expression as an autonomous object! But perhaps the teacher took the decision unconsciously, without realizing the discrepancy. Obviously it would be best to determine whether the reversal error is a writing error caused by an informal, procedural conception of symbolic expressions, or whether it is an error due to incorrect reasoning. One way to find out the nature of the mistake is to ask the learner to interpret the expression globally (for example, "Do you know which player has the most points?"). In the experiment we found that children interpret the relation correctly but sometimes write the expression down incorrectly – or what would be called algebraically incorrect. Errors of notation can be easily revealed by substituting numbers for the variables, but students did not think of checking their answers, in spite of their familiarity with sums and inverse sums from earlier tasks. Teaching children to be critical of their work and check their answers will help to solve this problem. Yet, prevention is better than cure. The reversal errors described above indicate that shortened notations are especially troublesome when the meaning of the letters is no longer clear in the formula. Thinking again of the student-professor problem, we conjecture the error is more likely to occur when the letters are used as if they are object-related, when in fact they are quality-of-object-related. Ideally, mathematical activities should instigate students to create shortened notations that describe the correct relation unambiguously.

reasoning

In this section we discuss higher order reasoning competence like reasoning about symbolic expressions, variables and unknowns. In *Playing Cards* the students are asked to reason which player has more points, and to explain which formula has to be used first in order to calculate a certain player's points and why. For example, test 1 at the end of *Playing Cards* includes a problem where students have to reason consecutively about two expressions (figure 5.25 for two student answers).

The soccer clubs De Hoekschoppers (*H*), Aanvalluh! (*A*) and FC Penalty (*P*) participate in the highest league. They started two months ago and have scored reasonably until now. After 8 weeks we can say about the number of goals:

$gH = gA \times 4$ $\quad\quad$ $gP = gH + 5$

Can you say which of these 3 clubs has the most goals?
Yes, FC Penalty.
Why do you think that?
Because De hoekschoppers have 4 × as much as aanvalluh, and FC Penalty 5 more than De Hoekschoppers each time.
(student 1)

Can you say which of these 3 clubs has the most goals?
FC Penalty.
Why do you think that?
If a has 1 g and H has 4 g and then H's points + 5 = 9 so that will be the g of p.
(student 2)

figure 5.25: two students reasoning about formulas

The aim of this problem is to determine whether students can apply their knowledge of reasoning with expressions in a slightly different situation (using two expressions instead of one). We believe that a student who has only learned to reproduce knowledge or who has relied on rote skills or tricks will not succeed at this task.
Figure 5.25 illustrates that the questions can be answered at different levels: the first student reasons in general terms, whereas the second student reasons for a specific case, by choosing a random number for one of the variables and calculating with it. In one class 17 students solved the problem by reasoning correctly, but in the other class only 10 students did. Five students assumed the number of goals were constant – using one of the entries from the table from the previous question – and three students contended that it is impossible to conclude anything using just the formulas. The remaining students did not answer the question in a recognizable way. Looking at the notations of figure 5.25, there are a few observations to be made. The first student uses mostly rhetorical notation and never mentions the word *doelpunten* (Dutch for 'goals'); the calculations are free of the context. The second student has chosen for syncopated notation but clearly has difficulty to do so in a consistent way. She uses the letter *g* as abbreviation for 'goals' instead of using it as a variable representing the *number* of goals. We already encountered this conflict of meaning frequently in previous activities. The word 'points' (*punten*) seems to be short for 'goals' (*doelpunten*) but it could also be a matter of habit from class activities on playing cards.

restriction problems

The restriction problems (linear problems in two unknowns) on shopping are not different from the activity sheets' version, but in this experiment students already encountered restriction problems in the orientation task and in combination with drawing rectangular bars. We expected students to be able to make the transfer. However, during classroom observations we found that there was a great difference in this respect. The students that remembered solved the two-stage-problem on Michael and Rose (see p. 104) easily, although mostly by seeing the answer immediately and not by generating possibilities for one condition first. The other students were overwhelmed by the information and tended to give up quite soon.
The unit ends with a task of making up your own problem of recovering unknown amounts of money using two restrictions, of which one example has already been discussed (Sander's problem on the blackboard, see p. 136). Table 5.3 and table 5.4 show a categorization of the types of problems that students made up, as indicator of the level at which they understood this task.
The students were meant to make up a problem with two restrictions, given either consecutively or simultaneously, but only seven students succeeded (level 1). Twenty-one students interpreted the task as a call for reverse calculations, six of whom

used only very simple calculations (levels 2 and 4 respectively). Some students missed the point of recovering unknown information entirely, and based their problem on relations with known numbers (level 5). Perhaps they reasoned that any type of problem from the instructional unit would be all right. We also know that the teacher of class 1 instructed some weak students to pick an easier type of problem. The last category includes problems that did not resemble the mathematical content of the unit at all. From these results we can conclude that students who made up problems that can be solved arithmetically by reverting the operations in the problem did not really understand the concept of problem solving with two restrictions. Apparently this type of problem needs to be characterized explicitly in the learning strand for students to be able to recognize its features.

		boys			girls		
level/type of problem	test level	good	average	weak	good	average	weak
1	2 restrictions	1		2	1	1	1
2	1 restriction + reversed calculations	2			2	1	
3	1 restriction, many solutions			1	1	1	
4	simple reversed calculations		1			2	1
5	1 or 2 relations with known numbers	1					1
6	nothing/unclear		1	1	1	1	3

table 5.3: level and type of own productions in class 1

		boys			girls		
level/type of problem	test level	good	average	weak	good	average	weak
1	2 restrictions					1	
2	1 restriction + reversed calculations	2	2	1	4	1	
3	1 restriction, many solutions				2	1	
4	simple reversed calculations	1					1
5	1 or 2 relations with known numbers						
6	nothing/unclear	2		4	1	1	2

table 5.4: level and type of own productions in class 2

Test 1 included a restriction problem with four symbolic expressions: $cA = cP \times 2$, $cD - cP = 8$, $cP + cA = 33$ and $cD = cH + 5$. The expressions describe how many cards (red or yellow) the four soccer clubs were given for foul play so far in the soccer league. Students are asked to calculate, using the formulas, the correct numbers of cards for each club. The aim of this problem was to find out whether students could link two topics from different units: restrictions and reasoning with formulas. Moreover, the problem lays the cross-bar one step higher: students need to decide which two formulas need to be combined first to get the solution process going. We anticipated correctly that only a small number of students might succeed at this problem: 4 students did, and 3 more solved it partially.

summary activity

The instructional unit *Playing Cards* ends with an activity that reflects the core of the mathematical content. Figure 5.26 shows how a student made sense of several formulas (in the student material a formula was called a 'rule' or 'arithmetic rule'). The activity fulfills three important conditions: it asks for a student's own productions, it turns the student's perspective and it has the student reflect on the mathematical content. Own productions are valuable for giving the student an opportunity to rise to his own challenge, and for demonstrating the student's level of learning. Instead of starting with the situation and deducing a notation, this activity demands that students look from the other side. They have to imagine how the formulas become meaningful and in doing so they have to think about what quantities can be represented by letters. Indeed, we found that some students made up the most ridiculous situations, which made clear to us how little they understood the concept 'variable'.

Let us take a closer look at what the student wrote. His descriptions for the first two formulas imply that he had a notion of the concept 'variable'. His letters represent measurable or countable quantities. However, for the third formula he does not define the quantity correctly: does he measure the area, or the length? He is not completely aware of the meaning of the letters. In the second description he makes a reversal error, and his reverse formula is hard to decipher. His notes have also been corrected: an arrow to interchange the names 'Jeroen' and 'Richard', and the term '+ 5' between brackets. It is unclear whether he finally understood this formula or not. His answer on convenience of the short notations indicate that he understands that the formula should contain the essential information in the problem. His answer on inconvenience could mean that a long and complicated problem is hard to compress, or otherwise it does not make sense.

The second class of students were given a slightly different version of the task because we found that some students in the first class had trouble translating the formula without a numerical interpretation.

We expected students to get a better grip on the formulas if they were also able to compare the numbers. Knowing by now that students tend to do draft calculations in their head rather than on paper, we added a table for each formula to be filled in first.

Every rule has a story
Jeroen's rules are convenient for calculating the numbers of points everyone has. There are many other situations where rules can tell something about quantities or numbers. Which stories can you make up for the rules below? Give an example to show the meaning of the rule and also write down the inverse rule.

1 gW = gF + 4
Willem has four grams of candy more than Frank
gF = gW – 4

2 leJ + 5 = leR
The length of Jeroen is 5 cm longer (greater) than Richard
leR – 5 leJ (+ 5)

3 j = 4 × m
The jungle is four times as large as the marsh
m = :4 j

Sometimes it can be convenient to have a rule for calculating. Why?
To recover something in it.

But sometimes a shorter way of writing is inconvenient. When, for instance?
When the problem is long.

Why is it, that you can turn a rule around and it still holds true?
Because you get two are almost the same like + – or × :

figure 5.26: summary questions

written test 1
The majority of questions in the test are isomorphic to the classroom activities with the aim of evaluating what the students have learned. A few related questions were added to see if students could take their knowledge one step further, like the questions in figure 5.25 and the restriction problem. All the tasks are situated in the context of a Dutch soccer league. In analogy with the notations used in *Playing Cards*, the number of goals scored by a given team are represented by the variable *gH* (*g* for goals, *H* for the name of the team).
The data were studied both quantitatively and qualitatively. Student answers were assigned four different variables: answer (right/wrong), strategy, explanation (yes/no), and notation. The purpose of these variables was to learn more about student achievement than scores alone; we also wished to distinguish between the strategies that students used and qualitative data like notations and writing down one's reasoning. Student answers were assigned quantitative scores based on correctness (correct/partially correct/incorrect) and completeness (with/without calculations or an explanation), but incorrect notations were not penalized. The students' total scores (in percentages) have been categorized into five levels: very good (> 85%), good

(70% – 85%), adequate (50% – 69%), poor (40% – 49%), fail (< 40%). We give a summary of the results most informative with respect to the hypothetical learning trajectory.

gender differences

Although generally speaking Dutch boys perform better at mathematics than Dutch girls (Van den Heuvel & Vermeer, 1999), in this test the girls scored better than the boys in both classes; in one class as many as seven girls outperformed the best boy. International research has shown that differences in mathematical achievement between boys and girls can be attributed to factors like attitude, working style, cognitive abilities, topics of interest and societal expectations (ibid., 1999). For this learning strand, classroom observations suggest two explanations: that girls are more willing to show calculations or give an explanation, and that the boys, who tend to be impatient and less precise, underestimate or misinterpret the tasks. The boys generally preferred to take the shortest route: a literal description instead of a normal sentence, a formula rather than a description in words, a result without explanation, thereby carrying out the task incorrectly.

Based on gender-related research results (ibid., 1999), we suspected that girls might perform better in this project than the boys since the material is very linguistic in nature, and many of the activities are word problems. An analysis of levels of strategy has not shown a notable difference between girls and boys. Another interesting observation is the fact that a few good male students performed quite poorly and gradually became agitated. Possibly the demands in this experiment are so different from what is usually asked of the students – not technical competence but matters of attitude like showing strategies, explaining your reasoning, changing perspectives, making assumptions about unknown or variable quantities, etcetera – that capable students can become insecure and slow or shy learners can gain confidence because they can demonstrate other competencies.

differences between the two classes

The second class achieved slightly better results than the first class, except for the top students, but we can only speculate about the reasons. Possibly the second class had the advantage of slightly improved materials and some didactical changes, or perhaps they profited from talking with the other class about what they had done. On the other hand, the other class had better moments of reflection. In one class the grade 6 students scored slightly better than the grade 5 students at all levels (good – average – weak) with an average of 68% and 62% respectively. In the other class there was a much greater difference, also at all levels, with respective averages of 65% and 47%.

representing a relation

The majority of students transformed a symbolic formula into a tabular and a visual representation correctly, but a substantial number of them gave a very literal, static description of the relation in words. Even the good students had trouble to construct a natural sentence; the girls with the best scores tried so hard to write a lengthy description that they enervated their answer. In one of the classes as many as six boys described the symbolic expression in a word formula instead of a sentence. A few students with low scores showed a procedural conception of relations, and were not able to express relations symbolically. In one class students performed significantly better at using formulas to calculate numerical values, whereas the class activities on this topic had been satisfactory in both groups. The test also included a question on interpreting a tabular representation and recognizing a pattern; of both classes only ten students were able to express the regularity (the relation between the quantities) correctly in words, but thirty students expressed it correctly in a symbolic expression. This result seems to be in contradiction with earlier findings that symbolic expressions are more difficult to construct than a description in words, which we cannot explain.

notations

Approximately half the number of students described their reasoning rhetorically, the other half used syncopated notation (a combination of words, symbols and abbreviations). We found no striking differences with respect to gender or cognitive level. At all levels we see that students use letters inconsequently: capital and lower case letters, different meanings for the same letter (variable as well as label) and an unnatural choice of symbolism. Two students constructed a shorthand symbolic expression where the variable and the operation term were written the wrong way around, for example $dB = +3\ dA$ instead of $dB = dA + 3$. This is an error we have already seen in the first experimental try-out. One student even made up a completely irrelevant meaning for the letter *D* in the optional restriction problem. Sometimes students use different styles of writing within the same problem, for example in writing down calculations, or they interpret the equal-sign arithmetically – to announce a result – instead of algebraically. Other significant observations on notation have already been mentioned in previous sections. In summary, a notable number of students chose freely to use syncopated notations, but it does not strike us as a natural way of writing.

reversal error

Students made mistakes of interpreting the relation the wrong way around in primarily two situations: when they were asked to formulate in words a relation given in tabular or symbolic form, or when they applied or rewrote a symbolic expression. This so-called ‘reversal error’ appeared sixteen times in the first situation, amongst sixteen different students, and twenty-six times (amongst twenty-two students) in the second situation. Ten students made the mistake in both situations. This means that more than 25% of the students made an error of interpretation once, and more than 15% made it twice or more! This result is quite disconcerting, especially knowing that the program aimed at avoiding interpretation errors, through a) presenting relations in context situations, b) linking symbolic expressions to numerical counterparts (sums and inverse sums), and c) developing a qualitative way of looking at relations at the same time as the quantitative point of view. It is therefore important that we attempt to explain how students might get entangled in reversal errors, especially in a symbolic medium.

lack of meaning

Perhaps the condensed form of syncopated and symbolic representations causes students to lose contact with the numbers and consequently with the meaning of the expression. None of the students made draft numerical calculations in the test to investigate a formula or check a solution. During classroom observations we also found that students made little use of the numerical counterpart of the formula, in spite of the orientation activities on sums and inverse sums. In one of the classes the teacher related these sums several times to the calculations in *Playing Cards*, but this has not effected a significant difference in achievement (nineteen errors versus twenty-three). The absence of numerical checks could imply students do not conceive the variables as substitutes for varying number values, but as abbreviations of certain constant quantities that cannot be changed. It would also explain why students do not fall back on substituting a numerical value, or at least not on paper, because we have not been able to check any unspoken thoughts and ideas students might have had.

shorthand vs. generalized arithmetic

A second explanation could be, that students do understand how the quantities are related, but that it is hard for them to write it down in symbolic form correctly, with the variables and the operation in the right places. In fact, we believe students see symbolic expressions as shorthand notation instead of an extension of arithmetic. Take, for example, the test task of constructing a symbolic expression from a tabular representation. A correct train of thought might be: "*dB* is always 3 more than *dA,* I have to add 3 to *dA* to get *dB*, so I must write $dB = dA + 3$", or simply "*dB* is equal to *dA* after I add 3 to it". But another student might reason as follows: "*dB* is 3 more than *dA* so *dB* is plus 3, $dB + 3$". This kind of reasoning was observed in the first experimental round of interviews with students. A correct understanding of a tabular representation can easily lead to a wrongful symbolic representation, which is in fact an error of notation.

Dutch school books reflect the belief that it is easier to construct a formula if you have a procedural point of view 'add 3 to *dA* to get *dB*', although they do not put the result *dB* at the end of the expression: $dA + 3 = dB$. An explanation in terms of an operational versus a static perception is not in question; only two students described the relation as a procedure, one correctly and the other incorrectly. The test results do not enable us to determine whether students generally had an operational or a static conception of the relation, nor whether the operational conception worked in the students' favor.

attitude

To finish off we present two results related to student attitude. First of all, we were quite discontent with the way students elaborated their answers. In spite of the teacher's instructions during the lesson series to show calculations or give an argument, students did not develop this kind of attitude.Some of them did not give an explanation even when they were specifically asked to do so, which brought down their score. Perhaps it was not a lack of care but more a matter of insecurity or know-how. In one of the classes the better students gave better explanations and more often than

the weaker students, but this correlation does not hold in the other class. Similarly we saw a discrepancy between the relation gender-explanation: in one class there was no significant difference between girls and boys, whereas the other class showed better explanatory notes by the girls than the boys.
The second observation is one of stamina and motivation: we were disappointed to see about 20% of the students give up on the tasks. Especially the low achievers left some straightforward questions unanswered without making an attempt. It is an observation that agrees with some classroom observations towards the end of the experiment: a loss of interest and motivation.

5.4.2 A small group of students: experiences and results

Three students were selected from each class to conduct the second half of the program. The selection was based on two conditions: an even distribution of pre-algebraic ability and gender, and a positive attitude with respect to interaction. With the help of the teachers we selected three girls and three boys whom we expected to be interested in continuing with the program and to feel secure and confident enough to engage in group discussions.

preliminary

continuity

Pre-algebraic elements like relating quantities, developing notations and reasoning by substitution which are dominant in the instructional materials *Pocket Money* and *Playing Cards,* are developed further in the context of fair trade. The mathematical content as shown in table 5.2 cannot be easily described in consecutive categories, because in the units *Marbles* and *Barter* most abilities are developed simultaneously. For example, the investigation of the barter context involves comparing trade values, constructing and using trade 'equations' (terms of trade), establishing conventions for notations, reasoning about consecutive trades, producing multiples of trade equations and substituting terms in expressions, as summarized in table 5.2. Nevertheless, we try to keep to the same structure as much as possible.

small adjustments

As mentioned before, the activities on trading marbles were relocated in the learning strand, where they would be an appropriate starter for *Barter.* Most of the activities have the same mathematical content as the case studies' activity sheets, but some of them were rephrased and edited. The activities on generalizing, which form a part of the topic *global view*, have been placed at the end of the lesson series because of the abstract reasoning involved, for which reason they may also be allocated to the topic *reasoning*. The cartoon problems were removed and reserved as additional tasks. In the following we describe how the students performed and what they learned.

starting activity

reflection on the test

The experiment started with a reflective session to refresh the students' memories and revert their perspective. In particular we wanted to confront students with the most notable shortcomings observed in the experiment so far: inconsistent notations, reversal errors, and wrongful interpretations of expressions. They were asked to take a look at various student answers from test 1 and comment on them. The activity was meant to have students look at notations and explanations to answers from the on-looker's point of view and hopefully become aware of the importance of conventions and clear reasoning.

The activity was an eye-opener to the children; they were obviously surprised at the variety of mistakes and they enjoyed playing the role of teacher. They were also quite inventive, and suggested a plausible explanation for each answer. The discussions on using a lower case letter both as a label (unit) and the variable were very animated. The meaning of the equal-sign provided a good opportunity to talk about conflicts between mathematical words and every-day language. The reversal error was quite hard to detect; Hans and Robert noticed it first and unfortunately spoiled the task for the others. Merrill and Renske did seem to profit from the researcher's suggestion of checking the formulas with numbers and sums, but their remarks also indicate a tendency to just remember the term 'the opposite'. Jacqueline participated very well at times but frequently made needless mistakes in writing down her answers. Group discussions did result in general agreement of what is mathematically most suitable, with the exception of Robert who continued to defend his own personal preferences.

The value of this reflective activity for the teacher-designer was to make the children active participant in the process of giving feedback. Unfortunately it was not always clear whether the level of understanding surpassed the level of trickery: "reverting the letters and the operations".

switching between representations

orientation task

The marble orientation task consists of constructing symbolic expressions for fair trading – in words, symbols or pictures. The task is perhaps more an activity of symbolizing than switching between representations, but it is more practical to discuss it at the beginning of the evaluation. Besides, one aspect of the task is interpreting each other's productions.

The students were asked to make a trading system for the rest of the class using five types of marbles of different value – super (10), bam (5), glimmer (3), speckle (2) and unit (1). The values and names were written on cardboard cards which could also be used for manipulation. The context immediately appealed to the students; they commented on the values and told about their own marbles. Also the concept of trading fairly was immediately linked to equal value; there were no hiccups this time about damaged marbles and personal preferences. Like before, the activity was intended as a rich, problem-oriented introduction in which the students play an active and constructive role. The students' own productions would form the framework of reference for the activities to follow. Given the emphasis on abbreviations in the previous two units, we expected that students would choose a syncopated or a symbolic form of notation, rather than a picture representation.

representing trade relations

The students worked in two groups of three. They were instructed to write the trade terms in a convenient way, but without mentioning any kind of notation in particular. None of the students used the concrete materials. Hans, Robert and Renske constructed horizontal expressions with symbols and abbreviations (see figure 5.27, bottom) whereas Jacqueline, Esther and Merrill chose for a tabular kind of representation (top of figure 5.27). After about fifteen minutes each group presented the list of trade terms and explained their strategy. They both claimed to have used a systematic method of making combinations – starting with the highest value (super) and then matching the other types of marbles – but not very strictly. We can tell this from their expressions but also from the fact that Esther's group was not sure that they did not miss out a combination; apparently they were not aware that a systematic approach is a guarantee for finding all possible combinations. We see that Robert's group made up expressions with larger coefficients (multiples of the simplest ratio expression), which were also checked numerically. The conjectured issues of clear and consistent abbreviations, the meaning of mathematical symbols and equal value arose naturally and formed a good basis for the activities to come in the last booklet.

1. The value has to stay the same.

2. Super = 10p.	bam = 5p.	glimmer = 3p.	speckle = 2p.
2x bam	2x speckle	1x speckle	2x unit
1x bam	1x unit	1x unit	
1x glimmer	1x glimmer	3x unit	
1x speckle	1x speckle		
5x speckle	5x unit		
10x unit	2x speckle		
3x glimmer	1x unit		
1x unit			
1x unit			
2x speckle			
1x bam			

1 su = 1b + 1 g + 1 sp
1 b = 5 u
1 b = 1 s + 1 g
1000 su = 10000 u
1 b = 1 sp + 1 g
10 g = 1 su + 1 b + 1 g + 1 sp + 10 u

= equals trading for

figure 5.27: two systems for trading marbles

descriptions

After the orientation task the students were asked to fill in a table on trading speckles for units (number of marbles). The table precedes a task on completing descriptions in words about the numbers of marbles and about their relative value:

one speckle is worth .... units, so
1 speckle for .... units, or
... times as many units as speckles, or
... times as many speckles as units.

The students were deliberately confronted with both perspectives to prepare a future task on the conflict between value and number (see heading *meaning of notations*). The last sentence also serves to reflect on inverse operations. Although the concept of inverse sums and operations came to mind fairly quickly, it was hard for students to explain why the number 2 changes to the number $\frac{1}{2}$. The discussion (see figure 5.28) ran aground on the fact that the students thought in terms of value rather than number, which distracted them from the idea of changing perspective. We had expected students to come up with a statement like 'depends how you look at it'.

Esther: Two times as many units as speckles.
Researcher: Yes, and then there is one more sentence. Hans?
Hans: Half times as many speckles as units.
Merrill: Two, two, two, a half.
Researcher: That is strange! Each time it is two, except for the last one. Why is that?
Robert: Because they are turned around.
Researcher: What is turned around?
Robert: The speckles and the units. First it was so many speckles for so many units, and now it is so many units for a speckle ... no ...
Hans: How many speckles for a unit.
Robert: Yes, it is hard to explain.
Hans: How many speckles for a unit, so that is half a speckle.
[... discussion on the meaning of half a marble ...]
Researcher: You have filled in a few lines. One speckle is worth 2 units, and twice as many units as speckles.
Robert: Yes, that's right.
Merrill: Yes.
Hans: Two times as many units as speckles, yes, that's right.
Researcher: And it is clear why it is 2 and $\frac{1}{2}$ in the descriptions ... on what does it depend?
Robert: The position of the marbles.

figure 5.28: change of perspective

comparing quantities

picture representation

The second task in the instructional unit *Marbles* deals with a picture representation of trading marbles, as the visual counterpart to the previous rhetoric, symbolic and numerical representations. Not only does it facilitate the more visually-oriented style of learning, but it is also an appropriate medium for presenting consecutive trade terms as a way of reasoning (see figure 5.29). The student explained her reasoning orally. However, we did not foresee that students would remember the numerical values of the marbles from the starter activity. These numerical values interfered with the idea of reasoning since it made reasoning superfluous. For some students the purpose of the problem was therefore not clear. They claimed that you could never know the value of a bam without knowing the numerical values of a speckle and a unit; they did not recognize the line of reasoning in the picture on their own.

However, a short discussion of the problem cleared up their confusion. The same kind of reasoning task appeared in the second lesson on *Barter*, which the students did solve as intended.

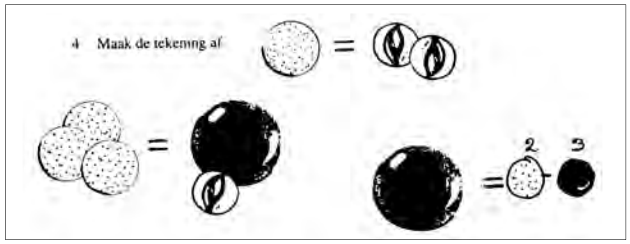


figure 5.29: finding the value of a marble by reasoning

Figure 5.30 illustrates how Renske compared two quantities of marbles by cancelling equal values. The lines connect marbles with equal value (the word *hetzelfde* means 'the same'), and the numbers above the marbles represent their values. On the lines she teams up the marbles with words. Renske's notations are an example of schematizing as well; they helped her organize her thinking. Similarly Hans drew circles around groups of marbles of equal value.

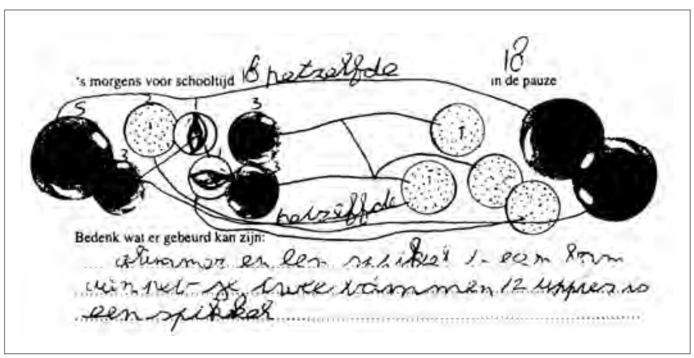


figure 5.30: comparing marbles by cancelling

In *Marbles* students can choose two strategies of making quantities comparable: using the absolute values of the marbles (number of points) or using the relative values (one type of marble in terms of the other). We expected the first strategy to be easier and more popular, which proved to be correct. Both Hans and Renske used the numerical values of the marbles to support their reasoning. The marble activities were intended to be a running start for the comparison problems in *Barter*, which can be solved only by comparing relative values (substitution). Indeed we found that all students succeeded at these tasks, although the answers were not always clear of errors.

The experiences with these children underwrite results in the *Mathematics in Context*-project that grade 5 students already respond well to this kind of reasoning with pictures.

schematizing

The introduction task to *Marbles* instigated students to construct horizontal symbolic expressions, marble combinations in columns and a few drawings. The use of tables and abbreviations as a means of organizing information efficiently continued to improve in *Barter*. The first two unit sections are about investigating the context: recognizing trade terms in the opening story and reasoning with picture trades.

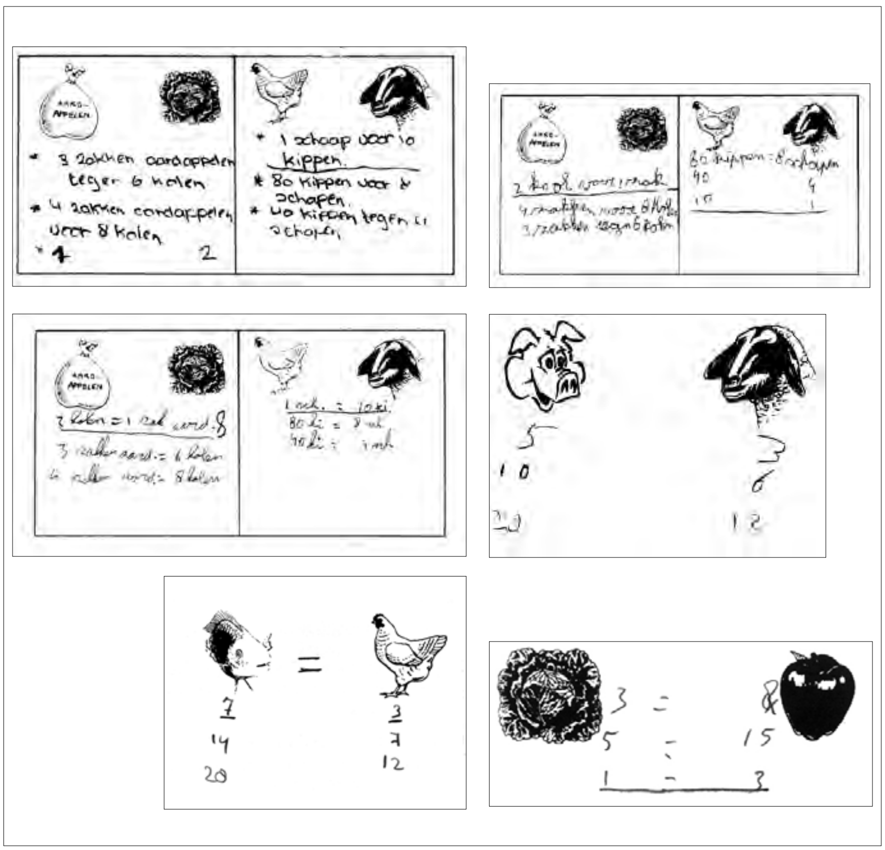


figure 5.31: organizing trade terms

The third section begins with a problem on dishonest trading on the market. An inspector is sent to check that the villagers don't violate the agreements, and he has made a list (in words) of all the trades he has seen. The students have to organize the trades in boxes that display two kinds of food at a time and then decide which trades are dishonest. Figure 5.31 is a collection of student work to show the variety of notations. In each box the lowest ratio is underlined and identified as the *trade law*; expressions with larger numbers are named *trade rules*. We immediately notice some differences. Esther used rhetoric notations and words like (trading) 'against' and 'for'. Hans and Renske began with words but switched to abbreviations and numbers during the task. Jacqueline and Robert used numbers and equal-signs in columns, whereas Merrill used only numbers.

equivalence

After the task of schematizing, the students have to study the descriptions more closely. The first purpose is to check for equivalence (identifying laws and rules), the second is to change perspective and repair the incorrect expressions. The latter task can be done in different ways, by changing either side of the relation. Comparing the numbers to the smallest ratio and finding the mistakes was no problem for the students. Merrill and Renske, as we discovered, had not paid close attention to marking the incorrect expressions, which led to mistakes and confusion in applied problems later. At the end there was a group discussion to reflect on the shortened notations in the boxes; there was some disagreement on how to recognize what the abbreviations mean. Hans and Robert appreciated letter notations and the equal-sign immediately, but Renske and Merrill demonstrated an unreasonable objection to a notation other than their own. Obviously the teacher has to guide this process with a firm hand.

The case studies in the previous try-out showed that students find it difficult to include symbolic expressions in their reasoning. They naturally explain their reasoning rhetorically. For this reason we decided to insert a loose activity sheet, which combines a number of pre-algebraic elements, right after the schematizing task. It starts with a repetition of the picture of trading goods similar to figure 5.29, which the children have already solved. Right below the picture is a frame with symbolic trade terms similar to their own productions, which correspond with the steps in the reasoning process. Students are required to compare both types of representation, match the drawn trades with the correct symbolic ones, figure out the meaning of the remaining expressions and then place all the expressions in a logical order. As it turned out, students reacted that they were solving a problem they had already solved; the goal of using symbolic expressions was not meaningful to them. The motivation was low, which reduced the output even further. It was interesting, though, to see that all the students except Robert first wrote down the three expressions of the picture. A professional algebraist would identify intermediate trades as constituents of the reasoning process. As a consequence the remaining expressions were not placed in what an outsider would call a logical order, but rather as an unconnected left-over. This type of task does not match the student's order of reasoning and therefore clearly misses its aim.

meaning of notations and symbols

The meaning of letter notation in a trading context was self-explanatory for students, especially after their own productions in the marble orientation task and the schematizing task. It was clear from the start that numbers in the expression refer to the number of items, the letters refer to the objects and these letters are not replaceable by number values (in contrast to the formulas in *Playing Cards*). Theoretically the letters could also be object-related, representing the *value* of the items instead of the items themselves. This perception is based on a value medium rather than a medium of quantities. However, the students never spoke of the expressions in this way. It was accepted in the group that some students used one letter or two if necessary, whereas others always used two letters for reasons of identifiability. The role of the letters was not made explicit at this time because it was not an issue and would therefore not have been meaningful.

math symbols

The mathematical symbols in the trade expressions took a longer time to become a custom. For example, in the marble activities Renske used the sign / to replace the word 'for', but also the equal-sign. Jacqueline used the plus-symbol for the word 'and' as we use it in normal language. The other students complained that these symbols were not appropriate. Hans objected to the use of the plus because it made the expression look like a sum, but Jacqueline answered that there was no equal-sign in it. A short discussion on other differences between normal language and mathematical language made the students stop to think about issues of standard norms and mutual understanding in the classroom.

minus sign

In one of the lessons we came to discuss the possibility of rewriting the expression $3\ \mathit{chi} + 12\ \mathit{ap} = 10\ \mathit{fi}$ into its reverse; the students all agreed that it is possible. Jacqueline proposed swapping the two left terms, to which most students objected. Renske suggested $10\ \mathit{fi} - 3\ \mathit{chi} = 12\ \mathit{ap}$. Hans could only explain the meaning of this expression by changing chickens to fish and then subtracting; none of the students was able to explain the meaning directly. Given the tip to think in terms of "the value of 10 fish minus the value of 3 chickens ...", Renske succeeded to finish the sentence and even substituted the original formula into the reversed one as follows: $3\ \mathit{chi} + 12\ \mathit{ap} - 3\ \mathit{chi} = 12\ \mathit{ap}$.

mutual agreement

We had conjectured that the equivalence task – at the latest – would facilitate a discussion on what type of notations are suitable for describing trade terms and how we can achieve consensus. After a short struggle by Renske and Merrill, the group agreed to use abbreviations – in principle the first two letters of the word – and the equal-sign. After completing the activities on reasoning with expressions, they received a small note-book (the so-called 'book of law') in which to write all the trade rules and laws encountered in the activities, as well as a few new rules for each law to practice notations and ratio expressions. This small note-book was intended to be an aid to memory, but in practice the visual overview of trade laws (see figure 5.31) was more convenient because it enabled students to reason by substitution with most of the trade possibilities on one page. For this reason it is important to keep both the visual and the written approach to trade relations available.

generality

One of the students achieved a more advanced conception of letter use than the others, which we can trace back to the first task in *Marbles*. During the discussion on describing the numbers of marbles in a general way (see figure 5.28), Robert mentioned the word 'formula' a few times. He connected the description '2 times as many units as speckles' and its inverse with the formulas in *Playing Cards*. In his opinion sums and inverse sums demonstrate that formulas are correct "because you can use [the formulas] for everything". He also commented that tables are formulas in a way, because 'for example it is all times two'. He appears to identify a formula with the concept of relation. In summary we can say that he shows an informal understanding of general versus specific representations.

variables

Roberts little note-book shows that he has independently constructed formulas with variables instead of trade rules with abbreviations. Initially he wrote the word formula *bags of p* × 2 = *cabbages* (*p* for potatoes), but from the second page on he used the same letters in his formula as the trade term, for example the trade term 1 *bp* = 3 *bf* (*bp* for bags of potatoes, *bf* for bag of flour) followed by the formula *bp* × 3 = *bf*. It is interesting to see that he always placed the operation behind the first variable on the left-hand side of the expression, and never in front like *bf* = 3 × *bp*. Perhaps the order still enables him to fall back on the idea of performing a procedure and getting a result. Still, he reads his expressions aloud as "the number of bags of potatoes times 3 is the number of bags of flour". We noticed in test 1 that expressions in this form often provoked a static description of the formula. It is not clear to what extent Robert is able to interpret his formulas naturally, but considering he also made very complex formulas like $fi : 2\frac{1}{3} = ch$, it appears that he did not use meaningless tricks.

change of medium

Letters undergo a change in meaning when a trade equation (an expression for fair trading) is rewritten as a formula expressing the number of goods. For example, the trade expression 1 *f* = 4 *a* can be read as a trading procedure 'trade 1 fish for 4 apples' or as a static description of value '1 fish is worth as much as 4 apples'. In the latter case, the medium is one of value (worth) and the letters refer to objects. If the expression is rewritten as the formula $f \times 4 = a$, the new medium is one of number (amount) and the letters represent the number of objects. (Yet another meaning of the letters has been mentioned in section 3.4.2 with respect to the historical Chinese barter problem, where a letter could either stand for the object or for the money value of the object.) This change of meaning has been included explicitly in the mathematical content, but in tabular form instead of symbolically. We discuss this activity in the section on *reasoning*.

repairing inequality

The topic of making two quantities equal has been worked out into context problems on making an unfair trade or distribution fair. The last task in *Marbles* requires students to distribute a given number of marbles fairly amongst three imaginary children. We expected to see two approaches: direct or indirect. In the indirect approach, the problem is split up into two parts: first determine the total value of the marbles and divide it into three equal portions, and then select marbles to match the values. The second strategy is to distribute the marbles directly, without calculating the average value first.

conflict with reality

The total value was deliberately made incompatible in order to create a situation of conflict, but the students' objections caught us by surprise. We had expected the students to distribute the marbles as far as possible, either with or without calculating the numerical values, and come with a suggestion for settling the remainder. Instead, they protested to the answer of $9\frac{1}{3}$ points because this could never really exist. Consequently the lesson degenerated into a hilarious discussion on partial marbles. Even the instruction to think of a solution to this problem – a way to make even distribution possible – led to an untenable situation: the students could not agree on a fair way to obtain the missing unit marble. The unexpected strong affinity with reality warned us for similar problems in *Barter*. Students have to learn when a mathematical point of view should prevail.

minus sign

The inspector's list of unfair trades already gave ample opportunity for students to adjust expressions of inequality. In the previous experiment students responded well to the idea of debt and negative quantities, and so we anticipated that another context problem might facilitate the construction of a trade expression involving debt. Indeed, Hans immediately suggested that the trade would be fair if the customer paid the shortage back later or the next time. The other students agreed that this would make the trade fair. The next question asked students to construct a trade rule to express the debt, and how you can recognize the debt. Hans and Robert discussed using a symbol; they decided not to use a minus sign because it would give the suggestion of taking away, like in a sum. Apparently they conceive debt as something that still has to be paid and not taken away. During the group discussion one of the students suggested an *X*, but Hans decided this looked too much like a multiplication sign. The group agreed unanimously that the best notation would be adding the word 'debt' or an abbreviation for it. So despite an earlier encounter with a negative term, symbolic expressions with a minus sign were not feasible for this group of students.

reasoning

aims of the task

One of the tasks on abstract reasoning in *Barter* is concerned with distinguishing between two different media for comparing quantities. As described before, a trade relation can be viewed in terms of value (trade terms with abbreviations) or in terms of quantity (formulas with variables). It was decided to present the problem in tabular form because we reasoned the numerical values would limit the abstractness. The problem starts off with the ratio table 5.5 already filled in with numbers of apples and fish satisfying the trade relation ‘1 fish for 4 apples’, followed by the following questions: “which food is worth more, an apple or a fish?” and “how many times as much?” The group discussion that follows should suffice to change the students’ perspective from *numbers* of fish and apples to their relative *values*. Ratio table 5.6 is based on *values* of an apple and a fish: if the value of an apple is 1 (guilder, quarter, ...), the value of a fish will be 4 (guilders, quarters, ...). After a few more combinations of tables, some free of choice, the students are asked to write down what they have noticed and to try to explain it. So instead of writing a (word) formula on the numbers and values (as Robert has done) and reasoning about the changed position of the numeral, students have to reason why the numerical values of the variable quantities themselves are reversed in the tables.

number of apples	4	8	16	20	40	100
number of fish	1	2	4	5	10	25

table 5.5: number of items

value apple	1	2	4	5	10	25
value fish	4	8	16	20	40	100

table 5.6: value of items

partial understanding

Two students interpreted the table independently of the trade term because they looked at the entries from left to right. This was really an eye-opener for us, and of course the whole task is based on looking the right way. Renske made the reversal error in the first question, which Hans immediately corrected. From that moment on she appeared to be on guard. Jacqueline needed a moment longer to understand the changed point of view. The concept “whenever you have more of an item, its value is less”, did not become explicit, though. Perhaps the step of generalization is premature. For instance, although all the students agreed in the preceding task that in this story a sheep is worth more than a sucking-pig – by comparing their values in terms of apples or fish – they were not able to deduce it from the expression 3 *sh* = 5 *sp* directly. Hans calculated that 1 *sh* = $1\frac{2}{3}$ *sp*, but this was already too abstract for everyone but Robert.

number vs. value

We had expected the better students to be able to interpret the numbers in table 5.6, but they were not able to change their perspective. Every attempt to explain the numbers 4 and 1 led to confusion because they clash with the trade expression. One example was enough to instigate the moment of insight, after which the other tables were filled in correctly (with the exception of Merrill, who first filled out two tables exactly the wrong way around, and then produced two tables with different pairs of goods instead of the same ones). Although the students succeeded at inverting the tables, they never showed to have true understanding of the reason. Their comments stayed at a level of describing what happens, not explaining it. For this reason one question on this topic was included in test 2, which will be discussed later for each student individually.

predictions

The activities on making predictions are embedded in real life situations, to find answers to questions like: “How can the trade terms be adjusted if a general means of payment is introduced”, and “What is the effect of bad weather on the value of the goods?” The tasks also call on competencies like generalizing and looking qualitatively, for which reason they have been named in the category ‘global view’ as well. We aimed to have students realize that in certain situations qualitative reasoning makes calculations superfluous.

adjusting a value

In answer to the first question, the students suggested unanimously to use the ‘cheapest’ item on the market to determine the other values. In response we purposely proposed a value of an apple which leads to fractional values, deliberately causing a conflict with reality. We intended students to object to fractional values and then search for a way to avoid them, preferably by reasoning but otherwise by trial-and-adjustment. Mathematically this means multiplying the proposed value of an apple by 3 because the value of one of the food items contains the fraction $\frac{1}{3}$. However, as soon as the girls found a fractional value, they became confused, lost their confidence and stopped prematurely. The instruction to think about changing the value of an apple only resulted in more tiresome calculations. The children then suggested introducing a smaller currency, accepting debts or adjusting the trade terms, instead of adjusting the value of an apple. After a while Hans proposed multiplying by 3 and the others agreed this would work, but they did mention general applicability. We can conclude that the higher meaning of the problem and its similarity with other activities (like the table of incompatible marbles or predictions with formulas in test 1) did not come through.

generalizing

The experiences in this group on generalizing about how the weather can influence food prices were similar to the case studies. The students succeeded to reason about the values of one or two items, especially in combination with calculations, but reasoning in general terms was a lot harder for them. There was also confusion about the relative values, which should stay the same, whereas the absolute prices are doubled. Only Renske openly admitted she would like to check her prediction with numbers; the others thought they would succeed just by reasoning. To be honest, a few overestimated their capacities for they could not explain the mathematical meaning of the price changes in their own words.

restriction problems

The topic of solving problems with restrictions is revisited in *Marbles* by way of tables with incompatible trades, right after the initial activities on representing relations. The activity is very similar to one of the activity sheets from the case studies experiment (see figure 5.5); it has children practicing tabular and descriptive representations with an additional factor of difficulty. Depending on the value, some combinations of marbles are not so easy to match. For example, trading glimmers (worth 3 points) fairly for supers (worth 10 points) requires at least 10 glimmers and 3 supers. The problem is restricted by the compatibility of the numbers; mathematically we can describe the task as finding the smallest common factor. The second part of the task consists of deducing from the table a trade description with three types of marble, first with words and then in shortened notations. Students are guided towards a trade term with abbreviations (like 3 *su* = 10 *gl*) instead of a word formula relating the *numbers* of marbles, which they do first (see figure 5.28). Such a word formula is very abstract in situations when both numbers in the reduced trade relation (the trade law) are unequal to one.

mis-judgements

The better students had no trouble with the table, and they were challenged by the task. The weaker students understood the principle but ran up against the demands of having to remember the values of the marbles – which are no longer visible – as well as finding their smallest common factor. We found that three marbles was really a lot tougher for these children than two. They frequently made errors of calculation which in turn frustrated their confidence and motivation. Moreover, it was a case of poor timing to place the task so early in the lesson series because, looking back, it is clear that the visual and descriptive comparison problems are much more accessible to students.

The hypothetical structure of consecutive tasks on tables and descriptions does not weigh up against the natural accessibility of context problems. And contrary to our expectations, it was not so obvious to students that they could use the numbers in the table for the trade term; some felt they had to return to the values of the marbles. Perhaps the meaning of the table was lost in the difficult process of calculating the values. The content, order and structure of these activities have been looked at critically when the instructional materials were adjusted.

finding combinations

The restriction problems in *Barter* are not different to the previous try-out. They are built up as follows: students work out all possible combinations of cabbages and bags of flour that amount to a total of ten fish, which is one condition, and then we give them a second restriction: the villager wants at least 6 cabbages and 4 bags of flour. The students then have to decide which combinations still comply. The results with this group of students are similar to the case studies results: students had trouble remembering the restrictions in addition to the trade rules and any other information given in the problem. They also tried to remember everything instead of making draft notes or developing a smart strategy. Only Hans found all the combinations in a logical order and discovered the pattern of exchange (cabbages for bags of flour) and the others drowned in the calculations. Robert misunderstood the question and did not attempt it again, but he demonstrated his understanding during the group discussion.

different types of tables

An additional difficulty is the changed meaning of a table in this situation. In order to organize the student answers and also guide them towards a more static approach a table was written on the black board, but without realizing the different interpretation. In a ratio table a number of goods are traded for a number of other goods, but in a so-called 'combination table' the number of goods need to be added because they make a combination. The students made many mistakes and were confused time and again about the relation between the numbers.
In other words, the task is too demanding in this form. The concept of solving problems with restrictions is feasible for most students in an arithmetical setting – with direct number relations – but not in story problems. In the next version of the design, this task is preceded by simpler restriction problems, especially to give students more opportunity for learning to schematize the information. In an arithmetical situation the restrictions are easy to remember and the solution is often directly visible, but story problems are more accessible after they are mathematized and schematized.

schematizing as a tool

Students have shown an acceptable understanding of tables and abbreviations as a convenient or efficient notation, as we have already illustrated. But schematizing as a tool takes it one step further. Not only is a table a well-organized way of writing down possible answers, but it can also instigate a more structured way of thinking. For instance, as in the case of the restriction problem described above, a table can elicit a pattern that is otherwise hard to notice. And the ordered nature of a table also helps to track all possible solutions to a problem by reducing the chance of omission. None of the students thought of using a table for this purpose, nor did they explicitly mention that the table helped to solve the problem. Hans did find the possibilities in logical order but not with an identifiable scheme; he appears to have structured the problem mentally. And although all students claimed to have used a structured approach to constructing trade terms in the orientation task on marbles (see figure 5.27), we cannot determine how thorough they were. We cannot ascertain that their schemes (abbreviations and column notations) were a tool for them.

5.5 Evaluation and reflection

learning trajectory

In summary we can say that the pilot experiment – the tests and the classroom observations – has shown again that the reasoning stream in the program is more accessible to students than the schematizing stream. The map in figure 5.32 illustrates the anticipated learning trajectory as envisioned prior to the pilot experiment. The encircled topics are unsuccessful elements in the learning trajectory at this stage of the project, due to unsatisfactory student performance or inappropriate mathematical activities. The topics with broken lines are only partially successful. The structure of the map – the order of and the connections between pre-algebraic elements and the increase in their complexity – still holds true, although some aspects of global view and meaning of notations are more complex than the map lets us believe, especially in situations where students are required to *reason* about *meaning of notations*, in other words, where both strands meet. For the average student the two strands do not interact other than in the initial stage of comparing and relating quantities, let alone stimulate each other. For the next round of design it is essential that some activities are concerned primarily with bringing the two streams together.

