K.P.E. Gravemeijer

DEVELOPING REALISTIC MATHEMATICS EDUCATION

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CIP-GEGEVENS KONINKLIJKE BIBLIOTHEEK, DEN HAAG

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Developing Realistic Mathematics Education / Koeno Gravemeijer - Utrecht: CD β Press, Center for Science and Mathematics Education, Freudenthal institute, Research Group On Mathematics Education, Utrecht University (CD-β series on research in education; 14). - Met lit. opg. - Met samenvatting in het Nederlands.
ISBN 90-73346-22-3
Trefw: rekenonderwijs; methodiek / wiskundeonderwijs; methodiek / onderwijs en onderzoek.

Cover: OMI Utrecht University
Translation: Hetty Vonk (chapter 2 and 3), Ruth Rainero (chapter 5 and 6)
Press: Technipress, Culemborg

ISBN 90-73346-22-3

DEVELOPING REALISTIC MATHEMATICS EDUCATION
ontwikkelen van realistisch reken/wiskundeonderwijs
(met een samenvatting in het Nederlands)

Koeno Gravemeijer

Tevens verschenen als dissertatie ter verkrijging van de graad van doctor in de sociale wetenschappen aan de Universiteit Utrecht (promotor: A. Treffers)

Preface

This study has its roots in the developmental work at the project OSM (Education and Social Environment) that finally resulted in the publication of the textbook series 'Rekenen & Wiskunde'. Succeeding in making a mathematics curriculum that embodies the main characteristics of 'realistic mathematics education' could only be possible, under the given circumstances, with the help of a fabulous team. The heart of that team was formed by Frans van Galen, Jean-Marie Kraemer, Toon Meeuwisse, Willem Vermeulen and Lida Gravemeijer. Notably thanks to their enthusiasm and comradeship, the period of developing 'Rekenen & Wiskunde' has become a period on which I look back with fondness.

It were the discrepancies between the overall concept of 'realistic mathematics education' and the educationalist view on curriculum development, implementation and evaluation, which prevailed at the project OSM, that triggered the deliberation and elaboration of a 'realistic' alternative for the general-educationalist approach. The intent of it all was to explain, justify, and work out, a domain-specific approach of the development of realistic mathematics education.

This endeavour, of course, was not undertaken in isolation, for there is a vivid community of researchers and developers that struggle with similar questions. Particularly the collective knowledge of, and the collaborative interaction with my colleagues at the Freudenthal Institute shore the outcome of this study. It has been the embeddedness in this community, that fostered the elaboration of the concept of developmental research. It is impossible to thank everybody by name, but I have to make an exception for Adrian Treffers, with whom I shared many inspiring discussions. I also want to thank Rob de Jong, who's timely incitements helped me finish this study.

The ultimate goal of developing realistic mathematics education is to change educational practice in schools. Therefore, I am grateful that I could extend my experiences as developer/researcher in a research project on the use and effects of mathematics textbook series (the MORE-project). I want to thank the whole team, and in particular Marja van den Heuvel, for all the hard work that had to be done to make this project a success.

I give thanks to Sylvia Pieters and her colleagues for all their efforts to get the manuscript camera ready, and to Els Feijs for her proofreading of the manuscript.

And last but not least, I want to thank my family for their support and consideration, during all the years that I have invested in 'the development of realistic mathematics education'.

Contents

On the texts

Chapter 1 is a reproduction of an article submitted to *Educational Studies in Mathematics.*

Chapter 2 is a reproduction of 'An Instruction-theoretical Reflection on the Use of Manipulatives' in: L. Streefland (ed.) (1991). *Realistic Mathematics Education in Primary School, On the occasion of the opening of the Freudenthal Institute.* Utrecht: CD-ß Press.

Chapter 3 is a reproduction of a chapter in T. Nunes and P. Bryant (eds.) (in press), *How do Children learn Mathematics.* Hove: Lawrence Erlbaum Ass.

Chapter 4 is a reproduction of an article accepted for publication in the *Journal for Research in Mathematics Education* (in press). National Council of Teachers of Mathematics. Used with permission.

Chapter 5 is based on the final report of an extensive research project on the implementation and the effects of a realistic and a mechanistic textbook series, under the leadership of the authors: K. Gravemeijer, M. van den Heuvel-Panhuizen, G. van Donselaar, N. Ruesink, L. Streefland, W. Vermeulen, E. te Woerd and D. van der Ploeg (1993). *Methoden in het reken-wiskundeonderwijs, een rijke context voor vergelijkend onderzoek* (Textbook series in mathematics education, a rich situation for comparative research.) Utrecht: CD-ß Press.

Chapter 6 is particulary written for this book.

Some adaptations of the original texts have been made to adjust for overlap. Further, U.K. English is changed into U.S. English, but differences in style are not eliminated.

Introduction

As is observed by many, it is clear that the field of educational research is in motion and that a multitude of research approaches is emerging. One can try to bring some structure in this multitude of approaches by making global distinctions. For instance, many of the present approaches can be captured under one of the two headings, 'explanation' or 'understanding' (Bruner, 1994). Here one may think of explanation in terms of causal relations between dependent and independent variables, and of understanding as making sense of what is going on. Apart from these two main perspectives we can discern a third, which might be labelled 'transformational research' (Research Advisory Committee NCTM, 1988). Research that does not focus on 'what is' but that deals more broadly with 'what ought to be'. This involves, for instance, research that addresses the question of how to constitute education that meets certain pregiven standards or ideals. This study falls into the latter category; it focuses on the development of what is called 'realistic mathematics education'.[1]

The label 'realistic' refers to the approach of mathematics education that is developed in the last two decades in The Netherlands. The name is taken from a classification by Treffers (1987). He discerns four approaches in mathematics education: mechanistic, structuralistic, empiristic and realistic. The decisive criteria for this classification are horizontal and vertical mathematisation. The first stands for transforming a problem field into a mathematical problem, the second for processing within the mathematical system.

In realistic mathematics education both horizontal and vertical mathematizing are used to shape the long term learning process. The mechanistic approach is the opposite of the realistic approach; it is characterized by the weakness of both the horizontal and the vertical component. The structuralistic and the empiristic approaches are somewhere in between; with at one hand the structuralistic approach emphasizing vertical mathematization, and on the other hand the empirist approach stressing horizontal mathematization.[2]

The reliance on horizontal and vertical mathematization makes the development of realistic mathematics education rather complicated. The students start out with situated, idiosyncratic, informal knowledge and strategies. From there they have to construct formal mathematics by mathematizing contextual problems (horizontally) and by mathematizing solution procedures (vertically). How fitting mathematics education ought to be constituted is not self evident. That generates the central question of this study: *How does one develop realistic mathematics education*? That is to say, the actual focus is on curriculum development, comprising the development of individual instructional activities, prototypical courses, textbook series and such. The point of this developmental work, however, is to shape the mathematics education that is to be realized with these materials. The way in which this question is answered is by reconstructing and reflecting upon practice; the practice of curriculum developers, the practice of curriculum implementation and evaluation, and above all the practice of developmental research. The first objective is to describe and clarify these practices. The second objective is to learn from these practices, to reflect upon them and to redescribe them in such a way that guidelines for improvement become available. It is in essence a bottom-up approach, where deliberation on practice is taken as a starting point for the constitution of a description on a more theoretical level.

setting

This study has its roots in the development of a grade one through six mathematics curriculum. From 1977 through 1986 the author was coordinator of a team of curriculum developers responsible for the hereafter mentioned curriculum. A curriculum that was developed within a curriculum project that based itself on what is called instructional technology. In our perception, however, the curriculum had to be oriented towards guided discovery, or better guided reinvention, and teacher autonomy.

The project was the OSM (Education and Social Environment) project in Rotterdam (The Netherlands), that aimed at improving the learning results of students of inner city schools. The basic assumption of the project management was that these children would profit of highly structured educational programs, that at the same time would address cognitive strategies, would link up with the students lived experiences, and would be activity oriented. Curriculum materials and curriculum implementation programs for mathematics, language and social development, were seen as the key vehicles for educational improvement. The strategy for mastery learning (Bloom, 1976) was taken as an example of what a highly structured curriculum might look like.

The structuring should also embrace the behavior of the teacher. What one was looking for were teacher proof curricula. At that time the research-development-diffusion (RDD) model was still en vogue, just as the strategies for instructional design. Gagné and Briggs (1974) had recently published their principles of instructional design. In line with these dominant views, the developers were expected to use instructional design strategies, starting with an operationalization of their terminal goals, and consequently deriving learning sequences and instructional activities.

This instructional design strategy was orthogonal to the general philosophy of the mathematics curriculum that inspired the curriculum developers. Their source of inspiration was the work of the IOWO [Institute for Development of Mathematics Education]. The IOWO was the national institute for the development of mathematics education in The Netherlands from 1971 through 1980.[3] Its educational philosophy was based on Freudenthal's concept of mathematics as a human activity

(Freudenthal, 1973). This approach to mathematics education became known as realistic mathematics education later on. The key idea here is that students should be given the opportunity to reinvent mathematics under the guidance of an adult. In this view mathematics education would be highly interactive, for the teacher would have to build upon the ideas of the students. This is only possible if the teacher reacts to what the students bring to the fore, and this does not fit with the idea of a teacher proof curriculum. But there were more discrepancies.

The technological approach asked for a complete description of the educational goals in terms of instructional objectives. In the above mentioned philosophy on mathematics education, however, the instructional objectives were not the main concern; the main goal was to establish a certain form of educational practice. In other words, the focus was not on the product but on the process. What objectives had to be reached was – at least for the time being – less important than the way in which they would be reached. Moreover, more abstract and rather global aims like 'a mathematical attitude' were to be strived for. In short, these goals for mathematics education did not fit the instructional technological approach of the project. Or to put it the other way around: the instructional design strategies did not seem applicable for this kind of mathematics education. It is this struggle between educational technology and a non-corresponding educational philosophy that ignited the search for an alternative for the technological approach of the seventies.

main theme

In short, the problem the group of developers were confronted with was the following. We did embrace a philosophy of mathematics education that was incompatible with the instructional design theories at hand. This triggered the question: How does one develop realistic mathematics education? A question to which this study tries to give an answer. Here developing realistic mathematics education is taken in a broad fashion. This study addresses three core elements:

- developmental research
- textbook development
- implementation and evaluation.

Mark that the question '*How does one develop realistic mathematics education*?' can be interpreted in two ways: as a call for description and as a request for guidelines. In this study both elements are combined in the sense that the actual practice is taken as a point of departure for reflection and reconstruction, and from there guidelines, heuristics, or points of attention can be constituted. Here 'actual practice' concerns the practice of doing developmental research, developing textbooks, training (prospective) teachers, giving teacher support and executing evaluation research.

Fortunately, the author had direct access to most of these practices as an inservice teacher trainer; as researcher/developer in a project on the professional development of inservice teacher trainers; as developer and developmental researcher at the OSM-project, at the Freudenthal Institute, and at Purdue University; and as implementation and evaluation researcher in a textbooks research project that was funded by the Institute for Educational Research SVO. This does not mean, however, that this study merely reflects personal experiences. Many practices are better understood as a participant observer. Apart from this, the work of the Freudenthal Institute as a whole – whether available in documents or in personal communication – formed the core of the practice that is reflected upon in this study. As such the study represents also a deliberation on the work of the Freudenthal Institute.

set up

This book consists of six chapters that are written as independent articles – with exception of the last chapter which is written as a closing chapter.

The set up of the book reflects in a way the chronology of the process of finding out how one develops realistic mathematics education. The first chapter contains a reconstruction of the development of the first grade curriculum. This reconstruction incorporates both the developmental process and the underlying theory. As such, it offers an experiential base for an alternative approach to curriculum development, from which a practice of developmental research follows naturally. One of the issues that emerges from this reconstruction is the central role of instruction theoretical deliberations. The thought experiments of the developers that underlie the design of the instructional activities take the form of micro-theories on the learning processes of the students.

The second chapter that deals with the role of manipulatives (e.g. tactile and visual models) in mathematics education can be seen as an extension of these deliberations. This chapter reveals the limitations to the alleged role of manipulatives as mediating tools to gain mathematical insight.

Chapter 3 is a follow-up in the sense that it focuses on alternative ways to fulfil this mediating function. At the same time this chapter elaborates on the theory of realistic mathematics education and the heuristics that can be employed when developing instructional sequences.

In chapter 4 the shift from curriculum development to developmental research is made. The basic idea of developmental research is illuminated and illustrated. Developmental research is placed in the context of educational development that is thought to be broader than mere curriculum development. Educational development encompasses the whole innovative process – from idea through actual change in the classroom, and all means that are employed to establish this change. At the end of this chapter the wider feedback loops incorporating everyday practice are brought to the fore.

Those two elements, actual change and feedback, are the central issue in chapter 5. This chapter focuses on implementation and evaluation in the context of this innovative endeavor. The basis for this review is an extensive research project on the implementation and the effects of a realistic and a mechanistic textbook series in grades 1 through 3.

Finally, the book concludes with an analysis of the ways in which developmental research can legitimize its own results. Much of what is discussed in the earlier chapters is brought together, not as a summary, but as a base for further reflection.

notes

1 This study concentrates on the development of realistic mathematics education in primary school, although it should be mentioned that realistic curricula for secondary education are developed within The Netherlands as well (De Lange, 1987; Kindt, 1993; Team W12-16, 1992).
2 In connecting 'realistic' to the Dutch approach it is not claimed that no similar approaches are developed elsewhere. Take, for instance, the work of Kamii (1993), Whitney (1985).
3 In 1980 the IOWO was terminated, but the research activities were continued in the Research Group on Mathematics Education and Educational Computer Center (OW&OC), that was renamed Freudenthal Institute in 1992 to honor its founder Hans Freudenthal.

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1 Instructional design as a learning process

reconstructing the development of an elementary school mathematics course

introduction

According to many specialists in the field, curriculum planning should be approached in a professional, scientific way. In the 1970s many instructional design theories and principles were proposed for this purpose, of which Gagné and Briggs' 'Principles of Instructional Design' is probably the best known (Gagné and Briggs, 1974). These design principles are characterized by a means-ends rationality and by rational methods of task analysis. First, the targets are to be operationally defined, followed by a division of learning tasks into a number of smaller, hierarchically ordered steps. Application of these principles to mathematics education leads to tightly pre-structured courses, modelled on what is called the mechanistic approach of mathematics education (after Treffers, 1987). In this context Goffree (1986) speaks of one-dimensional learning processes which are the result of one-dimensional design theories.

The significance and usefulness of such instructional design theories has been put into perspective in the last few years. Merrill, who has been one of the prominent theoreticians in this area, is doubtful about the potentiality of such design theories (Merrill, Li and Jones, 1990a and b). Merrill et al., summarize the deficiencies of what they call 'first-generation instructional design theories'. They characterize the deficiencies in question as a lack of coherence, of utility and of comprehensiveness, in the following terms:

- coherence: the instructional analysis and design focuses on knowledge and skill components in isolation, and not on the integrated wholes necessary for understanding of complex dynamic phenomena;
- utility: the prescriptions for pedagogic strategies are either superficial or lacking altogether;
- comprehensiveness: existing theory does not provide any means of incorporating finegrained expertise about teaching and learning, gained from research, and applying this in the design process (Merrill et al., 1990b, p. 26).

Pieters (1992) endorses these objections, but regards as unjustified the hope which Merrill et al. (1990b) attach to the second-generation design model they themselves propose. This model assumes a completely transparent knowledge base on the part of the designer and is based on the principle that designing is a strictly logical process. Research has shown, however, that 'we should regard the expert designer as somebody who operates from a rich rather than from a logically well-organized knowledge base' (Pieters, 1992, p. 35). An expert designer is a 'reflective practitioner' (Schön, as quoted in Pieters, 1992), in whom the processes of creating a design and reflecting on that design directly influence each other.

According to Merrill et al., however, first-generation design theories offer too few possibilities for utilizing finegrained expertise. This applies both to the use of previously available expertise and to the learning process of the designer. The models offer little room for the accommodation of new knowledge (Merrill et al., 1990a).

In reports on developmental work there are, in general, few traces of 'finegrained expertise'. This is probably the result of the dominance of technical instruction-based design models under which influence 'professional developers' feel obliged to justify their designs in terms of such models. This chapter examines what form of 'finegrained expertise' constitutes the basis for a particular instructional design, how the expert information is developed, and to what extent it is reflected in further developmental work. The design in question is contained in the Dutch textbook series 'Rekenen & Wiskunde' for grade 1 (Gravemeijer, Van Galen, Kraemer, Meeuwisse and Vermeulen, 1983). The investigation concerns two aspects of the design, namely its content and the process by which it is created.

The content involves the expertise itself, which is manifested in the developments, adjustments and revisions of the curriculum. In order to provide a point of reference for the reader we shall first reconstruct the global theoretical framework on which the curriculum is based. The process concerns the development process; that is, the interaction between the framework and the finegrained expertise on the one hand, and the experiences gained in the developments, the adjustments and the revisions, on the other.

The content analysis results in what might be called the theoretical product of the development work. In our example, this is a local instruction theory for initial arithmetic. The process analysis serves to detect emerging patterns of activities which can be used for a more detailed characterization of this type of development work.

The chapter takes the form of a reconstruction. In what follows, we shall describe the development process in chronological order and, simultaneously, reconstruct the theoretical basis of the development work. These descriptions of the development work will be interspersed with 'reflections' in which salient characteristics regarding content and development will be discussed. The fact that we are dealing with a reconstruction implies that the theoretical basis is dated.

At the end of the chapter this basis will be supplemented with the insights that have been developed since the period of analysis. The chapter concludes with a discussion of the development strategy.

1.1 development context

The New Math movement in the sixties was watched with great suspicion in The Netherlands. This critical attitude resulted in 1970 in the foundation of the Institute for the Development of Mathematics Education (IOWO). Under the direction of Hans Freudenthal, IOWO focused on 'educational development in consultation with educational practitioners'. This refers to the broadly based innovation approach of incorporating curriculum development pre- and in-service teacher training, educational research, and feedback from the schools. The design and discussion of inspiring examples ('prototypes') formed the core of the innovation strategy. The prototypes that were developed by IOWO served as a source of inspiration for textbook authors. This resulted in four textbook series in which these new ideas were given concrete form (De Jong, 1986).

The textbook series developed by the present author and others[1], which is published under the Dutch title 'Rekenen & Wiskunde', belongs to this group. It has been adopted by approximately one-third of Dutch primary schools and is given a positive evaluation by specialists in the field (Feijs, De Jong, De Moor, Streefland and Treffers, 1987). Formative and summative evaluations showed that the first grade curriculum which was developed within the Education and Social Environment (OSM) project, worked well.[2] The OSM project was strongly oriented towards educational technology and targeted towards inner city schools (Slavenburg, Peters and Van Galen, 1989). The project's approach consisted of a scientific analysis of target group problems and a scientific approach to curriculum development. A problem analysis for students from underprivileged backgrounds provided an overview of the conditions which curricula had to meet. These included a well-structured organization of the subject matter that would result in dividing content to be learned into small learning steps and accompanying this with highly structured forms of instruction.

In practice, this methodology amounted to the use of a curriculum development strategy similar to the first-generation design models. As will become clear, such a strategy was not followed in practice when developing 'Rekenen & Wiskunde' (see Gravemeijer, 1983).

frame of reference

As was mentioned earlier, Merrill et al. (1990b) refer to 'finegrained expertise about teaching and learning' linked to research. Presumably, however, the types of informal knowledge that will be used or applied in curriculum development practice will include informal knowledge in the form of ideas about teaching and learning, about the school subject matter, and about the potentialities for teachers and students. In the case of 'Rekenen & Wiskunde' (for grade 1) this frame of reference can be described globally by means of the following summary.

The common ideas of those who worked within the framework in and around the IOWO at the start of the development work – in the early '70s – included:
- the conception of mathematics as a human activity, and the associated ideal of learning mathematics as a 'reinvention process' (Freudenthal, 1973);
- domain specific instructional theories, such as Van Hiele's level theory (Van Hiele, 1973), Treffers' analysis of the mathematical thinking process (Treffers, 1978), Goffree's analysis of the counting process (Keijnemans, Jansen and Goffree, 1977), and Freudenthal's (1973) phenomenological analysis of the concept of number;
- Van Gelder's 'traditional' method of arithmetic instruction (Van Gelder, 1969);
- general instruction theories, including the theories based on activity theory (Van Parreren and Carpay, 1972; Gal'perin, 1972; Davydov, 1972), and on cognitive psychology (Ausubel, 1968; Skemp, 1972).

Apart from the theories just mentioned, developers had existing textbooks and model courses at their disposal, the latter including in particular the prototype courses developed at the IOWO.

In the following sections, the way in which these theories, ideas, and models are integrated into a development process is reconstructed. The guiding principles are, the idea that mathematics is a human activity, and that the learning of mathematics is a process of learning through reinvention. This view of mathematics education forms the core of the instructional design.

1.2 basic assumptions

philosophy of education

Any philosophy of mathematics education can, according to Thompson (1984), be divided into the following three parts: beliefs about mathematics, beliefs about teaching and learning, and beliefs about mathematics education itself. However, these beliefs are not independent. Particularly in curriculum development, ideas about mathematics as well as ideas about teaching and learning are bound to be strongly interdependent and difficult to separate. Curriculum development involves a conglomeration of ideas, theories and notions.

With 'Rekenen & Wiskunde' (hereafter called R&W), we chose to emphasize instruction that provided room for considerable personal contributions from the students, and for learning theories which recognize the importance of mental constructs. Following Freudenthal (1971, 1973), the emphasis was on the idea of mathematics as a human activity:

> 'It is an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter. This can be a matter from reality which has to be organized according to mathematical patterns if problems from reality have to be solved. It can also be a mathematical matter, new or old results, of your own or others, which have to be organized according to new ideas, to be better understood, in a broader context, or by an axiomatic approach.'
>
> (Freudenthal, 1971, p. 413-414)

This organizing activity is called 'mathematizing'.[3] Mathematics education for young children, according to Freudenthal (1973), has to be aimed above all at mathematizing everyday reality. Besides the mathematization of problems which are real to students, there also has to be room for the mathematization of concepts, notations, and problem solving procedures. Treffers (1987) makes a distinction in this connection between horizontal and vertical forms of mathematization. The former involves converting a contextual problem into a mathematical problem, the latter involves taking mathematical matter onto a higher level. Vertical mathematization can be induced by setting problems that allow solutions on different mathematical levels.

The accompanying reinvention principle[4] (Freudenthal, 1973) is regarded by the developers as a suitable alternative to the sequencing principles which characterize most of the first-generation design models. According to the reinvention principle, a learning route has to be mapped out that allows the students to find the result by themselves. To do so, the curriculum developer starts with a thought experiment, imagining a route by which he or she could have arrived at a solution him- or herself. Knowledge of the history of mathematics may be used as a heuristic device in this process. The emphasis is on the nature of the learning process rather than on inventing as such. The idea is to allow learners to come to regard the knowledge they acquire as their own, private knowledge; knowledge for which they themselves are responsible.

On the teaching side, students should be given the opportunity to build their own mathematical knowledge on the basis of such a learning process.

reflection on the basic assumptions

Unlike the first-generation model designers, the R&W developers did not seek a basis for the new curriculum in general instruction theory. Neither did they take concrete objectives as their point of departure. Rather, the basis for the new curriculum was in the philosophy of education, which itself was inextricably linked to the idea of mathematics as a human activity. The learning of mathematics was seen as an active process and the teaching of mathematics as a process of (guided) reinvention. In consequence, the designers' aims shifted from concrete objectives to the educational process itself.

Note, however, that this does not imply that concrete objectives have lost their significance, nor that the value of a curriculum cannot be measured in terms of goals to be attained. Surely, the relevance of specific skills can be a possible source of disagreement, but no one will dispute that every elementary school mathematics curriculum, in the end, has to result in the ability of students to deal with mathematical aspects of real-life situations in a satisfactory manner.

Another choice the development group made concerned the priority given to the design of learning materials. Here, priority could have been given to the process of influencing the beliefs held by teachers. However, these beliefs were relegated to second place. The reasoning behind this choice was that it makes no sense to try to rouse teachers' enthusiasm for new ideas if there are no suitable instructional materials available. This principle was supported by personal experiences with in-service courses organized by the IOWO, at which participants had expressed that they did not feel competent to make adequate changes in the textbooks they used.

The choice of priorities just outlined was, however, chiefly due to the particular context of the development. The OSM project was strongly oriented towards educational technology and favored a traditional 'research-development-diffusion' approach: the ultimate goal was the development of teacherproof curricula. In the views of the R&W developers, however, there was no place for a pre-programmed teaching-learning process, since the whole process would depend on the individual contributions of students and had to be interactively constituted between teacher and students. Acceptance of the ideas, as well as the knowledge and insights underlying these intentions, was considered essential. As a result, the developers invested a great deal of effort in informing and convincing teachers. This happened, among other things, by means of in-service teacher training and systematic clarification of the intentions behind the instructional activities in the teacher guides. However, all this took place against the judgement of the OSM-project leaders who clung on to the idea of teacherproof curricula and who wished to aim the in-service training at providing technical information about the implementation of the curriculum.

1.3 global framework

Van Hiele's levels

In the conceptualization of the first-grade curriculum Van Hiele's level theory (1973, 1985) played an important part. Van Hiele introduces his idea about different levels of thinking as the explanatory framework for many problems in mathematics education. He analyzed the communicative process between the teacher and student, and observed that the concepts used by the teacher and students are different in meaning. Although the same words are used, their meanings are based on different frames of reference. Teachers have a content-specific framework of relations at their disposal, students do not. It follows that discussions based on arguments which presuppose the existence of such frameworks are impossible. Only if both parties have

the same framework at their disposal a consensus can be reached on the basis of argumentation. Content-specific frameworks constitute the core concept in Van Hiele's definition of the following three levels of thinking:
- the ground level
- the first thinking level
- the second thinking level.[5]

At the ground level relational frameworks are as yet non-existent. Exploration of a subject matter area at ground level may lead to the formation of fundamental relations, which may, in turn, be interconnected in such a way that a framework is created. As soon as the student has established such a framework the first thinking level has been reached. The next level is unlocked when first-level processes are accessible for reflection and thus become thinking objects for the second level.

This distinction in levels is not an absolute one. For example, the second arithmetical level may, according to Van Hiele, be conceived of as the ground level for algebra. Van Hiele calls this 'level reduction'.[6]

the concept of number

Van Hiele elaborates the level theory for the development of the concept of number in the following way (Van Hiele, 1973, p. 182-183):
- At ground level numbers are still tied to observable quantities, and to actions involving physical entities.
- At the first level the relations between numbers and quantities are the object of investigation, and a relational framework is being created. As Van Hiele puts it:

  > 'Whereas at ground level the concept 'four' may be tied to visible entities, e.g. to the vertices of a square, and features as a word in the series 'one, two, three, four, five ...', on the first level it is a junction in a relational framework. On this level it might be two plus two, or two times two, or possibly five minus one. In any case it has already disengaged itself from the realm of the concrete.'
  >
  > (Van Hiele, 1973, p.182)

- On the second level the relations themselves are the object of investigation. Connections are made which allow for the construction of a logical and meaningful system.

For the authors of R&W, the significance of the level theory did not reside in its theoretical use, for example in a sharp classification into levels, but in its practical implications. First, mathematics has to start on a level at which the concepts used have a high degree of familiarity for the students, and, secondly, its aim has to be the creation of a relational framework. The selection of Van Hiele's level theory also had consequences for the curriculum goals: rather than aiming for isolated skills or basic facts, courses would be aimed at the creation of relational frameworks. In more concrete terms, numbers up to 20 would eventually have to function as junctions in a relational framework.

structuring

One of the principles of the OSM project was a pronounced structuring of learning content in order to meet the specific learning processes associated with underprivileged learners. In the OSM framework structuring was linked to the idea of task analysis (Gagné, 1977). However, from our earlier discussion it follows that it is difficult to reconcile Gagné's task analyses with a process-oriented learning theory such as the level theory. Consequently, the R&W development group looked for an alternative sequencing principle. This was found in Skemp's views on mathematics education (Skemp, 1972). Skemp builds on the notion of learning on the basis of schemata. The word schema refers to the way in which knowledge is stored in memory, as a coherent system of elementary knowledge items. Skemp bases his theory on cognitive psychology and adopts two main functions of schemata: first, a schema integrates existing knowledge, and, secondly, a schema functions as a mental aid in the acquisition of new knowledge. Skemp links learning on the basis of schemes to 'relational understanding', which he opposes to 'instrumental understanding' (Skemp, 1976). Relational understanding refers to what we take the term 'understanding' to mean in mathematics: knowing *how* something is done and knowing *why* it can be done in that way. In daily life the term 'understanding' has additional, different meanings. For example, understanding a rule or procedure can be taken in an instrumental sense: knowing what to do, without knowing why.

Given these distinctions, the creation of a relational framework can be interpreted as the accretion, restructuring or tuning of a schema. The development of a relational framework can be characterized as a fairly gradual process of growth, that is, a process that can be phased. Such phasing constitutes a basis for structuring the learning process. In other words, structuring is interpreted as the gradual construction of a relational framework.

These two principles, the level theory and relational understanding, form the global background for curriculum design. A more concrete characterization depends on Freudenthal's analysis of the concept of number.

phenomenological aspects of the concept of number

Freudenthal begins his analysis of the concept of number by indicating that the term 'concept of number' is, in essence, misleading. In his view it would be more accurate to use the plural form 'concepts of number': 'There are many number concepts, both as regards content and form, from methodological, genetic, and didactic viewpoints.' (Freudenthal, 1973, p. 170). He addresses different forms of access to the concept of number: '(...), we will discuss the question of its access. How do numbers emerge, and how are their domains and operations extended and restricted.' (op. cit., p. 170). He makes five distinctions in the use of the term number, labeled as: reference number, counting number, numerosity number, measuring (or: proportional) number and reckoning number.

- *Reference number.* The only significance of reference numbers is that they are used as a name or form of reference, as for example in referring to a bus service as 'bus number 14'.
- *Counting number.* Counting number refers to the counting process; that is, in the verbal control over a number word sequence, including the ability to count forward and backward. Counting numbers develop separately from the process of resultative counting. The number word sequence may be learned as the text of a playing song, for example as part of the game of hide-and-seek.
- *Numerosity number.* What Freudenthal calls numerosity number is, broadly speaking, equivalent to the notion of cardinal number or 'amount'. However, Freudenthal also refers to the associated concept of equipotency. Sets are equipotent when they contain the same number of elements. Equipotence does not necessarily involve counting. It implies the existence of one-to-one correspondences between sets. Young children are often able to compare quantities before they can count. In this context Freudenthal describes how his grandson Bastiaan was able to recognize the number of people present by means of the equality between the configuration of six berries on his spoon and the people around the table: 'That's how many there are of us'. He then provided the necessary evidence by referring to 'Grandma and Granddad', 'Mum and Dad', and 'Bastiaan and his sister'.
- *Measuring/proportional number.* This is the most frequently used type of number. In using numbers to describe the world around us, we mostly use the measurement aspect of numbers. For example, we might say, 'Four dollars for a pound of tomatoes? That's expensive.' This measuring function is immediately obvious in the expression 'one pound', but measuring is also involved in the expression 'four dollars'. This expression does not refer to a number of dollars as separate entities, rather we are using the dollars as a measurement unit. This example shows that measuring numbers have a very special function: they are used to represent proportions. Therefore, one also uses the term 'proportional numbers' to indicate this property.
- *Reckoning number.* In arithmetic books this is the number aspect that gets the most attention. It involves the ability to work with numbers within a system of conventions and rules, such as 'In multiplication you can exchange the numbers' (i.e. $16 \times 2 = 2 \times 16$). Knowledge of these types of rules simplifies working with numbers. The result of 16×2 can easily be derived from $2 \times 16 = 16 + 16 = 32$. However, badly understood rules can only lead to confusion: $16 \div 2$ does not yield the same result as $2 \div 16$.

The reckoning number was seen as a separate category. Following Van Hiele (1973), it might even be placed at the highest level of thinking. In R&W, however, 'reckoning number' was used as a label for the first level, that is, to numbers as junctions in a relational framework. In order to reach this level, the students would have to start with an exploration at ground level.

Adding the distinctions in the concept of number made by Freudenthal would provide a wider context for this exploration. Whereas Van Hiele concentrates mainly on pure arithmetical relations, Freudenthal's analysis adds an extra phenomenological dimension to the concept of number. Below we shall explain this in greater detail.

reflection on a global framework

In R&W the idea of reinvention is worked out in such a way that the creation of a relational framework is linked to relational learning. This approach differs fundamentally from traditional concept formation theory. The curriculum goal is formulated in terms of a relational framework (which can also be conceived of as a cognitive structure (Ausubel, 1968)). However, unlike Van Hiele's analysis, the intended relational frameworks are not confined to relations between numbers. The integration with Freudenthal's phenomenological analysis of numbers also adds connections with everyday life. This fits the idea of reinvention. If we assume that mathematics has its origin in the need to solve everyday problems, then such problems should also be the starting point in the reinvention process.

This integration is expressed in the structure of the curriculum, which is two-dimensional. The first dimension expresses the various distinctions in the concept of number, the second dimension forms the developmental stages of a relational framework. The starting point here is that different aspects of the concept of number describe the ground level where exploration can take place. The other assumption is that all these aspects contribute to the structuring of a relational framework and that they will be integrated in the reckoning number on the first thinking level.[7] Structuring is characterized as a process of phased growth in the development of the relational framework. In this sense, too, the approach being sketched here differs fundamentally from the traditional one, which divides the subject matter into separate learning steps which are to be connected only at a later stage. Most first-generation design models use a top-down strategy in designing a learning sequence. The intended end goals are analyzed and divided into small steps which determine the prerequisites for the learning of a particular skill. The same approach is applied to the prerequisites themselves, and eventually leads to basal enabling behavior. Subsequently, the student covers the learning sequence thus designed in reverse order.

The proposals by Van Hiele, Freudenthal, and Skemp follow a different route. It is the students' solution that anticipates what is to come. The problems are selected in such a way that promising approaches to a solution are teased out. In one form of reinvention or another, by a process of orientation, organization and reflection, students form the relationships which are necessary for the construction of a relational framework.

1.4 learning structure

Given Van Hiele's, Freudenthal's and Skemp's approach, phasing the construction of a framework becomes possible by restricting the exploration of the intended framework to a small area, for example the numbers one through six. The various number relations can then be explored one by one in such an area (which, in turn, provides a further structuring). Subsequently, the area can be extended to include numbers one through twelve. Finally, another extension covers the remaining numbers through twenty. It goes without saying that exercises have to be added to these explorations in order to internalize the previously formed relations.

resultative counting

The division sketched above may create the impression that various aspects of the number concept are to be developed independently from each other. However, this impression is false. Counting number and numerosity number especially, are strongly interwoven in resultative counting.

Against the background of Freudenthal's analysis, the process of resultative counting can be seen as a synthesis of the development of counting number and numerosity number. Resultative counting requires that countable objects are mapped one by one on the number word sequence, and that the last number is conceived as a cardinal number. In other words, in resultative counting the student is required, first, to exhaustively match one set of objects with a corresponding set of numbers, and, secondly, to go through a number word sequence plus the objects that are to be counted in a systematic fashion, and, thirdly, the student is required to know that the result of the counting process is independent of the manner of counting.[8]

Although the aspects of number mentioned here are interwoven in the proposed design, the theoretical division will be retained for purposes of clarity of description, especially with reference to the numerosity-number strand and the counting-number strand.

the numerosity-number strand

Two tracks are followed in the learning of addition and subtraction. One track follows the counting-number strand, with addition and subtraction as counting-on and counting-up/counting-back. The other track involves the numerosity-number strand, with the process of number-structuring is the most important principle. We shall begin with the latter.

The construction of a relational framework, as Van Hiele points out, is dependent on the formation of relationships at ground level. In the context of activities involving the structuring of quantities, Van Gelder (1969) gives some concrete suggestions for such explorations at ground level. Structuring includes the process of 'ordering' (i.e. dividing into equal groups, for example, into pairs), 'dividing' (into a number of equal groups), and 'splitting up' (into two equal or unequal groups of arbitrary size). The structuring of quantities typically precedes addition and subtraction (or 'taking together' and 'taking away'). In addition to the use of problems involving invisible quantities the use of Cuisenaire materials was considered for these types of activities. Notice that what is in fact involved here is, that the structuring of quantities has to serve as a basis for the mental structuring of numbers.

As mentioned earlier, the exploration of relations between numbers occurs in phases. The first phase involves the construction of a local relational framework with numbers one through six, followed by the first exploration of numbers one through twelve at ground level, and, finally, an extension to twenty. The structuring of numbers one through six is immediately followed by automatization of the operations (but without the use of operator signs). However, before the automatization of operations for numbers one through twelve comes into effect, procedures for addition and subtraction are introduced. These procedures are based on an approach proposed by Davydov (1972).

Davydov distinguishes three stages in the learning process involving addition (and subtraction). The core of his approach consists of the transition from re-counting to successive counting (i.e., count-on and count-back). In re-counting the sum of, for example, 5 plus 3 is found by first counting both numbers as sets of concrete objects and then counting the total number of objects. By contrast, in the case of successive counting only the second number (i.e. 3) has to be represented in concrete form. The first numerosity (i.e. 5) does not have to be the result of counting; the student starts with the number 5 and, after a process of curtailment, resumes counting ('six, seven, eight').[9]

In order to stimulate the transition from re-counting to successive counting the first number of a series is represented by a small box. The student can still go through the whole series preceding the first number by tapping on the box. In the case of 5 + 3 the student applies five taps, counts 'one, two, three, four, five' and successively counts 'six, seven, eight'. Tapping is gradually abandoned and the final number of the first series is automatically conceived as a manipulative object in the successive counting process problem. According to Davydov the concrete counting numerosity is 'carried along in the mind'.

Incorporating Davydov's procedure has several advantages. First, training for quick answers is postponed for a while, which prevents degeneration into mindless memorization. Secondly, Davydov's procedure allows the student to become familiar with using the number word sequence as an aid for addition and subtraction. (More about this later). Thirdly, it contributes to the development of resultative counting by linking counting number to numerosity number.