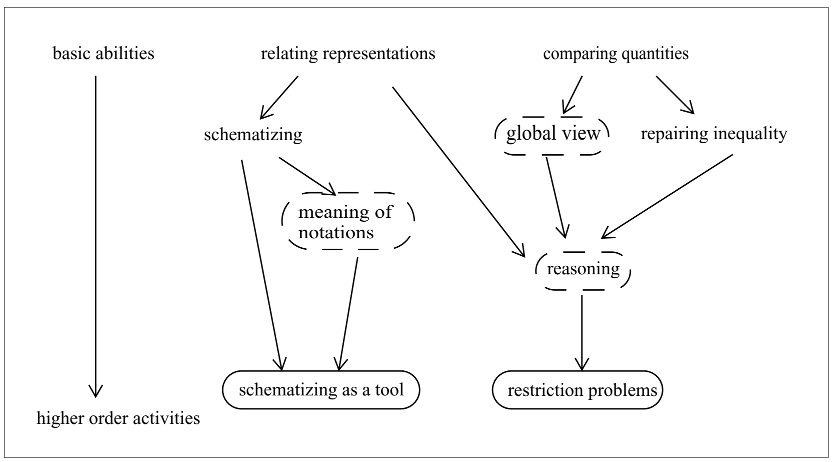


figure 5.32: evaluation of the learning trajectory

reasoning stream

Students achieved a moderate level of reasoning about quantities in context situations; the better students were able to conduct a longer reasoning process. Inverse operations and repairing inequality were two successful topics. Restriction problems were too ambitious for the lower performers and required more instruction than intended, but especially reasoning about patterns of change and different media of expressions (value versus number) turned out to be too abstract. Nonetheless, Robert's success at transforming a trade relation into a formula with variable quantities illustrates that this topic is not entirely misplaced, and perhaps a revision of the activities can give better results in the future. The intended development of the bar (strip) as a mathematical model of and for a relation between quantities did not occur; the bar appears in three different settings, but neither coherently nor with a clear target.

schematizing stream

The first level of schematizing – interpreting representations and organizing information – was achieved by all the students. They were capable of working with tables, the rectangular bar and barter notations. Most students performed well at moving from one form of representation to another. The group of six students certainly developed a better sense of meaning of notations, for example the use of math symbols and norms for consistency and uniqueness of letters. However, the formal notations used in *Playing Cards* have proven to be unfeasible for the majority of students, as well as the idea of using a table and a rectangular bar as problem solving tools. And students did not mathematize a description like 'two more than half of the total amount', for example by symbolizing or visualizing the information to get a better grip on it. They each attempted to solve it mentally, and only Hans succeeded. Merely when the task specifically asks for 'convenient notations', students suggest using abbreviations and math symbols or making a table or another scheme.

shortening notations

Very important is the consolidation of earlier tentative suppositions that introducing formal algebraic notations – with letters as variables – at this stage of early algebra learning may not be feasible. Students' preference to keep a link with the context, their inconsistent or undesirable symbolism – which clashes with mathematical conventions – and the reversal error are three legitimate reasons for postponing the introduction of algebraic symbolism. It appears that the proposed learning strand fails to develop the concept of letter notations; the confrontation with different roles of letters does not lead to discussion followed by insight and consensus as conjectured, but to confusion, disagreement and errors. The introduction of variables – of letters that can have varying number values – has shown to be incompatible with the students' natural application of letters. In other words, the current trajectory on shortening notations seems to diverge students' productions instead of converging them towards a starting-point for symbolic algebra. We expected students to construct their own mathematics in a sense-making way, but instead we found that every two steps forward were followed by – in the eye of the professional – three steps backwards.

attitude

In general the children were very sensitive to context issues and easily distracted from the mathematical interpretation by practical problems or personal objections, especially the group of six students. The advantage of such an attitude is the ease of introducing a new context, but a serious drawback we encountered several times was the students' blockade to looking beyond the outskirts of the problem and dealing with the mathematics. In some cases the activities created a conflict with reality that was not intended at all, which had a negative influence on the learning environment. Other characteristics of student attitude prerequisite for a realistic mathematics learning environment – in particular guided reinvention – are active participation, interest in new solution strategies and reflection. These aspects are not self-evident, as we found out. Of course it is the designer's task to create activities that are challenging for students, but an investigative mind and the perseverance to keep trying are matters of norms, values and character. The teacher is a key player in establishing clear social and math norms and a positive learning environment, stimulating and guiding the learning process and keeping students involved. Somehow a number of students was unable to accept the norms set by the proposed learning program: listen to alternative strategies, participate in classroom discussions, accept democratic decisions, and be prepared to investigate a problem. Especially the open resistance demonstrated by some students to illustrate an answer with calculations or other arguments we found very disconcerting. The current generation of Dutch teenage learners is more obstinate and product-minded than we had anticipated.

material revision

The lesson materials are quite different from the regular mathematics text book. In the eye of the arithmetic student, mathematics is about calculating a numerical solution. A lot of the activities in the proposed learning strand have another purpose than students expect, for example studying different kinds of notation to describe one situation, or solving a problem in a new, different way. Apparently this can make them nervous, insecure or even unmotivated. Some students actually mentioned that they felt they had to solve the same problem twice. It will therefore be essential that the tasks become more problem-oriented and challenging to students, and that we try to make every activity 'new'. One of the elements that should be given more priority is that of organizing and mathematizing. Students are less competent at translating a story problem into mathematics than we had anticipated. If we aim to teach 'schematizing as a tool', we should create opportunities for students to become more skilled at schematizing first. Another general shortcoming is the limited number of opportunities to differentiate, both in complexity and rate. Apart from their own productions, students all make the same tasks. In some lessons the high performers were slowed down too much by the rest of the class, and sometimes the cognitive capacities of the moderate and low performers were so limited that a large part of the class never got round to learning. If problems are too difficult or too easy for students, they lose interest. For this reason we should strive for adaptive teaching, and develop additional activities on two levels – exercise and enrichment.

5.6 Peer review

The ambivalent results in the pilot experiment in primary school necessitated a review of the research project with a panel of colleagues – math educators, educational designers and mathematicians. The aim of the meeting was threefold: 1) to get feedback on the proposed pre-algebra program and its premises 2) to discuss the shortcomings of the latest design and 3) to exchange ideas of possible improvement. Fourteen participants were provided with the teacher and student materials and a summary of the project's theoretical framework – research aims and questions, the hypothetical pre-algebra learning trajectory and methodological intentions for the remaining duration of the project.

The goals and organization of the peer review resemble a research method called the Delphi method (see also section 4.4.6). The panel was a group of critical experts representing all the different factions involved in the project. They were selected purposefully in order to

- evaluate and revise the design of a teaching instrument;
- obtain field legitimization for further research;
- explore solutions to research problems;
- compare different points of view.

The review set off as a frank and open exchange of ideas and then converged towards a more structured discussion of specific problems. It was our intention to incorporate the suggestions for improvement and then consult the participants again individually, but not all the participants were able to take part in the second round. The discussion produced some new essential ideas that caused the project to take a very different turn.

aims and validity

breach between theory and practice

Some of the peers commented that the research targets and aims as stated in the research plan are not identifiable in the research produce. They did not see how the proposed learning trajectory, and the lesson series in particular, would provide answers to the research questions posed. For example, what is the role of the history of mathematics in the project, and in the program? The historical elements in the project are hardly visible. Similarly the cognitive discrepancy between arithmetic and algebra has not yet been clarified. Which idiosyncratic aspects can be identified that might be responsible for the inaccessibility of algebra? What is it that makes a mathematical activity 'algebraic'? Two generally accepted properties of algebra, the reduction to standard forms and the letter representation of unknowns, are obviously problematic in the current design. In other words, the theoretical framework appears to be in dissonance with the instruments that have been developed to collect data, and therefore the validity of the project might be at stake.

history of mathematics

history not visible

A majority of peers questioned the relevance of the historical component in the research project. In fact, one colleague even suggested removing it altogether because he felt the program would be overloaded. The role of history has until now been limited to being a guide and a source of inspiration for the designer (for example, with respect to the choice of context and the preference for word problems, see also section 3.4.2). Unless the historical development of algebra is reflected more in the hypothetical learning trajectory, the current application of history is insufficient for comparing philogenetic and ontogenetic developments. And since history is an explicit topic in only a fraction of the lessons, we are not be able to gather enough information on the effect of using history in math education. In other words, the research question on the feasibility of history of mathematics as a didactical tool will only be relevant if we make the history more apparent in the learning trajectory as well as the lesson materials. Further advice regarding the implementation of history in math education and educational research included: use original sources whenever possible, and be cautious in suggesting (simplistic) parallels between western and non-western notations.

letter notations

meaning of letters

Another main issue of the discussion was the proposed style of letter notations. Most experts were strongly opposed to the use of a pair of letters as one variable, either because it is not compatible with algebraic convention or because it seems unnatural. One person suggested that good students might even object to such a context-bound notation. Letter notations illustrate well the ambiguity of 'guided reinvention' as a design heuristic, or as one of the project initiators phrased it: 'How do you provide students with the freedom to construct their own notations, while at the same time guiding them in one direction?' Someone suggested, that if students are not ready for symbolic notations, it would be better to keep with syncopated forms like word formulas. An historical input along these lines: context-dependent words to represent the unknown can be a suitable alternative to a literal symbol.

It was agreed that the essential issue of letter notations is the meaning of the letters, in particular the distinction between numbers and magnitudes. Make students aware of the difference: when are letters numbers, and when are they magnitudes? Letters cannot be considered variables unless the quantities they represent truly vary. One of the objections to the context of scores in *Playing Cards* was the fact that the number of points are artificial magnitudes. They are not variable but constant in each new situation, and hence the relation between the letters is determined by the number values instead of vice versa. Another colleague emphasized the fact that letter manipulations are different from number manipulations, and that they must not be confused. Last, someone mentioned to beware of dimensions when operating in a symbolic medium.

choice of context

contexts criticized

The more structured evaluation of the lesson materials that followed focussed primarily on context relevance. The activities in *Pocket Money* need to become more problem-oriented, to make general descriptions worthwhile. For example, for the activity on Mark and Eelco who have 4 guilders and 8 guilders allowance respectively, one of the colleague designers proposed to give a fixed relation instead, like "Eelco has twice as much as Mark", and then formulate the question "Can Mark have 10 guilders more than Eelco?" Although the students were quite motivated by the context of playing cards, the panel of experts were not so pleased with it. They felt that activities on describing and calculating scores might have more meaning in a context which is less artificial than a game of cards, for example if linear relations were used to compare achievements at an athletic meet. The problems should give rise to a natural need for general forms. The contexts in the other instructional units were seen as sufficiently challenging, although the abbreviations used in barter trade terms –

where letters are referents to objects rather than quantities – were not considered appropriate by everyone.

revision of student units

The panel suggested to integrate the mathematical content of the instructional units *Pocket Money*, *Playing Cards* and *Marbles* into one, new problem-oriented unit. *Barter,* and the secondary school units should be revised and restyled where necessary but the mathematical content was considered qualitatively acceptable. Two important points of criticism put forward by a fellow designer concern the lack of structure in the instructional units. The new materials should give students more opportunity to mathematize their constructions: horizontally (schematizing, constructing models of activities) and vertically (abstraction of notations, generalizing solution strategies, developing models for mathematical reasoning). The students also need a theoretical framework to focus their attention and organize their thinking. A more general point of advice was to draw on existing materials for support and inspiration.

research plan

Towards the end, the discussion focussed on the next classroom experiment. We spoke about methodological issues: the set-up (at least two schools at both levels, not a comparative study) and data collection (observations, audio-visual recordings, interviews, written tests). And finally some researchers mentioned the relevance of classroom organization (individual work, group work, classroom discussions) and the role of the teacher (to direct the learning process and provide structure).

5.7 Conclusions

The first two design cycles and the peer review have led to a number of conclusions and points of action regarding student results, the mathematical content of the learning strand and the theoretical framework of this study. In this section we confine ourselves to the first two items; theoretical and organizational issues of the study in reaction to the peer review are discussed at the beginning of chapter 6. First we present a compact list of student performance trends observed in the various try-outs (ordered by topic), followed by the most noteworthy conclusions and points of action regarding the mathematical content in general.

attitude

1 Generally speaking the students did not attempt to elaborate their answers, despite clear teacher instructions.
2 Especially the low ability students tended to lose interest and motivation, giving up even on some straightforward tasks.

shortened notations

1 Students in grade 5 preferred to use rhetoric notations, organizing the problem and writing their answers in full. In grade 6 we observed students who enjoyed and chose to use shortened notations, but we also saw students who continued to write in the vernacular.

2 Most primary school students were not yet ripe for algebraic symbolism; they had trouble interpreting and writing syncopated and symbolic notations in spite of their own productions in preceding activities.
3 Syncopated and symbolic notations appear to be responsible for the occurrence of the reversal error in situations of numerical substitution as well as global interpretation.
4 As the primary school students became more confident and inventive at symbolizing, they tended to suggest unconventional, counter-productive notations, i.e. notations that diverge from the algebraic symbolic language that we wish them to reinvent.
5 It is the teacher's task to look out for opportunities to discuss the meaning of symbols, why they might have been invented and why we find them useful.

switching between representations

1 Difficulties of translating a static, symbolic expression into a dynamic statement in the vernacular appear to be caused by conflicts contained in the process versus product perception of algebraic expressions.
2 The primary school students worked well with tables representing a *ratio* between two items, but when a table was used to organize *combinations* of goods totalling a given value, they had trouble looking at the numbers in the table from a new perspective. Apparently these students do not learn to use tables in different areas of application.
3 Students were found to be more capable of translating a ratio table to a static trade description (like "1 banana is worth 2 apples") than to a static description of number of items (like "twice as many apples as bananas"), which requires yet another change of perspective.
4 Complex ratio tables for trading goods – where trades are sometimes not compatible – were more difficult for the primary school students than expected.
5 In the pilot experiment the primary school students had no trouble constructing a trading system for marbles based on equal value, while the same activity failed in the case studies. It seems that the units *Pocket Money* and *Playing Cards* has helped students to develop a mathematical outlook on equivalence.
6 In the pilot experiment students had little trouble switching between the rectangular bar (a visual presentation) and a description or a numerical representation.

schematizing

1 Schematic diagrams were used primarily as a calculational or an organizational tool, but not as tools for mathematical reasoning (when unknowns are involved).
2 In the pilot experiment the rectangular bar emerged naturally as an abstraction of paper money to represent a given amount of money visually.

3 At elementary school level the rectangular bar did not develop into a model for mathematical reasoning. Students struggled to accept its indeterminate character and most problems did not really require a visual representation; the problems that did were usually attempted with mental arithmetic and then left unsolved.

repairing inequality

1 The pilot experiment showed that the recurring theme of inverse operations appealed to students and was conceived by them as useful and well worth learning.
2 A procedural conception of expressions seems to support activities of making unequal amounts equal, but students struggled to symbolize their proposals; the combination of inequality and shortening notations is therefore not appropriate.

global view

1 In both experiments primary schools students were able to reason qualitatively about formulas – like determining which variable takes on the highest value – but they did not use it to check their results. In fact, students who made the mistake of misinterpreting the relation became confused when they were asked to reason about the quantities.
2 Contrary to our expectations, the pilot experiment showed that reasoning about invariance of relations between two given quantities was not understood by the students. For example, the quantities 4 and 16 satisfy the multiplicative relation '4 times as much' as well as the additive relation 'plus 12'. Generally speaking students were not interested to know and explain why, when 4 and 12 are halved three times, the relation '4 times as much' continues to hold and the relation 'plus 12' does not. The task did not motivate them and it also caused various practical problems.

reasoning

1 Given the second conclusion on 'global view', it is not surprising that predicting the behavior of variables *without* calculating with specific numbers has proven to be too complex for most students. Manipulating barter expressions – where the letters in the expressions are object-related – posed little of a problem, except for two cases: when students were asked about the influence of changing the value of all the products, and when the medium of the expression changes and the letters get a new meaning (from number of items to the value of an item, for instance).
2 Reasoning qualitatively with two consecutive formulas is something students had more success with.

reversal error

1 Misinterpretation of relations – also referred to as the reversal error – appeared more often when students switched between a symbolic and a written description than in other situations of transformation.
2 The reversal error also occurred when students were asked to rewrite a formula (making the dependent variable independent and vice versa).
3 Students who use equal-signs to give intermediate outcomes, not only violate the symmetry and transitivity of the equal-sign but also appear to make a reversal error. Such an expression can be meaningful if it is conceived procedurally, but from a static point of view the operations no longer comply.

making assumptions about unknowns

1 Representing an unknown quantity or magnitude by a visual model like the rectangular bar clashed with students' intuition to make only precise, correct drawings on scale.
2 The try-out *Making up your own problems!* (see section 5.3.1) has shown that students do not object in principle to assuming a certain value for variables or to making up a restriction for the problem. Apparently it matters whether the variable actually varies or whether the variable is a fixed number determined by the problem's conditions. The girls were more prepared to make assumptions than the boys, but this may be because they are generally known to be more compliant and do their school work as they feel is expected of them (Van den Heuvel-Panhuizen & Vermeer, 1999).

dynamic versus static conception, meaning of letters

1 Problems of acquiring both a dynamic and a static conception of relations appear especially in situations involving symbolic notations.
2 A few students have shown to be able to interpret expressions both as a procedure and as a static product. In the case studies, one grade 5 student generalized the specific situation 4 = 8 – 4 to a static expression involving the variables *zg M* (meaning 'Mark's pocket money') and *zg E* (Eelco's pocket money), namely *zg M = zg E – zg M.* Robert, one of the students in the pilot experiment, constructed formulas for trading barter goods as an alternative to the usual trade expressions. In Robert's formulas the letters represent variable quantities of goods, while the letters in trade expressions refer to objects. For most students the trade expression $3b = 5a$ represents a dynamic process, 'trade 3 bananas for 5 apples', while for Robert the formula $a \times \frac{5}{3} = b$ describes the state 'five-thirds times the number of apples is equal to the number of bananas'. Hans, another student in the pilot teaching experiment, switched very easily between the media 'value' and 'number'. He was the only student who could explain the change of perspective

in the written test 2, but perhaps more significantly, he applied the inverse relation to solve a system of equations: if 5 melons are worth 2 pineapples then the values are in the ratio of 2 and 5 units respectively. We believe that free productions like these can provide opportunities for mathematical discourse on different meanings of letters and the dynamic – static duality.

problem solving/restriction problems

1 In the try-out *Making up your own problems!* (see section 5.3.1) we observed that students barely use visual means to investigate problems.
2 Normal life situations for developing the concept of equivalence – value, fair trade, balance – are meaningful and real to students and therefore suitable, but subjective aspects like damage and personal taste can interfere with the strict, context-free mathematical norms of equivalence.
3 Children do not self-reliantly search for a an efficient, systematic strategy of problem solving; they tend to be satisfied with trial-and-adjustment approaches.
4 Primary school students can cope with restriction problems in a transparent arithmetical setting, but word problems with restrictions tend to be too complex because students are not able to mathematize (schematize or visualize) the problem to make it more accessible.
5 Explaining one's reasoning, either orally or on paper, is more difficult for students than carrying out the task, particularly where it concerns algebraic competencies like equivalent expressions or the meaning of letters. The instruction 'explain your reasoning' is taken very literally: students use full sentences, not symbolic expressions. Their explanation usually reflects a procedural approach to the problem: "first you do this, then you do that", etcetera, rather than using the different states of the problem as consecutive steps in the solution process.

mathematical content in general

procedural approach

With the exception of trade expressions in the unit *Barter*, where letters take on only one role, symbolic notations have been removed from the primary school instruction materials. Especially the combination of a static perception of formulas and their symbolic appearance has been found to be beyond the bounds of this study's target group. A more rhetoric and procedural approach to formulas (with word formulas, for example) seems more natural at primary school level. The experiences with Robert – one of the primary school students in the pilot experiment – imply that a combination of a dynamic and a static conception (dynamic trade expressions written as static word formulas) is not necessarily out of range for every student. Still, we have chosen not to incorporate formal symbolic expressions in the learning strand because our learning strand is not intended for the high achievers only. A procedural approach to relations also concurs better with the Dutch secondary school algebra curriculum, where formal notations and structural aspects of formulas are postponed until grade 9 and 10. Indeed, the essential aim of the research project is to investigate the learning and teaching of *pre*-algebra, not algebra, using students' arithmetical foundation. We have utilized the historical development of algebra to inspire and direct us in developing mathematical activities that expose connections and discrepancies between arithmetic and algebra (see also section 3.4).

schematizing

The decision to remove symbolic expressions other than trade terms from the primary school instruction materials does not have any consequences for the combined development of reasoning and symbolizing competence. We can maintain the core of the hypothetical learning trajectory as before. Other notions of schematizing – such as visualizations and tables – are still valuable component of the program, certainly as tools for organizing and problem solving. The materials for secondary school students will not be influenced by these changes, either, because they can also be used as an independent learning trajectory in which notations have been given another emphasis.

6 Final phase of the study

6.1 Introduction

three levels of research results

In this chapter we present the results in the final phase of the research project. The term ‘research results’ refers to different types of results at various levels, as shown in figure 6.1. At ground level we mean the field test results obtained from student work, questionnaires, observations, and so forth. On a higher level we mean research results regarding the research project as a whole: the researcher’s reflections on her personal learning process, implications and ideas for improving the designed prototype and the answers to the research questions. Finally, on the third level we reflect on the relevance and implications of the research results in the discussion, ending with some recommendations for other educational designers (both in chapter 7).

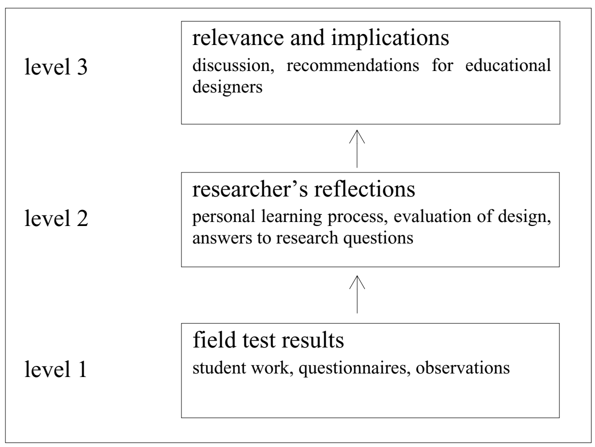


figure 6.1: three levels of research results

Before we turn to the field test and the results, we summarize the state of affairs in this final phase of the study and recapitulate the research questions and sub-questions formulated in section 4.2.

6.2 State of affairs after the peer review

The peer review at the end of the exploratory phase (see section 5.6) marked an important turn-about of ideas and, if not a new beginning, then at least a significant restart. The study refocused on its two principal aims:

- building on students’ intuitive strategies and symbolizations to help students overcome some of the syntactical and semantic obstacles in the passage from arithmetic to algebra;
- integrating historical problems and methods considered pre-algebraic as teaching and learning tool for algebraic symbolizing and problem solving.

turn-about of the study

We summarize the most important changes to our educational design in five different topics: combining theory and practice, integrating history, symbolizing, problem solving and structure.

1 *Combining theory and practice*
The educational design should focus more on continuities and discontinuities between arithmetic and algebra in order to determine whether and how pre-algebra can bridge the gap. And if we wish to investigate the role of history as a didactical tool, we must make sure that history is also given more priority in the mathematical activities.

2 *Integrating history*
The history of algebra can be made more prominent through the use of authentic sources – problems, methods and developments – which will give us the information we need to assess the effect of history on students and on the teaching-learning process in the classroom.

3 *Symbolizing*
A very important result of the study up to this point is, that algebraic symbolizing with letters does not come naturally to most students and should therefore not be forced upon them. A good alternative is the use of (word) formulas. We should certainly not use two (or more) letters to represent the variable – despite the fact that students suggested this themselves – because it will stimulate students to perceive letters arithmetically (for instance, as a unit or as a label of an object). The mathematical activities should also bring out the different roles of letters: as labels (representing *objects)*, as unknowns (fixed *quantities*) and as variables (representing *magnitudes).* Students should encounter variables in meaningful contexts where the numbers vary in a truthful and sensible way.

4 *Problem solving*
The mathematical activities should become more meaningful and challenging to students by giving them problems which they can organize and solve at different levels. These productions can then be the starting-point for a process of progressive formalization. The experimental learning strand has been focussed too much on re-organizing activities which were already organized in advance.

5 *Structure*
The educational design has been a collection of seemingly incoherent mathematical activities. It requires more structure if we wish students to develop (and perhaps even formalize) their symbolizing and reasoning abilities. Here we can think of making explicit the advantages of certain methods and representations which should emerge from the mathematical activities that students do.

These points of action will reappear in the discussion when we reflect on the field test results, in order to establish what has been the effect of these changes.

6.3 Research questions

The main research questions and sub-questions – discussed already in section 4.2 – are repeated here to bring them to the reader's attention before we present the results. The most relevant issues addressed by these two questions have been formulated as sub-questions to concretize the final phase of the study. The research data presented in this chapter provide a direct answer to some of these sub-questions, which in turn enable us to answer the main research questions.

main research questions

1 When and how do students begin to overcome the discrepancy between arithmetic and algebra, and if they are hampered, what obstacles do they encounter and why?
2 What is the effect of integrating the history of algebra in the experimental learning strand on the teaching and learning of early algebra?

sub-questions

1 With respect to the discrepancy between arithmetic and algebra:
   a How do students conceive symbolic notations as a mathematical language, which type of shortened notations do children use naturally, and how do we obtain an acceptable compromise between intuitive, inconsistent symbolizations and formal algebraic notations?
   b How can students actively take part in the process of fine-tuning notations and establishing (pre-)algebraic conventions?
   c To what extent and in what way can students become aware of different meanings of letters and symbols?
   d Is there a correlation between the form of notation (rhetoric, syncopated, symbolic) and the level of algebraic thinking?
2 With respect to the didactical value of history of mathematics:
   a What is the effect of integrating history in the mathematical classroom on the students, in particular their motivation and their learning process, and what is the possible influence of age, gender, intellectual level and the teacher?
   b How does the learner's symbolizing process compare with the historical development of algebraic notations?
   c Which parallels, if any, do we observe between the development of algebraic thinking amongst individuals and the epistemological theory?

6.4 Pre-algebra units revised

Chapter 5 is largely an account of the design process prior to the final version of the lesson materials. Two cycles of try-out and revision produced a global learning trajectory on a pre-algebraic approach to equation solving (see figure 5.32), which laid the foundations for the last leg of the project. Critical reactions from the field (see *peer review,* section 5.6) motivated us to reconsider certain didactical choices and instigated conducting new thought experiments on equation solving. As a result another diagram of skills and insights was developed to give direction to a) a more thorough learning trajectory, and b) the design of a connected program of mathematical activities. In this section we will first present this diagram of skills and insights, followed by the revised learning trajectory – specific activities designed to develop these skills and knowledge – and an overview of the program spread out over four learner units.

refined map of abilities

As we have mentioned in chapter 2, we perceive the (pre-)algebraic content in the learning strand to be mostly 'advanced arithmetic', with a strong component of 'study of relations and variables'. From this perspective the term 'pre-algebra' refers to the transition zone of informal explorative activity from arithmetic into elementary algebra, specifically the algebraic topic of equation solving. The diagram of abilities in figure 6.2 shows which knowledge and competence we consider to be prerequisite to equation solving. Each 'shell' represents a new layer of related competencies, from most immediate (inner shell) to remotely connected ones (outer shell). It is this feature of layering that makes it a more refined diagram than its predecessor (figure 5.3) in chapter 5.

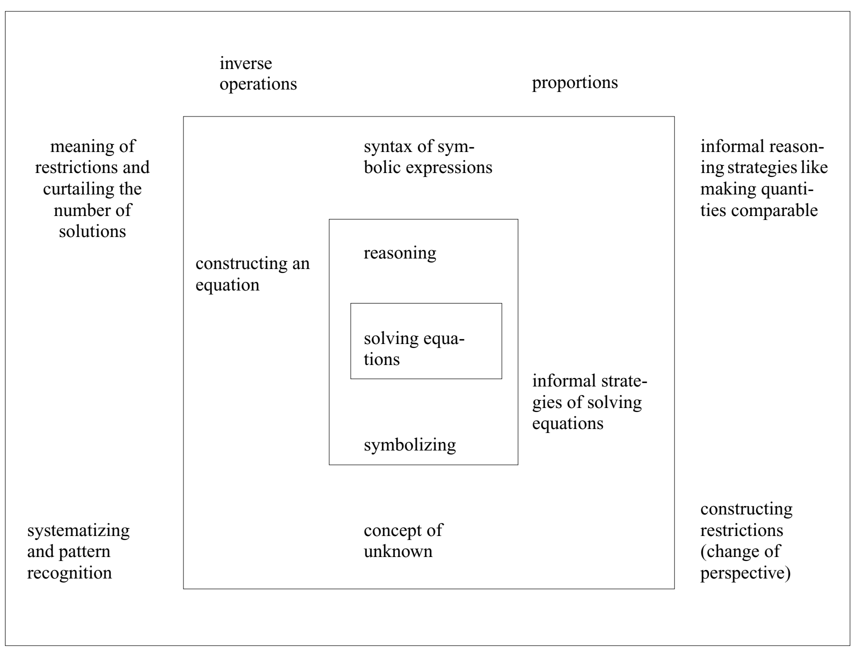


figure 6.2: diagram of shelled skills and knowledge

In the centre of the map we find the program's endpoint, 'solving equations': constructing equations in a given problem situation and solving them. The first, most urgent pre-conditions for competent equation solving are reasoning and symbolizing abilities. The term 'reasoning' involves recognition of relations between quantities in a problem situation, 'symbolizing' means expressing these relations symbolically. Prior to that, a student needs to understand the concept of 'unknown' and algebraic syntax (rules of symbolic notation and manipulation). In an earlier stage students should learn informal strategies for solving equations, to create a foundation of meaning and insight. And at the very beginning students can start with reasoning about unknown quantities: solving restriction problems (embedded equations) with simultaneous and subsequent restrictions, comparing quantities by substitution, proportions and equivalent expressions, inverting operations, recognizing and continuing a pattern and creating your own restriction problem (inverse task). Some of these early activities have an entirely arithmetical setting but nonetheless help to prepare students for a transfer to algebraic reasoning and symbolizing.

connected pre-algebra program

Figure 6.3 is the result of developing the diagram in figure 6.2 into a connected pre-algebra program. The shelled abilities have been rearranged into three parallel learning strands starting in arithmetic – *comparing quantities*, *ratio* and *inverse calculations* – each leading hypothetically to a type of symbolic expression: a system of equations, a trade term and an equation in one unknown. The legenda shows which elements in the learning strands are based on an historical problem or method, and which competencies are assessed at primary school level and at secondary school level. The double arrows in the diagram represent a restriction to one particular aspect (for instance, the Diophantine problem is a specific type of restriction problem), while the single arrow means 'leads to'. The encircled terms 'primary' and 'secondary' in the diagram indicate where a distinction between the two levels is relevant.

Having decided upon the mathematical constitution of the experimental pre-algebra program, we were required to translate it into mathematical activities. The two main streams of *schematizing* and *reasoning* abilities continued to characterize the core of the program as the student materials were rewritten. The three most important adjustments are the use of more appealing and challenging problem settings, short summaries for reflection and a stronger presence of historical elements. The first instructional unit *Exchange* is the result of combining revised activities from *Pocket Money* and *Marbles* with a number of new problems. *Time Travelers*, the fourth and last unit in the series, contains examples of ancient methods and problems from early algebra (two sections added in the appendix). The content of the eventual units *Barter* and *Fancy Fair* is largely unchanged compared to earlier try-outs; only the structure has been adjusted to concur with the other two. Figure 6.4 shows a diagram of the learning strand as a whole, which we describe very briefly below. The most influential changes in the units will become apparent as we illustrate (below) and analyze (in section 6.5) test tasks and classroom activities.

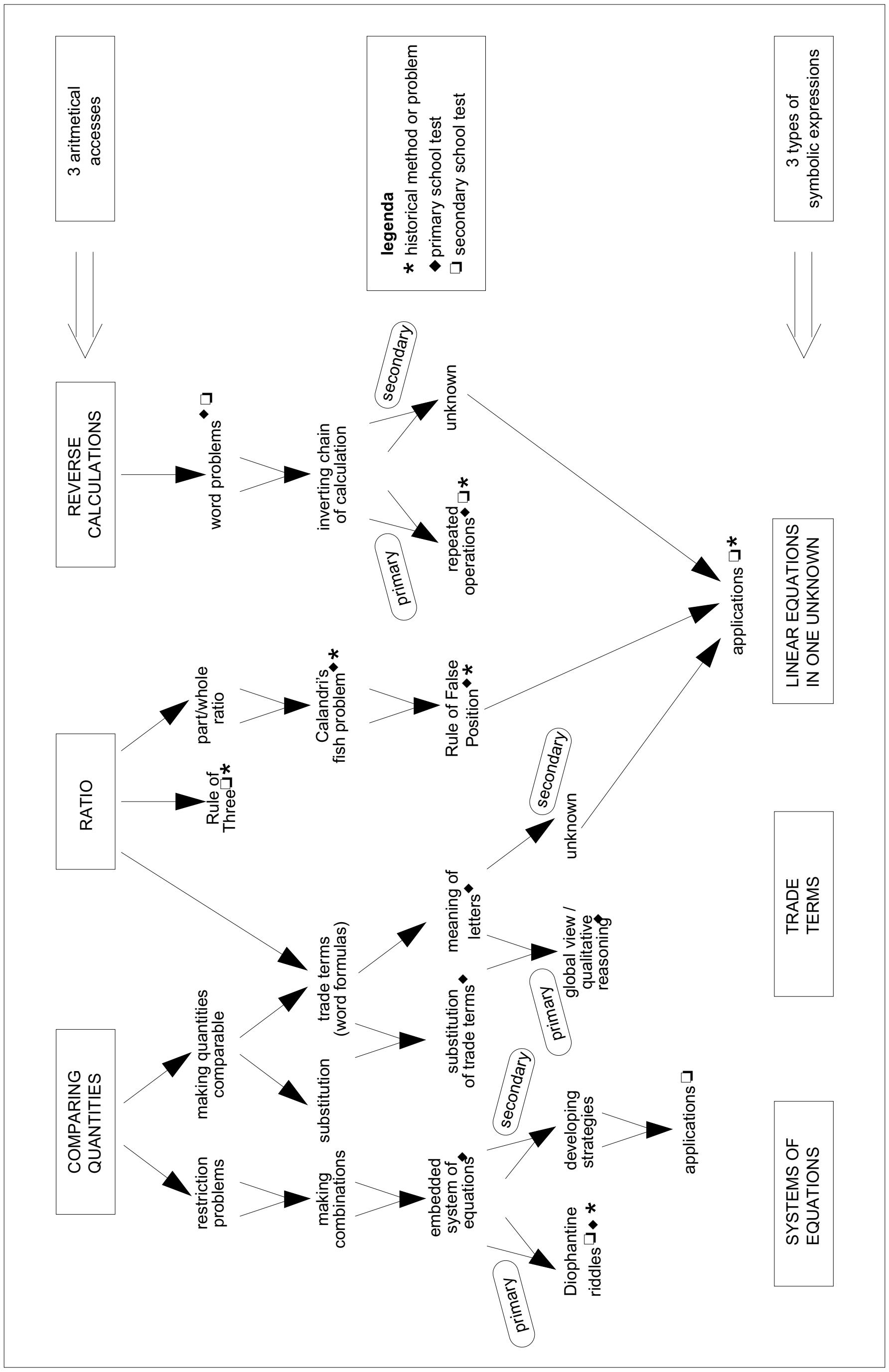


figure 6.3: directed graph of mathematical content

outline of the program

Figure 6.4 shows the connection between the four consecutive units and the continuity of skills and competencies in the learning strand: *Exchange*, *Barter*, *Fancy Fair* and *Time Travelers* (from left to right). Reading downwards we see thirteen themes or sub-strands of mathematical content (*representations* through to *reflection on algebra*), placed in order of increasing complexity. Reading across from left to right we see a global description of activities for each theme in the different teaching units.

Growth of competence and understanding has been facilitated by a variation of situations and representations in consecutive teaching units. In some cases the left-right direction does not show a hierarchy of activities, so that cognitive growth is a matter of attaining a wider perspective or a more global understanding (for instance, for *representations* or *comparing quantities*). The learning activities *schematizing, shortened notations, system of equations, reasoning* and *manipulating expressions*, however, do reflect a hypothetical path of progressive formalization in consecutive teaching units.

6.5 Field test

collecting data

In chapter 4 we mentioned that the field test results are based on four sources of data: written tests, student instructional units, classroom observations (including video and audio recordings) and student questionnaires. All four sources are used to evaluate the learning trajectory foreseen by the designer prior to the experiment, to test the validity of the conjectures we have formulated, and to answer the research questions as formulated in section 6.3. However, we have deliberately chosen to place the emphasis of our field test analysis on the tests, for the following reasons. First, we have already paid much attention to the teaching-learning process observed in the previous two classroom experiments, *case studies* and the pilot experiment, so in the final experiment we confine ourselves to a brief description (see section 6.5.2). Second, we preferred to limit the researcher's influence on the teaching-learning process as much as possible in this final classroom experiment, which means that we did not collect enough data to report on individual learning moments during the lesson series. In our opinion the written tests form a sufficiently objective and informative source of data on which to base our qualitative analysis.

6.5.1 Experimental groups

One hundred and thirty-four students from five different schools participated in the experiment, as shown in table 6.1, but about half of them only for the first part. Forty-one grade 6 students from schools A and B completed the primary school program; school C withdrew from the experiment halfway. School E tried out both secondary school units, while school D completed just one unit and the corresponding written test.

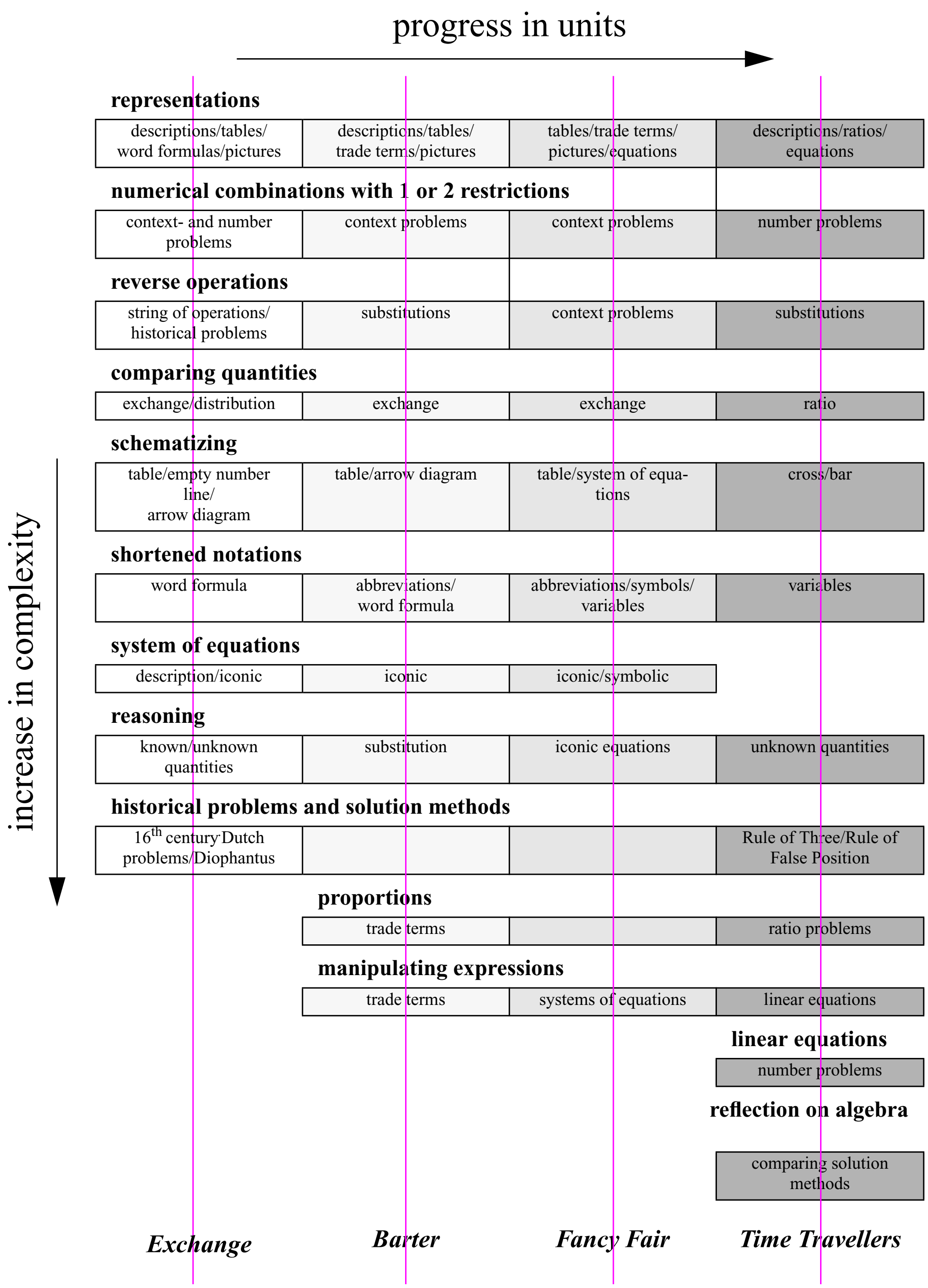


figure 6.4: experimental pre-algebra program: an overview

school background

The background of the different primary schools are quite diverse, but they all work with a recent mathematics textbook written according to the principles of RME (one of the criteria described in section 4.4.4). School A is a small urban white school with an experienced teacher who favors and practices the RME approach. The teacher typified the group of grade 6 students as above average ability. School B is a black urban school where a teacher trainee – recommended to the researcher by a teacher employed at the teacher training college – replaced the regular teacher as part of her teacher training. This college student was also involved in the study to observe lessons in the secondary school lessons. Her lack of teaching experience was largely compensated by her creativity and her ability to carry out the RME ideals. The regular teacher observed the trainee’s lessons and took notes. The grade 6 students in the experimental group have been described as being capable of working independently as well as in groups, of average ability. At school C, an urban white school, the experimental lessons were conducted by an experienced part-time teacher who sometimes felt aggravated by her and her colleague’s shared responsibility. Her attitude towards mathematics teaching and learning did not comply with the RME approach like the other teachers. The teacher described her grade 6 class as just above average ability.

School D is an urban white secondary school, while school E is a regional white school. The mathematics lessons were conducted by two full-time experienced teachers who volunteered to participate in experiment; they were not selected according to particular criteria. The teacher at school E was particularly interested in history of mathematics. Both schools use a mathematics textbook which is common in the Netherlands, and which is designed with the intention to enable students to work on the tasks with little teacher intervention. Both groups of grade 7 students were above average ability.

	school	**# boys**	**# girls**	**assess1, primary**	**assess2, primary**	**assess1, secondary**	**assess2, secondary**
primary level	A	12	11	23	22		
	B	4	14	18	18		
	C	12	21	30	–		
secondary level	D	16	14			29	–
	E	14	16			27	30
	total	58	76	71	40	56	30

table 6.1: number of students per school and per test

6.5.2 Execution of the teaching experiment

Sometimes we can ascribe differences in student results between schools to the influence of didactical circumstances: the role of the teacher, social and didactical norms in the classroom, the school background, unexpected incidents etc. It is therefore important that we describe briefly how the teaching experiment was carried out at the various schools.

the lessons

The lessons at primary and secondary school level were conducted as intended at schools A, B and E. School C and school D ran out of time and managed to complete just one instructional unit (school D, *Fancy Fair*) or not even one unit (school C, *Exchange* except for the second part of the last section). According to the teachers, the students appeared not to be disturbed by the presence of an observer. At primary school students take mathematics every day, so we decided to alternate the experimental lessons with regular arithmetic lessons to reduce the risk of loss of interest. In retrospect it may have been better to keep the period as short as possible, for instance by presenting the experiment as a thematic project, because in the current form the experiment lasted too long to keep students interested. The time schedule was fairly well predicted ; the lessons usually lasted for as long as the students held their concentration, which was between 45 and 60 minutes. At secondary school the grade 7 groups completed the project in one go, at a rate of four lessons a week. Most lessons lasted between 40 and 45 minutes. Classroom activities were observed and summarized in compact protocols by four observers (the researcher, two senior students from a teacher training college and the regular teacher of school B), and if a lesson took place unobserved the teacher reported unexpected or noteworthy incidents.

teaching and learning differences observed in the classrooms

We confine ourselves to a description of a few differences observed in the classroom which we think might be important for our analysis and interpretation of student work.

Primary school

structure

The teachers at school A and school B spent more time on classroom discussions of strategies and symbolizing than the teacher of school C, who was more concerned with allowing students to work at their own pace. After three weeks the teacher at school B decided to reorganize the lessons slightly by confining classroom discussions to the beginning and the end of the lesson. She preferred to reflect with students in groups, depending on their level of understanding and their working pace. This teacher also invented new settings for some of the activities. For instance, she introduced the students to barter trading by acting out a market place in the classroom, assigning a different role (salesman, customer, etc.) to each student. At school A the teacher saw and used opportunities of extending activities, for example by linking them to tasks in the regular program, by giving students more background information on some of the contexts and by presenting some activities of the unit *Ex-*

change as a challenge. In other words, the lessons were not only more structured and coherent but also more personal and challenging at schools A and B than at school C.

social norms

At all three schools we observed a stimulating and safe learning environment where individual contributions were encouraged and making mistakes was perfectly acceptable, but the students at the white schools were more comfortable to share their solutions with the group than the students at the black school. As far as student attitude is concerned, we noticed important differences. The teachers at school A and B were very outspoken towards the students regarding attitude, involvement and motivation. It was always clear to the students what was expected of them and which type of behavior would or would not be tolerated. Both teachers emphasized the value of taking part in such an innovative project. The teacher at school C tended to take a more subservient role.

The motivation of the teachers themselves also varied considerably. The teachers at school A and B were always convinced of the surplus value of taking part in the experiment, and they believed in the idea of easing the transfer from arithmetic to algebra by introducing pre-algebra at primary school. At school C the teacher's effort and support of the study decreased with time. We believe that eventually the teacher at school C lost interest in the lesson series herself, and this must have influenced the students as well. After two weeks she even asked a group of students whether they wanted the class to continue to participate or not.

teacher qualities

The teacher at school B was very conscientious in carrying out the lessons as intended and she had a natural ability to improvise, but she sometimes lacked the didactical experience to guide students in their development of progressive formalization. The teacher at school A, on the other hand, decided for himself which activities in the learning strand constituted the heart of the program. In most situations his judgement was correct, but the historical elements in the instructional units were sometimes passed unnoticed. In addition this teacher was not attentive of a correct use of the equal-sign by himself as well as by the students. At school C the teacher lacked the ability or awareness to guide classroom processes towards reaching a general consensus, sometimes favoring contributions by high achievers to smoothen the lesson instead of using informal or incorrect strategies to have students reflect on their work. On the other hand, compared to the teachers of school A and B who sometimes interpreted and symbolized on the blackboard what the students said, she had very little influence on the students' symbolizing activities. She usually let the students write their answers on the blackboard, which is a good approach until the moment arrives that free productions should be structured.

Secondary school

Fancy Fair

We do not have much information on the didactical differences between the teachers because the researcher was unable to observe the lessons on the instructional unit

Fancy Fair herself. Any comments with respect to social norms in the classroom, the attitude of the students or the role of the teacher at school D are based on observations done by the two students of the teacher training college. The teacher at school E used one more lesson than we had planned due to some organizational problems. The most remarkable differences are related to teacher influence on students' use of a formal method and representation for solving equations. Apparently the teacher at school E encouraged students who used a formal level of solving and symbolizing systems of equations. At school D the teacher did not favor any particular contributions made by students, either at a formal or an informal level.

Time Travelers

The researcher did attend all the lessons on the unit *Time Travelers* at school E, and they were conducted as intended. However, the time schedule was too tight. The teacher felt rushed at three particular moments of the lesson series. Students did not have enough time to investigate the very open orientation task for themselves. The third section of the unit – which deals with four variations of the Rule of Three and its generalization to a word formula – turned out to be more difficult for students than expected. And in the final section, where informal methods are intended to be formalized, the end of the school year forced the teacher to miss out one lesson. The historical elements in this unit were given due attention by the teacher and in a humorous and personal way. In this lesson series the teacher was quite reserved with respect to students' symbolizing activities, not pushing his own preferences to the foreground.

written tests

premises for assessing student work

Starting-point for the analysis of student results is student performance on some of the test tasks, comparing algebraic and arithmetical solution strategies and notations in order to establish to what extent students have succeeded to overcome discrepancies. We remind the reader that not only failure or success but also the strategies students use indicate their level of understanding and competence. Notations inform the researcher about the student's initial ideas, his or her thinking process, and the solution strategy used. Draft notes disclose how well a student can mathematize the information – mentally (few or no notes), visually (symbols, drawings), schematically (tables, diagrams) or analytically (in words).

It is not easy to determine a hierarchy for different types of notations, especially because students might associate certain problems with certain representations from experience. But since the main objective is to distinguish arithmetical from algebraic thought processes, we have chosen to concentrate on notations that facilitate and report the solution process instead of the representation of the end product (the answer). If a student has left the designated draft area empty, we look to see if the answer includes an elaboration. If there is no explanation at all, the notations are described as 'answer only'.

Tests 1 and 2 are individual written tests for the primary school units *Exchange* and

Barter respectively. In the analysis we have included seventy-one students who took the first test and forty students from school A and B who took the second.

6.6 Results of the *Number Cards* task

6.6.1 Purpose and expectations of *Number Cards*

purpose

Since restriction problems form a prime theme in the proposed learning strand, we decided to include at least one in the test. But there is more. By choosing addition and subtraction as the relation between the quantities, the task becomes isomorphic to the number riddles, which gives it a second meaning. If students recognize it as a number riddle, we should be able to identify specific strategies which emerged in the corresponding lesson. In order to reduce the chance of reproduction – the algorithmic strategy, in particular, can be easily learned as a rote skill – we decided to present the problem visually.

At the first level students interpret the task as a restriction problem with two conditions, so we can expect them to have access to strategies like trial-and-error and trial-and-adjustment. Moreover, the classroom activities should have shown students that a systematic approach and schematic notations are useful tools for investigating a restriction problem. At the second level, if the problem recognition is complete, the student will solve the task as a number riddle on sum and difference using whichever strategy is preferred. For most students this will be the strategy that worked for him or her in class. None of the strategies are explicitly taught to the students but the teacher is advised to stimulate systematic approaches, and the strategy 'adjusting the difference symmetrically' is mentioned in the summary. The teacher can decide to introduce it if students remain at an arithmetical level but if students invent their own successful method, the strategy 'adjusting the difference symmetrically' will not have any value. Still, we expect to see some students use an empty number line or a systematic approach in a table. And of course there will be some students who do not succeed, perhaps unable or unwilling to keep trying and not susceptible to a systematic way of thinking.

Apart from this pedagogical aim, we expect the *Number Cards* task to deliver valuable data for the feasibility of the proposed learning trajectory as a whole. We feel the task is sufficiently rich for producing solutions at different levels, from where we can adjust and improve the intended route of vertical mathematization discussed below. For example, we will be looking out for recapitulation of the Babylonian method (see also section 3.4 on implementing history). And if a student solves the problem with another strategy than the designer had in mind, it is also a valuable piece of information. Each 'new' strategy contributes to a better understanding of what comes to students naturally, and why one pre-algebraic strategy is successful and the other is not. Theory formation on one specific problem on embedded equations will help to evaluate other parts of the hypothetical learning process dealing with the same issues.

6.6.2 The analysis: strategies observed

classification

In order to make a global comparison of arithmetical, pre-algebraic and algebraic styles of problem solving, the strategies have been re-organized into 6 categories: algebra, pre-algebra, arithmetical, answer only, incorrect and nothing. The schools have first been combined to get a larger number of data, followed by a comparison between the schools. Data on boys and girls have been sorted to look for gender-related trends which might be expected on account of educational research results on gender differences and mathematics learning (Van den Heuvel-Panhuizen & Vermeer, 1999).

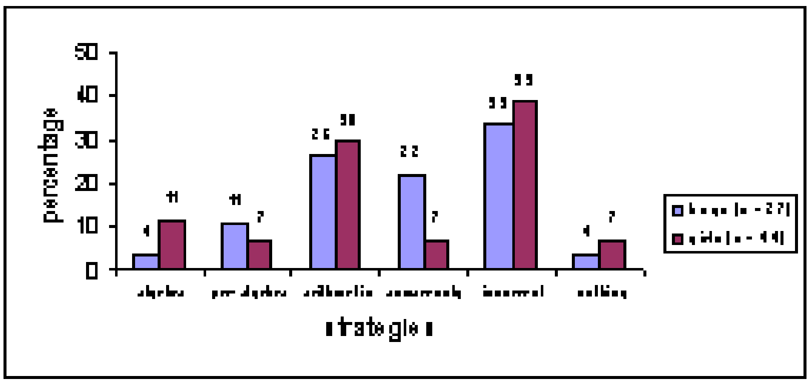


figure 6.5: *Number Cards* task, (pre-)algebraic versus arithmetical strategies

Adding up the first four strategy categories in figure 6.5 (reading from left to right), we find that a total of 63% of the boys and 55% of the girls succeeded at solving the task (calculation errors tolerated). We found 6 algebraic and 6 pre-algebraic solutions, each accounting for just over 8%. Just more than a quarter of the boys and 30% of the girls solved the task arithmetically.

gender differences

We observe a number of differences between male and female performance. First, the girls scored better in the algebraic category than the boys (11% versus 4%), due to five girls at school C using the algorithmic strategy. In the pre-algebraic category we find a slightly larger percentage of boys, spread evenly amongst the schools. Second, a slightly larger percentage of girls failed to solve the task – 39% incorrect and 7% no attempt, compared to 33% and 4% of the boys respectively. Note that rounding errors may occur in the percentages. And third, six boys (22%) – all from school C – solved the problem correctly without an explanation, compared to three girls (7%). This may be explained by the fact that when students waiver supporting their solution with calculations or an explanation, girls tend to write down at least a little whereas boys are known to be more indolent (Van den Heuvel-Panhuizen et al., 1999). An analysis of similar tasks in the unit *Exchange* is needed to substantiate these gender differences.

frequent errors

Twenty-one students (of whom ten come from school C and six from school B) calculated 200 – 68. The majority of students gave 132 as the answer, the others got stuck and did not write down a solution. One possible explanation is the persistence of arithmetical thinking: a capacity to calculate and reason with known numbers, not with unknown numbers. The minus sign in the second ‘equation’ somehow has precedence over the addition sign in the first equation; only two students tried 200 + 68 during their investigation of the task (see figure 4.7 and figure 4.9 in chapter 4). Perhaps, in spite of the confusion, these students did realize that both numbers on the cards must be less than 200, and that addition will only result in values higher than that. Six students (two at each school) missed the meaning of simultaneous ‘equations’ (conditions). They either gave a combination of numbers that satisfies only one equation instead of both, or they wrote down a combination for each condition in turn, as shown in figure 4.9. We will investigate whether the students’ work in *Exchange* also indicate a likeliness of these two types of errors.

comparing schools

Comparing the scores of different schools (see figure 6.6), the most remarkable observation is that the students at school B performed very poorly compared with the rest, at all levels of competence.

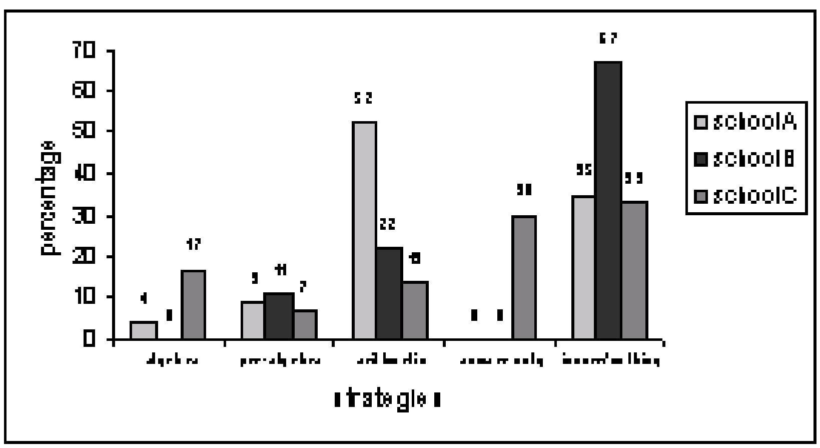


figure 6.6: comparing schools for student achievement

We observed no algebraic solutions at all, and only one boy and one girl (which is 11% of the class) succeeded at solving the problem pre-algebraically, by adjusting

the difference symmetrically. Four girls solved the task arithmetically (22%). The remaining 67% of the class was unable to find the solution; two girls did not even make an attempt. This means that, in comparison, nearly twice as many students failed to solve this task as at the other two schools. Second, school C performed better with respect to algebraic strategies (in particular the algorithmic strategy), while school A scored particularly well in the arithmetical category. And third, it appears that the students at schools A and B were more capable and/or willing to write down their calculations; nearly a third of the students at school C wrote only an answer. These differences between the schools formed a third argument to check how students coped with similar problems during the lesson series.
The test scores per school also show slightly different gender trends than for the entire experimental group. At school C, the girls scored slightly better than the boys: ten of the thirteen girls who answered correctly used a clear strategy (five algebraic, one pre-algebraic and four arithmetical), whereas six of the seven boys who solved the problem scored in the category ‘answer only’, and only one used a pre-algebraic strategy.

6.6.3 Notations

Student notations for the *Number Cards* task have been categorized as ‘number line’, ‘table’, ‘calculations’, ‘answer only’ and ‘none’. ‘Table’ here stands for any type of tabular diagram or schematic representation, and ‘calculations’ can be in column (vertical) form or written horizontally with operator symbols and an equal-sign. In situations where students used more than one kind of notation, we chose the most sophisticated one and otherwise the most representative one.

classification

Again the classification into levels is an equivocal matter. The table is a more sophisticated type of representation than calculations. In the first place it is an organizational tool, but on top of that it can be used for investigating a pattern, which is certainly an algebraic application. The number line is a semi-abstract representation and as such also more sophisticated than calculations; its undetermined length and scale are algebraic properties. The categories ‘answer only’ and ‘none’ are of the lowest level because they reveal nothing about the student’s thinking process.
It frequently occurs that notations are difficult to typify or that they belong to more than one category.
To illustrate the complexity of judging student work, we take a look at figure 4.5 again. The notes show a mixture of notations: symbols, a table, a schematic list of horizontal calculations and column calculations. In order to do justice to the student’s systematic approach of the problem, this work has been allocated to the category ‘table’. The symbols have not been newly constructed by the student and as such are not considered part of the solution process. Of course if a student does invent a new symbolic representation for the problem, an additional category ‘symbolic’ would be in order, but none of the students in the experiment did so.

general results

Figure 6.7 clearly shows that the majority of students expressed their reasoning in calculations (52% of the boys, 61% of the girls). Many of the students who wrote horizontal calculations maintained the same structure as the task, i.e. they wrote a system of ‘equations’ with the unknowns substituted by numbers (see also figure 4.6, left example).

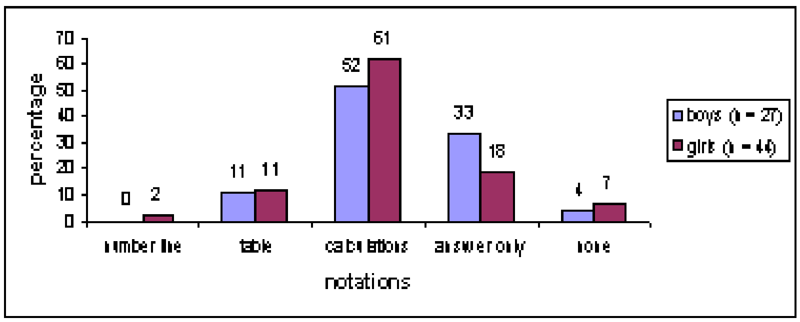


figure 6.7: *Number Cards* task, notations

This representation stays close to the task, and helps to visualize and interpret it. Column calculations, on the other hand, require a mental translation of the two expressions first: ‘the sum of two numbers is 200’ and ‘the difference between the numbers is 68’. Similarly, reasoning about the sum and the difference in a table also involves a mental step in the mathematization process. Only three boys (11%) and five girls (11%) – of whom four come from school A – used a table of some kind to solve the task either pre-algebraically or arithmetically. Just one girl (school B) drew a numberline, which automatically corresponds with the strategy of adjusting the difference symmetrically.

(pre-) algebraic vs. arithmetical notations

In other words, if twelve students solved the task (pre-)algebraically while only four students used a table or a number line to do so, then there is a substantial number of students who solved the problem (pre-)algebraically while their notations were arithmetical. This can be for a number of reasons. First, (pre-)algebraic notations are not suitable for every strategy. A tabular form benefits the approaches of trying numerical values and adjusting the difference symmetrically, whereas the empty number line is relevant only for the corresponding method. Second, the choice of notation often depends on student and teacher contributions in the lessons. Neither the table nor the empty number line are integrated into the lessons as compulsory notations; it is up to the teacher to introduce students to these tools when the situation calls for it. Clearly it is unlikely that a student who has never seen a problem solved with the empty number line, will spontaneously use it for the first time on a test. Similarly we can reason that a student who quickly accomplished the algebraic algorithm, might not have encountered reasoning with numbers in a table. Trends of strategy use and favored notations during the lesson series are discussed hereafter. Third, the problem itself is also responsible; student work in the unit *Exchange* indicates that tables are more commonly used when the relation is multiplicative (as ratio tables).

gender differences

Comparing the boys with the girls, we see some differences amongst the distribution of ‘calculations’ and ‘answer only’. This discrepancy is of course linked with the gender differences in strategy use as discussed before, because the strategy ‘answer only’ always coincides with the notation ‘answer only’. In other words, the exceptionally high percentage of the strategy ‘answer only’ amongst the boys at school C explains the gender contrast between ‘calculations’ and ‘answer only’. Comparing the schools for differences in notation use, we notice that five of the seven students – four girls and one boy – who used a table go to school A.

6.6.4 Summary of *Number Cards* results

The test results on the *Number Cards* problem show a few notable differences between the schools and between boys and girls:

1 School B scored poorly compared with schools A and C;
2 There is a large variety in strategy use amongst the schools;
3 The number line and table are used hardly at all;
4 Girls are less successful at solving this type of restriction problem than boys, with the exception of the algebraic, algorithmic strategy;
5 The boys are less cooperative in supporting their answer with calculations or otherwise than the girls, in particular at school C;
6 Students make two major thinking errors: they consider just one condition instead of two, or they apply the relations to the given numbers instead of the unknowns.

More data have been collected from the student instructional units and classroom protocols to substantiate and possible find an explanation for these observations.

6.6.5 Comparison with unit tasks

similar tasks in the unit

Four isomorphic problems in section 3 of *Exchange* were selected to compare the schools to one another with respect to students’ individual and collective learning. Problems 2 and 6 are restriction problems with simultaneously an additive and a multiplicative relation, such as ‘five times as old’ and ‘28 years older’; problems 9 and 10 are Diophantine number riddles like the one in figure 4.4. The first two tasks are strictly speaking not mathematically equivalent to the test task, because the (pre-)algebraic strategies described for the problems on sum and difference are not applicable. However, the general character of the problems is the same, and they can each be solved arithmetically – trying numbers – or by reasoning with the relations between the unknowns. For instance, for problem 2 several students reasoned with proportions: ‘the mom is 5 times as old, and the boy is already one time, so 28 must be four times, so the boy must be 7’. In our opinion these four problems are valid material for investigating progress in mathematical thinking and notation use. Problems 9 and 10 will be considered separately for a more precise comparison with the test results.

1 School B scored poorly compared with schools A and C

achievement

In order find a possible explanation for the fact that school B scored so poorly, we first investigate how students performed at solving the selected problems in their booklet. In order to avert student answers copied from a class mate or the blackboard, we only consider answers that include draft work or an explanation. Figure 6.8 shows the average success rates – per school and per gender – for the four unit problems. For instance, the first column ‘38%’ means that on average 38% of the boys from school A can solve these kinds of problems. Comparing school B with the other schools, we see that these data do not comply with the test results. The students from school B were even slightly more successful at solving these problems than the others.

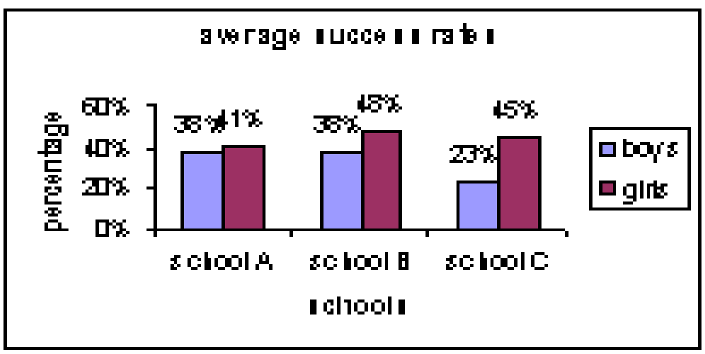


figure 6.8: average correct answers on four unit problems

interpreting success rates

But these scores may be deceiving. First, we need to look at how the averages are compiled, because problems 2 and 6 are not as good an indicator for the *Number Cards* problem as problems 9 and 10. Doing so we find that in comparison the school B students score better on problems 2 and 6, equally well on problem 9 and slightly worse on problem 10. Secondly, we must compare the levels at which the students solved these problems and the amount of teacher aid. One can imagine that if students from school B were given half an hour to solve one task by trial-and-error methods, the success rates in figure 6.8 will give the wrong impression that these children are as competent as the others. Classroom observations indicate that the students of school B and school C were given a similar amount of time for the tasks, but many students of school C completed them very quickly. We have no information on school A, but the total amount of time planned for this particular section was not exceeded. As far as the solution strategies are concerned, we have recorded three occasions of an algebraic solution to problem 9, all solitary cases. In addition we have counted seven students (nearly 40%) who solved problem 9 and 10 correctly. It is fair to assume that none of the students of school B accomplished an algebraic level of understanding during the lesson series, and more than half the class never understood how to solve this type of task. This seems to be a fair explanation for the underachievement on the *Number Cards* task of school B.

2 We observed a large variety in strategy use amongst the schools

progressive formalization

Classroom observations and student answers to the four selected problems point out that there are a few notable differences regarding progressive formalization. We use the term 'progressive formalization' to describe the process of vertical mathematization that a student goes through (see also section 4.5.3). If a student first uses arithmetical strategies of trial-and-error, followed by a more systematic approach of trial-and-adjustment, ending with a method of reasoning, the student passes through different levels of mathematical thinking. In other words, the ultimate progressive formalization is the development from an arithmetical to an algebraic way of thinking. This development will for many students not be a linear, one-way process; learning is usually a combination of moving ahead and falling back. Sometimes we see that a student shows instances of systematizing or reasoning but then falls back to a lower level strategy; in these cases we cannot be sure that the formalization process has actually set in. And when a student adopts a more sophisticated approach or representation than his own from another student, and proves to be competent with it, then we can say this student has also demonstrated mathematical growth. Indeed, if the learner does not really understand the higher level mathematics, he can only use it artificially and will not be able to make the transfer.

school B

First, not one student at school B discovered an algebraic strategy for solving Diophantine problems, and only one girl progressed from trial-and-adjustment in problem 2 to a method of reasoning with proportions in problem 6. Most students applied arithmetical strategies of trial-and-error and trial-and-adjustment. As the problems become increasingly complex, these strategies become more and more tedious, and so after a generous amount of time the teacher handed various students the method of adjusting the difference symmetrically for problem 10. According to the answers in the instructional units, four students seem to have profited from the teacher's help. Students who succeeded at solving problem 9 arithmetically continued in the same way and did not adopt the more advanced strategy. In turn, the teacher did not push them. A few individual students of this group incidentally rose to a level of reasoning, but the data do not reveal how these moments of insight came about and whether they might have led to a better strategy if there had been more problems of the same kind. Of the four students at school B who were recommended to use a number line, no one completed the summary question with the number line. In other words, the students of school B appear not to have formalized their skills and competencies regarding number problems with two conditions, and their success rates seem to be the result of a generous amount of time for trial strategies and in some cases teacher aid.

school A

In the other two schools, algebraic solution strategies did play a big role. In school A about half the students demonstrated progressive formalization as they worked through the unit section. Most students applied arithmetical strategies for problem 2 and 6, but a group of four girls and a few individuals used reasoning strategies at the start of the section, like ‘halving the difference’ in a simple context problem on sum and difference. This reasoning strategy was not shared with other class mates until later on. One student’s proportion strategy for problem 2 – ‘the whole is 5 times, which is the same as 4 times more, and 4 times is equal to 28’ – did have a special effect. The teacher tried to visualize the proportion to the rest of the class by drawing an arch on the black board, but it did not seem to catch on. At the end of the class discussion the teacher praised the students who used a systematic approach, but the students were especially impressed by the proportion-method. Or, as some students said, ‘very clever’, ‘I think it is really algebra’, ‘top method’. It is unclear when and how the algebraic algorithm for problems 9 and 10 appeared for the first time, but two variations were adopted by approximately half the class. Some students used the algorithm ‘halving the difference’, others first subtracted the difference from the sum and then divided by 2. It is therefore quite remarkable that none of the students of school A solved the *Number Cards* task in this way. Finally, five boys produced their own Diophantine problems and a few even completed the extension task, checking the general applicability of the Babylonian algorithm for problems on sum and difference.

school C

More than half the class (66%) acquired a (pre-)algebraic strategy in school C, the girls in particular (75%). This high percentage agrees with the relatively high success rate of the girls compared with the boys (see figure 6.8). From the classroom observations it appears that a group of three or four girls discovered the algorithm of halving the difference. This solution appeared on the black board during the discussion of problem 7, and it actually became the standard method for problems 9 and 10. Reasoning strategies for problems 2 and 6 were not seen so frequently; only two girls reasoned with proportions and five students generated multiples systematically in a tabular form until they found the correct difference. The teacher also played a role here; she suggested making a table if the trial-and-error method did not work. In other words, it seems that for most students the moment of insight occurred in the second half of the unit section. However, there is also another possible point of view, namely that higher order reasoning skills for problem 2 and 6 are easier to perform mentally. There is a chance that a larger number of students solved problems 2 and/ or 6 (pre-)algebraically because about 47% of the answers was without an explanation! This is significantly more than the other schools (36% and 33%). From classroom observations it is clear that at least a few students used strategies of reasoning with half the difference (problem 1) and with proportions (problems 2 and 6) without making their method explicit.

individual maintenance of strategy level

When we look at how well students were able to maintain their level of strategy during and after completing their classroom work, we notice that there is a significant regression from the algebraic level to a lower level at all the schools. For example, figure 6.9 shows an overview of individual achievement for students at school A. Horizontal arrows represent a maintenance of level, while arrows pointing up and arrows pointing down mean a progression and a regression respectively. Comparing performance of problem 10 with respect to problem 9, we see that most students were able to maintain their strategy level, whereas the majority of students was unable to reach their classroom standard in the test task. At school B we observed 7 instances of regression (39%) and 9 (50%) students who maintained their classroom level, of understanding, while at school C the figures are 14 (42%) and 13 (39%) respectively, and 18% inconclusive. From this perspective, student achievement on the *Number Cards* task is rather disappointing. We return to this matter in section 6.6.6 and section 6.6.7.

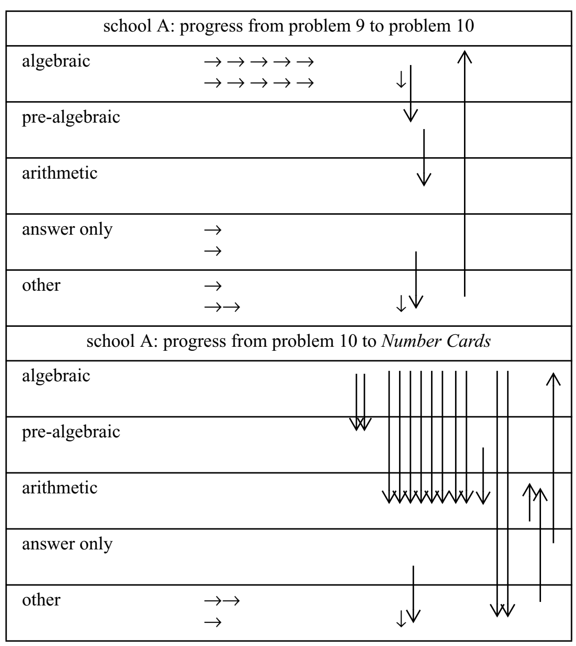


figure 6.9: progress in strategy use of school A students

3 The number line and tables were hardly used as problem solving tools

The explanation for the limited use of schematic or visual representations in the *Number Cards* task has already become apparent. In schools A and C the algorithmic strategy prevailed towards the end of the section, making the number line and the table superfluous. One student from school A compressed the algorithm in problem 10 to just the numerical outcomes for each step: '85 $42\frac{1}{2}$ $107\frac{1}{2}$'. The strategy 'adjusting the difference symmetrically', either using the number line or a table, did not play a role at all. And although the table is very suitable for listing values in problems 2 and 6, only the students from school A chose to use it. Students from schools B and C intuitively preferred calculations, and decided otherwise only on the teacher's advice. In other words, students don't even think to use a table as a neat and efficient way to organize numerical information, and without this habit it certainly becomes very difficult to discover and appreciate more advanced applications like pattern recognition.

4 Girls were less successful at solving this type of restriction problem than boys

gender differences

Compared with the test scores, the numbers in figure 6.8 suggest quite another gender trend: not the boys but the girls performed better. Indeed, the scores for only problem 9 and 10 are even more pronounced: 39% correct and 26% incorrect answers amongst the boys, versus 53% correct and 14% incorrect answers amongst the girls. The good female performance can be attributed to school C, while the highest male contribution of correct answers comes from school A. Clearly the girls performed more poorly on the test than might be expected from their classroom work.

5 The boys were less inclined to support their answer with calculations or otherwise than the girls, in particular at school C

We have already mentioned in point 3 that, compared with the other schools, the students from school C wrote less calculations for their solutions. In the test problem the boys particularly wrote down only the answer, but for the selected problems from the unit the difference between the sexes is not so pronounced. For problems 2 and 6 we see a higher percentage of boys than girls, but for problems 9 and 10 the figures are similar. These findings do not contradict the earlier proposal that the gender contrast in the test might be explained by the fact that boys are more reluctant to write their answer elaborately. But there is no indication why the students of school C in particular are so unwilling to show their work. The teachers of all three schools have emphasized repeatedly the importance of writing down draft calculations, and in each class we observed a few boys and a few girls who protested against having to write down an explanation.

6 *Students made two major thinking errors: they considered just one condition instead of two, or they applied the relations to the given numbers instead of the unknowns*

The error of not satisfying both conditions simultaneously occurred especially in problem 6, often in combination with the other error. Problem 6 in particular seems to provoke an incorrect first numerical attempt, at schools A and C much more so than problem 2 on age. The context of problem 6 is more abstract in comparison: '2 numbers added up gives $22\frac{1}{2}$, and one of these numbers is 4 times as big as the other'. Students cannot rely on real life experience to judge what is a reasonable first guess. Moreover, the problem involves fractions for which reason the solution is not so easily found without a systematic approach or reasoning. A typical train of thought is, to divide $22\frac{1}{2}$ by four and then subtract the answer from $22\frac{1}{2}$. In doing so, only the condition of the sum is satisfied, and not the muliplicative relation. Schools A and C show a much lower number of errors for problems 9 and 10; only school B shows a more or less constant percentage of errors amongst the four problems. This contrast between the schools is quite easily explained because many students in school A and C applied an algorithmic strategy in problems 9 and 10, avoiding wrongful numerical attempts. The majority of students in school B, on the other hand, continued to use arithmetical methods where the error of relations is much more likely. Similarly the relatively infrequent use of reasoning in problems 2 and 6 explains why the error occurred here more often.

6.6.6 Theoretical reflections

Reflections on the analysis phase of the test task *Number Cards* lead to the preliminary formulation of theoretical ideas. Some of these reflections are very specific for the task, while others are concerned with algebraic thinking and symbolizing in a wider perspective. The more general ideas laid the foundations for a number of conjectures to be discussed in section 6.6.7. First we evaluate the learning effect of the experimental lesson series on restriction problems, and number riddles in particular.

learning trajectory

In retrospect, when the intended global learning trajectory for number riddles described in figure 4.10 is compared with the attained learning trajectory, it is clear that some matters did not go as planned. We will link the observations to the corresponding phases (arrows) in the global learning trajectory when possible.

strategy use

Comparing the test results with classroom work, we have observed less flexibility in the use of pre-algebraic (at school B) and algebraic strategies (at schools A and C) than expected. At all levels students performed below their level of competence displayed in the unit problems, especially the girls. One explanation for both the loss of previous abilities and the prevalence of arithmetical strategies could be, that only a small number of students recognized the type of task they were dealing with. The girls in particular had little practice with trial-and-adjustment strategies because during the lessons they opted for the algorithmic strategy more quickly than the boys. Or perhaps we have underestimated the impact of changing the problem representation, from the descriptive form of the Diophantine riddles in the unit to drawn symbols representing or hiding number values in *Number Cards*. On the other hand, a number of students have demonstrated that this type of task can be solved without passing through the pre-algebra stage. In conclusion: for this type of task a learning path from arithmetic straight into algebra is feasible although the strategy, and the pre-algebraic level of problem solving is less effective as an intermediate station than assumed (arrows 3 and 4).

algorithmic strategy

In schools A and C the algebraic strategy 'algorithm of halving the difference' has been more dominant than expected prior to the experiment, but a remark must be made. Although quite a large group of students understood this strategy in the lessons, not so many students succeeded in applying it to the test task. It is possible that after a while the learners' understanding was replaced by rote skill, blocking the way back to a less advanced strategy (arrows 6 and 7 reversed).

schematizing as a problem solving tool

Student use of the number line and tabular forms as tools for mathematical reasoning did not answer our expectations. The designer's trajectory of open problems (with little instruction how to tackle them) has not led students naturally to search for increasingly effective strategies. As a result we have seen relatively few applications of systematic approaches like the pre-algebraic methods 'adjusting the difference symmetrically'.

gender contributions in the classroom

In two of the three primary school classes taking part in the teaching experiment we have seen that girls have played an equally active (if not more active) role in the reinvention and popularization of (pre-)algebraic strategies as the boys. This result does not agree with what might be expected on the basis of research on gender differences in learning mathematics (see also section 7.2).

other strategies

The *Number Cards* task and similar problems of two restrictions may also be solved using visual representations or systematic methods other than the ones we have proposed. The learning trajectory has shown some limitations on how to proceed to reasoning with an unknown magnitude. For instance, it might be more suitable to adjust the bar model by using dotted lines instead, to indicate that the length is indeterminate (see also section 7.2).

parallels with history

The algorithmic and elimination methods students invented show similarities with the Babylonian method of solving tasks on sum and difference.

global view of restriction problems

The intention of developing competence and insight through 'variation of restriction problems' throughout the first, second and third instructional unit appears not have been successful for the first part. The change in context and structure of the tasks in the unit *Exchange* in section 1 (distributing candy, dimes and quarters, magic balls and bars) section 2 (2 restrictions for bags of candy) and section 3 (age riddles and Diophantine problems) has not led to a spontaneous recognition of this theme by students. This means that students have not been able to acquire a global view.
We decided to test the following trends with respect to early algebra learning for other types of problems than *Number Cards*:

- Students appear to be able to reinvent reasoning strategies which can be characterized as algebraic, where parallels with the historical development of algebra are not unthinkable.
- When students solve tasks in new situations they fall back on arithmetical strategies, in spite of multiple use of algebraic methods in the lesson series (see figure 6.9).
- Students have a clear preference for calculations to demonstrate their solution strategy; the development of schematic notations or models for problem solving seems to be unnatural; when students use tables, they do so especially for tasks involving multiplicative relations (ratio's) rather than additive relations.
- Students do not self-reliantly symbolize unknown quantities.
- Relations between unknown quantities are frequently linked to the given quantities in the problem.
- Girls appear to have more trouble applying newly acquired knowledge to a new situation than boys.

6.6.7 Formulation of conjectures

The theoretical reflections described in the previous section have resulted in three conjectures on algebraic thinking, symbolizing and typical errors. These conjectures constituted the starting-point for the next round of analysis and theory formation, which turned out to be a very short one. The first two test tasks selected to test the conjectures, *Pocket Money* and *Birthday*, made clear that the formulation of conjecture 'reasoning versus symbolizing' was not employable because we were unable to gather relevant information for it. The conjecture needed to be rephrased in order to make it compatible with the data. This interaction between analysis and theory formulation led to a second version of the conjecture, which is actually the version used for the remainder of analysis. The other two conjectures have not been altered.

conjecture *reasoning versus symbolizing*, first version

Students appear to proceed more easily to a a higher level of mathematical thinking than a higher level of symbolizing. Results on the Number Cards task indicate that pre-algebraic strategies which rely on the use of schematic or visualized notations seem not to emerge naturally as a way of formalizing arithmetical strategies. Students either discover or adopt from each other mental (reasoning) strategies, or they continue with trial-and-adjustment methods. Furthermore, it is not self-evident that students are inclined to search for more accurate and efficient methods like systematization and recognizing (number) patterns. This means that the proposed trajectory of reinventing algebra may not be as feasible as expected. On the positive side, there are indications that students are able to find another way. In conclusion:

conjecture

1 *(Pre-)algebraic notations are not prerequisite to algebraic thinking;*
2 *The reinvention of algebra as advanced arithmetic is hampered by the fact that students are not as susceptible to systematized notations as expected.*

conjecture *reasoning versus symbolizing*, second version

employability

The results of the *Pocket Money* and *Birthday* tasks pointed out that this first version of the conjecture *reasoning versus notations* is not employable for the analysis, particularly the second part. If a task does not succeed in provoking pre-algebraic symbolizing as a stepping stone for higher level strategies, we cannot say the reinvention process is hampered for that reason. It is prerequisite for the conjecture that the task gives students a reason or a need for pre-algebraic notations (schematization or symbolization), otherwise the data are inconclusive.

In addition it seems that the present conjecture does not cover the relation between reasoning and symbolizing sufficiently. Having compared levels of strategy with levels of notation for several test problems and classroom activities, we see basically three kinds of student behavior towards this pre-algebra material: 1) students who find their own way to a level of algebraic thinking, solving the problem mentally or with calculations without structuring the problem in a diagram or with shortened notes 2) students who profit from pre-algebraic notations to solve the problem, regardless of the strategy level, and 3) students who fail to solve the problem arithmetically and still do not switch to a more systematic approach, probably because they don't know how. At this point we saw the need for rephrasing the conjecture in such terms that sufficient relevant data could be collected and the three dominant types of student behavior were relegated. Therefore the conjecture on reasoning versus symbolizing/schematizing has been changed into:

conjecture adjusted

1 *It is possible for students to invent or take a path from arithmetical to algebraic methods independent of intermediate pre-algebraic strategies and/or representations;*
2 *Students who have difficulty in progressing from an arithmetical to an algebraic mode of thinking tend to remain at an arithmetical level of notation;*
3 *(Pre-)algebraic notations are not prerequisite to algebraic thinking, but they appear to be effective in solving (pre-)algebraic problems.*

conjecture *regression of strategy level*

Flexible application of knowledge can be identified by comparing student work in the test with earlier work on similar tasks in the instructional units, in particular horizontal mathematizing activity and the level of strategy used to solve the problem. Mathematization which is typical for a specific task is a sign that the student recognizes the problem, but flexible application also involves picking a correct procedure for *solving* it. When the student chooses a strategy suitable for the task – i.e. a strategy which worked appropriately in classroom activities –, earlier ability is used again. If a student does not recognize the isomorphism, he or she will resort to a lower level strategy not particularly fit for solving equations (like trial-and-adjustment or incorrect strategies).

Results on the *Number Cards* task show that a substantial number of students fall back from an algebraic level to an arithmetical level of understanding when they find themselves in a slightly different problem situation. The apparent formalization that is observed during the lesson series does not reach further than one particular type of problem. Or perhaps the algebraic solution easily becomes an artificial, rote skill for those students who adopt a higher level of strategy from a classmate but who are not cognitively ready to take that step, in other words, we notice a problem of vertical mathematization. Students seem to focus on specific characteristics of the problem and struggle to make the transfer to a new situation, especially the girls (who tend to stick with one particular strategy which works for them). Most students who succeed with an algebraic strategy seem to lose previous abilities; only a small number of students can fall back on an informal, pre-algebraic method for solving the problem. In conclusion:

conjecture

1 *The general applicability of algebraic strategies enlarges the risk of superficial understanding, causing an apparent regression of strategy level in a new problem situation;*
2 *The chance of regression is greater for girls than for boys.*

conjecture *understanding relations*

The first research results (the pilot experiment included, see chapter 5) indicate that an arithmetical perspective of how quantities are related hinders the development of an algebraic way of thinking about variables and unknowns. Generally arithmetic does not involve operating or reasoning with an undetermined number or the idea of more than one solution. Instead, typical arithmetical problems are based on a direct approach using fixed numbers leading to a single answer, like add-end sentences or word problems involving reverse calculations. This arithmetical view of relations between two quantities causes students to misinterpret the problem situation and operate on the known numbers instead of the unknowns. In conclusion:

conjecture

*Arithmetical notions of numbers and relations between numbers hinder the emergence of an algebraic conceptio*n.

In the current form these conjectures enable us to answer the first of the two main research questions. The results expose a number of essential differences between arithmetic and algebra which have been discussed in chapter 2. Some sub-questions and the second main question on history of mathematics have not been addressed directly by the three conjectures stated above. These questions will be answered using data from the pilot experiment, questionnaires and tasks from the instructional units not discussed in relation to the test.

6.7 Results of student work

In the final phase of the research project, the research questions have been operationalized step by step in order to make them more accessible. In section 6.3 we explained that the main research questions have been translated in terms of more detailed sub-questions. These sub-questions, which concentrate on issues of algebraic thinking, symbolization and the integration of historical elements, have been taken as points of departure for the field test analysis. In the initial phase of analyzing student work, we focussed on the theoretical ideas even further by formulating some *conjectures*.

operationalizing the research questions

In other words, there have been three steps of 'zooming in' more closely on the issues that concern us (see figure 6.10). When the analysis has been completed, we return to the starting-point – the main research questions – by 'zooming out' from our findings in the reverse direction. The conjectures and discoveries on algebraic learning provide (some) answers to the sub-questions, which in turn enable the main research questions to be answered.

The analysis of student work has two components: testing the conjectures, while simultaneously keeping an eye open for other results. In section 6.7 we present student results which either confirm or contradict certain conjectures, while other findings can be qualified as unexpected. First we present an overview of the student results, followed by a qualitative and quantitative argumentation for each (except result 8) in separate sub-sections. Results 1 through 4 are concerned with bridging the gap between arithmetic and algebra (research question 1) while results 5 and 6 address the issue of history of mathematics as a didactical tool (research question 2). Result 7 brings both theoretical themes together, and result 8 is beyond the range of this study.

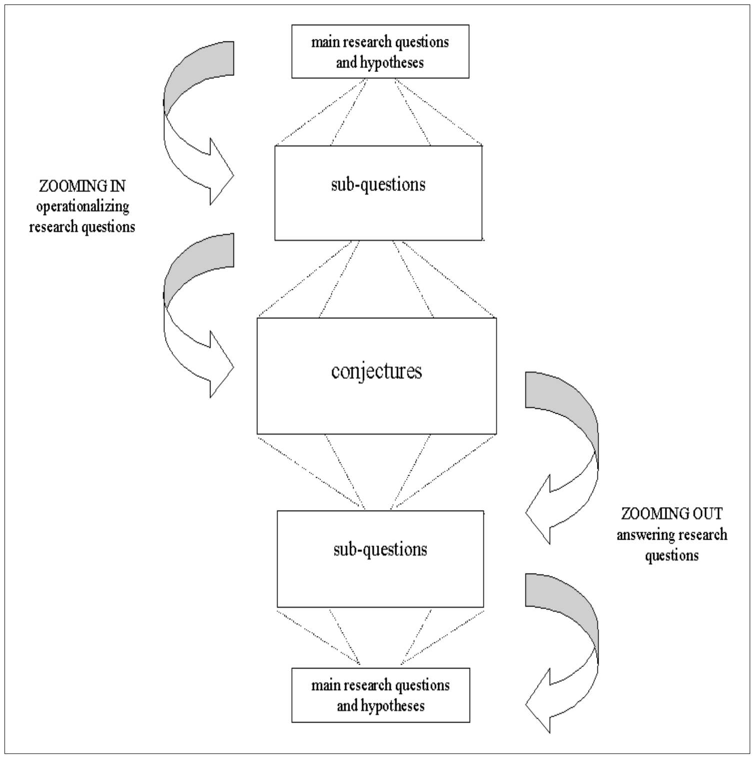


figure 6.10: accessibility of the research questions

6.7.1 Overview of results

1 reasoning versus symbolizing

Algebraic *reasoning* appears to be more accessible to learners of early algebra than algebraic *symbolizing*. Furthermore, in student work they have been identified as independent aspects of algebra learning. For instance, algebraic thinking does not necessarily rely on algebraic notations, nor does an algebraic representation imply the use of a higher strategy. However, higher level notations do seem to correspond with better achievement. Three parts of the conjecture *reasoning versus symbolizing* have been justified:

1 Some students are able to invent or follow a path from arithmetical to algebraic methods independent of intermediate pre-algebraic strategies and/or representations;

2 Students who have difficulty in progressing from an arithmetical to an algebraic mode of thinking tend to remain at an arithmetical level of notation;
3 (Pre-)algebraic notations are not prerequisite to algebraic thinking, but they appear to be effective in solving (pre-)algebraic problems.

2 schematizing as a problem solving tool

Results from previous experiments in the research project already suggested that students are less susceptible to schematizing activities than expected (see section 5.7). The development of models, diagrams and tables as tools for mathematical reasoning does not concur with the arithmetical capacities of the primary school students. Indeed, it is not self-evident that students are inclined to search for increasingly efficient strategies. As a result the application of so-called restriction problems as an informal approach to solving systems of equations has not been effective.

3 regression of strategy use

Based on the results of the pragmatic test task Number Cards, we formulated a third conjecture on regression of the level of strategy students used in the test compared to their classroom work. First, we hypothesized that the general applicability of algebraic strategies enlarge the risk of superficial understanding, causing an apparent regression of strategy level in a new problem situation. Second, we anticipated that the chance of regression would be greater for girls than for boys. At present our sources have not disclosed sufficient data to draw a valid conclusion on this conjecture. In those cases where we have detected a regression in strategy level, it has been both arithmetical as well as algebraic strategies; in other words, it cannot be ascribed to the nature of algebra. Furthermore, we have found no evidence for the second part of the conjecture amongst the other tasks we analyzed.

4 conceptions of relations

Arithmetical notions students have of numbers and relations between numbers have been found to hinder the emergence of an algebraic conception. For instance, when a mathematical problem represents an embedded system of two equations, an arithmetical view of how quantities are related causes some students to operate on the known numbers instead of the unknowns. The support for this conjecture is very convincing, as shown in table 6.2.

5 effect of history of mathematics on learning (pre-)algebra

At primary school level the Diophantine problems on sum and difference led to the reinvention of the strategy of halving the difference, and the tasks on inverting a chain of calculations have shown some promising applications of schematizing.The pre-algebraic methods Rule of Three and Rule of False Position created opportunities for secondary school students to reflect on their own strategies. The main drawback of implementing source material is the language barrier. Amongst students there was a mixed appreciation of historical problems and it does not seem to be gender-bound. Some students experienced these tasks as a pleasant alternative to routine exercises, while others – especially low attainers – found them confusing and difficult rather than interesting. Student reactions are more positive when the teacher himself has a good feeling about integrating history. For the designer/researcher the greatest value of using history has been the implications for a suitable learning trajectory.

6 role of the unknown

When equations arise in an informal textual environment, the unknown is relevant and effective for problem identification and organization but not for the problem solving procedure itself. Frequently the unknown appears early on during horizontal mathematization activities, and as students begin to solve the problem, the unknown disappears from the setting. We can distinguish here between an arithmetical and an algebraic role of the unobserved unknown. First the arithmetical role: if the unknown can be determined by just inverting a series of operations, it does not need to be a part of the calculational procedure; it has become superfluous. The unknown can also be latent in an algebraic process of reasoning: if students focus their attention on the *coefficients* of the terms, they do not need the unknown to be present all the time. Usually the unknown reappears when the answer is written down. In addition we see that students keep the symbolic system of equations separated from the calculations, almost in two columns, as if the calculations are not a part of the system itself.

7 recapitulation opposed to reinvention

One of the major aims of this study has been to establish to what extent the historical developments of algebra reflect a natural path for learning early algebra. To do so, we have looked for signs of *recapitulation* – where historical developments of the subject matter show parallels with the cognitive growth of the individual learner – and *reinvention* of mathematics, where the natural development of the individual may diverge from the ontological growth of the subject to follow another – more efficient or natural – route. The analysis of student work in the final teaching experiment indicates that there are certain similarities between informal symbolizing of students and developments of algebraic notations over time. In addition we have seen a few students solve linear equations in the same manner as Ahmes did many centuries ago. On the other hand we have observed the reinvention of solution strategies for Diophantine problems on sum and difference and a variant of the Rule of False Position.

8 gender differences

Although the number of students participating in the field test is rather small for announcing a result on gender differences, we have found some evidence that girls responded more actively to the experimental learning strand than they generally do in mathematics (Van den Heuvel-Panhuizen et al., 1999). We have observed situations where girls played at least an equally active role in reinventing and popularizing (pre-)algebraic strategies as boys. However, we have been obliged to focus our attention on the main research questions, and so we leave the issue of gender differences to a possible continuation of this study. For this reason we have not included a separate section on this topic at the end of the section.

6.7.2 Separate developments ofalgebraic reasoning and symbolizing

The analysis of a wide range of written tasks has demonstrated that students proceed more easily to an algebraic mode of thinking than to a higher level of symbolizing. Initial results on the *Number Cards* task initiated the presumption that reasoning and schematizing are not necessarily dependent competencies (see section 4.5 for an elaborate account). For instance, pre-algebraic strategies which rely on the use of schematic or visualized notations seem not to emerge naturally as a way of formalizing arithmetical strategies. We also found that students either invent or adopt from each other mental (reasoning) strategies, or they continue with trial-and-adjustment methods. Apparently it is not self-evident that students are inclined to search for more accurate and efficient methods like systematization and recognizing (number) patterns. Consequently the proposed trajectory of reinventing algebra encounters obstacles we had not anticipated. On the positive side, it appears that students are sometimes able to find another route from arithmetic to early algebra.

conjecture *reasoning versus symbolizing*

This conjecture claims that algebraic reasoning is not dependent on algebraic symbolizing, but students can profit from schematic notations in their acquisition of an algebraic way of thinking. We distinguish three components:

conjecture

1 *Students are able to invent or follow a path from arithmetical to algebraic methods independent of intermediate pre-algebraic strategies or representations, or both;*

2 *Students who have difficulty in progressing from an arithmetical to an algebraic mode of thinking tend to remain at an arithmetical level of notation;*

3 *(Pre-)algebraic notations are not prerequisite to algebraic thinking, but they appear to be effective in solving (pre-)algebraic problems.*

Table 6.2 shows the support for this conjecture amongst a broad selection of tasks for primary and secondary school students, which we will discuss more closely in chronological order in the remainder of this section. For practical reasons the three parts of the conjecture will from now on be referred to as *skipping pre-algebra* (part 1), *arithmetical notations* (part 2), and *(pre-)algebra as a tool* (part 3).

test (number - level)	task	conjecture *reasoning versus symbolizing*		
		skipping pre-algebra	arithmetical notations	(pre-)algebra as a tool
I primary	Pocket Money	+	+	+
	Number Cards	+	+	+
	Birthday	+	+	+
	Dutch Past	–	+	+
II primary	Trading Stamps	+	+	+
I secondary	Flowers & Cabinets	+	INC	+
II secondary	Human Body	+	+	+

table 6.2: support for the conjecture 'reasoning versus symbolizing'
(+ = confirmative, INC = inconclusive, – = not applicable)

***Pocket Money* test 1 primary school**

The *Pocket Money* task deals with three relations and four unknowns:

> Dean, Martin, Sabrina and Josy have 28 guilders altogether. Dean has the same amount as Sabrina. Martin has three times as much as Josy.
> 1. How much could each kid have?
> 2. Suppose Dean has 7 guilders. Then how much do the other kids have?

This problem belongs in the theme of combination/restriction problems and concurs with the activities of the first two sections in the unit *Exchange.* Question 1 of the *Pocket Money* task can be interpreted by students at two levels, namely 'give an example of how much each kid might have' and 'give different possibilities of how much each kid might have'. It is relevant for the discussion to report that for question 1 we have classified two strategies as algebraic and two as arithmetical; we have not identified a strategy at a pre-algebraic level. For question 2, on the other hand, there is just one pre-algebraic strategy and no algebraic ones.

purpose and expectations

The *Pocket Money* problem belongs to the theme of combination/restriction problems and concurs most with activities of the first two paragraphs in the unit Exchange. The purpose of the task is to have students show how they reason about relations between unknowns, and whether they make use of schematic or symbolic notations in the process. The expectations in advance were, that there would be a group of students who can solve the problem mentally – with little draft work – by reasoning how to combine the relations in a direct, efficient manner, and a group of students who need to try numerical values to get started. On account of classroom activities on making combinations – intended to provoke students to develop a systematic approach – our expectations prior to the experiment were that few students would use a random trial-and-error strategy, especially for question 2. It was also foreseen that some students might write the relations in shorthand notations, due to the natural emergence of abbreviations and teacher appraisal in the classroom.

Question 2 is a specific case of question 1 but a complex one, intended to determine which students could actually cope with the 1 : 3 ratio and trying to reduce trial-and-error strategies to a minimum. However, the complexity of this final part of the solution has been underestimated; the number of students which succeeded to divide an unknown into two parts with ratio 1 : 3 was much lower than expected. This issue is discussed in more detail in section 6.7.5. It was also not foreseen that the designer's choice of value in question 2 corresponded with the most natural choice for students, because for these students question 2 was no more than a repetition of question 1. For question 1 we supposed there would be a substantial number of students giving more then one solution, even though they were not required to do so. However, classroom observations prior to the test indicated already that students are not inclined to pursue a systematic approach, nor do they do more than what is asked of them. It certainly means a misjudgement on behalf of the designer with important consequences. We will go into this matter of attitude in section 6.9, too. Last but not least, we assumed students to have become accustomed to the request to show their thinking, either with calculations or with words.

strategies question 1

The highest level algebraic strategy for question, 'reasoning with variables', involves the calculation of several solutions: the quantities are variables taking on a range of values. The second strategy level, 'reasoning × 3 and equally much' is also considered to be algebraic because students are required to make an assumption about an unknown quantity (see figure 6.11).

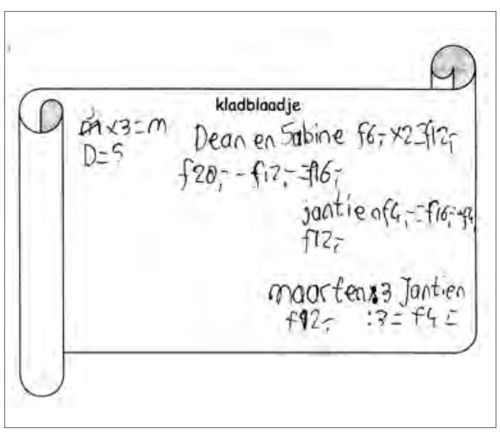


figure 6.11: *Pocket Money*, question 1: algebraic reasoning and symbolizing

The quantities can be calculated in a different order: either Martin and Josy first (the relation ‘3 times as much’) or Dean and Sabrina (the relation ‘equally much’) first. The former choice is more convenient than the latter because dividing any particular number by 2 is easier than dividing it by 4. In other words, if students deliberately turn the order around compared to what is stated in the problem, they show a higher level of thinking. For the strategy of numerical attempts we have not distinguished between trial-and-adjustment and trial-and-error strategies since the limited range of values makes it harder to recognize random attempts. The majority of numerical efforts show an improvement in consecutive attempts anyhow. The most frequent incorrect strategy is that of ignoring the given relations and dividing 28 by 4 instead. There are no strategies for the ‘pre-algebra’ category.

strategies question 2

For question 2 we have typified the highest level of solution as pre-algebraic rather than algebraic because since one of the values is set, the first part of the solution is purely arithmetical. Only the second step in the solution involves some reasoning, namely that there are 14 guilders left for Martin and Josy in the ratio 3 : 1. If a student then clearly demonstrates the calculation 14 : 4, the sample belongs in the highest category, otherwise it is considered to be a case of numerical attempt. The second arithmetical strategy represents students who take the first step successfully but then stumble on the division 14 : 4, either getting stuck on the arithmetic or saying it is not possible.

notations

Like before, only the notations used in the solution process will be considered. The notation categories are the same for both questions, but in fact the mathematization and symbolization of the problem situation primarily occurs in answering question 1. The representation ‘(word)formula’ is judged to be more advanced than ‘syncopated notation’ because of the presence of an equal-sign. If a table is used as a tool for listing values it is characterized as ‘arithmetical’, while if the table helps students recognize a pattern we consider it to be of pre-algebraic use. The two arithmetical types of representation are hard to place in a hierarchal order because it is quite likely but not demonstrable that the students who give a prescription solved the problem by calculations first. However, if the prescription were in general terms, it would be of a general nature and therefore more formalized than calculations.

task results

The students from schools A, B and C performed quite differently at the *Pocket Money* task, and the test results are far below expectations. As shown in figure 6.12, nearly half the students at school A (n = 23) succeeded to answer question 1 with a correct strategy, of which 22% algebraically.

For example, the student whose work is shown in figure 6.11. mathematized the task using (word)formulas (algebraic notations) and then solved the task by algebraic reasoning. Note that the he nearly made the error of reversing the relation ‘three times as much’; such errors will be discussed in more detail in section 6.7.5.

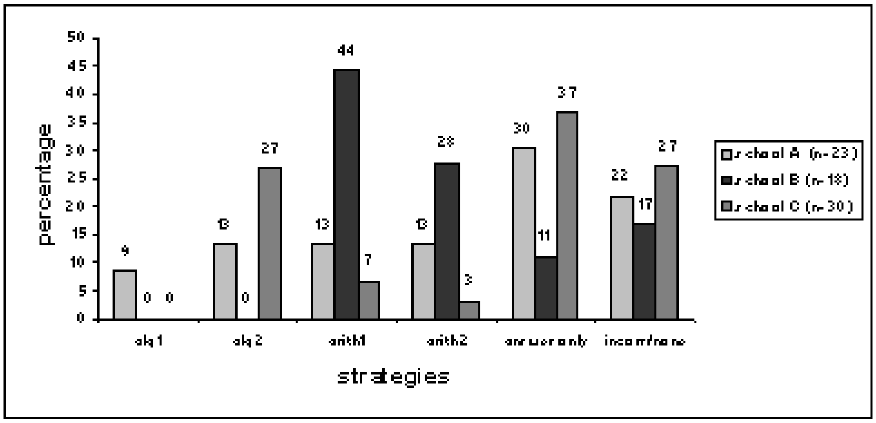


figure 6.12: strategies used for *Pocket Money,* question 1

algebraic reasoning

Just four students – three of whom at school A – gave more than one solution to question 1, interpreting the unknowns as variables, which in our opinion is the most advanced conception of the situation. However, since they wrote only an answer (without explanation), their contribution is not included in the algebraic component. There are a few possible explanations for this low number.

First, there may have been more students than these four who realized the possibility of multiple solutions, except they did not see the necessity to write more than one down. Indeed, they are not requested to do so, and it is generally known that students tend to stick with the task very rigidly.

Second, it might have taken the student longer than intended to unravel the problem, causing him or her to leave it at one solution. This is quite a likely explanation for those students who calculated by coincidence the values that are sought in question 2.