Another issue concerns the introduction of the operation signs ('+' and '–'). These are introduced in the context of a story involving a city bus, in which the student has to check ongoing changes in the number of passengers. (This context was taken from a prototype course by Van den Brink, 1974). Passenger entering and leaving a bus provides a situation in which addition and subtraction emerge as natural activities. The situation is used to introduce some sort of written language for the description of quantitative changes. The entering and leaving of passengers is described by means of what is called a 'bus chain' (fig. 1.1).

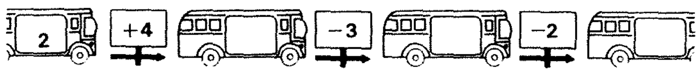


figure 1.1: bus chain

On the basis of this context-bound mathematical language a semi-formal arrow language is to be developed that can also be used in other situations (fig. 1.2).

$$2 \xrightarrow{+4} 6$$

figure 1.2: arrow language

In the curriculum the arrow language is eventually replaced by standard forms of notation ('2 + 4 = 6') after the equal-sign has been developed in situations involving static comparison (see Van den Brink, 1984). Also, at this stage the development of the relational framework is supported by the structuring of numbers, among other things, with the aid of Cuisenaire materials. In extending the relational framework, strategies are co-developed which form the foundation for the basic number facts. However, the automatization process itself is regarded as a separate component.

Knowledge of number relations, and the ability to form relations between numbers are distinguished from the ability to provide prompt answers to sums presented in standard '+' and '-' notation without the aid of countable objects. In other words, there is a difference between the ability to explain why 7 + 8 equals 15, and the ability to provide a prompt answer. However, the latter ability does not imply that facts such as 7 + 8 necessarily have to be memorized. It is sufficient that the answer is provided quickly using a speedy reasoning process. Automatized solution strategies of this kind have the advantage that the connection with the underlying number relations are maintained.

the counting-number strand

The number word sequence is not only important as a prerequisite for resultative counting, it also supports the process of arithmetic itself. Freudenthal (1973) points out that the child develops the number word sequence independently of the cardinal aspect. This not only means that most children can recite part of the number word sequence from an early age, it also means that they quickly learn the properties of the number word sequence. For example, they get to know the relative order of the numbers (e.g. 8 comes somewhere before 15), and are able to count-on and count-back. The following anecdote illustrates this:

> Six-year old Onno says to his sister, who is two years older: 'When I'm eight, you will be ten'... 'And when I am ten you will be twelve.'

Knowing the number word sequence is clearly also important in operations involving cardinal numbers, such as addition and subtraction. From prototypical materials developed by Van den Brink (1974), the developers adopted the idea to use the number line for addition and subtraction. It was hoped that the use of the number line would enable the students to use counting number properties for operations involving cardinal numbers.

An abrupt introduction of the number line would bring along the danger that the students would experience acting on the number line as a mere trick. The distinction made by Freudenthal between counting number and numerosity number shows that there is a problem in this area: The number line represents the number word sequence, and therefore the counting number, whereas the process of addition is related to the number of bus passengers, that is, to the numerosity number.

The solution to the problem is found in Davydov's procedure for the learning of addition and subtraction, which was discussed earlier (Davydov, 1972). In this procedure, counting-on and counting-back are explicitly linked to working with quantities, beginning with the most elementary form of addition, namely exhaustive re-counting. A process of shortening and internalization of the action subsequently leads to the aimed final operation.

If we look at this process of shortening in greater detail we see that the act of counting becomes more and more detached from concrete objects. The objects that are to be counted disappear into the background, and successive counting is gradually replaced by counting-on and back (in jumps) within the number word sequence. Davydov's procedure offers a conscious transition from re-counting based on resultative counting to the skillful use of the number word sequence as such.[10] As we observed earlier, the number line represents the number word sequence. This fact has to be understood by the student. In the curriculum design the process of understanding is anticipated by means of forward and backward counting. This allows for a mental image of the number word sequence to be constructed as a series of ordered numbers, which can obviously be represented by way of a number line. The number line subsequently serves as a material basis for the execution of operations. Van den Brink (1989) calls this use of the number line a 'working model', to be distinguished from the use of the number line as 'reflection model'. The latter use represents a later stage in which a process of reflection takes place on the relationship between addition and subtraction by means of visualizations along the number line (fig. 1.3).

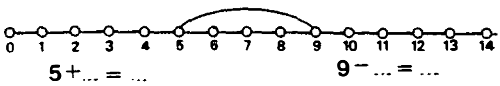


figure 1.3: adding and subtracting on the number line

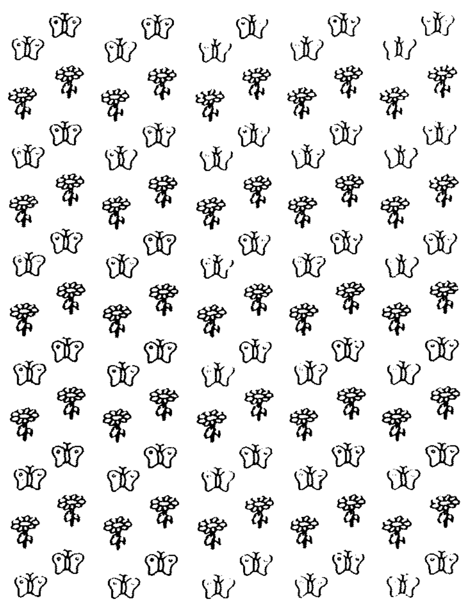
It is clear from the above that the counting number does not only form the basis for resultative counting in the curriculum design, but that there exists a longitudinal strand that continues to be significant in which the number word sequence as a basis for various counting activities. The counting-number aspect forms an important aid in the learning of addition and subtraction; first in the procedures of counting-on and counting-back, and, subsequently, in the use of the number line as a work and reflection model.

measurement (or proportional) numbers

Since this chapter is concentrated on initial arithmetic, we shall disregard pure measuring activities as well as geometry, and restrict ourselves to the concept of proportions. The introduction of a qualitative concept of proportion is linked to 'proportion fidelity' in pictures and photographs. Even young children realize ('intuitively') that proportions have to be right. In the IOWO prototypes this intuitive grasp of proportion is brought to consciousness by means of perceptual contradictions and extreme differences, in thematic subjects such as 'Madurodam' and 'Duimeliesje' (fig. 1.4) (see Van den Brink and Streefland, 1979).

figure 1.4: Duimeliesje

The next step in the design involves the arbitrary – but proportionally faithful – assignment of numbers to distances and strips. The notion of proportions also plays an important role in qualitative counting (Keijnemans et al., 1977), which involves, among other things, comparisons on the basis of density, patterns, or groups (fig. 1.5).



More butterflies or more flowers?

figure 1.5: qualitative counting

The significance of this kind of activity lies not only in the subject matter but also in the possibilities for individual approaches and different solution levels. Together with one-to-one matching, qualitative counting is subsumed under the heading of 'counting strategies'.

The focus that is attached the notion of proportion is also reflected in the choice of Cuisenaire materials. The question of which numbers are represented in these materials is determined by the length of the bars. Since, in contrast to MAB and Unifix, the Cuisenaire materials are not articulated, counting is impossible and the student therefore has to go by length or color. In Cuisenaire's original conception, color is crucial. In our design the notion of proportion is given prominence. For example, it uses graphically represented Cuisenaire problems (fig. 1.6).

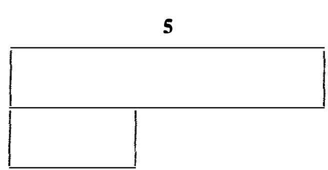


figure 1.6: graphically presented Cuisenaire problem

In fig. 1.6 the number 5 represents a sum that has to be split into two. The bar proportions make it clear that the solution is $2 + 3$.[11]

reference and reckoning numbers

The role of reference numbers is restricted. In the course design reference numbers only appear in the learning of numerals. Within that context, the specific significance of referential numbers as labels is stressed. This did not cause any problems for the majority of first-grade students. The reference number however can create confusion for young children as long as it is not distinguished from other number aspects.

Finally, reckoning numbers do not fit in with the ground level. They are part of the relational framework that has to be created, along with the subsequent process of reflection. This implies that rather than having reckoning numbers distinguished by themselves, they coincide with the development of the relational framework.

overview

The learning structure is divided into a number of sections, which are, in turn, subdivided into learning steps that form a hierarchical structure. Fig. 1.7 represents an overall outline of the structure in question in which the strands sketched above are easily recognized.

reflection on the learning structure

The influence of available educational designs comes clearly to the fore in the preceding reconstruction. The reconstruction does not provide a formal derivation from operational goals as is the case in many concrete instructional design theories. On the contrary, the concrete goals remain in the background. The basis of the course is determined by a selection from the (sets of) instructional activities known at the start of the development work. It is notable that this selection involves a number of widely different sources. We find ideas by Van Gelder, Davydov and Cuisenaire, next to the prototype arrangements by IOWO members like Van den Brink, Goffree and Streefland. However, these educational arrangements are adjusted and fitted into an overall design that fulfills the initial condition of 'mathematics as a human activity'. To put it differently, the available educational arrangements are used as (malleable) building stones for a structure of educational activities that is intelligible to the developer. The selection of the building stones, the manner of incorporation, and the adjustment, are determined by the principle of reinvention, which functions as the guiding principle for the developers. The reinvention principle is well illustrated by Van den Brink's city-bus course, in which different descriptive instruments are used to describe the quantitative changes. Subsequently, the descriptive instruments become more and more central and increasingly detached from reality. There are two mathematical processes involved in this process, namely, formalization and generalization.

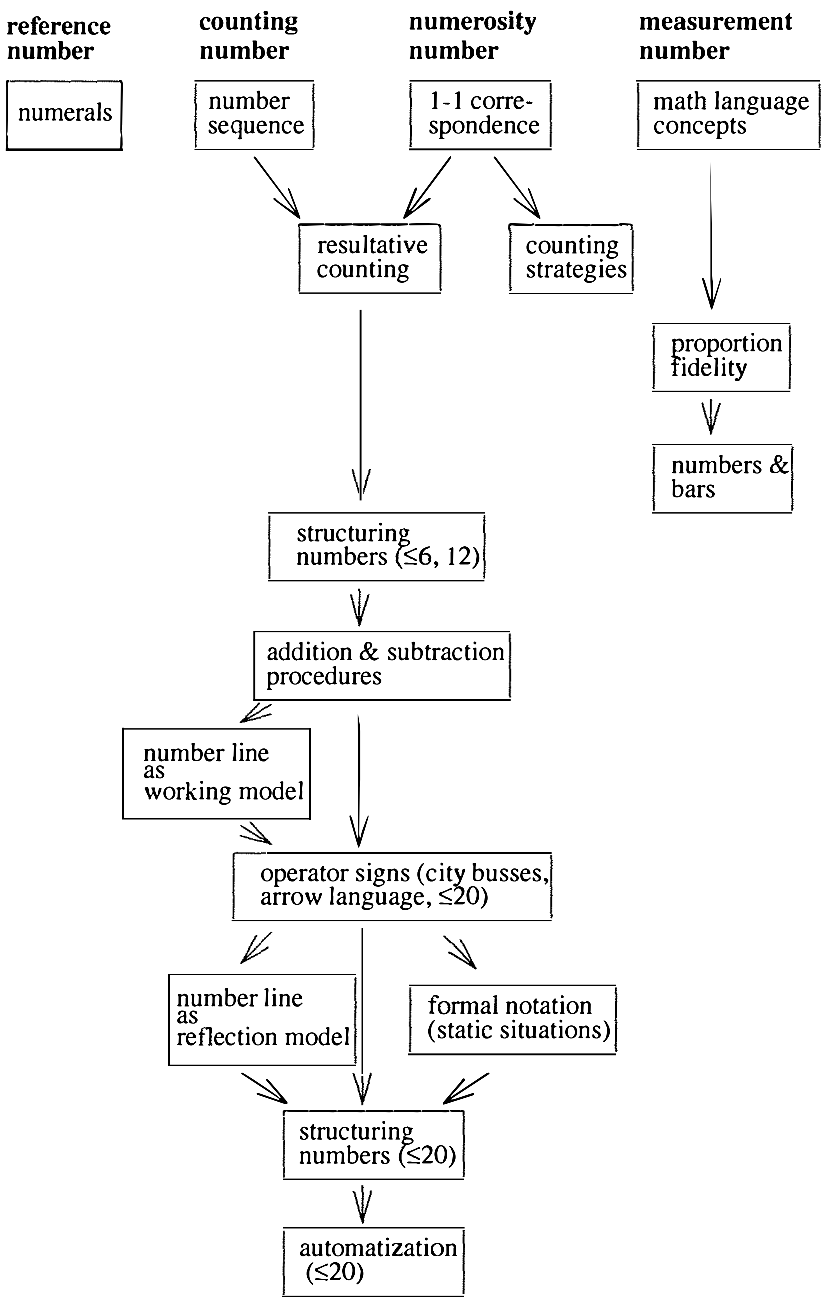


figure 1.7: learning structure

Formalization concerns the process of changing from 'everyday language' to the formal language of mathematics. In the case of the city buses the number of passengers is first represented in ordinary language, which is followed by busstop signs and a line of buses, or 'bus chain'. The chain is subsequently schematized as a bare arrow language. Once the students have become familiar with the equal-sign they are in a position to handle a formal language in which even the last visible reference to events or dynamic situations (as represented by arrows) has disappeared; a formal language which is also suitable for the description of static situations.

Generalization refers to the extension of the area in which routines or a special language can be used. Once the students have gained some familiarity with the (pictures of) busstop arrows, these arrows are also used in different situations, such as, the game of skittles, or the number of waiting ticket-buyers, and so forth (fig. 1.8). The process of generalization is accompanied by a certain formalization of the language. The stop-sign poles disappear and the arrows are no longer interpreted as events at busstops.

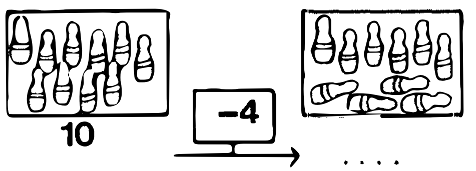
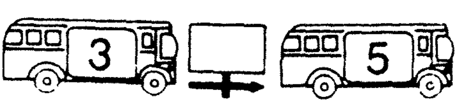


figure 1.8: pictures and arrows

The importance of the reinvention approach becomes clear if we compare the 'dot problem' 3 + ... = 5 with the corresponding city-bus problem (fig. 1.9).



What has happened?

figure 1.9: realizing

The second problem is much more accessible. The strength of the city-bus context lies in the fact that the students become conscious of what is happening. Thanks to the reinvention process they are now able to imagine a real-life context from the more formal notations; constructing a formal notation through reinvention automatically makes backtracking possible. To put it differently, the city-bus context gives meaning to the mathematical activities (also see Van den Brink, 1989).

1.5 development and testing

connected to practice

Just like other first-generation design models, the development strategy for the OSM project requested completely developed educational materials be made available at the start of the trial implementation. This was not the case with the trial of R&W. At the start of the trial period there were instructional materials for only three to four weeks. This was a situation, which in fact, did not change at all. During the entire project the preparation of instructional materials never preceded classroom implementation by very long. This is as some call 'if developing with one's back against the wall' (Shoemaker, personal communication). The time pressure was considerable, but it had one advantage: experiences in the classroom could be incorporated immediately. Consequently, practical experience assumed great significance as a 'feed forward'; the components of the curriculum which were still to be developed could be adjusted directly on the basis of classroom experiences.

As a result of such adjustments, formal evaluation lost much of its significance. Before the final evaluation of a particular school year was possible, the curriculum would already have been adjusted in many respects. At the same time, other kinds of criteria came into the picture. Not only test results, but also classroom observations provided important indications about the learning process. The areas of observation include:

- the interest and commitment shown by students
- the presentation of context-specific arguments
- the emergence of different, or, by contrast, very specific solution strategies
- discoveries, sudden insights, and so on.

It goes without saying that these processes were not regarded as being independent of the instructional behavior of the teacher. Consequently, interactions between teacher and students also formed an area of observation. In addition, classroom observations and discussions with teachers showed whether the teachers understood or shared the intentions of the developers, and to what extent they considered these intentions practicable. The teachers actively participated in the evaluation. They functioned as participating observers and commented on instructional activities.

It should be mentioned that not all components of the development work were equally strongly linked to actual classroom implementation. The initial design of learning structures and the overall curriculum structure, for instance, usually took place earlier. In addition, there were regular discussions concerning the structure and form of manuals, student materials, mixed ability teaching etc.

property arithmetic

The core of the design for first grade mathematics may be defined as: *the phased development of a relational framework in which the numbers 1 through 20 form the* *junctions, and which incorporates the phenomenological aspects of the number concept.*

The design did not contain a structure for exploration. That is to say, the initial idea was to focus on relations in various situations. In the development and testing of the curriculum this approach turned out to lead to rather aimless explorations. In the case of the numbers 1 through 6 this problem was not particularly serious since the number of relations is restricted. However, for the higher numbers it produced a situation in which the various number relations stayed too isolated from each other and occurred too infrequently to lead to a cumulative learning effect.

To counteract this disadvantage of the explorative method the number line was inserted in the first design (following Van den Brink, 1974). It was thought that counting-on and counting-back would lead to spontaneous shortening, such as counting in jumps and the use of 'anchoring points'. However, in our experiment this approach did not work, firstly, because in practice the number line invited rote counting rather than shortened counting (e.g., $6 + 3$ is solved by finding 6, moving up three places and reading off the result). In addition, the detour from a real-life problem to the number line and back proved too long for the students in the experimental classes.[12] They got stuck. The result was that a new approach had to be developed.

The solution was sought in 'property arithmetic'. By taking the logical properties of addition and subtraction as starting points, the exploration of the intended relational framework could be structured. These properties can be exploited in flexible arithmetic, as advocated by van Gelder (1969). Flexible arithmetic is distinguished from standard approaches to arithmetic by a flexible use of available knowledge. For example, the solution to $6 + 7$ is easily found by adding 1 to the result of $6 + 6$.
In fact, the example illustrates the use of the associative property. These, and other properties were utilized in the design to structure the development of relational frameworks, as shown in fig. 1.10.

the commutative property:	$a + b = b + a$	$2 + 7 = 7 + 2$
the associative property	$a + (b + c) = a + b + c = (a + b) + c$	$5 + 4 = 5 + 2 + 2$
with the following special cases:	$a + (a + 1) = (a + a) + 1$	$6 + 7 = 12 + 1$
	$(10 + a) + b = 10 + (a + b)$	$12 + 3 = 10 + 2 + 3 = 10 + 5$
the inverse relation	$a - b = c \Leftrightarrow c + b = a$	$9 - 2 = 7 \Leftrightarrow 7 + 2 = 9$
with the following special case:	the inverse of $a + a$,	$12 - 6 = 6$
	the inverse of $a + (a + 1)$	$13 - 7 = 6$
	and the inverse of $(10 + a) + b$	$16 - 4 = 12$
the cancelling out	$(a + p) - (b + p) = a - b$	$9 - 7 = 4 - 2$
with the following special case:	$(10 + a) - (10 + b) = a - b$	$15 - 12 = 5 - 2$

figure 1.10: properties used in Rekenen &Wiskunde

To facilitate property arithmetic, various contexts were incorporated in the curriculum which served as models for the different properties. For example, the commutative property was highlighted in the possible divisions of a number of passengers on a double-decker bus (fig. 1.11) (De Jong, Treffers and Wijdeveld, 1975).

Six upstairs and two downstairs is the same as two upstairs and six downstairs.

figure 1.11: double-decker bus

The double-decker representation was also used to illustrate the connection between doubles and their closest neighbors, the 'near doubles' (fig. 1.12).

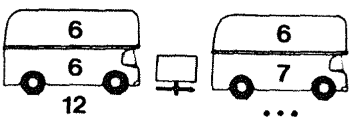
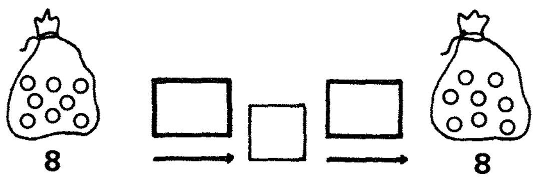


figure 1.12: 'near doubles'

The inverse relation was linked to the game of marbles (fig. 1.13).



If you break even you have gained as much as you have lost.

figure 1.13: game of marbles

The approach above implied that the number relations on which attention was focused were selected on base of a theoretical analysis. The only exception in this respect was the use of doubles (3 + 3 = 6, 4 + 4 = 8, etc.). The developers knew from experience that most children acquire doubles more quickly than other basic facts.[13]

reflection on development and testing: micro-didactical deliberation

The described details of the course show how it is developed through interaction with practical experiences. This method does not involve a fully worked out curriculum which is subsequently tested in practice. In the trial period, sufficient curriculum materials are prepared for a few weeks of teaching. This leads to the feedforward process mentioned earlier, and to the erosion of formative product evaluation. The course is adjusted before test results can be collected. As new components of the curriculum are elaborated, classroom factors are constantly taken into account. As a consequence, process evaluation gains in significance. However, process evaluation is not only aimed at the achievement of learning results, but also at the implementation of the program. Through the close connections with classroom implementation, the manageability of the instructional activities are taken into account from the start; indeed, under the influence of the feedforward mechanism, the effects on classroom implementation are anticipated from the start.

The basic philosophy governing the program naturally remains the leading principle. The practical development work is in general determined by the learning structure for grade 1 and the philosophy of mathematics education. To make detailed choices the developers have to fall back upon this philosophy. However, we may wonder whether such a general philosophy is sufficient to develop an internally consistent curriculum. The question becomes even more interesting when it is realized that the R&W textbook series is generally appreciated because of its consistently sound structure (Feijs et al., 1987). This prompts the question of what makes this consistency possible.

The context of the OSM project undoubtedly has contributed to it. The project strategy lays down the requirements to which the curricula must conform in order to be effective for students with poorer socio-economic backgrounds. The requirements in question concern, inter alia, the structuring of subject matter by means of small learning steps. This requirement forces the developers to carefully think through the kind of learning process intended in the instructional program. These processes of reflection and structuring assume a specific character, since the structuring process is linked to the IOWO philosophy. A synthesis is attempted by organizing relational learning as the stepwise construction of a relational framework. This means that the developers have to possess a fairly detailed picture of what goes on in the minds of students. In this sense *structured relational learning* forms a full-fledged alternative of the task-analysis approach.[14]

The practical details of the relational approach depend on a clear, common philosophy of the development team. In this case the philosophy is inspired by the IOWO, which leads to what is nowadays called a realistic approach (Treffers, 1987). At the same time, the R&W team, influenced by the OSM project, also developed a pragmatic approach. This *pragmatic-realistic approach* resulted in what might be called the core principle of this particular development activity, namely, the principle that *reflection and justification (within the development team) always takes place in terms of 'micro-theories' about the mechanism of the proposed teaching activities.* Within these micro-theories, assumptions about what goes on in the minds of students occupy an important place. These assumptions depend, firstly, on the frame of reference of the developers, the set of ideas, theories, and notions mentioned earlier; and, secondly, they depend on the developers' detailed knowledge of the students' previous learning experience.

1.6 revision

Revisions were further removed from teaching practice than trial implementations. The curriculum was mostly revised from behind office desks on the basis of experiences with trial implementations. Contacts with the schools were maintained by school counsellors, who passed on their own experiences, as well as remarks made by teachers, on to the developers. Observation forms and the like were used in the process, which were discussed at monthly meetings (see Vermeulen, without year). On the basis of this information the curriculum was revised a second time, followed by yet another revision on behalf of the commercial version.

automatization

Shortly after the course was introduced into more project schools, following the trial round, criticisms were voiced by the schools concerning the late introduction of a formal notation. This late introduction restricted teachers considerably in their ability to make students practice. There was also criticism of the slow start of the automatization process. It was felt that basic number facts should be systematically introduced as early as possible, in the first grade.

In order to meet these wishes, a book with exercises was developed which contains a number of problems which could be used to work on number relations before the operator signs were introduced. Most part of the book, however, contains hierarchically ordered exercises for the learning of the (majority of the) addition and subtraction facts up to 20. The hierarchy is based on property arithmetic, as incorporated in the curriculum. The reason for this structure is that not all number facts need to be memorized, provided that they can be derived quickly. In order to achieve this, property arithmetic is introduced.

The exercises are ordered according to the prerequisite knowledge, that the learner must acquire to be able to apply a particular strategy to a given problem (fig. 1.14). For example, in order to be able to solve $6 + 7$ by means of $6 + 6 + 1$ the student must know that $6 + 6 = 12$. One drawback of the design of the exercise book is the fact that the hierarchy more or less predetermines which strategy the students has to apply to a given problem.

type of problem	example	strategy
a-b, a>10, b≤10, a-b>10	18-3	8-3=5 -> 18-3=15
a+b, a>10	12+3	2+3=5 -> 12+3=15
a+b, a=10	10+5	
a-b, a=2b or a=2b+1, b<10	9-4	5+4=9 -> 9-4=5
a+b, |a-b|=1, a,b<10	6+7	6+6=12 -> 6+7=13
a+b, a<10, b=a	4+4	
a-b, 6<a≤10,b>a-b	9-6	3+6=9 -> 9-6=3
a=b+.., 6< a<10	9=5+...	
a-b, 6<a≤10, b=(0),1,2,3	7-2	5+2=7 -> 7-2=5
6<a+b≤11, a=(0),1,2,3	2+6	2+6=6+2
6<a+b≤11, b=(0),1,2,3	6+2	
a-b, a≤6	6-4	2+4=6 -> 6-4=2
a+b≤6, a<b	1+3	1+3=3+1
a+b≤6, a≥b	4+1	

figure 1.14: global learning hierarchy for the basic facts

reflection on the revision

The significance of learning-theoretical reflection becomes clear in the approach of automatization. The mental activities which lie at the basis of the mastery of the basic number facts – such as the derivation of facts in addition to the reproduction of known facts – are explicitly included in the design. This process of reflection also underlines the importance of knowing facts as a basis for the derivation of (new) facts. The structure of the automatization program therefore acquires some features of task analysis – even though the individual steps remain significant for the students. As a consequence, the approach to basic facts clearly deviates from the rest of the curriculum. Seen against the basic curriculum philosophy, the basic number facts program is tightly structured and the solution strategies are highly preprogrammed.

On the other hand, it seems essential that flexible arithmetic is not only a matter of thinking strategies but also of adequate basic knowledge. The application of informal strategies for the derivation of number facts presupposes knowledge of more basic number facts from which the intended knowledge can be derived. It is precisely this principle that is made visible by the learning hierarchy that has just been outlined. At this point, we reach the boundaries of the ideal of problem-oriented instruction. At some moment in time, progress in the learning process is bound to be seriously impeded if instruction does not allow room for the internalization of specific knowledge.

1.7 implementation

After the course was tried out in three classes at two schools, the revised version was implemented at some 20 project schools. These schools were in a different situation than the experimental schools. The practice-connected nature of the development work meant that the teachers in the experimental classes were closely involved in the development work. It also meant that these teachers acquired the necessary expertise in the area of innovation. Such expertise could not be assumed to exist in the new schools. For this reason, in-service training courses were developed for these schools, followed by targeted guidance.

In-service training sessions were interspersed with periods in which the teachers had to carry out observation assignments and practical tasks. Following Goffree (1979), a great deal of attention was given to mathematical tasks for the teachers' level: the teachers' reflections on their own solution processes and their own classroom experiences are assumed to foster adaptive and thoughtful use of the textbook series (Vermeulen, without year). Once the commercial version of the book was distributed, such elaborate forms of in-service training were no longer possible. However, the original in-service training courses were reworked into a more concise introductory course (Vermeulen, 1986), which is still in use.

reflection on the implementation

The implementation of the program has been a constant source of dispute between the R&W group and the OSM project management. The latter took the view that the curriculum should not be too teacher dependent, for the following two reasons. Firstly, the dissemination process would be jeopardized, since wide-scale distribution of textbooks would fail if their use was dependent on extensive in-service teacher training. Secondly, teacher dependency would lead to problems with the summative evaluation in that high teacher dependency would result in an undesirable variation in the use of the curriculum.

The developers objected that this type of mathematics education was teacher dependent. The key idea is that teachers build upon the students' responses and initiatives, for which they must rely on their own judgement. Use of the instructional material as intended is, therefore, only possible if teachers share the body of ideas adopted by the developers. The developers were also not convinced by the evaluation argument either, since the project contained no control groups. How can the effects of the curriculum on learning achievements be determined if one does not allow for any variation in the utilization of those curricula? The learning results cannot possibly be explained statistically by means of an independent variable without any variance.

With hindsight we can see that the assumed teacher dependency did not impede commercial distribution, since R&W has been acquired by about one-third of all Dutch schools. However, the in-service teacher training which the authors considered of such importance did not materialize. It is not inconceivable, therefore, that teacher dependence may obstruct the successful implementation of the intended form of pragmatic-realistic mathematics education.[15]

1.8 in retrospect

The result of the development work is a course in simple arithmetic, in which the aim for the learner is to create a relational framework with numbers as junctions. Following Van Hiele's level theory, an exploration of number relations at a level accessible to the learner serves as a starting point of the learning process. The scope of the exploration is, inter alia, determined by Freudenthal's analysis of the various aspects of the number concept. The course is structured by means of phasing the exploration of the framework under the following conditions:

a taking into account number size and the phenomenological aspects of the number concept,
b letting the structure of the framework be supported by solution strategies which are based on property arithmetic, and
c constructing a learning hierarchy for addition and subtraction facts up to 20.

Following the idea of guided reinvention, explorations of number relations, properties and notations are introduced to the students by means of real-life contexts which allow those relations, properties and notations to be (re)invented. This concerns local as well as global processes of mathematization. An example of the latter is the development of the formal notation by means of the arrow language in the bus context, whereas an example of local mathematization can be found in the double-decker bus context.[16]

In addition to its merits, this particular design, of course, has its drawbacks also. The first point, which was already apparent during the finishing of the commercial version, is the lack of attention for informal strategies. Especially Ter Heege's (1985) research pertaining to multiplication tables was enlightning to the designers.

Ter Heege found, while going back to Brownell and Chazal (1935), that many students develop arithmetical shortcuts to derive the table products. These findings fit nicely with the instructional approach chosen for R&W grades 2 and 3, where attempts are made to stimulate the students to derive the table products. The important difference between the two, however, is that the strategies and anchoring points used in R&W are created by the developers, whereas, Ter Heege presents the strategies of the students themselves. The developers realized that they had neglected the children's own solution strategies up to that point. This is also true for the addition and subtraction facts up to 20. Already at an earlier stage, researchers from the Dutch 'Kwantiwijzer' project pointed to the fact that there are students who systematically use the number five as reference point (Van den Berg and Van Eerde, personal communication; see also Van den Berg and Van Eerde, 1985). They attribute this strategy to finger arithmetic, which stimulates the practice of structuring sums around the numbers five and ten; for example 8 is represented as 5 plus 3 and 9 as 10 minus 1. Finally, the clinical interviews conducted by Groenewegen and Gravemeijer (1988) show that the R&W students sometimes use self-invented strategies which are not dealt with in the instructional activities. At the same time, it also appears from literature studied within the same context, that a number of strategies offered in the course fit in well with what children do spontaneously. However, this does not apply to the use of the inverse relation.

Internationally there has been interest for some time in informal strategies for arithmetic up to 20. Research shows that students make intelligent use of the opportunities offered by numbers and contexts (Ginsburg, 1977). Addition is often executed as forward counting, with some of the students consistently taking the greater number as a starting point (Groen and Parkman, 1972; Resnick and Ford, 1981). The tasks are interpreted in such a way that a minimum of counting steps suffices; which also points to the students' spontaneous use of the commutative property (also see Baroody, Ginsburg and Waxman, 1983).

A similar shortcut is used in subtraction, which is solved by the alternative use of counting-back and counting-up. Depending on the numbers involved, the interpretation is chosen which yields the least work in counting (Woods, Resnick and Groen, 1975). In fact, this latter process involves a broad interpretation of subtraction. For example, once the problem $9 - 7 = ...$ has been solved by means of counting-up, the student has in fact solved the problem $7 + ? = 9$. This can be interpreted as a comparison: How much is 9 more than 7?

In R&W, it is hoped the students will use the inverse relation between addition and subtraction, but in practice they rarely do. It also appears that they do not use the possibility of counting-up (Groenewegen and Gravemeijer, 1988). The research literature also shows that students rarely spontaneously use the inverse relation (e.g. see Baroody et al., 1983). Besides the formal character of this relation, the difficulty could lie in the requirement to search for an addition fact whose first term is unknown; that is, a – b = ? is converted by inversion into ? + b = a. It is possible that solving b + ? = a is easier. Indeed, this is what students do when they use simple forward counting.[17]

The fact that R&W students do not use counting-up as a strategy in this context could be explained, on the one hand, by the emphasis that is given to inversion. On the other hand, it means that subtraction is interpreted too narrowly (see also Gray, 1991). Following the Davydov approach, subtraction tied to counting-back and counting-up is omitted. Moreover, the type of contexts that are presented in R&W do not evoke this strategy. In other research it appears, for example, that the count-up strategy is even used spontaneously for two-digit subtraction problems, provided the problems are presented in suitable contexts (Carpenter and Moser, 1983; Gravemeijer et al., 1993; Vuurmans, 1991). In other words, there exist specific contexts which invite counting-up. However, such contexts are absent from R&W for grade 1. Comparative contexts which give rise to question such as 'how much more' or 'how much less' were removed from the course in view of the fact that students could not handle the language used. Not only were the words 'more' and 'less' confused, but the associated concepts were also confused with 'most' and 'least'. An interesting alternative in this context could possibly be the concept of 'two-sided subtraction' (Veltman, 1993), in which subtraction can be explained either as 'taking away from the beginning', or as 'taking away from the end'. Students are confronted with either of these two forms in splitting a number of beads on a chain (fig. 1.15).

figure 1.15: double sided subtraction: 9 – 7

The oversight with regard to informal strategies is further manifested in R&W when one-sided contexts are used. Nearly every contextualized problem is constructed with a prior motive: the student is supposed to learn something specific every time. The number of problems which allow for a variety of solution strategies is restricted. As a matter of fact, the learning track is almost completely fixed. This is shown in its most extreme form in the hierarchy for the basic number facts. This hierarchy is incompatible with the idea of flexible arithmetic and the students' own contributions. The 'guided' in 'guided reinvention' dominates 'reinvention' with a vengeance in this context. In fact, the whole process is distinctly Socratic in character (Freudenthal, 1973).

In the last few years, a concentrated search has been conducted for ways to stimulate the use of shortcuts and anchoring points in calculations up to 20, while at the same time leaving space for individual solution strategies. An interesting proposal in this connection is the one by Treffers (Treffers, De Moor and Feijs, 1989; Treffers, 1990), which involves a so-called arithmetic rack with sets of five colored beads (fig. 1.16).

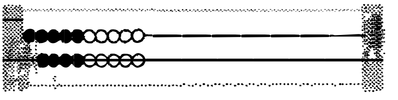


figure 1.16: the arithmetic rack

Numbers up to 20 are represented by shifting the required number of beads to the left. This can be done in two ways: by filling the upper bar, or by filling both bars. In the first case the fives configuration can be used, in the second case doubling or near- doubling prompts itself as the obvious procedure (whether combined with the fives patterns or not). A comparison can also be made with what is left on the other side. All these references can be used in addition and subtraction, with the aid of the arithmetic rack. This makes a flexible use of reference points possible (see chapter 2).

1.9 discussion

Looking back, we see that the development work was guided by a particular philosophy of mathematics education. What mattered in the end is that the course be organized in such a way that a student's conviction about a particular solution strategy is based on his or her own judgement. This philosophy manifests itself in the reinvention principle, in relational learning, and in the use of levels and didactical phenomenology.

The content of the curriculum is obtained by using sets of instructional activities that are already available. As a result, the first stage of the development work consists of selecting, fitting and adapting the available instructional activities. For the most part, the selection involves prototypes specifically developed by the IOWO, but it also includes activities from completely different sources which also fit.

The elaboration of the design has mostly taken place during the stage of the trial implementation. Already at that stage, the initial design had been adjusted where necessary. The short period between development and trial made it possible for practical experiences in the trial classes to be reflected in the development work. In connection to this we spoke of a process of feedforward, in which experiences with certain education activities have consequences for activities that are still to be developed.

assimilation, adaptation and adjustment

The formative evaluation, unlike that of first-generation design models, is not formal in character. As a consequence of the theory-oriented nature of the development work, the empirical information is interpreted in terms of that theoretical framework, and no formal rules are followed in the decision making process.

In first-generation design models, empirical testing fulfills the role of a referee. Decision rules are to be constructed in advance with regard to a subsequent formative evaluation. These rules determine what has to be done with the curriculum in the case of certain empirical findings, usually in the form of test results. In the development work sketched in the previous sections, the influence of empirical findings is more indirect, in that, the findings are interpreted first. Explanations are sought for unexpected results, so that empirical findings provide arguments for adjustments.

In practice, the findings are more often due to classroom experiences – as in the case of problems with number line arithmetic – than to testing results. That is, with intensive observations and productive contacts with experimental schools, there is a great deal of information obtained prior to when tests are administered. Sometimes this leads to the postponement of tests with problematic components. The tests often serve as checks on the components which are assumed to be sound.

Another characteristic of curriculum development which is based to a large degree on arguments and various theories, is that it is open to new ideas. New insights from literature or other sources can be accommodated quickly if they appropriately fit with the overall philosophy of the developers. This is certainly true for those ideas that are developed within the wider circle of realistic mathematics education. A good example of this connection is the way in which Ter Heege's (1985) research results are received.

This whole process of assimilation, adaptation, adjustment and interpretation may be seen as a learning process for the developers. It may perhaps best be characterized as the 'gathering' of knowledge. It is a process that takes place not only within one development project, but one that transcends individual projects. It should, therefore, be seen in a wider historical framework. In this way realistic mathematics itself builds on older teaching methods in arithmetic. For example, Freudenthal's division of various aspects of the number concept is based on an earlier division first proposed by Diesterweg and Heuser in 1830 (Radatz and Schipper, 1983). On the other hand, we see how developments in R&W are reflected in arithmetic textbook series as well as in prototype development. In this sense, this type of development work differs fundamentally from first-generation design models, which are often largely based on the idea that development models determine the quality of development work. It is not surprising, therefore, that these models are by nature so inflexible. This fits with the idea that the quality of the development work can be guaranteed by the quality of the design model.