Third, the apparent understanding of the mathematical meaning of restriction problems is a misjudgement on behalf of the researcher; students may have just gone along with the teacher's or other students' conclusion or they might have forgotten it. If the question had specifically asked for more than one solution, there would probably have been more multiple answers, but this would have made the question biased.

At school B (n = 18) we see that more than 70% of the students used an arithmetical strategy opposed to none algebraic, while just more than one third of the students from school C approached the problem correctly (27% algebraic). For the second part of the task we observe only seven instances of pre-algebraic reasoning, compared to twenty-four correct applications of arithmetical strategies and thirty-two incorrect answers. The students from school B in particular performed significantly worse than before. Presumably quite a number of students misunderstood that the relations between the unknowns also applied to the second question. All in all we can say that the task has been quite difficult for students, and arithmetical strategies prevailed over (pre-)algebraic ones.

classroom differences

These results can be explained by the teaching-learning process observed in the different classrooms. In school A and school C there were (groups of) students who were able to reason about the relationships – dividing by 5 if the relationship is '4 times as much' – while the students at school B worked more at a level of trial-and-adjustment.

use of notations

Student use of notations varies amongst the schools as well. Ten instances of (pre-) algebraic notations (seven syncopated expressions and three (word)formulas) were scored for school A, which is 44% of the class. These notations are found in all strategy categories, and the answers are all correct. The combination of shorthand notations–'answer only' strategy seems to be a contradiction, but it means that there is not enough evidence (calculations, or descriptions) of how the student has found the solution. In addition there are two students who expressed their algebraic reasoning in terms of calculations. At school B none of the students solved the task algebraically, but all five applications of the table as calculational tool led to a correct solution; only two students used syncopated notations. Of the eight algebraic solutions at school C, only one includes a table and five include a description. In comparison, half the students used arithmetical notations – including six descriptive ones – and 38% gave only the answer.

support for the conjecture

The results on the test task *Pocket Money* support the conjecture *reasoning versus symbolizing*. Firstly, we have observed that students are capable of *skipping the pre-algebraic phase* as they solve this type of problem. It appears that algebraic thinking does not necessarily rely on (pre-)algebraic symbolizing (note that we have not identified a pre-algebraic strategy for question 1, and for question 2 the pre-algebraic strategy is the highest level strategy students used). Second, when we look at students who find it difficult to reason correctly about simultaneous relations, our conjecture that they tend to remain at an *arithmetical level of notations* holds true for this part of the analysis. Alternatively, regarding part three of the conjecture – on *(pre-)algebra as a tool* –, the task results suggest that when a student uses a (pre-) algebraic representation, it does not necessarily imply the use of a higher order strategy, nor does it guarantee success. On the other hand, even though advanced schematizing or symbolizing is not prerequisite for algebraic thinking, it does seem to correspond with better achievement.

skipping pre-algebra

If we compare students' level of strategy with their level of notation for question 1 (figure 6.13), we notice that reasoning and symbolizing are independent competencies. For this task, apparently, algebraic reasoning does not require advanced symbolizing (school C), nor does advanced symbolizing imply a higher level of reasoning (school A). Altogether twelve students applied an algebraic strategy in the absence of any pre-algebraic schematizing (table) or symbolizing (syncopated

notations). Of this group, eight students operated at an arithmetical level of reasoning in the classroom, which means they jumped from an arithmetical to an algebraic level of reasoning. If we were to include the data we considered invalid (fourteen instances of ‘answer only’ strategy), it would also strengthen the argument that algebraic reasoning can occur in the absence of pre-algebraic symbolizing.
Since question 2 can be solved without algebraic reasoning – calculating only with known quantities – it is not relevant for the discussion on skipping pre-algebra.

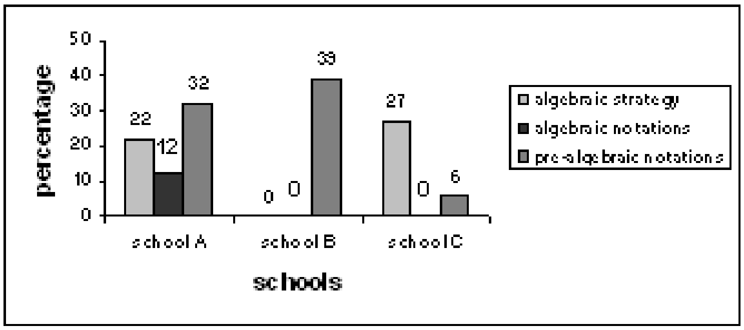


figure 6.13: *Pocket Money* task question 1, reasoning versus notations (n = 71)

arithmetical notations

For the justification of the second part of the conjecture *reasoning versus symbolizing*, on the restraining influence of *arithmetical notations*, questions 1 and 2 of the *Pocket Money* task were combined with another test task, *Birthday.* In this way we obtained three comparable problem situations. Students who used incorrect strategies for two of the three problems were selected to test for a correlation between poor achievement and level of notation. In this way we found seventeen students who scored poorly on restriction problems where the relations between the quantities are presented in words, using just arithmetical notations during the solution process. Alternatively we found no counter-examples, meaning there were no students who used an incorrect strategy twice in combination with pre-algebraic notations. In other words, we have found support for the conjecture that students who struggle to reason algebraically about quantities and the relations between them, tend to remain at an arithmetical level of notations, or vice versa.

(pre-)algebraic aid

With regard to the functionality of *pre-algebra as a tool,* the third part of the conjecture *reasoning versus symbolizing*, we can say that the data agree. They imply that students who use a (pre-)algebraic notation like a formula, table or syncopated notations often – but not always – have more success at solving the problem. For instance, after investigating the occurrence of the combinations (pre-)algebraic notations/correct solutions and (pre-)algebraic notations/incorrect solutions, the figures were quite convincing. For the first question of the task eighteen cases of (pre-)algebraic notations led to a correct solution, and none to an incorrect solution; for the second question the numbers are six and one respectively. If we compare these results with the frequencies of the combination arithmetical notations/incorrect solution, we find sixteen cases for question 1 and thirteen for question 2. Consequently we can conclude that schematizing and/or symbolizing has a positive effect on solving restriction problems such as the task *Pocket Money.*

Number Cards **test 1 primary school**

support for the conjecture

An elaborate discussion on the results of this test task has already been given in section 6.6.2. Here we merely summarize our findings with respect to the conjecture reasoning versus symbolizing. The justification of the part on *skipping pre-algebra* follows from the way in which the elimination and algorithmic strategies – characterized as being algebraic – originated during the test and in the classroom. We know from analyzing the source problems in the unit *Exchange* and from classroom observations that some students invented the strategy of halving the difference and others adopted it, without using the pre-algebraic approach of adjusting the difference symmetrically first. For the justification of the second part, on sticking with *arithmetical notations,* the Number Cards task was combined with two source problems in Exchange to establish whether poor understanding is related to primitive symbolizing. Indeed, eleven students applied an incorrect strategy in at least two of the three problems, and they all used calculations to mathematize the problem. None of these students took a schematic approach, meaning they have not been able to organize their work in a pre-algebraic manner. This part of the conjecture, therefore, clearly holds. Finally, there is also support for the component of the conjecture on *(pre-)algebra as a tool.* Classroom results in school B show that, when (pre-)algebraic strategies do not emerge spontaneously, students can profit from schematic notations. The empty number line enabled a few students to make progression. Furthermore, six of the nine applications of a tabular representation in the test task led to a correct solution. However, the relatively limited use of pre-algebraic notations altogether for Number Cards also indicates that pre-algebraic notations are not prerequisite for all students.

Dutch Past **test 1 primary school**

task results

Dutch Past consists of seven questions on two different mathematical problems, the second of which will be used only to supplement the discussion. Some of the task questions are also used to evaluate the effect of integrating history in an early algebra lesson series (see section 6.7.6). Problem 1 is taken from a Dutch arithmetic book by Abraham de Graaf (1672), and it tells of a man in an apple orchard who has to hand over some apples each time he meets a gatekeeper. There is a small textual question to get warmed up, followed by a summary of the problem:

> The man has to hand over half of his apples plus one more apple. The question is: if he has one apple left after the third gate, how many apples must he have had?

purpose and expectations

With this task we aim to determine the learning effect of the classroom activities on a chain of reverse calculations, in particular the ways in which students mathematize the problem information. In the unit *Exchange* students encounter inversion problems in an historical context as well as a mathematical context (‘guess my number’). This type of problem requires only arithmetical thinking: there is no reasoning with unknowns, systematizing or generalizing involved. However, students use various strategies at different cognitive levels: inverting the operations mentally, working directly in the backward direction; representing the forward actions on paper before or after inverting them (for support); or the method of trial-and-error. During the lesson series students are introduced to arrow language as a convenient tool for fixing the operations in the description: which operations to be carried out in which order and how often. The chain of arrows can also support the mental activity of inverting by reminding the student whether he is going in the forward or backward direction. We expected students to recognize the problem very quickly, particularly by its historical context, and have generally little trouble solving it. The operations (division by two and subtracting one) are easy to invert, but there may be a few students who have difficulty interpreting ‘half of his apples plus one more’, despite their experience with two very similar problems in the classroom. The strategies will probably cover the whole range of levels, but the learning effect is deemed satisfactory if more than half the students can carry out the inversion of the operations in a recognizable way (not trial-and-error or an answer without explanation). The use of arrow diagrams is not compulsory in the program; students are free to write the solution any way they want, as long as it explains their thinking. Based on the classroom experiences, we expected to see mostly arrow diagrams or string calculations, perhaps accompanied by words or symbols to indicate the three gates in the orchard.

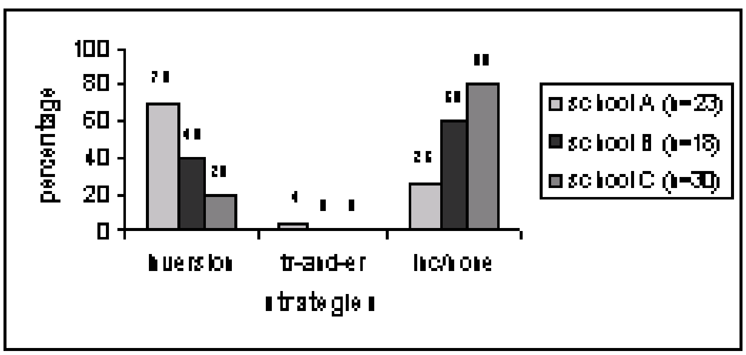


figure 6.14: strategies used for *Dutch Past,* problem 1

task results

The first, most obvious observation is the poor success rate amongst two of the three schools (see figure 6.12). No more than thirty students out of seventy-two solved the problem using a correct strategy, 17 of which from school A (which is a 74% success rate). The percentages in figure 6.14 show that the students at school B and C performed far below expectation. In school A the strategy of mental inversion was clearly the most natural approach for students at all levels of competence; none of the students first noted down the forward chain of operations but a few used it to check their solution. And contrary to school C, not one student at school A left the problem untouched. At school B we observe all three inversion strategies (only amongst the better students) but slightly more examples of an incorrect strategy (44%). Most students at school C had very little understanding of the problem, because only six students showed an inversion strategy (all mental). There may be different explanations why one school does so much better than the others, and why so many students had difficulty in solving this problem. For starters we can say that the poor results for school C are probably due to lack of practice, because the last paragraph of the unit *Exchange* was skipped. We will shed some light on this issue as we discuss notations and errors that occur in the students' work and the classroom experiences with the unit *Exchange*.

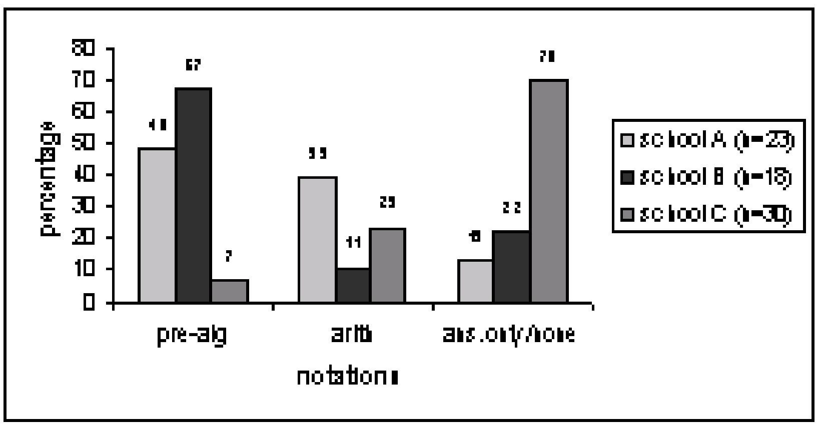


figure 6.15: comparing schools on notations for *Dutch Past*, problem 1

notation use

About one third of the number of students – both high and low achievers – used pre-algebraic notations like arrow language or another type of schematic diagram, but again not evenly spread amongst the schools, as shown in figure 6.15. In the total student population, arrow language or schematic notations is used by 35% of the students and calculations or descriptions by 25%, leaving 40% for only the answer or no answer at all. The algebraic component is significantly larger than in the other test tasks we have discussed, but not so large as to suspect that students are trying to please the teacher. Arrow diagrams, schematic representations and calculations appear at all levels of cognition, and only at school A we see that girls use slightly more calculations than the boys (see appendix for some examples of notations). The occurrence of pre-algebraic notations is very different for the three schools, as can be seen in figure 6.15. At school A we see some more students use visual notations (arrow diagrams 42%, schematic 8%) than a series of calculations (38%), and 13% of the students wrote only an answer. At school B more than half the students drew arrows (58%), and at the bottom end three students (16%) failed to write anything at all. Classroom observations reveal that only in this class the students seemed convinced that it is not correct to write equal-signs between intermediate answers, and that an arrow diagram is better. The students of school C used very little notations in general; just nine students (30%) showed their draft notes, compared to 27% 'answer only' and 43% 'none'. Again we see how little effort the students from school C make to show their thinking process.

partial support for the conjecture

It must be said at forehand that the task *Dutch Past* is only partly relevant for confirming the conjecture *reasoning versus symbolizing* (see also table 6.2). The task is not applicable for the first part of the conjecture because solving it does not require any (pre-)algebra. In fact, an arithmetical approach is much more suitable, which means that *skipping pre-algebra* is not under discussion. However, its opportunities for schematizing make it a suitable case study for the issues *arithmetical notations* and *(pre-)algebra as a tool.* Apparently visual diagrams like a chain of arrows can be an effective problem solving tool, although it does not guarantee a reduced chance of error nor does it imply a higher level of problem solving.

arithmetical notations

We have found that results of the test task *Dutch Past* underline the conjecture *reasoning versus symbolizing* with respect to the use of arithmetical and pre-algebraic notations. Research data on similar problems in the unit *Exchange* strengthen our belief that students who have trouble inverting the operations fail to exceed the lowest level of notation (the conjecture part referred to as *arithmetical notations*). In addition it seems that students who make an error in their reasoning tend to use the arrow diagram superficially, i.e. without meaning.

First we consider problem 1 of the task. We note eleven cases of incorrect strategy combined with pre-algebraic notations on the one hand, and fourteen instances of the combination incorrect strategy/arithmetical-or-other notation on the other. Figure 6.16 shows an example of each: incorrect operations in the arrow diagram at the top (in the top right-hand corner we see '× 2 – 1', so the in the middle of the arrow diagram '2 × 1 –' should be read literally from right to left), and repeated calculations 'minus 1.50' below. Analyzing the work more closely, it appears that most students who do not succeed with the arrow diagram, have not put it to good use. In ten of the eleven occasions the student attempted to solve the problem by reverting mentally (like in figure 6.16), while the real profit comes from expressing the forward sequence first, as shown in figure 6.17. In other words, these students have probably not yet progressed to the pre-algebraic level of symbolizing and are really still on an arithmetical level; the algebraic notations have a superficial character. We have observed the same trend amongst corresponding unit problems solved in the lesson series. Most students were able to invert the operations mentally; the arrow diagram was used primarily as a tool for writing the answer instead of solving the problem. For the second problem of the task a similar analysis produces inconclusive data.

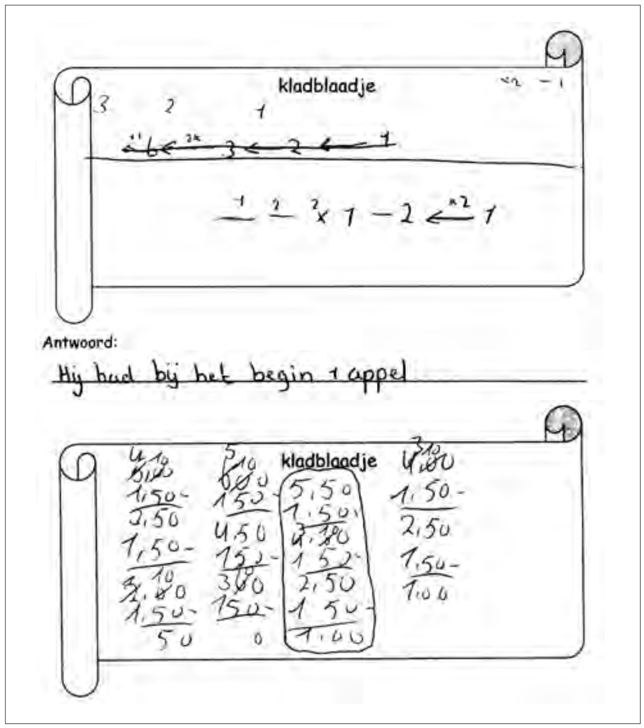


figure 6.16: incorrect strategies for the task *Dutch Past*

pre-algebraic aid

Where *pre-algebra as a tool* is concerned (the third part of the conjecture *reasoning versus symbolizing*), we found that pre-algebraic notations like the arrow diagram can fulfill a supportive role for students performing reverse calculations. The use of visual reasoning strategies – writing down the forward chain of operations too – was compared with the use of visual diagrams (arrow language or schematic notation) on one hand and arithmetical notations on the other. Figure 6.17 shows, for instance, how the forward chain first supports the student's reasoning (it was initially empty, judging from the dotted lines) and then it enables the student to double-check her answer.

Eight instances of the combination visual inversion/visual diagram and sevenmore cases of mental inversion/visual diagram were counted. This means that fifteen learners used pre-algebraic notations in a way that helped them. Comparing these figures with the eleven cases of incorrect strategy/pre-algebraic notations, we can conclude that pre-algebraic notations seem to be a effective for learners who understand how to profit from them. To give an example, in figure 6.18 we see a student who uses numerical notations for a trial-and-error approach first, before switching to an arrow diagram. The arrow diagram then enables him to solve the problem.

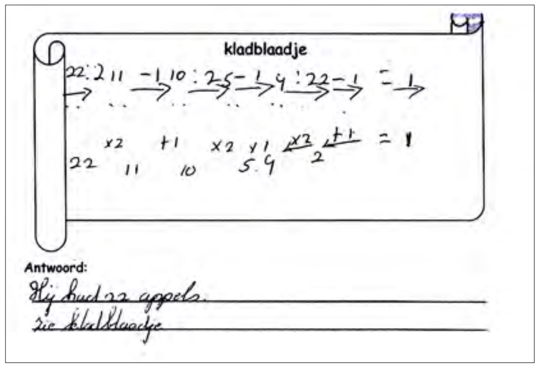


figure 6.17: the forward arrow diagram, also as a checking device

Again the forward chain is used to check the answer. This observation is supported by the respective numbers for the second problem in the task (nine cases of correct strategy/pre-algebraic notations opposed to none incorrect strategy/pre-algebraic notations). Classroom work on inverting operations does not reveal a clear outcome on this matter. In conclusion, the results suggest that it is worthwhile stimulating the use of schematic notations for solving reversion problems as in *Dutch Past*.

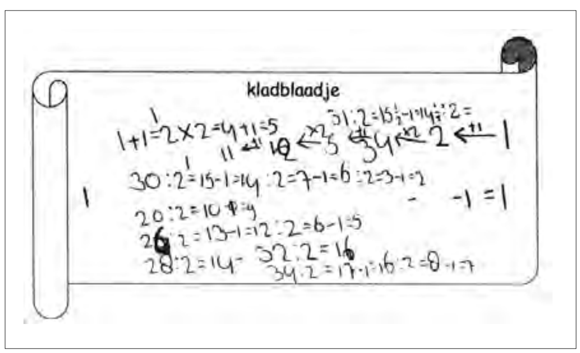


figure 6.18: arrow diagram as problem solving aid

closing remarks

In addition we have investigated a possible link between the mental inversion strategy and notation. The test data indicate that advanced symbolization is not prerequisite for advanced arithmetical thinking (six combinations advanced strategy/pre-algebraic notation opposed to fifteencombinations advanced strategy/arithmetical notation), as illustrated in figure 6.19. Especially the frequent use of mental strategies at school A for the classroom activity ‘Guess my number’ underlines this result. In other words, although the *Dutch Past* task does not provide any support for the conjecture’s part on *skipping pre-algebra* – which states that *algebraic* reasoning can take place without (pre-)algebraic notations – it does make a strong point for the feasibility of *advanced arithmetical* thinking.

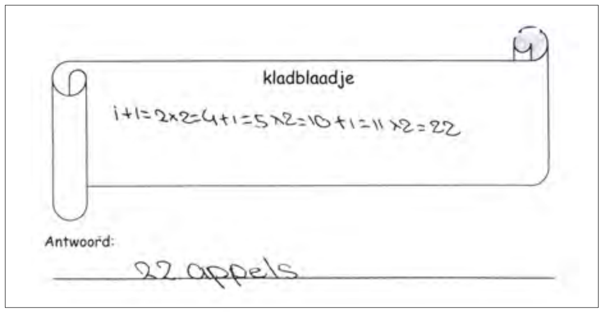


figure 6.19: mental inversion with arithmetical notations

Birthday test 1 primary school

The *Birthday* task is a restriction problem with three restrictions and three unknowns:

> “Happy birthday to you, happy birthday to you.” Mirjam’s voice carries across the lawn. It’s her mom’s birthday, and so she baked a delicious cake for her. Her mom has trouble blowing out all the candles in one go. “You are now exactly three times as old as me, mom!” Mirjam's dad adds in: “Your mom and I are now 88 years old together”. Mom replies: “And the three of us are 102 years old”.

Mathematically this test problem is similar to the second part of the *Pocket Money* task, except for the context. For this reason we also expected similar results, and so we were quite surprised to discover that the student scores for this problem are much better than the *Pocket Money* task. The problem situation appears to have been easier for students to understand and remember while working at it, because we see less draft work than in the case of the *Pocket Money* task. Since many students solved the task mentally, without any draft work, we cannot say whether strategy use is also more advanced in comparison.

strategies

The most advanced strategy to solve this problem, ‘reasoning with relations’ (pre-algebraic level), involves comparing the last two restrictions to determine Mirjam’s age, which is 14. The other two ages follow immediately from there. After the first value is found, the problem is very similar to the second question in the *Pocket Money* task, and therefore the categorization of strategies is very similar. However, if two other relations are combined first instead, the problem becomes much more complex and the student can be at a dead end very easily. At the arithmetical level we observed two different strategies: 'reasoning with 1 condition' and 'trial-and-adjustment'. Finally there are also incorrect strategies and 'none'. The most frequent errors will be discussed in section 6.7.5.

notations

The categories for written student work are also the same as the *Pocket Money* task: (word) formulas and syncopated symbolism (algebraic), tabular and schematic notations (pre-algebraic), calculations (arithmetical), 'answer only' and none. The vast majority of students wrote either calculations or just the answer, but we also observed several students who copied down the restrictions in the allocated draft area or underlined them in the text. Five students used a schematic representation – diagram or summary – to clarify the problem situation to themselves, like in figure 6.22. It seems this student developed his diagram after his numerical attempts proved to be incorrect (44 in the right-hand corner, and 51, 33 and 11 on the right-hand side, although these values are not so clear). So just like the *Pocket Money* task, only a few students thought of using a pre-algebraic organizational tool to support the solution process. Whenever students used a table, it served a calculational purpose (for example, to generate the table of 3) or it contained the answer, rather than facilitating a shortcut of consecutive numerical attempts. In other words, the table only had an arithmetical status for this problem.

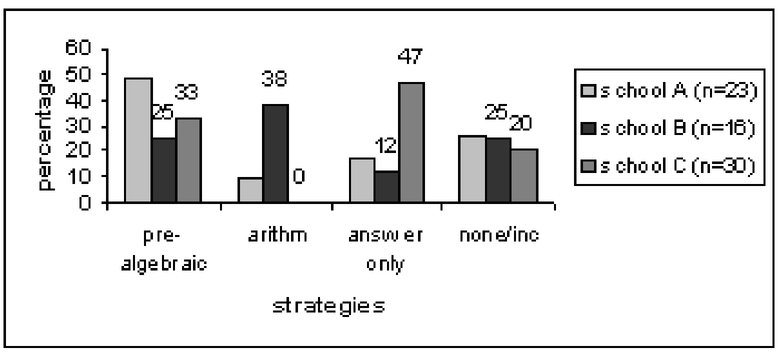


figure 6.20: strategy use for the task *Birthday*

task results

At first glance we see a large variety in strategy use at the three schools. The success rates (correct strategy) are quite similar, but there is a wide distribution of the categories 'pre-algebraic' (reasoning with the relations), 'arithmetical' (trial-and-adjustment) and 'answer only'. At school A nearly half the number of students succeeded to solve the problem by reasoning, but for schools B and C the respective figures are 25% and 33%. Even though the students from school B did not perform as well at the highest strategy level, 38% of the students successfully solved the problem arithmetically (compared with 9% for school A and 0% for school C). At school C the largest group of students solved the task mentally.

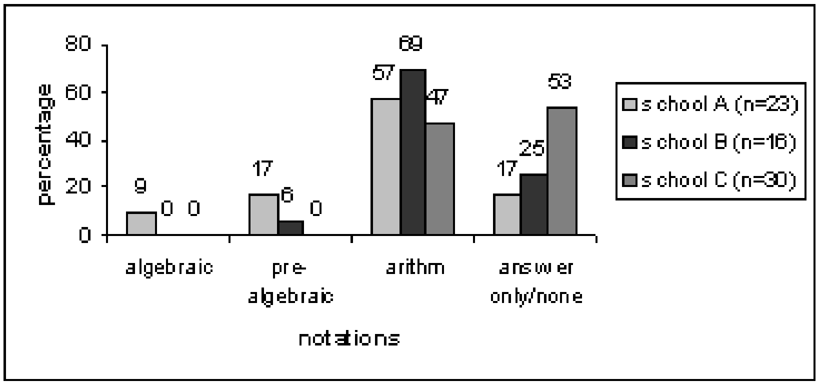


figure 6.21: *Birthday*: comparing the use of notations

partial support for the conjecture

The results of the task *Birthday* partially support the conjecture *reasoning versus symbolizing*. The data are inapplicable to the first part of the conjecture dealing with *skipping the pre-algebraic phase* because solving the problem does not require any algebra. By combining two of the three relations, the unknown quantities in the problem can be calculated in a direct manner. The highest strategy level is pre-algebraic, so a passage from arithmetic to algebra is not in order. However, the rest of the conjecture on *arithmetical notations* and *(pre-)algebra as a problem solving tool* can be justified.

arithmetical notations

The second part of the conjecture *reasoning versus symbolizing* deals with the role of arithmetical notations in relation to students' trouble to formalize their reasoning.

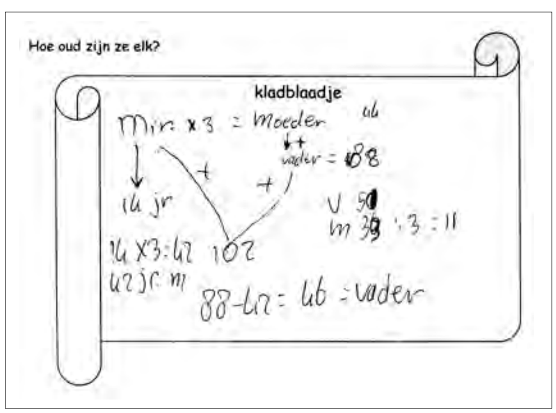


figure 6.22: *Birthday* task, schematic and syncopated notation

We have already mentioned in our discussion on *Pocket Money* that a significant number of students who struggled with two of the three questions, seemed to remain at a lower level of symbolizing. Taking just the *Birthday* task by itself, we count seventeen students who failed to solve the task, and all the incorrect answers but one correspond with the lowest levels of notations. Considering that only one student used a diagram in the solution process, it raises the question whether these students might have performed better if they had found a way to structure the problem on paper (like in figure 6.22), which we discuss next.

(pre-) algebraic aid

The frequent occurrence of correct reasoning without a sign of schematizing implies that the supportive role of pre-algebraic notations is not automatically a sure fact. If we also take into account the twenty correct answers without explanation, it seems even more evident that students do not rely on schematic notations to support them becomes even stronger. Still, in certain situations students such mathematizing activity can have a positive effect. Seven students opted for a (pre-)algebraic notation, either to represent the relations in the problem such as in figure 6.22 (two students), to guide the trial-and-adjustment process (two students), or to present their answer in a tidy fashion (three students). In six of the seven cases the solutions were correct. Despite the small number of data, the figures point out that students seem to profit from translating the problem into a schematically or symbolic format first. In conclusion, (pre-)algebraic symbolizing or schematizing is not required for a good understanding of problems dealing with relations between quantities, but it can fulfill a supportive role.

Task 1 Trading Stamps

1. Which stamp is worth the most? And which the least?
2. Can you trade (picture of lighthouse stamp) fairly for (picture of sailor stamp) and how does that work?

figure 6.23: *Trading Stamps* task, test 2 primary school

Trading Stamps **test 2 primary school**

purpose and expectation

The task *Trading Stamps* consists of an iconic trade situation and two questions (see figure 6.23) and its purpose is to assess students' reasoning skills with respect to making quantities comparable. These skills appear throughout the units *Exchange* and *Barter Trade* in different contexts and representations. We have deliberately chosen for the iconic representation because the visual effect is stronger than a description in words. At the same time the stamps themselves are not so practical for writing down the reasoning process, and will therefore provoke students to mathematize the situation. In this way we can get a good idea of the student's symbolizing activity. We expected to see primarily syncopated notations or new iconic expressions (pre-algebraic level), calculations or descriptions in words (arithmetical).

strategies

The first question is intended to give students the opportunity to investigate the situation using qualitative reasoning, but calculating the relative worth of each stamp is of course also possible. Qualitative reasoning has been integrated in different problem situations throughout the program, but is not presented explicitly to the students as a particular strategy. Instead it is more a logical way of thinking. The answers to this question will inform us about a student's intuitive or preferred solution strategy. The second question is expected to yield a variety of strategies ranging from an algebraic level of reasoning (substitution of expressions) to an arithmetical level. Note, however, that any suitable strategy involves reasoning, so with the categorization at hand there is not an arithmetical minimum (like trial-and-error) for the student to fall back on. In other words: it is almost an all-or-nothing situation.

level	alg-ar	strategy	description
1	alg1	substitution	reasoning by substituting relative values, 'variables'
2	alg2	assigning a value	calculating through, 'variables'
2*	pre	global reasoning	qualitative reasoning, not with exact numbers
3**	pre2	partial substitution	rounding off 'inconvenient' numbers, or using three types of stamps
	pre2	trial-and-adjustment	reasoning and trying new quantities
4	ar	estimation	mention of substitution method, not calculating
5		answer only	strategy not clear, answer correct
6		incorrect reasoning	reference to postal value, no comparison
7		none	

table 6.3: classification of strategies for *Trading Stamps* (* = just question 1, ** = just question 2)

task results

The task *Trading Stamps* requires students to compare quantities by reasoning with and substituting trade relations. In general the task has not been solved as well as expected. The first question in particular produced a lot of incorrect strategies at both schools, on which we elaborate in section 6.7.5. Note that the classification 'incorrect' was also used whenever the first answer was correct but without an explanation – 50% of the cases – given that incorrect reasoning for the second answer outweighs presumably correct reasoning for the first. Only six (out of forty) students gave the right answer to both parts, three of which without any notes. Contrary to our expectations, not a single student was able to reason qualitatively and write it down comprehensively. We also encountered 6 occasions where the student referred to the postal value on the stamps instead of the trade value, which we had not envisioned at all. The results for the second question are better at both schools, but still below expectation (see figure 6.24); (pre-)algebraic and incorrect answers occur in nearly equal proportions.

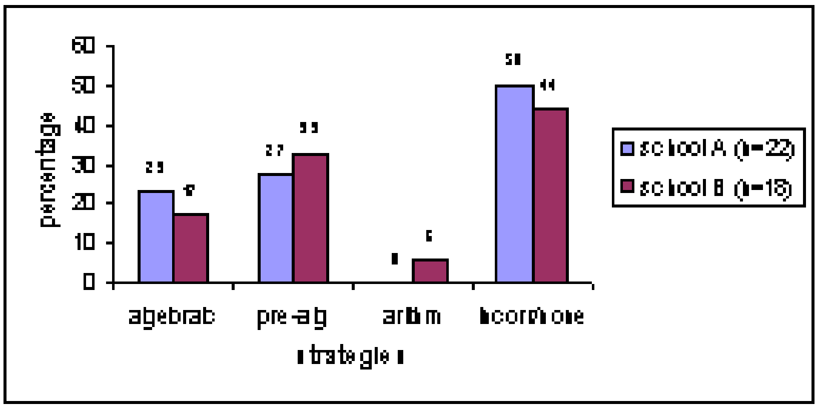


figure 6.24: strategies used for question 2 of *Trading Stamps*

When it comes to symbolizing, we observe some differences between the two schools. The students at school A showed more of their work (arithmetical and pre-algebraic in equal proportions) than school B, where 50% of the students did not mathematize the problem at all. On the other hand we observe three applications of an arrow diagram only at school B, while both teachers used this representation in a similar way: as a convenient but not compulsory notation. Before we move on to the conjecture on the relationship between reasoning and symbolizing, we discuss three examples of notations to illustrate why we have specified them as pre-algebraic and how they can make a difference.

pre-algebraic notations

The first student (figure 6.25) uses rhetoric notations and arrow diagrams to explain her thinking, where she numbers the stamps from 1 to 4 for convenience (lighthouse is 1, people is 2, sailor is 3, horse is 4). It is not so clear how the arrow diagram helps her, but the rhetoric notes in the allocated draft area don't suffice. In the middle we see part of the reasoning process, followed by two attempts at the bottom. Translation: 'suppose you have 2 samples of st(amp) 1, then you have 8 of st. 2, then you have 16 of st. 4, then you still can’t trade fairly’; the numbers 12 and 24 indicate she also considered 3 samples of stamp 1. In fact, this may have induced her to draw the arrow diagram at the top of the scroll. It is an efficient visualization of the trade exchanges and it signifies the changing character of the quantities. In the unit *Barter Trade* the students encountered arrow diagrams where the trade quantities were variables, i.e. ‘number of so-and-so’. Even though this student has left out the general terms and substituted the numerical values immediately, her work demonstrates how the arrow diagram can take on an algebraic meaning.

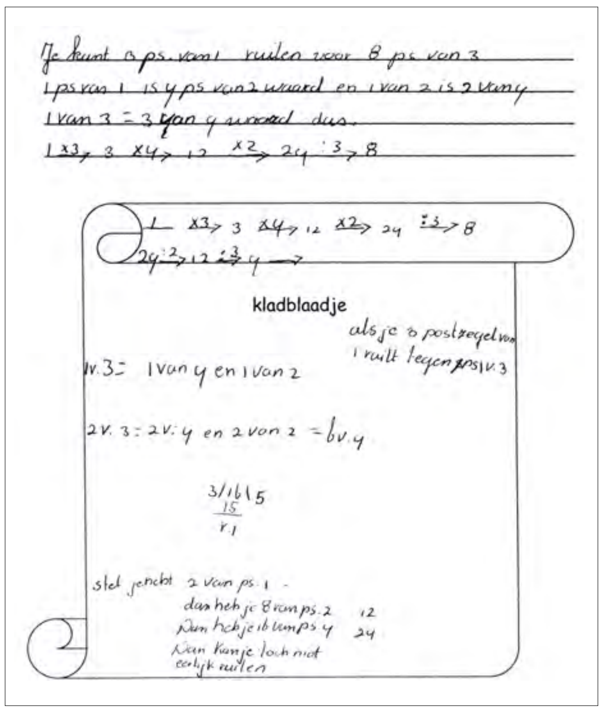


figure 6.25: *Trading Stamps*: use of an arrow diagram

Figure 6.26 shows an example of partial reasoning using syncopated notation. The expressions involve letters and one mathematical symbol, the equality-sign. In spite of this rather arithmetical application of symbols it has been characterized as syncopated, because the student made a special effort to schematize the expressions – numbers and symbols in line – and be consistent about the abbreviations. Listing the expressions neatly one below the other helps to structure one’s reasoning. The solution process consists of two parts: symbolizing and reasoning with the first two trade expressions (first three lines), and assigning a value to the stamps (four lines at the bottom). Since multiples of the numbers 10 and 3.75 are not so easy to equate at first glance, the student decides to include a third type of stamp in the trade (2 × 3,75 + 2,5 = 10; in the Netherlands a comma is used instead of the decimal point).

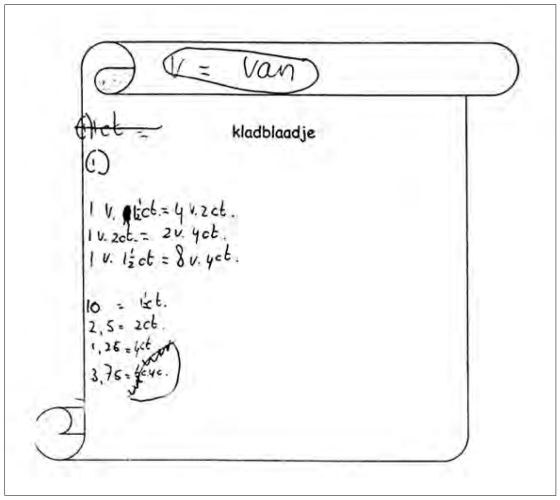


figure 6.26: *Trading Stamps*: use of syncopated notation

In figure 6.27 we see an iconic representation of the trade terms with signs of syncopation. The stamps have been replaced by sober icons, and the expressions are compressed to a more mathematical format involving the multiplication symbol. From the answer 'no, because you get something with a half' we can deduce that this student has reasoned correctly but does not know how to adjust the numbers of stamps to get equal values.

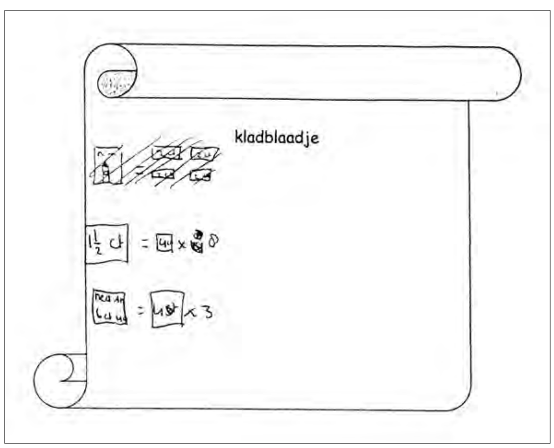


figure 6.27: *Trading Stamps*: use of iconic expressions

support for the conjecture

The results of the task *Trading Stamps* support (in various degrees) the conjecture *reasoning versus symbolizing*. The mental act of reasoning about quantities appears to be easier for students than symbolizing it, where advanced mathematizing is an effective but not prerequisite tool. Our conclusions are based only on the results of question 2 because most of the answers to question 1 have no written notes of the solution process. Perhaps this lack of communication is the most convincing evidence how difficult it is for students to explain their reasoning on paper!

skipping pre-algebra

Student answers to question 2 of *Trading Stamps* show that it is possible for learners to *skip the pre-algebraic phase* in their development of algebraic thinking. Six students succeeded to apply correct algebraic reasoning in the test task without using any supportive symbolizing. An analysis of their work on three similar tasks in the unit *Barter Trade* shows that they had not worked with pre-algebraic strategies such as partial reasoning or trial-and-adjustment. Three of the six students did use pre-algebraic notations in the unit tasks, but these did not lead them to a correct solution. Consequently we can say that their success in the test is not a result of prior competence at informal, pre-algebraic strategies or notations. In other words, they have skipped the phase of pre-algebra.

arithmetical notations

The research data make a stronger case supporting the conjecture's part on the prevalence of *arithmetical notations*. It must be said though, that we have considered only the hampered progression from an *incorrect* strategy to a (*pre-*)algebraic level of understanding, because in our categorization the arithmetical level of reasoning plays only a minor role. Nineteen students used an incorrect strategy. Only three of these students succeeded at symbolizing the problem; five students used calculations and eleven had no draft work at all. In other words, nearly 85% of the students who struggled with the (pre-)algebraic reasoning was not able to clarify the task for themselves through symbolizing. Additional data from classroom work on three similar problems show that students who failed to solve even one substitution problem intelligibly indeed did not exceed the arithmetical level of notations.

pre-algebraic aid

The third and last part of the conjecture *reasoning versus symbolizing,* dealing with *pre-algebra as a tool,* is also justified for this task. The effectiveness of symbolizing activities can be deduced from their functionality in pre-algebraic and arithmetical strategies. Students at this intermediate level of understanding interpret and investigate the problem using symbols, arrow diagrams or iconic representations, more so than the users of algebraic and incorrect strategies. Twelve students used a pre-algebraic strategy, meaning partial success with substituting the trade terms; nine of them mathematized the problem symbolically and appears to have profited from it (see figure 6.25, for example). Group work and teacher aid make it difficult to determine the value of symbolizing for the individual in a classroom situation.

Flowers **and** ***Cabinets*** **test 1 secondary school**

We have chosen two test tasks typical for the mathematical content of 'equations in a context':

> Sacha wants to make two bouquets using roses and phloxes. The florist replies: "Uhm ... 10 rozes and 5 phloxes for f15,75, and 5 roses and 10 phloxes for 14,25; that will be 30 guilders altogether please".
> What is the price of one rose? And one phlox? Show your calculations.

and

> John's office will be fitted with cabinets along two walls. One wall has a total length of 4.80 meters, the other one is 2.80 meters long. John can choose two colors: light and dark wood. A light cabinet is longer than a dark one. John has done some measuring: 'If I put 4 dark and 4 light cabinets against the long wall, and 3 light ones and 1 dark one against the other wall, it will fit exactly'.
> How wide is a light cabinet? And a dark cabinet?

(The *Cabinet* task also shows a floor-plan of the office, measurements included).

purpose and expectations

Both tasks are intended to provide sound data to evaluate the learning effect of the equation solving activities in the unit *Fancy Fair.* If the teacher has used it in the lessons, students will know the term 'equation', but it is not mentioned anywhere in the unit. Instead, equations are called 'combinations' of 'unknown numbers'. So when we speak of 'systems of equations' we mean any representation of two combinations of unknowns; it is only for convenience that the formal term is used here.

strategies

Flowers is a step-in problem. Since the students are familiar with embedded equations in a money context, it should be quite easily recognized by most students.

level	code	alg-ar	strategy	description
1	E	alg1	elimination	calulation with coefficients
2	C	alg2	combinations	comparing equations and constructing new equations
3	P	pre1	pattern	recognizing pattern of exchange
4	RT	pre2	reason and trial	more than trial-and-error: includes reasoning
5	TE	ar	trial-and-error	numerical attempts
6	AO		answer only	strategy not clear, answer correct
7	INC-R		incorrect reasoning	system of equations correct, but incorrect method
8	INC-ER		incorrect equations + reasoning	incorrect or no equations, no understanding
9	N		none	

table 6.4: classification of strategies for solving (embedded) systems of equations

The double combination of numbers (5 and 10) enables students to solve the task at different levels: by elimination, combinations or pattern continuation (see the classification and descriptions in table 6.4). The *Cabinets* task is a little more difficult: the information in the problem is more hidden and the context is changed from money to measurement. Moreover, the equations do not show a pattern, which means we can really test for algebraic competence: translation of the problem situation and algebraic reasoning to solve it.

task results

In both classes the final results are above expectations. In general the high achievers used algebraic strategies more often than not, and certainly more than the low achievers. The *Flowers* task in particular has been very well done: only one student in each class was not able to solve the problem. There is a remarkable difference in strategy use, though, between the schools. In school E algebraic strategies (mostly 'elimination') just outnumber the pre-algebraic method 'pattern continuation', but at school D the problem was solved by no other strategy than continuing the pattern! Comparing the strategies used for the *Cabinets* task, we see a very different distribution (see figure 6.28).

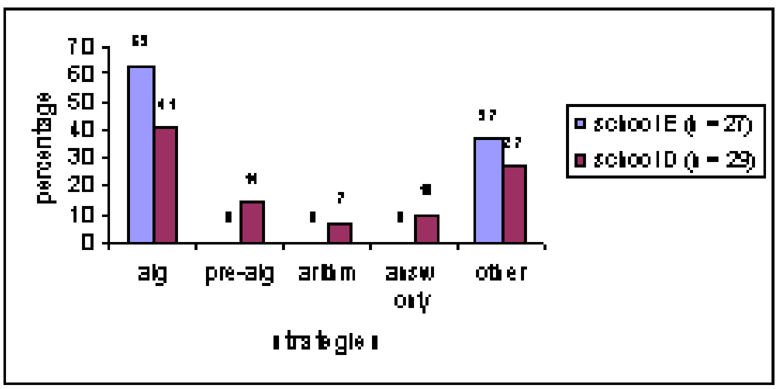


figure 6.28: percentages strategy use for the *Cabinets* task

For school E we see a large percentage of algebraic solutions at one end and incorrect methods at the other end, with a big gap in between. This would suggest that pre-algebraic strategies are not accessible for students who are not competent with either the elimination or combination strategy. However, at school D some students did succeed using a method of reasoning and trying ('reason and trial') or even trial-and-error. In fact we see a wide spread of strategy use, including a strong algebraic component. Comparing notations (see table 6.5 for a classification and description) for the *Cabinets* task we observe that students at school E represent the system of equations mostly symbolically (over 70%), whereas most students at school D (47%) continue to write a combination of abbreviations, symbols and words (i.e. syncopated notation). All in all the results indicate a more formal level of equation solving at school E. It must be said, however, that the students at school E spent one lesson longer to practice solving equations, which perhaps just made the difference:

level	code	alg-ar	notation	description
1	E	alg1	equations	correct equation (math. symbols and letters)
2	SYN	pre1	syncopated	shortened notations (abbreviantions and/or math. symbols)
3	IC	pre2	iconic	drawn symbols (realistic/mathem)
4	T	pre2	table/schematic	listing of coefficients (no unknowns)
5	C	ar	calculations	calculations prevalent
6	D	ar	descriptive	with words, no effort to shorten
7	A		answer only	
8	N		none	

table 6.5: students' notations for (embedded) systems of equations

support for the conjecture

Test results and classroom work on solving equations support two of the three statements of the conjecture *reasoning versus symbolizing.* The results on both test tasks indicate that algebraic equation solving need not necessarily develop synchronously with algebraic symbolization.

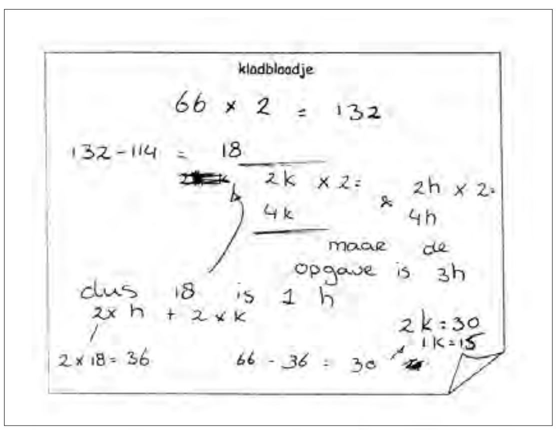


figure 6.29: level of reasoning versus level of symbolization

For instance, figure 6.29 illustrates how the level of reasoning can be higher than the level of symbolizing. This student solves the system of equations

$$2 \times h + 2 \times k \; = \; 66$$
$$3 \times h + 4 \times k \; = \; 114$$

by doubling the first equation and then subtracting the second from it. First he deals with the right hand sides of the equations (66×2 and $132 - 114$). In between the two horizontal lines we observe how he multiplies the terms with the unknowns. Then he writes 'but the task says $3h$ so 18 is 1 h'. Finally he substitutes the value 18 to solve for k. Here we see a remarkable contrast between the levels of reasoning and symbolizing. This student successfully applies a formal algebraic strategy of eliminating one unknown by operating on the equations, while his symbolizing is still at a very informal level. The unknown is only partially included in the solution process; it appears only where necessary. There is a parallel here with the historical development of symbolizing the solution. In the rhetoric and syncopated stages of algebra the unknown was mentioned only at the start and at the end of the problem; the calculations were done using only the coefficients.

The component *arithmetical notations* can only be justified partially because the data from both schools do not agree. Contrary to expectations, learners at school D who continued to use an informal strategy – not formalizing their activities to an algebraic level – did develop meaningful symbolic language. The third part of the conjecture, on *pre-algebra as a tool,* holds true also in the case of solving systems of equations. Even though algebraic symbolizing is not prerequisite to algebraic thinking, it is quite certain that students from both schools have profited from representing the problem systematically prior to solving it. Vice versa we have not found an indication that students who schematize their work are more likely to formalize their mathematics than the others. Students sometimes use symbolic equations without having a clue how to solve them, which goes to say that one must not be fooled by appearance.

skipping pre-algebra

At school E we have found evidence that students can proceed from an arithmetical to an algebraic method without acquiring the intermediate informal level of pre-algebra *(skipping pre-algebra*). Fourteen students achieved a definite algebraic competence (two consecutive applications of a certain strategy) of whom only two used a pre-algebraic strategy in the test. From this group, nine students who solved both types of tasks (story problems and context-free problems) in the test algebraically, set aside the informal strategies very rapidly during the lessons. Some children were unable to use the strategy 'continuing the pattern' even once in the classroom, which means they skipped the informal level altogether. In summary, for a notable number of students at school E the pre-algebraic phase was not prerequisite for their progression. At school D, however, we see only two cases of certain competence at elimination, and twelve instances where algebra is preceded by a solid base of pre-algebra. Especially the strategy of continuing the pattern caught on well.

It is not possible to conclude whether algebraic equation solving is accessible to students with no pre-algebraic representations at all, because syncopated notations and iconic expressions form an integrated part of the program. (For reasons explained in chapter 5 the symbolizing stream of the program is not entirely open but partly structured.) However, student work seems to indicate that informal symbolizing plays an important role. At school D, in particular, syncopated and iconic notations have been used quite intensively and in a meaningful way (see figure 6.30). Only four students show a probable preference for literal symbols, and these were neither context-free nor syntactically correct. From lesson observations we know that the teacher at school E reacted very positively to symbolic notations, which means we must consider that students made a socially desirable choice.