We already mentioned in the beginning of the chapter that Goffree (1986) characterizes these first-generation design models as one-dimensional design theories that leave little room for the personal views of the developer.

development of ideas

By contrast, in Goffree's alternative model, the learning process of the developer is central. Goffree (with reference to Schön) argues in favor of 'reflection-in-action'. That is, during the development process the developer reflects on the development work, which benefits the developer, the development process and the development product. Reflection-in-action fits with the type of development process which Goffree characterizes as 'development of ideas' or 'idea analysis', as follows:

> 'In the case of idea analysis (...) the structure of the subject matter does not guide the development, but rather the following three factors:
> - the intuitive notions and informal strategies of the students
> - the full-blown mathematical concepts and procedures that fit with such notions
> - situations in which notion and informal work methods can be developed into full-blown concepts and procedures.'
>
> (Goffree, 1986, p. 15)

The concept of idea analysis comes from Lesh, who conducted research on domain-specific problem-solving strategies (Lesh and Landau, 1983). He examined ways in which mathematical ideas function in students. His assumption is that ideas develop; that is, they do not come out of the blue. His research shows that students who are confronted with a problem try to get insight into the context of that problem. Such conceptualizations of problem situations appear to undergo certain developments. The students start off with some vague, unstable concepts, which develop further and further (Lesh, 1985). The position of the developer at the start of the development work is comparable to that of the problem-solving student. The developer, too, starts with vague, unstable notions. During the development work, a refinement of these vague notions is effected by means of reflection-in-action. In this process of development (or analysis) of ideas, enough room is left for the utilization of notions about instructional methods or pre-scientific experiences; for the use of reality in the form of rich contexts and for the possibility of making pedagogical inventions. This concept of idea analysis fits well with the development process described in this chapter.

The different approaches examined (the design-model approach and the idea analysis approach) can be seen as examples of top-down and bottom-up procedures respectively. The technological instructional design models emphasize the idea of a design model. A formal, rational analysis provides a general design model which is subsequently applied to concrete contexts. The quality of the development work in this approach mainly depends on the quality of the design model. Consequently, progress is sought in attempts to improve the design model.

The idea analysis approach clearly works from the bottom-up. Whereas knowledge of the subject, knowledge of specific educational arrangements, and knowledge of local instructional design theories are taken for granted in the rational-technological approach. These forms of 'domain knowledge' constitute the hard core of the idea analysis: they are gradually, step by step, extended during the development process. In other words, the rational technological approach aims for a comprehensive, generally valid and definitive solution, whereas the idea analysis has a much more modest profile. Moreover, the latter approach fits much better with what developers actually do. In this sense, too, it is a bottom-up approach. Instead of telling developers what to do, the starting point is the way in which developers work in practice.

from development practice to development strategy

As Goffree's analysis shows, the R&W group's approach is not unique in The Netherlands. A good deal of development work carried out in The Netherlands in the area of realistic mathematical education is comparable to the R&W approach (Goffree, 1985; Kindt, 1993; De Lange, 1987; Nelissen, 1987; Streefland, 1990; Team W12-16, 1992). The current Mathematics in Context project, in which curriculum materials are developed in cooperation with the University of Wisconsin-Madison, is also conducted along the same lines (Romberg, in press).

The above-mentioned development activities vary from research and theory development to craftsmanship development work, and are all accompanied by lively exchanges of ideas and experiences. In this way prototype designs can, through the use of textbooks, be reflected in educational practice and in the same way experiences from everyday teaching practice can benefit further prototype development. The development approach outlined here is a modest one. It is restricted to a specific subject matter area, within which gradual progress is aimed for. The strength of the approach lies in its gradual nature, which guarantees continuity and a cumulative build-up of knowledge. Furthermore, the considerable number of associated development activities makes it possible for a particular development strategy and its associated educational theory to be made into an object of reflection (for example, see Gravemeijer, in press; De Lange, 1987; Streefland, 1990; Treffers, 1987 and 1991a).

The development work outlined in this chapter combines a phenomenological analysis, the reinvention principle and Van Hiele's level theory.[18] (In fact, only Van Hiele's ground level and first level are part of the course. But a basis is laid for the second level.) Since then, these three elements – phenomenology, reinvention and level structure – have been elaborated further (Treffers, 1987), and worked out as a heurisitics for development work (see chapter 3).

Taking into account these elaborative efforts, we may conclude that the bottom-up approach has developed into a full-fledged strategy for the development of realistic mathematics education. This does not mean, however, that such a strategy is capable of being developed for other subjects, nor that all strategies should be molded to the same pattern.

notes

1 The original development group consisted of F. van Galen, J.M. Kraemer, T. Meeuwisse, W. Vermeulen and K. Gravemeijer (coordination and final editing). K. Buys cooperated on the commercial version.

2 The formative evaluation showed that in general more than 80 percent of the students had mastered at least 80 percent of the items in each of the different learning steps (De Bondt, 1979).
The summative evaluation showed that 90 percent of the students solved the final test problems correctly (Slavenburg, 1986). Additional research (Groenewegen and Gravemeijer, 1988) showed that mastery of the various types of 'basic facts' lay between 71 percent and 98 percent.
Further, the learning hierarchy for these facts was confirmed on its principal points by this research with the aid of a hierarchy validation procedure developed by Novillis (1976).

3 Mathematizing literally means making more mathematical. In this context more mathematical can be related to the characteristics of mathematics itself: generality, certainty, exactitude and conciseness.

4 In subsequent publications, Freudenthal uses the term 'guided reinvention' to express more clearly the fact that teachers and textbooks have a clear role in the learning process. In practice, there will often be an area of tension between guidance and reinvention.

5 We will use the division into ground, first and second levels, although Van Hiele also uses the terms first, second and third levels.

6 As seen from the point of view of the learner.

7 Whereas Piaget's work led to a one-sided emphasis on cardinal numbers in the so-called New Math textbooks, a much broader conception of number is chosen here under Freudenthal's influence. This broader view was manifested especially in renewed attention to counting and measuring.

8 In practice, different strategies will undoubtedly be followed. For example, small numbers are usually identified directly, whereas for the larger numbers, richer strategies can be used. However, what concerns us here is a content analysis of the principle that forms the basis of resultative counting.

9 At that time, the R&W group was not familiar with the large amount of research in which similar strategies are put forward (e.g. Groen and Parkman, 1972; Resnick and Ford, 1981).

10 In retrospect we can state that here the verbal number word sequence functions at first as a model of the objects to be counted. Later, the number word sequence functions as a model for counting in jumps as a method for determining either the sum of two numbers, or their difference.

11 This solution can be checked by means of real objects, for example by arranging bars or 'whities' (small unit size cubes). The assignments are, therefore, always doable for the students and the material is self-correcting. This was regarded as important for the target group. The use of Cuisenaire materials also appears to fit in with the structure proposed by Gal'perin (1972), which procedes from material actions, via verbal actions to interiorization. It is also assumed that learners will readily abandon the fairly sophisticated materials when they no longer need them.

12 With hindsight we can state that the problems with the number line were partly caused by the type of number line and by the way in which it was used (see Treffers, 1991b and Gravemeijer, 1994).

13 This is confirmed in the research literature (Groen and Parkman, 1972).

14 Note that whereas the emphasis in structured relational learning is on reconstruction, it is on reproduction in task analysis. In a general sense, the former fits with Freudenthal's idea of constitution of mental objects as an alternative for concept attainment (Freudenthal, 1983).

15 Research with users of comparable textbooks shows that this type of education makes high demands on micro-didactic and pedagogical skills (Gravemeijer et al., 1993). Furthermore, other research (Desforges and Cockburn, 1987) shows that there are social processes with even this type of education. This is presumably due to the fact that the didactical contract (Brousseau, 1990; Elbers, 1988) between teachers and students is altered without the students having been informed. In order to meet this problem the approach of Cobb, Yackel and Wood (1992) might be incorporated. This approach focuses explicitly on the development of social norms that fit in with the new type of mathematics education.

16 Note that horizontal as well as vertical mathematization are involved here. The context consists of a story about real double-decker buses which is made real in the form of a staged 'math play' (Van den Brink, 1989). In this play the students crawl under or on top of a table, which serves as a double-decker bus. Realistic drawings and diagrams of double-decker buses and their passenger distributions are the initial models of this real-life context. Here we have a case of horizontal mathematization.
However, after frequent use the double-decker diagrams assume their own significance and they start functioning as models for logical argumentation about number relations. This transition may be characterized as a vertical mathematization process. The process can be clarified by the way in which the commutative property is brought forward in this type of context.
In the beginning, the learner has to gain the insight that if a number of passengers can be divided into an x number upstairs and a y number downstairs, there is also a possibility of an x number downstairs and a y number upstairs. Only after the context is interpreted as an addition involving x and y does the commutative property become visible. This property really begins to function if the learner solves $x + y$ through $y + x$. The idea is that this latter process is facilitated by thinking about the double-decker bus, which is a model for the following inference: 'If you calculate 6 + 2 instead of 2 + 6 you get the same result, because it does not matter for the result whether you exchange 6 and 2'.

17 A broad interpretation of subtraction can possibly have a facilitating effect, since in 'taking away from above' the inverse of $a - b = ?$ is $? + b = a$, whereas in 'taking away from below' it is $b + ? = a$.

18 We interpret the level theory in a very global sense in this context and ignore the question of whether horizontal or vertical 'décalage' takes place (allowing ourselves the use of some neo- Piagetian terms (Case, 1980)). Also, the fact that the theory as such admits of a structuralist interpretation is hidden by the link with the phenomenological aspects of the concept of number.

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2 An instruction-theoretical reflection on the use of manipulatives

introduction

From earliest times, the use of manipulative material has played an important part in theories regarding mathematics instruction. In each theory the interpretation of how such manipulatives exactly ought to be used, varies. And each time there is the endeavor to give the theoretical grounds for the use of such manipulative material. Notably cognitive psychology and action theory are focused on an instruction- theoretical foundation, while constructivist psychologists tend to approach the effect of manipulatives from an epistemological point of view. In realistic instruction theory the use of manipulatives does not hold a very prominent position. Yet, here also we can indicate an explicit view of the function of manipulatives. In analyzing the various theories we are trying to find an answer to the question if and how manipulative material can be deployed in a significant manner.

2.1 action psychology

From action psychology we know Gal'perin's theory of the stepwise formation of mental actions (see Van Parreren and Carpay, 1972). The major difference in regard to the use of manipulatives is the notion of a complete orientation basis, the principle of shortening the action and making the distinction of different parameters in the development of the action. It should, moreover, be mentioned that for Gal'perin the manipulative action is not necessarily carried out with manipulative material; symbolic representation can also be employed (also referred to as materialized action). Characteristic for action psychology is the attention which is devoted to mental activity. Mastering the action is defined as 'internalizing'. The aim of the Gal'perin procedure is the forming of a well formed mental action. To achieve this it is essential that the manipulative action is isomorphous with the pursued mental action. This precondition gives us a criterion by which to judge the use of manipulatives: is the manipulative action isomorphous with the intended mental activity?

Working with manipulatives does not automatically fulfill this requirement. This may be shown with the following example. In developing the textbook series 'Rekenen & Wiskunde', the use of the abacus as a concrete preparation for column addition and subtraction was analyzed. A discrepancy appeared to exist between the manipulative action and the intended mental action.

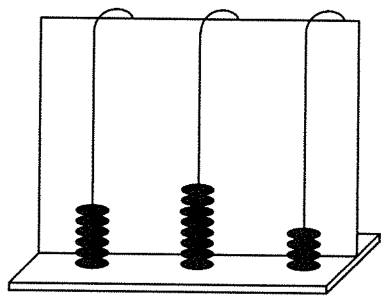


figure 2.1: 684 on an abacus with 2 × 10 beads per rod

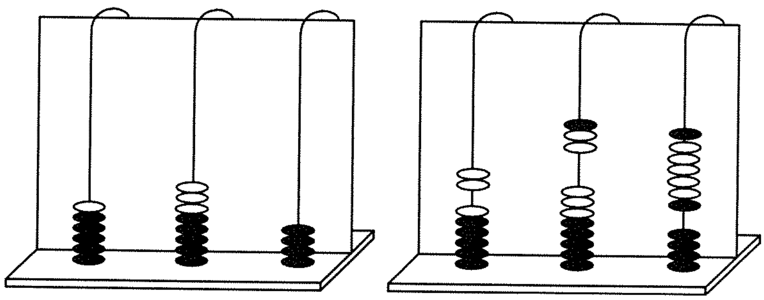


figure 2.2: 684 and 684 + 237 respectively on an abacus with 4 × 5 beads per rod

Because of the large number of similarly colored beads on each rod the students have to repeatedly count the beads one by one (fig. 2.1). In consequence the basic facts for addition and subtraction up to 20 are not used. However, the written algorithm, for which the abacus is a preparation, rests entirely on the use of these basic facts. There are no beads left to count. The manipulative action was therefore not isomorphous with the intended mental action. To overcome this problem the beads on the abacus were divided into groups of five.

This allowed the students to 'read off' the numbers and set them up without counting (fig. 2.2). Experiments, however, proved that the students again developed strategies here that were specific to the device (Van Galen, without year). The basic facts were again not used, but now the quinary structure was employed to facilitate calculations.

If the student has four beads and needs to add seven, she or he sees that one more of the same color is needed, then a group of five of the other color – that already makes six – so that only one more bead is needed to make seven (fig. 2.3).

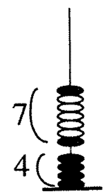


figure 2.3: 4 + 7 as 4 + [1 + 5 + 1]

No complete isomorphism was therefore reached between the abacus action and the final action. It is for this reason that the period of time in which the abacus is used in 'R&W' has been kept short. The abacus serves primarily as a thinking model. The use of the abacus as a working model might accustom the child to action structures which do not correspond with the mental action that must be conducted when doing written arithmetic. To make this more clear, we will first briefly explain the concepts 'thinking model' and 'working model'. In speaking about a working model we refer to a tactile or visual model that can be used to arrive at the answer to a calculation. In this sense manipulative material often serves as a working model. students use the material to find the answer to a calculation, but that does not mean to say that they are aware of what they are doing. The material serves as a primitive calculator as it were. Notably the use of MAB-materials can work this way, as is evidenced by the observation of Resnick and Omanson (1987), for instance, that students did not find it obvious that the calculation with blocks and the calculation on paper would produce the same answer. Manipulation with material is sometimes a rather meaningless procedure.

The problem of the difference between the material (or materialized) action and the mental action is a problem that arises in more places. We also encountered this problem when using the number line as an aid in support of addition and subtraction up to 20. When adding 5 + 4 the students were able to find five on the number line, count four positions further, and read out the answer. The problem, however, is that on the number line the students count out 'one, two, three, four' and read out 'nine', whereas if they have to solve the problem mentally they have to count 'six, seven, eight, nine' and at the same time keep up with the steps ($6 \rightarrow 1$; $7 \rightarrow 2$; $8 \rightarrow 3$; $9 \rightarrow 4$;). This is an entirely different action!

Through analyses like these, action psychology can help us to considerably improve the effectiveness of working with manipulatives. We are, however, left with another problem, the problem that is as it were illuminated by action psychology itself: What exactly is the isomorphism between the material action and the (full) mental action? Should we imagine that in his or her mind the student is manipulating with concrete material? That would not seem a very efficient nor very flexible mental action. In an application situation the students must first translate the problem to the manipulative material, then carry out the operation in their mind with this concrete material, and finally interpret the solution back to the original context.

In his 'building block model' Van Parreren (1981) shows that something other than strict isomorphism between the actions at the beginning and at the end of a learning process is possible. He sees different actions as building blocks, as separate entities, which can be used in various 'constructions'. The integration of a number of sub actions into a new action which can be called upon as one, he calls a shortcut. Van Parreren distinguishes three types of shortcuts:

- the forming of perceptive actions;
- the automation of motor skills;
- the restructuring of a task.

It is this last type of shortcut that we are interested in. Restructuring means that in the course of the learning process the student switches from the one to the other action. At a certain moment the student discovers that you can replace one action (recounting, for example) by another action that gives the same result (counting on, for example, the use of a property, or of a memorized fact), and the student dares to trust in this at a certain stage. Thus, Van Parreren gives further substance to the idea of the interiorization of the action as we know this from Gal'perin. And this gives us a different, that is clearer, image of the mental act which is ultimately formed.

With Gal'perin's approach it seems that the student keeps on thinking about concrete material. Van Parreren shows us that ultimately the student can let go of every reference to the material source. The building block model offers the student precisely the possibility to call up and make a complete 'construction' as a whole, without consciously having to execute the various subactions. The building block model also indicates that a number relationship or an operation with numbers can ultimately be set free of thinking about concrete quantities. This does not, however, explain how this step is actually achieved.

2.2 information processing

It is precisely this problem of transition that the mainstream information processing psychology is running up against. We will elaborate on this in the following, but first let us discuss this information processing approach.

Information processing psychology is characterized by the conception that knowledge is stored away in the memory as an organized entity of elements of knowledge; usually indicated as a schema, or as a cognitive structure. Learning is considered as an active process:

- whereby expansion of knowledge generally takes place by fitting in new elements of knowledge into an existing cognitive structure (assimilation);
- but whereby the cognitive structure must sometimes be completely reorganized to make room for the new knowledge (accommodation).

Subsequently, the acquisition of knowledge is described as information processing. The cognitive structures appear to play an important role in the interpretation of new information, in remembering and recalling information. The cognitive structures of experts and beginners are analysed to be able to give direction to the learning process. It is, therefore, not surprising that within cognitive psychology there is an important movement which is involved with an advanced form of task analysis (see Schoenfeld (1987)). The task analytical approach as we know it from Gagné (1977) has been stripped of its behavioral traits because one no longer stops at making an analysis of the externally perceptible behavior. However, further refinement of task analysis does in this manner lead to very complex models. Models which – it is hoped – can be tested notably through computer simulation. Aside from this, one is of course experimenting with this task analytic approach in education.

What has not changed in comparison with the old behavioral task analysis is the top-down strategy that is followed. The pursued action, the expert behavior, forms the starting point for the analysis. This focus on the pursued action makes that the expert model and the procedures to be learned are so much at the centre of things that the aim towards acting with understanding suffers in consequence. The computer metaphor is so dominant that it seems as if the only question that is being asked is, how to get students so far that they will exhibit the discovered model behavior, without asking oneself if the students understand what they are doing. Illustrative is the multiplication model that Greeno (1987) used to solve the following problem:

> Dr. Wizard has discovered a group of monsters living in a dark cave in South America. He has counted seven monsters, and there are eight fingers on each monster. If there are four fingers on each monster hand, how many monster hands did he find?

The solution of the problem is outlined by Shalin (Greeno, 1987) in the following manner (fig. 2.4).

The focus on the general solution model causes him to overlook the simplest solution: from the number of fingers you deduce that there are twice as many hands as there are monsters, hence $2 \times 7 = 14$ monster hands.

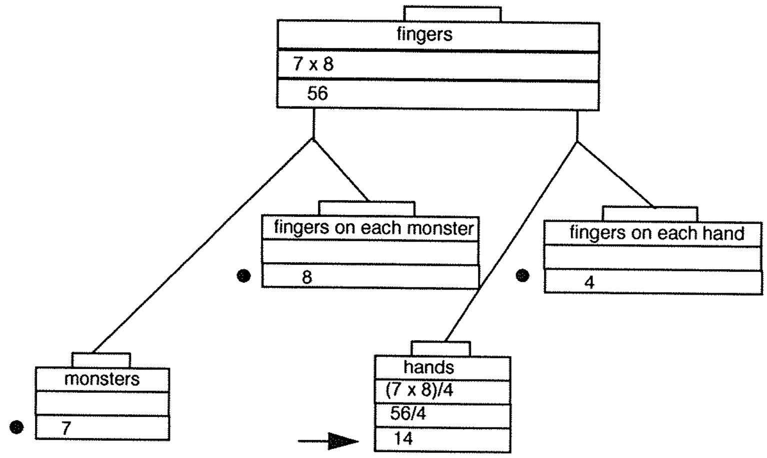


figure 2.4: Shalin's outline

It is noteworthy that in information processing psychology the approach to arithmetic is much the same as the one arrived at by Van Erp (1988) and Gal'perin (1989) from action theory. E.g. this is evident from the work of Resnick and Omanson (1987).

The study by Resnick and Omanson was directed at the potential cause of the so-called 'buggy algorithms', systematic deviations from the standard algorithm that result in a recognizable pattern of errors. They based themselves on Brown and Van Lehn in this regard, who suggest that the student reverts to 'repairing algorithms' when stuck. The repair algorithms can, for instance, comprise:

> (...) performing the action in a different column, skipping the action, swapping top and bottom numbers in a column, and substituting an operation (such as incrementing for decrementing).
> (Resnick and Omanson, 1987, p. 45)

The result of actions like these are then assessed by the student on the basis of a kind of self-constructed list of criteria which the answer to this type of problem should meet. For example, that there must be something in each column, and that no two numbers are allowed in the same column. The supposition of the researchers now is that instruction that is directed at the basic principles which lie behind the column algorithm and the applications thereof, solves the problems, or at least diminishes them. In an exploratory study it has been established that the cause of the problems should not be sought in the lack of the prerequisite knowledge of the relevant principles, but in an inadequate connection of these principles to the symbols and the syntax of the written algorithms. In experimental instruction it must be attempted to bring about this connection by means of so-called 'mapping instruction':

Mapping instruction requires the child to do subtraction problems both with the blocks and in writing, maintaining a step-by-step correspondence between the blocks and written symbols throughout the problem. (Resnick and Omanson, 1987; p. 71)

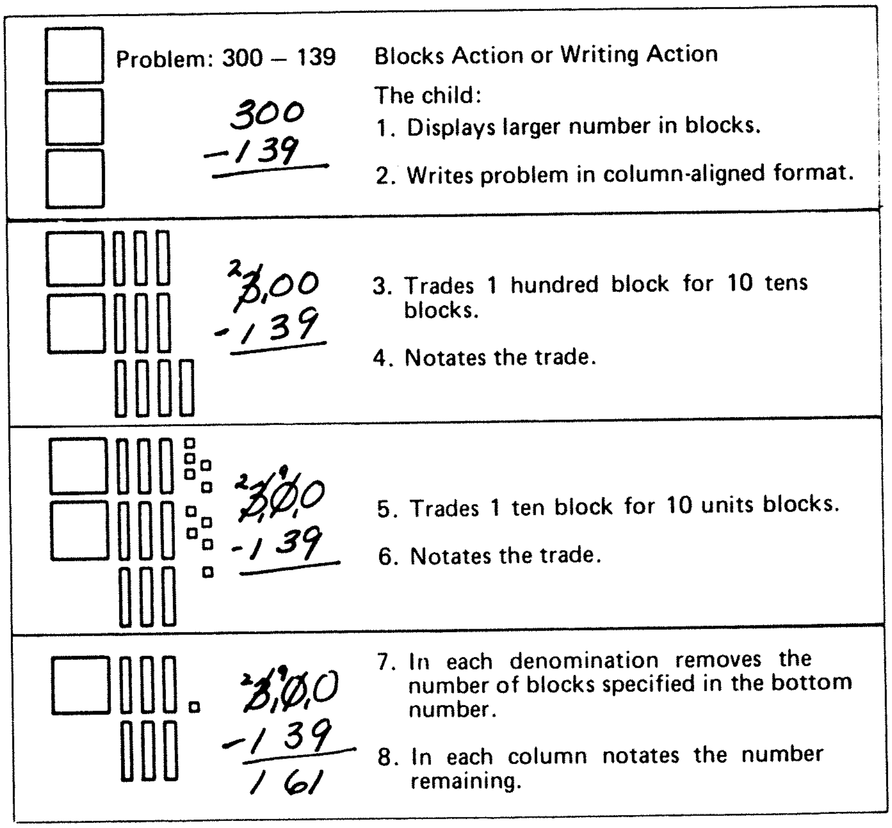


figure 2.5: mapping instruction (Resnick, 1987)

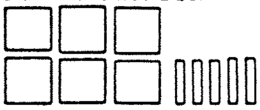
The blocks referred to are Dienes blocks with which the calculation can be made concrete. The mathematical relationships are thus embedded in manipulative materials. And the connection between the mathematical principles and the execution of the column algorithm is in this set up replaced by a connection between working with manipulative material and working on a symbolic level (fig. 2.5). Characteristic in this approach is that the blocks must be handled according to rules set by the researchers. The small blocks stand for the units, the bars for the tens and the squares for the hundreds. The compensation principle must be observed when changing the blocks: one bar is changed for ten small blocks and one square for ten bars.

Subtraction is done in columns, from right to left. This implies that first it is tried to take away the correct number of small blocks, then the bars and finally the squares.

The children often become confused. That already starts when determining the number of blocks (fig. 2.6).

b) Jane

E: Good. So how much do you think this would be?



S: (Touching the hundreds blocks) 100, 200, 300, 400, 500, 600 . . . (touching the tens blocks) 700, 800, 900, ten hundred, eleven hundred.
E: Are these (tens) worth 100?
S: I count them all together.
E: But these (tens) aren't hundreds.
S: I am counting these like tens.
E: OK. But how much would these (tens) be worth then?
S: Oh. 10, 20, 30, 40, 50 . . . 50 dollars.
E: How much would this (entire display) be worth altogether?
S: 600 . . . wait! It's 5 and 6.
E: But how much is it altogether? This (hundred) is 6, right?
S: Eleven hundred.

figure 2.6: examples of difficulties children encounter (Resnick, 1987)

This, in our view, is where the price is paid for the fact that no distinction has been made between the objects that are counted and the representation of the number. The blocks are both the objects to be counted as well as the representation of the result of that count. As a result, the differences between the mathematical concept 'ten' and 'hundred' and the visual representations of these numbers become unclear. The results of the experimental program were disappointing. Only two of the nine students did the borrowing correctly on the test which was given immediately after the instruction. On the basis of data analysis which shows that the nature of the verbal interaction is important, Resnick and Omanson consequently arrive at the conclusion that what is needed is a learning process of some other order (p. 90):

> Instead of attention to the blocks as such, it seems to be attention to the quantities that are manipulated in both blocks and writing that produces learning.
> (...) Perhaps any discussion of the quantities manipulated in written arithmetic, without any reference to the blocks analog, would be just as successful in teaching the principles that underlie written subtraction.

Nevertheless, the conviction remains that working with the blocks is extremely worthwhile:

> We believe, however, that mapping between blocks and writing may play an important role in learning by helping children to develop an abstraction – a higher level of representation – that encompasses both blocks and writing.

Just like Van Erp (1988), Resnick and Omanson share the conviction that the connection between working with manipulatives and doing written arithmetic must bring the solution to the problems. Resnick and Omanson do, however, realize that raising of level is essential:

> If the analogy between blocks and writing is clear, as it is likely to be when a step-by-step mapping is required, then a condition is created in which it is reasonable to construct a new cognitive entity that is neither blocks nor writing, but could be used to characterize both.

2.3 criticism of the task-analytical approach

Cobb (1987) directs his criticism on the information processing approach precisely at the forming of abstract mathematical objects. According to him, this task-analytical approach falls short at this point.

> (...) the lack of an appropriate explanatory construct to account for the transition from concrete action to abstract, conceptual knowledge such as an objectified part-whole structure is apparent. In lieu of an explanation, it is implied that students *will come to 'see'* various abstract, arithmetical relationships.
> (Cobb, 1987; p. 18)

Notably, the arithmetic teaching method of Resnick and Omanson falls short according to Cobb. The analogy between working with the blocks and executing the written algorithms 'is spelled out in detail', as Greeno calls this. But this analogy is only clear to the designer, because he created the units of ten or a hundred as mathematical objects. For the student, who does not yet have this mathematical knowledge, there is nothing to see!

Characteristic is the fact that the decimal structure is not respected. Exactly the same problem as is observed by Labinowicz (1985). When Dienes blocks are used, the children often count the small blocks as tens, while another time the bars are counted as units (see the example in fig. 2.7).

It is presumed that the students will immediately recognize the bars as 'tens' but that appears not to be so easy. According to Cobb (1987) this is due to the fact that the mathematical concept 'ten' is not such a simple concept for children. He refers to Steffe and Von Glasersfeld who, in a long and detailed observation study, have identified six levels in the construction of 'ten' as a mathematical object that can be both one 'ten' as well as 'ten ones'. From there it can be derived that the distance between the lowest level 'ten as a perceptive unit' and the highest (abstract) level is not easily bridged. The children that do 'see' the relationships between the tens and the ones in concrete material are, according to Cobb, the children that have already construed 'ten' as an abstract object. Or, in other words (Cobb, 1987; p. 19), 'those that have got it, get it'.

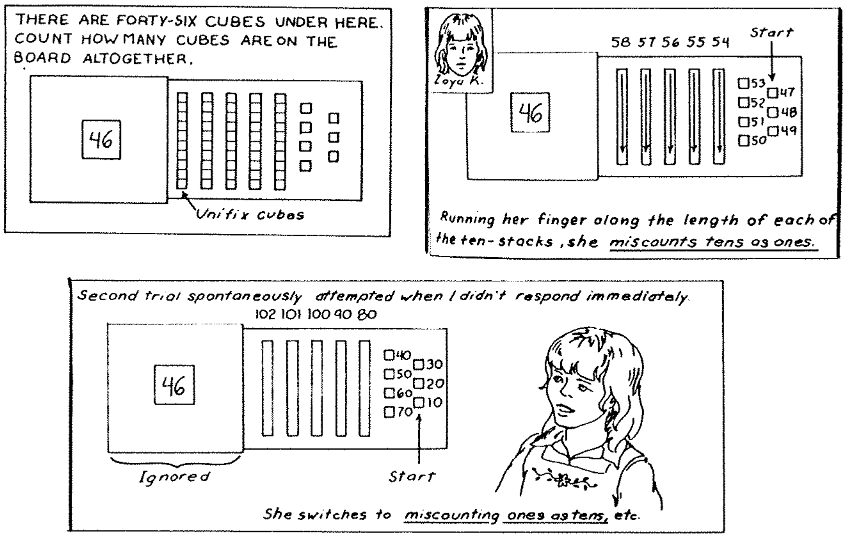


figure 2.7: Zoya's confusion between tens and units (Labinowitz, 1985)

According to Cobb, distinction must be made between an 'actor's point of view' and the 'observer's point of view'. One should be able to look at the world through the eyes of the student in order to judge the significance of learning material. The misunderstandings surrounding the concept 'ten' are also caused because we already have the abstract mathematical object and therefore fail to see the problems of the children who do not have it.

The problems become more clear when we regard working with concrete representations in the same manner as children do, namely as working with concrete material and not as working with the incorporated mathematical relationships. Then we see that this will lead to a mental action which consists of imagining a material act.

Cobb (1987) points out that the word 'representation' can be confusing here. Representation can stand for a mental representation in the mind of the child ('Vorstellung') and for a didactical representation in the form of concrete material ('Darstellung'). The fact that no clear distinction is generally made in this case goes back to the 'observer's' standpoint. For the adult the mental representation is already there and this person 'sees' it in the material as well.

To illustrate this, Cobb brings up the experience of Holt, who in first instance is most enthusiastic about the Cuisenaire material. The relationships between the material and the world of numbers are so evident that it would appear that working with this material would afford the students a wonderful entry into the world of numbers. However:

> The trouble with this theory was that Bill and I already knew that the world of numbers worked. We could say, 'oh, the rods behave just the way numbers do.' But if we hadn't known how numbers behaved, would looking at the rods have enabled us to find out?

By not making a clear distinction between internal and external representation it goes unnoticed that one is mixing up the time order: the student needs the mental representation which he or she must construe in order to be able to interpret the concrete representation!

This issue reflects the same communication mix up between teacher and student that Van Hiele (1973) observed in secondary education. He explains the problem in a discussion about the geometric concept 'rhombus'. The students only recognize a rhombus by its shape, not by its properties. A square is not recognized as a rhombus, unless you place the square on its tip (fig. 2.8).

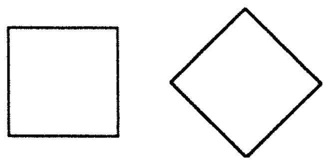


figure 2.8: recognition of a square as a rhombus

For the teacher the rhombus is a collection of properties: an equilateral parallelogram, perpendicular diagonals, etc. For the teacher it is evident that a square is a rhombus. He or she 'sees' that from the properties of the square. But an explanation at that level will not be understood by the students as long as they do not have the mathematical object of the rhombus.

After an analysis of this matter, Van Hiele arrives at the distinction in three levels of concept forming (see chapter 1).

According to Van Hiele, instruction must start at the ground level. By experimenting at ground level the student can discover relations and in that way the student will build up the relation network her- or himself. That, according to Van Hiele, is also the only way: the student must build up the relation network by him or herself, no teacher can talk a student into this knowledge. Working at a concrete level – and in the case of plane geometry this also means working with concrete material – therefore also for Van Hiele forms the basis for understanding, just as for Resnick, Gal'perin, Van Erp and others. Cobb (1987, p. 14) even says: 'Sensory-motor action is a primary source of mathematical knowledge.'

But how can the problems be avoided that are encountered precisely in doing so? Van Hiele has really already given the answer to that: by placing the initiative with the student.

Formally, that is also one of the points of departure of cognitive psychology. And also Gal'perin pursues this in his proposal for a complete orientation basis. In practice, these good intentions are often not realized because the designers are insufficiently aware that they are taking the 'observer's' point of view.

2.4 constructivism

We can guard against this, according to Cobb, by adopting a constructivist standpoint. Constructivism departs from the idea that there is no strict logical way to know 'objective reality'. Radical constructivism purports that you cannot even know if there is an objective reality. Radical constructivists call the reference to 'genuine reality' metaphysical realism. It is precisely the reference to the reality 'out there' that causes the misunderstanding. One must continually keep in mind that one is talking about constructions, and that these constructions are idiosyncratic. Only through social interaction, through consultation and negotiation, one can try to attune the various constructions as much as possible.

In education one must provide students with the opportunity to build up their own knowledge by themselves. According to constructivists, every individual will try to build a theory of reality that is acceptable to him or her, and children try this as well. Constructivists find proof for this in the so-called 'misconceptions' (or 'alternative conceptions').

Examples of misconceptions are also found in optical illusion (fig. 2.9), naive expectations in physics (fig. 2.10a, b) and in the own solution strategies of young children.

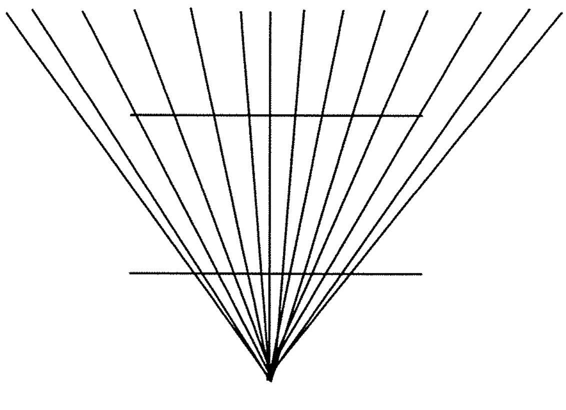


figure 2.9: optical illusion

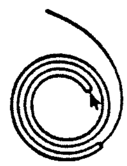


figure 2.10a: naive expectation of the path of a bullet as it is shot into a spiral shaped tube at great speed

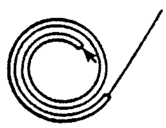


figure 2.10b: actual path of the bullet

If this interpretation is correct, this means that children will try to interpret their experiences in a logical manner. In education we can make good use of this, although we must be aware that the logic children discover will differ from ours.
Learning-theoretical concepts corresponding with this approach are widely adopted in cognitive psychology, following Piaget, whereby the focus is on concepts such as assimilation and accommodation.

This process of acquiring knowledge displays a clear similarity to the development of scientific knowledge, as described by Kuhn (1970) and Lakatos (1978). Main elements are consistency and the not immediate rejection of an accepted 'theory' when unexpected results are encountered. Scientists do not give up their theories so easily. According to Kuhn no less than crisis and scientific revolution are necessary to make that happen. And also in our everyday life we do not give up our theories about reality so easily; that much is proven by the existence of stubborn preconceptions.

2.5 realistic instruction theory

According to Van den Brink (1981) one can induce children to discuss their theories by creating conflict situations. For example, a conflict can be created by comparing the number of boys and the number of girls in a class, with the help of a conflicting graph (fig. 2.11).

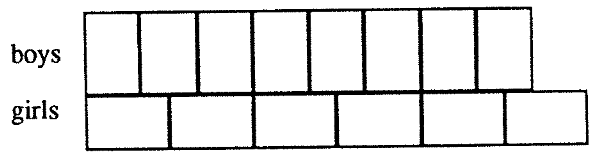


figure 2.11: graph

From the graph it looks as if there are more girls, while actually there are more boys. For a good understanding of the conflict situation it is necessary that we are aware that the conflict does not arise because one can prove by counting that there are more boys. For the students that does not have to lead to a conflict. The counted number is yet too far removed from the quantity number and the perception of quantity. The solidification of the knowledge about the number of boys and girls in the childrens' own world causes the conflict: the students know – from physical exercise class for example, from games – that there are more boys than girls in their class. The result of the graph conflicts with this self developed knowledge.

We can ask ourselves, however, whether conflict situations are really necessary for learning the algorithms for column addition and subtraction. It remains to be seen whether the detected misconceptions constitute a necessary phase in the learning process. It would sooner appear that the misconceptions have been caused by education itself. Or, in other words, that the misconceptions are the result of the 'observer's' expectation that the material will 'show' the mathematical structure.

In realistic mathematics education it is attempted to prevent the forming of misconceptions by following the reinvention principle (Freudenthal, 1973). In this approach the student follows a learning route that takes its inspiration from the history of mathematics. In the case of column addition and subtraction the symbolical representation of large numbers in a denary structure, the positional notation, and the use of the abacus are important milestones. If the student discovers the meaning of these devices by him- or herself in problem oriented instruction, the occurrence of misconceptions can be avoided.