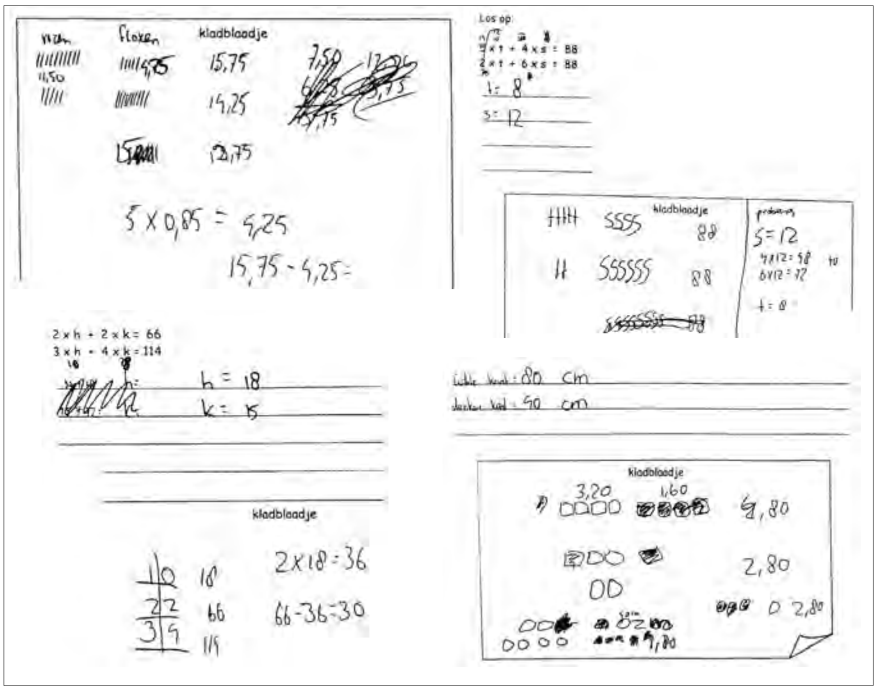


figure 6.30: development of pre-algebraic notations

arithmetical notations

For the second part of the conjecture we must divide the learning trajectory into two stages: from arithmetic to pre-algebra and from pre-algebra to algebra. Students who remain at an *arithmetical* level of thinking and fail to understand the informal strategies referred to as pre-algebraic appear to have an artificial understanding of (pre-)algebraic notations. For example, they might set up a correct system of equations and then have no clue what to do next. Hence they are really at an arithmetical level of notation; the conjecture holds true for this part of the transition. On the other hand there are students who cope very well at the *pre-algebraic* level, and who use syncopated notations in a way that is meaningful to them (i.e., that the notations are an integrated part of the solution process). For instance, six students were able to solve the *Cabinets* problem without algebra, mathematizing the problem with icons or symbols and manipulating these expressions to solve the problem, as shown in the last sample of figure 6.30. None of these students resorted to an arithmetical level of notations (rhetoric or calculations). In conclusion, the second part of the conjecture is true only for the passage from an arithmetical to a pre-algebraic mode of thinking.

(pre-) algebraic aid

Syncopated and symbolic notations appear to have a positive effect on the development of (pre-)algebraic strategies. At school E we see that the strategy of elimination appears mostly in combination with symbolic notations, where the unknowns are involved in the solution process until the last stage (see also section 6.7.7, where we discuss the role of the unknown). At school D we see much more informal strategies and notations while the success rates are almost the same. Some students even use a variety of representations, as shown in figure 6.30. We can say that the better students at school D have learned how to apply representations as a tool for mathematical reasoning. Looking at the symbolizing process globally, we see that notations tend to be streamlined over time – as in figure 6.30 (from top left to top right to bottom left to botom right) where only the iconic representation at the end is a small step backwards – and also as students formalize their mathematical activities. For instance, algebraic solutions to the *Cabinets* are more frequently accompanied by symbolic and syncopated notations than schematic and iconic ones.
However, this does not mean that advanced notation implies higher order thinking. Symbolic notations often but not always imply the problem is also solved algebraically. First, we have observed student work where a correct symbolic system of equations was followed by an incorrect or lower order strategy, or where the student proceeded with the solution process rhetorically, as in figure 6.31.

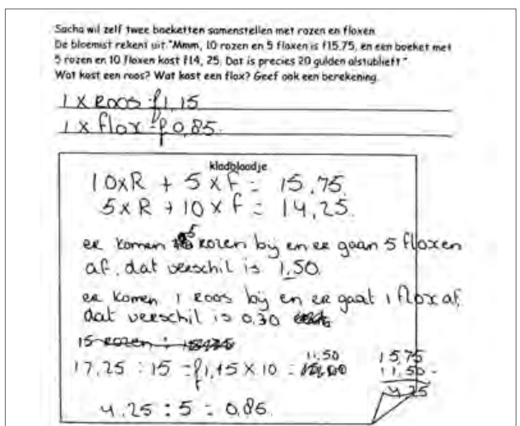


figure 6.31: symbolic representation but rhetoric solution

This student mathematizes the problem by constructing a system of equations, and then applies an informal, pre-algebraic exchange strategy which is developed in the unit. Below the equations she writes: 'We get 5 roses more and 5 phloxes less, the

difference is 1.50. We get 1 rose more and 1 phlox less, the difference is 0.30.' The calculations show that she continues the pattern to get 15 roses for the price of 17.25 guilders, and then she determines the price of 1 rose and 1 phlox. The level of symbolizing may appear to be high at first due to the presence of symbolic equations, but the student does not operate on the equations. The equations may have helped her structurize the problem but they are not a part of the solution process. And even though the unknown numbers of flowers are an integral part of her reasoning, the letters representing them are not needed in the calculations.
Second, writing the problem down schematically does not guarantee success. We have found that students who struggle with the strategy 'making new combinations' use systematic notations equally frequently as not. Perhaps a schematic representation helps students to recognize a pattern but not sufficiently to focus their attention on the coefficients (numbers of unknowns), giving the combinations a clear target. In other words, although advanced notations seem to support the development of algebraic thinking, there is no reason to believe that they can bring insight, especially when students encounter such notations prematurely (without meaning).

Human Body test 2 secondary school

The task on proportions of the human body consists of a description and two questions:

> The proportions of your body change as you grow older. A baby has a relatively large head. The head of a baby just born fits 3 times into the rest of the body. A grown-up, on the contrary, is about 8 times as long as his or her head. The legs of an adult are relatively long. The lower part of the adult leg (from the ground to the knee) is about one fourth of the total body length.
> 1. A baby measures 48 cm at birth. How long is its head?
> 2. Your math teacher is in a male fashion shop to buy a long raincoat. In order to determine the right size, the salesman takes his measurements: 112.5 cm from his shoulder to his knee. How tall is your teacher?

We can distinguish the following strategies:

algebraic strategies

algebraic 1: 'equation solving'
The student constructs and solves an equation in one unknown, conceiving and manipulating it as an abstract entity free of any contextual meaning. Theoretically this strategy may be accessible to students as a formalization of earlier symbolic manipulation in the unit *Fancy Fair.* It is most likely that the student will define the unknown as the total quantity (length) as the unknown rather than one of the parts, because all the different fractions are then easily determined; the latter is perhaps more suitable when the problem involves only one fraction (as in question 1 and in paragraph 6 problem 4 of the unit *Time Travelers*). However, it must be said that solving an equation is not the most suitable strategy for question 2 of this task because the

fractions make it a complex expression. Still, it can be effective as a way of structuring the problem. Since we have not seen this strategy being used at all, it has not been included in the classification of strategies used to discuss the results.

algebraic 2: 'reasoning with the whole' (figure 6.32)
The student reasons with the fractional parts in terms of one unknown: the total value (or the part). In other words, the head is referred to as '$\frac{1}{8}$ of the body' and 'the lower leg' is called '$\frac{1}{4}$ of the body'. Or, from the opposite perspective, this strategy also includes conceptions like 'body = 4 × head' where the word 'head' has the role of unknown. From this more general formulation it is only a small step to the next level strategy (constructing an equation in one unknown).

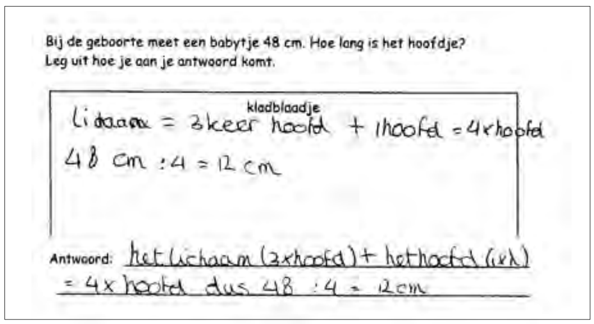


figure 6.32: reasoning with the whole using syncopated notations

pre-algebraic level

pre-algebraic 1: 'false position' (figure 6.33)
This strategy is the simple version of the ancient Rule of False Position when algebra was essentially advanced arithmetic. It portrays various algebraic characteristics while at the same time the calculations involve only known numbers.

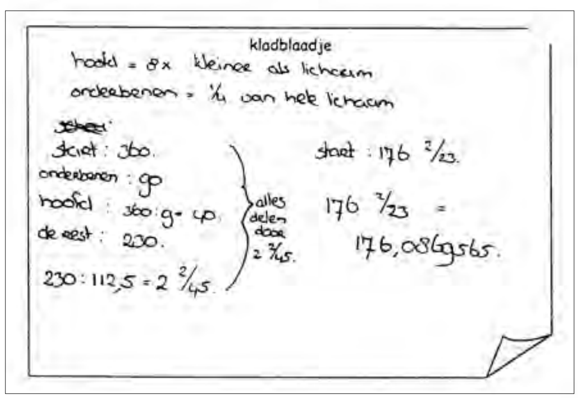


figure 6.33: strategy 'false position'

The student chooses an initial value for the unknown length (360 in the example), treating it as if it were a number; this is an algebraic conception of unknown quantities. She makes the error of dividing 360 by 9 instead of 8, which makes the correction factor (230: 112.5, which gives $2\frac{2}{45}$) rather nasty, but she seems not to be deterred. We consider this strategy to be more advanced than the trial-and-adjustment approach shown in figure 6.34 since it is applicable to any kind of number (even irrational numbers) and it can be generalized into a formula, whereas the other cannot. Besides, amongst historians of mathematics the strategy is qualified as early algebra.

pre-algebraic 2: '*correction with proportions'* (figure 6.34)
The strategy meant here is a variant of the method 'false position'. Instead of correcting the presumed value (1.80 m in this case) by multiplying with a certain factor, the initial attempt is adjusted by adding or subtracting using proportions. This particular student – making the same error as the student we discussed previously – finds as the first attempt 1.20 m for the body from the knees to the shoulders (1.80 m – 60 cm in the lower right hand corner of the draft area). She divides 1.20 m into smaller parts – dividing 120 cm first by 12 and the answer then by 4 to get 2.5 cm – and then she calculates 120 : 10 × 11 + 2.5 to get the 112.5 cm. The other body parts are adjusted with the same procedure to find the answer. This strategy always involves an adjustment by determining the *difference* with the actual length, instead of the multiplicative relation of 'false position'. When a student tries various attempts by studying the error qualitatively (too high/low) we also qualify the method as correction with proportions.

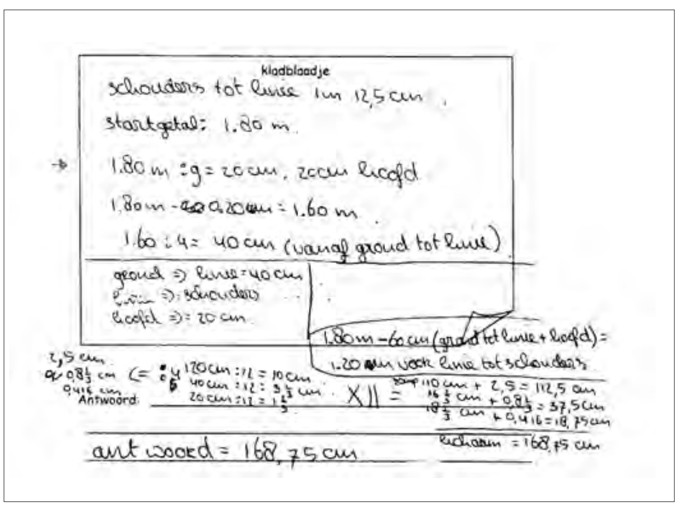


figure 6.34: strategy 'correction with proportions'

arithmetical: reasoning with parts

The arithmetical version of reasoning with the given fractions differs from the algebraic counterpart in terms of the conception of the unknown. We suspect that a student who continues to see the different portions of the total quantity as separate parts (as shown in figure 6.35) does not realize that the problem deals with only one unknown: the total length. Note how the student gives a rhetorical explanation of the procedure, while the problem is solved using calculations (including a lot of incorrect 'stringing' of intermediate answers and the equal-sign).

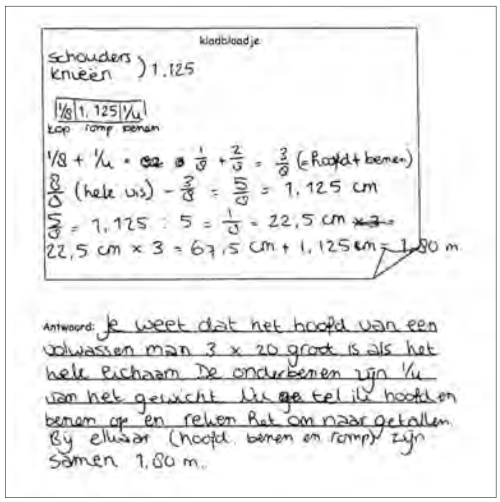


figure 6.35: reasoning with body parts

from arithmetic to algebra

In order to proceed to the algebraic conception, the student needs to conceive each portions as part of the whole, which makes the parts comparable and operable (addition will lead to the total). Eventually this will result in the construction of linear expressions in one unknown. In other words, in passing from an arithmetical to an algebraic perspective the attention has to switch from the parts to the whole, and in this respect the strategy Rule of False Position can be helpful as an intermediate phase.

task results

First question. This orientation question can be easily solved arithmetically using only the given quantities. Even so three students used the algebraic approach 'reasoning with the whole', as shown in figure 6.32. They expressed the body length as '4 times the length of the head' in a word formula and/or equation, where the word 'head' behaves as a word variable. The strategies 'false position' and 'correction with proportions' did not occur at all. Half the class used the correct ratio (strategy named 'reasoning') while the other half made the error of dividing 48 by 3. This error is discussed in more detail with respect to the corresponding unit tasks. We must say that these results are far below expectations. With regard to notations we observe that the better students used mostly descriptions while the weaker students primarily used calculations.

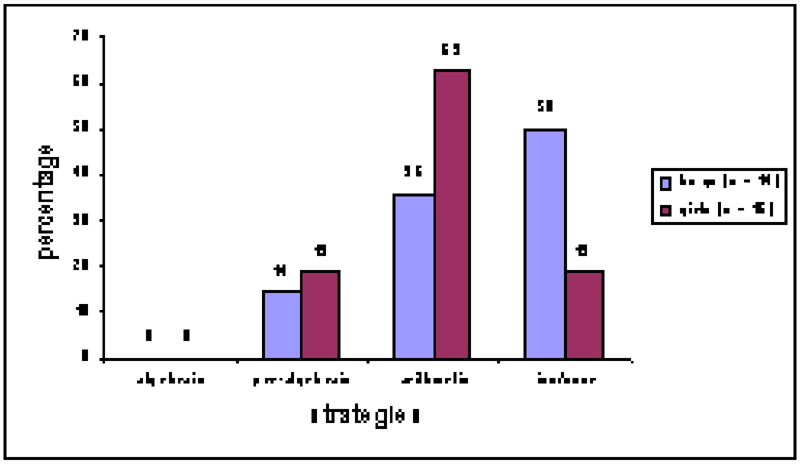


figure 6.36: strategy use for the second question of the task *Human Body*

Second question. Figure 6.36 shows an overview of how the different strategy categories are distributed, differentiating between boys and girls. Quite remarkably the girls scored much better on this problem than the boys (50% versus 15% correct respectively). None of the students applied the algebraic strategy ‘reasoning with the whole’ convincingly. Three students described separate parts of the body in terms of the whole body, but not for adding all the parts together. Five of the better students chose a fictional first value for the unknown quantity (pre-algebraic level), of whom three applied the strategy ‘false position’ such as in figure 6.33. The other two students used a similar strategy involving ratio which we have named ‘correction with proportions’ (see figure 6.34).

The majority of students solved the task successfully at an arithmetical level, but even more remarkable is the fact that one third of the class (even 50% of the boys) failed to solve it. Reasoning with parts of the body, illustrated in figure 6.35, is the most frequently used strategy (37%) of the class, while 4 students (13%) used partially correct reasoning. By ‘partially correct’ we mean a solution which is based on incorrect proportions caused by not including the head as part of the body. A strategy is classified as ‘incorrect’ if the method of reasoning is wrong, for instance if the head is not included at all or if 112.5 is divided by the ratio’s denominator instead of the numerator (8 for the head, 4 for the lower part of the leg).

notations

A notable majority of students (70%) used descriptions in their solutions, often combined with calculations. Five students – of whom four low achievers – used only calculations and one boy used syncopated notations. Finally, three girls drew a bar and seven students (at all levels of competence) drew a human body to organize the problem.

support for the conjecture

If we include isomorphic problems in the unit *Time Travelers*, we find that the conjecture that algebraic *reasoning and symbolizing* can develop independently is supported also by proportion tasks like *Human Body*. Many students were able to reason algebraically about the unknown quantity without needing to symbolize it. It also seems that students who have not attained such understanding also lack the ability to structurize or visualize the problem. Such mathematizing activities have shown to be effective tools for mathematical reasoning for some students.

skipping pre-algebra

In the instructional unit *Time Travelers* we have found support for *skipping the pre-algebraic phase,* but not amongst the test task *Human Body*. To be honest, this task does not provoke students to use the algebraic strategy 'reasoning with the whole'. The first question can be easily solved mentally with arithmetical means, so students need not mathematize the problem first (drawing a bar or representing the problem symbolically). Without this act of mathematizing the algebraic strategy is difficult to identify. For the second question the strategy of reasoning with parts of the body is the most logical and practical. Indeed, the expression 'five eighths of the total body length' is more than just 'five eighths' or 'the rest'. In the unit *Time Travellers* we see that the number riddles lack context terminology like 'head' or 'lower leg' which makes it more worthwhile to express the quantities in terms of the whole. Even so there are five students who demonstrate a nearly algebraic way of thinking. In order to determine whether these students have *skipped* the phase of *pre-algebra* (either strategy or notations), we have studied their work on corresponding tasks in section 5 of the unit *Time Travelers*. Two boys indeed made a direct jump to the (partially) algebraic way of thinking, two girls used pre-algebraic representations and one student has been categorized as inconclusive.
A second source of data for this part of the conjecture is section 6 of *Time Travelers*, where we see the progression from arithmetic to algebra problems: reasoning in terms of the unknown as well as symbolizing problem in an equation. For instance, problem 6.4 – also found in *Rhind Papyrus* – is a linear problem in one unknown:

> 'A quantity whose seventh part is added to it becomes 19.'
> Which equation can you construct for this problem? Solve the equation, too.

For the part of the conjecture on *skipping pre-algebra* the question is: which students use the algebraic strategy 'reasoning with the whole' without using a pre-algebraic approach or notation in the process? We found that six students showed a consistency of algebraic *reasoning* for two number riddles like this without using a pre-algebraic strategy or notation.

conclusion

The proposition that students can develop an algebraic way of thinking about unknowns without passing through a phase of pre-algebraic thinking or symbolizing has been strengthened.

arithmetical notations

We have found that results on the *Human Body* task underscores the part of the conjecture on *arithmetical notations* The difficult transition from an arithmetical to an algebraic mode of thinking indeed appears to be linked to the level of notations that the learner uses. Classroom achievements on task 6.4 in the instructional unit show that eight out of ten students who were unable to reason with the unknown, did previously understand the arithmetical reasoning strategy with parts. So eight students failed to proceed from the arithmetical to the algebraic level of thinking. Checking these students' use of notations in the last two unit sections, we found that only two of them frequently used a visual representation (the bar), the others did just once or not at all. It is therefore not unreasonable to conclude that students who have trouble proceeding to the algebraic level of reasoning tend to use only arithmetical symbolism.

(pre-) algebraic aid

As far as the *supportive role of (pre-)algebraic symbolizing* is concerned, we can say a few things about the functionality of the bar and the symbolic (or syncopated) equation as a problem solving tool. In classroom work on the unit *Time Travelers* some students drew pictures of fish portions just before the bar was introduced, which means it may have emerged naturally as a model in due time. In section 6.7.3 we illustrate how the bar might develop from a drawing of a fish to an abstract mathematical representation. Visual support soon became redundant since most students were able to solve later fish problems without drawing a bar. As a model for reasoning the bar was only partly successful in new situations, because although it was used fourteen times altogether for solving the number riddles in the last unit section by nine girls, in the test task the bar appeared just three times.

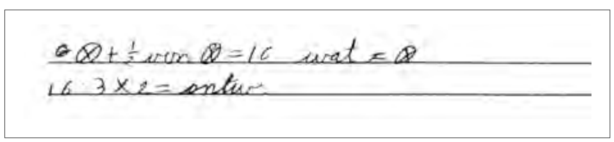


figure 6.37: syncopated equation

The beneficial effect of the (semi-)symbolic equation is more difficult to determine because there was no opportunity during the lessons to ask students about their choice and opinion on this matter. We can only judge their ability on the basis of their written work. Theoretically an equation can be helpful by organizing the information in the problem in a compact and structured way, but only for someone who understands it. From the classroom observations we know that only four students were able to construct a symbolic equation on their own and two students produced a syncopated expression such as in figure 6.37, which increased to half the class in the second task (following a classroom discussion). In many of these equations the unknown is left out of the broken term (see also figure 6.38), a phenomenon to which we return in section 6.7.7. In spite of this extra stimulant, the same number of students solved the task with curtailed notations as without, which means the data are inconclusive. All in all we can say that the bar appears to have been effective for some students during the classroom activities on linear problems in one unknown.

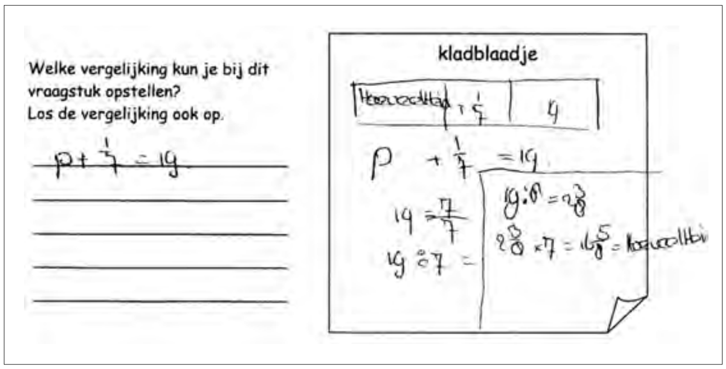


figure 6.38: equation missing the unknown

6.7.3 Symbolizing

In the final experimental version of the learning strand, we have aimed at integrating the reasoning and symbolizing streams of the program. Earlier classroom experiments (see chapter 5) showed that mental reasoning comes more easily to students than symbolizing, and students are not as susceptible to tabular representations as expected. Hence the program was revised to make schematizing and symbolizing a part of the problem solving process, to facilitate a more natural and meaningful development of algebraic notations.

different types of symbolizing

We opted for the integral use of tables, arrow diagrams, the empty numberline, symbolic expressions and the rectangular bar, each of which can function on an arithmetical and a (pre-)algebraic level. At the arithmetical level, a schematic or symbolic representation helps students organize the problem situation and/or calculate numerical values. The table is an efficient way of listing numbers or calculating proportions, the arrow diagram simplifies the inversion of operations, the number line visualizes symmetrical calculations, and the rectangular bar and symbolic expressions structurize and organize relevant information. In other words, these notations show activities of *horizontal mathematization*. At the (pre-)algebraic level, the representation becomes a tool for problem solving: recognizing or continuing a pattern in a table or a system of equations, generalizing an arrow diagram (numerical entries replaced by variables), formalizing a series of actions on the empty numberline, or visualizing a linear relation in one unknown by a rectangular bar. In these situations we recognize *vertical mathematization,* namely formalization of mathematical activity.

In the previous section we have already discussed symbolizing and schematizing in relation to the level of reasoning. In this section we focus more specifically on how students have succeeded at mathematizing – either horizontally or vertically – various problem situations using the representations mentioned before.

letters and symbols

primary school level

Especially at primary school level we find that symbolizing is not something students do readily, and when they do symbolize they often choose unconventional notations. A large number of students display a clear preference for calculations or rhetorical notations. In the second primary school test the task *Trading Stamps* challenges students to organize the problem using symbolism. Earlier in this chapter we displayed two samples of student work where students appear to profit from this symbolism in their reasoning process (figures 6.20 and 6.21). Nearly half the number of students translated the picture representation of the task into semi-symbolic trade terms involving icons, symbols and abbreviations. The icons and letters all refer to objects: the stamps. We also observed algebraic symbolizing once at primary school level, though. Figure 6.39 shows a visual representation for the unknown in one of the supplementary tasks – a restriction problem with two restrictions and two unknowns – of the unit *Exchange*. The problem says: ‘A full grown dog eats 5 times as much as a puppy. Together they eat 3 kilos of dog biscuits in a week. How much more does a full grown dog eat than a puppy?’ The student has combined both restrictions to obtain a linear problem in one unknown.

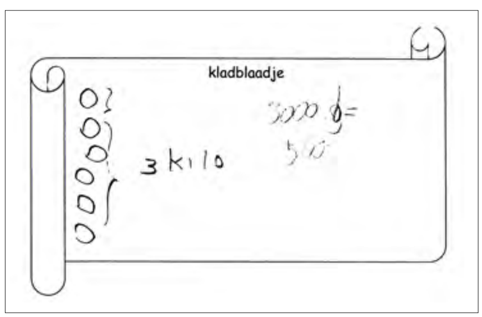


figure 6.39: spontaneous invention of a symbol for the unknown

In figure 6.40 we see a curtailed representation of a solution to the task *Dutch Past*. It should be interpreted as ‘1 apple gives 4, 10 apples and eventually 22 apples’, which is a typical, arithmetical conception of the equal-sign.

We have already mentioned that one of the issues of the study has been to confront students with different roles of letters: as objects, as unknowns and as variables. At primary school level this is done in an informal way, by presenting students with situations where the same letters can have different meanings, depending on the representation that is used (trade term, arrow diagram or (word) formula). Generally speaking this algebraic conception of letters was not successful, but as a matter of exception we want to mention here one example taken from the pilot experiment (section 5.4.2) to point out that a spontaneous change of perspective can take place.

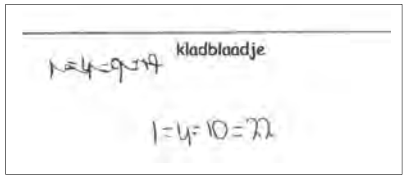


figure 6.40: arithmetical conception the equal-sign

One student used trade terms and formulas simultaneously in his ‘book of law’, for instance 4 *ap* = 1 *vi* (4 apples for 1 fish) followed by the formula *ap* : 4 = *vi* (see figure 6.41). He read aloud the latter expression as ‘the number of apples divided by 4 is the number of fish’, so he perceived the variables as magnitudes. This student shows the capacity to switch flexibly between the dynamic, arithmetical perception of the trade term and the static, algebraic perception of the word formula.

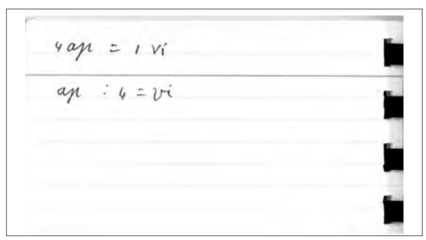


figure 6.41: double role of letters

secondary school level

At secondary school level we see a nearly equal distribution of symbolic and syncopated notations for solving simultaneous equations in two unknowns, and mostly syncopated notations for linear problems in one unknown. The analysis of the test tasks *Flowers* and *Cabinets* shows that at school E 46% of the students used symbolic notations consistently throughout the tasks compared to only 14% at school D; the percentages for syncopated notations are 31% and 26% respectively. The largest group of students at school D (41%) did not have a clear preference, using symbolic, syncopated, iconic and/or rhetorical notations alternatingly. The symbolizations at secondary school level are therefore still of a pre-algebraic nature, but they appear to be more natural to these students than their younger. Of course there are also students who continue to describe their solutions and solution procedures in full sentences, but they are mostly low achievers in this experiment and relatively few in number. We have seen a wide variety of symbols – drawings, icons, letters – for the unknown after instructions with the unit *Fancy Fair*, especially when students were asked to produce their own equations (in the competition). In the unit *Time Travelers* students are asked again to symbolize the unknown; only a few students were able to make a proper algebraic expression (involving an unknown, an equal-sign and operations, also in the correct order). Some of the descriptions and symbols students used to represent the unknown are the word ‘something’, the Egyptian symbol of a book scroll, the question mark, invented symbols and syncopated notations (see also figure 6.37 and figure 6.42 below). All in all few students thought of symbolizing the unknown like they had done in the first instructional unit.

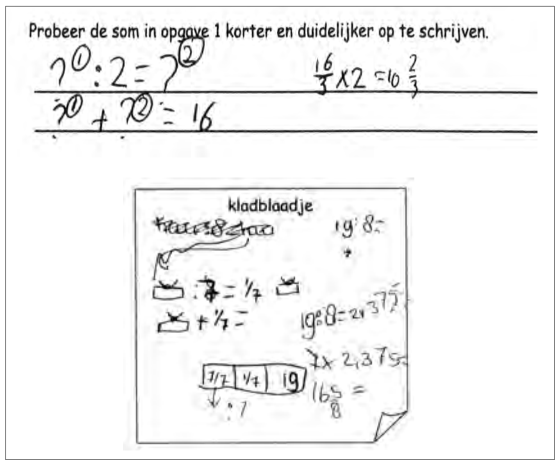


figure 6.42: two different students using symbols for the unknown

roles of letters

However, there are a few examples of symbolizing that we wish to share. The student whose work is shown in figure 6.43 decided to use letters in a new type of task, namely on the Rule of Three, although the symbolism itself is still informal (letters as objects).

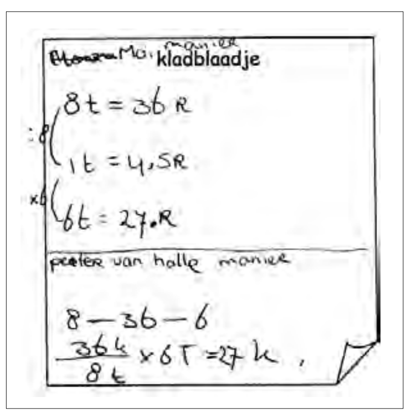


figure 6.43: spontaneous use of letters

At the top we see 'her own' strategy (referred to as 'moi manier'), at the bottom we see the Rule of Three according to the 16th century mathematician Peter van Halle ('peter van halle manier'). We believe the letters support the student because they enable her remember which number is which in her calculations.
A few students invented what we have called 'word variables' to mathematize the first question of the *Human Body* test task, as shown before in figure 6.32 Another student used letters to express intermediate answers in a problem (see figure 6.44) His description of the solution procedure using the letters displays a shift towards generality.

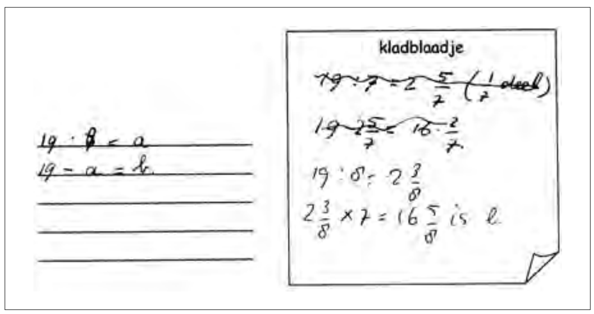


figure 6.44: letters for intermediate answers

Finally, we discuss in a separate section below the role of the unknown in solution procedures.

tabular representations

Few students are inclined to search for effective methods like schematizing to structure their approach, for instance to be certain that all possible combinations of values which add up to a certain amount, have been found. In this study we found that primary school students use tables primarily as a calculational tool (like a ratio table) and not so much for problem solving (recognizing a pattern, generating values to make a prediction, etc.). When students make a table it is usually for generating values when the relationship between them is multiplicative ($y = ax$ if the relation was represented symbolically) rather than additive ($y = x + a$) or totalling a fixed amount ($x + y = a$). This means that especially the solving of restriction problems with two or more restrictions – which rely on a structural approach – continued to be a laborious matter. At secondary school level the table was not relevant as a tool for reasoning for most activities, although the ratio table was frequently used as a tool for calculating. Only in the unit *Fancy Fair* we included an activity with a so-called combination table, in order to visualize patterns of exchange before moving to an iconic and syncopated environment. We cannot conclude anything on the role of a tabular representation on the basis of so little data.

empty numberline

The empty numberline can be used to support reasoning about an unknown quantity, for instance in case of restriction problems with sum and difference, previously also called Diophantine problems (see section 4.5). In the elementary school unit *Exchange* the number line is connected to one of the related strategies; if this strategy is not needed or if it does not catch on, the number line becomes superfluous by consequence. We already discussed in section 4.5 that in just one of the schools the number line was used by a few students in the lessons, but this competence was not transferred to the test problem *Number Cards.* In other words, it may have worked as a model of the Diophantine problems in the classroom, it has not succeeded as a model for slightly different problem situations.

arrow diagram

Looking back on the field test, we must conclude that the arrow diagram has not been effective as a visual representation, particularly at an algebraic level. Many primary school students (like the one whose work is shown in figure 6.45) used an arrow diagram to represent a chain of calculations either in the forward or the backward direction, but not a diagram for each. This means that the diagram's surplus value – associating each operation in the forward chain with its inverse in the backward chain, thus relieving the short term memory – was not recognized, needed or perhaps not appreciated by these students. In some cases the student realized this advantage just in time: figure 6.18 shows how a student switched from a trail-and-error strategy to reasoning with inverses.

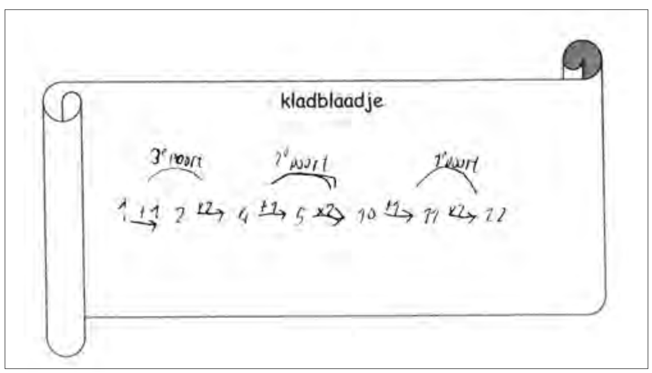


figure 6.45: forward arrow diagram with reverse calculations

The formalization of arrow diagrams at primary school level is anticipated in the instructional unit *Barter*, where the input and the output of the arrows are variables (number of goods) instead of numbers. For example, the trade '5 tomatoes for a melon' which has been symbolized as 1 m = 5 t, can be represented by the arrow diagram $m \xrightarrow{\times 5} t$, which requires a new interpretation of the letters *m* and *t* (from labels to variables, see also the discussion on symbols and letters below). The arrow diagram is a more visual and dynamic counterpart of the word formula *number of tomatoes = number of melons* × 5. Word formulas are introduced in the unit *Barter* to show students a representation which might be more convenient for calculating numbers of goods directly, but most students were confused by it. Apparently the primary students in the field test were not ready to proceed to algebraic conceptions of the trade relationship. Just once we saw a student who chose to use an arrow diagram for calculating numbers of goods in a trade situation (figure 6.25 in the discussion of the test task *Trading Stamps).* She did not write a general arrow diagram (with variables) but it is clear that she has broadened her understanding of the arrow diagram since the unit *Exchange*. At secondary school the arrow diagram could not compete with the ratio table, even though these students are familiar with the very similar 'machine diagrams' (input, operation, output). It seems that the transition from a convenient and familiar representation (barter term, ratio table) to a new representation requires more attention than we anticipated.

rectangular bar

In *Time Travelers* a rectangular bar was introduced as a model for the unknown, in this particular case the length of a body. Students' work shows that this representation was well received and applied at different levels of abstraction.

1 The rectangle stands for the body as a whole, and proportional subdivisions are made (figure 6.46).

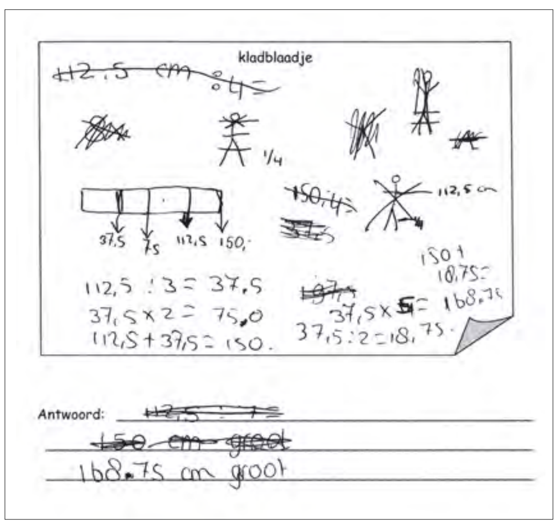


figure 6.46: from body parts to a bar

2 We notice progressive schematization, from a bar completed with ratios and specific lengths (figure 6.47 top), via an empty bar (figure 6.47 middle) to numbers on a line, where the bar has disappeared (figure 6.47 bottom)

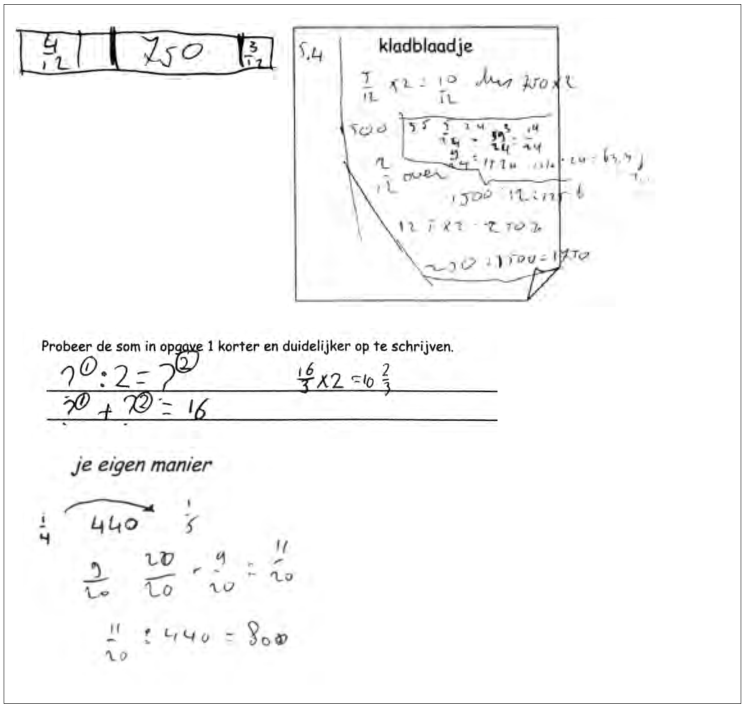


figure 6.47: three developmental stages of the bar

3 Working on a problem from *Rhind Papyrus* a student makes a scale drawing of the bar, and uses this in combination with False Position, to solve the problem.

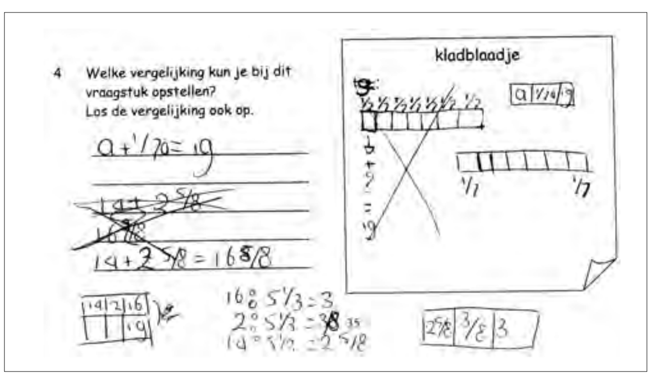


figure 6.48: from scaled to indeterminate length

4 Various students use bars to represent numbers in general, irrespective of their actual value (as in figure 6.49).

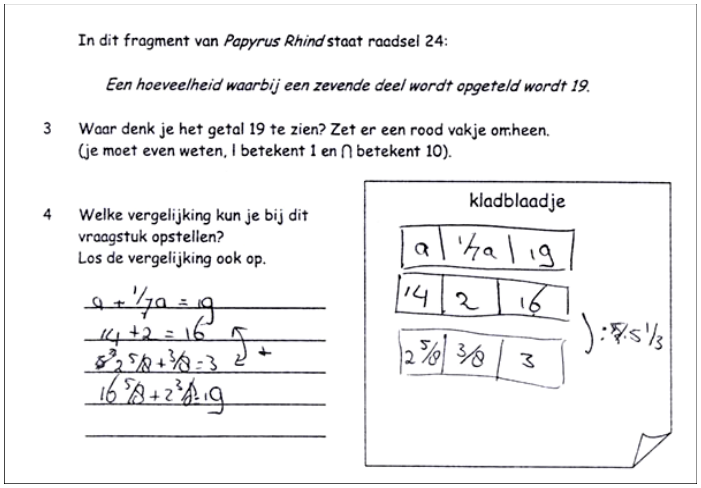


figure 6.49: bar as a tool for mathematical reasoning

6.7.4 Regression of strategy use

In this section we present the research results on artificial formalization of strategies. We will assume that the degree of transfer can be deduced from consistency in student work with respect to initial mathematization and the level of strategy use.

conjecture *regression of strategy use*

The analysis of student solutions of the *Number Cards* task led to the following conjecture:

conjecture

1 *The general applicability of algebraic strategies enlarge the risk of superficial understanding, causing an apparent regression of strategy level in a new problem situation*

test (number - level)	task	part 1	part 2
I primary	Number Cards	+	+
	Dutch Past	+	–
II primary	Trading Stamps	–	–
I secondary	Flowers & Cabinets	+	–
II secondary	Human Body	+	–

table 6.6: support for the conjecture ‘regression of strategy level’
(+ = confirmative, – = not confirmative)l

2 The chance of regression is greater for girls than for boys

Table 6.6 shows that we have found supportive data only for the first part of the conjecture, and even then only partially (for certain strategies).
These results are elaborated on in the following discussion.

Dutch Past **test 1 primary school**

In spite of its arithmetical nature, the task *Dutch Past* is included here because the results point out that a regression of strategy use can occur also for arithmetical strategies. The results on the first problem of *Dutch Past* will be compared with how students performed at the second problem to establish consistency of test level, and we will discuss a reflective question on the task and relevant unit problems to establish transfer of knowledge.

consistency

The second problem in the test task is also situated in an historical context but there are two basic differences with the apple orchard problem. First, students may have been put off by the long original text and the introductory questions right behind it. The large percentages of 'no attempt' indicate that a considerable number of students were too demotivated to continue (see figure 6.50). The other difference is concerned with the chain of operations. Instead of a repeated chain of two alternating operations mentioned once, the second problem involves a series of mostly different operations each mentioned separately. The results in figure 6.50 indicate that student achievement is quite similar from problem 2; it only brings out the differences between the schools more clearly. On an individual level the strategy level is generally consistent too, and therefore we may say that the results for problem 1 are representative for the students' level of understanding for this type of task.

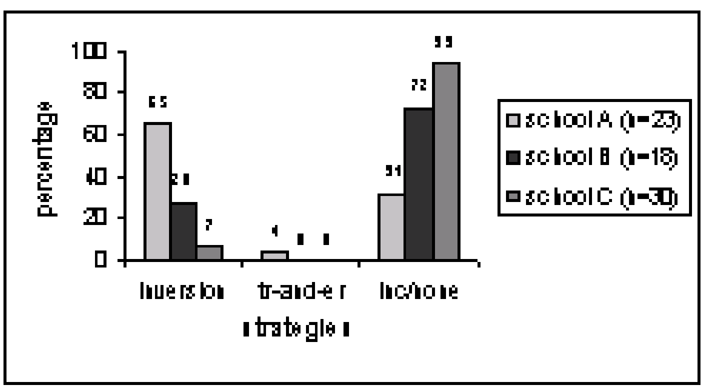


figure 6.50: strategy use for *Dutch Past*, problem 2

transfer

In the unit *Exchange* students encounter inversion problems in a mathematical context ('guess my number') and more elaborately in an historical context. Both activities will be treated as classroom data since we have found it difficult to determine individual competence for two reasons. First, the authentic old texts are a little too complex for students to deal with individually, so the historical problems are studied by the class altogether. Second, the game 'guess my number' was played in pairs but student strategies and notations are hard to distinguish. Sometimes students wrote in each others notebook or they performed most of the calculations mentally. Therefore we will discuss the classroom observations and a selection of student work from each class.

The task *Dutch Past* concludes with a reflective question on problem recognition: 'Does the problem of the servant (i.e the second problem) remind you of another problem you know? Explain your answer'. We expected students to respond a reference to either the historical nature, reverse calculations, the unit *Exchange* or a combination of these. The results are quite remarkable: only twenty-two students answered they recognize the problem, which is barely 30%. Even at school A only just over half the students (seven boys, six girls) responded positively, which means that students interpreted the question very literally. Instead of recognizing general characteristics, some students concentrated on 'servant' and the actual numbers involved. However, the low percentage of recognition in school B (33%) and school C (10%) helps to understand why students perform less successfully than expected. It appears that transfer of knowledge might be a bigger problem than we think.

guess my number

Inverse calculations in a mathematical context appears to have been no problem for most students in all the schools. Students are able to formulate why inverting the chain of operations will always lead to the answer. There are a few differences between the schools, though. Students at school A were clearly very enthousiastic about the activity; they were challenged to produce complex chains and even though most students could solve most problems mentally, they used visual diagrams or calculations to find or check the solution. Moreover, the teacher continued to play the game every once in a while even after the experiment had ended. The students at school C have written very little in their booklet. Although it is difficult to determine their level of understanding and their mathematizing skills, the reflective question produced good answers by several students. Only at school B we observe a few students who appear to have difficulty with inverting the operations, judging by their messy notes. In addition we see mostly very simple chains. In other words, the students at school B achieved a lower level of competence in the classroom activity than the other students.

historical unit problem

The most appropriate unit problem for determining the effect of the historical context and the repetative chain unfortunately gives very little information. Due to unforeseen circumstances the task has not been done at school C at all, which means these students have had only one experience with an historical problem in the unit *Exchange*. This probably explains why they performed so poorly on the task *Dutch Past*. At school A all the student answers are based on numbers different from those stated in the problem. The researcher is no longer in a position to check for what reason the teacher has changed the task. If we were to accept this circumstance, the majority of students appears to have achieved a good understanding of this type of task prior to the test. Judging from the classroom observations and student work, the students at school B were hampered by the old text too much to make a serious attempt at solving the problem. The majority of students use trial-and-error strategies or nothing at all. Amongst the successful solutions we see mostly good examples of mathematization (forward arrow diagrams or syncopated notes).

summary

Summarizing the results, we can say that for school A students' strategy use at the test task are consistent with those seen in classroom activities, but not for school B and school C. Prior to the test the students at school A showed a strong preference for mental inversion, and calculations and arrow digrams were equally effective. They were especially competent in the mathematical context, but probably also in the historical one. Students at school B reached a lower level of understanding in the historical context and it is therefore not surprising that they performed less successfully at both test problems. Nonetheless, their work on 'guess my number' indicate that at least a number of students did not equal their classroom level. We propose two explanations for this regression: the historical context put the students off, or student understanding of reverse problems has been overestimated. The former will of course play a role but we feel it cannot be the only cause for such poor results. And even if students at school C cannot be expected to make the transfer based on just one classroom experience with historical problems, their apparent understanding of reverting operations in general should have led to some successful attempts at least.

partial support for the conjecture

The research data indicate a regression of strategy use for two of the three schools, but with no indication of a significant gender difference. The regression observed here may also be a result of superficial understanding, just like the algebraic strategies for the *Number Cards* task. The correspondence of correct horizontal mathematization with correct solutions in school B support the impression that perhaps only some students achieved true understanding. We suspect that where students were able to rely on their memory in simple 'guess my number' activities in the classroom, appearing to be competent, they were not able to cope with the more complex historical problems. Perhaps some of these students did not invert the operations but performed add-end problems, making trial-and-error calculations at each step in the chain. The strategy of mental inversion gives little insight into a student's true level of competence, and it can easily cause misjudgement on behalf of the teacher. In the same way we expect a number of students at school C to have demonstrated an artificial understanding of inverting operations, although their limited experience with historical situations make it difficult to conclude anything.

Trading stamps test 2 primary school

comparison with classroom activities

Three unit tasks have been selected to compare the test results with levels of strategy and symbolizing during classroom activities. The problem situations are not really similar to the iconic expressions of *Trading Stamps*, and we can use only one of the tasks for the role of arrow diagrams because students were not yet familiar with them. One task is a story of trading soccer cards, where the trade values are presented as word formulas. The second problem is presented as a series of weighing scales in picture form. The comparison task in *Barter* asks students to compare the shopping list for two recipes, with two verbal lists of goods. Each task is based on the principle of expressing compiled quantities in terms of one item. Students worked together and profited from classroom discussions, so we will interpret the data on a classroom level instead of individually.

no support for conjecture

There are no signs of support for the idea that students show a lower strategy level at the test task than what might be expected based on their classroom work. Lesson observations and written student work in the units *Exchange* and *Barter* show that the majority of students struggle with complex tasks on making quantities comparable and substituting expressions. Only about one third of the students demonstrate correct use of the substitution method for two out of three problems, and nearly half of them made an error in their reasoning or calculations. Use of syncopated or schematic notations do not help to avoid these errors. A little more than one third of the students succeeded at solving one problem correctly, and the rest answered incorrectly or with an unclear strategy. In other words, we do not see a higher level of understanding in the classroom activities, even though students were allowed to work together.

Flowers and *Cabinets* test 1 secondary school

Two more test tasks, *Flowers* and *Cabinets*, and classroom work have been included in the analysis to test the validity of this conjecture.