In order to develop the denary system with the children, an apocryphal shepherd appears on the scene. This shepherd keeps track of the number of sheep he has by putting aside a stone for each sheep. At a certain point, however, the shepherd has so many sheep that the sack of stones is becoming a burden to him. The problem of the shepherd is made the problem of the children. How does he solve it? When the solution of the shepherd is finally presented, it is also experienced as a genuine solution for a real problem. If the number of stones becomes too great, the shepherd changes ten stones (as many as he has fingers) for one colored stone. This process of making groups of ten and the representation and interpretation thereof is re-enacted with tokens. In this example, the function of the material is different from the Dienes blocks. In a (too limited) introduction of the Dienes blocks the agreement about grouping on the basis of ten is communicated by the material. In the case of the shepherd, a very conscious agreement is made to solve a certain problem. In that phase the work is still with unstructured material. Later on, the position system is construed in a similar manner. Only at a later stage, materials and contexts are introduced where the denarity has been solidified, such as working with money or with the decimal system.

The introduction of the abacus follows largely the same method as the case of the shepherd: concrete material is used to symbolize quantities situated in a context. The idea of repeated grouping on the basis of ten is again picked up in the story of the sultan. Whenever the fancy strikes him, the sultan wants to know how many gold pieces he owns, and to make counting them easier, the coins are grouped as stacks of ten and bundles of a hundred (fig. 2.12).

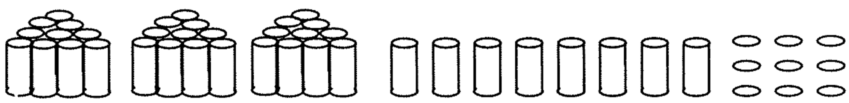


figure 2.12: the sultan's pieces of gold

In class, this story is retold with checkers, and changing and grouping is practized with drawings. Later, the abacus is introduced, a device that also played a major role in the history of the algorithm. The beads on the different rods refer to loose pieces of gold, the stacks of ten and the bundles of a hundred. The principles of changing, borrowing and carrying are developed now against the background of packing and unwrapping pieces of gold. Only then there is the transition to something like the written algorithm, which first leaves ample opportunity for writing down the interim steps or interim scores, and which is only later abbreviated to the standard algorithm.

The principle difference with the Dienes blocks or other base ten material is again the relative unstructured character of the material. Concepts such as tens and hundreds are not illustrated by the material. The context provides a situation model in the story of the sultan and the method of wrapping the gold pieces. In so far as the material is structured, the structurization is directed at eliciting certain mental activities. For that reason, the quinary structure is put on the abacus as a visual support for the setting up and reading off of the number of beads per rod. While the bars on the abacus in turn are an aid to differentiate between the units, tens and hundreds.

Naturally, also in this approach, there is the danger of trickery action. First of all, the handing in of one bar can be done without thinking about the context or meaning.

In addition we have already observed that students develop own solution methods which are not isomorphous with the actions needed to do column arithmetic on paper. This is the reason why actions with concrete material must, in our view, be regarded especially as a transition phase. The manipulations on the abacus, together with the sultan context, must offer a framework of reference for arithmetic on paper. Purpose then is, that the student, in thinking about the abacus has the global structure in mind. It is not the intention that the students add or take away beads in their mind on the basis of the denary structure. What must be prevented, therefore, is that these kind of actions become routinized.

As an alternative, precisely the opposite route might be followed, taking the informal arithmetic methods of the students as point of departure. Especially when the quinary structured abacus is used long in advance of doing arithmetic up to 20, the basic facts can possibly be developed from informal arithmetic methods of the children. If the sultan's story and the 'arithmetic' on the abacus is not introduced until thereafter, the chance of too strong a binding with concrete manipulation is much smaller. With such a prominent position for the self invented arithmetic methods of children we add a new element to the realistic approach.

In the 'traditional' realistic approach, the paths along which the algorithms are developed, are to a great extent, predetermined. The instruction is designed thus that the student makes discoveries himself, but what is discovered and in what order has been determined in advance by the constructor of the course of instruction. He tries to achieve this by way of a didactical series of problems and by eliciting the corresponding discussion and reflection.

Meanwhile, (realistic) developmental research, such as conducted by Ter Heege (1983) and Streefland (1988), has been the cause for an awareness that the children themselves invent alternative solution procedures which are as good, or even better suited for lining out the course of instruction. In this respect Streefland mentions solutions which 'anticipate' and which act as 'road signs' for the developer.

Designing a possible learning route on the basis of the own solutions of children can be regarded as a further refinement of the re-invention principle. This principle does not only praise the history of mathematics as heuristic, it also refers to a certain manner of learning: the student who globally follows the historical course of instruction, reconstructs the thus discovered mathematics.

This idea of the self (re)construction of mathematical knowledge is much more fundamental than the historical aspect. Freudenthal chooses for the reinvention principle from his idea about how one, as a mathematician, adopts new mathematical knowledge. The history of mathematics certain can help one to find a fitting course of learning, but as it appears, so can the own solutions of the children. Recently, Treffers c.s. (1988) also applied this principle to the basic facts and column addition and subtraction.

Research shows that children can spontaneously come up with a number of informal strategies to arrive at the basic facts for addition and subtraction (see Groenewegen and Gravemeijer, 1988, for example). First, most answers are still found by counting. Then, the children develop ways of counting and calculating efficiently to shorten the counting activity. At the same time, it appears that the doubles (ties) and the quinary/denary structure are often used as points of reference. Of course not every child will develop efficient strategies with the same ease. Hence the search for concrete material that will elicit the development of such habits. Suitable tools here would appear to be the arithmetic rack and the bead string (fig. 2.13a and b).

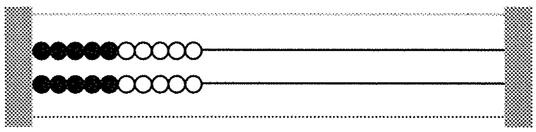


figure 2.13a: arithmetic rack

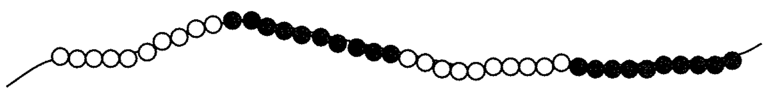


figure 2.13b: bead string

We will not elaborate on both learning expedients here, but restrict ourselves to a brief description of working with the arithmetic rack. For a detailed description we refer to Treffers (1990). On the arithmetic rack the beads that have been moved to the left count as the numbers that are being worked with; the beads to the right do not count. The quinary structure in first instance only offers visual support in quickly overseeing the numbers. As such this structure also provides support to the discovery and remembering device-restricted number relationships such as 'five is three plus two', 'five and two is seven' and 'six and six is twelve'. These are precisely the number relationships which the quick student will spontaneously use as points of reference. Also, the use of these anchoring points in finding basic factis is facilitated by the device. In this way, you can read $6 + 7 = 6 + 6 + 1$, or $6 + 7 = 5 + 5 + 1 + 2$ at a glance (fig. 2.14).

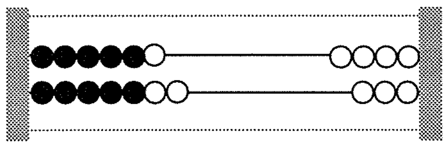


figure 2.14: six plus seven

Finally, the device provides opportunity for different strategies. The sum 13 – 7, for instance, can be solved in various manners (fig. 2.15).

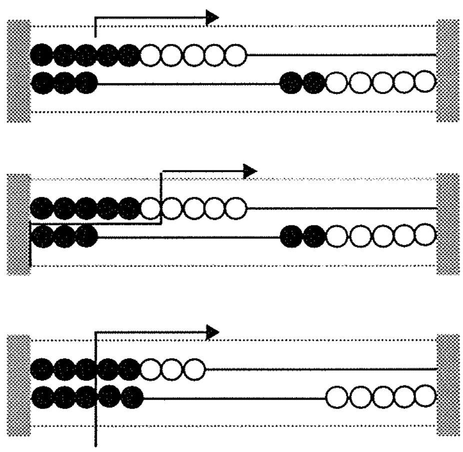


figure 2.15: various solutions to 13 – 7

So much for the informal strategies for doing arithmetic up to 20. The arithmetic methods invented by children themselves can also be used to outline column arithmetic as we mentioned earlier.

Treffers (Treffers c.s., 1988) shows that many students develop an informal procedure in which subtraction is done in columns from left to right. Contrary to the standard procedure, the children consciously employ the position values of the numbers: in 675 – 482, 7 – 8 is read as 70 – 80. The problem is solved by first subtracting the hundreds: 600 – 400 = 200. Then the column of the tens follows. Here, the students end up ten short. This can be settled straight away: 200 – 10 = 190 (some students use an intermediary notation and calculate this later).
The units are next, 5 – 2 = 3 and the answer is compiled: 190 + 3 = 193.

These informal solution methods can be seen as an intermediary form between mental arithmetic and column arithmetic, and can ultimately be abbreviated to an arithmetic method.

Turning back to the fundamental criticism by Cobb (1987) of the implicit 'observer's' standpoint that is often taken, we can establish that recent developments in realistic mathematics education place the 'actor', even more than before, at centre stage. If one wants to adopt the viewpoint of the student, the solution procedures of the children have to be taken very seriously. That would seem the best route to follow to avoid misunderstandings. However, some kind of tension will continue to exist between the following and the guiding of children. In this sense, the realistic approach differs from constructivism.

Constructivism is still primarily a research approach that is directed at analyzing micro didactical situations and the actual theories of children. Realistic instruction theory is directed on long term learning processes and tries therein to do justice to the own contribution of the students.

2.6 conclusion

Both from action psychology as well as from information processing psychology, it can be understood that the danger exists, that working with manipulative material does not prepare for working without manipulatives.

Action psychology makes us aware of the possible differences between the external action, the mental action and the pursued structure of action. On the one hand the danger lurks of a manipulative action without insight that does result in the requested action result. On the other hand there is the problem of the transition from thinking about material to thinking in terms of mathematical relationships and concepts. Cognitive psychology makes us aware of the fact that students interpret new information, therefore also the use of manipulatives, from their own knowledge. Cobb points out that as a consequence manipulative material must be regarded from the standpoint of the student. The student only sees the manipulative material and not the mathematical relationships which adults recognize in it. In this connection he refers to the mixing of the intended internal (mental) representation and the actual external (concrete) representation. This occurs with Dienes blocks where ten, a hundred and a thousand are concretely presented as perceptive units, while it is expected that the students are using mental mathematical objects.

Cobb's distinction between the point of view of the child (actor) and that of the outsider (observer) is induced by his theory of constructing knowledge. The conception that everyone forms his or her own image, his or her own theory about reality, makes Cobb realize that the reality of the student is a different reality than that of the developer/researcher. Children construct their own theories about reality and will in general tend to hold on to these theories.

In this sense, realistic instruction theory can be regarded as in concordance with the constructivistic approach. The reference to conflict situations as a means to further learning already points in this direction. But especially the idea of re-construction of knowledge relates closely to constructivism. In realistic education theory, two sources are tapped in designing instruction courses which are meant to elicit this reconstruction process: the history of mathematics and the spontaneous, self invented arithmetic methods of children.

The use of manipulatives is thus placed in a different perspective. It is not the material that transmits certain knowledge. In the 'historical' elaboration of realistic instruction theory, material is only an aid to solve certain practical problems in a certain context. In this approach, understanding and insight are supported by the context, which can serve as a situation model. In the 'informal solution' variant of realistic instruction theory, the material is used to elicit (mental) arithmetic actions which other children have previously developed themselves. Close study of the actual occurrence of such acts is necessary.

In a general sense we can draw the conclusion that it must not too readily be assumed that instruction activities and visible learning behavior will lead to the intended learning result. And even though the realistic approach seems to offer solutions to prevent discrepancies, a study of the actual solution process of the children and of the actual forming of mathematical concepts and relationships, remains essential.

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3 Mediating between concrete and abstract

introduction

One of the problematic issues in mathematics education is the question of how to teach students abstract mathematical knowledge. In the mainstream information processing approach, one usually presents concrete models to help students acquire this abstract knowledge. However, 'concrete' in the sense of tangible does not necessarily mean 'concrete' in the sense of making sense. This observation is in line with research findings (which will be presented later) that the use of manipulatives[1] does not really help students attain mathematical insight. Moreover, even if a certain mastery of procedures is attained, their application appears to be problematic. The manipulatives approach fails, probably because – although the models as such may be concrete – the mathematics *embedded in the models* is not concrete for the students. Or, to put it another way, the manipulatives approach passes over the situated, informal knowledge of the students. Alternative approaches depart from the idea that situated, informal knowledge and strategies should be the starting point for developing abstract mathematical knowledge.

In this chapter we will present such an alternative to information processing approaches, known as *a domain specific theory for realistic mathematics education*. This is an approach for mediating between concrete and abstract based on self-developed models. We may characterize this approach as bottom-up, since the initiative is with the students. As such, it is in contrast with the top-down character of the manipulatives approach, where models (e.g. manipulatives) are derived from abstract mathematical knowledge. Both approaches will be described and explained from a learning sequence on long division. It will be argued that the realistic approach deals with problems starting from students informal knowledge, whereas a top-down approach relies on the idea of transfer. The bottom-up character of the realistic approach is expected to guarantee insightful mathematical knowledge, and the realistic concept of generalizing is presented as a bottom-up alternative for the top-down concept of transfer.

3.1 long division with manipulatives

In the mainstream information processing approach, formal crystallized expert mathematical knowledge is taken as a starting point for developing instructional activities. In general, representational models and manipulatives are designed to create a concrete framework of reference in which the intended mathematical concepts are embodied. That is to say, that abstract mathematical knowledge and procedures are introduced, exemplified and learned with manipulatives. This approach is based on the idea that everyday life is not pure enough for the learning of mathematics. Everyday situations are thought to be too complex; there is too much distraction from the mathematics embedded in them (see for instance Gagné, 1965). Therefore, an artificial environment is created where no frills distract from the mathematical content.

For instance, base ten blocks, often referred to as Dienes blocks, constitute such an environment. The blocks exemplify the base ten position system, and the students will learn how to deal with this system by working with the blocks in a prescribed manner. The characteristics of the positional system are incorporated with rules such as:

- exchange groups of ten blocks for higher order blocks if there are more than ten of a kind,
- notate the number of blocks in a strict order, corresponding with an increase in value.

In general, the main concern of this form of teaching is not with the place value system as such, but with the written algorithms. Therefore, other rules are added.

For addition and subtraction, the students must start with the smallest blocks: first the ones, next move to the ten, etc. This is required because working from right to left, which implies starting with the smaller units, is the standard procedure for column algorithms.

Division is built on the idea of division as fair sharing. One starts with small numbers. A problem like $84 \div 6$ will be thought of as dividing 84 blocks among 6 persons. First 6 of the 8 ten-rods are distributed, then the remaining tens are exchanged for ones, and the resulting 24 ones are divided, each getting 4. Usually problems with a remainder are avoided in the beginning of instruction.

Next, to make the shift from dealing with blocks to the paper and pencil algorithm, a procedure like the one outlined above is soon replaced by procedures executed with imaginary blocks, using a standard form that resembles the written algorithm. This shift is also necessary when larger numbers are introduced. Moving to larger numbers, larger dividends come first and larger divisors follow later. When dealing with larger numbers, working with blocks becomes unfeasible.

Take $1476 \div 24$ as an example. The students would have to exchange a one thousand block and four hundred squares to get 147 rods that will have to be divided over 24 people (fig. 3.1).

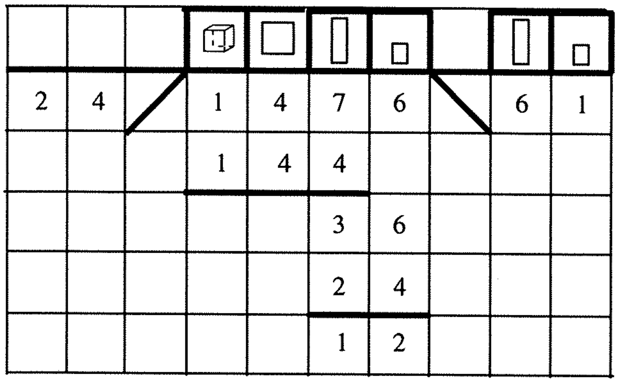


figure 3.1: 1476 divided by 24

Thinking along the lines of fair sharing, this procedure makes sense. In applications, however, long division does not necessarily have to involve fair sharing. Suppose, for instance, that the problem 1476 ÷ 24 was about 1476 bottles which are to be packed in crates that hold 24 bottles each. In this case, it does not make much sense to consider fair sharing or distribution.

Moreover, the procedure becomes rather confusing for reasons outlined below.

- Executing the procedure sketched above might make one wonder. Why should 1476 be represented with blocks and 24 not? For addition both numbers are represented with blocks, for division apparently only the dividend is represented with blocks.
- Compared with the other operations there is also a change of strategy; where previously one always worked from small to large, one now starts with distributing the largest blocks.
- If the blocks are thought of as a representation of the applied situation, other inconsistencies arise. In the problem above, the blocks at first represent the bottles: 1476 bottles represented by 1476 ones, organized in one cube of a thousand, four squares of a hundred, seven rods of ten and six ones. After the execution of the division procedure, the answer is represented with six rods and one unit block. But the blocks now represent 61 crates! So at first the blocks represent *bottles* and in the end they represent *crates*. But if we consider the remainder, 12, these 12 blocks still represent bottles!

3.2 research findings

It is now possible to say that research has shown that this approach does not work. Firstly, students do not gain much insight from it (Labinowitcz, 1985; Resnick and

Omanson, 1987). Thus, students may succeed with manipulatives and fail with the paper and pencil algorithm. Secondly, even if students learn to master the concepts and procedures that are taught, they lack the capability to use them in applications (Schoenfeld, 1987).

Careful analyses from a constructivist point of view have resulted in a convincing explanation of what goes on in regular classrooms and what goes wrong with conveying knowledge with the help of manipulatives (Cobb, 1987). In short, the mathematical concepts embodied in the didactical representations are only there for the experts who already have those concepts available to be able to recognize them in the materials. For the students, there is nothing more to see than the concrete material. In other words, concrete embodiments do not convey mathematical concepts.

At the same time, novice-expert paradigm inspired research generated a growing awareness of the importance of domain specific, situated, informal knowledge and strategies. Research illustrates that in everyday situations people are quite able to use whatever holds the situation offers in order to develop rather efficient strategies (Brown, Collins and Duguid, 1989; Lesh, 1985; Nunes, 1992). Furthermore, Carpenter and Moser (1983, 1984) found that young childrens' performance on word problems is far superior to their performance on bare sums, due to the use of informal strategies. (We may note in passing that this contradicts the suppositions of the information processing approach, where applications are postponed.)

In conclusion, we may say that shortcomings of a manipulatives based learning sequence are exposed as lack of insight and problems with applications, and we may also conclude that these problems stem from ignoring the importance of informal knowledge and strategies.

3.3 new approaches

Educationalists attach different consequences to the recognition of the importance of informal strategies and situated cognition. In this chapter three current approaches that recognize the existence of childrens' informal strategies will be described, followed by a detailed exploration of the approach at the Freudenthal Institute, the realistic mathematics education approach.

The *Cognitively Guided Instruction* project (CGI), for instance, opts for innovating mathematics education by informing teachers about informal strategies (Carpenter, Fennema, Peterson, Chiang and Loef, 1989). The idea of this project is to present research findings on informal strategies to teachers to help them construct a referential framework. With the help of this framework the teacher can then guide the spontaneous learning process of the students.

Brown, Collins and Duguid (1989) propose a teaching model which they call *cognitive apprenticeship*. They base this model on the assertion of the indexical character of all knowledge; '(...) knowledge is situated, being in part a product of the activity, the context, and culture in which it is developed and used.' (ibid, p. 32) All words, they argue, are at least partially indexical. Pure indexical words such as 'this', 'here' and 'now', can only be interpreted in the context of their use. However, they claim all words are at least partially context-dependent: 'the meaning of a word cannot, in principle, be captured by a definition, even when the definition is supported by a couple of exemplary sentences.' (ibid, p. 33)

They conclude that new knowledge and skills have to be developed in applied situations. Since most applied situations are too complex for novices to handle, they propose the idea of cognitive apprenticeship by analogy to apprenticeship in vocational training. Keywords in this approach are coaching, scaffolding and fading.

They point out Schoenfeld's (in press) and Lampert's work (1986) for concrete elaborations of similar instructional concepts. If we look at Lampert's work (1989) as an example, we can see that she uses a top-down design strategy. She describes an instructional experiment on decimals in which the system of decimal fractions and its notation system are taken as given, and the instructional activities focus on *connecting* the informal, situated knowledge of the students with this pre-existing system. A clear advantage of this approach is that negotiations about interpretations and meanings are made explicit and placed at the centre of the instructional process. From a constructivist point of view, a drawback, however, is that expert knowledge is taken as an immediate goal for instruction. This implies that the students are not free to construct their own solutions, but that they have to find out what the teacher has in mind.

Socio-constructivists argue that all knowledge is self-constructed, thus mathematics education should acknowledge idiosyncratic constructions and foster a classroom atmosphere where mathematical meaning, interpretations and procedures are explicitly negotiated. According to Cobb, Yackel and Wood (1992) the teacher should at the same time stimulate a process of acculturation into the practices and interpretations of the wider community. In this socio-constructivist teaching approach, the self-invented problem solutions of the students are to be framed as topics for discussion, to function as the starting points for the acculturation process of the classroom community. Socio-constructivism does not as such however offer heuristics for developing instructional activities that are compatible with a constructivist epistemology. Albeit curricula have been developed in the Problem-Centered Mathematics Project at Purdue University.

For mathematics education a suitable domain specific instruction theory can be found in the theory for *realistic mathematics education* (Treffers, 1987). This theory will be the focal point of this chapter. In the following, we will introduce this theory briefly first, then we will present a description of a realistic course for developing long division. This example will be used as a concrete base for a more elaborate description of the key principles of the realistic theory. Finally, we will discuss whether this approach supports applicability.

3.4 realistic mathematics education

Realistic mathematics education is rooted in Freudenthal's interpretation of mathematics as an activity (Freudenthal, 1971, 1973). Freudenthal takes his starting point in the activity of mathematicians, whether pure or applied mathematicians. He characterizes mathematical activity as an activity of solving problems, looking for problems and organizing a subject matter – whether mathematical matter or data from reality. The main activity, according to Freudenthal, is organizing or mathematizing. Interestingly, Freudenthal sees this as a general activity which characterizes both pure and applied mathematics. Therefore, when setting 'mathematizing' as a goal for mathematics education, this can involve mathematizing mathematics and mathematizing reality.

One could remark that Freudenthal's concept of mathematics as a human activity is mainly concerned with the individual, in what Ernest (1991) calls 'the private realm'. There are other scientists who stress the opposite, 'the social realm' (ibid), where mathematics comes to the fore as mathematical discourse (Balacheff, 1990; Schoenfeld, 1987). However, the social interaction is not neglected in realistic mathematics education. It is essential to the teaching and learning process, which we will discuss later.

Freudenthal uses the word 'mathematizing' in a broader sense than simply an indicator of the process of recasting an everyday problem situation in mathematical terms. It is also employed within mathematics. In Freudenthal's view, mathematizing relates to level-raising – in a mathematical sense. The idea of level-raising is at the heart of Freudenthal's (1971, p. 417) concept of mathematics learning: the activity on one level is subjected to analysis on the next; the operational matter on one level becomes a subject matter on the next level. Level-raising is obtained when we promote features that characterize mathematics, such as generality, certainty, exactness and brevity. In order to clarify what we mean by mathematizing, we can look at strategies we use to promote these mathematical characteristics:

- for generality: generalizing (looking for analogies, classifying, structuring);
- for certainty: reflecting, justifying, proving (using a systematic approach, elaborating and testing conjectures, etc.);
- for exactness: modelling, symbolizing, defining (limiting interpretations and validity);
- for brevity: symbolizing and schematizing (developing standard procedures and notations).

In realistic mathematics education, mathematizing mainly involves generalizing and formalizing. Formalizing embraces modelling, symbolizing, schematizing and defining, and generalizing is to be understood in a reflective sense. It refers to a posteriori constructions of connections rather than a premeditated application of general knowledge. There is little explicit attention to proving, but reflecting and justifying are central to the course.

From the students' point of view generalizing and formalizing are not central issues; they are mainly guided by considerations of efficiency.

Freudenthal suggests that mathematizing is the key process in mathematics education for two reasons.

Firstly, mathematizing is not only the major activity of mathematicians. It also familiarizes the students with a mathematical approach to everyday life situations. Here we can refer to the mathematical activity of looking for problems, which implies a mathematical attitude, encompasses knowing the possibilities and the limitations of a mathematical approach, knowing when a mathematical approach is appropriate and when it is not.

The second reason for making mathematizing central to mathematics teaching relates to the idea of reinvention. In mathematics, the final stage is formalizing by way of axiomatizing. This end point should not be the starting point for the mathematics we teach. Freudenthal argues that starting with axioms is an anti-didactical inversion; the process by which the mathematicians came to their conclusions is turned upside down in education. He advocates mathematics education organized as a process of guided reinvention, where students can experience a (to some extent) similar process as the process by which mathematics was invented. In the following sections, the realistic mathematics education approach will be illustrated using the teaching of long division as an example.

3.5 developing long division

In the realistic approach contextual problems are used as a starting point, preferably problems that allow for a variety of informal solution procedures. So, *applied problems precede instruction on the algorithm.* The instructional sequence on long division can start, for example, with the problem described by Dolk and Uittenbogaard (1989), where children of about 8 or 9 years were asked to solve the following:

> Tonight 81 parents will be visiting our school.
> Six parents can be seated at each table.
> How many tables do we need?

The teacher gave the students a cue by drawing a few tables (fig. 3.2) on the board.

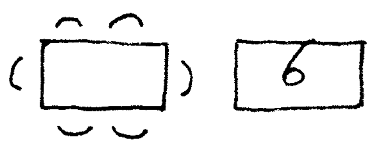
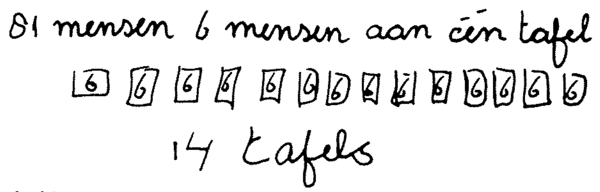


figure 3.2: setting out tables

The students produced all kinds of solutions:

- some used repeated addition: $6 + 6 + 6 + ...$, or stepwise multiplication, probably based on addition, $1 \times 6, 2 \times 6, 3 \times 6, ...$, some only wrote down the resulting sequence 6, 12, 18, ...;
- some used 10×6 as a starting point, in order to continue by multiplication or repeated addition;
- one student knew $6 \times 6 = 36$ by heart, which was doubled to get $12 \times 6 = 72$, one 6 was added, and finally one more 6.

81 people; six at each table (tafels)



each pot holds seven cups, how many pots (koffiepotten) are needed?

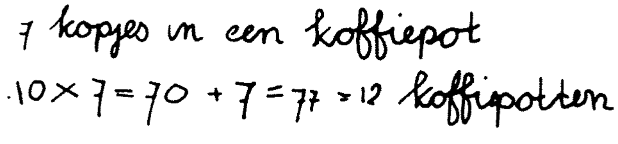


figure 3.3: the solution procedures of one of the students

The teacher stimulated the students to compare their solutions. Obviously, most found the first jump to 10×6 a nice short cut. When a similar problem (concerning the same night at school) was administered afterwards, it appeared that a substantial number of students imitated the ten times short-cut spontaneously (see fig. 3.3). The problem read like this:

> One pot serves seven cups of coffee; each parent gets one cup.
> How many pots of coffee must be brewed for the 81 parents?

Only one child stuck to the single step method. Thirteen used ten times, compared to six in the first round. From the work of three children it was not clear how they arrived at their answers.

It should be noted that the teacher did not ask the students to use ten times in the coffee task. He expected that the children who saw its advantage and trusted it would adopt it on their own initiative.

The procedure that is employed here to solve what in principle is a division problem can be labelled as compounding. One tries to approach the dividend as closely as possible by adding multiples of the divisor. As a matter of fact, people often prefer this strategy for mental divisions. For instance, in The Netherlands, where gas consumption is measured by kilometers per liter of fuel, the easiest way to find the average fuel consumption of one's car is to reset the odometer when the tank is full and compare the number of kilometers with the amount of fuel needed to refill the tank next time you go to the gas station.

We can even try a more precise estimate while driving away from the gas station. Suppose you used 34.09 liters for 466.8 km, which would require the division $466.8 \div 34.09$. To keep it simple, let us do $467 \div 34$. Obviously, 34 goes at least ten times into 467. Ten times 34 gives 340 to start with. Two more times? No, three more times gives $340 + 102 = 442$, which means 13 km per liter. Or more precisely, one decimal at least: with 25 left by $467 - 442$ do $0.5 \times 34 = 17$, $0.7 \times 34 \approx 17 + 7 = 24$. So our estimate would be 13.7 km per liter.

If we compare this calculation with the column algorithm, it appears that our mental arithmetic resembles the standard procedure (fig. 3.4a). In fact, the standard procedure can be translated into the mental procedure (fig. 3.4b).

```
34 / 467 \  13.7
     34 -
     127
     102 -
      250
      238 -
       12
```

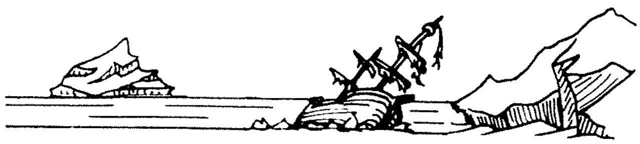
figure 3.4a: standard procedure

```
34 / 467.0 \ 10+3+0.7=13.7
     340 -  [10 x 34]
     127
     102 -  [3 x 34]
      25.0
      23.8 - [0.7 x 34]
       1.2
```

figure 3.4b: interpretation

However, the algorithm is so condensed that one hardly realizes that in the first step $10 \times 34 = 340$ rather than 34 is subtracted. On the other hand, it is not so difficult to recognize the underlying repeated subtraction behind this procedure: after any subtraction of a multiple of the divisor, one concentrates on what is left. In fact the column algorithm of long division is nothing but the most abbreviated manner to perform a division by counting how often the divisor can be subtracted from the dividend.

In realistic mathematics instruction, the standard procedure is taught by letting it evolve from informal ones, in a learning process which starts in a situation where the mathematical model of repeated subtraction offers itself in a natural manner. Rather large numbers can be used in the assignments right from the start. This is the case in the following problem that is presented in the broader context of a story about Dutch sailors whose ship was stranded on the isle of Nova Zembla (fig. 3.5).



The captain of the stranded ship is told that there are 4000 biscuits left.

The crew consists of 64 members. Each man gets 3 biscuits a day,

which means 192 biscuits a day for the whole crew.

How long will this supply last?

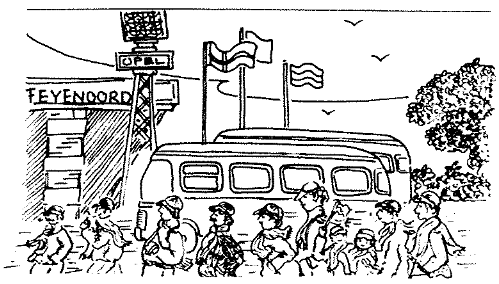
figure 3.5: overwintering in Nova Zembla

Identifying with the situation, we can almost see the supply of biscuits diminish day by day, every time a ration is consumed. What makes this problem interesting is the variety of solving procedures on different levels. Some of the students may continue to subtract 192 one at a time. Others may use multiples of 192, such as decuples, or doubles (fig. 3.6) but use 192 also.

4000		4000		4000	
192	- 1 day	192	- 1 day	1920	- 10 days
3808		3808		2080	
192	- 1 day	384	- 2 days	1920	- 10 days
3616		3424		160	
192	- 1 day	768	- 4 days		
3424		2656			
192	- 1 day	1536	- 8 days		
etc.		etc.			

figure 3.6: repeated subtraction of smaller or larger quantities

With appropriate contextual problems, one can induce children to use decuples such as in the problem in fig. 3.7.



1296 supporters want to visit the away soccer game of Feijenoord.

The treasurer learns that one bus can carry 38 passengers and

that a reduction will be given for every ten buses.

figure 3.7: Feijenoord

The information about the reduction can work as a suggestion to calculate the number of reductions. It will call the students' attention to the opportunities offered by the decimal system. Even then, various solutions are possible (fig. 3.8).

38 / 1296 \	36 / 1296 \	36 / 1296 \
380 - 10x	380 - 10x	1140 - 30x
916	916	156
380 - 10x	760 - 20x	152 - 4x
536	156	4
380 - 10x	76 - 2x	
156	80	
38 - 1x	76 - 2x	
118	4	
38 - 1x		
80		
38 - 1x		
42		
38 - 1x		
4		

figure 3.8: various levels of curtailment

Such steps on the way to the column algorithm are opportunities for students to make discoveries at their own level, to build on their own experiential knowledge and perform shortcuts at their own pace. Working with realistic problems also implies a different approach to the problem of the remainder, i.e. as a real life phenomenon that calls for practical solutions, rather than as a peculiarity of non-terminating divisions which must be justified by formal arrangements. If the context is taken seriously, then '34 rem. 4' is not an acceptable answer. What can we do with these four supporters? Well, there are several possibilities, distribute them over the other buses, order an extra bus (or a car), or speculate on the withdrawal of at least four supporters at the last moment.

The continuation of this instructional sequence will include fractions and decimals. The fuel consumption problem (discussed earlier) shows how the solution procedure can be extended to incorporate decimals (or fractions) in the quotient. The same situation suggests how to deal with decimals in divisor and dividend: we are dealing with a ratio. A rate of fuel consumption of 34.09 liter for 466.8 km is the same as that of 3409 liter for 46680 km. In other words, the division can be freed from decimals by multiplying divisor and dividend with the same factor; the ratio stays the same.

3.6 teaching-learning process

Although we briefly mentioned the whole class discussion of solution procedures for the table setting at the parents' meeting, we concentrated so far on the nature of contextual problems and individual solution procedures. It must now be emphasized that class discussion of solution procedures and problem situations is one of the core activities in realistic mathematics instruction. These discussions centre around the correctness, adequacy and efficiency of the solution procedures and the interpretation of the problem situation. In this context the socio-constructivist assertion must be taken into account: there is no such thing as *the task*. A so-called taken-as-shared interpretation of a task is interactively constituted in the classroom community. This incorporates implicit negotiation of notions such as what counts as a problem and what counts as a solution (Cobb et al., 1992; see also Yackel, 1992).

Specifically for realistic mathematics instruction this interpretation of a task must be directed at the real life character of contextual problems, since there is always a tension between practical solutions and a mathematical interpretation of the task. One must acknowledge that one cannot bring the reality into the classroom. Although students will be able to identify with well chosen contextual problems, these will never become real life problems. The extent to which practical considerations are valued is part of what will be established as the classroom culture.

Note that those notions of what counts as a problem and what counts as a solution are not self-evident. This was clearly illustrated when we presented 12 year old low attainers with the following problem about a school party. There are 18 bottles of cola for 24 students and the bottles must be distributed over the tables fairly, taking into account the different numbers of students at each table (tables with 1, 2, 4, ... students). What was intended as the task was the production of equivalent ratios (bottles per students). Some students, however, did not want to interpret the task in this way. They thought equivalent ratios of the bottles per students was inadequate because 'Some students don't drink cola' and also 'They don't drink the same amount'.

A whole class discussion in realistic mathematics education differs to some extent from mathematical discourse. Mathematical discourse is identified with conjecturing, justifying and challenging. Although some of the activities in realistic mathematics education are of this type, others have a different focus. Part of the whole class discussion refers to the interpretation of the situation sketched in the contextual problem. Another part of the discussion focuses on the adequacy and the efficiency of various solution procedures. This can involve a shift of attention towards a reflection on the solution procedure from a mathematical point of view. The latter discussion closely resembles what is viewed as a mathematical discourse.

To elucidate mathematical discourse, one can think of Lakatos' (1976) famous reconstruction of the coming about of the Euler formula as a paradigm. However, this may lead to an overestimation of the importance of the discourse as a mathematical activity, since this was a process of a different magnitude – the elaboration of the Euler formula was enacted over many decades. Moreover, mathematical discourse represents the practice of the mathematical research community, it does not cover applied mathematics. In realistic mathematics education, practices of applied mathematicians are thought to be more relevant for primary school mathematics.

Realistic mathematics education places the student in quite a different position than traditional educational approaches. Students have to be more self-reliant. They cannot turn to the teacher for validation of their answers or for the directions for a standard solution procedure. Research by Desforges and Cockburn (1987) shows that it is difficult to implement a problem-solving approach. Students seem to feel insecure and keep asking for directions and approval, while teachers find it much easier to deal with a class which is executing routine tasks than with students who are left to their own devices to solve problems. These problems could in part be due to a change in the so-called 'classroom social norms' (Cobb, Perlwitz and Underwood, 1992).

Cobb and his colleagues argue that classroom social norms need to be explicitly re-negotiated. The students have to become aware of the change in what is expected of them in mathematics lessons. They are no longer expected to simply produce correct answers quickly by following prescribed procedures. In realistic mathematics, like in inquiry mathematics, they have other obligations, such as explaining and justifying their solutions, trying to understand the solutions of others, and asking for explanations or justifications if necessary.

This change in social norms corresponds to another role of the teacher. The authority of the teacher as a validator is exchanged for an authority as a guide. He or she exercises this authority by way of selecting instructional activities, initiating and guiding discussions, and reformulating selected aspects of students mathematical contributions.

3.7 key principles of realistic mathematics education

A teaching strategy that leads to comparing and explaining solutions by students is only possible when the learning sequence consists of contextual problems that give rise to a variety of solution procedures. It is the variety that allows for discussions about adequacy and efficiency, which in turn lead to a reflection about these procedures from a mathematical point of view. This brings us back to the learning sequence and its underlying theory. In the following section the realistic approach will be further explained by elaborating three key principles that can be seen as heuristics for instructional design.