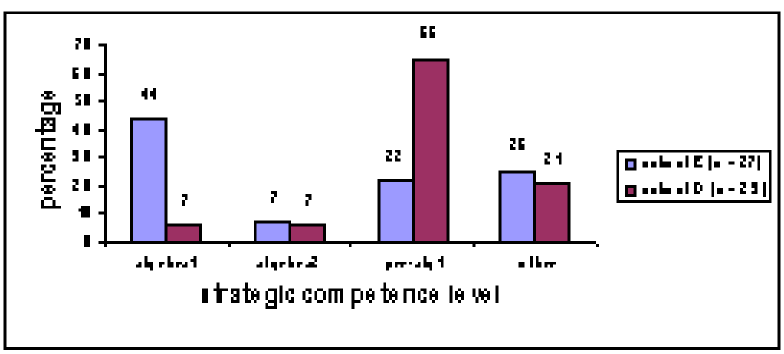


figure 6.51: strategic competence levels for systems of equations

For each student we have drawn up a list of strategy use on the four test tasks to determine the minimal level of understanding ('elimination' = algebra1, 'combinations' = algebra2, etc.). We say that a student is 'algebra2-level' if we observe the algebra2 strategy twice, and no more than one unclear or incorrect method. Students whose strategy use is very inconsistent have been rated 'unclear'. For example, if a student used the pre-algebra1 strategy twice, algebra2 once and 'answer only' once, the scoring is 'pre-algebra1'. If a student has a pre-algebra1 level twice and both algebraic levels once, he is certainly beyond the pre-algebra1 level and therefore rated as algebra2. Figure 6.51 shows the distribution (in percentages) of strategic competence levels at school D and E. As we can see, there are more students from school E at the highest level, whereas most students of school D have a pre-algebraic level of understanding. In each school there are two weak students who are at the lowest level of competence (poor understanding).

classroom contest

Two classroom activities on own productions are available for establishing strategy and notation use in the regular lessons: a problem solving contest and writing an instruction book (summary). Unfortunately we cannot look at an individual level for a number of reasons: students work in pairs or larger groups, they can copy each other's work or they might have been aided by the teacher. We can never be sure that a student's work is exemplary for what the student really can do. The contest has two functions: to turn around the student's perspective (own production of a combination problem, perhaps even the discovery of a dependent system), and to stimulate the development of ('better') strategies during the contest. Neither of these aims have been fulfilled. In one class the teacher already revealed what happens with a dependent system of equations in the class, and the contest environment did not provoke more efficient strategies in either group. Eight girls and one boy were already very competent with elimination at the start of the contest, and none of the other students formalized the combination strategy. In fact, some students were still not competent even to begin reasoning. Protocols of lessons not visited by the researcher reveal that in school E one girl presented the elimination strategy to the class, explaining how it works. There is no sign of explicit reflection on this strategy by the teacher, so if it has become a standard strategy for students, then it was only by their own choice.

instruction book

Writing your own instruction book is intended as a reflective activity which can be done at each student's own level of competence. This, in turn, will tell the researcher to what extent the newly constucted mathematical knowledge has obtained meaning and value for the student. Unfortunately most students did not get round to this particular task, and only a small number of students finished it. Presumably the teachers ran out of time. Moreover, the work of a few good students tells us that we have underestimated how difficult it is for students to communicate their thinking. In other words, the activity has not given us the information that we expected and will therefore not play a role in the following discussion.

consistency of strategy use

Draft notes of the contest illustrate that at both schools the students did not underperform at the written test compared to what they did in class. Generally students solved the test tasks in the same manner as they did towards the end of the lesson series: at the same strategy level, using the same kind of mathematizing notations. In both schools we see students apply the methods of elimination and combinations in *Cabinets* task, indicating that these strategies have been formalized with understanding, not superficially. There are a few students at both schools who are marked ‘consistent’ at the pre-algebraic level but who show algebraic reasoning in the test once. Looking at the male and female students separately, there are no significant differences. Six students in school D voluntarily dropped back. They have shown to be capable of making combinations but prefer the method of pattern continuation, using combinations only when necessary (in the *Cabinets* task). Apparently it is not self-evident that a student will always opt for the most advanced approach. Actually, having a variety of strategies at hand may be more advanced than just the strategy of elimination because it indicates flexibility and it reduces the risk of rote skill without understanding.

partial support for the conjecture

The conjecture only holds true for the strategy of combining equations. First, two boys of school D applied the combination strategy in classroom activities but resort to pattern continuation in the test. They were not able to solve the *Cabinets* task. And second, there are four students who unexpectedly were unable to attempt any reasoning for the *Cabinets* task. There is no sign of regression for the algebraic strategies in a new problem situation, and the majority of students maintained their highest competence level. Also, there is no indication that the girls are any different in this respect from the boys. Nonetheless we notice that students do not necessarily solve each problem with the most advanced strategy, and so lower order strategies can continue to be effective. It is not problematic that students fall back from an algebraic to a pre-algebraic level, as long as the regression is not to the arithmetical level.

Human Body test 2 secondary school

partial support for the conjecture

Comparing the test results on this task with similar problems in the unit *Time Travelers*, we see a regression in strategy level only for the arithmetical strategy ‘reasoning with parts’. A large number of students who used an incorrect strategy in the test task had shown a good understanding of the method ‘reasoning with parts’ in the classroom activities. It is not valid to consider the algebraic strategy ‘reasoning with whole’ here because it is not a logical choice: question 1 can be solved more easily arithmetically, and for question 2 it is especially appropriate to reason with (body) parts.

problem recognition

There is no indication of poor problem recognition amongst students, except, maybe, for some uneasiness with the different context (humans instead of fish). The children used either their intuition and common sense or a strategy known from previous tasks.

algebraic level

Regression within the algebraic level cannot be attested because the *Human Body* does not call for the approach of ‘reasoning with the whole’. The students could have chosen the method ‘false position’, but classroom observations have established that classroom experiences with this particular method were not favorable. The other pre-algebraic method, ‘correction with proportions’, was used by five students when working on section 6 in the instructional unit *Time Travelers*, against three students who used it in the written test. This is not a strong regression, either.

arithmetical level

The arithmetical strategy ‘reasoning with parts’ was chosen frequently, but as we already discussed, many mistakes were made in carrying it out. Only 40% of the students applied ‘reasoning with parts’ correctly in question 1, and no more than 37% did so in question 2.

6.7.5 Understanding relations between quantities

In this section we discuss typical errors of interpreting the relation between two or more quantities (magnitudes or unknowns). The arithmetical perspective of how quantities are related hinders the development of an algebraic way of thinking about and symbolizing variables and unknowns. Generally arithmetic does not involve operating or reasoning with an undetermined number or the idea of more than one solution. Instead, typical arithmetical problems are based on a direct approach using (a) fixed number(s), like an open sentence (3 + ? = 7) or a word problem involving reverse calculations, leading to a single answer. This arithmetical view of relations between two quantities causes students to misinterpret the problem situation and operate on the known numbers instead of the unknowns. Another typical consequence of a procedural conception of relations is the violation of the symmetrical and transitive properties of the equality-sign.

conjecture *understanding relations*

The analysis of the *Number Cards* task in section 6.6 led to the following conjecture:

conjecture

Arithmetical notions of quantities and relations between quantities hinder the emergence of an algebraic conception.

test (number - level)	task	conjecture *(mis) understanding relations*
I primary	Number Cards	+
	Pocket Money	+
	Birthday	+
	Dutch Past	+
II Primary	Trading Stamps	+
I secondary	Flowers & Cabinets	+
II secondary	Human Body	+

table 6.7: : support for the conjecture *understanding relations* (+ = confirmative)

Table 6.7 shows a convincing number of tasks at primary and secondary school level which provide evidence that the conjecture holds true. We now describe and discuss for each of these tasks except *Number Cards* the kinds of errors that students made.

***Pocket Money* test 1 primary school**

The discussion of this conjecture is based on three points of observation:
– common errors involving the relation ‘3 times as much’;
– low achievement on question 2;
– few occurrences of multiple solutions for question 1.

Let us look at the task again:

> Dean, Martin, Sabrina and Josy have 28 guilders altogether.
> Dean has the same amount as Sabrina. Martin has three times as much as Josy.
> 1. How much could each kid have?
> 2. Suppose Dean has 7 guilders. Then how much do the other kids have?

support for the conjecture

Although the task in the present form does not bring out the student’s actual level of understanding regarding restriction problems, the limited presence of simultaneous solutions and the errors students make support the conjecture that an algebraic conception of numbers and relations between numbers contradicts the arithmetical conception.

errors

In all three schools we have observed two major errors related to the interpretation of ‘3 times as much’. First, there is the error of incorrect strategy where the student does not fully understand the problem situation (see table 6.8 for the percentages). In most cases the multiplicative relation is ignored and the student simply applies ‘equal sharing’, dividing 28 by 4. There are a few students at each school who are confused by the problem and then use an incomprehensive strategy.

school	question 1	question 2
A (n=23)	22%	39%
B (n=18)	17%	44%
C (n=30)	20%	10%
total (n=71)	20%	47%

table 6.8: incorrect strategy or no answer for the *Pocket Money* task

The second type of error is of a more operational nature. The student gets 7 for the two equal quantities, leaving 14 for the remaining two quantities which have to satisfy the multiplicative relation. However, although aware of the condition ‘3 times as much’, the student then decides that 14 is 3 times as much as *the* number which is sought, rather than reasoning that 14 has to be split up into *two* numbers in the ratio 1 : 3. In other words, it is a thinking error based on the arithmetical perspective that only one number is sought and that it can be found by direct calculation, just like the error observed in the analysis of the *Number Cards* task. Five answers to question 1 and fourteen answers to question 2 (7% and 20% respectively) involve this error, where the student does not succeed at finding the correct values. In most cases the student settles for '4,67 and 14' or '4,67 and 9,33'; a handful of students declare that the task cannot be solved.

question 2

The high percentages of incorrect strategies for question 2 are also caused by the thinking errors discussed above. The operational error plays only a minor role because the corresponding strategy has been categorized as arithmetical, not incorrect. The high percentages in the last column are all related to misunderstandings of the problem situation from the start. School B, in particular, shows a very low score compared with the achievement on question 1. Perhaps it has not been clear for these students that question 2 deals with the same problem situation as question 1, i.e. the restrictions still count. If a child believes the starting situation is new, then everyday life experience with pocket money might induce the student to choose for a fair solution: equal sharing. Moreover, the student might think the problem situation no longer holds since the problem has already been solved in question 1. Since there is no similar problem in the unit *Exchange* with which to compare these results, it is difficult to look for clues. The high frequency of answers without explanation for school C agrees with earlier observations on the *Number Cards* task.

Dutch Past test 1 primary school

> The man has to hand over half of his apples plus one more apple. The question is: if he has one apple left after the third gate, how many apples must he have had?

support for the conjecture

The results of the first problem of the task *Dutch Past* support the conjecture that arithmetical conceptions hamper the development of an algebraic notion of relations between quantities. In this case we look at thinking errors concerning the operations as well as incorrect use of the equal-sign. Students tend to make three types of thinking errors: they choose the wrong order of operations (twelve times), they invert the operation incorrectly or inconsistently (six times), or they misinterpret the formulation 'half his apples plus one more' (three times). Just the third type can be attributed to an undesirable arithmetical influence.

order and inversion of operations

If a student reverts the chain of operations starting with the operation '× 2' instead of '+ 1', it is probably a matter of carelessness. Students at all levels make this mistake, so it is not typical for weaker students. We already mentioned before that visual diagrams do little to prevent thinking errors if the student uses a mental strategy. For the first two types of mistakes there is a very simple remedy: just write down the forward chain of operations first to make sure the order is correct – which is how the arrow diagram was intended. Writing down the forward operations will probably also reduce the risk of inverting correctly, because we reckon the inversion itself is not difficult for these students. For instance, in some cases we see students use the operation ‘× 2’ and ‘: 2’ alternatively (like in figure 6.52), which we interpret as not remembering the ‘direction of calculation’.

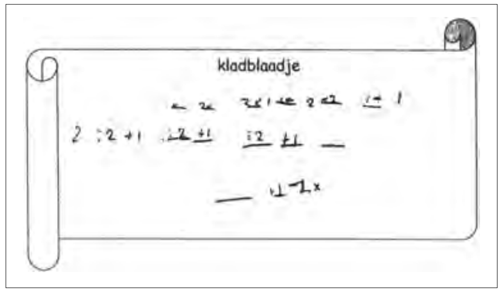


figure 6.52: alternating operations in *Dutch Past*, problem 1

However, incorrect inversion of ‘– 1’ can be caused by misunderstanding the expression ‘plus one more’. A student can focus so much on the word ‘plus’ that the forward chain is taken to consist of ‘: 2’ and ‘+ 1’. This would automatically lead to ‘– 1’ in the reverse order, in which case it is not an error of inverting but an error of misinterpretation (type 3). We expect that some of the inversion errors are in fact misinterpretation errors.

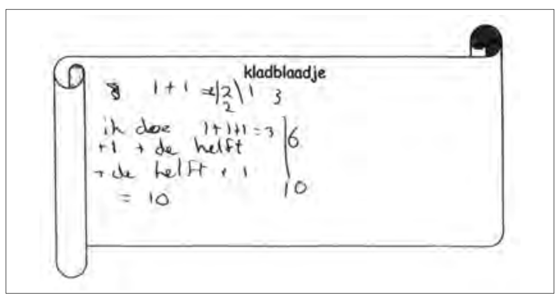


figure 6.53: misinterpreting ‘+ 1’ in *Dutch Past*, problem 1

mis-interpretation

Perhaps the two errors discussed above are caused by other factors. One can wonder if they are typical for inversion problems with repeated operations. Perhaps the repetition is a complicating factor, or maybe it causes students to underestimate the task. The second problem in the test task is slightly different in nature: it involves not two but five different operations, only one of which is repeated. The data of school A and B point out that mistakes of inversion do not occur at all. This can be due to the fact that less students were able to perform the inversion (in particular at school B). Still, just over half the students at school A inverted the operations mentally and this time no one made an error, compared to seven in the other problem.

Another possibility is the presentation of the problem. Perhaps, since each operation is mentioned separately this time, it is easier to deal with step by step.
We have observed three occasions where students calculated with the fraction $\frac{1}{2}$, as illustrated with a drawing in figure 6.54. We attribute this error to a strong arithmetical conception of numbers and relations, rather than careless reading. It is our conjecture that some students might have certain expectations of 'half' and 'plus', causing an error of interpretation. The word 'plus' is naturally associated with addition, not subtraction as in this special case. For the word 'half' this means that a student will see it as an isolated number, $\frac{1}{2}$, instead of as a part of the relation 'half times a quantity'. Perhaps the terminology unintentionally contributes to the confusion, and might 'half the number of apples' been more appropriate. We have opted for the (mathematically incorrect) phrase because in the Dutch language 'number of' sounds more formal. Another explanation for the error could be, that the student expects to deal only with known, fixed numbers and straightforward numerical results. For example, in the classroom we observed a student who insisted you should begin at the beginning, that it is strange to begin at the end. And if you don't know the number at the beginning, you should make a guess. This conception clashes with the algebraic notion of reasoning with unknowns and general expressions involving a variable or an unknown, or expressions where the numbers are objects to be left alone. A student who cannot cope with 'half the apples' because it cannot be calculated, can decide to work with the number $\frac{1}{2}$ instead.

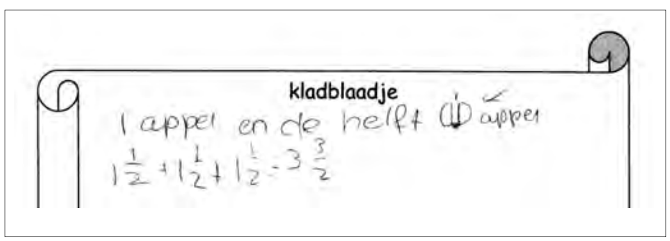


figure 6.54: misinterpretation of 'half the apples'

equal-sign

The syntactical error of stringing equal-signs is very likely in reverse calculations tasks. Of course the student intends the equal-sign to announce intermediate results, its relational meaning (equivalence) is irrelevant here. Besides, in their opinion the result is all that matters. It is not fair to expect a student to be aware of the violation. Instead, it would be more valuable and appropriate to make students aware of their imprecise notations in a setting where symmetry and/or transitivity of the equal-sign have meaning and importance, like when both ends of the expression are compared. One teacher who devoted some time to the issue managed to convince a few students, but most students reacted like they did not see a problem. The reason we discuss the error here is to evaluate the role of the arrow diagram. One of its purposes is to avoid the stringing error: prevention is better than cure. But the real classroom situation does not live up to our expectations. Many students still prefer calculations (particularly in the second problem and in school C) and the syntactical error occurs even in combination with the arrow diagram. We can only conclude that students at this time of their mathematical career are not susceptible to syntactical rules and symbolic conciseness.

Birthday **test 1 primary school**

> "Happy birthday to you, happy birthday to you." Mirjam's voice carries across the lawn. It's her mom's birthday, and so she baked a delicious cake for her. Her mom has trouble blowing out all the candles in one go. "You are now exactly three times as old as me, mom!" Mirjam's dad adds in: 'Your mom and I are now 88 years old together'. Mom replies: "And the three of us are 102 years old."

support for the conjecture

We have found further evidence for our conjecture on arithmetical versus algebraic notions of relations. In the *Birthday* task data we observe thinking errors reflecting typical arithmetical influences, and we feel that they interfere with a correct understanding of the problem situation. In addition we have noticed inaccurate use of the equal-sign, reflecting an arithmetical notion of the equal-sign. Even though this is also or perhaps more a matter of algebraic symbolism than reasoning, we choose to discuss it here, and it will become clear why.

taking the average

Eight students wrongfully interpreted '88 years old together' as a sign to take the average, dividing 88 by 2. Three of them realized that the other conditions would not hold true, and proceeded to solve the problem by trial-and-error. Two other students made the same mistake with 'the three of us are 102 years old', calculating the average values 102 : 3 and even 88 : 3. Taking into account the prevalence of equal sharing strategies for the *Pocket Money* task, it seems reasonable to say that certain students naturally or automatically opt for the strategy of 'taking the average', ignoring the other restrictions in the problem. Thus the error of taking the average automatically leads to the error of not fulfilling all the restrictions in the problem, as we saw before in the *Number Cards* task.

equal-sign

A seemingly very different type of error is concerned with conventions of algebraic notation. Algebraic and arithmetical notions and meanings of the equal-sign are a well-known source of conflict in the teaching and learning of equation solving (see section 2.4). Four students wrote their intermediate numerical results horizontally in a string, without satisfying the equivalence relation, but having no trouble to keep track of which number (age) belongs to which referent (person) and solving the problem correctly. In other words, the usual argument of not knowing which value we are dealing with in each phase of the solution process, is not so much a problem for the student but more for the teacher or the professional algebraist. In a similar fashion the student is never troubled by the inequality, because the equal-sign has a different meaning to him or her, namely to announce an outcome. Similarly it is the perspective of most advanced mathematicians and curriculum developers to think that a student must learn to cope with algebraic symbolism before algebraic reasoning. However, we believe that these errors with the equal-sign are not related at all to the level of mathematical reasoning. In fact, it serves to show that competence of reasoning do not rely on symbolizing skills.

concluding comment

But, one of our aims following the pilot experiment was to reduce writing errors with the equal-sign, and for this reason we introduced arrow language. However, the arrow diagram is not a logical representation for this problem because it is static in nature; there are no actions involved, like in the reverse calculation problems. It is quite likely that arrow language is strongly associated with experience and recognition, and is to be expected only in analogous situations. This means that there are other types of problems where the arrow diagram cannot prevent violation of the symmetry/transitivity of the equal-sign.

We go into the need for conventions for algebraic symbolism when we present recommendations for designers in section 7.3.

Trading stamps test 2 primary school

support for the conjecture

Test results show that students generally make three types of error, and indeed they appear to be related to the dominance of arithmetical conceptions. The most remarkable error students make in question 1 of the task concerns the determination of 'least value'. Reference to postal value instead of trade value is a common mistake amongst the lower achievers, and the third type of error is related to a limited perception of ratio. The first two situations imply that students concentrate too much on quantitative characteristics; they miss the capacity of qualitative or global thinking. The latter error may caused by unvaried experience with ratio problems.

least value

Sixteen students at all levels of achievement failed to determine the least valuable stamp while succeeding to determine the most valuable one. Unfortunately we have only answers to go on, so we must resort to our own interpretation. The correct answer of the highest value implies that the student can reason to a certain degree about number and value in trade terms. The first risk factor of interpretation is the meaning of 'third stamp'. Some students numbered the stamps without explanation.

However, we observed that students tend to number the stamps one through four by going down first, which makes the stamp with the horse number four. We assume, therefore, that students mean the stamp with the sailor. If the third stamp is said to be the least valuable, the student is probably biased by the problem presentation: 'there is a pattern in the trade terms – highest value stamp in the first row, the next highest in the second, and so the third highest in the next row again, and since it is the last row it would be the lowest value stamp'. If instead the student answers with the second stamp (i.e. the 'people' stamp), it is probable that he only considers the stamps on the left-hand side of the expressions, perhaps because the 'horse' stamp is not presented by itself. Another possibility is, that since the horse stamp is the 'unit stamp' (the other stamps can be expressed in terms of this one), it is not expressed in terms of itself and therefore left out of the comparison. A more global perspective might prevent students from fixing their attention on one aspect (the pattern, the left-hand side) and think about general aspects first, like 'how many stamps are there to consider'. Another argument for this fixation on quantitative characteristics is the fact that some students give numerical solutions there is no evidence of qualitative reasoning like 'since $a > b$ and $b > c$ then a = max and c = min'.

postal value

Low achievers at school B in particular tend to base the trade value on the postal value printed on the stamps. They reason the trade is not fair because the stamp with the sailor is worth 10 cents while the stamp with the lighthouse on it is worth only 1,5 cents. The iconic trade terms play no role in their solution. Moreover, such reasoning is based on trading one stamp of each, even though the task explicitly says that it is not necessarily one-for-one. They either do not read the question well enough or they are distracted by the two single stamps placed in the question. In summary, the students who refer to the postal values instead of the relative values fail to oversee the trading situation (global perspective) and fix their attention on the numbers instead. Possibly it is again a matter of arithmetical influence.

unusual ratios

The third type of error appears when students compare and substitute trade values correctly until they discover that the numbers do not match conveniently, as we have seen in figure 6.25. Some students decide to involve a third type of stamp, perhaps out of convenience or because they think it is not otherwise possible. Other students stop halfway and say it cannot be done, or they approximate the numbers. These results indicate that students may be unfamiliar or uncomfortable with ratios where both numbers are bigger than one, or that they have trouble adjusting the numbers to resolve fractions.

Flowers and *Cabinets* test 1 secondary school

support for the conjecture

The tasks on solving systems of equations display a number of typical obstacles of algebra that students encounter, but mostly concerning symbolizing and not so much students' conception of relationships between quantities. After all, students encounter reasoning with unknowns early on in the lesson series, so conflicts of conventions are already resolved during the lessons. Still, since the errors of symbolizing can also be ascribed to the discrepancy between arithmetical and algebraic notions of symbols, we feel that these data are relevant for testing this conjecture.

errors of symbolism

Incorrect use of symbolism occurs in three different ways: violating the transitive and symmetrical characteristics of the equal-sign (by stringing calculations), not including unknowns in the equations and inconsistent use of letters (as the unknown but also as a unit of measurement). The first error is observed four times in the *Flowers* task at all levels of reasoning and particularly in combination with syncopated notations; seen in this light it is quite likely that these students still have a dynamic, procedural conception of the equal-sign (announcing a result instead of a state of equivalence). Inconsistent use of letters, the third error, occurs five times in the task *Cabinets.*

Human Body test 2 secondary school

conjecture supported

The analysis of the test task *Human Body* gives further evidence in support of the conjecture that differences between arithmetical and algebraic conceptions of (relations between) quantities hamper the student's progression to algebraic reasoning. We observed incorrect interpretations of ratio which originate from mixing up *part : whole* and *part : remainder*.

Mistakes in interpreting sentences like 'the head fits three times in the remainder of the body' are frequent. The focus is strongly on the number 3, whereas the unknown quantity cannot directly be found via this number. In the same way the statement "An adult is about 8 times as long as his or her head" in the second problem causes difficulties of getting the ratios right. Three students divided the length in 1 + 8 = 9 parts, which means that they did not consider the head as part of the body. Six other students made the same type of mistake while solving isomorphic problems in the instructional unit *Time Travelers*. In addition to misunderstanding the ratios there also seems to be a language problem involved here. Apparently the distinction 'with or without head' was not clear to the students.

6.7.6 Effect of history on the learning and teaching of early algebra

Throughout the learning strand we have integrated historical problems and methods to facilitate the reinvention of pre-algebra (see also chapter 3). In this section we present the results in four parts:

- history of mathematics' contributions to the learning of (pre-)algebra;
- students' personal reactions to historical elements;
- teacher reactions to the use of history;
- design and research value of history of mathematics.

contributions to the learning of pre-algebra

Diophantine problems

primary school level

The Diophantine problems in *Exchange* are part of the theme on embedded equations, also called restriction problems. The purpose of these tasks in the learning strand is twofold: to further student understanding of restriction problems, and to enable the reinvention of a systematic approach for one particular type of restriction problem, namely on sum and difference. The elaborate discussion of the *Number Cards* task in section 4.4.4 has shown that the number riddles have not been successful in easing the transfer from arithmetic to algebra. Arithmetical or incorrect strategies dominated in school B, and those students in school A and C who did achieve an algebraic level skipped the intermediate pre-algebraic stage. In spite of a high success rate in the classrooms – correct answers as well as positive student reactions – we have noticed little formalization of mathematical thinking. Students did not outwardly make the connection with earlier sections in the unit, nor did they attempt to systematize their arithmetical strategies during the lessons. However, the number riddles also showed an unexpected large group of students who reinvented part of the Babylonian algorithm ('sum minus difference divided by two'). In other words, the Diophantine problems appear to be a suitable constituent of the restriction problems theme and they enable the reinvention of algebraic methods, but they do not bring forth pre-algebraic strategies that reduce the cognitive gap between arithmetic and algebra.

secondary school level

In the secondary school unit *Time Travelers* the Diophantine problem plays only a marginal role. Its main purpose is to bring together the symbolic and the reasoning streams in the learning strand. Its second aim is to show students an alternative approach – one which played a big role in the history of algebra. Unfortunately the designer misjudged the situation in four ways. First, the sample was selected deliberately for the author's treatise of the solution method, making the prerequisite of challenge second priority. We found that most students solved it simply by observation or by immediately halving the difference; it did not call for an investigation. So again the pre-algebraic phase was omitted. Secondly, the author's solution involves introducing an unknown and is needlessly complex in comparison, which means the equation is not a useful tool in this case. In fact, if a student were to discover the limitations of his or her informal method, he or she would appreciate the equation's assumed superiority – general applicability. The lesson material should create a *need* for algebra. And third, having found the solution already, the average student does not appreciate or understand the idea of reflecting on it. In conclusion, the current application of the Diophantine problem in the unit *Time Travelers* is not satisfactory and must therefore be revised. Its learning effect for the secondary school student is inconclusive in this experiment.

Apple Orchard

The problem of the *Apple Orchard* appears in a sixteenth century arithmetic book by Bernard Stockmans (1595). It serves as orientation task for the section on reverse calculations and the arrow diagram in the unit *Exchange*. In an attempt to make the problem more accessible, a schematized diagram was added to visualize how the unknown quantity of apples changes (see figure 6.55). The diagram accentuates the string of operations embedded in the problem and might initiate students to try inverting it. Some students might use the trial-and-adjustment approach to get used to the diagram first.

results

Classroom observations indicate that the authentic text and the unusual pronunciation have caused students to lose sight of the mathematics. Although the historical elements were expected to be a complicating factor, we had not foreseen that the mathematical content – inverse calculations – might become subsidiary to the problem representation. In addition, the didactical choice to schematize the problem appears to have worked counterproductively. In order to understand the diagram at all and complete it, students are required to interpret the problem in a forward direction, reasoning with the unknown, which is known to be a cognitive obstacle. In fact, the long struggle with the concept of unknown throughout history caused arithmetical methods to sustain until the seventeenth century. Filling in the missing numbers was clearly an underestimated task. Only a few students actually tried a numerical value, but they worked towards the solution they spotted in the original text and made several errors along the way. So instead of facilitating the strategy of inversion, which is also the more natural approach, the diagram distracted and confused the students. Indeed, the mathematical goals of reverting a string of calculations and how to use an arrow diagram to do so were achieved in the problems following the *Apple Orchard* task.

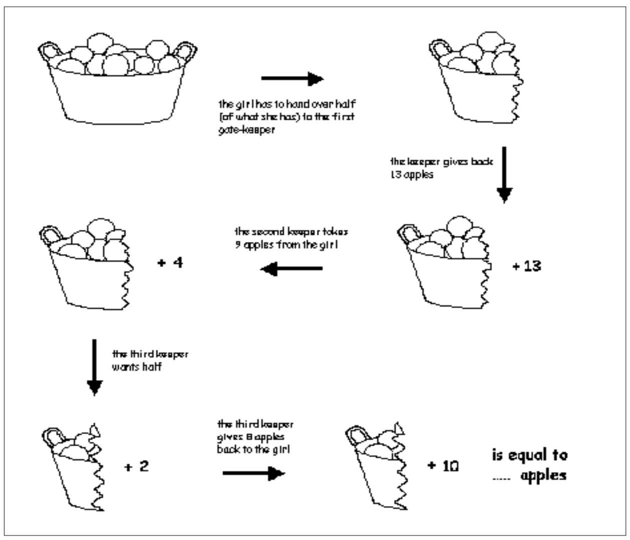


figure 6.55: schematic diagram of the *Apple Orchard* problem in the unit *Exchange*

Has the *Apple Orchard* problem not contributed at all then to the algebraic learning process? We find this question unanswerable because although the problem itself has not been altered, the designer's additional information has been too influential. We presume that part of the students will be able to identify the task as being of the inversion type – as we have read in the tests and the questionnaires – and so we can say it is relevant for this particular part of the program. But the timing of the problem is premature for reflections on advantages and disadvantages of calculating in a forward or a backward direction. The reasoning process in the diagram was found to be too complex for a retrospective view later on, when students are asked to comment on a similar situation in the hermit problem (see below).

revision

Looking back it might be better to change Apple Orchard into an unstructured task and see what students come up with. Or, perhaps the problem is not suitable for orientation purposes and the problem on the hermit should be used instead due to the repeated operations and the role play. Another possibility is to postpone the task to a later stage of algebra learning, when students have learned to solve equations. It is indeed a wonderful example to show how algebraic problems can be solved (more easily, too) without algebra. In a later stage it can even be used to have students reflect on the process of equation solving, how each manipulation corresponds with a reverse operation in the arithmetical string.

The Hermit

The last section of the unit *Exchange* is based on an historical problem about a hermit exchanging money with three saints. Mathematically the task is similar to the apple orchard problem in test 1 discussed above, dealing with the inversion of a repetitive series of operations. The teacher of school A reports that most of the 23 students responded well to the hermit problems; no special difficulties were observed. The tasks did prove to be difficult at school B, though, as mentioned before in section 6.7. School C did not have time to do this set of problems.

meta-cognition

The second part of the unit section contains a few meta-cognitive questions. After the *Hermit* problem is solved, students study the author's own solution – which merely states the answer and demonstrates its correctness – and are asked to comment on it as follows:

> Suppose you have to solve a similar problem, where the hermit has 38 pennies left instead of nothing at all. Does mister Van Varenbraken's solution help you with that?

We intended and expected the question to provoke a classroom discussion on the limitations of such a solution. After all, forward calculation applies only for the one correct answer, and without mention of method it has no value in other situations. Such a discussion can include looking back to the schematic diagram of the apple orchard, where the trial-and-adjustment approach is also less suitable than the method of inverting the operations.

It appears that this question was either not understood or not read properly, or perhaps the students have no experience of this type of reflection. We have no observations of this lesson at school A, but the instructional materials show that many individual answers which were affirmative at first (saying that you can always invert the calculations) were corrected; only two students had a different, personal answer. Classroom protocols give little information on how the second part of the section proceeded in school B, but the written work in the unit *Exchange* indicates very poor understanding (five affirmative answers, twelve no answer). In the absence of personal observations in the classroom, we cannot conclude anything at this time.

The section ends with a few extension tasks to have students explore the mathematics further and possibly lift it to a more formal level, for example by way of the following question:

> How many pennies at least must the hermit have at the beginning in order to make a profit?

This question has been answered by nearly half the number of students at school A and B, at a very concrete level (trial-and-adjustment). We observed no abstract reasoning about earlier outcomes. The students at school B scored very poorly on this question because there were only two answers with computations.

free productions

Finally the students are asked to construct their own hermit problem. At school A five students produced a correct story problem with two repeated operations. Four students constructed a 'bare' version, giving only the operations, while 7 students made up an arrow diagram. In other words, the mathematical content of inverting a chain of operations has been well understood by 16 students (70%). At school B only 6 students were successful at producing their own hermit problem involving two operations; 5 students made an incorrect attempt and the remaining 7 wrote down nothing at all.

summary

Comparing the two schools, the results are clearly very different. From the test results we know that the majority of students at school A (70%, see figure 6.14) solved the reversion problem successfully, versus only 40% for school B. The reflective questions and students' own productions, too, indicate a discrepancy between the two groups. The question on the author's solution is certainly the least effective, although there is a chance that students misread it. For the students of school A we can conclude that the historical context has played a positive role in the learning process on inverting operations. On the other hand, it is also clear that a large number of students at school B have not done well by the historical context, and most students were unable to take a more global view of these types of problems (recognizing certain properties or limitations).

Rule of Three

analysis

The third section in the unit *Time Travelers* focuses on the ancient method of proportions known as the Rule of Three (see also section 5.3.3). In order to determine the extent of student understanding, a number of interrelated tasks and investigative questions have been analyzed for signs of consistency. Questions which have been influenced too much by classroom discussions are not included. The results are based on the performance on two productive tasks (to investigate and describe the Rule of Three for the Indian and the Dutch problem) and three applications. A student is considered competent when a) one of the productive tasks and b) two of the three applications are carried out correctly. A student is qualified 'incompetent' when the Rule of Three is not used properly even once. The Indian Rule of Three should be expressed as a calculational algorithm in general terms, whereas the Dutch version should clarify the calculations implied by the lines in the numerical representation (either in words or with operations). Application of the Rule of Three is deemed ‘correct’ if the notational form is included and the solution strategy reflects the right procedure, i.e. the operations comply with the algorithm. Since the Rule of Three is explicitly requested, the use of proportions – in a table or otherwise – is not satisfactory.

results

In this way we obtain the following results: twelve students appear to understand the Rule of Three, nine students do not and nine are inconclusive. Five inconclusive cases seem able to apply the rule but have not been successful at formulating it themselves. One girl adopts the word formula representation of the Rule of Three and applies it correctly throughout the second part of the section, which means she is able to think in terms of variables. In comparison, only two students succeeded in formulating the Rule of Three in general terms for the indian problem, like ‘multiply the second number by the third and divide the answer by the first number’, where the variables do not yet have a universal character. Eleven students remained at an arithmetical level, identify the correct algorithm for a specific case (the Dutch problem).

Generally speaking the Rule of Three did not appeal to students. Nearly half the class – including five competent students – thinks the method is less convenient than the proportion table for different reasons: it requires careful thinking, is less surveyable, confusing or difficult. Only two students are positive about it and ten students have mixed feelings. In other words, the section on the Rule of Three has shown students how longtime favorites are eventually overtaken by superior methods, which is a valuable result.

Reflection on symbolic notations

We cannot determine what students think of the development of symbolism because we have no information on this issue. There are no records of a classroom discussion on the symbols mathematicians centuries ago. The teachers did not ask students to comment on the pieces of text either, and one teacher even skipped the part on the plus and the minus sign altogether! Personal responses to the use of history are described further on in this section.

Rule of False Position

We mentioned before that the strategy ‘false position’did not appeal to students because they found it laborious, and choosing a suitable starting number is not always easy. Indeed, students struggled with the calculations whenever the correction factor was a fraction with a numerator larger than 1. The advantages and disadvantages of

this strategy were a point for further questioning of the students, by way of a questionnaire. Figure 6.58 clearly shows that 'Calandri' was not such a popular topic. In order to determine the method's contribution to crossing the gap between arithmetic and algebra, we looked for instances where students progressed from 'reasoning with parts' to 'reasoning with the whole'. The results indicate that 'false position' was not helpful. Maybe the switch to the mathematical context in the final section (see also the appendix) is made too abruptly. Another reason may be the lack of a transparent form of representation for the strategy 'false position'.
On a more positive note we can mention that the creation of a related strategy, 'correction with proportions', indicates that for some students the method 'false position' does have merit. According to De Groot and De Jong (1928), this advanced trial-and-adjustment approach was also used by Ahmes in *Rhind Papyrus* (ca. 1650 BC) (see also section 6.7.8).
Reflecting on the experiment one can say that students were introduced in a relatively successful way to a non-regular pre-algebraic solution strategy and that they were able to reflect on their own methods. A limited number of students used the pre-algebraic strategy to solve the related test task. We cannot establish that the transfer from arithmetic to algebra was made easier by using this strategy. However, the fact that students reinvented another, similar strategy is encouraging.

Babylonian number riddles
As indicated before, 43% of the students showed convincingly that they could solve the number riddles from *Time Travelers*. They used the algebraic strategy 'reasoning with the whole' . The transition from 'reasoning with parts' to 'reasoning with the whole' has been successfully made by the students. The students were captivated by the context, even though it was nearly the end of the school year. Number riddles can form an important part of a learning trajectory on linear equations with one unknown.

The big question that remains is: how to continue? How can we make the last and vital step, combining reasoning with symbolizing? We discuss this difficult issue in chapter 7.

personal reactions of students

Students' personal reactions to the use of history in the mathematics classroom are taken from four different sources of data: questionnaires, part 2 of the secondary school test of the unit *Time Travelers*, observations of lessons and discussions, and personal comments in the instructional materials. We have made a brief overview of our findings below.

Questionnaires, primary school

The questionnaire contains a general section with multiple choice questions on attitude towards school and school mathematics and a series of open questions on various matters of the experiment.
A total of seventy questionnaires from three schools has been analyzed. Of these, thirty students had worked only in the unit *Barter*, the other forty had done the unit *Exchange* as well. The questions were completely open, no written instructions or sample answers were presented. Table 6.9 gives a survey of the subjects mentioned by students. Some students gave more than one answer (the numbers between brackets indicate the 'surplus' of answers)

which topic did you enjoy?	***frequency***	***which topic did you find hard?***	***frequency***
Diophantos	4	Diophantos	0
number riddles	3	number riddles	0
barter	11	barter	15
old Dutch	7	old Dutch	10
arrow diagrams	2	arrow diagrams	5
soccer cards	2	soccer cards	7
marbles	2	marbles	1
sweets	14	sweets	1
(almost) all	3	(almost) all	8
incorrect	12	incorrect	6
none	7	none	15
other	7	other	4
total	74 (4)	total	72 (2)

table 6.9: enjoyable and difficult topics in *Exchange* and *Barter* (n = 70)

The reactions about history are organized in two categories, i.e. problems from the 'Dutch past' and 'Diophantus'. The researcher has classed the answers as follows:

challenge: you really have to use yor brains
puzzle: every time the solution is different
play: you can act out the problem in a small play
reverse calculation: solving the problem by calculating backwards
unclear: the reason is not given or is not understandable
old text: the sixteenth century Dutch language is mentioned
mathematical content: the mathematics is mentioned specifically

Opinions vary: some students are attracted by the history, one even mentions both topics. The old text leads to complaints rather than to pleasure. For some students history only makes mathematics harder.

	which topic did you enjoy?		*which topic did you find hard?*	
reason	Diophantos	old Dutch problems	Diophantos	old Dutch problems
challenge	1			
puzzle	2 (1[a])			
play		2		
reverse calcu-lations		1[a]		
unclear	1[b]	4 (1[b])		6
old text				4 (1[c])
mathematical content				1[c]
total	4	7	0	11

table 6.10: reasons enjoyment and difficulty historical topics in *Exchange* (n=70);
(1[a], 1[b] and 1[c] are second topics mentioned)

Finally students were asked what subjects they would like to return to at a later stage in school (see table 6.11)

which topic would you like to spend some more time on?	***first choice***	***second choice***	***third choice***	***total***
Diophantos	1	1		2
number riddles	2		1	3
barter	9	2	3	14
old Dutch	6	8	4	18
arrow diagrams	2	2		4
soccer cards	4	3	2	9
marbles	3	3	1	7
sweets	6	3		9
equal sharing	3			3
comparing picture quantities		1	1	2
age riddles	2	3		5
hermit	2	1	1	4
incorrect	3	4	1	4*
none	22	34	51	20*
other	5	5	5	15
total	70	70	70	*

table 6.11: primary school students' choice of topics for additional lessons (n=210)

The * in the righthand column indicates that there were four students who answered all three choices incorrectly, and twenty students who answered “none” three times. The Dutch arithmetic problems from a few centuries back is clearly an item that students would like to try again, followed by barter and a number of topics from the unit *Exchange.* The old Dutch problem *Hermit* is mentioned explicitly a few times, which means the share ‘old Dutch’ is actually 21%. The Diophantine problems constitute 2% of the answers.

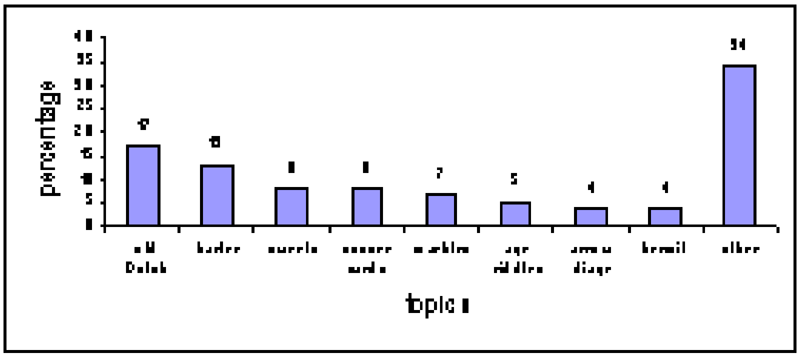


figure 6.56: frequently mentioned topics for additional lessons, primary school (n = 107)

Table 6.12 shows the numbers of students who would like to have additional lessons on the historical topics ‘Diophantos’, ‘old Dutch problems’ or ‘hermit’, and for which reason. In the column ‘old Dutch problems’ we see that three students give two reasons: ‘hard’ combined with either ‘fun’, ‘interesting’ or ‘boring’. The combination ‘hard and boring’ as a motivation for more of the same is unusual at least ... Apparently the primary school students are quite motivated to have another go at solving the more difficult problems; a few students even indicate that difficult problems are more enjoyable than easy ones. One student would like to have both the Diophantine riddles and old Dutch problems again.

reason	Diophantos	old Dutch problems	hermit
fun	2	6 (1^a)	2
interesting		3 (1^b)	
play			2
weird and boring		1	
boring		1^c	
hard		9 (1^{abc})	
poor explanation		1	
total	2	21 (3)	4

table 6.12: reasons for wanting more lessons on historical topics, primary school (n=70) (1^a, 1^b and 1^c are double answers)

In order to get a better idea of the student's general attitude towards history and mathematics, the questionnaire included a question on favorite school subjects. Thirty-six students mentioned mathematics as one of their three favorites, thirty-one mentioned history, and nine mentioned both. The students are also asked whether or not they are interested to learn about historical methods and problems of mathematics. Thirty-one students (44%) answered positively, compared to thirty neutral and nine negative answers (43% and 13% respectively). Eleven students who enjoy history at school indicate that they are interested to learn about history of mathematics, four of whom are also positive about mathematics (see figure 6.57).

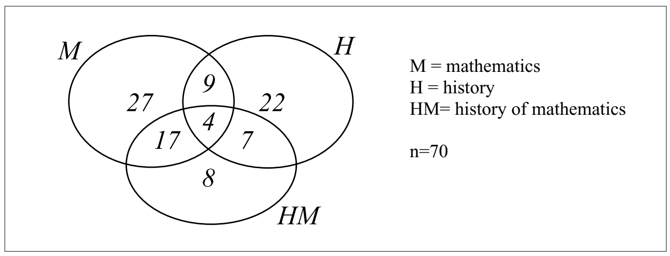


figure 6.57: students interested in history, mathematics, history of mathematics

In summary, the general attitude towards learning about history of mathematics is not below expectations. Nearly half the number of students indicate they are interested. Still, only five students mentioned an historical topic as the one they enjoyed most, while eleven students gave a positive reason (fun or interesting) for wanting to spend more time on the historical problems in the units.

which topic did you enjoy?	***frequency***	***which topic did you find hard?***	***frequency***
chips	7	chips	3
barter	6	barter	1
equations	20	equations	21
rule of three	3	rule of three	8
Calandri	3	Calandri	4
proportions	5	proportions	1
Time Travelers	6	Time Travelers	6
Fancy Fair	3	Fancy Fair	1
incorrect	1	incorrect	1
none	3	none	9
other		other	2
total	57	total	57

table 6.13: enjoyable and difficult topics *Fancy Fair* and *Time Travelers* (*n*=57)

Questionnaires, secondary school

The questionnaire for the secondary school students is very similar to the one for the primary school students, the only difference being that both the multiple choice section and the open questions section were shorter. Fifty-seven students filled out the form. The twenty-eight students from school D – who participated only in the first part of the experiment – based their answers on just the unit *Fancy Fair*. Table 6.13 shows the topics that students mentioned as being fun (left column) or difficult.

Ten students from each school answered that they enjoyed solving equations, making this the most popular topic by far. However, since the students from school D did not do any tasks from the unit *Time Travelers*, this is not a representative outcome.Taking aside the group of twenty-nine students from school E, we find that the five most enjoyable topics are: 'equations' (10), *Time Travelers* (6), 'proportions' (5), 'rule of three' (3) and 'Calandri' (3) (figure 6.58 shows the respective percentages). In summary, seventeen students (59%) chose topics from the unit *Time Travelers*, of whom six (21%) explicitly mention an historical one.

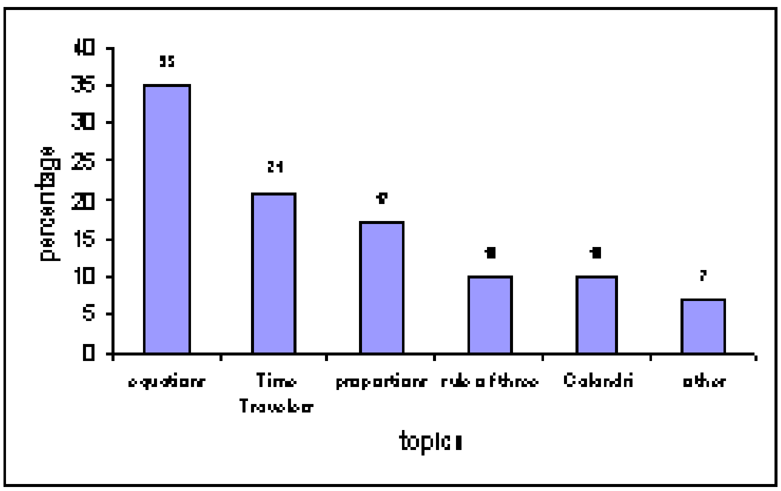


figure 6.58: most enjoyable topics, secondary school E (n = 29)

The right hand column in table 6.13 shows that equation solving is also most frequently mentioned as the most difficult topic, but again this is not a representative outcome. After all, the students from school D have no experience with the topics from the unit *Time Travelers*. For these students equation solving was the final part of the experimental program. Due to the increasing complexity of the tasks in the unit *Fancy Fair*, it is logical that many school D students nominate 'equations' as the most difficult, thereby bringing the outcome grossly out of balance. But if we consider the students from school E separately again, we find that only four students named a topic from *Fancy Fair* and the most frequently mentioned topic is 'rule of three' (eight times). *Time Travelers* and 'Calandri' occurred six and four times respectively. Five students indicated they did not find any topic difficult, including one student who failed both tests and four students who performed poorly on the second one.

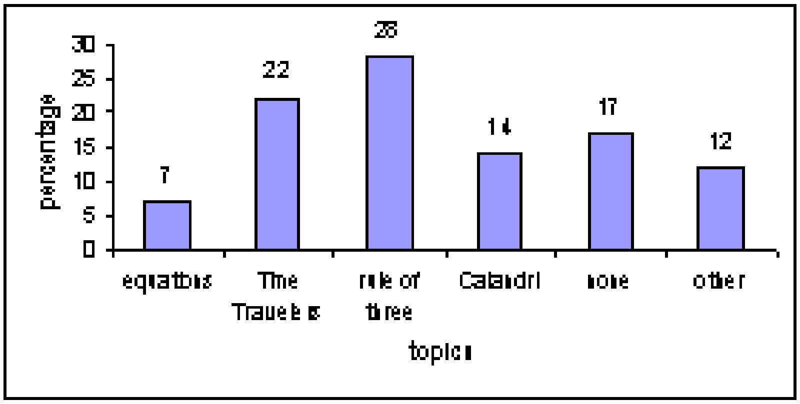


figure 6.59: difficult topics, school E (*n*=29)

Next we take a closer look at the reasons given for enjoying the historical topics 'rule of three' and 'Calandri' and the unit *Time Travelers* (table 6.14), followed by the reasons for finding them difficult (table 6.15). In both tables the reference to *Time Travelers* does not specify which element is decisive: the historical character of the unit, or other elements like proportions or arrow language. Four students mention an historical topic explicitly, though, appealing to them for different reasons: interesting, puzzling, or understandable.

reason	rule of three	Calandri	Time Travelers	total
challenge			2	2
interesting	2	1		3
puzzle	1			1
easy				
understandable		1	2	3
hard				
unclear			1	1
other		1		1
total	3	3	5	11

table 6.14: reasons for enjoying historical topics in *Time Travelers* (n=29)

The tasks on the Rule of Three were found difficult primarily because of the use of old measures (a *hekat* of grain, a *dou* or *sheng* of rice, a *pala* of saffron). The troubles that students had with the Rule of False Position (Calandri in the questionnaire) have been mentioned before. A few students found the mathematical content confusing in general, and most of the other explanations are not clear.

reason	rule of three	Calandri	Time Travelers	total
old measures	4		2	6
confusing	2	1		3
puzzle	1			1
text				
unclear	1	2	4	7
other		1		1
total	8	4	6	18

table 6.15: reasons for finding historical topics in *Time Travelers* difficult (n=29)

Lesson observations and discussion

Student reactions to integrating history are deduced from their personal comments and expressions written in the instructional units and spoken aloud in the lessons.

primary school

In general, the activities on Diophantos were a success in the sense that many students appreciated the riddles, and the strategy of halving the difference occurred regularly. However, it remains unclear whether the positive reactions were due to the recreational character of the tasks or the historical component. It seems that the role of the teacher in this matter must not be underestimated. For the *Apple Orchard* problem the experiences were less favorable. The text was too difficult for the students and did not stir their interest. In fact, quite a few students were demotivated by the historical context; they were very answer-minded and neglected the setting. For instance, at just one primary school the students took the questions on the characteristics of the text seriously. However, student responses to the *Hermit* problem varied from relatively positive to very positive.

secondary school

In the instructional unit *Time Travelers* a major – and also inevitable – stumbling block for the students were the unfamiliar measures in the section on the Rule of Three. This additional complexity sometimes caused students to respond in a negative way, like complaining about the tediousness or giving up altogether. Sometimes the teacher spent quite some time exploring and explaining historical facts and tried to create an interest by integrating historical facts himself, with a positive effect. For instance, quite a few students became more involved when they were invited to bring an abacus into the lesson and learned about the relation between the historical abacus and its contemporary counterpart. However, links to modern symbolism were hardly made or mentioned.