The first principle is termed *'guided reinvention'* and *'progressive mathematizing'*. According to the reinvention principle, the students should be given the opportunity to experience a process similar to the process by which mathematics was invented. The history of mathematics can be used as a source of inspiration for course design. The reinvention principle can also be inspired by informal solution procedures. Informal strategies of students can often be interpreted as anticipating more formal procedures. In this case, mathematizing similar solution procedures creates the opportunity for the reinvention process. In a general way one needs to find contextual problems that allow for a wide variety of solution procedures, preferably those which, considered together, already indicate a possible learning route through a process of progressive mathematization.

The second principle relates to the idea of a *didactical phenomenology* (Freudenthal, 1983). According to a didactical phenomenology, situations where a given mathematical topic is applied are to be investigated for two reasons. Firstly, to reveal the kind of applications that have to be anticipated in instruction; secondly, to consider their suitability as points of impact for a process of progressive mathematization. If we see mathematics as historically evolved from solving practical problems, it is reasonable to expect to find the problems which gave rise to this evolving process in present day applications. Next, we can imagine that formal mathematics came into being in a process of generalizing and formalizing situation-specific problem-solving procedures and concepts about a variety of situations. Therefore, the goal of our phenomenological investigation is to find problem situations for which situation-specific approaches can be generalized, and to find situations that can evoke paradigmatic solution procedures that can be taken as the basis for vertical mathematization.

The third principle is found in the role which *self-developed models* play in bridging the gap between informal knowledge and formal mathematics. Whereas manipulatives are presented as pre-existing models in the information processing approach, in realistic mathematics education models are developed by the students themselves. This means that students develop models in solving problems. At first, a model is a model of a situation that is familiar to the student. By a process of generalizing and formalizing, the model eventually becomes an entity on its own: It becomes possible that it is used as a model for mathematical reasoning (Gravemeijer, 1994; Streefland, 1985; Treffers, 1991). This transition from model-of to model-for is similar to the theoretical reconstruction of the genesis of subjective mathematical knowledge by Ernest (1991):

> 'What is proposed is that by a vertical process of abstraction or concept formation a collection of objects or constructions at lower, preexisting levels of a personal concept hierarchy become reified into an object-like concept, or noun-like term.' (p. 78)

In the following paragraphs, these key principles of realistic mathematics education will be illuminated with the sequence of long division as an example.

3.8 reinvention/mathematizing

The difference between mathematics instruction according to a realistic approach versus an information processing approach is most apparent in the way applications are dealt with. The information processing approach views mathematics as a ready-made system with general applicability, and mathematics instruction as breaking up formal mathematical knowledge into learning procedures and then learning to apply them. Within the realistic approach, the emphasis is on mathematizing. Mathematics is viewed as an activity, a way of working. Learning mathematics means doing mathematics, of which solving everyday life problems is an essential part. A variety of contextual problems is integrated in the curriculum right from the start.

These two fundamentally different views about mathematics and mathematics education imply essentially different mathematical learning processes. If mathematics is viewed as a formal system, its applicability is provided by the general character of its concepts and procedures, and thus, first of all, one must adapt this abstract knowledge to solving problems set in reality. One has to translate real life problems into mathematical problems. This can be visualized as shown in fig. 3.9.

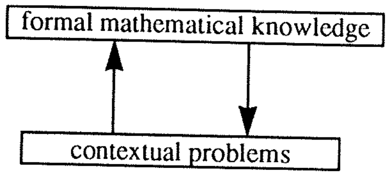


figure 3.9: application of formal mathematics

The model describes the process of solving a contextual problem with the help of formal mathematical knowledge. First, the problem is translated; it has to be formulated in mathematical terms, as a mathematical problem. Next, this mathematical problem is solved with the help of the available mathematical means. Finally, the mathematical solution is translated back into the original context. Transformation of a contextual problem into a mathematical problem implies a reduction of information; many aspects of the original problem will have been obliterated. The translation from mathematics back to the original problem, therefore, involves an interpretation of the mathematical solution within the context of the original problem. The aspects that were obliterated have to be taken into account again. But it may be the case that the original problem does not allow for the exactitude which is suggested by the mathematical solution. On the whole, the translation described above boils down to recognizing problem types and establishing standard routines.

In contrast, if we choose to teach mathematics as an activity, problem-solving takes on a different meaning. Teaching becomes problem-centered, which means that the problem is the actual aim, rather than the use of a mathematical tool. Even if problem-solving passes through the same three stages of describing the contextual problem more formally, solving the problem on this (more or less) formal level, and translating the solution back to the context, the character of these activities is now fundamentally different. Rather than aiming at fitting the problem into a pre-designed system, one tries to describe it in a way that allows us to come to grips with it. Through schematizing and identifying the central relations in the problem situation, we come to understand the problem better. The description we provide can be sketchy and using self-invented symbols (fig. 3.10); it needs not be presented in commonly accepted mathematical language.

figure 3.10: a student's description of the number of tables needed for 81 parents

The description does not automatically answer the question, but simplifies the problem by describing relations and distinguishing matters of major and minor importance. Solving the problem as it is stated at this more or less formal level differs greatly from applying a standard procedure. It is a matter of problem-solving as well. Translating the final solution does not differ that much from translating a solution which is produced by a standard procedure. But translation and interpretation are now easier because the symbols are meaningful for the problem-solver, who is the one who gave them their meaning (fig. 3.11).

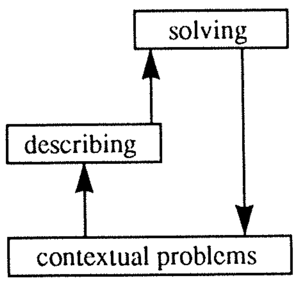


figure 3.11: realistic problem-solving

Within this type of instructional program, students learn to mathematize contextual problems. The sequential solution of similar problems induces another process. Problem descriptions develop into an informal language, which in turn evolves into a more formal, standardized language, due to a process of simplifying and formalizing. This is also a process of mathematizing, albeit stretched over a longer period of time. Something similar happens to the solving procedure. In the long run, solving some kinds of problems may become routine, i.e., the procedure is condensed and formalized in the course of time. Genuine algorithms can thus take shape (fig. 3.12).

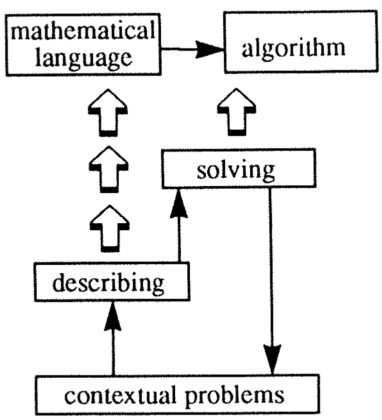


figure 3.12: vertical mathematizing

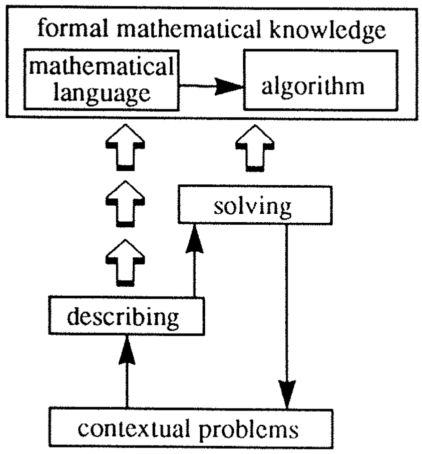


figure 3.13: reinvention

This is a learning process by which formal mathematical knowledge itself can be (re)constructed (fig. 3.13). Following Treffers (1987), the latter process – the mathematization of mathematical matter – is called vertical mathematization. This is distinguished from horizontal mathematization, which is mathematizing contextual problems. Freudenthal (1991) characterizes this distinction as follows:

> 'Horizontal mathematization leads from the world of life to the world of symbols. In the world of life one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly: this is vertical mathematization. The world of life is what is experienced as reality (in the sense I used the word before), as is symbol world with regard to abstraction. To be sure the frontiers of these worlds are vaguely marked. The worlds can expand and shrink - also at one another's expense.' (p. 41, 42)

As Freudenthal indicates, the boundaries between what is to be denoted as horizontal mathematization, and what as vertical mathematization, are ill defined. The crux is in what is to be understood as reality. Freudenthal (1991) elucidates: 'I prefer to apply the term 'reality' to that which (one K.G.) at a certain stage of common sense experiences as real.' (p. 17)

Reality is understood as a mixture of interpretation and of sensual experience. This implies that mathematics too can become part of one's reality. Reality and what one counts as common sense is not static but grows under the influence of the learning process of the person in question. This is also how Freudenthal's statement about 'Mathematics starting at, and staying within reality' (Freudenthal, 1991, p.18) must be understood. One might argue that this idea is better expressed by 'common sense mathematics' than by 'realistic mathematics'.

This progress is supported by suitable contextual problems. These problems can facilitate certain interpretations and strategies leading to horizontal mathematizing processes. The distribution of rations in the Nova Zembla story, for instance, is easily interpreted as a situation for repeated subtraction. The reduction given for every ten buses in the supporters problem can function as a hint for using decuples.

In designing the sequence, the reinvention principle is used as a guidance by asking: 'How could I have invented the standard procedure for long division?' The answer is based on the recognition that long division is based on repeated subtraction, which in turn is related to informal strategies of adults or students (like the one used for calculating fuel consumption). This sort of consideration allows the designer to find contextual problems that can function as anchoring points for a learning sequence.

Summarizing, in realistic mathematics education mathematics is primarily seen as a process, a human activity. At the same time the reinvention principle means that this activity results in mathematics as a product. Vertical mathematizing is in the core of this process. Vertical progress is reflected in a sequence of gradually more formal symbolizations and solution procedures. This is shown in the long division sequence, which starts with informal notations that lead to more formal schemes resembling the standard written algorithm. Meanwhile elaborate solution procedures such as repeated subtraction are shortened through the use of decuples and centuples, ending in the maximally curtailed standard algorithm.

3.9 didactical phenomenology

The understanding of division as distribution does not require explanation. Even without instruction young children know what to do in sharing situations. This can be illustrated by the solutions of 8 and 9 year olds for the problem of dividing 36 by 3 (Galen et al., 1985). They had never done multiplications with numbers bigger than 10, let alone performed the inverse division procedure. Of course, rather than using a formal representation like 36 ÷ 3 = ..., the contextual problem was presented as shown in fig. 3.14.

Three children shall divide 36 sweets. How many will each of them get?

figure 3.14: dividing sweets

The students invented all kinds of solving procedures:

- dividing on a geometrical basis (fig. 3.15a) by dividing the area of the square with 36 sweets into 3 equal parts;

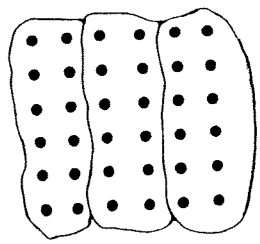


figure 3.15a: geometric division

- distributing one by one (fig. 3.15b), by crossing each one out of the total and adding it to one of the rows (the students even tried to copy the children's portraits faithfully);

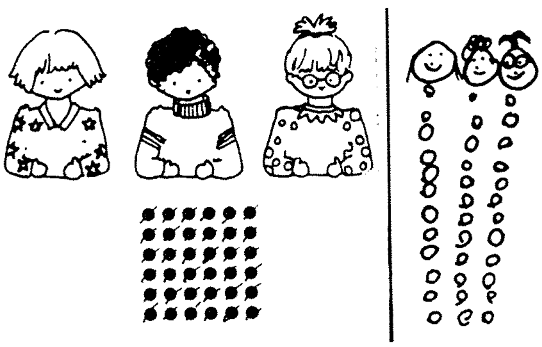


figure 3.15b: piece wise distribution

- grouping in triads (fig. 3.15c), probably by reasoning that each time a sweet is distributed to each of the children, the supply diminishes by three; thus, they figured out how many groups of three one could create;

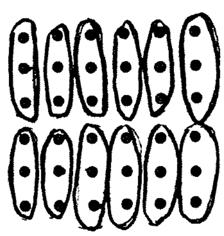


figure 3.15c: grouping in triads

- using multiplication facts (fig. 3.15d), by a method of grouping (the students were likely to know that $12 = 3 \times 4$; perhaps 3×4 was recognized in the pattern; of each twelve, four can be given to each child; doing this three times gives each child twelve sweets).

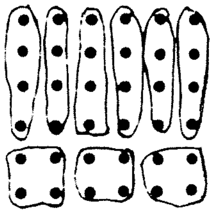


figure 3.15d: division based on $3 \times 4 = 12$

Implicit in all these approaches is the conception of division as repeated subtraction, fair sharing or distribution, and as the inverse of multiplication. Repeated subtraction in divsion can be conceived as the counterpart of repeated addition in multiplication (see also Freudenthal, 1983). Repeated subtraction and distribution are also referred to as ratio division and distribution division, respectively (fig. 3.16). The distribution division appears most clearly in the geometric solution and in distributing one sweet for each child at a time, when the students interpreted the problem by creating three equal groups.

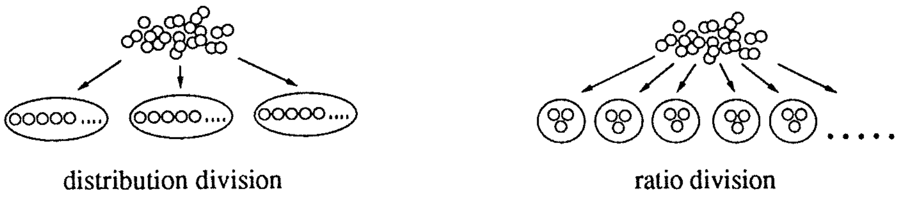


figure 3.16: two basic forms of division

Some students approached the problem by asking how many groups of three can be made. This corresponds to a ratio division. The ratio approach involves a translation of the problem, because the original contextual problem raises the question of a distribution division. A ratio division can be suggested more directly by a problem such as:

> A net can hold three balls.
> How many nets will be needed to pack 36 balls?

Taking our point of view on multiplication, the two types of division can be seen as two different inverse operations. In a times b equals c, the two factors have different roles; a is taken b times. In applications the difference is especially clear when b is a magnitude (Freudenthal, 1983). Take for instance:

3 × 12 sweets = 36 sweets,
with its inverses:
36 sweets : 12 sweets = 3,
and
36 sweets ÷ 3 = 12 sweets.

Besides this, there are other applications where both factors are magnitudes, as in:

quantity × unit price = total price,
time × speed = distance etc.

The problem for the educational designer is to decide which type of division should be chosen as the starting point for the development of long division. Learning sequences for both division types have been worked out in realistic mathematics education.

The distribution division was initially elaborated in a sequence; where the piece wise distribution was supported with drawings of hands or cups, for example; very similar to the drawings of the children in the sweets division problem. These elaborate descriptions are needed to promote the step from distributing one by one to distributing larger portions at a time (fig. 3.17).

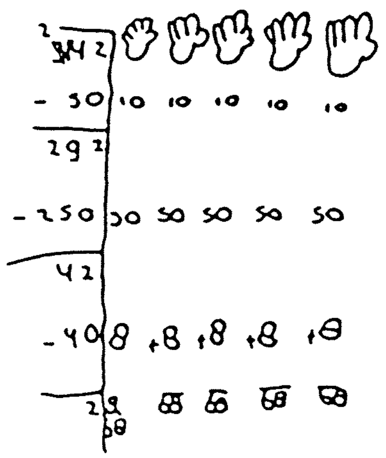


figure 3.17: a scheme for distribution division

These supports were abandoned towards the end of the sequence. Eventually, the children drew only one column, because all the columns were the same.

Although comparative research shows that such a distribution division sequence was far superior to a traditional approach (Rengering, 1983), there are good reasons to switch to ratio division.

- Ratio division can more easily be connected to the column algorithm. Solving the contextual problem does not have to be reformulated in terms of repeated subtraction, and no sub-stage is needed where the complete division process is represented with hands or cups.

- The ratio division has the advantage that many applications correspond with ratio division. Moreover, children instructed in the traditional way find it difficult to relate the long division algorithm to the ratio division type of applications. Research by Hart (1981) shows that, rather than using the column algorithm, students are inclined to solve applied (ratio) division problems by repeated subtraction.
- The problem about transporting football supporters in buses constitutes a good argument for choosing the ratio division. Due to its dynamic character, the problem may function as a paradigmatic situation. Thinking of buses will enable the students to concretize a division problem which is presented numerically.
- It is not so difficult to extend the ratio division procedure to distribution problems (as will be shown later). And whatever division type is chosen as a starting point, both have to be integrated in the learning sequence.

In addition to the distinction between ratio division and distribution division, other phenomenological variations must be taken into account. Some of these variations are illustrated by the diversity of meanings which can be attributed to the remainder. Treffers, De Moor and Feijs (1989) list the following examples for the division $26 \div 4$:

'1. 26 passengers have to be transported by cars. Each car can carry 4 passengers. How many cars will be needed?	[7]
2. A rope of 26 meter is cut into pieces of 4 meters. How many pieces does one get?	[6]
3. If 26 bananas are to be fairly divided among 4 people, how many bananas will each one get?	[$6\frac{1}{2}$]
4. A 26 km walk is divided into 4 equal stretches. How long is each of them?	[6.5]
5. A rectangular pattern of 26 trees with 4 trees per row, how many rows will there be?	[??]
6. A rectangular terrace with a size of 26 square meters has a width of 4 meters. How long is this terrace?	[6.5]'

The interpretation of the remainder largely depends on the situation in which the result of $26 \div 4$ has to be used. It will be clear that a formal treatment of the remainder, detached from applications, will not prepare the students for this wide variety of interpretations. The student will have to encounter all kinds of situations, not only to become familiar with different interpretations, but also to learn to attune the interpretation of the remainder to its meaning in the problem situation.

3.10 self-developed models

At first glance, models seem less prominent in realistic mathematics education than in manipulatives-based mathematics education. However, the models differ mainly in role and character. This can be elucidated by analyzing the position and role of models in relation to formal and informal knowledge in both approaches.

In the information processing approach, expert mathematical knowledge is embodied in concrete models, through manipulatives. We may characterize this as a top-down strategy, because the formal expert knowledge is taken as the source for the didactical models. Furthermore, the implicit supposition is that formal mathematical knowledge, once acquired, has general applicability. For this reason, initially no attention was given to application in this approach. When it showed that applicability was a problem, separate models were developed to support application, however, without abandoning the idea of general applicability.

On the whole, models are primarily used to constitute a concrete point of departure for developing formal mathematics. This implies, however, that no explicit connection is made with the informal, situated knowledge of the student. In more recent cognitive approaches this omission is corrected. In these approaches, models are used as mediating tools to bridge the gap between situated knowledge and formal mathematics. An objection to an intermediary model would be that there is still a top-down element in this approach; the formal knowledge is treated as a given and the intermediate model is derived from this formal mathematical knowledge.

In all those cases the label 'model' refers to concrete models such as manipulatives and diagrams. However, we can also use a broader concept which includes situation models and mathematical models. In realistic mathematics education paradigmatic situations can develop into 'situation models'. In the learning strand on long division, repeated subtraction can be seen as a mathematical model.

Following the reinvention principle, a bottom-up approach is pursued. The idea is that the students construct models for themselves and that these models serve as a basis for developing formal mathematical knowledge. To be more precise, at first, a model is constituted as a context-specific model of a situation. Later, the model is generalized over situations. Thus, the model changes in character; it becomes an entity on its own. In this new shape it can function as a basis, a model *for* mathematical reasoning on a formal level. The bottom-up character of this approach is prominent in the nature of the models; in realistic mathematics education the models are inspired by informal strategies, whether used by students or in the history of mathematics.

In realistic mathematics education we can distinguish four levels: situations, model of, model for, and formal mathematics (fig. 3.18).

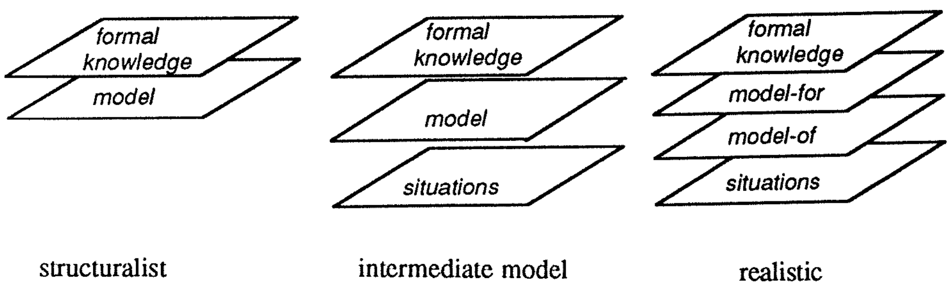


figure 3.18: models

Considering long division, the first level is associated with real-life activities, such as sharing sweets among children – not with paper and pencil. Here, the students bring in their situational knowledge and strategies and apply them in the situation. The second level is entered when the same sweets division is presented as a written task and the division is modelled with paper and pencil. A characteristic problem for this level is the one about busing the Feijenoord supporters. To solve this the students create a model of the situation: filling buses is modelled by repeated subtraction. However, the situation still pervades the solution process. The third level marks the next step in the development of long division: the focus is shifted towards strategies from a mathematical point of view. Taking the optimal centuple or decuple becomes a topic for discussion with the question, What is the biggest, neat portion that one can take away at once? Now the student is just dealing with numbers, without thinking of the situation. The fourth level, finally, would contain the standard written algorithm for long division.

Note that the term model should not be taken too literally. It can also concern a model situation, a scheme, a description, or a way of noting. In the example above of long division the situations for application of long division are modelled with repeated subtraction. It is this procedure of repeated subtraction that legitimizes the formal long division algorithm.

The levels can also be described in more general terms (fig. 3.19):

- the level of the situations, where domain specific, situational knowledge and strategies are used within the context of the situation (mainly out of school situations);
- a referential level, where models and strategies refer to the situation which is sketched in the problem (mostly posed in a school setting);
- a general level, where a mathematical focus on strategies dominates the reference to the context;
- the level of formal arithmetic, where one works with conventional procedures and notations.

This general description is more adequate because not all learning sequences relate either to a model-of or to a model-for description.

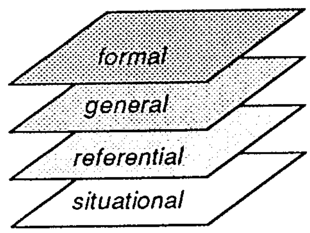


figure 3.19: levels

In realistic mathematics education, models are placed at an intermediary level between situated and formal knowledge. According to the 'model-of' versus 'model-for' distinction, this intermediary level is then separated into a referential level and a general level. The referential level contains the models, descriptions, concepts, procedures and strategies that refer to concrete or paradigmatic situations. As a result of generalization, exploration and reflection, this level is developed in order to make reflection about the strategies more dominant. The formal level can be seen as a formalization of the general level. In other words, the general level functions as the referential level for the formal level.

We should remark that the levels that are discerned are local levels; they are tied to a specific topic, basic facts, two digit addition and subtraction, written algorithms, fractions, and so forth. The levels are not absolute in another sense; the different levels are not segregated. The idea is that the student should be able to revert to a lower level. The lower levels are meant to be incorporated into the higher levels. The general level – with the model-for function – is not detached from the originating contexts: e.g. the formal algorithm for long division is justified as a procedure for repeated subtraction, and this notion of repeated subtraction in turn relies on knowing that division can be interpreted as repeated subtraction in contextual problems.

3.11 discussion

In this chapter we reviewed two diverging approaches to the teaching of mathematics. The mainstream information processing approach, which proposes embedding the mathematics in concrete models, was considered to result in insufficient understanding and lack of applicability of the learned procedures. It was suggested that a bottom-up approach is needed to address the problem of insufficient understanding. Realistic mathematics education represents a bottom-up approach. But can this approach also help improve applicability? The answer to this question lies in the role of *generalizing* in realistic mathematics education.

generalizing vs. transfer

It could be argued that the situatedness of knowledge learned within a realistic approach might cause barriers for transfer; what is learned in one situation may not seem applicable in other situations. However, when working towards generalization in realistic mathematics education, we are not looking for the application of a procedure to a new situation, as is often meant by the notion of transfer (De Corte, Verschaffel and Schrooten, 1991). When we pose non-routine problems to students in realistic mathematics education, the situation is a genuine problem-solving situation to which the students will have to bring all the informal knowledge and strategies they possess. The new situation is not a situation for transfer in the sense of finding out what routine or strategy has to be used. It is about a realistic problem that has to be solved within a context in which the students can use their common sense. By generalizing we do not mean the application of a routine procedure, but rather an a posteriori construction of connections between various situations.

Related to this is Steffe's distinction between 'being efficient in-action' and 'being efficient prior-to-action' (Steffe, personal communication). Students may solve a problem with the same procedure. However, being efficient-in-action one student may (re)invent the procedure while solving the problem, and another student may apply the procedure knowingly. In the latter case, the student has to anticipate the applicability of this particular solution procedure before deciding to apply it – prior to action. If the student in the first case realizes that his/her solution is similar to the solution to other contextual problems solved earlier, this student is generalizing. In the second case, the student already knows the algorithm; in this case one could speak of transfer of the written algorithm.

This interpretation of generalizing can be exemplified in the context of division. It is not self-evident that the repeated subtraction procedure is applicable in a situation of sharing and distribution. However, it has already been noted that some of the solutions given by nine year olds for the problem of sharing 36 sweets among three children used the repeated subtraction of threes. The next step, of course, is the use of multiples of ten and hundred. The choice of the contextual problem is essential, as in all other learning sequences. Dividing money, for instance, provokes an interpretation of the situation in terms of bills of one dollar, two, five, ten, twenty, one hundred dollars etc.

When dividing $10,000 among 17 people it make sense to give each participant $500 first. This leaves 1500 dollars (10,000 – 8500) to be divided, etc. The whole story can be notated in a subtraction scheme that is very similar to that of the ratio division $10,000 ÷ $17.

100000		10000	
8500 -	500 (dollars)	8500 -	500 x
1500		1500	
1360 -	80 (dollars)	1360 -	80 x
140		140	
136 -	8 (dollars)	136 -	8 x
14		14	
...		...	

figure 3.20a: $10,000 ÷ 17. figure 3.20b: $10,000 ÷ $17

As shown in fig. 3.20a and b, the difference between the two situations is reflected in the notation. In the distributive division 17 × 500 is subtracted in the first step, in the ratio division this is 500 × 17. Knowing that 17 × 500 and 500 × 17 can be seen as two manifestations of the same multiplication, the student can connect the same scheme for repeated subtraction with the two kinds of division problem.

As these examples from division problems illustrate, generalizing is fostered by looking for similarities, which enables one to classify problem situations as belonging to the same type. At the same time, the solution process can be structured and thus, generalizing takes shape as an organizing activity, as a form of mathematizing. By mathematizing the situation students may find its similarity to other situations and realize that earlier invented solution procedures could be used in the new one. The way in which students in the previous examples approach applied problems is fundamentally different from the situation where they have been taught a procedure which should now be applied.

This does not mean that direct application is excluded. However, application presupposes the primacy of the 'theory' which is to be applied. An actual reconstruction of the situation to make the procedure applicable will often be needed. That is to say, one decides to try to apply a certain procedure and, next, one tries to interpret the situation in such a way that this procedure actually can be applied. In practice, it will often be a matter of commuting between the interpretation of the problem and a review of possibly suitable strategies or procedures. In simple cases, application will demand only recognition; in more complex cases, it takes some effort to find cues that might suggest trying a specific procedure. We suggest that if a solution procedure is rooted in a generalization over various situations as described above, there will be a wide range of situations where application will be relatively easy.

notes

1 In mathematics education, the term manipulatives is used as a collective noun for tactile material and graphical representations, that function as models.

2 However, base ten blocks are just one category of the MAB blocks Dienes invented. In his view one should vary over various number systems (e.g. base three, base six and base eight, next to base ten) to account for mathematical variability.
3 In the following we will use the terms 'situated' and 'situations' in a restricted sense, referring to the kind of situations where students develop informal strategies, e.g. situations that are personally meaningful for the students.
4 Realistic mathematics education will not be compatible with constructivism in every shape or form. However, socio-constructivists accept endpoints that fit with the realistic approach (Cobb, 1992a). Moreover, they accept a certain amount of guidance by the teacher. In our opinion, realistic mathematics education can be made compatible with socio-constructivism if notions like 'negotiation of meaning' are integrated in the realistic approach. This idea is being elaborated by Cobb (Vanderbilt University), Erna Yackel (Purdue University Calumet) and the author.
5 Contextual problems describe situations where a problem is posed. More often this will be an everyday life situation, but not necessarily so; for the more advanced students mathematics itself will become a context.
6 Examples taken from a Dutch textbook series, 'Rekenen & Wiskunde' (Gravemeijer et al., 1983).

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4 Educational development and developmental research in mathematics education

introduction

Research on mathematics education is subject to many changes; views on mathematics education are changing and at the same time research paradigms are shifting. In the community of mathematics educators the view of mathematics as a system of definitions, rules, principles, and procedures that must be taught as such is being exchanged for the concept of mathematics as a process in which the student must engage. In the United States this view of mathematics education is eloquently advocated in the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 1989).

These changes being promoted have their consequences for curriculum studies and curriculum design. First of all, there is a practical problem: there are few, if any, textbook series or other forms of curricula that fulfill the requirements of the NCTM Standards. Therefore, new curricula will have to be developed. Second, the question arises whether the curriculum strategies that are in line with the traditional research-development-diffusion (RDD) model are apt to develop a curriculum that fits the NCTM Standards. Nevertheless, together with the changes in mathematics education, new approaches in educational research are developing. The positivist research paradigm is losing ground in favor of an interpretative approach (Walker, 1992). Large survey studies and studies that focus on general trends are no longer 'in vogue', and the new trust is in interpretative case studies. Their aim is to find out and make sense of what is going on in classrooms. Some of the representatives in this line of research are integrating research and instructional design. This varies from using a teaching experiment as an expedient in psychological research (Cobb and Steffe, 1983; Steffe, 1983) to using teaching experiments to answer questions of instructional design (Lampert, 1989, 1990).

In this way, educational research approximates practices in curriculum development projects, which do not follow a (strict) RDD-tradition. The Madison Project and the University of Chicago School Mathematics Project (UCSMP) in the United States and the School Mathematics Project (SMP) and the work of the Shell Center in England are notable examples of such projects.

Here, the developmental work more often includes informal research and theory development. In general, however, those basic research efforts are not elaborated in a systematic way, nor are they reported in research journals.

In this chapter I discuss curriculum development and educational research in The Netherlands, where the concept of integrating design and research has a long-standing tradition. In The Netherlands, the development of a type of mathematics education similar to the ideas presented in the NCTM *Standards*[1] has been pursued for the last two decades. Here, curriculum development is embedded in a holistic framework, defined as 'educational development', which embraces all the developmental activities and interventions between the initial idea and an actual change in educational practice[2].

I shall show that curriculum development in this context can be described as purposeful and sensible tinkering of sorts (referred to as 'theory-guided bricolage': a process that is guided by a theory and also produces theory). Curriculum development as such does not necessarily produce new theories: one may only expect this if the researcher pursues theory development as a specific goal. Therefore, I must distinguish between developmental work and developmental research. The latter is not yet well established as a research discipline, and so this chapter aims to explicate its characteristics and to discuss its methodological aspects. In this context, qualitative research is considered as a source for methodological guidelines. Here the issue of the interconnectedness of empirical and theoretical justification arises and is explored by means of a model for two-digit addition and subtraction. The goal of the discussion is to illuminate the characteristics of the theoretical and empirical components of the justification, that underpin the local instruction theory incorporated in this learning sequence. The chapter concludes with some remarks on the relation between educational development, developmental research, and evaluation.

4.1 educational development

In general, curricula are developed to change education, to introduce new content or new goals, or to teach the existing curriculum according to new insights. Notwithstanding the final goal of implementing change, development and implementation are completely separated in the conventional RDD model, which was in vogue in the '60s and '70s.

In opposition to this approach, Freudenthal (1991) set out his concept of 'educational development'. This is more than just curriculum development; it also contains the end goal of changing educational practice.[3] Educational development not only implies that the implementation of the curriculum is anticipated from the outset, it also implies that preservice and in-service teacher training, counseling, test development, and opinion shaping are incorporated in the development work. Starting in 1970, all of these activities were carried out by the Institute for Development of Mathematics Education (IOWO), under Freudenthal's directorship.[4]

The kind of change pursued is one towards what is now called 'realistic' mathematics education: mathematics education that is compatible with the idea of mathematics as a human activity (Freudenthal, 1973). Such a goal differs from those usually elaborated in the RDD model. In this model, one usually starts off the development phase with concrete operational goals, and one builds on the educational research executed in the research phase. The change in mathematics education as set out by Freudenthal is, however, so innovative that more general educational research does not offer much support. In addition, when IOWO began, little content-specific research was available. Consequently, questions of how to develop instruction had to be answered during the process of educational development. This highly exploratory character of educational development, inspired by a philosophy of mathematics education, shapes, then, the developmental work, the implementation, and the research. Note that there is no sharp demarcation between development and research; curriculum development may encompass research elements and developmental research relies on developing prototypes of instructional sequences. Curriculum development, however, is primarily product-oriented, whereas developmental research is theory-oriented. In addition, curriculum development is self-contained and time restricted, whereas developmental research accumulates knowledge in a long-term research process.

curriculum development

In Freudenthal's philosophy of mathematics education, mathematics ought not to be associated with mathematics as a well-organized deductive system, but with mathematics as a human activity (Freudenthal, 1971, 1973). A mathematician himself, Freudenthal claimed that for mathematicians, mathematics is an activity of doing mathematics; an activity the greater part of which consists of organizing or mathematizing subject matter. The subject matter can be taken from reality and it must be organized according to mathematical patterns when solving problems from reality. Students should begin by mathematizing subject matter from reality. Next, students should switch to analyzing their own mathematical activity. The latter is essential since it contains a vertical component, which Freudenthal in reference to Van Hiele (1973) describes thus: 'The activity on one level is subjected to analysis on the next, [and] the operational matter on one level becomes a subject matter on the next level.' (Freudenthal, 1971, p. 417)

Unlike Van Hiele, Freudenthal has numerous minute levels in mind. The key principle is reflection (mathematizing one's own mathematical activities) that brings about these levels. For curriculum development this implies that the instructional activities should capitalize on mathematizing as the main learning principle. Mathematizing enables students to reinvent mathematics.

RDD-inspired curriculum development strategies would not appear suited to the development of realistic mathematics curricula, because most of these curriculum strategies are based on the notion of 'capitalizing on existing knowledge' (specifically scientific knowledge, preferably underpinned by quantitative empirical research (e.g., Gagné and Briggs, 1979; Romiszowski, 1981)). And, what is more, most strategies for instructional design are grounded in an empiricist means-end rationality. One starts by identifying in detail the goals to be reached by the curriculum, in order to determine the means to reach these goals. That is to say, one designs the course in such a way that one can be sure that this is an appropriate means for attaining those goals according to the knowledge available. In short, these strategies suffice only in cases where the goals can be expressed precisely and without distortion, and where there is a great deal of relevant research knowledge.

For the first realistic textbook series which were developed in the late '70s and early '80s, these conditions were not fulfilled. The central goal, mathematics as a human activity, was only vaguely defined and as such hard to measure. There was no consensus about the concrete objectives among the proponents of realistic mathematics education. In addition, only a little a priori knowledge about adequate instructional arrangements in realistic mathematics education was available. Because a new curriculum cannot be designed from scratch, the developer looks for examples of instructional activities that can be adapted to his/her overall concept of mathematics education and can be fitted into a total structure.

The characteristics of such a design process are exemplified by the learning strand for elementary arithmetic in one of the first realistic textbook series, *Rekenen & Wiskunde*, grade one (Gravemeijer, 1983), where instructional activities designed by Davydov are integrated. Davydov (1972) elaborated activity theory concepts in an instructional design on addition and subtraction, aiming at the transition from material action to mental action. These instructional activities are integrated in the realistic learning strand, however, without adopting the activity theory framework. Davydov's instructional activities are primarily seen as a way to integrate ordinal and cardinal number concepts.

Compiling a course is to some extent comparable to solving a jigsaw puzzle (although what is different is that most puzzles have a fixed number of rigid pieces and only one solution). At one time the pieces may be chosen on the basis of their color, at another on the basis of their shape or because of a recognizable image. In a similar manner, the deliberations of the curriculum developer will vary. Psychological considerations will be alternated with mathematical arguments, instructional options, or implementational concerns. In this context it can be argued that instructional activities can be thought of as detached from the teaching-learning theory in which they are presented.

theory-guided bricolage

The thinking process involved in the sort of curriculum development just described resembles the thinking process that Lawler (1985) characterizes by the French word *bricolage*, a metaphor taken from Claude Lévi-Strauss. A *bricoleur* is a handyman who invents pragmatic solutions in practical situations that can differ greatly from what a professional would have chosen. The bricoleur is used to undertaking all kinds of jobs, thereby differing from the technician to whom acceptability of a task depends on the availability of appropriate tools and materials. Having limited means available, the bricoleur has become adept at using whatever is available. The bricoleur's tools and materials are very heterogeneous: some remain from earlier jobs, others have not been collected with a certain project in mind but because they might come in handy later. We can imagine that a bricoleur who starts a new project begins by figuring out how the problem can be tackled with the materials available. The technician will do something similar, but will be more inclined to look elsewhere for other tools and techniques.

Lawler envisions this concept of bricolage as an adequate characterization of human activity. Thinking, too, can be seen as a kind of bricolage. This gives a description of thinking as an activity that is highly compatible with other human activities. Lévi-Strauss (cited in Lawler, 1985) characterizes bricolage thus: 'In the continual reconstruction from the same materials, it is always earlier ends which are called upon to play the part of means.' (p. 52) The same idea of bricolage as a process of growth is also found in Jacob (1982), who refers to evolution as a kind of bricolage, for which he prefers the English expression 'tinkering':

> 'Evolution behaves like a tinkerer who, during eons upon eons, would slowly modify his work, unceasingly returning to it, cutting here, lengthening there, seizing the opportunities to adapt it progressively to its new use.'
>
> (cited in Lawler, 1985, p. 253)

This idea of adapting, improving, and adjusting continuously is characteristic of educational development, where the development never stops. Educational development, however, is more conscious and more goals oriented than evolutionary tinkering: it is better described by the term bricolage. Moreover, the bricoleur's flexible and creative way of dealing with means corresponds well to the work of the curriculum developer.