In brief, judging from the classroom data, the historical aspect of the mathematical activities clearly has some appeal, but not so much for the primary school student and it is dependent on the teacher's attitude as well.

teacher reactions

We have no information on how integrating history has affected the teachers personally. The questionnaire we asked them to fill out did not ask specifically for a reaction on using history, and we have not observed any particularities during the lessons. We only know that the teacher of school D expressed an interest in history of mathematics prior to the experiment.

design and research value

The integration of history of mathematics in education has been valuable for conducting research and design both at the start of the study, at intermediate moments of reflection, and towards the end. Since we elaborated on the first two stages in chapter 3 and chapter 5, we confine ourselves to the final phase of the study in this section. We have already discussed which lessons can be learned from history: it is best to conduct a careful and unforced introduction to algebraic symbolism, integrating reasoning and symbolizing is a complex issue (for instance, reasoning with symbolic expressions), and although some problems and methods were successful, reflecting on them and generalizing them was difficult.

There is also a disadvantage to searching for possibilities of reinventing the subject. In the case of this study we have focussed on problems and methods which support the learning of algebra as advanced arithmetic, which means we have neglected other openings to the learning of early algebra. For instance, it might have been productive to deviate from the ontological development to investigate the feasibility of geometrical-visual means.

6.7.7 Role of the unknown

When equations arise in an informal textual environment, the unknown is relevant and effective for problem identification and organization but not for the problem solving procedure itself. Frequently the unknown appears early on during horizontal mathematization activities, but as students begin to solve the problem, they follow an arithmetical procedure (not operating with or on the unknown). We can distinguish two different situations. First, there are cases where the solution strategy used is not algebraic. Since the actual level of understanding of the problem is pre-algebraic at most, we can say the algebraic symbolization is artificial. The unknown appears to have no relation with the problem. We have already discussed an example of such student work in section 6.7.2 (see figure 6.31). Another example is shown in figure 6.60 below, where the linear equation $x + \frac{1}{7}x = 19$ is solved arithmetically.

Since we cannot be certain to what extent this student operates on the unknown or just on the coefficients, we have not qualified it as (pre-)algebraic. The unknown is represented by an encircled cross, but in the draft area the student refers only to a number of parts instead of number of parts *of the unknown.* He explains: “19 is equal to $\frac{8}{7}$ part so $19 \div 8 = 2\frac{3}{8}$ that is $\frac{1}{7}$ part but you have to know 1 so you do $2\frac{3}{8} \times 7 = 16\frac{5}{8}$ $16\frac{5}{8} = \frac{7}{7} = 1$”.

In the last expression the student of course means to say that seven seventh part *of the unknown* is equal to $16\frac{5}{8}$.
In the case of simple equations of the form $ax + b = c$ the unknown is actually superfluous because no manipulation is necessary; the unknown can be determined by just inverting a series of operations.

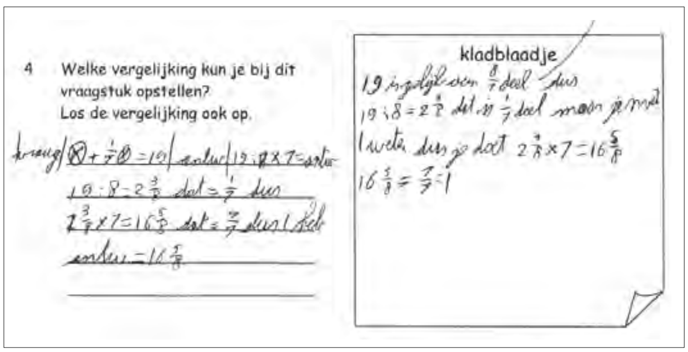


figure 6.60: unknown not involved in solution procedure

Second, we have observed student work where the unknown disappears from the solution procedure when the solution method is algebraic but the unknown is not involved. This phenomenon has already been discussed using figure 6.29 in section 6.7.2.

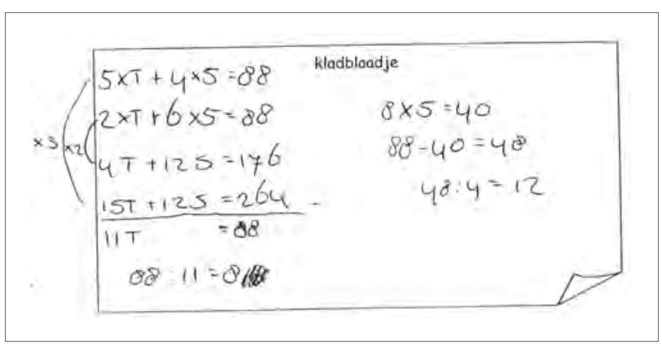


figure 6.61: equations and calculations separated

The unknown can be latent; if students focus their attention on the *coefficients* of the terms, they do not need the unknown to be present all the time. Usually the unknown reappears when the answer is written down. In addition we see that students keep the symbolic system of equations separated from the calculations, almost in two columns, as if the calculations are not a part of the system itself (note: the letter *s* in the equations must not be mistaken for the number 5) (see figure 6.61). These habits of ignoring the unknown and writing the calculations separately were not uncommon in the early days of algebra. Tropfke (1980, p.374) describes which symbols were used long ago to represent the unkown, at first without the possibility of transforming them in calculations (as we do, for example, when we speak about $\frac{1}{2}$ x). In Mesopotamia there was a preference for geometrical concepts, but these were also applied in a more abstract sense. For instance, adding terms of different geometrical dimension was commonly done. According to Tropfke, Diophantus was the first mathematician to really calculate with the unknown (Tropfke, 1980, p. 378). Remarkable is the fact that quadratic problems are frequent in Mesopotamia, sometimes in terms of rather complex systems of equations, whereas linear equations are rarely found (ibid., p. 386). Probably people thought linear problems in one unknown too obvious to elaborate on.

6.7.8 Recapitulation versus reinvention

It is not an unequivocal matter to determine if or when the observed learning trajectory demonstrates either a *recapitulation* of historical developments – showing clear parallels between the developments of the individual and the subject matter – or a *reinvention* of mathematics, where the natural development of the individual may diverge from the ontological growth of the subject to follow another – more efficient or natural – route. In most situations where we see a parallel, we also observe counter-examples or deviations, as illustrated below. Since we have already provided many examples of student work on the various topics in the sections 6.5.1 through 6.5.7, we have decided to discuss the matter in general terms as much as possible in this section.

(pre-)algebraic notations

Informal notations which students use to mathematize word problems show certain parallels with the historical development of algebra:

- use of abbreviations and symbols to describe the unknown, where the meaning of the symbols is ensured through the link with the context;
- non-linear progression of notations for equations but generally speaking we observe a trend of increasing efficiency and abstraction;
- pseudo-presence of the unknown: the unknown may appear at the beginning of the problem but is not an integral part of the solution process.

reinventing strategies for solving Diophantine number riddles

We have observed primary school students who reinvented the strategies ‘elimination’ and ‘halving the difference’ for the Diophantine number riddles (see section 6.7.2). Both strategies have been categorized as algebraic and show similarities with the Babylonian approach (as explained in section 4.5), but they are very different from the method Diophantus himself suggested (allocating a letter to the smallest number, expressing the second number in terms of the first using the given difference and then constructing an expression for the sum.) The result is a linear equation in one unknown, which can then be solved. A large number of secondary school students, too, has self-reliantly reinvented the strategy ‘halving the difference’ or a variant of it (determining first ‘sum – difference’, followed by taking a half), so this method appears to be well-suited for a pre-algebra trajectory.

linear equation in one unknown

Some strategies used by secondary school students to solve a linear equation in one unknown resemble methods employed by Ahmes nearly 4000 years ago. De Groot and De Jong (1928) observe that Ahmes tackles problem 24 in *Rhind Papyrus*, ‘A quantity whose seventh part is added to it becomes 19’, in the following way. He supposes the number is 7, calculates that $(1 + \frac{1}{7}) \times 7$ equals 8, en remarks that it must become 19. Then he multiplies 8 as often as needed to obtain 19, which happens to be $(2 + \frac{1}{4} + \frac{1}{8})$ times. Therefore the quantity must be $(2 + \frac{1}{4} + \frac{1}{8}) \times 7$. This method is the same as the student strategy solution ‘correction with proportions’ shown in figure 6.34. Sometimes Ahmes used another trial-and-adjustment method, namely by writing the problem as $(a + \frac{1}{7})\, x = p$ and then multiplying the number $(a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \ldots)$ as many times as needed to obtain p. The student solution in figure 6.49 is a combination of both of Ahmes’ methods. The first attempt is 14, after which the student observes that the result is 3 less than required, which means she needs to divide 14, 2 and 16 by $5\frac{1}{3}$. Just like Ahmes in his second method, this student retains the structure of adding up the parts. Ahmed used yet another method (which we have called ‘reasoning with parts’ in this chapter): determine the sum $a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \ldots$, divide p by the numerator and multiply this result by the denominator.

In other words: the methods of reasoning we observed amongst the students were also known thousands of years ago.

Generally speaking the secondary school students preferred to use their own arithmetical solution to solve linear problems of the type $ax = b$ rather than the Rule of False Position, although we have seen several instances of correction with proportions instead of a factor. The question we can ask here is: do students find this variant more natural, or do they find a method they invent for themselves more meaningful?

schematizing

We can speculate also on comparative roles of the arrow diagram and the cross diagram for the Rule of Three; both emphasize the dynamic, calculational aspects of the solution procedure ‘do this first, then do that, etcetera’. It is possible that these diagrams act as an intermediate learning phase in the development from arithmetic to algebra.

reinvention of old problems

Finally we can say that reinventing historical problems such as the *Hermit* task can liven up the classroom and really let students enjoy mathematics. We can only speculate whether such tasks were also acted out in role play and whether it had the same positive effect on the learners centuries ago.

6.8 Answers to the research questions

At the beginning of section 6.7 we operationalized the research questions by zooming in twice: from main research questions to sub-questions, and from sub-questions to conjectures. In this section we zoom out again.

6.8.1 Conjectures

Reasoning versus symbolizing
Algebraic reasoning and symbolizing have been found to develop independent of each other, whereby reasoning skills are more accessible to early algebra learners than symbolizing skills. The use of algebraic symbolism does not imply algebraic reasoning nor vice versa. Although pre-algebraic strategies and symbolizing as an intermediate level between arithmetic and algebra are not necessary for every student, we have found that they can help students to progress from arithmetic to algebra. Students who have difficulty taking this step seem incapable of symbolizing the problem effectively and tend to remain at an arithmetical level of notations.

Understanding relations
The emergence of an algebraic conception of numbers and relations between numbers is hampered by prevailing arithmetical notions that students have. In particular we have seen that students tend to reason with the given terms in the problem where they should be reasoning with the unknown, they violate the equivalence relation (symmetry/transitivity) in expressions, they misunderstand the meaning of letters or they make errors of inverting operations. In brief, students display an attitude of ‘doing something with the given numbers’ instead of penetrating the problem.

Regression of strategy use
We have not been able to conclude in general that students are more prone to a re-

gression in strategy use at an algebraic level than otherwise. There are some indications of a regression for certain particular methods of problem solving, though, but there seems to be no difference between boys and girls in this respect.

6.8.2 Sub-questions

The answers to the subquestions are based on the results numbered 1 through 8, including the outcome of the conjectures. Questions 1a through 1d deal with the gap between arithmetic and algebra, while questions 2a through 2c are concerned with the didactical value of history of mathematics. The answers are given in this order, immediately proceeding the question:

1a How do students conceive symbolic notations as a mathematical language, which type of shortened notations do children use naturally, and how do we obtain an acceptable compromise between intuitive, inconsistent symbolizations and formal algebraic notations?

Letters as abbreviations are very natural to students, but informal notations are not easily proceeded by formal notations. The results imply that the students at this age are not yet susceptible to formal symbolism. At the primary schools, for instance, we observed that students interpreted symbolic expressions as shorthand instead of an extension of the common arithmetical notion of numerical expressions. Symbolic language of secondary school students tends to be cognitively sound (correct interpretation of unknowns and relationships) but mathematically incorrect (like unconventional semi-symbolic expressions, or not writing down the unknown).

1b How can students actively take part in the process of fine-tuning notations and establishing (pre-)algebraic conventions?

At primary school level the students in this study were generally not able to formalize schematic and curtailed notations to a level where they support mathematical reasoning. And although these students agreed that their symbolism was sometimes unsuitable or ambiguous, they were not capable of changing their habits. Only the activities of constructing trade terms gave primary as well as secondary students the opportunity to participate in a democratic process of reaching agreement on symbolic conventions. In one class the seventh grade students also took active part in the formalization of systems of equations, perhaps due to the stronger guidance by the teacher. We must conclude that opportunities for involving young students in establishing algebraic conventions are limited.

1c To what extent and in what way can students become aware of different meanings of letters and symbols?

The present study shows that when students take active part in the development of a system of notations, these notations become meaningful to them, but the concept of variable has proven to be too ambitious all the same. Even when the activities are placed in a meaningful context (like exchanging coins and barter trading), primary school students are confused to see that letters can play different roles, and they do not see the relevance of finding multiple solutions or a generalized expression. In other words, at primary school level most students only perceived letters as objects, not as magnitudes.
In the secondary school activities the letters first refer to objects and then to unknowns; judging student achievement on solving systems of equations, this change has apparently not caused many problems. However, the symbolic representations students constructed do not always reflect this understanding (letters are sometimes used as unknowns and as objects at the same time).

1d Is there a correlation between the form of notation (rhetoric, syncopated, symbolic) and the level of algebraic thinking?

The conjecture *reasoning versus symbolizing* implies that for most students their level of symbolizing cannot compete with their level of reasoning. Classroom experiments indicate that students had little difficulty to reason about quantities and the relationships between them in a rhetorical manner. Alternatively, some students have shown to be very comfortable with symbolic notations – double perspective of letters by one student in the pilot experiment, barter trade terms and syncopated and symbolic notations for systems of equations – but have not succeeded in reasoning with and about these notations. So even though we cannot conclude that a lower level of notations automatically corresponds with a lower level of reasoning, we have found evidence that there are signs that the level of symbolizing makes up arrears as children mature.
At secondary school some students have shown a combination of algebraic reasoning and algebraic symbolizing, while symbolic notations at primary school level rarely reflected an algebraic level of reasoning.

2a What is the effect of integrating history in the mathematical classroom on the students, in particular their motivation and their learning process, and what is the possible influence of age, gender, intellectual level and the teacher?

To some students the historical problems were an eye opener and a nice change of scenery, while other students – mostly low achievers and students who enjoy traditional mathematics – did not find it interesting but confusing instead. At the primary schools the *Hermit* task was appreciated very much, while the secondary school students did not mention one activity in particular. Apparently the teacher influences to what extent the students find history helpful, because most grade 7 students responded positively while the majority of primary school students did not. We have not observed any notable differences between girls and boys at his time.

2b How does the learner's symbolizing process compare with the historical development of algebraic notations?

We have observed primarily rhetorical and syncopated (including iconic) notations; symbolic notations were often unconventional. Generally speaking the individual learner displays a gradual progression of symbolizing. Comparing symbolizing activities with the historical development of algebraic notations, we notice the following parallels and differences:

– Reasoning with the unknown precedes symbolizing and manipulating the unknown.
– Symbolic expressions, and the unknown in particular, is usually not integrated in the computational solution process.
– Problem presentation and the solution process are treated as separate parts of the problem.
– Modern day students accept letters and symbols in mathematics at an earlier stage than mathematicians did in the past, probably because they encounter signs at school and in the community at a very early age.

2c Which parallels, if any, do we observe between the development of algebraic thinking amongst individuals and the epistemological theory?

The long and slow development of algebra required the conquest of a number of cognitive obstacles that the individuals in this study also struggled with:

– Intuitively the students in this study tended to reason in an arithmetical manner, for example by reverting computations, using trial-and-adjustment methods, reasoning with proportional parts, i.e. reasoning with and about numerical values but not with unknowns or variables.
– The primary school students struggled to accept the notion and the visualization of an indeterminate quantity.
– At secondary school level the concept of unknown was more readily accepted but students have problems manipulating it, especially in new situations (like moving from simultaneous equations to a single-letter linear equation).
– Various cultures used the Rule of False Position in different forms and some extended it to include multiple unknowns; some secondary students in this study reinvented a variation involving proportions similar to the method Ahmes used in *Rhind Papyrus* (1650 BC).

6.8.3 Main research questions

The results summarized above in the conjectures and the sub-questions can now be used to answer the two main research questions:

1 *When and how do students begin to overcome the discrepancy between arithmetic and algebra, and if they are hampered, what obstacles do they encounter and why?*

The confirmation of the conjecture on reasoning versus symbolizing indicates that the gap between arithmetic and algebra can be bridged with pre-algebraic strategies and symbolizing – particularly for the topics of systems of equations, Diophantine number problems and the Rule of False Position – but not by all students. Some students are able to make the jump to the algebraic level without the support of a pre-algebraic phase. When students are not capable of symbolizing or schematizing the problem, or when they are not convinced of its surplus value, they tend not to be able to cross this gap. On the other hand, there is evidence that students do not always choose to solve problems algebraically if an informal, pre-algebraic strategy (for solving systems of equations) or sometimes even an arithmetical strategy (for linear problems) works equally well. In other words, a good understanding at pre-algebraic level does not automatically result in a progression to algebra.

The obstacles that students may encounter are

- *using symbolizing as a tool for mathematical reasoning*: Students do not automatically search for efficient, structurized methods and therefore not only fail to see a need for them but also they cannot make the step of reinventing symbolizations and models. They tend to get stuck on the mental processes involved because they lack the ability to use visual support;
- *applying newly acquired competence to another type of problem*: Particularly at primary school level students act as if each problem is a new problem: they have trouble recognizing problem characteristics and generalizing their approach. Perhaps we cannot expect novice learners to see the difference between the particular and the general at this stage;
- *symbolizing and operating on the unknown*: The indeterminate magnitude of the unknown is a real cognitive obstacle for students at primary school, which is why the rectangular bar has not been successful for them as a model. At secondary school level we have seen that students are quite successful at reasoning with the unknown but they have trouble expressing this reasoning on paper – as a result the unknown is usually only latently present;
- *misconceptions of numbers and relations between numbers*: This obstacle is related to several differences between arithmetic and algebra. First, the dilemma of a procedural (operational) versus a structural perception: the arithmetical, procedural notions that students have of numbers and relations caused them to interpret static expressions dynamically, making the reversal error and violating the

symmetrical and/or transitive characteristics of the equal-sign. Second, the arithmetical expectation to calculate with knowns directly causes students to operate on the given terms in the problem instead of the unknown;

- *switching between different meanings of letters*: Results from both the pilot experiment and the field test have shown that students struggle to have multiple perceptions of letters: as objects, as unknowns, as variables. Where students appear to understand the different meanings, they are not sure of themselves and they cannot explain their reasoning.

Many of these obstacles are described in table 2.1 in chapter 2.

2 *What is the effect of integrating the history of algebra in the experimental learning strand on the teaching and learning of early algebra?*

The teaching and learning effect of historical methods and problems in primary school is of a different order than in secondary school . At primary school level reflection on the problems and methods is not made explicit in the instructional materials but should be done by the teacher. In the field test the teacher at school A unfortunately did not pay much attention to the unspoken, implicit values of integrating history, and so we cannot conclude what the effect has been. At secondary school level students are asked more directly to comment on the historical aspects. Before the experiment started the teacher at school E expressed a special interest in using history. His enthusiasm and appreciation of integrating history has probably contributed to the positive responses amongst the students. The teacher has not mentioned his own experiences with the historical activities in the questionnaire. The learning effect of integrating history for the students is positive, although more so at secondary school level than at primary school level. Some problems and methods led to the emergence of pre-algebraic strategies which helped the progression from arithmetic to algebra – Diophantine problems: algorithm of halving the difference, Calandri’s Fish problem: (variants of) the Rule of False Position – but reflecting on these methods and problems turned out to be a major obstacle for the primary school students.

6.9 Personal reflections

At the end of the study it is a good thing to look back at how the study took its course and formulate a few personal observations on the results:

1 In retrospect, early algebra appears to be a very complex field of mathematics education for which to design a learning trajectory according to RME theory. The conflicts in the combination reasoning–symbolizing, the balance between accepting informal symbolism on the one hand and guiding towards a formal notion of symbolism on the other, and the confrontation between arithmetical and algebraic conventions make it especially difficult to apply the design heuristics at hand – stimulate the development of mathematics as a human activity, use in-

formal knowledge and methods, facilitate the emergence of models and progressive formalization, etcetera.

2 The turn-about of the study which was brought on by unproductive preliminary results and unforeseen personal circumstances has fortunately had a positive effect. The adjustments we made to the research questions and the educational design have produced a number of relevant and interesting results, recommendations and questions for further research.

3 The role of the teacher in offering structure, in stimulating students to reflect and in exploiting opportunities of further exploration must not be underestimated. For instance, the didactical qualities of the teacher at school A showed us the potential of the learning trajectory which we had not foreseen ourselves. Figure 6.62 shows a nice example of such an exploration of the first activity in the unit *Exchange.* The students are given a list with prices of 20 different candy bars (ranging between 5 and 95 cents), and are asked to write down what can be bought for precisely 1 guilder (100 cents).

> Several students, working on their own, reckon it is possible to know for sure, but it will take a long time.
> Observer: How do you know you haven't missed one out?
> A girl replies that in that case you are doing it wrong. Another girl replies that she would start at the top of the list, take one item and check all the possibilities, and then take the next item from the top of the list, and so on.
> Class discussion. The teacher asks for answers; some students give a numerical answer.
> Teacher: How do you know there are so many?
> Student: At some time there will be an end to all the possibilities.
> The class investigates all the possibilities in combination with potato chips; there are too many to write down.
> Teacher: How many possibilities altogether, do you think?
> A boy replies: 400.
> He then explains: he compared the problem with a comment the teacher made a week earlier, that there are as many as 520 possible simple sums with the first 20 natural numbers! And so, he concludes, there must be at least 400 in this case.
> Other students then suggest more than 1000 possibilities, but they would like to hear the exact number from the author of the booklet!

figure 6.62: how many combinations

Many students realize that the answer will require a lot of paper and decide to use abbreviations. Immediately there is an opportunity to talk about effective mathematical notation (letters, syllables, operator symbols, tabular forms). In the next question, students are asked to comment on a disagreement between two imaginary students: “I found all the possibilities for 1 guilder!”, one says; “But you can never know that for sure!”, the other says. In school A this activity instigated a lively discussion on the total number of possibilities (see figure 6.62). This example illustrates how an open problem can lead to a spontaneous exploration of the problem and the connection of mathematical knowledge.

4 We have been very surprised and concerned about primary school students' attitudes towards explaining one's answer or describing the solution procedure. Perhaps it is not only a matter of reluctance but also of inability, which means it should be given more priority in primary school teaching. Speaking from the researcher's experience as practicing mathematics teacher in secondary school, it would be a great improvement if students who start in the seventh grade already have experience in writing explanations with their answers.

7 Discussion and recommendations

7.1 Reflections on the study's turn-about

The peer review marks a crucial division of the study 'Reinvention of algebra' into an exploratory and a final phase. In the exploratory phase we stumbled upon certain limitations of our theoretical ideas and the educational design. For this reason we decided to organize an evaluation by a panel of experts. A number of interventions that this panel advised were carried out prior to the field test. In this section we summarize the effects that these adjustments have had, some of which also appear in the discussion.

1 *Connecting theory and practice*
The focus on continuities and discontinuities in arithmetic and algebra has led to the design of new pre-algebraic activities as well as an analytical framework. Some of the pre-algebraic activities have shown to be fruitful in helping students overcome the gap between arithmetic and algebra, and others turned out to be less successful, but either way they have provided us with good data. Our emphasis on contrasting properties of arithmetic and algebra has resulted in the formulation of some conjectures which have directed the analysis of student work and which have enabled us to answer the research questions.

2 *Integrating history*
Direct use of historical problems and methods in the instructional materials has enabled us to investigate the effect of integrating history in the classroom. At primary school level it has in some cases led to an increase of personal involvement in the mathematics lesson, but it has also caused confusion and loss of interest. At secondary school level most students responded positively. Through the reflection on historical developments and methods the secondary school students have also learned about advantages and disadvantages of arithmetical and algebraic problem solving.

3 *Symbolizing*
The decision to confine letter use to barter expressions has caused the conflict between static and dynamic conceptions of symbolic expressions to diminish, and as a result also the amount of unconventional symbolizing. Errors in symbolic expressions and stringing calculations using equal-signs are rooted in certain dominant arithmetical conventions, and they continue to be a problem.

4 *Problem solving*
Generally speaking the new activities succeeded in challenging the students, although the total length of the lesson series was too long. Eventually this caused some students to become saturated, particularly at primary school level. The more open character of the activities led to the invention of unexpected strategies

and symbolizations, and in some cases even opportunities of progressive formalization.

5 *Structure*
Although the coherence of activities and unit sections has become more obvious for ourselves and the teachers, most students did not recognize the ways that certain problem types reappeared in new settings and with new appearances at different moments in the lesson series. In addition the teachers remarked that the learning strand handled too many topics with too little room for practice, which also explains why students struggle to get a helicopter view. However, the unit summaries – which were greatly appreciated by the teachers – did succeed in giving students some structure.

7.2 Discussion of the results

For the present study we have narrowed our perspective on research on the learning and teaching of early algebra in three ways: a) by conducting a particular type of research: developmental research b) by adopting a specific approach to mathematics education: RME, and c) by choosing a confined research topic: the transfer from arithmetic to early algebraic equation solving, and the didactical value of history of mathematics in making this transfer. In the discussion we broaden our point of view again by reflecting on the results and comparing them with other findings in the field.

7.2.1 Symbolizing

Generally speaking, when students are asked to describe a problem situation using curtailed notations, they use a combination of words, abbreviations and symbols. At primary school level rhetoric notations are certainly most natural to students, while the secondary school students in this study displayed a better understanding of algebraic symbolism. These findings support Harper's classification of rhetoric-Diophantine-Viètan algebra according to the age of the students (Harper, 1987).

prior to the peer review

The present study indicates that nudging primary school students to use symbolic formulas is not productive, not even if it is done in a tentative and well-considered way. In the first phase of the study – prior to the peer review – the notations we proposed to the students did not lead to a (pre-)algebraic conception of quantities and the relations between them as we had anticipated, even though these notations were based on students' free productions. These students interpreted symbolic language as shorthand notations instead of an elaboration or generalization of the language of arithmetic. For example, students tend to construct expressions like $dB = +3\ dA$ instead of $dB = dA + 3$. Apparently students do not connect the state '3 more than' with the action 'adding 3', which is a necessity if students are to accept the simultaneous existence of processes and products in algebra. Although the former expression shows an attempt to symbolize the state '3 more than ', it is not a promising first step towards algebraic symbolism. In addition we have noticed that certain common errors of learning early algebra like violating the transitivity and symmetry of the equal-sign and the reversal error are also related to symbolic representations in particular. In other words, symbolic expressions do not imply a natural progression from arithmetic to algebra. Moreover, if students perceive symbolic language as shorthand notation, the static-dynamic controversy of algebra and arithmetic continues to be a problem.

after the peer review

In the final phase of the study the use of letter notations at primary school level was confined to trade terms, where letters refer to objects instead of magnitudes. Apparently symbolic trade terms do bring out the dynamical aspect of relationships between quantities (i.e. trading so much of this for so much of that), which corresponds more with arithmetical conventions. One obstacle of symbolizing that we decided to keep in the learning strand even after the peer review deals with changing meanings of letters. In order to investigate to what extent students can reason about different roles and meanings of letters and word variables – depending on the medium in which they are used – some activities on 'value of goods' versus 'number of goods' were included, of course not in a static, symbolic environment. Unfortunately we must report that even in a dynamic setting such as trading goods or money the switch between expressions of value (1 pineapple for 5 bananas) and expressions of number (number of pineapples = number of bananas : 5) continued to be an obstacle for students. We know of only one student who was able to perceive letters from two perspectives: as objects in the trade term, and as variables in a symbolic expression on the numbers of goods (see figure 6.41). This student created his own rule of thumb for the change in the numbers of goods, on the basis of a barter rule.

secondary school

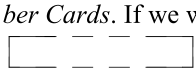
Students at secondary school level seem to be more susceptible to symbolizing activities. In fact, symbolizing has been quite a successful element of the learning strand, especially for solving systems of equations. Syncopated notations seem to be most natural to these students, although at one school symbolic equations (involving letters) were equally common. Results of classroom experiments in the *Mathematics in Context* project have indicated a similar spread of symbolizing amongst students solving systems of equations (Van Reeuwijk, 2000). Some students preferred a visual representation of the problem while others used letters, abbreviations or complete words. Many students invented their own shortcuts, which implies that symbolizing for these students was a natural activity.

symbolizing the unknown

At primary school level we have seen only a single spontaneous invention of an unknown, which has been a rather pleasant surprise. This particular student chose to express two restrictions with two unknowns in terms of just one unknown. It shows that some students might be receptive to symbolizing an unknown even at primary school level. Obviously we cannot expect students to invent and use informal notations if they have never done so before. At secondary school level students generally had little trouble inventing a symbol for the unknown, for which we can give two possible reasons. First, being one year older and having had one more year of mathematics, their mathematical knowledge is more developed. Second, students in Dutch secondary schools are introduced to simple letter manipulations in the seventh grade, which means that the experimental group had already encountered letters in mathematics before the field test began. Although most students accepted the idea of symbolizing an unknown quantity, many did not did not include the symbol in their solution procedure. In other words, we noticed a cognitively correct but mathematically incorrect use of the unknown. We have already observed in chapter 6 that this result agrees with the historical development of solving equations. Harper, too, found evidence that pre-algebra students (who have no symbolizing experience) tend to write their solution without a symbol for the unknown (Harper, 1987).

schematizing and modeling

The field test results show that students are not inclined to search for effective, structured methods of problem solving involving schematic diagrams or models. The empty number line, arrow diagram, bar and table helped only some of the students in solving test tasks; a similar number of students did not reach for these tools and failed. Some of these representations were useful during the lesson series as a model of specific situations (table, bar, arrow diagram) but did not emerge as model for mathematical reasoning in a convincing manner. One of the reasons for this poor result can be found in the students' arithmetical background. It seems that students have little experience in schematizing or visualizing problems. Despite the growing influence of RME, which advocates the use of models and solving open problems, the teachers admit that students only practice these abilities in familiar, typical situations. In other words, these students did not have the mathematizing competence we had expected.

Another reason could be the passive role of the students in the origination of the model. After all, various educational researchers have pointed out the importance of active participation in the emergence of models (Streefland, 1985; Treffers, 1987, Gravemeijer 1994, 2001, Van den Heuvel-Panhuizen, 1995). In other words, we need to look for opportunities of involving and stimulating students to organize problems using schematic representations. For instance, we might search for a geometrical setting to make the bar a more meaningful model, or to search for another linear model altogether. We have already suggested an adjustment to the bar in our reflection on the hypothetical learning trajectory for Diophantine riddles like *Number Cards*. If we were to include a phase where the bar has dotted lines, as follows: , the length of the bar has become indeterminate. This makes it possible to visualize strategies based on reasoning about an unknown length or quantity, such as 'halving the difference'.

other accesses to symbolizing

In conclusion, certain elements of the experimental learning strand have supported the development of algebraic symbolizing and do not require a substantial adjustment. Here we can think of reverse calculations, barter trade terms (to a certain extent), systems of equations and linear problems. However, since the dual static-dynamic perception of word formulas and symbolic expressions forms a major obstacle in the learning of early algebra, we must look for other accesses to (pre-)algebra for this part of the program.

1 We might base the introduction of general expressions on the dynamic, procedural action language which is common in arithmetic. This approach has been worked out in the algebra unit *Expressions and Formulas* of the *Mathematics in Context* project (*Mathematics in Context* Development Team 1998). In this unit arrow diagrams and tree diagrams are used to practice compiled computations and prepare students for the introduction of word formulas. The algebra program developed by the W12-16 algebra working group (see section 2.9) also advocates an approach to early algebra based on a dynamic perception of quantities and the relationships between them, for instance by generalizing arithmetical computations. Perhaps the unnatural, static conception of expressions has also been caused by the use of static contexts.
2 We should use spontaneous symbolizations and solution strategies as the foundation for our educational design. Students' free productions can act as signposts, anticipating a suitable learning trajectory (Streefland 1988, 1995abc, 1996a; Streefland & Van Amerom,1996). This means we should look amongst the less successful parts of the learning strand for signs of improvement. From this point of view, we have chosen to elaborate on the transition from pre-algebraic to algebraic reasoning on linear problems in the instructional unit *Time Travelers* in section 7.2.5.

7.2.2 Symbolizing versus reasoning

We have found substantial evidence that symbolizing and reasoning develop separately in the learning and teaching of early algebra. Some students can attain an algebraic level of reasoning – reasoning about unknowns– while the notations continue to be arithmetical. On the other hand, a student's level of symbolizing can be artificial, for instance when the mental processes are still arithmetical. The researcher's intention to design mathematical activities which facilitate the development of both symbolizing and reasoning has been partially effective, as we discuss in section 7.2.5. The differences in need of symbolic representations agree with Krutetskii's observations that some students are visually inclined while others prefer to use only mental processes (Krutetskii, 1976). In his analysis of mathematically gifted students we read that their level of reasoning is usually higher than their level of symbolizing, and only some students find a visual-pictorial approach helpful. Algebraic reasoning indeed appears to be more natural to students than symbolizing, which means we would be wise to exploit this ability.
However, the interaction between symbols and their meaning (see section 2.5) sug-

gests a need for a combined development of symbolizing and reasoning. After all, symbols obtain their meaning through using them in problem solving situations (Freudenthal, 1983; Sfard, 2000). Sfard suggests that the interactive process of symbolizing and mathematical discourse stimulates their simultaneous development. Meira, too, reports that children's competence at designing inscriptions emerges in their interaction with classroom circumstances (Meira, 1995). Mathematical tools like tables are transformed and adjusted according to the possibilities and limitations of the situation. According to Meira, "... a display designed on paper has the important function of shaping its designer's activity at the same time that the designer shapes the display itself" (ibid., p.310). The phenomena of evolving and shifting meanings of symbolism also underscore the importance of interaction between symbolizing and social practices in the classroom.

In other words, in spite of the natural mental approach to problem solving displayed by students, there are arguments in favor of an approach which incorporates activities of supplementary symbolizing.

7.2.3 Regression of strategy use

In this study we observed a regression of the use of certain types of strategies for solving (embedded) systems of equations and linear equations in one unknown. This regression occurred specifically in situations where students appeared to understand the most formal, algebraic strategy suited for the problem. A significant number of learners was not able to fall back on a less formal strategy to solve the task. This result underlines the principle of RME that the learning trajectory should enable the learner to retrace the route of development to a lower level of understanding. For the case of solving equations in particular, Kieran (1990) reports that several research projects have shown the importance of staying at the stage of informal solution strategies. According to Whitman (1976, in Kieran 1990), students who learn only informal methods for solving equations achieve better results than students who learn to solve equations formally as well. Apparently the formal manipulations have an impeding effect on earlier, informal abilities. The results are not unambiguous, however, since Petitto (1979, in Kieran 1990) found that students who were competent with both informal and formal strategies performed better than students who understood only one of the two, and this seems to contradict the outcome of Whitman's research. Finally, Lewis (1980, in Kieran 1990) observed that learners who once used trial-and-error methods to solve equations, stopped to use it as a checking device once they had learned a formal method. In other words, if students are flexible in the application of problem solving strategies, they are more likely to be successful than if they focus on only one particular strategy.

7.2.4 Discontinuities between arithmetic and algebra

problem solving strategies

Arithmetical and algebraic modes of problem solving differ in the way students represent and handle the given, the unknown and/or the general numbers in the problem, and the relations between these numbers. Booth (1984) distinguished six properties of arithmetical strategies which hamper the development of an algebraic outlook: arithmetical strategies are 1) intuitive 2) primitive 3) context-bound 4) they involve little or no symbolism 5) they are based on basic operations and 6) they usually involve only whole numbers. The third and fourth property provide new input for the discussion in section 7.2.3 and section 7.2.2 respectively. Regression of strategy use is often related to the difficulty of recognizing specific characteristics of a problem, knowing when to apply newly acquired abilities. We have observed that especially the primary school students had great difficulty recognizing isomorphic problems and generalizing their approach. Second, the lack of symbolism in arithmetical solution strategies helps to explain why students do not naturally engage in symbolizing activities to solve problems: they do not have much experience in using symbolic representations. Bednarz and Janvier (1996) describe how their research team's analytic framework helps to characterize arithmetical and algebraic problems and how this enabled them to compare arithmetical and algebraic reasoning processes. Bednarz and Janvier suggest that arithmetical problems are 'connected', meaning that the student can reason from the known to the unknown data directly with arithmetical reasoning. Algebraic problems, on the contrary, are labeled as 'disconnected' because they require reasoning with unknowns: the given information does not enable direct bridging between the knowns. They identified various types of arithmetical reasoning which are all directed at creating direct links between the data (given numbers or relations) in the problem: a) linking up a given number in the problem with relationship which applies to the unknowns b) creating a starting state by inventing a fictitious number and using the error to make a better attempt (2 variants) and 3) reasoning about the problem as a whole with the structure in mind. These types of reasoning were also encountered in this study. The first resembles the erroneous strategy we observed in the test tasks *Number Cards* and *Human Body*, the second is a trial-and-adjustment method like the ones we have seen for various tasks, and the third corresponds to the strategy 'reasoning with parts' for the task *Human Body*. Van Dooren, Verschaffel and Onghena (2001) conducted a study on the use of and attitude towards arithmetical and algebraic problem solving strategies amongst teachers in training. Their research results comply with the analytic framework proposed by Bednarz and Janvier. In other words, arithmetical and algebraic reasoning appear to be essentially different and can cause serious obstacles for the passage from arithmetic to algebra, as we discuss next.

typical errors of reasoning and symbolizing

The present study indicates that some typical errors of reasoning demonstrate an arithmetical conception of relationships between quantities where an algebraic one is needed. Bednarz and Janvier (1996) point out various obstacles that students encounter in interpreting algebraic reasoning. First, they report that using one unknown to generate directly the other quantities in the problem is very difficult for students. Second, students struggle to express one unknown in terms of the other (substitution). Third, Bednarz and Janvier remark that some students refused to operate on the unknown because they would not treat it as if it were known, and fourth, some students represented the relation the wrong way around (the reversal error).

conception of the unknown

The third and fourth obstacle appear also in our study but mostly at primary school level, for instance, the reluctance to use the rectangular bar as a linear model. At secondary school level, the reversal error occurred only in the simple ratio task of *Human Body*. Operating with and on the unknown in systems of simultaneous equations and in the linear problems ('number riddles') of the unit *Time Travelers* was not a problem for the majority of students, although the symbolic representations of these problems were frequently mathematically incorrect (which we discussed in section 7.2.1). We believe that the pre-algebraic strategies in *Fancy Fair* on simultaneous equations and the learning trajectory on linear equations in *Time Travelers* have helped students to take a valuable step towards algebraic reasoning by accepting the notion of an unknown number and being able to reason with it (see also section 7.2.5).

substitution

The first and second obstacle Bednarz and Janvier mention – expressing one unknown in terms of the other and reducing two operations to one – are also not so common in this study. First of all, the primary school students responded very well to the substitution of consecutive trade values and were able to simplify expressions to their simplest form. Of course we realize that the role of the letters can play a crucial role here; the letters in the trade terms are maybe arbitrary labels instead of unknowns, which means the students are dealing with concrete numbers instead of unknowns. But we like to point out that in the test task on the soccer competition in the pilot experiment, various students were able to combine two expressions (by substituting one of the variables) and then reason qualitatively about the relative values of the variables, without using numbers. At secondary school level some of the fish problems in *Time Travelers* required substitutions to reduce the number of unknowns to one, which were then solved arithmetically, but a few students applied the strategy 'reasoning with the whole' (i.e., reasoning in terms of one unknown) in the test task *Human Body.*

In conclusion, we recognize some of the obstacles of algebraic reasoning pointed out by Bednarz and Janvier (1996) but this study also indicates that certain informal, pre-algebraic activities might help students to overcome certain obstacles.

static versus dynamic perception of relations between quantities

It is generally known that students struggle to proceed from an arithmetical to an algebraic outlook on relations between quantities, in particular in symbolic expressions. Not only do students have trouble in understanding different roles and meanings of letters, they also have to deal with procedural (or operational) and relational aspects of the equal-sign, as we described in chapter 2. In this project we found that difficulties with letters occurred especially when primary school students interpret and use letters very literally as shorthand notation, as we observed in the first two classroom experiments. In such cases students produce unconventional (semi-)symbolic expressions. Procedural conceptions of the equal-sign in situations where a relational perception was required, was a problem both at primary school and at secondary school level, particularly in linear expressions in one unknown. Carpenter and Levi (2000) propose to tackle this problem by introducing young children to algebraic activities based on their understanding of number sentences. They have found that through the generalization of number sentences even young learners can begin to obtain a relational perspective of the equal-sign.
However, many grade 7 students also displayed correct symbolism and understanding of relations in the activities on systems of equations. Consequently we can say that the secondary school students are beginning to obtain a structural conception of quantities and the relations between them.

7.2.5 Indications of a suitable learning trajectory: crossing the gap

Even though a pre-algebraic phase is not required for the progression from arithmetic to algebra for every student, we have found that informal, pre-algebraic strategies and symbolizing can be helpful. The teaching-learning process observed in the classroom indicate a few promising and successful activities which anticipate a path of progressive formalization. For instance, the sub-strand on systems of equations has led to a flexible use of informal strategies and symbolism as well as the emergence of the formal method of elimination. And in section 4.5.3 we presented a hypothetical learning trajectory for Diophantine riddles based on the analysis of the test task *Number Cards*. We will now describe in more detail a possible development of algebraic reasoning for linear problems in one unknown.

informal strategies

In the unit *Time Travelers* students developed some informal strategies for solving linear problems in 1 unknown which we called ‘fish problems’ (see also section 6.7.2 where we discussed the test task *Human Body)*. Most students instinctively chose a strategy of arithmetical reasoning referred to as ‘reasoning with parts’. Another possibility is a trial-and-adjustment approach, where the adjustments are qualitative in nature (next attempt smaller or larger). A few problems later the students were introduced to the historical, pre-algebraic strategy ‘false position’, which is also a trial-and-adjustment approach but the adjustment is done with a factor of multiplication. A third strategy, ‘correction with proportions’, was invented by some students as a variation to ‘false position’. In the final section of the unit we take the step to early algebra: the *reasoning* needed for a linear problem is combined with *symbolizing* the problem as a linear equation. The context for this activity is also drawn from history, namely number riddles like problem 24 in *Rhind Papyrus*: “A quantity whose seventh part is added to it becomes 19”. At first the problem will generate an equation of the type $x + ax + bx + ... = s$, which can then be simplified to $kx = s$ and solved for x.

algebraic reasoning

Prior to the field test we envisioned a progression from an informal, meaningful strategy at arithmetical level (reasoning with parts of a fish) to reasoning about an unknown number at an algebraic level. The algebraic aspect is contained in the manipulation of the expression, as we will show using the student work in figure 7.1. The explanation in the draft area reads “19 is equal to $\frac{8}{7}$ part so $19 \div 8 = 2\frac{3}{8}$ that is $\frac{1}{7}$ part but you have to know 1 so you do $2\frac{3}{8} \times 7 = 16\frac{5}{8}$ $16\frac{5}{8} = \frac{7}{7} = 1$.” In section 6.7.7 we qualified this solution as arithmetical (reasoning with parts), based on the fact that the computations involve only the coefficients of the equation (the unknown does not appear in the solution process). This interpretation is a rather careful one; of course we do not wish to judge student work too highly. But now we reconstruct the student’s reasoning process from another perspective in order to illustrate the algebraic potential (for practical reasons we use formal symbolism):

a the student realizes that $x + \frac{1}{7}x = \frac{8}{7}x$;
b the student reasons that if $\frac{8}{7}x = 19$ then $\frac{1}{7}x = 19 \div 8$
c the student realizes if $\frac{1}{7}x = 2\frac{3}{8}$ then $x = 2\frac{3}{8} \times 7$.

It is the first step that clearly shows an algebraic manipulation, namely the addition of two terms with the unknown. In step b we might even place an intermediate step: if $\frac{8}{7}x = 19$ then $8 \times \frac{1}{7}x = 19$, which would indicate a second manipulation with the unknown. The cover-up method (covering up the unknown) then leads to the division 19 : 8.

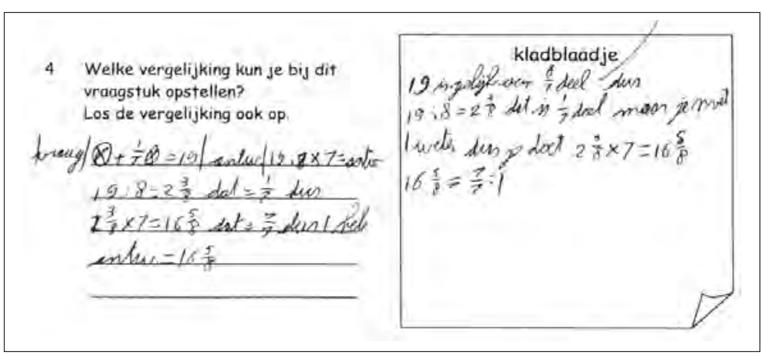


figure 7.1: reasoning with the unknown

In summary, we see here a possible progression from the arithmetical strategy 'reasoning with parts' to an algebraic counterpart. The direct relation between the two methods ensures that the algebraic strategy is meaningful, which makes it a suitable alternative to the more common, formal strategies of solving linear equations like performing the same operation on both sides or transposing ('change side – change sign').

7.2.6 History as a didactical tool

History contributes to the learning and teaching of early algebra by offering a number of pre-algebraic methods that can intermediate in the transition from arithmetic to algebra. It can also be used to let students reflect on their own methods. On the issue of student reactions we found that there is a mixed appreciation of historical problems which is not gender-bound. On the one hand history is welcomed as a change in routine exercises, on the other the low attainers mostly find it confusing and difficult, rather than interesting. Students are generally more positive when the teacher himself is enthousiastic about integrating history in this way.

Although many researchers investigated the use and value of history in mathematics education, and mathematics teachers have become increasinly interested, not much systematic work has been done on the assumed positive effect on the quality of learning and teaching (Gulikers & Blom, in press). In a survey of recent literature on the use and value of history in teaching geometry Gulikers and Blom conclude that the contributions to the discussion appear to be isolated. There are many proposals, but they do not fit into a larger framework and they lack a legitimation of ideas and suggestions. According to Gulikers and Blom, there is a discrepancy between theoretical (historians) and practical (teachers) perspectives and most authors emphasize the mathematical content without justifying their method (choices they made, didactical strategies they used).

Our contribution to methodological and theoretical issues of integrating history in the mathematics classroom is the following:

1 The combination with special activities, such as the dramatic acting out in the *Hermit* problem, is a strong asset;
2 Differentiation must be envisaged. Activities which have a strong appeal for high attainers, such as deciphering old documents, can totally obscure the underlying mathematics for the lower attainers;
3 As in most subjects, it is very important to stimulate the teacher and keep him well-informed. A teacher who believes in integration of historical elements (as in school D) can do wonders;
4 History should not be the ultimate goal; in the design, if it leads to problematic texts, it is better to change to a non-historical alternative.

7.2.7 Reinvention versus recapitulation

Having compared individual developments of students with the historical development of algebra, we have found evidence of a few parallels between the two. Modern students show preferences, progression and limitations with respect to symbolizing which can also be found in history. Some of the problem solving strategies, too, have been recapitulated by students in the field test, while other strategies took a slightly different turn when they were reinvented.

Harper (1987) conducted a study to investigate whether the solution strategies used by students to solve Diophantine problems on sum and difference coincided with the three historical periods of algebra (rhetorical-Diophantine-Vietan). Harper found that the pupils' responses show a preference for solution type which agrees with the historical development of algebra. Younger students who have no algebraic experience tend to solve the task rhetorically: with the method of halving the sum and the difference written out in longhand for a specific numerical case, not using unknowns. Slightly older students, who have just a little experience with symbolic algebra, use either the rhetorical method or one or two equations with letters for the unknowns. Only some of the oldest students were able to symbolize a general method of equation solving, using letters also for the given numbers as Viète did. If we compare these results with the findings of this study, we see that the rhetorical solution of Harper's pupils coincides with the strategy 'algorithm of halving the difference' and their notations are also of a similar nature. We have also found a higher level of symbolizing at secondary school level – in particular with respect to the acceptance of a symbol for the unknown – but we cannot conclude anything on spontaneous strategies at this level.

7.2.8 Further research

The study 'Reinvention of early algebra' has produced a number of results on continuities and discontinuities between arithmetic and algebra with respect to reasoning and symbolizing, some of which have a more solid foundation than others. It might be worthwhile to continue the study to look for new evidence by carrying out new cycles of theory formation and classroom experiments. We have also discussed some of the weaknesses in the learning trajectory which would need to be revised; these adjustments also call for further research.

We have listed some conjectures, ideas and issues of dispute for further research in arbitrary order below:

1 Conjecture: When equations arise in an informal textual environment the unknown is relevant and effective for problem identification and organization, but not for the problem solving procedure itself.
2 Is it possible to tackle the problems concerning strings of calculations and incor-

rect positioning of the operator in a symbolic expression simultaneously?
This was a research question which arose after the pilot experiment, but due to the rigorous adjustments made to the learning trajectory this question was no longer relevant for this study.

3 What kind of teacher skills and knowledge are prerequisite for the successful implementation of the proposed program?
We hypothesize that the teacher a) must be competent in the fields of arithmetic and algebra and the learning and teaching of these b) should advocate RME theory c) be interested in the use of history of mathematics and d) be tolerant of the idea that in a pre-algebra phase informal, unconventional symbolizations are acceptable.
4 Does the experimental learning strand lead to an improved understanding of solving equations formally at a later stage, as we believe it should?
5 International research reports that there is evidence that girls prefer to use and perform better using standard strategies, where boys are more capable of intuitive problem solving strategies (Van den Heuvel & Vermeer, 1999). In the present study we observed a few trends of gender differences that have not been discussed in the results for lack of evidence, but which could be investigated further:
   a Girls have shown to be more consistent in the use of notations and strategies than boys, and they choose to use the same strategy more often than boys;
   b Is it possible that this preference for one strategy causes girls to be less flexible in new situations, resulting in a regression of problem solving ability?
   c Can the didactical choices of the teacher influence developments of symbolizing and reasoning amongst boys and girls, for example, by stimulating either a systematic or a creative and flexible approach?
   d It seems that girls have more courage to symbolize freely when the lessons are not so structured, as if the teacher's strictness causes especially the girls to opt for an algorithmic strategy.

7.3 Recommendations for teachers and educational designers

As a supplement to the discussion we present a list – in random order – of recommendations for the teachers and educational designers. These recommendations are based not only on the results of this study but also on the teacher's own teaching experiences and discussions with the participating teachers and fellow researchers.

1 We advise designers and teachers not to start with symbolic algebra too quickly, because formal symbolism is difficult for the novice learner and informal symbolism can interfere with algebraic conventions. It is recommendable to let students develop symbolizing and meaning simultaneously; meaningless symbolizing appears to stimulate problems like shorthand notations, the reversal error and

stringing calculations. Starting from informal knowledge and methods will give symbols meaning, but the teacher should be responsible for guiding informal notations towards a generally acceptable (pre-)algebraic form; acceptable to the teacher, to the students, and to professional practitioners of algebra. It is of course undesirable that students develop habits that need correcting at a later stage.

2 We were surprised to see that students think a longhand explanation is more clear to the reader than the (semi-)symbolic representation of their actual reasoning, as shown in figure 7.2. Learners need to learn and practice how symbolism can be an integral part of their reasoning process, in order for them to interact in their dual development. It would be good to design activities where symbolizing the solution procedure is at least as important as finding the solution. As professionals we must not underestimate the difficulty and try to keep it simple at first, like organizing the numerical computations of a typical arithmetic problem in a logical order.

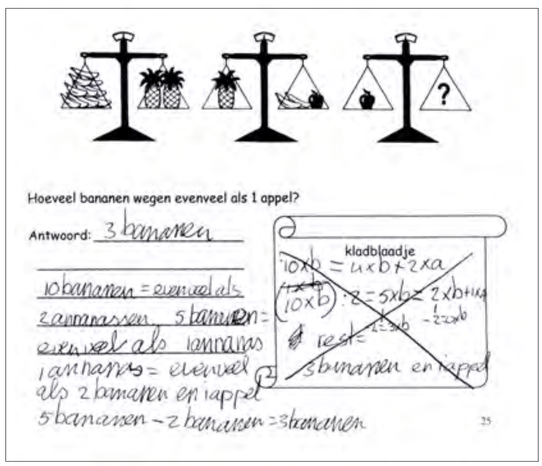


figure 7.2: symbolic notations for reasoning versus rhetorical explanation

3 We advocate the design of mathematical activities which make structurizing and schematizing meaningful, useful and perhaps even necessary. This will make students realize that models and schemata can be helpful in solving problems.

4 We recommend teachers and designers not only to utilize students' natural reasoning abilities, but also to provide structure for these abilities by making students reflect on their mental strategies. Algebraic reasoning is accessible through the development of many pre-algebraic skills like comparing and substituting quantities, reverting calculations, etc., but if the instructional materials are too open and if they lack coherence, these abilities will not help the students formal-

ize their mathematics. We feel that more priority must be given to the development of socio-math norms in order to create a situation where students' strategies are lifted to a higher level. We suggest that the organization of such reflective activities takes place in classroom discussions as much as possible, particularly at primary school level, because we have found that some students do not know what is expected of them and many others take reflective questions too lightly. Contrary to written tasks, a classroom discussion enables the teacher to guide and stimulate students in this process.

5 We propose to confront students deliberately with certain well-known obstacles of learning algebra – like the interference between natural language and algebra, the reversal error, different meanings of letters and the equal-sign, and other discontinuities between arithmetic and algebra – because it seems impossible to avoid or to prevent them. Especially the relational conception of the equal-sign is a persistent learning difficulty. Carpenter and Levi (2000) have promising results with classroom actvities on generalizing number sentences, where young students acquire a relational perspective of the equal-sign. In addition students should be made aware of the difference between strict mathematical equivalence and the frequently subjective interpretation of equivalence in daily life. This seems to be prerequisite to understanding the formal meaning of equivalence in the context of equations.

6 In the classroom try-outs we observed that most students are satisfied when they solve a problem and they trust that they have done it correctly. Students were often surprised or annoyed to be asked to check their solution, for example by testing the values with the relations at hand. Apparently students are not used to this very simple and direct type of feedback. Double checking is particularly useful for solving equations and for avoiding reversal errors when constructing (word) formulas. The experience of success may also give weaker students more security and confidence.

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Samenvatting

Dit uit twee delen bestaande boek vormt de afronding van het aan het Freudenthal Instituut uitgevoerde ontwikkelingsonderzoek 'Reinvention of algebra'. Het eerste deel schetst de achtergrond van het onderzoek, met een beschrijving van de probleemstelling (hoofdstuk 1), de leerproblemen rond algebra (hoofdstuk 2), de geschiedenis van de algebra (hoofdstuk 3) en methodologische kwesties (hoofdstuk 4). Het tweede deel doet verslag van het ontwerpproces van de experimentele leergang (hoofdstuk 5), de resultaten (hoofdstuk 6) en een discussie van deze resultaten (hoofdstuk 7).

Hoofdstuk 1 belicht de overwegingen die aan het onderzoek 'Reinvention of algebra' ten grondslag lagen.
Het algebraonderwijs staat momenteel sterk in de belangstelling. Docenten, onderzoekers, ontwikkelaars en opleiders vragen zich onder meer af waardoor het komt dat zoveel kinderen moeite hebben met het rekenen met letters. In een poging om meer vat te krijgen op deze problematiek, is inmiddels al een jarenlange discussie gaande. Om de leerproblemen op te lossen boog men zich over kwesties als: Waarin verschilt het vak algebra van rekenen? Welke algebraïsche onderwerpen zou men, op welk moment, op welke wijze aan leerlingen moeten aanbieden? Op deze vragen is tot nu toe geen eenduidig antwoord gegeven, en dat vormde de motivatie om te onderzoeken of er misschien een nieuwe onderwijsaanpak mogelijk is waarin het verband rekenen-algebra centraal staat.
Een tweede motivering van dit onderzoek komt voort uit de toenemende belangstelling voor het gebruik van geschiedenis van de wiskunde in de klas. Hoewel er aanwijzingen zijn dat historische elementen in het onderwijs een positieve uitwerking hebben op de interesse en betrokkenheid van de leerlingen, is het nog niet duidelijk of de geschiedenis ook een rol van betekenis speelt bij het ontwikkelen van wiskundig inzicht.
Het project 'Reinvention of algebra' heeft deze beide aspecten – leerproblemen op het gebied van aanvankelijke algebra en de didactische waarde van de geschiedenis van de wiskunde – gecombineerd in één onderzoek. Aan de hand van een lessenreeks over vergelijkingen voor 11- tot 14-jarigen is ontwikkelingsonderzoek uitgevoerd rond de overgang van rekenen naar algebra en de rol die de geschiedenis daarbij zou kunnen spelen.

Hoofdstuk 2 beschrijft de problematiek rond het leren van algebra en in het bijzonder de overgang van rekenen naar aanvankelijke algebra.
Afgelopen decennia hebben verschillende onderzoekers geschreven over de aard van algebra, welke onderdelen van algebra (minimaal) in een leerplan thuishoren en

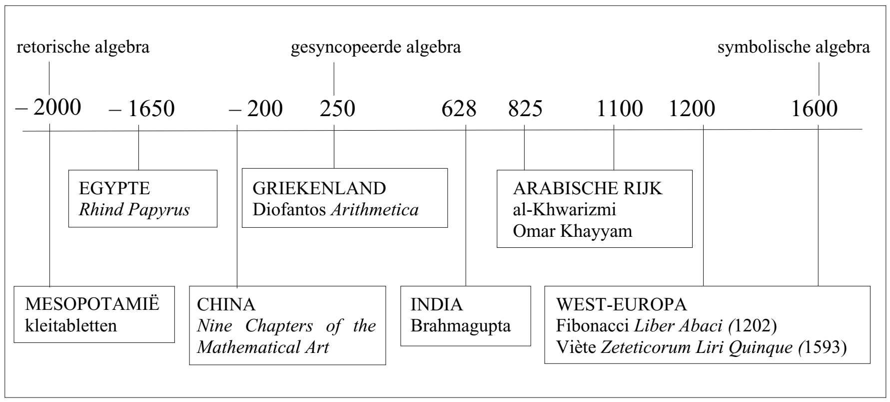
op welke wijze leerlingen met algebra aan de slag zouden moeten gaan. Uit werk van Kieran en Bednarz blijkt dat over deze kwesties geen eenduidige mening bestaat, maar dat een classificatie van algebraïsche kenmerken wel zinvol en nuttig is voor een discussie over algebraonderwijs. Bednarz verdeelt de te leren algebra in vier verschillende stromingen: generaliseren, probleemoplossen, modelleren en werken met functies. Het leren van algebra door te generaliseren richt zich op uitbreiding van de rekenkundige kennis van getallen en getallenrelaties. Bij een benadering van algebra vanuit probleemoplossen ligt de nadruk op het opstellen en oplossen van vergelijkingen. Bij modelleren gaat het vooral om de wiskundige beschrijving van dagelijkse verschijnselen, en komen wisselingen tussen representaties – grafieken, tabellen, formules – veelvuldig aan bod. Relaties tussen veranderlijken spelen hier – net als bij het werken met functies – een belangrijke rol. Bij algebra als studie van functies wordt gekeken naar de didactische mogelijkheden van computers en rekenmachines. Bednarz is wel van mening dat deze classificatie star en onvolledig is, en dat elk van de vier thema's op school zal moeten worden aangesneden.

De voornaamste leerproblemen van leerlingen bij aanvankelijke algebra hebben betrekking op de overgang van aritmetische naar algebraïsche conventies, de betekenis van letters en de herkenning van structuur. Veelgenoemde obstakels zijn de betekenis van het gelijkteken, het omzetten van een beschrijving in een vergelijkingen of een stelsel vergelijkingen, en het manipuleren van symbolische vormen. Sfard schrijft de leerproblemen gedeeltelijk toe aan het feit dat wiskundige denkbeelden op twee fundamenteel verschillende manieren kunnen worden opgevat: als *processen* of als *objecten*. Zij maakt daarmee onderscheid tussen respectievelijk een zogenoemde 'operational conception' en een 'structural conception'. In de algebra moeten symbolische vormen enerzijds als processen en anderzijds als objecten worden gezien. Zo kan bijvoorbeeld een vorm als $3x + 5$ betekenen 'vermenigvuldig x met 3 en tel daar 5 bij op' (een proces), maar tegelijkertijd is $3x + 5$ ook de uitkomst van de berekening (een object). En aangezien een leerling bij rekenen alleen de operationele opvatting tegenkomt, is er hier sprake van een obstakel in het leerproces.

In het huidige reken-wiskundeonderwijs op de basisschool in Nederland wordt in enkele gevallen voorzichtig een aanzet gegeven tot pre-algebra, maar dit gebeurt niet op systematische wijze. Een rekenkundige ingang van algebra sluit aan bij het niveau van de leerlingen van groep 8 en maakt het mogelijk om historische bronnen te gebruiken. De historische ontwikkeling toont aan dat algebra lange tijd beoefend werd als 'advanced arithmetic', en dat rekenkundige vraagstukken aanleiding kunnen geven tot algebraïsch redeneren. Ook leert de geschiedenis dat symboliseren en algebrataal geen noodzakelijke voorwaarden zijn voor algebraïsch redeneren, dus kunnen leerlingen zonder veel voorbereiding hieraan beginnen en zodoende zelf tot verkortingen komen.

Uitgangspunt voor het ontwerp van de experimentele leergang zijn activiteiten die de overgang van rekenen naar algebra zouden kunnen vereenvoudigen. Wat het niveau betreft, is gekozen voor een beperking tot lineaire verbanden en het oplossen van stelsels vergelijkingen die ‘verpakt’ zitten in een plaatje of een beschrijving. Onder meer worden vaardigheden ontwikkeld als hoeveelheden vergelijken, symboliseren, operaties omkeren, patronen herkennen en verschillende representaties gebruiken. Bij het aanleren van algebraïsche notaties, het symboliseren, is het van belang dat de nieuwe symbolen van meet af aan betekenisvol zijn voor leerlingen.

In het *derde hoofdstuk* wordt de geschiedenis van de algebra besproken.
Bij de keuze om geschiedenis van de wiskunde in te zetten als didactisch gereedschap is verondersteld dat het gebruik van geschiedenis een positieve uitwerking heeft op het onderwijs. De geschiedenis plaatst de leerstof in een breder perspectief. Leerlingen kunnen ontdekken dat wiskunde door toedoen van de mens verandert en groeit, en dat zij zelf dus ook een bijdrage (kunnen) leveren. Door naar de geschiedenis van de wiskunde te kijken, zien leerlingen hoe culturele en sociale aspecten de ontwikkeling van de wiskunde beïnvloeden, en hoe wiskunde met andere disciplines samenhangt. Niet alleen leerlingen maar ook docenten, ontwikkelaars en onderzoekers kunnen profiteren van enige historische kennis. Daarmee kunnen zij soms anticiperen op de moeilijkheden die leerlingen te wachten staan. Maar ondanks de groeiende belangstelling voor de integratie van geschiedenis in de wiskundeles en de positieve ervaringen, beschreven in de ICMI studie over geschiedenis in het wiskundeonderwijs, is er nog niet veel bekend over de feitelijke opbrengst van geschiedenis voor het onderwijsleerproces. Deze studie poogt te onderzoeken welke rol de geschiedenis kan spelen bij de ontwikkeling van wiskundig inzicht op het vlak van aanvankelijke algebra.
In de loop der jaren hebben verschillende ontdekkingen en veranderingen een bijdrage geleverd aan de ontwikkeling van de algebra. Men onderscheidt doorgaans drie fasen die overeenkomen met de soort notatie die werd gebruikt: retorische (beschrijvingen in de natuurlijke taal), gesyncopeerde (beschrijvingen vermengd met afkortingen en wiskundige symbolen) en symbolische algebra (de moderne algebraïsche symbolentaal). In onderstaande figuur is een globaal verloop van de ontwikkeling van de algebra te zien. Diofantos (250 na Chr.) noteerde als eerste de onbekende met een eigen symbool, waarmee hij rekende als ware het een getal. Vanaf de dertiende eeuw werd de gesyncopeerde algebra in West-Europa verder ontwikkeld; zo stelde Robert Recorde in 1557 voor om ‘gelijkheid’ aan te geven met het hedendaagse gelijkteken ‘=’. Het duurde tot het eind van de zestiende eeuw voordat Viète op het idee kwam om niet alleen onbekenden maar ook gegeven getallen te symboliseren met letters, zoals wij dat nu gewend zijn.



De algebra in de oudheid zou men kunnen typeren als een geavanceerde manier van rekenvraagstukken oplossen. De Egyptenaren en de Babyloniërs kenden enkele rekenmethodes om problemen met één onbekende aan te pakken, bijvoorbeeld de 'regel van drieën' voor verhoudingsproblemen en de 'regel van onjuiste aanname' (*regula falsi*) voor vraagstukken die te herleiden zijn tot de vergelijking $ax = b$. De Chinese tekst *Nine Chapters on the Mathematical Art* (ca. 200 voor Chr.) is tot op heden de oudste wiskundige verhandeling waarin een algemene methode wordt beschreven voor het oplossen van een stelsel van n vergelijkingen met n onbekenden (n = 2, 3, 4, 5). Indiase en Arabische wiskundigen – waaronder Al-Khwarizmi, wiens werk *Hisab al-jabr w'al-muqabalah* (ca. 825 na Chr.) de oorsprong is voor het woord 'algebra' – hebben een belangrijke rol gespeeld bij de verdere ontwikkeling van een systematische aanpak voor het oplossen van vergelijkingen. Na 1600, toen getallen eenmaal in letters konden worden uitgedrukt, stond algebra niet langer alleen in dienst van het oplossen van wiskundige vraagstukken, maar werd zij een op zichzelf staand vakgebied in de wiskunde.

De geschiedenis van de wiskunde is een belangrijke bron van inspiratie geweest voor dit promotieonderzoek. In de experimentele lessenreeks zijn enkele belangrijke elementen uit de historische ontwikkeling van de algebra gebruikt.

1 De nadruk van de leergang ligt op het oplossen van woordproblemen.
2 De activiteiten zijn te omschrijven als 'geavanceerd rekenen' en sluiten goed aan bij informele oplossingsstrategieën die leerlingen op de basisschool al beheersen.
3 Ruilhandel speelt regelmatig een rol als natuurlijke en realistische context voor het vergelijken van hoeveelheden.
4 Historische methodes en vraagstukken nodigen leerlingen uit om na te denken over de voor- en nadelen van verschillende manieren van oplossen.

Hoofdstuk 4 gaat in op het onderzoeksplan en de methode van onderzoek die zijn gehanteerd. De onderzoeksvragen zijn:

1. Op welk moment en op welke wijze in de experimentele leergang overwinnen leerlingen 'tegenstrijdigheden' tussen rekenen en algebra, en indien dit niet lukt, welke obstakels komen zij tegen en waardoor worden die veroorzaakt?
2. Welk effect heeft de verwerking van geschiedenis van de wiskunde in de experimentele leergang op het doceren en leren van aanvankelijke algebra?

Met deze vragen als leidraad worden twee hypothesen gesteld.

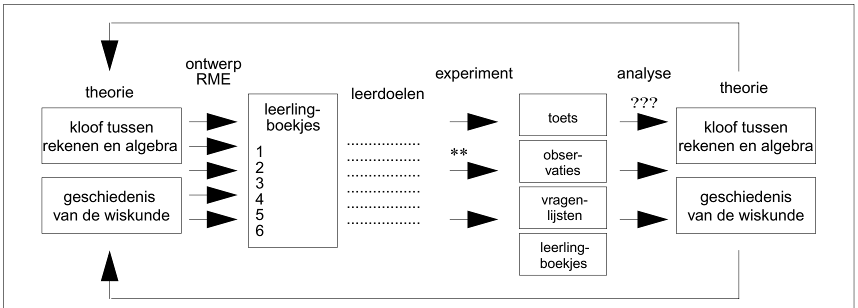
1. *Leerlingen zijn in staat om via pre-algebraïsche activiteiten de kloof tussen rekenen en algebra te overbruggen.*

   Deze hypothese is gebaseerd op eerder onderzoek en een vooraf bedacht leertraject.
2. *De geschiedenis van de wiskunde heeft een positieve uitwerking op het leren en doceren van aanvankelijke algebra, niet alleen vanwege de bredere kijk die het leerlingen geeft op de lesstof, maar ook vanwege het intermediërende (pre-algebraïsche) karakter en de meta-cognitieve toepassing van de geselecteerde historische methodes en vraagstukken.*

   De onderzoeksvragen zijn vervolgens opgedeeld in enkele deelvragen die aan de hand van de resultaten in hoofdstuk 6 worden beantwoord. De deelvragen bij onderzoeksvraag 1 hebben betrekking op algebraïsch symboliseren en de samenhang tussen het niveau van symboliseren en het niveau van redeneren; bij onderzoeksvraag 2 wordt gekeken of factoren als leeftijd, geslacht, intellect en de docent van invloed zijn, en of er overeenkomsten zijn tussen de algebra die leerlingen ontwikkelen (het individuele leerproces) en de historische ontwikkeling van algebra als wiskundige discipline.

   De studie gaat uit van de onderwijstheorie die bekend staat als Realistic Mathematics Education (RME). RME kenmerkt zich onder meer door de opvatting dat wiskunde wordt gezien als menselijke activiteit, dat men in het wiskundeonderwijs moet voortbouwen op wat leerlingen al weten en kunnen, en dat wiskundige problemen op meerdere niveaus moeten kunnen worden opgelost. Bij het ontwerpen van lesmateriaal spelen principes als 'guided reinvention' en didactische fenomenologie een belangrijke rol.

   De aard van het onderzoek is te typeren als ontwikkelingsonderzoek. Het is een combinatie van onderwijskundig onderzoek en leerplanontwikkeling die resulteert in een onderwijstheorie. In een cyclisch proces van theorie en praktijk wordt de leergang meerdere malen uitgeprobeerd en aangepast; bij de rapportage hierover is het van belang dat buitenstaanders het leerproces van de onderzoeker kunnen nagaan. Het onderzoek 'Reinvention of algebra' omvat drie volledige cycli van ontwerpen en uitproberen en nog enkele aanvullende groepsactiviteiten. Onderstaande figuur illustreert de opbouw (gelezen van links naar rechts) van de laatste onderzoekscyclus.



** observaties in de klas en gesprekken met leerlingen en docenten tijdens het experiment

Het project laat zich het beste omschrijven als ‘exploratief’ onderzoek aangezien de nadruk ligt op de experimentele fase van ontwikkelingsonderzoek, met een voorlopig ontwerp, kleinschalige experimenten en de aanzet tot enkele theoretische ideeën. Er zal meer onderzoek nodig zijn om tot een definitieve leergang en een empirisch onderbouwde onderwijstheorie te komen.
Aan het eind van het hoofdstuk krijgt de lezer de gelegenheid om aan de hand van een uitgebreide analyse van een toetsopgave in de huid van de onderzoeker te kruipen. Het geheel van oplossingsstrategieën voor deze opgave leidt vervolgens tot een mogelijk leertraject voor dit type vraagstukken.
De doelgroep voor de resultaten van de studie ‘Reinvention of algebra’ bestaat uit onderzoekers, ontwikkelaars, docenten en opleiders uit zowel basis- als voortgezet onderwijs.

In *hoofdstuk 5* wordt het ontwerpproces van de experimentele leergang beschreven. De eerste ideeën voor het lesmateriaal zijn uit drie bronnen afkomstig, te weten de geschiedenis van de wiskunde, de algebra-leerlijn in het project *Mathematics in Context* en enkele onderwijsexperimenten van Streefland. Een mathematisch-didactische analyse van het onderwerp vergelijkingen oplossen heeft een kern van vaardigheden en inzichten als vereiste voorkennis voor het leren van algebra opgeleverd, waarbij vervolgens een reeks wiskundige activiteiten is ontworpen voor leerlingen in het basisonderwijs en in de brugklas. In het eerste experiment zijn de activiteiten uitgeprobeerd met tweetallen leerlingen uit groep 8 en daarna in aangepaste vorm verwerkt tot een aantal leerlingenboekjes. Deze boekjes zijn vervolgens in twee klassen getest en grondig geëvalueerd. De eerste ronde van uitproberen was vooral bedoeld om feedback te krijgen over de haalbaarheid en geschiktheid van de activiteiten, terwijl de bedoeling van de tweede cyclus was het beoogde leerproces te vergelijken met het feitelijke leerproces dat in de klassen heeft plaatsgevonden. Tegelijkertijd vonden in de brugklas enkele lessen over het oplossen van stelsels verge-

lijkingen plaats, en over enkele pre-algebraïsche methodes en problemen uit de geschiedenis van de wiskunde.
De meest opvallende conclusie van de onderwijsexperimenten op de basisschool luidt, dat algebraïsch redeneren en symboliseren zich als twee losstaande vaardigheden bij leerlingen hadden ontwikkeld. Het redeneren gaat de meeste leerlingen beter af dan het symboliseren; vooral activiteiten over het vergelijken van hoeveelheden en het substitueren van ruilvoeten verlopen goed. Daarentegen blijken leerlingen symbolische vormen – die aan eigen producties ontleend zijn – te interpreteren als een soort steno in plaats van een notatie voor gegeneraliseerd rekenen. Zo wordt het verband ‘3 meer dan’ in symbolen uitgedrukt als $dB = +3\ dA$ in plaats van $dB = dA + 3$. Uit onjuist gebruik van het gelijkteken en omgekeerde interpretaties van bewerkingen blijkt dat de meeste leerlingen blijven steken in een procedurele manier van denken. Met tabellen werken is lang niet zo vanzelfsprekend voor leerlingen als verwacht, zodat het inzetten van tabellen als hulpmiddel voor redeneren niet van de grond komt. Ook het redeneren over relaties tussen hoeveelheden blijft voor de meeste leerlingen buiten handbereik.
De ervaringen in de brugklas zijn positiever, alhoewel ook hier allerlei onvolkomenheden van symboliseren zijn opgemerkt. Leerlingen zijn niet zo snel bereid om onbekenden te symboliseren als verwacht; meestal wordt de onbekende gewoon weggelaten. Er bestaat een sterke voorkeur voor afkorten tot een lettergreep, en het gelijkteken wordt voornamelijk in procedurele zin (als aankondiging van een resultaat) opgevat. Slechts enkele leerlingen behalen het formele niveau van symboliseren. Het experiment heeft echter ook aangetoond dat informele strategieën van vergelijkingen oplossen, zoals herhaald inwisselen en nieuwe combinaties samenstellen, een overgang van een rekenkundige strategie (handig proberen) naar een formelere aanpak mogelijk maken. De pre-algebraïsche rekentechnieken uit de geschiedenis worden door sommige leerlingen als nuttig en/of bijzonder ervaren, maar de meeste leerlingen geven de voorkeur aan een hedendaagse aanpak.
Na afloop van de tweede onderzoekscyclus heeft met deskundigen uit het veld een evaluatie van de onderzoeksopzet en het experimentele lesmateriaal plaatsgevonden. Deze bijeenkomst is van grote invloed geweest op het verdere verloop van het onderzoek. Er zijn belangrijke beslissingen genomen op vier verschillende punten.

1 De onderzoeksdoelen en -vragen waren voor sommige deskundigen onherkenbaar in de uitvoering.
2 Om de didactische waarde van de geschiedenis van de wiskunde als onderzoekscomponent te kunnen handhaven, zou de geschiedenis veel explicieter in het lesmateriaal aanwezig moeten zijn.
3 Het in het lesmateriaal voorgestelde gebruik van letters stuitte op groot verzet, omdat het (in de algebra zo essentiële) onderscheid tussen getallen en grootheden niet duidelijk werd gemaakt.
4 De activiteiten waren onvoldoende probleemgericht.

Deze aanwijzingen zijn in de laatste versie van het lesmateriaal verwerkt, waardoor zowel de wiskundige inhoud als de leerlingactiviteiten ingrijpend zijn veranderd.

De laatste fase van het onderzoek staat beschreven in *hoofdstuk 6*.
Er zijn drie niveaus van onderzoeksresultaten te onderscheiden: leerlingenresultaten, reflecties van de onderzoeker, en relevantie en gevolgen van de bevindingen. Hoofdstuk 6 richt zich op de eerste twee categorieën, de laatste staat in hoofdstuk 7. De leerlingenresultaten bestaan uit schriftelijke toetsresultaten, werk uit de lesboekjes, lesobservaties en individueel ingevulde vragenlijsten. Uit de analyse van de antwoorden van de leerlingen op de toetsopgave *Getallenkaartjes* zijn drie vermoedens voortgekomen, die vervolgens aan de praktijk zijn getoetst met andere vraagstukken:

1 Redeneren tegenover symboliseren

a Leerlingen zijn in staat om zelf een weg te kiezen of te ontdekken van rekenkundige naar algebraïsche manieren van oplossen, los van tussenliggende pre-algebraïsche strategieën en/of representaties;

b Leerlingen die moeite hebben om van een rekenkundige tot een algebraïsche manier van denken te komen, blijken veelal op een rekenkundig niveau van notaties te blijven steken;

c (Pre-)algebraïsche notaties zijn niet noodzakelijk voor algebraïsch redeneren, maar zij lijken het oplossen van (pre-)algebraïsche problemen wel te ondersteunen.

2 Terugval in niveau van oplossen

a De algemene toepasbaarheid van algebraïsche strategieën vergroot het risico van oppervlakkige kennis, waardoor leerlingen bij een nieuw vraagstuk terugvallen tot een lager niveau van oplossen;

b De kans op terugval is groter voor meisjes dan voor jongens.

3 Begrijpen van verbanden

Rekenkundige denkbeelden over getallen en verbanden tussen getallen bemoeilijken de ontwikkeling van een algebraïsche opvatting.

De uitkomst van deze vermoedens geeft, samen met andere bevindingen in het leerlingenwerk, antwoord op de deelvragen en daarmee ook de twee hoofdvragen van het onderzoek.
De drie onderdelen van de veronderstelling over redeneren en symboliseren worden alle door de analyse onderschreven. Leerlingenwerk laat zien dat *algebraïsch redeneren* meer toegankelijk is voor leerlingen dan *algebraïsch symboliseren*. Bovendien is vastgesteld dat zij zich als twee onafhankelijke vaardigheden manifesteren bij het leren van algebra. Zo blijkt dat de ontwikkeling van algebraïsch denken niet noodzakelijkerwijs afhangt van algebraïsche notaties, en dat de aanwezigheid van algebraïsche notaties nog niets zegt over het niveau van oplossen, alhoewel geavan-

ceerde notaties vaker wel dan niet gepaard gaan met een goede oplossing. Overeenkomstig de bevindingen uit eerdere experimenten, blijken leerlingen nauwelijks gebruik te maken van *schema's en tabellen ter ondersteuning van hun redeneringen*. Het gevolg hiervan was dat een deel van de leergang voor de gemiddelde leerling niet haalbaar bleek te zijn. In tegenstelling tot de vooraf geformuleerde vermoedens komt een *terugval in niveau van oplossen* niet aantoonbaar vaker voor bij algebraïsche strategieën dan bij rekenkundige, en ook zijn er geen verschillen waarneembaar tussen jongens en meisjes op dit gebied.

Er zijn enkele typische fouten in het leerlingenwerk geconstateerd die doen vermoeden dat enkele *misvattingen over verbanden* tussen hoeveelheden veroorzaakt worden door sommige conventies binnen het rekenen. Doordat leerlingen gewend zijn om met vooraf vastgestelde hoeveelheden te rekenen, hebben zij moeite om te redeneren over (verbanden tussen) onbekenden. Vooral basisschoolleerlingen hebben bij gebruik van het gelijkteken weinig oog voor de vereiste van gelijkheid. Hiermee is de derde veronderstelling voor de analyse wel aangetoond.

De meerwaarde van *geschiedenis van de wiskunde* voor het leren van aanvankelijke algebra was in de brugklassen beter zichtbaar dan op de basisscholen. De activiteiten rond de regel van drieën en de regel van onjuiste aanname, hebben brugklasleerlingen aangezet tot nadenken over hun eigen strategieën. Verder hebben sommige activiteiten – de Diofantische raadsels over som en verschil en het visprobleem van Calandri – pre-algebraïsche methoden en notaties voortgebracht die de overgang bevorderen van een informele naar een meer algemene aanpak. Wat de reacties van leerlingen betreft, oordeelden de brugklasleerlingen beduidend positiever over de historische vraagstukken dan de basisschoolleerlingen. Vooral de zwakkere kinderen werden door voor hen ongebruikelijke formulering van de vragen afgeleid of ontmoedigd.

Uit leerlingenwerk is gebleken dat de *onbekende* een opmerkelijke functie heeft bij het informeel oplossen van (stelsels) vergelijkingen. Letters en symbolen helpen leerlingen om de informatie in een probleem te organiseren, maar de onbekende speelt geen rol van betekenis in het oplossingsproces.

Een van de speerpunten bij dit promotieonderzoek was het *recapituleren dan wel het heruitvinden* van algebra langs historische lijnen. Naast de gewoonte om de onbekende niet bij de berekeningen te betrekken, zijn er nog twee overeenkomsten aan te wijzen tussen de historische ontwikkeling van de leerstof enerzijds en het leerproces van het individu anderzijds: beschrijvingen met woorden en afkortingen hebben bij aanvankelijke algebra de overhand, en lineaire vergelijkingen met één onbekende kunnen worden opgelost met gebruik van verhoudingen. Daarnaast hebben leerlingen zelf enkele strategieën bedacht voor het oplossen van vraagstukken over som en verschil die afwijken van de aanpak van Diofantos.

De analyse van leerlingenwerk en leerprocessen in de klas levert de volgende antwoorden op de onderzoeksvragen op:

1 De kloof tussen rekenen en algebra kan met behulp van sommige pre-algebraïsche strategieën en informele manieren van symboliseren gedeeltelijk worden overbrugd, maar niet door alle leerlingen. Het is ook mogelijk dat leerlingen de pre-algebraïsche fase overslaan. Soms grijpen leerlingen bewust terug naar een informele aanpak, vooral in situaties waar deze (bijna) even effectief is als een meer geavanceerde aanpak. Met andere woorden, pre-algebraïsche bekwaamheid leidt bij leerlingen niet automatisch tot verdere formalisering. Kinderen die het niet lukt om qua redeneren het rekenniveau te ontstijgen, blijven ook vaker dan anders hangen op een lager niveau van noteren.
Enkele hindernissen die leerlingen moeten nemen bij de overgang van rekenen naar algebra zijn het gebruik van schema's en andere representatievormen ter ondersteuning van wiskundig redeneren, het herkennen van isomorfe problemen, het symboliseren van de onbekende, onjuiste opvattingen over (onbekende) hoeveelheden en de verbanden daartussen, en een flexibele kijk op letters.

2 De integratie van geschiedenis van de wiskunde in de experimentele leergang heeft in de brugklas meer effect dan op de basisschool. In de brugklas zijn – onder meer dankzij de grotere betrokkenheid van de docent – meer momenten van reflectie en hebben de leerlingen een meer actieve leerhouding dan in groep 8. Sommige vraagstukken en methoden uit de geschiedenis van de algebra hebben spontaan succesvolle pre-algebraïsche strategieën doen ontstaan in de klas. Enkele overeenkomsten die zijn vastgesteld tussen de historische ontwikkeling van de leerstof en het individuele dan wel collectieve leerproces in de klas, leveren weer nieuwe ideeën op voor docenten, ontwikkelaars en onderzoekers.

Tot slot reflecteert de onderzoeker op de moeilijkheid om algebraonderwijs volgens de principes van RME in te richten, de rol van de leerkracht en de houding van leerlingen ten aanzien van het opschrijven van uitwerkingen.

In *hoofdstuk 7* worden de resultaten van het onderzoek besproken, gevolgd door enkele aanbevelingen voor verder ontwikkel- en onderzoekswerk.
Het is in dit onderzoek duidelijk geworden dat verkorte notaties op de basisschool allerlei complicaties met zich meebrengen. Algemeen bekende problemen als een onnauwkeurig gebruik van het gelijkteken, de zogenaamde 'reversal error' en de betekenis van letters zijn ook in deze studie aan het licht gekomen. Veel leerlingen lazen de formules als een soort steno. Kennelijk wordt de toestand '3 meer dan' niet in verband gebracht met de actie '3 optellen', wat voor een dubbele visie – als proces en als object – toch een vereiste is. Met ruilvoeten lukt het wel om het dynamische aspect van verbanden tussen hoeveelheden op een correcte manier te symboliseren, waarschijnlijk vanwege de activiteit van het ruilen, maar dan opereren de letters niet op algebraïsch niveau (als veranderlijken) maar op rekenkundig niveau (als afkortingen). Wellicht is het mogelijk om formules met dynamische, procedurele actietaal

– zoals pijlennotatie – in te leiden, om beter aan te sluiten bij de rekenkundige voorkennis. Verder suggereert het matige functioneren van schema's, tabellen en strookjes als redeneergereedschap, dat leerlingen in het vervolg actiever betrokken zouden moeten zijn bij de introductie en de ontwikkeling van dergelijke hulpmiddelen.
In het voortgezet onderwijs hebben zich minder problemen met symboliseren voorgedaan. Overeenkomstig de bevindingen van Harper, staan deze iets oudere leerlingen meer open voor algebraïsche opvattingen. Meestal leidt het spontaan verkorten van notaties tot een combinatie van symbolen, afkortingen en woorden, maar op zich vormt het symboliseren van een onbekende hoeveelheid voor de brugklasleerlingen geen bezwaar. Het meest opvallende resultaat is, dat de onbekende vervolgens niet of nauwelijks in de berekeningen wordt betrokken – een gebruik dat ook in het verre verleden niet ongewoon was. Af en toe worden symbolische notaties op formeel niveau gemanipuleerd.
De observatie dat symboliseren en redeneren zich als min of meer onafhankelijke vaardigheden bij leerlingen ontwikkelen, sluit aan bij de bevinding van Krutestkii dat sommige leerlingen visueel ingesteld zijn, terwijl andere voornamelijk gebruik maken van mentale processen. Hij heeft bovendien vastgesteld dat hoogbegaafde leerlingen beter kunnen redeneren dan symboliseren. Tegelijkertijd pleiten verschillende onderzoekers voor wiskundige activiteiten die beide aspecten combineren, omdat symbolen juist betekenis krijgen wanneer ze worden ingezet bij het probleemoplossen.
Het feit dat een opmerkelijk aantal leerlingen uit het onderzoek een lager niveau van oplossen laten zien in de toets dan ze eerder in de klas hebben bereikt was vaak, maar niet altijd een teken van zwakte. Met name leerlingen die de aanpak 'elimineren van de onbekende' in de lessen onder de knie lijken te hebben, zijn tijdens de toets ineens niet meer in staat om het probleem op te lossen of doen dit alleen door middel van trial-and-error. Kennelijk kunnen deze leerlingen niet teruggrijpen op een pre-algebraïsche oplossingsmethode. Slechts af en toe kiest een leerling bewust voor een informele aanpak. Vanuit het oogpunt van betekenisvolle wiskunde is het belangrijk dat de mogelijkheid bestaat een stapje terug te doen. Ook uit andere onderzoeken over vergelijkingen oplossen is gebleken dat leerlingen eerdere vaardigheden soms weer afleren, en dat kinderen die behalve de formele ook nog informele strategieën beheersen, een beter resultaat boeken.
Uit de onderzoeksresultaten blijkt dat de overgang van rekenen naar algebra wordt belemmerd door verschillen tussen rekenkundige en algebraïsche manieren van oplossen, door typische fouten in de omgang met de variabele en door de tegenstelling van een statische dan wel dynamische opvatting van verbanden tussen hoeveelheden. Booth beschrijft zes eigenschappen van rekenkundige strategieën die de ontwikkeling van een algebraïsche denkwijze in de weg staan, waaronder de nadruk op het specifieke in plaats van op het algemene en het uitblijven van symboliseren. Bednarz en Janvier merken in hun analyse van reken- en algebravraagstukken op, dat re-

kenvraagstukken samenhangend zijn – leerlingen kunnen direct met de bekende gegevens redeneren om de onbekende te bepalen – maar de algebravraagstukken niet. Als gevolg van dit verschil gaan leerlingen met de verkeerde gegevens in het probleem rekenen; dit is een van de fouten die in het onderzoek duidelijk naar voren zijn gekomen. Twee andere door Bednarz en Janvier genoemde obstakels die ook bij dit onderzoek een rol spelen, zijn de weerzin om een onbekende hoeveelheid te symboliseren (op de basisschool) en de zogenaamde 'reversal error', waarbij het verband tussen twee hoeveelheden precies andersom wordt begrepen. Daarentegen is het de brugklasleerlingen tot op zekere hoogte wel gelukt om met symbolen te manipuleren, en hebben zowel de basisschool- als de brugklasleerlingen met succes substituties uitgevoerd en symbolische vormen vereenvoudigd (met ruilvoeten en lineaire vergelijkingen met één onbekende).
De in dit onderzoek geconstateerde moeilijkheden met betrekking tot de statische en dynamische (procedurele) kijk op hoeveelheden en de verbanden, komen overeen met de in hoofdstuk 2 besproken leerproblemen van algebra.
De intermediërende werking van sommige pre-algebraïsche oplossingsmethoden en notaties biedt aanknopingspunten voor een leertraject dat de kloof tussen rekenen en algebra dient te verkleinen.

1 De voor deze studie ontworpen leerlingactiviteiten over stelsels vergelijkingen zetten aan tot een flexibel gebruik van informele strategieën en notaties en een natuurlijke overgang naar de methode van elimineren.
2 Het totaal van aanpakken gesignaleerd bij de toetsopgave *Getallenkaartjes* vormt een leertraject op kleine schaal.
3 De leservaringen met de opgaven over de regel van onjuiste aanname suggereren een opeenvolging van activiteiten die uiteindelijk kunnen leiden tot het inzichtelijk oplossen van een lineaire vergelijking met één onbekende.

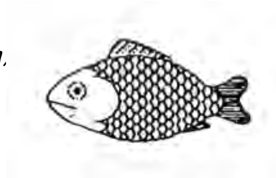
Naar aanleiding van de onderzoeksresultaten doe ik enkele aanbevelingen aan ontwikkelaars en docenten. Symboliseren zou op de basisschool meer aandacht moeten krijgen, vooral ter ondersteuning van mentale processen. Wat verkorte notaties betreft is het goed om informeel te beginnen en bij de rekenkennis aan te sluiten; formele letternotaties botsen te sterk met rekenkundige conventies. De nadruk zou moeten liggen op dynamische actietaal. Ik pleit ook voor meer activiteiten die een beroep doen op vaardigheden als structureren en schematiseren, opdat tabellen en andere schema's zich kunnen ontwikkelen tot hulpmiddel voor wiskundig redeneren. Leerlingen zouden meer oefening moeten krijgen in het reflecteren op hun eigen werkwijze zodat de drang naar verbetering groter wordt. Ga de leerproblemen van algebra niet uit de weg, maar grijp ze aan als mogelijke punten van discussie; maak leerlingen bewust van de conflicten die zich voordoen. Met betrekking tot de integratie van geschiedenis van de wiskunde stel ik voor om te differentiëren naar leerlingniveau, om speciale groepsactiviteiten in te lassen en niet de geschiedenis, maar

de leerdoelen als uitgangspunt te nemen.
Voor verder onderzoek komen in aanmerking de onderdelen van de leergang die niet aansloegen, de vermoedens over verschillen tussen jongens en meisjes die in dit onderzoek niet konden worden bevestigd, de vraag welke eisen de voorgestelde leergang stelt aan de docent, en de vraag of de experimentele leergang uiteindelijk leidt tot een verbeterde deskundigheid op het vlak van formeel vergelijkingen oplossen.

5 Italian fish

Paolo en Lisa walk through the library, giggling and chuckling.
'We'll see if the cook can solve one of our ancient sums'.
'I bet he will have quite a bit of trouble with it'.
They find a problem by the Italian mathematician Philipo Calandri from 1491. It goes something like this:

The head of a fish weighs $^1/_3$ of the whole fish, the tail weighs $^1/_4$ and the body weighs 300 grams. How much does the whole fish weigh?



1 How much does the fish weigh? Give a smart approximation.

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Fish sticks?

2 The cook is quite clever! He makes a drawing first: 'This is the fish. The head on the left, the tail on the right. And now we write in the numbers ...'

3 Find out how much the fish weighs exactly.

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draft

4 The body of a larger specimen weighs 750 gram.
What is the weight of this fish?
Draw a rectangular bar on the right first.

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An eel has a much longer body proportionally. The head weighs only $^1/_{12}$ part of the whole fish and the tail just $^1/_8$ part.

5 If an eel's body weighs 1520 grams, what is its total weight?

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Here is another eel:
6 How heavy is its body?

= 720 gram

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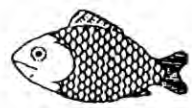
notes

<u>Venice</u>

'Quite handy, those bars the cook used,' Paolo says, 'I have to remember that
'But maybe mister Calandri also used a clever method' Lisa replies.
She looks at the problem again:

> The head of a fish weighs $^1/_3$ of the whole fish,
> his tail weighs $^1/_4$ and his body weighs 300 grams.
> What does the whole fish weigh?

'Maybe he solved the problem in a very different way ... but on the other han
maybe not ... we'll have to find out!'
One snap of the fingers, a flash, and the children are seated in a 15th century
gondola in Venice.
'Welcome to Venice!' the captain says cheerfully.
'Who ... uhm ... who are you?' Lisa can only stutter.
'I am Luca Pacioli, and I teach mathematics at the university'.
'That must be a clever man', Paolo thinks to himself.
During the first part of the boat trip he tells the children the first part of
Calandri's calculations:

Assume the fish weighs 120 grams.


Then the head weighs 40 grams,


the tail weighs 30 grams


and the body 50 grams.


7 How might the rest of the solution method go?

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Then mister Pacioli tells them how Calandri proceeds:

> *The body's true weight, 300 grams, is 6 times as much as the answer we found, 50 grams.*
> *So the assumed weight, 120 gram, must to be multiplied by 6.*
>
> *Answer: the total weight is 720 grams, the head weighs 240 grams, the body weighs 300 grams, and the tail weighs 180 grams.*

'Wow, that's quite different from the approach we use, don't you think?' Lisa nods, she has to agree with Paolo.

8 'Why does he start with 120?' Paolo wonders. What do you think?

handwriting by Paolo Dagomari (ca. 1281-1370) who solved the same fish problem many years earlier

9 'And then Calandri probably uses the 120 gram to calculate the weight of the head, the tail and the body.' Show how he does that:

10 'Finally he multiplies the assumed weight with 6. Of course!'
Why does Paolo say 'of course', do you think?

draft

A false rule?

'Calandri's solution method is known as the 'rule of false position'. Did you know that the Egyptions also used this rule 4000 years ago?'
Lisa and Paolo shake their head. They have almost completed the gondola trip through Venice, so mister Pacioli quickly talks on.
'The rule works like this: you assume that a certain conveniently chosen number, the 'starting number', is the solution. With this starting number you calculate the information in the problem. Most likely the starting number will not give you the right answer, and so you need to adjust your initial choice. This is done by looking how many times the result of your calculation fits into the given value. That is the number with which to multiply the starting number, and then you have the solution.'

Which name would you give to this rule?

Back on the quay Paolo asks: 'Do you think Calandri's method is handy at all, Lisa?'
Lisa shrugs: 'I'll have to compare it with my own method first'.

11 Try to solve the next problem in both ways: your own method on the left (bar, shortened notations, ...), and with the rule of false position on the right.

> The head of a fish weighs $^1/_4$ of the whole fish, the tail weighs $^1/_5$ and the body weighs 440 gram. How much does the fish weigh?

your own method	*Calandri's method*
	choose a starting number: …………

notes

12 Here you see problems 5 and 6 once again. Investigate whether Calandri's method also works for these problems.

problem 5	problem 6
An eel has a much longer body proportionally. The head weighs only 1/12 part of the whole fish and the tail just 1/8 part. If an eel's body weighs 1520 grams, what is its total weight?	Here is another eel: How heavy is its body? 

notes

The rule of false position is also used with two starting numbers instead of one. Often the first attempt will result in a value which is too low, and the second attempt a which is too high. That's why this method is also called the 'rule of surplus and deficiency'. It states how the surplus and the deficiency can be used to find the solution. But we won't do that here.

Summary

Math problems like the Calandri fish problem can be solved in different ways. One student may decide to draw a diagram first, another student might prefer to try different numbers, and yet another one may choose to solve the problem by reasoning.
A long time ago people used the so-called rule of false position: think of a possible solution, calculate the problem through and depending on how much your answer differs from what it ought to be, the solution must be adjusted.

13 What do you think of the rule of false position (usefulness, convenience)?

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6 Number riddles

Egypt

Lisa and Paolo are back in Itapisuma; their time travel is over.
"We have to go back to school soon !" Lisa can hardly believe it.
"But we looked up a few number riddles for you. Did you know that long ago people solved mathematical puzzles like Calandri's fish problem for fun? Sort of as a hobby. . Even in ancient Egypt!"
Let's see if you can solve them …

1 "A quantity whose half is added to it becomes 16. What is the quantity?"
Clue 1: try a number
Clue 2: draw a strip first

notes

"In the past people used a lot of words in mathematics, you have seen that a few times already. I think that's pretty inconvenient, çause sometimes I can hardly see what the task is!
Don't you think it's a hassle?"
"That's because 'they didnt use symbols for their calculations at that time, like +, -, x, :, and =", Paolo explains. "But we do!"

2 Try to rewrite the problem in task 1 in a shorter and clearer way.

In the unit *Fancy Fair* you learned that the unknown can be represented by a letter. With this letter you can construct an equation to solve the problem. The Egyptians called the unknown quantity 'hau', which meant 'heap', and they used a hieroglyphic in the shape of a scroll to symbolize it: 

In this passage from *Papyrus Rhind* you see problem 24:

A quantity whose seventh part is added to it becomes 19.

3 Where do you see the number 19? Put a red frame around it.
(you need to know, l means 1 en I means 10).

4 Which equation can you construct for this problem? Solve the equation too.

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notes

Paolo has also selected problem 33 from the *Papyrus Rhind*:

> *A quantity whose $^{2}/_{3}$ part, $^{1}/_{2}$ part, and $^{1}/_{7}$ part are added to it becomes 33. What is this quantity?*

5 "Can you construct an equation for this one, too?" Lisa wonders.

notes

Comparison with Diophantus

"Approximately 250 years after Christ there lived a Greek mathemati-cian named Diophantus of Alexandria. His book *Arithmetica* consisted of 13 volumes, but only 6 of these have been recovered. What makes *Arithmetica* so special is the fact that it contains abbreviations and symbols for the first time. And look, the numbers 1 through 10 are written with letters."

A' B' Γ' Δ' E' F' Z' H' Θ' I'

Paolo also discovers that it is not easy to determine the very beginning of equation solving.
Especially the development of notations took a long time.
"Thank goodness those people invented symbols, otherwise we might still have been calculating with words nowadays!" exclaims Lisa.
"Diophantus described different types of riddles. For each type he gave an example and the standard method for solving it. For instance, a riddle on difference: "to divide a given number into two numbers with a certain difference between them".

6 Which two numbers is Diophantos looking for in the following case?

Divide the number 100 into two numbers, such that the difference between those numbers is 30.

kladbladje

"According to Diophantus, what should you do next?"
Lisa sits down at the table, and together they read on.

> *Namethe smallest of the two numbers ς, then the bigger one is ς + 30, and the sum 2ς + 30 = 100."*

7 How does Diofantos' solution continue? You may write the letter s instead of ς.

8 "A number riddle on proportions? That's old stuff for us."
Lisa is no longer interested.
"No, it looks kind of different", replies Paolo.

> *To divide a given number (60) into two numbers that satisfy a certain proportion (3:1).*

kladblaadje

"Here you see how Diofantos does it".

Name the smaller number ς, then the larger number is 3ς, and then the sum is $4\varsigma = 60$, $\varsigma = 15$.

"Perhaps you can use this method for the other problems as well?" Lisa wonders.
"You can't possibly compare them, they are so different!"
Lisa disagrees.
"Take Calandri's fish problem, for example, the one you can solve with the rule of false position. I think it contains an unknown quantity, too."

9 Choose a task from section 5 and find out whether Lisa's hunch is right. Think carefully which quantity you should represent by a letter (because one quantity is more convenient than the other …).

notes

10 Now make up a riddle of your own which requires an equation to solve it.
Calculate it through yourself first, on one of the empty pages in the back.
Write the riddle down below and ask another student to solve it.

Riddle: ____

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notes

Solution: ____

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Curriculum Vitae

Barbara van Amerom was born in Dordrecht, The Netherlands, in 1969. She completed her secondary education at the United World College of South East Asia in 1988 in Singapore and then moved to Groningen in 1989 to study mathematics at the Rijksuniversiteit Groningen. In the final stage of her studies she specialized in the disciplines of history of mathematics and didactics of mathematics, resulting in a thesis on the design and trying out of a lesson series based on the history of calculus. After receiving her doctorandus degree in 1994, she entered the teacher training program at Utrecht University to obtain her qualification as secondary school mathematics teacher. In september 1995 she started her work on the project 'Reinvention of algebra' at the Freudenthal Institute, and since 1996 she has combined her research with a part time teaching job. The project 'Reinvention of algebra' has led to the publication of this book. She is presently working as a secondary school teacher and taking care of her newly born son.