Curriculum development can very well be seen as constituting a composition of instructional activities that makes sense to the developer. Making sense in this case means that the set of instructional activities has the character of a theory on how to ensure that the students learn what is intended. If that is true, what matters is not how the instructional activity was seen by the original author, but if and how this activity can be fitted into the new composition in a sensible fashion.

In constructing a set of instructional activities that makes sense, the developer of realistic mathematics education is guided by beliefs about what mathematics is, how it is learned, and how it should be taught. This belief system of the developer functions as a background theory by which all instructional activities are evaluated.

This is why one may speak of 'theory-guided bricolage'. At first, this 'theory' is more like a philosophy, a vision. However, this philosophy has a theoretical core. Underpinning the idea of mathematics as a human activity is a theory of learning in which the mental activity of the student is at the center. The 'set of instructional activities that make sense' makes sense because it incorporates a learning route – a learning route in the form of a well-considered composition of learning processes in terms of mental activities.[5] Moreover, the core theory is embedded in a framework of theories or theoretical notions on learning, instruction, and instructional design. It is this set of theoretical notions that guides the developmental work from the outset – not just in thinking out the appropriate instructional activities but also in relation to classroom trials.

4.2 developmental research

The elaboration of an educational design is, in practice, constituted via a thought experiment. Before the instructional activities are tried out in the classroom, the developer will try to envision how the teaching-learning process will proceed; it is on these thought experiments that the instructional design is founded. Freudenthal (1988) points to the function of thought experiments in physics. According to him, physics, as it is known to us today, did not emerge as a theory nor purely as a result of experiments, yet it did develop in a rational way.

> 'One did not experiment at random, nor did one theorize left and right, hoping that experiments would lead the way or that one of the theories would do so. The development was guided by what Mach called a thought experiment.'
>
> (Freudenthal, 1988, p. 51)

As an example, Freudenthal mentions a thought experiment of Galileo. The question whether heavier objects would fall faster (in vacuum) can be answered by thinking of dropping two objects from the same height, disconnected or joined. By gluing the two objects together (or cutting one object into two pieces) one may come to realize that the speed will be the same in both cases.

Freudenthal (1988) thinks it self-evident that thought experiments are important in educational development. The developer will envision how the teaching-learning processes will proceed, and afterwards he or she will try to find evidence in a teaching experiment that shows whether the expectations were right or wrong. The feedback of practical experience into (new) thought experiments induces an iteration of development and research.

This cyclic process is at the center of Freudenthal's concept of developmental research. Unlike what is suggested by the RDD approach, development practice depends upon a cyclic alternation of development and research: a cyclic process that is more efficient when the cycle is shorter. What is invented behind the desk is immediately put into practice; what happens in the classroom is consequently analyzed and the result of this analysis is used to continue the developmental work.[6]

This process of deliberating and testing results in a product that is theoretically and empirically founded – well-considered and well-tried.

theory development

On a micro-level, this interaction between development and research is self-evident. However, according to Freudenthal (1991), often it is not acknowledged that such a relation also holds on a macro-level; nevertheless, theory is produced in a similar manner. The cyclic process that Freudenthal discerns can also be seen as a learning process of the developer.

Returning to the theory-guided bricolage, we may say that a global a priori theory – which Freudenthal would prefer to call a philosophy – guides developmental work. This theory functions as a basis for a learning process by the developer that is nurtured by the cyclic alternation of thought experiment and practical experiment. Such a learning process can be interpreted as theory development: each course can be seen as the concrete sediments of a local instruction theory.

However, we cannot stamp every learning process of a developer as theory development, unless we are willing to stretch our concept of a theory so far that it loses meaning. A local instruction theory requires at least a broader embedding in a more general theoretical framework. Certain requirements of coherence and consistency must be fulfilled, and the theory should also include the mental activity of the students. A learning sequence based solely in terms of observable behaviors will not do. A theoretical consideration of the mental processes necessary for progress and integration is needed also.

Therefore, we will have to make a distinction between curriculum development and developmental research. To clarify this distinction we may remark that research can be distinguished from development by its goal: the building of justified theory, not just for the private realm of the researcher but to be put before the research community. This indicates research activities like theory development (whether adding new elements, elaborating on accepted theories, or abandoning falsified theories), corroboration, and reporting. The last of these brings with it the obligation to present research findings in such a way that the research community can grasp the arguments and weigh the empirical evidence.

The distinction between research and development also involves a difference in conditions. In curriculum development, the focus is on the instructional activities that embody the educational change; the emphasis is on the product, not on the learning process of the developer. On the whole the knowledge that is gained will remain implicit, tacit knowledge. In developmental research, knowledge gain is the main concern. The focus is on building theory, explicating implicit theories. In curriculum development, especially in textbook development, other priorities hold; there are deadlines to be met, and any elbowroom that the deadlines permit will be used to improve the product, not on reflection.

When textbook development is supported and inspired by prototype development – as carried out by the IOWO – better conditions for developmental research result. Within prototype development, careful experimentation, systematic description, and reportage will get more attention. Moreover, there will be more room for elaborating and extending ideas and there will be more time for reflection. Note that both bricolage and the cyclic process of deliberating and testing receive different interpretation here. Here, theory plays a more important role than in textbook development, which implies that the bricolage will be more consciously theory-guided. In a similar fashion, the process of deliberating and testing will be more consciously theory-oriented.

What is more important, however, is that in the case of curriculum development, the bricolage concept illuminates the idea of using what is available and adapting those means to one's momentary goals. In the case of developmental research, the evolutionary aspect is much more important, not in the sense of a random process channeled by natural selection, but as a goal-oriented process of improvement and adjustment: a process that is guided by a theory that grows during the process.

At the start, this theory consists of a global framework, with key concepts such as mathematics as a human activity, mathematizing and reinvention. This global theory is elaborated in the prototypes that represent local theories (e.g., local instruction theories on fractions, addition and subtraction, or written algorithms).

In other words, global theory is concretized in local theories. Vice versa, the more general theory can be reconstructed by analyzing local theories. In this manner, Treffers (1987) (re)constructed a domain specific theory for realistic mathematics instruction. What he did was to try to make sense of twenty years of development work, carried out inside and outside IOWO and its successor OW&OC. In this way, he was able to trace five characteristics of 'progressive mathematizing', as he denotes the actual elaboration of the reinvention principle. Progressive mathematizing, in turn, could be embedded in Van Hiele's level theory (Van Hiele, 1973, 1985) and Freudenthal's didactical phenomenology (Freudenthal, 1983). I sketch Treffers' 'realistic' instruction theory in the following way.

Van Hiele distinguishes three levels of thought, which Treffers denotes as: an intuitive phenomenological level, a locally-descriptive level, and a level of subject-matter systematics (the level of mathematics as a formal system). These levels, which are subject-matter dependent, can be used for the global organization of an instructional course. Amidst these macro-levels we may situate Freudenthal's minute levels, which are attained by subsequent mathematization of these levels (i.e. reinvention by progressive mathematization). This process is characterized by five types of activity:

1 *Phenomenological exploration*: In line with the basic ideas of Freudenthal's di-

dactical phenomenology, emphasis is laid on a phenomenological exploration – '(...) starting with those phenomena that beg to be organized and from that starting point teaching the learner to manipulate these means of organizing.' (Freudenthal, 1983, p. 32)

2 *Bridging by vertical instruments*: Broad attention is given to models, model situations, and schemata that, rather than being offered right away, arise from problem-solving activities and subsequently can help to bridge the gap between the intuitive level and the level of subject-matter systematics.
3 *Student contribution*: The constructive element is visible in the large contribution to the course coming from the student's own constructions and productions.
4 *Interactivity*: Explicit negotiation, intervention, discussion, cooperation, and evaluation are essential elements in a constructive learning process in which the student's informal methods are used as a lever to attain the formal ones.
5 *Intertwining*: The holistic approach, which incorporates applications, implies that learning strands can not be dealt with as separate entities; instead, an intertwining of learning strands is exploited in problem solving.

Notice how the relation between theory and development in realistic mathematics education differs from the traditional relation. The theory applied in curriculum development is not a well-defined, fixed theory. The initial theory is global, to some extent vague, and open for adaptation. Application of an a priori theory is not under discussion; the theory functions as a guideline and it inspires developmental research. The more refined theory is an a posteriori theory: it is the reconstruction of a theory in action.

To put it another way, global basic theory is elaborated and refined in local theories. At the same time, the basic theory itself is developing. The central idea – mathematics as a human activity – remains the same; the relating theories, however, are adapted continuously. In this sense, there is a clear kinship with the idea of a research program as identified by Lakatos. Lakatos (1978) states that a research program is progressive if there is theoretical and empirical progress. This theoretical progress is provided by the aforementioned theory development. We can speak of empirical progress if those theoretical extensions are empirically justified. However, we must realize that an empirical justification in educational development will differ from the empirical justification in the physics which was modeled by Lakatos.

deliberation and testing

According to Freudenthal (1991), one of the most important differences between physics and the social sciences is the possibility or impossibility, respectively, of replication. In the natural sciences, he argues, it is easy to present new knowledge as the result of an experiment, because such an experiment is easily repeated. In educational development, replication in a strict sense, is impossible. An educational experiment cannot be repeated in the same manner, under the same conditions. Therefore, new knowledge will have to be legitimized by the process by which this new knowledge was gained.[7]

To Freudenthal, this is especially important for the conveying an innovation. To consider only the courses that have been produced will not be sufficient, for such a product can be interpreted in a variety of ways. If one is to use an innovation sensibly, one must know how it was developed. For Freudenthal, becoming conscious of the developmental process and explicating this process is the essence of developmental research:

> 'Developmental research means: 'experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience.'
> (Freudenthal, 1991, p. 161)

Freudenthal demands a constant awareness of the developmental process, a state of permanent reflection, as much as possible, of which is reported in order to make it credible and transferable. As shown before, such a developmental process contains a theoretical component (deliberation) and an empirical component (testing). This implies that the performance of the students – independent of how they may have been assessed – must also be included in this report.

Teachers will have to be encouraged to confront the new ideas with their own practice theories and to try out the new ideas in their own classroom. If that works, the implementation of the curriculum will take on the character of ongoing developmental research.

The teacher, too, will pass through a learning process that is nurtured by experience and reflection, a learning process that can coincide with the implementation of the new curriculum. In this manner, one can avoid the pitfall of the RDD-model, where 'a teacher could use the materials and alter some teaching behaviors without coming to grips with the conceptions or beliefs underlying the change.'(Fullan, 1982, p. 30) The intended learning process of the teacher can be accommodated with in-service teacher training designed according to the concept of the reinvention principle, with room for working on ones own level, didactizing, key points for reflection, theory, information about learning and teaching, and so on (Gravemeijer, 1987).

justification

To the forum of researchers the justification of the new theory will be of major interest. But, although a distinction can be made between the context of discovery and the context of justification, discovery and justification cannot be separated in such a strict sense (compare Lakatos' (1976) typecasting of theory development in mathematics). In the case of developmental research, theory is not put to the test after the development has been concluded. Instead, it is the developmental process itself that has to underpin the theory.

In the cyclic process of development and research as sketched by Freudenthal, discovery and justification are closely interwoven. Discovery is not restricted to the thought experiment, and justification is not merely found in the results of the trials; some discoveries are made in the trials phase, and part of the justification is not empirical.

Justification is also found in the thought experiment; however, it is then justification based on arguments. In a positivist interpretation, justification is confined to empirical testing; in the case of developmental research, however, the rationale for ones choices and the interpretation of the empirical data are part of the justification as well. This is connected with a shift from what Habermas (1987) calls 'Zweckrationalität' (means-end rationality) to what he calls 'kommunikativer Rationalität' (communicative rationality). The positivist rationality which only takes into account means-end relations, is exchanged for a broader kind of rationality based on argumentation and comprehension.

As the deliberation contributes to the justification, so the trials phase can provoke discoveries. In the trials phase the focus is not only on justification, one also looks for discoveries that will enable theoretical progress. An example is given by Streefland (1990), who found that the spontaneous solution procedures of students solving fraction problems pointed to powerful instructional activities. It showed that the informal strategies of the students anticipated the formal procedures aimed for in the course, and by elaborating on those strategies the gap between informal strategies and formal procedures could be bridged.

qualitative research

One of the cornerstones of the positivist research paradigm is prediction: both in fundamental research, where experiments are used to test theoretical predictions, and in curriculum research, where experiments are used to obtain generalizable predictions about the effects of curricular interventions.

In developmental research, making sense of what is going on is more important than prediction. Here, the experimental experiences are subjects of an interpretive process. The researcher tries to make sense of what is going on in the classroom against the background of the thought experiments that preceded the instructional activities.

There is a strong similarity with qualitative research, where gathering data and analyzing data also often coincide. The empirical evidence in developmental research is more often qualitative in nature, in the sense as explicated by Smaling (1990). According to Smaling, the analysis of the data stays close to their original meaning. The data are not projected onto a mathematical-numerical system with the objective of doing analytical reasoning within that system, but -

> 'Analyzing more often is a process of interpreting, in which the data which are gathered in this research are compared with other data, in which each item is interpreted

in the light of the data as such. An important step in the transition from the data as such to the interpretation is the construction of categories of data and the construction of concepts.'

(Smaling, 1990, p. 5)

To the extent that developmental research is qualitative in nature, the norms and principles of qualitative research can be applied. How reliability, validity, and objectivity can be treated in qualitative research can be summarized as follows (cf. Smaling, 1990).

Reliability refers to the absence of accidental errors and is often defined as reproducibility. For qualitative research this means virtual replicability. Here the emphasis is on virtual, because it is important that the research is reported in such a manner that it can be reconstructed by other researchers. What is meant by this is aptly expressed by the term 'trackability', which is highly compatible with Freudenthal's conception of developmental research, because 'trackability' can be established by reporting on, 'failures and successes, on the procedures followed, on the conceptual framework and on the reasons for the choices made.' (Smaling, 1990, p. 6) Furthermore, internal reliability can be interpreted as intersubjective agreement among the researchers of the project.

Internal validity concerns the correctness of the findings within the actual research situation. Researchers can improve the quality of their judgements and interpretations by seriously searching for counterexamples, or by searching for alternative explanations. They could also ask fellow researchers to play the role of 'devil's advocate'.

External validity concerns the bearing of the results on other situations. In the case of qualitative research this external validity does not concern generalizability as such. Here a rather differentiated generalizability is more important; the question is how certain elements of the results will apply to other situations. This is exactly what developmental research strives for.

Reliability and validity are indicators of objectivity, but there is more. Smaling (1990, 1992) expresses this in the bootstrap conception that he developed for objectivity as a methodological norm:

'The researcher must strive to do justice to the object under study; this will be done in reference to a certain question, problem or goal, within a certain framework (like a certain culture or an underlying philosophy of mathematics education). In this instance doing justice has two important aspects: the positive aspect, which concerns the opportunity for the object to reveal itself, and the negative aspect, concerning the avoidance of a distortion of the image of the studied object.'

(Smaling, 1990, p. 7)

For this he points to the participation-dissociation balance, role-taking, and the triangulation principle. The participation-dissociation balance indicates that the researcher must find the middle course between too much dissociation and too much involvement. Role-taking implies taking the 'actor's point of view', as it is called by

Cobb (1987). Adequate role-taking will prevent one being too detached and the realization that the actor can be aware of his or her position can prevent too much involvement. The term triangulation comes from trigonometry. Just as is the case in trigonometry where two angles are needed to define the third angular point, two sources will tell more about a certain phenomenon. Smaling mentions several kinds of triangulation. One could combine different kinds of data, different methods, or the same data gathered by different researchers.

As the above analysis of Smaling indicates, the heart of the matter is in interpreting mainly qualitative data within a tentative theoretical framework. One of the main problems in reporting on developmental research is found in reporting on this complex process. If the whole learning process is explicated in detail, the report will be a thick, indigestible book. But, which experiences, conditions, and deliberations should be reported and which can be omitted? It is probably best to be guided by the intended audience for the report. Publications that focus on teachers should be different from those intended for researchers.

If we present results for researchers, next to accessibility, trackability and the possibility for verification are of utmost importance. Fellow researchers must be able to retrace the learning process of the developmental researcher in order to enter into a discussion. Just like the researchers in the research project, the research community must be in the position to come to inter-subjective agreement. The size of the community is essential here. Thinking of a domain-specific research program for realistic mathematics education, we might restrict ourselves in the first instance to the circle engaged in this program. This would have the advantage of reporting for insiders. One could take as shared a certain knowledge base, which could be used as a basis for discussion. This knowledge base will include, for instance, the domain-specific instruction theory and some local instruction theories. Furthermore, such a knowledge base would presuppose common notions about learning or epistemology. Moreover, one could consider domain-specific knowledge about instructional settings, methods, content and experiences, which can vary between researchers to some extent, but which will also have much overlap.

To inform outsiders, this knowledge base will have to be unraveled to some degree. This is the background theory that sets the stage for a description of a priori expectation and various key points in the learning process of the developer/researcher: adjustments, 'Aha' experiences, reflective moments, and the like. To be able to describe the learning process, the developmental researcher first must identify the yield of his or her own learning process. The local instruction theory that is embedded in the educational material will have to be made apparent and consideration must be given to questions such as: What is the essence of the extension to the knowledge that already exists? Why should one consider it to be true? The justification will include the relations with theory, notions, crucial moments in the development process, and so on. What makes matters complicated is that the theoretical arguments by which empirical data are valued evolve in the development process itself.

4.3 the empty number line: an example

Page constraints do not allow for a detailed elaboration of the dialectical relation between empirical and theoretical justification, so only the core-elements of a justification for the use of the empty number are described here. Notice that no systematic developmental research on the empty number line has taken place comparable, for example, to that by Streefland (1990) on fractions. On the other hand, one of the advantages of this example is its accessibility.

Moreover, it also shows developmental research as an ongoing process; local or domain specific instruction theories are never final. One should note, however, that such a thumbnail sketch can give a false impression, because careful argumentation according to the aforementioned guidelines is essential in developmental research.

The empty number line, on which the students only mark the numbers they need for their calculation, is proposed as a didactical tool for addition and subtraction up to 100 (Treffers, 1991; Treffers and De Moor, 1990).

Fig. 4.1a illustrates the use of the empty number line as a tool for solving 27 + 38 as curtailed counting. A student could, however, also use other solution procedures, which are illustrated in fig. 4.1b and c.

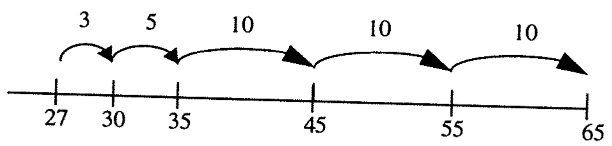

figure 4.1a: solving 27 + 38 as curtailed counting on:
27 + **3** = 30, 30 + **5** = 35, 35 + **10** = 45, 45 + **10** = 55, 55 + **10** = 65

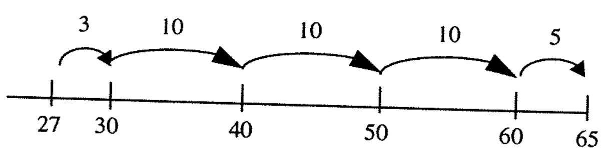

figure 4.1b: another way of solving 27 + 38:
27 + **3** = 30, 30 + **10** = 40, 40 + **10** = 50, 50 + **10** = 60, 60 + **5** = 65

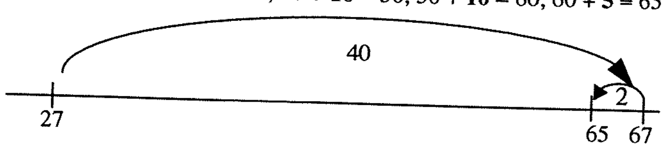

figure 4.1c: solving 27 + 38 by skillful calculation:
27 + **40** = 67, 67 – **2** = 65

For subtraction (e.g. 65 – 38) two different approaches emerge (fig. 4.2a and b). One approach is to take away 38 from 65 (fig. 4.2a).

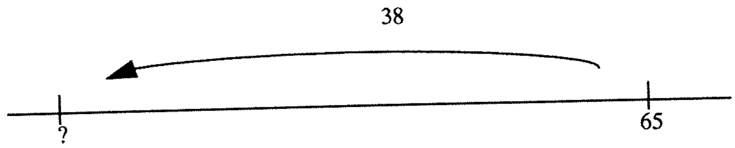

figure 4.2a: solving 65 – 38 as take away 38

A second approach is to compare 38 and 65, and then establish the difference (fig. 4.2b).

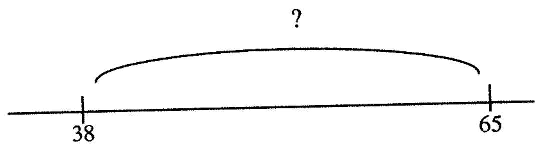

figure 4.2b: solving 65 – 38 by taking the difference between 38 and 65

The two approaches may be worked out in several different ways (fig. 4.2c).

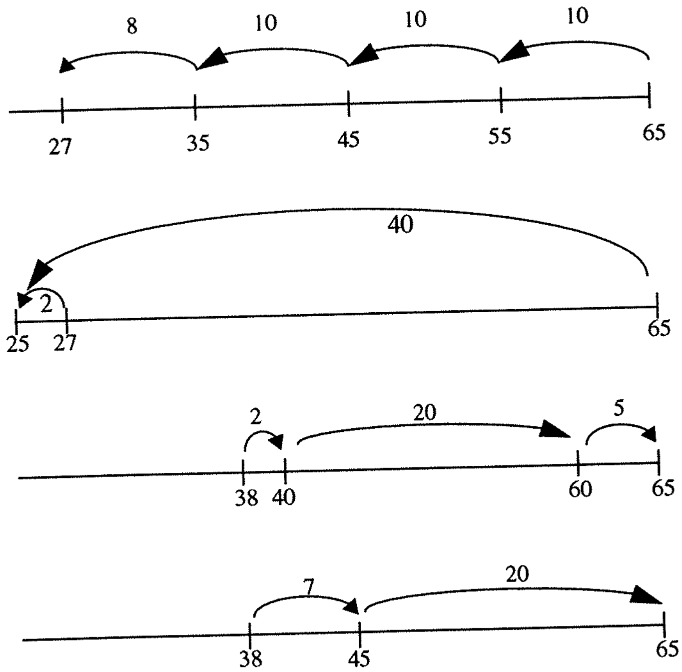

figure 4.2c: various strategies for solving 65 – 38

Treffers argues that when the empty number line is used in this way, it is a valuable supplement to the common use of base ten manipulatives like the Dienes blocks.

legitimizing

Having embraced the basic idea, developmental research is needed to (a) find out how a course could be set up to exploit this idea, and (b) see if the idea works. The answers to both questions constitute the justification of the initial idea, now elaborated in a local theory of how to employ the empty number line as a model in learning two digit addition and subtraction. Note that the justification might seem to be reconstructed logic, because it does not do much justice to the historical process and it could seem as if the idea to use the empty number line came out of the blue. One should realize, however, that many of the arguments are a priori arguments. These a priori arguments can be described in terms of a 'demon procedure', a concept used in artificial intelligence (cf. Lawler, 1985). The demon stems from a prior, frustrated objective. The demon procedure can be seen as a background program; it watches a limited set of conditions and leaps into action when those conditions indicate the opportunity is right to effect its objective. In similar fashion, the adoption of the empty number line can be interpreted as the result of a demon procedure looking for a 'realistic' complement to base ten blocks: an alternative model offering possibilities for informal strategies and for raising the student's level of sophistication in strategy use. One can infer that an openness to a number line-like model was triggered by the awareness of informal strategies like curtailed progressive counting.

As a way of structuring the justification for the idea, let us look at the following questions:

1 Why was there a need to develop something new? Or, to put it another way, what problem had to be addressed?
2 Why was the empty number line chosen?
3 How exactly was the empty number line approach elaborated and why in this particular fashion?
4 To what extent is there an experiential confirmation of the expectations?

1. What problem? It has become apparent that there are a number of problems concerning two-digit arithmetic. First, there is the rather common instrumental use of base ten manipulatives. Second, there is the common practice of misunderstood procedures, especially in performing the standard subtraction algorithms. For example, when working on an item like '65 – 38', the subtraction '5 – 8' may be answered with three or with zero. This type of error can be blamed on the didactical top-down strategy. In this strategy, the standard procedure – which is historically reached at the end of a long process – is taken as the starting point for the learning process. According to the realistic approach, students should be given the opportunity to develop (standard) procedures themselves. Therefore, alternative instructional activities are needed, which must offer students the freedom to develop their own solution procedures, and which must foster the development of more sophisticated strategies.

2. *Why the empty number line?* There are two reasons to choose the empty number line as a model. The first is grounded in phenomenological considerations. In a phenomenological analysis of number, Freudenthal (1973) discerns several aspects of number, which he denotes as: counting number, numerosity number, measurement number and reckoning number. Models such as Dienes blocks and other set-type representations clearly reflect the numerosity number. The counting number needs a linear representation. This distinction between set and linear situations is also found in applications. Besides the set types of situations dealing with quantities, there are linear types of situations, involving, for example, traveling distances. It is clear that a representation with Dienes blocks does not fit the latter type of situation. Here the empty number line model would seem more appropriate.

The second reason for choosing the empty number line has to do with informal solution procedures. Taking away tens and ones separately goes well with the numerosity number. More informal strategies, however, often go with the counting number. Strategies such as counting on and counting down are well documented for children working with small numbers (Carpenter and Moser, 1983). Let us take a closer look at informal strategies for subtraction.

Many young children discover that it is more efficient to solve certain subtractions by counting on instead of counting down (Woods, Resnick and Groen, 1975; Svenson and Hedenborg, 1979). Solving '9 – 7', for instance, requires seven counting steps when counting down: (nine), eight, seven, six, five, four, three, two, and only two steps when counting on: (seven) eight, nine. In the latter case, subtraction is implicitly interpreted as taking the difference, which is given by the number of steps. The advantage of (curtailed) progressive counting remains when problems with larger numbers must be solved. It appears that for most second graders the bare item '53 – 45' is harder than the situated problem:

> There are 53 beads in a jar. You need 45 beads for a necklace.
> >> How many beads will be left? (Gravemeijer et al. 1993)

Presumably this difference can be explained by the different strategies students might have used: curtailed counting on for the situated problem and curtailed counting down for the bare item. In the traditional approach, the second graders are expected to solve an item like '64 – 37' by first subtracting 7 from 64 and then subtracting the tens. In applied problems, however, students use other strategies, especially when the problem is placed in a linear type of situation:

> A book has 64 pages. You have already read 37 pages.
> >> How many pages are left to be read? (Vuurmans, 1991)

Most children employ a sort of curtailed counting on, such as: $37 + 10 = 47$, 47 + 10 = 57, 57 + 3 = 60, and 60 + 4 = 64, to conclude: that 27 pages are left.

The operation is taken apart in a sequence of 'add-ons', which mathematics educators associate with shifts on a number line. This suggests that it might make sense to have students symbolize this solution procedure with jumps on the number line.

This brings us to the third reason to choose the empty number line, its level-raising qualities. It was argued that a new model should leave students the freedom to develop their own solution procedures. However, this argument is not sufficient; there has to be room for improvement. Or, better, employing the model should foster the development of more sophisticated strategies. This is the case for the empty number line.

Looking at the role of the empty number line, we see that this model not only allows students to express and communicate their own solution procedures, it also facilitates those solution procedures, because marking on the number line functions as a way of scaffolding. It shows the (partial) results; it shows which part of the operation has been carried out and what remains to be done. Moreover, there is ample room for curtailment and increased sophistication, which can be stimulated by small group and whole-class discussions of the students' strategies. The most basic strategy would involve counting in ones. Next, the structure of the counting sequence may be used by counting in tens and ones. Knowing some basic facts permits counting in groups of tens and ones. This brings us close to the standard procedures; however those are just some of the options. In addition, the empty number line facilitates strategies for skillful calculation, such as 'compensating' (i.e. adding or subtracting too much and then compensating, for instance, in solving 76 – 49, first subtract 50 and then add 1). This strategy is supported by the visual image of the operations on the empty number line and the underlying structure of a bead string that facilitates counting by tens. This illustrates that the empty number line can foster level-raising in the sense of the Van Hiele levels.[8]

3. How was the approach elaborated? Not all experiences with the use of the number line are positive. The full number line (with all counting numbers shown) appears to evoke rather primitive counting strategies (Gravemeijer, 1991). When the full number line is used to solve '64 – 37', students may rely on counting 37 steps to the left and read off the answer (fig. 4.3). Therefore, it makes sense to opt for the empty number line, where the student merely fills in the numbers he or she needs.

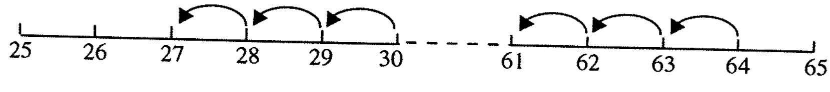

figure 4.3: elaborate counting on the full number line

Earlier experiments with the empty number line in the nineteen seventies failed because of the unwillingness of students to use it in a global, flexible manner. When solving a problem such as 412 – 379, students were not willing to place 400 somewhere between 379 and 412, since it probably would not be the 'right' place. Nor did placing 379 and 412 'somewhere' on an empty number line appeal to them. In hindsight, a possible explanation can be found in the way the number line was introduced. The number line was constituted in measurement situations and so associated with a rigid ruler with fixed, pre-given distances. Therefore, one must find another way to introduce the empty number line.

Following Whitney (1988), Treffers (Treffers and De Moor, 1990) opts for a structured bead string (fig. 4.4). The numbers are represented by placing a clothespin next to the last bead counted. The way the bead string is structured helps students find a given number. The tens can serve as a point of reference in two ways: there are six tens in 64 and there are almost seven tens in 69 (fig. 4.4).

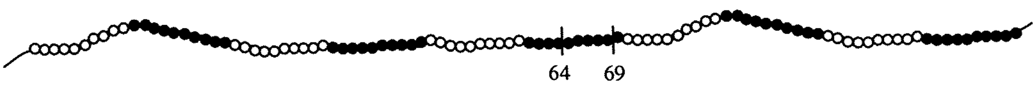

figure 4.4: marking numbers on the bead string

After working with the bead string, the empty number line can be introduced as a model of the bead string. To be more precise, it is the actions on the bead string that are modelled. The implicit structure of the bead string facilitates this modelling; it stimulates a person to fill in only the numbers that are essential (fig. 4.5). Note that there are no marks on the number line, the student places the marks that he or she chooses.

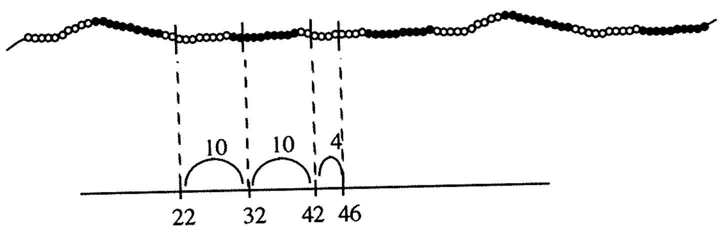

figure 4.5: modelling a bead string solution with an empty number line

4. *Experiential confirmation?* The first experiments with the empty number line were carried out with individual students. The results were encouraging: students were quite able to work with the empty number line and they displayed a large variety of solution procedures (Treffers and De Moor, 1990). This indicates that students do make sense of what they are doing.

These experiments were followed by a small educational experiment, which was carried out in three American second-grade classes (Gravemeijer, 1993). The research proceeded along the lines of developmental research; there was a quick alternation of designing and testing, in which a short course was developed. The experiment showed that the empty number line really does function as a flexible model that stimulates flexible arithmetic. The designed set-up, however, presented some problems for subtraction by curtailed counting on. As long as two numbers were given and the difference was asked for, all went well. But, problems arose when bare subtraction sums were presented. With a subtraction like 63 – 46, many students placed 63 at the beginning of their number line, which made it difficult to take away 46. This was a problem that could be overcome. More problematic was the approach of students who placed both numbers on the number line and did not know what to do next. Apparently these students knew only one interpretation of the subtraction sign: take away.

To address the second problem, another experiment was carried out in The Netherlands (Veltman, 1993). Here, the students were made conscious of the twofold interpretation of the subtraction sign. In a whole-class discussion, a subtraction problem was represented on a bead string. Then the teacher asked: 'What if we had taken this number from the beginning instead of from the end?' For a majority of the students it came as a surprise that the result was the same. However, they were able to justify this result after a moment's reflection. Ways of referring to these two kinds of subtraction were discussed and agreed upon, and consequently, these two interpretations could from then on play a role in the lessons. For instance, the students were asked to solve the same subtraction in two ways and to indicate their favorite approach with a flag (fig. 4.6). The result showed that some of the students varied their approach depending on the numbers involved, while others had a favorite strategy. One of the poorer students, who originally was not able to do any subtraction at all, was very pleased with the new approach. Now she could be effective, solving each problem by adding on.

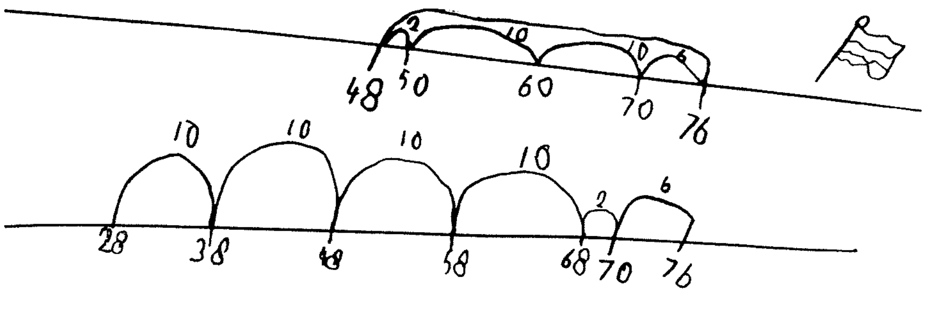

figure 4.6: two ways of solving 76 – 48

In this phase of the experiment, two questions that had come up earlier recurred (Gravemeijer, 1993): What strategies would the students use when solving context problems? Would these solution procedures be determined by the structure of the situation in the context or by the characteristics of the numbers involved? Take, for example the assignment shown in fig. 4.7.

> Gary is baking a pizza.
> The pizza must stay in the oven for 60 minutes.
> Gary sets the cook alarm for 60.
> When he looks again the arrow of the alarm points to 34.
> How long has the pizza been in the oven already?

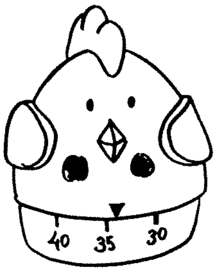

figure 4.7: a situation with an unknown subtrahend

Modelling the situation, the student will represent the subtraction as 60 – 34. Looking at the numbers, however, an interpretation of the subtraction as 34 + ... = 60 will be more likely. The worksheets of the students display both strategies. One of the students even reversed the direction of the number line to make it fit the situation (fig. 4.8).

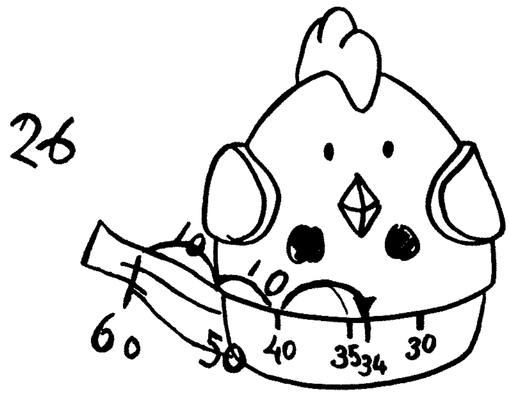

figure 4.8: creation of a situation specific number line

The fact that some solutions are guided by the numbers and not so much by the situation is very revealing. The student does not make a model of a situation, but for the calculation – a model meant to support the student's mathematical reasoning. In other words, the model is raised to a different level; it has attained an independent status.

Up to now, the teacher's viewpoint has not been discussed. One of the reasons is, that most of the experiments were carried out by researchers, members of a larger not so surprising since socio-constructivism and realistic mathematics education are highly compatible (Cobb, Yackel and Wood, 1992). The teachers saw this approach as a welcome supplement to their repertoire. And, although the empty number line was not integrated in their curriculum, they continued to use it on their own initiative the next year.

conclusions

This preliminary justification leaves many of the aspects that are essential for trackability and checkability undiscussed (these are addressed later). Yet, we can draw some conclusions:

- This schematic legitimation shows that the basic theory plays an important role. The theory provides the arguments for developing a new approach and it defines the constraints: there must be room for flexible use and for discoveries, and one must be aware of two different phenomenological aspects of number.
- The room for flexible use is also one of the criteria by which the prototypical course will be evaluated. Unlike, say, the way in which Dienes blocks are used, where the model imposes a certain manner of thinking, here, the model 'follows' the student's thinking. This is a theoretical justification which is supported by observations.
- Similar remarks can be made about the sophistication of the solution procedures and the process by which those procedures are formed. Both the quality of the solution procedures and their developmental process are parts of the theory which must be supported by observation. Unlike common practice, the number of correct answers is not the only or the most important aspect of evaluation.
- The empirical justification of the new approach is not confined to the experiences within the educational experiment; this justification also combines the empirical knowledge that was gathered earlier (knowledge about informal strategies, for instance).
- There is no room for a detailed description of the direct interaction between development and research in this educational experiment. However, a similar pattern can be seen on a larger time scale. This comes to the fore in three ways: the earlier experiments with the number line, the integration of the knowledge of informal strategies, and the design of the final version.
- Finally, it deserves mention that not only a piece of local instruction theory has been developed, but that at the same time a more fundamental principle is explicated, something which has received little attention up until now. By this I mean the double function of the model, which at first functions as a model of a situation and later as a model for formal arithmetic (Gravemeijer, 1991; Streefland, 1985; Treffers, 1991). This principle can be applicable in several learning strands. And, as such, it is a principle that rises above the level of a local instruction theory. It provides an extension on the level of the more general (although

still domain-specific) instruction theory. This shows how reflection on a local instruction theory lays the foundation for an ongoing development of the more general instruction theory.

As indicated above, the schematic description lacks many aspects essential for trackability and checkability. One might think of the following issues:

- For outsiders, it is important to know more about the context in which the observations were made. Were those informal solutions for '64 – 37' found with students who worked with realistic textbooks? Were they average students? How many students were involved and how many of them used this strategy?
- One educational experiment was carried out in a U.S. school, where the students had experienced almost one year of experimental – socio-constructivist – based instruction, following one year of instruction based on a traditional textbook (Cobb, Yackel and Wood, 1992). More information is needed to be able to appraise how this may have influenced the experiment.
- More information is needed about the clues (observations and interpretations) that indicate that the empty number line really functions as a means of support for student thinking, and not as a mere instrumental procedure.
- Information on the number of students involved and how many of them were successful is omitted in this sketch.
- The learning process of the students is indicated, however, a detailed analysis of the way student thinking develops is lacking.
- In connection with the introduction of the empty number line via the bead string, the question arises as to whether or not this is in accordance with the theoretical framework. In general, the realistic instruction theory proposes the use of contextual situations as an alternative to the use of manipulatives. This question has to be dealt with in view of the theoretical validity.

Of course these are essentials that have to be dealt with in a research report.[9] As such these are some of the pilings that will have to support the bridges between the communities of curriculum developers and curriculum researchers. Bridges can be built if curriculum developers become more aware of the research component in their work, and if researchers become open to developmental research. The latter may not be especially problematic, in view of the positive reception given to the work of researchers such as Steffe, Cobb, and Lampert. Moreover, we may argue that our concept of developmental research is in keeping with the recommendations of Romberg and Carpenter (1986).

Note that I am not making a plea for one uniform research paradigm; 'theory-guided' developmental research is but one promising research paradigm among several others (in particular psychological research and evaluation research). Psychological research can help to develop more sophisticated thought experiments and evaluation research is needed to get feedback from everyday practice.

4.4 research and development

On a small scale, the feedback loop of outcomes of developmental research into theory development is self-evident. However, we have not dealt with the larger feedback loop including textbook development and everyday practice in schools. This larger loop starts with the delivery of the results of the developmental research. The results must be described in such a way that they can be used and interpreted in other situations. In this manner, developmental research has a similar role as the more general educational research in the RDD model: the role of a research basis for the developmental trade. The character of the yield of developmental research makes this yield very adequate as a basis for theory-guided bricolage. In particular, the combination of theoretical and empirical justification gives curriculum developers the freedom to evaluate the prototypical courses by their own standards and to make well considered adjustments. That is to say, developers do not merely rely on the researchers claims of success, but value the prototypes themselves against the background of the local and general instruction theories, in connection with their validation of the theories as such. In other words, developers, too, have to explicate their theoretical standpoints, judgements, and purposes, which implies that the judgements and their criteria are open for discussion. For the textbook author, this creates the possibility of building upon the ideas and theories that are embedded in the prototypes. This will ensure that the author's bricolage is theory guided, not just a stacking of instructional activities. Furthermore, textbook series constituted within the same theoretical framework fit in the wider scheme of educational development. The adoption of the textbook series will profit from other efforts – in teacher training, counselling, test development, and communication of research findings in professional journals and conferences. This will help teachers to adapt the curriculum on the basis of the underlying ideas. Within this framework informal feedback is provided. Through conferences, journals, and other contacts, textbook authors, teacher trainers, counsellors, and teachers react to prototypes and ideas; they contribute their thoughts and experiences.

More formal feedback asks for an evaluation of the actual change in schools. However, this does not have to imply standard evaluation research. In positivist evaluation research, generalization in the sense of predictions on similarities over situations is one of the main concerns. That is its fundamental difference with evaluation in the context of developmental research. We can illustrate this difference considering what is understood as replicability.

Developmental research involves virtual replicability at first. After a short period, however, the developers strive for real, although more global, replicability. The highly innovative character of the educational development compels one to anticipate the implementation of the innovation. If a new idea is well established, it should be possible to follow the example set by the experimental instruction. However, unambiguous replication is not something to strive for. Strictly speaking replication is even impossible, because following the example is actually a matter of re-creation. More global replication or re-creation, however, is sufficient to enable evaluative research.

This closes the circle, because now the evaluative research, which is the end in most RDD models, is also fitted into the concept of developmental research. However, one must realize that in this case different goals and different criteria will have to be taken into account. The key elements of the empirical underpinning of the developmental research will have to be the essence of the evaluative research.

First, theoretical criteria that are brought to the fore in the developmental research are to be taken as essential criteria in evaluation research. With the empty number line, one must look for flexible use for indications of whether the model is used by students to express their own thinking or whether the model constrains or funnels their thinking. The focus will have to be on the quality of the solution procedures and on how these strategies come into being.

Second, in line with the theoretical orientation, an investigation of student interpretations and beliefs is indicated. Do they experience the empty number line as a flexible tool which they can employ in their own way? Do they take responsibility for their mathematics (Whitney, 1985), or do they rely on the teacher's authority? Is school mathematics connected with everyday life, or is it seen as detached from reality outside school?

Third, the evaluation will not do justice to the developmental research if it is confined to establishing results. An investigation into the ways local instruction theories are adapted to various situations, and into the relations with teacher beliefs is crucial. Interpreted in this way, evaluation produces insightful information that feeds back into theory development and educational practice.

In conclusion one can say that developmental research, theory-guided bricolage, teacher adoption as a learning process, and evaluation as outlined here, combined under the umbrella of educational development, offer an comprehensive alternative for the RDD-model.

notes

1 This is shown in the 'Mathematics in Contexts' project (Romberg, in press), in which Dutch materials are used as a basis for the development of a middle school curriculum that is consistent with the NCTM Standards (NCTM, 1989).

2 The innovations that have taken place in The Netherlands have been facilitated in part by the fact that The Netherlands is a small country. We recognize that this might make the success of similar innovations in the United States more difficult.

3 The term *educational development* is also used by Hemphill (1973) and Schutz (1970), however, they do not include the research component, which is essential for Freudenthal.

4 IOWO was succeeded in 1981 by the Research Group for Mathematics Education and Educational Computing Centre (OW&OC). In 1991, the Centre was renamed the Freu-

denthal Institute, to honor its founder.

5 In describing learning routes, concepts taken from activity theory such as mental activity, action structure (Van Parreren, 1981), and concepts taken from cognitive psychology (e.g. cognitive structure (Ausubel, 1968)) help describing the intended learning process as clearly as possible.

6 Streefland (1990) speaks of continuous shifts of roles among the instrumental, the design, and the learning process. On the one hand, design influences the learning processes of students, and on the other hand, the learning process influences the (next version of the) instructional activities.

7 Following Barnes (1982), we can question whether there is such a great difference with the natural sciences. In 'normal science' (Kuhn, 1970), an unambiguous interpretation of experimental results is only reserved for those who are enculturated in the research practice.

8 According to Van Hiele's (1973) elaboration of his level theory for number, numbers are connected with concrete objects at the lowest level and with number relations at the next level (e.g., 4 is connected with 2 + 2, 5 – 1, half of 8, etc.).

9 Many of these issues are addressed in the references: Treffers and De Moor (1990), Vuurmans (1991), Veltman (1993) and Gravemeijer (1993).

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5 Research on implementation and effect of realistic curricula[1]

introduction

More than twenty years ago the Wiskobas section of the institute then called IOWO began developing what we now call realistic mathematics education. Hans Freudenthal's ideas on mathematics education inspired the developers to create an alternative to 'New Math', which had spread to Europe from the United States. The Wiskobas group approached their task on many fronts; they developed and researched curricula prototypes, published background articles, developed materials for teacher training and organized conferences and in-service teacher training courses. All these activities were part of a broad strategy of educational reform, which focused on the furtherance of expertise, material development, and consensus formation.

These efforts resulted, among other things, in mathematics textbooks that have sufficiently incorporated the Wiskobas ideas (De Jong, 1986). More than three-quarters of the Dutch primary schools have, in the meantime, acquired a realistic textbook series and a good number of these schools is now using such a textbook series up through the sixth grade. The reform set in motion by Wiskobas appears, therefore, to have been put into effect via the school textbooks. This reveals the informal nature of the reform: the government has never made any formal resolutions regarding the introduction of realistic arithmetic and mathematics education. The schools, themselves, have been the ones who resolved to reform educationally, although said reform was, of course, initiated by an institute under governmental authority.

Moreover, the reform was legitimized after the fact by the attainment targets determined by the government. This does not alter the fact, however, that the reform itself came about in an indirect fashion: by influencing textbook authors (supply) and the market (demand). The new mathematics textbook series that sprang from these circumstances now determine the look of the reform. Does this mean, therefore, that the desired reform has, indeed, been a success? This is the question at the heart of this chapter. Succeeding sections will reveal that there are no simple answers to this question. The cause lies in the complexity of the reform. The central topic of this chapter is not a response to the research question itself, but a characterization of the research connected with the reform. The basis for this characterization is a curriculum research project conducted by the OW&OC and the Educational Research Department of Utrecht University (VOU). This is the Textbook Research in Mathematics Education project (hereafter referred to as the MORE-project), which was commissioned by the Institute for Educational Research SVO. The discussion on the set up and results of this project are based on the end report of this project (Gravemeijer et al., 1993).

The chapter is composed as follows. First, the discussion on (research into) the results of realistic curricula in The Netherlands is regarded briefly. We then turn our attention to the specific character of the reform. The issue here is the fact that the reform has primarily taken place through the introduction of new textbooks, even though the reform itself assumes an adaptive use of the curriculum. In this context we review the following: the issue of designing textbooks, the character of an implementation suited to the realistic concept, and the type of implementation process connected to such an implementation.

This is followed by a characterization of the research appropriate to this reform as expressed in the MORE-project. This characterization is structured as follows. A distinction is made between observational research and research from an innovation perspective. Both types of research are then again divided into implementation research and research into learning results. The observational research is discussed first, beginning with observational research into the implementation of realistic mathematics education. This is followed by a characterization of observational research into learning results. We then turn our attention to research from an innovation perspective. Indications provided by this research for reform practice are also discussed here. And, finally, keeping the characterization in mind, the various aspects of the research are again linked together and briefly summarized.

5.1 discussion on research results

The issue of the reform's success can be responded to in a number of ways. One answer has, in fact, already been given here: if we take the marketplace as our criterium (Hoeben, 1989), then the reform has, indeed, been successful. More than three-quarters of the primary schools have, after all, acquired a realistic method. Moreover, the teachers appear generally satisfied with the realistic textbooks.[2]

The judgement of the marketplace is not sufficient, however, which is why evaluational research is necessary. The first question to ask, then, is: does realistic mathematics education lead to the desired learning results?

This question has been answered in different ways depending upon the research project in question. A number of small-scale researches into the effects of specific textbook series (Van den Brink, 1989; Van Dongen, 1988; Rengering, 1983; Streefland, 1988; Veldhuis, 1981; Vink, 1978) and into specific effects (such as research into 'reflection' by Nelissen, 1987) revealed positive results. Harskamp and Suhre, on the other hand, found no difference between the effects of 'modern' and 'traditional' mathematics textbook series (Harskamp and Suhre, 1986; Harskamp, 1988). But, in the PPON-research (Wijnstra, 1988), students who were taught using realistic methods did distinguish themselves in the positive sense (Treffers, 1988).

Discussion on the explanation for these differences in research results focuses on the contents of the tests, the degree of implementation, and the specific educational setting. One argument, for instance, is that the structure of the standard final exam produced by the National Institute for Educational Measurement (CITO) is too coarse for measuring the differences between realistic and mechanistic education (Treffers, 1988; Van den Heuvel, 1990a). It is suggested that the PPON-tests and the tests in the small-scaled researches are more appropriate for measuring the essence of realistic mathematics education. Questions are raised as to whether generalizations can be made about the small-scale researches. The instructors in these cases are often very well informed and, in some instances, it is the researcher who actually gives the lessons. Harskamp tends to ascribe the good results to these circumstances. Moreover, according to Suhre (1988), there is a connection between student population and textbook series: schools with weaker students from disadvantaged environments are more likely (according to him) to choose a traditional method. A point of critique might be that no research into the degree of implementation was conducted in the PPON-project. Further, the method by which Harskamp and Suhre measured implementation is criticized (Van den Heuvel-Panhuizen, 1990a; Van den Oever, 1987; Verschaffel, 1989).

The main idea in this discussion, alongside the influence of students' and teachers' characteristics, is the following: different textbooks lead to different instruction, and different instruction leads to different learning results.

'Different learning results' is a more appropriate term here than 'better learning results'. 'Better' suggests that we are agreed on which criteria should be used to measure the learning results. All too often, however, the goals of traditional arithmetic education are the criteria used. One tends to take it for granted that the student must master the traditional arithmetic skills. But educational reform did not get started because traditional arithmetic goals were not being met sufficiently. It was rather the awareness that traditional merchant's arithmetic needed replacing and the conviction that another type of mathematics education was needed that together formed the basis for the reform. In addition, the fear of the importation of 'New Math' ideas and textbooks provided an impulse for the educational developments. As far as learning results are concerned, matters other than traditional arithmetic goals are involved here as well.

We may describe the reform in question as a pursuance of insightful, meaningful mathematics education which will lead to broadly applicable knowledge while also offering the necessary space for the students' own contributions. Central points are the concept of 'mathematics as a human activity' and the 'reinvention principle' (Freudenthal, 1973, 1991).

So the reform had a dual goal: different education and different learning results. This means that the evaluation must also cover how the reform has been implemented: how realistic is mathematics education now? Fullan's (1983) description of implementation as putting an idea into practice hits the mark exactly. This is, indeed, what it is all about. Before the question of learning results is investigated we must first find out what has come of the intended reform. Not only because educational results are dependent upon good implementation but, also, because the implementation of other education is a goal in itself. The following two questions are at the heart of an evaluation of realistic curricula: has the realistic mathematics education in question actually been implemented and, does realistic mathematics education lead to the intended learning results?

These questions are regarded within the context of the innovation. The actual situation is that the reform has to be put into practice via the textbooks. Therefore the following section first deals with this question of how reform might take place via the textbooks. The following issues are discussed: curriculum design, intended use of curricula, and the implementation process leading to the desired use.

5.2 reform via the textbooks

As a result of Wiskobas' innovative activities, there is now a receptive climate in The Netherlands for the reform of mathematics education. This is true even though the reform of mathematics education on the primary school level must, for the time being, be put into practice via the textbooks. This involves some limitations, but ones that can be partially counterbalanced by a suitable implementation process. We will first view the possibilities and limitations of (teacher's guides for) mathematics, after which we will focus on the implementation process.

A great variety of mathematics textbook series is available due to the way in which these textbook series came about. Even the four most prominent realistic textbook series – 'Operatoir Rekenen', 'Wereld in Getallen', 'Rekenwerk' and 'Rekenen & Wiskunde' (hereafter referred to as OR, WiG, Rw and R&W) – are all quite different.[3] This is partly due to when they appeared and partly to the history behind their appearance. Moreover, the varying beliefs of the different groups of authors are also reflected in the textbooks. The differences between the textbooks concern the subject matter content and its didactic expression on the one hand and, on the other hand, the design of the teacher's guide. The guides for the first edition of both OR and WiG were rather succinct.[4] R&W was the first to provide an extensive teacher's guide and was followed by Rw with a fairly extensive guide of its own. R&W's teacher's guide remains the most extensive and also provides the most explicit guidance, while Rw leaves more decision to the teacher.

Two of the three textbook series mentioned above were developed under the umbrella of a counselling agency. Presumably, this was not only because of the facilities available, but also because these guidance agencies would be able to help translate Wiskobas' ideas into feasible practice. Within these guidance agencies the necessary attention was paid to making these textbook series in such a way that they would be accepted and used by teachers who otherwise lacked enthusiasm for this approach. Concessions had to be made, of course, in order to reach large groups of teachers. This was the first step, but did not in itself complete the implementation process. We are dealing here with a quite complex innovation. It involves a substantive reform using numerous new courses for initial arithmetic, multiplication, algorithms, measuring, geometry, fractions, decimal numbers, percentages and ratios.

Moreover, all these textbook series assume interactive instruction. This requires an appropriate attitude, specific skills and knowledge as well as insight into the subject matter and pedagogic capability on the part of the teacher. The question is whether teachers can develop these qualities independently on the basis of information provided by the teacher's guides. There are a number of limitations inherent in using such guides to 'guide' teachers (Gravemeijer, 1987).

The issue of designing teacher guides was analyzed in depth by Walker (see Westbury, 1983). He distinguishes four problems in writing such guides. The first point concerns the clarity of the guide; the problem here is how to achieve sufficient clarity and specificity so that the users know exactly what is expected of them. Or, as Walker more accurately puts it, 'what they are being advised to do' (cited by Westbury, 1983).

The second point Walker mentions is the accessibility of the guide. It must be attractive, easy to read, and useful, otherwise it will be laid aside. A third problem is formed by the variety of situations in which the method will be used. How can one write a method so that it will be appropriate to a variety of situations? The fourth problem is that of achieving wide acceptance. The goal is, of course, to ensure that the suggestions, recommendations and ideas reach a broad public.

Attempts to meet these conflicting demands will soon lead to a dilution of the reform concept. But, even then, it remains difficult to write a guide in such a way that it be clear, accessible and applicable. The question is whether it is in fact possible to write a guide in such a way that teachers will precisely understand its intention. Harris poses this question as well, and adds, 'And what is more, can a guide be written so that a teacher who is so inclined can, from reading alone, appropriately relocate the type of practices described?' (Harris, 1983, p. 28).
Harris mentions the following limitations:

- the limitation of the written word when dealing with the transference of practical knowledge and skills
- a description of the practice is never exhaustive
- it is impossible to form a seamless connection to every individual user's knowledge.

The R&W authors attempted to resolve these problems by providing a variety in degree of concreteness in the levels of description: i.e. in terms of instructional activities, intentions, or underlying ideas.[5] The hope was that lack of clarity in the concrete descriptions could be compensated for by a clarification of the intentions and points of departure. And, reciprocally, the concrete activities could provide substance to the intentions and points of departures. This is true to an even greater extent when the lesson descriptions are actually enacted in the classroom. This may confirm Harris' (1983) expectation that users who are so inclined can develop approaches that mesh with the recommended ideas and procedures. Whether this does, indeed, take place depends upon the nature of the implementation process.

implementation

The implementation process should have the character of a learning process.This learning process should result in a form of implementation that is appropriate to realistic mathematics education. In reference to Fullan and Pomfret's (1977) distinction between '*fidelity*' and '*mutual adaptation*', the appropriate form of implementation was given the Dutch label '*fidele adaptatie*' (Gravemeijer, 1987), which can be translated as '*idea-consistent adaptation*'.

Fullan and Pomfret distinguish two approaches: the 'fidelity approach' and the perspective of 'mutual adaptation'; 'idea-consistent adaptation' is a third form. The fidelity approach assumes that the teacher will follow the guide's instructions precisely and use the material in the manner prescribed. The mutual adaptation approach assumes that not only does a teacher's behavior alter under influence of the curriculum, but that the curriculum itself will change as it is transferred from a document to classroom practice. Research from this perspective focuses primarily on the practicability, in this case the adaptability of the curriculum to the desires of the teacher. The adaptability model is linked to the idea that the teacher knows exactly what sort of instruction she or he wishes to give and, therefore, should be provided with a suitable curriculum, or one that can be made suitable (Creemers and Hoeben, 1988). The fidelity perspective, on the other hand, rests on a model in which teachers are demoted to implementors of 'teacher-proof' programs.

Neither one of these models would seem to be appropriate for the implementation of realistic mathematics education. The fidelity approach requires a guide containing an unattainable degree of detail. Moreover, a kind of docile, subservient behavior is required here of the teacher, which clashes with the degree of responsibility expected from the students. The adaptability model, however, assumes a degree of expertise with respect to the content of the reform that, on the whole, will not be present. The teachers are likely to be familiar with and appreciate the reform concept along broad lines, but their knowledge of specific content is probably limited. A synthesis of these two approaches is needed for realistic arithmetic education, the so-called 'idea-consistent adaptation'. Adaptation is unavoidable in realistic mathematics education because of the intrinsic goal of including a great deal of student contributions in the educational process. The teacher will need to interpret the guide's instructions flexibly in order to work in this fashion.

'Idea-consistent adaptation' assumes, therefore, that the adaptations instituted by the teacher will be appropriate to the concept of realistic mathematics education.

how to achieve idea-consistent adaptation?

In-service teacher training courses would seem to be an obvious resource, but here, in fact, lies a weak spot in the reform.[6] The necessary government support for such courses has never been obtained by the reform movement. The development of in-service teacher training courses by Wiskobas (De Moor and Treffers, 1975), the National Association for the Development of Mathematics Education (NVORWO) (Feijs, Gravemeijer and De Moor, 1986; Feijs, Gravemeijer, De Moor and Uittenbogaard, 1987a and b), and by a collaboration between the Institute for Curriculum Development (SLO), the Freudenthal Institute, the National Institute for Educational Measurement (Cito) and others (Vuurmans, 1991) have never been followed up by nation-wide courses. The sole in-service teacher training venture initiated by the government, called 'Speerpunt Rekenen' was cancelled at the last minute.[7] What remains are individual activities by school guidance services and teacher training colleges. For idea-consistent adaptation we must, therefore, fall back on the teacher's independent learning process. This learning process will have to take place during the implementation of the realistic textbook series.

Research into implementation has roughly revealed how implementation processes progress. The research distinguishes different levels of use (see, for instance, Hall and Loucks, 1977), which may be viewed as different phases in the implementation process (Hall and Loucks, 1981; Van den Berg and Vandenberghe, 1981).

At first, the curriculum is followed 'mechanically'; the teachers follow the guide quite closely, without having thoroughly comprehended the intentions of the educational activities. As the teachers become more familiar with the curriculum and understand the construction of the subject matter, they are able to make small adaptations to suit the particular classroom situation. The better the teachers understand the intentions of the textbook series, the more flexibly it will be used. Eventually, the teachers can make adaptations in the curriculum on the basis of now acquired insights and experiences.

Actual practice reveals that not all teachers experience the same development. The access points vary, as does the growth to 'higher' levels of use. No clear pattern can therefore be observed, although we can recognize the teacher's learning process in the description. The teachers learn from working with the textbook series and their grasp on working with it becomes firmer. These learning processes do not go without saying, however, and they do not all lead to higher use levels. The learning process is at the same time a choice process. The teacher determines which educational activities from the textbook to use and how to present them. Fullan (1984) calls this a 'long-term adoption process'. In other words, the decision to adopt is not made suddenly but, rather, stretches out over a longer period. The longer one works with the textbook series, the clearer one's view of it becomes. It also becomes clearer what one has chosen or, better, what one is choosing. It is, of course, possible that the teacher may not choose certain consequences. That is, one may use the textbook series without adopting the corresponding idea of realistic mathematics education, or without adopting all of its aspects. It is a form of learning by experience in which the teachers become familiar with the reform and learn to judge it. According to the concept of idea-consistent implementation, the teachers should learn to comprehend the new ideas, experiment with them and make well-founded choices. The teacher's guide should offer the necessary support as, for instance, is the case in the 'Rekenen & Wiskunde' textbook series, in which the educational descriptions are given on different levels (see above). The question remains, of course, whether use of the textbook series will indeed develop in this way. Research into this matter was conducted in the MORE-project, where the use of 'Wereld in Getallen' and 'Naar Zelfstandig Rekenen' was investigated. The research focused on three questions:

1 To what extent is the nature of the instructional practice – whether one is using a realistic or a mechanistic textbook series – determined by the textbook series, and to what extent by the user's views?
2 How do the user's views and the way in which the textbook series is actually employed develop during the first years of implementation of a new textbook series?
3 Do the two different types of education also lead to different learning results?

In other words, the implementation as well as the learning results were investigated. The research itself can be characterized as a combination of qualitative and quantitative research. The underlying idea was that the two research approaches would complement one another. For clarity's sake, the characterization is divided into two parts: one for observational research and one for research from the perspective of innovation. The observational research is dealt with first, beginning with research into the implementation. It should be noted that the distinction is made in order to simplify the characterization. In practice, there was absolutely no question of separated projects. A similar differentiation can be made for the distinction between qualitative and quantitative research. The distinction is not as sharp as the terms suggest; qualitative research sometimes involves counting, and the quantification in quantitative research is at times based on qualitative assessments. The terms qualitative and quantitative are used here primarily to indicate the nature of the research.

5.3 research into the implementation of realistic textbook series

As mentioned above, the MORE-project focused on two issues of implementation. The first issue concerned the relative influence of textbook series and beliefs on the actual education. The second issue concerned the teachers' learning process. Both issues are connected to the work of Fullan, who points out that educational change takes place on three levels (Fullan, 1983).[8] These levels have a bearing on changes in:

- use of materials
- educational activities and
- beliefs.

According to Fullan, true change is only possible when the views of the teacher also change. In this context, he also speaks of the teacher's learning process. If we follow Fullan's train of thought, we reach the natural assumption that the curriculum document will be followed in terms of the subject matter, but that the differences will primarily manifest themselves in the teaching-learning process. It is in the interaction between teacher and student that implicit and explicit beliefs will be of decisive significance (see also Thompson, 1984). In association with Fullan's ideas, a distinction has been made in the research between the content of the instruction and the nature of the instructional practice. The content of the instruction is understood to be the subject matter and how it is constructed. The nature of the instructional practice concerns the character of the teaching-learning process. Along with a difference between subject matter content and teacher-learning process we also have here the difference between macro structure and micro structure. The content concerns primarily larger subject matter units and broad lines of subject matter sequence, while the instructional practice involves micro-didactics. It is particularly on this micro-didactic level that beliefs may play an important role.

The most significant variables can be categorized in the following causal model:[9]

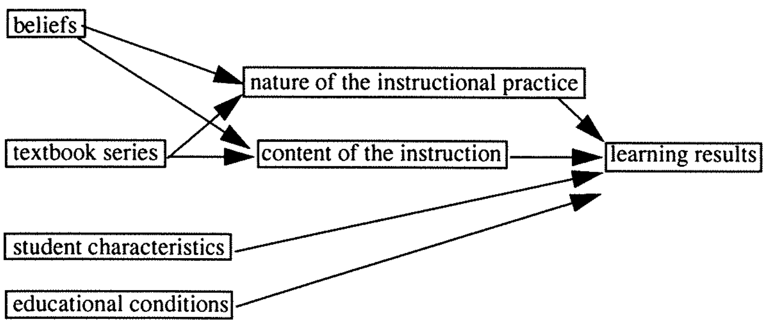


figure 5.1: causal model

In order to answer the research questions, education in eight schools where the mechanistic textbook series 'Naar Zelfstandig Rekenen' (NZR) was used was compared with education in ten schools which used the realistic textbook series 'Wereld in Getallen' (WiG). For this research, students and teachers at these schools were followed from the beginning of grade 1 through the end of grade 3 (fig. 5.2).

	number of schools	number of classes		
		grade 1	grade 2	grade 3
NZR	8	9	9	8
WiG	10	13	15	13

figure 5.2: research group

nature and content

Treffers' characterization of the realistic instruction theory (Treffers, 1987) was used as a reference framework for assessing the intended 'idea-consistent' implementation of realistic mathematics education. For the mechanistic approach, however, no domain-specific characterization is available. The mechanistic approach does de facto have more to do with a general theory of education applied to arithmetic and mathematics education. It did turn out, however, that Gagné's analytical approach (Gagné, 1969) contained the very same characteristics as Treffers attributes to the mechanistic approach. We therefore have used Gagné's theory as the mechanistic instruction theory. These two theories are used as a reference framework for analysis of the nature and content of the instruction. Analysis of the nature of the instructional practice focuses on its execution: the lessons. Analysis of the content concentrates, in this particular case, on analysis of the textbook series.

content of the instruction

A textbook series analysis was conducted to determine the content of the instruction.[10] This revealed clear-cut differences between the NZR and WiG textbook series (see also Van den Heuvel-Panhuizen, 1991). The following conclusions were drawn: NZR and WiG differ considerably in terms of supply and sequence of subject matter. The underlying instructional theories are expressed by a broader supply of subject matter in WiG (more attention to geometry and ratio, among other things) as well as a structural integration of applications. There are also related temporal differences in subject matter planning. While NZR passes quickly through the subject matter, providing a narrow supply of subject matter and a one-sided focus on drill and practice, WiG chooses a broader and more gradual set-up. Neither textbook series, by the way, proves to be homogenous. One of the WiG units is somewhat less realistic than the others, while one of the NZR units turns out to be more realistic than the rest of the textbook series.

nature of the instructional practice

Audio recordings were made of a number of lessons in order to determine the nature of the instructional practice. Transcriptions were then made of these recordings and these were assessed by experts in the field. Treffers' (1987) domain specific instruction theory for realistic mathematics education and Gagné's (1969) mechanistic instruction theory served here as a reference framework. This literature was used to make the two theories operational in a number of characteristics. For the realistic didactics:

- use of contexts
- use of models
- own constructions and productions
- interaction
- connection between learning strands.

For the mechanistic didactics:

- step by step construction
- bare sums first
- instrumental instruction
- fixed manner of working
- extrinsic motivation.

These characteristics were used to determine to what extent instructional practice corresponds to the theory.[11] It was clear from the average total score for the years in question that WiG and NZR teachers did, indeed, enact different pedagogies (fig. 5.3). According to the assessment of the experts, NZR-instruction was 'fairly' mechanical and not realistic; WiG-instruction was only 'partially' realistic, but negligibly mechanistic.

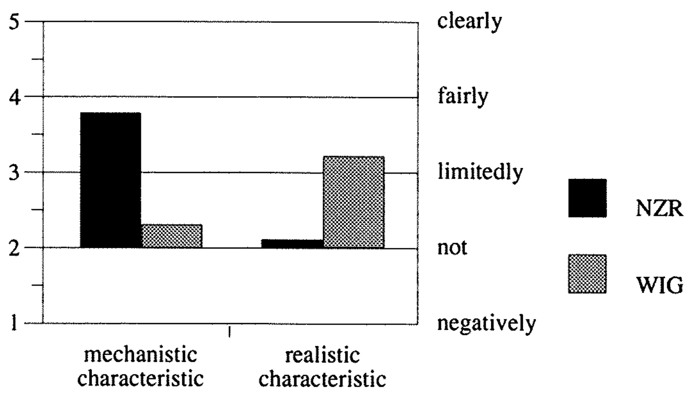


figure 5.3: realistic and mechanistic characteristics

This means that the mechanistic textbook series was implemented more in accordance with the intentions than was the realistic textbook series.

beliefs

A list of questions was used to measure the teachers' beliefs. The list contained both open questions and pre-coded questions. The open questions asked about considerations and arguments and requested comments. The pre-coded questions were primarily concerned with assessments, preferences and factual information. In order to obtain the maximum clarity, multiple use was made of examples and descriptions of instructional activities. For instance, the teachers were given concrete problems and asked for their comments (fig. 5.5).

> 15. Do you think the following assignments adequate to check whether the students can reason with numbers and concepts?
>
> Jan lives at a distance of 3 km from school.
> Lida lives at a distance of 5 km from school.
> What is the distance between the dwellings of Jan and Lida?
>
> adequate / not adequate, because / if:

figure 5.4

The teachers' answers were quantified according to their degree of agreement with each of the two educational theories. Then the questions were arranged along a line ranging from general to specific; at one end lay views regarding mathematics in relation to other school subjects and, at the other end, comments about specific problems. The following four levels were distinguished:

- subject level: questions on the specific character of the subject 'mathematics' in relation to other school subjects
- didactics level: questions on mathematics didactics in general
- curriculum level: questions on the construction of and approach to specific curricula
- problem level: views on the didactic choices surrounding specific problems.[12]

The analysis revealed that realistic beliefs were dominant across the entire spectrum, although the WiG teachers did reveal more outspoken realistic views than did the NZR teachers (fig. 5.5).

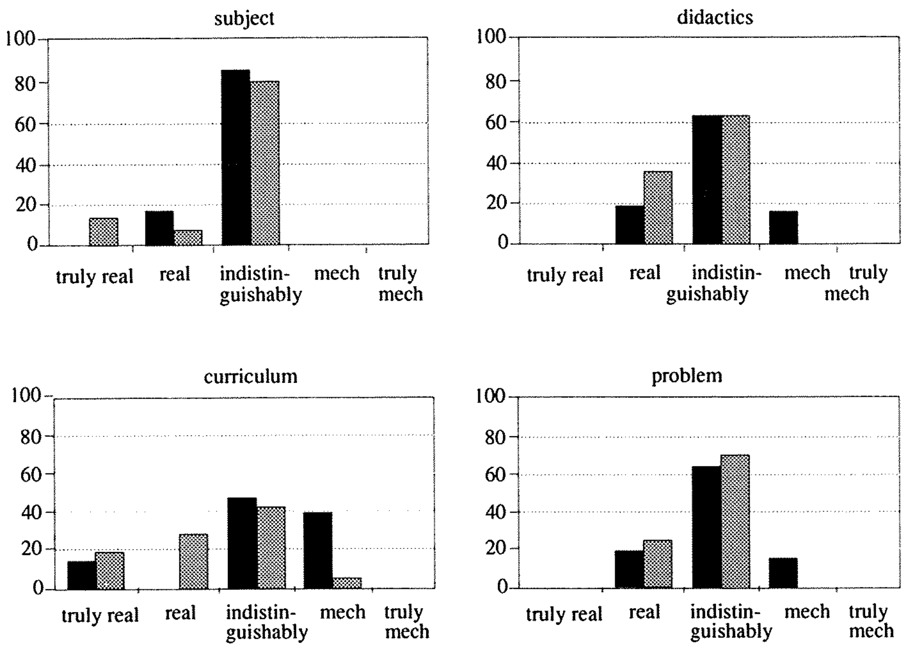


figure 5.5: beliefs according to the levels of subject, didactics, curriculum and problem

This leads to a curious paradox: While the WiG teachers more often endorse the ideas behind their textbook series than do their NZR colleagues, it appears that the instruction given by the NZR teachers better corresponds with the ideas underlying their textbook series. An explanation for this paradox could be that a mechanistic approach is more self-evident, or easier to deal with (see also Gravemeijer, 1992a).

connection

The issue of the connection between beliefs and the nature of the instructional practice could not be answered statistically, because the nature of the instructional practice correlated too strongly with the textbook series in question. Only the users of NZR exhibited a mechanistic instructional practice and only the users of WiG showed a realistic instructional practice (albeit to a limited degree).[13] The beliefs (in particular on the specific level) did correspond positively to the nature of the instructional practice, but it could not be ascertained whether this was due to the influence of the beliefs or to the textbook series itself. An analysis per textbook series made little sense as the execution per textbook series was fairly homogenous.

There was evidence, however, that the direction in which the instructional practice diverged from the textbook series was significantly connected to the teacher's beliefs (fig. 5.6).

connection between beliefs and more *mechanistic* use of the textbooks					
	beliefs (the higher the score the more *mechanistic*)				
actual use compared to the textbook	subject	didactics	curriculum	problem	specific beliefs
	mean st.dev.	mean st.dev.	mean st.dev.	mean st.dev.	mean st.dev.
less mechanistic	2.73 (.59)	2.76 (.57)	2.58 (1.01)	2.82 (.55)	1.65 (.48)
more mechanistic	2.83 (.40)	2.83 (.41)	3.50 (.54)	2.67 (.52)	2.00 (.54)
level of significance	.67	.77	.03	.52	.10

connection between beliefs and more *realistic* use of the textbooks					
	beliefs (the higher the score the more *realistic*)				
actual use compared to the textbook	subject	didactics	curriculum	problem	specific beliefs
	mean st.dev.	mean st.dev.	mean st.dev.	mean st.dev.	mean st.dev.
less realistic	2.85 (.38)	3.00 (.71)	3.23 (.60)	2.92 (.49)	1.95 (.40)
more realistic	2.71 (.26)	2.71 (.50)	2.52 (1.05)	2.77 (.55)	1.61 (.48)
level of significance	.61	.10	.02	.37	.01

figure 5.6: connection between beliefs and use of textbook series

learning process?

The MORE-research was not able to answer the question of whether the teachers had undergone a learning process. It was assumed that such a learning process would come to light upon comparing the lessons of teachers who had taught the same grade for some years in a row. But, unfortunately, it turned out that very few teachers had taught the same grade two or three years in a row during the research period. A total of only ten WiG teachers had taught the same grade two times in a row. The teachers' views hardy changed during the research period, nor were any significant differences to be found in the instructional practice (fig. 5.7a and b).

changes in the beliefs of the WiG teachers (n = 10)		
levels	1st year	2nd year
subject	2.60 (.84)	2.50 (.85)
didactics	2.70 (.48)	2.80 (.42)
curriculum	2.30 (.95)	2.10 (.99)
problem	2.70 (.48)	2.60 (.52)
specific beliefs	1.45 (.25)	1.40 (.21)

figure 5.7a: changes in beliefs.

changes in the instructional practice				
	WiG (n = 7)			
	mech.	st.dev	real.	st.dev.
1st year	2.19	(.23)	3.28	(.33)
2nd year	2.38	(.26)	3.23	(.44)
level of significance	.16		.80	

figure 5.7b: changes in the instructional practice

5.4 research into learning results

There are two points at issue in the research into the learning results: the instruction theories and the textbook series. In principle, we can evaluate textbook series without involving the underlying instruction theories. However, the textbook series can also be regarded as operationalizations of instruction theories. We can interpret an evaluation of a textbook series as the testing of an instruction theory. The instruction theory, however, is thereby tested indirectly: it is laid in a mathematics textbook series, instruction is then given using this textbook series, and the result is then measured. This obviously has consequences for the power of the test.

We have seen from the above that the incorporation of educational reform into a textbook series is far from simple. In practice, the reform, too, turns out to have been elaborated in different ways. In particular, the four realistic mathematics textbook series mentioned earlier differ both in the way in which the implementation was prepared as in the didactic elaboration. Moreover, as indicated above, it does not go without saying that the use of a realistic textbook series will result in realistic mathematics education.

Whenever is the intention to test the underlying instruction theory, the instruction concerned should itself be the independent variable. This is, indeed, the path followed by the MORE-project. The research focused on the direct connection between the actual education and the learning results. In terms of the influence model presented earlier, this has to do with the direct influence on the learning results (fig. 5.8).[14]

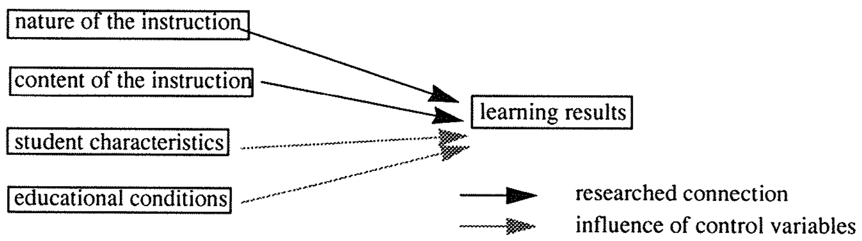


figure 5.8: direct influence on learning results

The distinction between the nature of the instructional practice and the content of the instruction is significant here, too. After all, in addition to the nature of the instructional practice, the content of the instruction can also be of influence. Concretely, this has to do with the influence of the textbook series' subject matter content and subject matter sequence on the learning results. In order to determine this influence one must, however, have adequate methods of testing at one's disposal.

Theory-evaluation means that said tests should be geared to the educational theory in question. It would be incorrect, for example, to measure the realistic theory of education according to the objectives of traditional arithmetic education. The point, after all, is the reform of mathematics education. This reform must be measured primarily according to its own objectives, after which the traditional objectives may have a turn. Eventually, a balance will have to be found between feasibility and the importance of old and new objectives. Administration of a norm referenced test is not sufficient for measuring and comparing the results of mechanistic and realistic textbooks. Popham (1975) contends that criterion referenced tests, rather than norm referenced tests, must be used for curriculum comparison. Norm referenced tests measure, to a certain extent, the same kinds of properties as intelligence tests. This, by the way, could be an argument for using norm referenced tests if one assumes – as does Lohman (1993) – that curriculum has an impact on intelligence test achievement.[15] However, Lohman also points out the problem of bias, which arises from the fact that not all the test problems are equally new to all students. And this problem exists precisely because the available norm tests are simply grafted onto traditional arithmetic. This means that the newness of the test items is not the same for all students and, therefore, that the tests themselves are not a correct measurement for transfer.

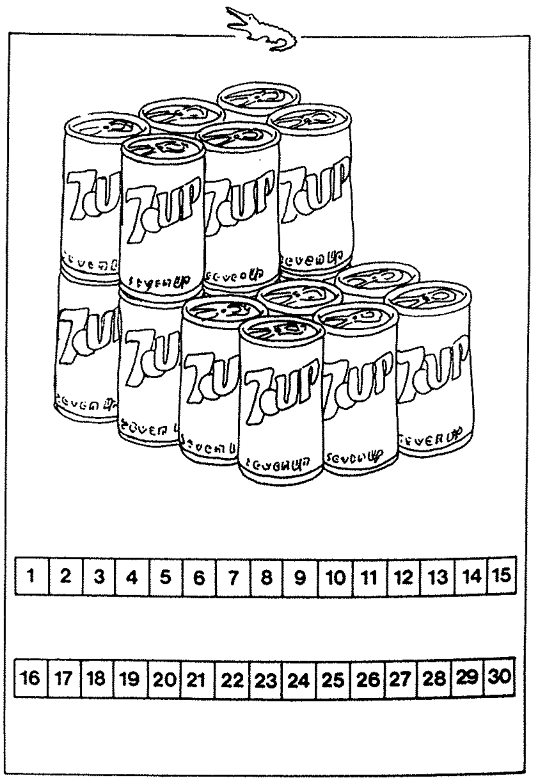
Many of the existing criterion tests are unsuitable as they, too, are based on traditional arithmetic textbooks. The tests developed in the framework of the PPON-research on arithmetic (Periodieke Peiling Onderwijs Niveau; Wijnstra, 1988) are an exception here. These tests, however, have not been released. In addition, the PPON contained no tests for first and second grade. New tests must, therefore, be developed for measuring the results of realistic mathematics education. Considering the inherent objectives, these tests should be extremely varied. De Lange (in press) distinguishes three levels here. The lowest level tests definitions, technical skills and standard algorithms. The middle level deals with 'making links', 'integration' and 'problem solving'. The highest level attempts to test such things as 'higher order thinking skills'. This is where one finds such things as reflection and creativity, as well as generalization and mathematization.

test development within the MORE-project

In general, we may state that test development has not kept pace with developments in mathematics education. This meant that the MORE-project had to develop its own tests in order to facilitate a comparison between the results of the NZR and WiG textbooks.

The MORE-project developed twelve written tests for the class as a whole (ten general math tests and two tests on 'basic facts'), and eight oral tests to be administered individually in grades one through three. The starting point in designing the written tests was to determine achievements with respect to generally accepted (communal) arithmetic and mathematics objectives. Points of reference here were the 'Specimen of a National Curriculum for Mathematics Education in Primary School' (Treffers, De Moor and Feijs, 1989; Treffers and De Moor, 1990) and the 'Provisional Attainment Targets for primary school mathematics education' (Projectgroep Voorlopige Eindtermen Basisonderwijs, 1989). Range of subject matter in both textbooks was taken into account when the tests were being designed. An attempt was made to prevent the students from being confronted with notation forms that were used in one of the textbooks but not the other.

The relative newness of the terrain created a situation where the development of effective written tests could grow into an independent project. A salient feature of this project was the endeavor to design the tests in such a way that the students could demonstrate their abilities (Van den Heuvel-Panhuizen, 1990b; Van den Heuvel and Gravemeijer, 1993). Stated differently: one of the aims was to provide room for a student's own contributions, solution strategies and choices. Following the example of Van Galen and others (1985), it was decided to use accessible, more or less obvious context problems, in order to provide the students with the opportunity of demonstrating their abilities. These contexts were evoked by using both text and drawings (fig. 5.9).The text was to be read aloud by the teacher:



'Here are a whole lot of cans of 7up. They're all stacked up, so you can't see them all. Can you still tell how many cans there are? Mark the correct number.'

figure 5.9: cans

In contrast to bare sums, the advantage of this kind of context problem is that the designer is not bound to the notations familiar to the students. One can thereby bypass unknown forms of notation and anticipate the subject matter that has yet to be handled in class. Then the students can really show their capabilities. Moreover, these context problems provide the students with more room to choose a solution procedure. Bare sum notations are often associated with standard solution methods. When this notation is missing, the tendency to apply such methods is also absent. Instead, the contexts often offer opportunities for informal solution strategies.

The results achieved by the students halfway through second grade on two similar test items confirm the notion that context problems better stimulate the use of informal strategies. When the students were told that a jar contained 47 beads, 43 of which were necessary to make a necklace, roughly 60% of them figured out that 4 beads would be left over. When presented with the bare sum 47 – 43, however, less than 40% of the students found the correct answer.

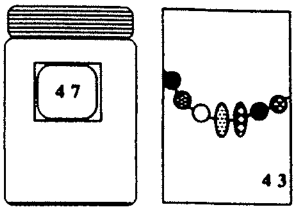


figure 5.10: beads

As mentioned above, test development grew into a project of its own. The goal of this project is to describe the specific characteristics of these tests and to expand them further (see Van den Heuvel, in press).

answering the question of effect

'Do the two types of education also lead to different results?'
This was the question of effect that the MORE-project attempted to answer. In order to determine the differences in learning effects between the two types of instruction it was necessary to conduct longitudinal research. To this end, the data collection was integrated with the implementation research then in progress in first, second and third grade. The students who were in first grade at the beginning of the 1987/1988 academic year were followed until the end of third grade (fig. 5.2). Discounting the initial level of the students was not the only reason for the longitudinal research. Differences in the nature of the learning path were also expected. Students educated according to the realistic approach would, for instance, first have to develop a broad range of strategies for deriving number facts before these operations could become automatic. The students who followed a mechanistic curriculum, on the other hand, would always have to use set procedures – such as 'bridging' ten by splitting the second addend. The research project examined how this arithmetical knowledge developed. Four times a year written tests were administered for this purpose. These were supplemented by the oral tests, which were administered individually to a limited number of students per class.[16]

research results

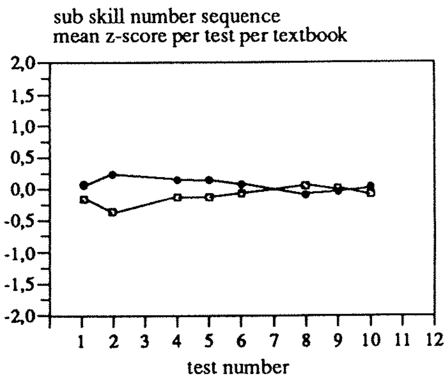
The textbook analysis revealed differences between the two textbooks with respect to the breadth of the range of subject matter, the sequencing, and the attention paid to informal strategies. The research examined whether these differences in textbook content were linked to differences in learning achievements. Differences in breadth and in sequencing would be reflected in the written test results. The use of strategy was measured through the oral tests. In order to facilitate an effective analysis between range and result, the test items from the 10 general tests were subdivided into separate skills. These skills are: familiarity with the number sequence, bare sums, context problems, ratio (& proportion) and geometry. Because these categories were only added after the fact, sometimes only a few items of a particular skill were found on a given test (fig. 5.11).

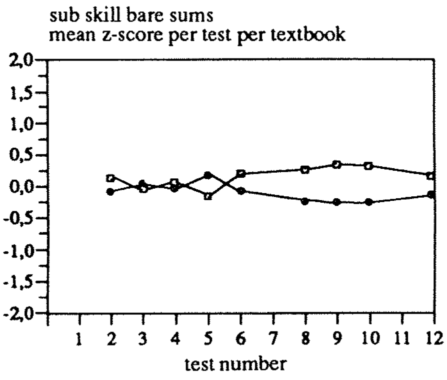
MORE general math tests												
	test nr.											
	1	2	3	4	5	6	7*	8	9	10	11*	12
subskills	number of items											
number sequence	8	6		7	7	5		3	3	2		
bare sums		6	8	20	20	18		20	20	23		19
contextual problems	12	15	16	10	10	14		16	16	14		14
ratio			2	5	5	2		6	6	2		3
geometry		1		6	6			2	2	2		4
whole test	28	28	27	43	43	38		42	42	44		37

)* basic facts test

figure 5.11: items arranged according to skills

There was also, moreover, some overlapping; the ratio problems were also part of the context problems, and many ratio problems also fell into the category of geometry. The results must, therefore, be interpreted with the necessary caution. Nevertheless, they do indicate certain general tendencies.





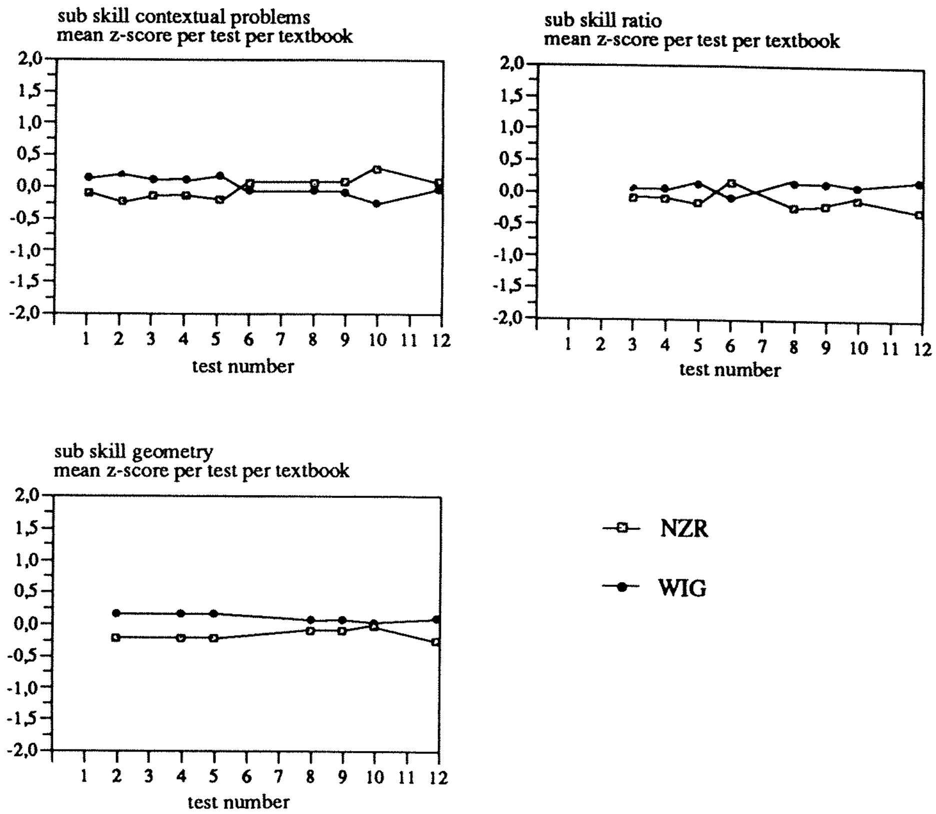


figure 5.12: links between textbook and sub-scores

For instance, the achievements on the sub-scores for the various kinds of subject matter form a strong link with the textbook's range of subject matter (fig. 5.12). The learning achievements also reflect the differences in subject matter sequencing. This is primarily true of the difference in tempo, where NZR is almost always ahead on the bare sums. WiG, however, is the first to extend the number sequence and to introduce calculation using relatively large numbers (to 100). This is also true of specific topics such as geometry and ratio (WiG sooner) and 'bridging' ten (NZR sooner) in first grade. The conclusion may thus be drawn that a direct relation does exist between textbook content and learning achievements. This is consistent with the findings of Walker and Schafferzick (1974).

As for the context problems – the pièce de résistance of realistic mathematics education – the picture is less clear. Probably, in some instances the basic arithmetic skills (NZR) proved more helpful and, in others, the familiarity with applications (WiG). On the last test, at the end of third grade, achievements on the context problems were equal. Strikingly enough, this was also true of the written algorithms on the same test. Clear differences can be seen, however, with regard to basic facts. Here the WiG students lagged behind the NZR students on addition and subtraction facts as well as on multiplication and division facts.

The remaining question concerns the influence exerted by the instructional practice. The inadequate realization of the realistic pedagogy in mind, however, impeded the establishment of a link between the instructional practice and the learning results. The number of cases were, moreover, too few to be able to correct for initial level and suchlike. Simple correlations did, however, show a positive relation between a realistic practice of WiG and the learning achievements.

5.5 research from the perspective of innovation

The MORE-project results presented up to this point have a particularly observational character. The research project answers the questions: 'To what degree has the intended implementation been put into practice?' and, 'To what extent do mechanistic and realistic education lead to different learning results?' But those involved will certainly want to know more. From the reform perspective, the question is still: 'Is the situation satisfactory', or, 'Can it improve?' How might this improvement be put into practice? Take, for instance, the evaluation of the textbook with regard to the instruction theory. In addition to instructions for textbook revision, a thorough textbook analysis also provides directions for those who use the textbook. For instance, it may turn out that there is a better approach to certain topics than the one in the textbook. And the answer to the question of the degree of implementation will also be accompanied by an evaluation. The questions regarding textbook adaptation and teacher support follow naturally here. Something similar is true of the research into the learning results. The research shall therefore have to anticipate this kind of practical question. Assessment is not sufficient. Practical links, too, will have to be included in the analysis:

- How do the presumed deficiencies of the textbook influence the instructional practice, and how does this affect the learning results?
- Can the lagging behind of the instructional practice be traced back to specific beliefs or specific educational conditions?

Moreover, the problems and potential barriers which arise when reform is put into practice will be viewed from this perspective. Something similar is true of the learning results. Here, too, an evaluation of the results plays an important role and, once again, one should take a look at potential influences.

- What links can be found between the nature and content of the instruction and the learning results (with respect to achievements, learning paths and solution strategies)?

One characteristic feature of all these practical questions is that no quantitative answers are to be expected. The nature of the assessments is not quantitative but, rather, qualitative. The question is not, what percent is the divergence but, rather, in what respects does that which has been found diverge from that which was intended? The same is true of the relationships. The question is not so much, what is the magnitude of the correlation, but, what is the nature of the relationship? The MORE-project did not focus directly on quantification when responding to this type of practical question. Quantification is necessary in order to statistically detect certain relationships, but this quantification was always embedded in a broader qualitative analysis. This is true of the textbook analysis, the instructional practice and the beliefs as well as of the learning results. This makes it possible to supply practical directions, primarily regarding the learning results and the implementation.

implementation of the reform

When we look at the instructional practice, it is clear that the reform as put into practice has lagged far behind what was intended. So, if we wish the reform of mathematics education to succeed, then something will have to change. Simply using a new textbook is obviously not sufficient. One cause of this lag may be the fact that the WiG teacher's guide offers only succinct instructions didactically.[17] Aside from this, however, it is, of course, true that the actual constitution of the instruction in the classroom must be left to the teacher, however detailed a textbook's guide may be. At issue is, ultimately, a living, interactive and creative process, and that can never be programmed in advance (Leithwood, 1981).

A qualitative analysis of lesson protocols, conducted in the framework of the MORE-project, revealed how demanding it really is to implement realistic mathematics education in the way it is intended (see also Streefland and Te Woerd, 1992). In mechanistic education, one can work according to a set plan. Moreover, the class follows a fixed routine of 'demonstrate-copy-practice', which can be entirely planned beforehand. It is expected of the realistic teachers, by contrast, that they adapt the instruction to the students' contributions. At the same time, however, potential problems must be foreseen, and the teaching-learning process must be streamlined in such a way that the students can glean the essence of the subject matter as concealed in the context problems. This requires not only pedagogical skills but also specific didactic know-how. The teacher must know what role a given problem (or given type of problem) plays in a course, what solutions are possible and how these relate to the various learning paths. In other words, the teacher must possess micro-didactic knowledge. This specific knowledge is part of the aspect of beliefs investigated in the MORE-project.

The beliefs of the WiG teachers seemed to correspond fairly well to the theory behind the textbook. A closer look, however, revealed a different picture. The correspondence was mainly on a rather general level and on the level of the textbook as a whole. On the lesson or problem level it became apparent that the realistic theory was not always endorsed by the WiG teachers. This was evident, for instance, where the teachers were asked their opinions on various ways of calculation under 100. They were given a number of different solution methods for the sum 45 – 28. Their reactions revealed a clear preference for the more traditional approaches. One-quarter of the WiG teachers felt that such a sum should be dealt with in a fixed manner. A completely natural manner such as interpreting 45 – 28 as a missing addend, to be solved by counting up from 28, was rejected by a good 45% of the WiG teachers. So, on the micro-didactic level, discrepancies do appear between the general beliefs and how they are put into practice. This unfamiliarity with realistic theory may lead to a loss of the effectiveness of the instruction because of too extensive involvement with the context. The story and the students' own contributions to it may become so dominant that they smother the actual mathematics.

On the other hand, the students' contributions may not be given recognition because the teacher doesn't know how to fit them into the educational process.

Micro-didactic knowledge is not, by the way, the only aspect involved here. General pedagogical skills are, as mentioned, also at issue. Research conducted by Desforges and Cockburn (1987) shows that certain mechanisms active in instructional practice tend to hinder more daring instructional activities. They observe, for instance, that most students do not like insecurity and would rather be told what to do. In practice, this means that students are constantly appealing to the teacher to tell them what they should do. It is also clear that a class is much less manageable when the students are given problem-oriented instruction than when they can work in a more routine fashion. Teacher support will therefore have to focus on two objectives: the reinforcement of the micro-didactic knowledge on the one hand and, on the other hand, the reinforcement of the general pedagogical skills.

The MORE-project offers little support for the idea that teachers will acquire these things automatically through a learning process that occurs simultaneously with the implementation. (This result may be specific for the textbook series WiG, because of its concise teacher guide). Neither a quantitative comparison between two successive academic years involving twenty teachers, nor a more qualitative analysis revealed any growth. It did show, however, that the more experienced teachers worked more realistically. Perhaps they were already better equipped when they began using the new textbook.

footholds for improvement

The main problem in enacting realistic mathematics instructional practice is the area of tension between 'letting them (re)invent it themselves' and 'guiding the learning process'.[18] Ideally, the guiding should be put into practice indirectly: by discussing solutions, clarifying solutions (or having them clarified), offering new problems, giving hints, posing critical questions, and so on (Goffree, 1979). The teacher chooses from such an array of tools depending upon what the student knows and is capable of, and the learning result intended. These two aspects determine a potential learning path, and the educational activities are then chosen to fit this path. This kind of guiding demands a great deal of micro-didactic knowledge on the part of the teacher. In the first place, the teacher must be aware of the potential learning paths but, moreover, must be able to recognize unclearly formulated or incomplete solutions. It should, in principle, be possible to impart such micro-didactic knowledge by way of courses for inservice teacher training courses. But, in addition to the fact that a rather extensive amount of specific knowledge is involved here, there is also the problem of its being theoretical knowledge. It would therefore be better if the teachers could develop this knowledge themselves. The teachers do probably already possess a great deal of informal knowledge that has the potential to be developed further. This knowledge can be made more conscious through well-focused reflection on the context problems.

> Take, for instance, a problem like the following:
> Swiss Cheese costs $1.20 per lb.
> What does 0.75 lbs. cost?
>
> This problem could provide the starting point for an assignment such as: Try and find as many different solution strategies as possible and use this knowledge to develop a lesson around this sum. American students who were given a similar problem presented a variety of solutions which, moreover, offered insight into the various learning paths (Gravemeijer, 1992b). A number of solutions emerged which made use of the relationship between 0.75, $\frac{3}{4}$ and 'the ratio of 3 to 4'. One solution was to break up $1.20 into quarters and nickels and then remove three quarters and three nickels. The relation to money also affected the rising awareness that 0.75 corresponds to $\frac{3}{4}$ (three quarters). Sometimes solutions were supported by a double number line or ratio table, such as: 'calculate the price of one and a half kilos and divide that by two' or, 'take the price of a kilo and of a half a kilo and calculate the amount in between'.

Teachers can probably develop the necessary micro-didactic knowledge themselves when they begin viewing individual solution strategies as research terrains. In-service teacher training and teacher support can be of assistance here, as can the teacher's guide for the textbook in question. Perhaps, with directed support, the teacher's learning process could get jump-started. The NVORWO is advocating an arithmetic coordinator in every school for this purpose; this would be someone who could initiate such a learning process and support it over the long term (Dolk, 1993).

The necessary pedagogical skills also demand a learning process that must take shape in the classroom. The great need for pedagogical skills springs from the above mentioned tension between 'letting them (re)invent it themselves' and 'guiding the learning process'. On the one hand, the students themselves have responsibility and, on the other, the teacher is still in charge. This may lead to lack of clarity, which was perhaps the cause of the problems observed by Desforges and Cockburn (1987). In traditional classrooms it is clear how things stand: it's the teacher's to know and the student's to find out. The familiar question-answer pattern fits this situation, in which the teacher asks a question, the student answers, and the teacher determines whether the answer is correct (Voigt, 1985). How specific this pattern is to education can be seen when we project it onto a familiar situation.

> Passer-by A: 'Can you tell me where Main Street is?'
> Passer-by B: 'That's the second street to the right.'
> Passer-by A: 'It *is* the second street to the right. Very good!'

In traditional mathematics education this type of pattern is quite normal. Evidently, there are implicit agreements regarding the path of an educational learning process (see also Wijffels, 1993). Such a collection of implicit agreements is sometimes called a 'didactic contract' (Brousseau, 1990). The students discover this contract themselves, without it ever being discussed explicitly. This means that the students may also interpret the contract in a way that the teacher had perhaps not intended. For example, the students may think that the object in math class is to quickly give the right answer. And since only the teacher knows what is right, you, the student, must guess what the teacher wants you to say. This lack of authonomy is often visible when a teacher repeats a question. In most cases, the students will not repeat their first answer, but will come up with a new one. They have of course discovered much earlier that repetition is a signal for an incorrect answer (cf. Yackel, 1992). Something different is expected of the students in problem-oriented mathematics education. But do they know that? It is probable that a transition to problem-oriented education will require explicit attention to the change in expectations. The students must learn that 'the correct answer' is not the point, and that it's OK if they make mistakes. In addition, the students must learn new obligations:
- the students are expected to justify their own solutions to themselves, and to explain and substantiate them to others
- the students are expected to try and understand the solutions of others and, when they do not, to discuss them.

The point here is not to learn new rules of behavior by heart. It has to do with:

> '... establishing a culture in the classroom. A big piece of teaching for understanding is setting up social norms that promote respect for other people's ideas. You don't get that to happen by telling. You have to change the social norms – which takes time and consistency.' (Lampert in: Brandt, 1994, p. 26)

Social norms are not, after all, explicit agreements but, rather, indications of implicit expectations of both teacher and students. A change in social norms must be made visible by an actual change in behavior. Concrete situations can be used here to make the new norms explicit. Gradually, this will create a situation in which realistic mathematics education can flourish. A learning process can be initiated in which the teacher increasingly learns how to manage problem-oriented mathematics education. It may be possible to combine this practical learning process with a learning process in which the teacher expands his/her micro-didactic knowledge. This expansion can take place partly through studying teacher's guides, for instance, but, in the first place, by anticipating and analyzing the students' responses. The foundation for this learning process lies with the teachers themselves. The teachers, like the students, must 'gain respect for their own ideas' (see Lampert, ibid.). The teacher's own reflection on the instruction then becomes the motor for his/her own learning process (see also Clarke and Peter, 1993). Footholds for in-service teacher training and teacher support can be formulated on the basis of the above. These footholds can then lead to a better implementation of realistic mathematics education.

learning results

Alongside the implementation research, an analysis of the learning results can also provide pointers for making the reform more successful. The slow start of WiG in the area of bare sums will not worry most reformers, particularly if it turns out that the achievements in the areas of algorithms and context problems are equal at the end of third grade. More worrisome is the matter of the basic facts. The results of the oral tests revealed that the NZR students more often used strategies for deriving basic facts than did the WiG students, who often resorted to counting. This is particularly striking, considering the fact that use of strategies is one of the objectives of realistic mathematics education. On the whole, the solution procedures used by the students in the other areas do conform to the textbook's approach. But the fact that NZR students used more advanced arithmetic strategies than did the WiG students, who counted more, is in direct conflict with the objectives of the realistic educational theory. This would seem to be a direct result of the specific design of WiG.

The disappointing results are, in fact, explained when one views the analysis of the textbook's contents. Informal strategies are indeed stressed in WiG, but little attention is paid to establishing the necessary elementary basic knowledge. In NZR, on the other hand, it is seen to that the students quickly get the elementary basic facts (such as $3 + 2 = 5$ and $3 + 4 = 7$) firmly in their heads. And this is why the second grade NZR students possessed the basic knowledge needed for deriving other facts while the WiG students did not.

The conclusion is therefore that the textbook is in need of some revision. The authors of WiG had already taken a step in this direction by publishing the so-called 'Tip booklets'. Our research showed, however, that these booklets were rarely used. Fortunately, the new edition of WiG has integrated the booklets into the main text. The MORE-project revealed the significance of this kind of textbook revision, for one of the conclusions was that learning results can be influenced by the subject matter in the textbook. This also means that the learning path can to a certain extent be planned. For teachers who are using the WiG version used in this research, it is recommended that they pay more attention to the elementary basic facts than the textbook indicates. The 'Tip booklets' mentioned above can be of service for structuring this learning process. It is interesting that a new edition of WiG is now available in which those very aspects criticized by our research have been improved. The teacher's guide for the new WiG is both clearer and more extensive and the course on basic facts has been thoroughly revised. It would seem that WiG is gradually taking its definitive shape. The question that now arises is, of course, how the new WiG will compare with NZR. This emphasizes once again the relative nature of this type of comparative research.

5.6 evaluation research: 'doing justice to the object'

The expansion from quantitative and observational evaluation research to qualitative and interpretational evaluation research leads us to more general questions regarding the place of evaluation research:

- What do you expect to accomplish with evaluation research?
- For whom is such research intended?

When we approach evaluation research from a purely theoretical perspective, these questions may either concern curriculum theory or implementation theory. The evaluation may also be approached more practically, as an evaluation for the benefit of groups for whom the results of such research will have practical significance. By this means, policy makers can be the consumers of evaluation research. Other potential consumers are, for example, teachers, textbook authors, teacher trainers and school guidance counsellors. Quantitative and observational research is certainly meaningful for these groups but, for the last group in particular, a formal assessment of the effectiveness of the objectives of the mathematics textbooks in question will not be sufficient. This group is more in need of instructions for interventions that can lead to better implementation or better results. Here we arrive at the above-mentioned innovation perspective as a specific evaluation perspective.

Curriculum evaluation is customarily seen primarily as the assessment of the curriculum's effectiveness. In cases where explanatory variables are included, this is done to correct for the influence of such variables, in order to facilitate general pronouncements. The reformers of mathematics education will approach the evaluation mainly from the perspective of change. This group is, naturally, also interested in the question 'what does it produce', but more so in the question that lies directly behind, namely, 'what can you do to improve the success of the educational reform'. So, in addition to assessments, qualitative relationships are especially important.

It is not the quantification of relationships that is of primary significance but, rather, the interpretation of the situation. Empirically determined statistical relationships are integrated into an interpretational analysis. Qualitative data and qualitative analyses can be extremely important. We saw earlier, for instance, how qualitative analyses of lesson protocols contributed to a better understanding of the implementation problem.

A qualitative analysis of beliefs revealed the problem located at the level of micro-didactic viewpoints. It is precisely these qualitative analyses that can offer footholds for improvement.

We can integrate these two approaches of the evaluation issue by taking the two-sidedness of objectivity as mentioned in chapter four (following Smaling, 1987). This consists of:
- avoiding distortion, and
- offering the opportunity for the object to reveal itself.

The former corresponds to the prevalent manner of objective assessment of the effects of a curriculum. For the latter, one needs to keep an open mind to the reformers' intentions, the teachers' beliefs and the students' ideas. The MORE-project met both aspects of objectivity. In this sense, the MORE-project is a good example of evaluation research that endeavors 'to do justice to the object of the study'. This does not mean that the project was perfect. The chosen approach was at once ambitious and experimental. Many new instruments had to be developed while these same instruments were needed for experimentation in the research. As a result, not only did the project yield a great deal of information, but it also provided directions for improving the design for subsequent research. It is now clear, for instance, that the instrument for protocol analysis was insufficiently tailored to individual lessons. The realistic categories in particular were more concerned with curriculum characteristics than lesson characteristics. Moreover, it became clear that a precise assessment of the nature of a particular teacher's instructional practice was impeded by the great variety of lesson content. It would be advisable in future research to work with preselected lesson content and to focus the measurements of the instructional practice on these particular lessons. Such a focus would also make it possible to link micro-didactic beliefs more directly to a teacher's behavior on a micro-didactic level. Research such as this can yield important instructions for a concrete interpretation of effective implementation support (cf. Van den Heuvel, 1993).

5.7 conclusion

The reform of mathematics education has a number of specific characteristics. These concern, among other things, the nature of the learning objectives, the change in education as an innovation objective, and the fact that the reform relies heavily on the textbooks. In determining the success of this reform, we must take these specific characteristics into account.

As contended earlier, this means that the implementation must be involved in the reform evaluation. It also means that appropriate tests must be developed. The MORE-project has shown that such an evaluation increases its significance when it is placed in the perspective of the innovation. In quantitative, empirical evaluation research, the degree of implementation and the learning results achieved with the new educational approach can, in principle, be objectively determined. In principle, because determination of the learning results assumes a sufficient implementation of the reform. But this was not the case; the implementation of the reform lagged behind the intentions. Qualitative, interpretational evaluation research that was part of the MORE-project provided footholds for a better implementation.

Furthermore, instructions could be given for improving the way in which the learning of the basic facts in the textbook in question (WiG) were structured. By connecting the evaluation from an innovation perspective to the specific manner in which the reform of mathematics education takes place – through the textbooks – it was possible, moreover, to provide recommendations for implementation support. In practice, this comes down to stimulating and supporting teachers in their passage through a learning process whose focus is on attaining general pedagogical and micro-didactic knowledge and skills. The general pedagogical skills involve alteration of the 'didactic contract' between teacher and student, the implementation of 'social norms' suitable to realistic mathematics education, and dealing with problem-oriented education. The development of micro-didactic knowledge and skills has to do with placing individual solutions within potential learning paths. This assumes a learning process that relies on analysis and reflection of students' solutions and on consideration of one's own teaching before and after the fact in relation to the studying of information on learning paths. With effective support, a long-term learning process could be realized that results in an actual implementation of realistic mathematics education.

notes

1 The research reported in this paper was supported by a grant of the Institute for Educational Research in the Netherlands (SVO). The opinions expressed do not necessarily reflect the views of the Foundation.

2 We must, however, also be aware that even the users may err in evaluating a textbook. In the nineteen-seventies, for example, strongly individualizing textbooks, such as 'Niveau Cursus Rekenen' were greeted enthusiastically. Since then, nearly everyone's enthusiasm has waned. This variety of 'programmed instruction' did not end up delivering what had been expected of it.

3 Kuipers, N. en E. de Groot (1978) (authors revised version). *Naar Zelfstandig Rekenen.* Groningen: Wolters-Noordhoff.
Working Group lead by G.W.J. van de Molengraaf (1981). *De Wereld in Getallen.* Den Bosch: Malmberg.
Gravemeijer, K., F. van Galen, J.-M. Kraemer, T. Meeuwisse and W. Vermeulen (1983). *Rekenen & Wiskunde.* Baarn: Bekadidact.
Vuurmans, A.C., W. Klukhuhn, S. Gribling and J. Nelissen (1986). *Rekenwerk.* Gorinchem: De Ruiter bv.

4 The latest editions of OR and WiG have more extensive teacher's guides.

5 The descriptions vary from: concrete descriptions of educational activities in practice, in the form of stylized accounts of lessons, to more general descriptions of the objectives of

the activities and the relation between activities, reported in bi-weekly summaries, to descriptions of didactical starting points and educational beliefs in various project publications (Gravemeijer, et al., 1984, 1986).
A form of presentation is sought here that is informative without being prescriptive. The teacher should read the guide as 'it could be done this way' and 'these were the authors' underlying intentions' and then draw his or her own conclusions.

6 Gearing the original training programs to the reform is also, of course, a possibility. This option is also used in this innovation (Goffree, 1982), but the effect of pre-service teacher training will only be noticed in the long run. Moreover, the Dutch training programs are beset with all sorts of problems as a result of mergers and suchlike.

7 The government has stimulated an extremely limited form of in-service training in the form of so-called 'introduction programs'. These are brief courses that familiarize school teams with the new (mathematics) textbook series during the initial phase of its introduction. (Vermeulen, 1987)

8 Since that time, Fullan has, for that matter, put the significance of isolated reforms into perspective. He has, therefore, shifted his attention to an integrated approach to teacher development (Fullan, 1991; Fullan and Hargreaves, 1992).

9 Note the absence of attention to the potential influence of student characteristics or educational conditions on beliefs, textbook, nature, and content. These influences may be present, but they were not a research aspect in the MORE-project.

10 Approaching the content of the instruction via a textbook-analysis assumes a close link between the textbook's content and the actual content of the instruction. This link is examined by, among other things, asking the teachers what they had altered. The answers showed that the teachers had, on the whole, followed the subject matter sequence supplied by the textbook.

11 For each of the characteristics, a five-point scale was used to indicate to what degree a realistic and to what degree mechanistic instructional practice was present. In the same manner, the evaluators gave a general assessment of the degree of realism and mechanism, respectively. The evaluators could choose from: negatively, not, limitedly, fairly or clearly present. At least three lessons given by each teacher were included in the assessment.

12 For both scales, a composite score was determined by taking the average of the five categories together with a general assessment. The averages per textbook series were determined on the basis of the scores of 26 NZR teachers and 39 WiG teachers. The analysis also distinguished, moreover, the category 'specific beliefs', which consists of a combination of the categories 'unit' and 'problems'.

13 The correlations between the mechanistic and realistic score and the textbook (NZR = 1, WiG = 2) were -.91 and .88 respectively.

14 The influence of the content of the instruction was, however, measured indirectly, due to the fact that a direct link was laid between textbook contents and learning results.

15 The more problem-oriented programs in particular turn out to have a positive influence on the development of intelligence. This agrees with the project's findings, namely, that the WiG students' score on the Raven (intelligence test) increased more between first and third grade than did the NZR students' score (from p = 63 to p = 71, and from p = 63 to p = 66 respectively).

16 Naturally, the most significant student and educational characteristics were included in this research. The students' initial state was determined by a written test that was administered during the three weeks following the summer vacation. The Raven-test was used to measure the students' intelligence. In addition, data was collected on the students' cultural background. As for the education, here the content, the nature of the instructional practice, and the effective learning time were examined. The effective leaning time was examined by taking into account both the total instruction time and the ratio between task-oriented and other activities per lesson (measured using a structured observational instrument).

17 In this respect, it would have been better to have examined the 'Rekenen & Wiskunde' textbook rather than WiG. RW is, namely, characterized by an extremely extensive teacher's guide. Moreover, this is the textbook in which the intentions of Wiskobas have best been elaborated (De Jong, 1986).The leader of the MORE-project, however, was also the main author of 'Rekenen & Wiskunde', which is why it was deemed preferable to choose a different textbook.

18 This field of tension lies within the fundamental idea of realistic mathematics education: 'guided reinvention' (Freudenthal, 1991).

literature

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6 Developmental research revisited

introduction

There was a time when 'instructional design' was a popular subject for research and theory forming (see Creemers and Hoeben, 1988). This subject has long since lost its popularity. Only as an auxiliary science in vocational training does instructional design still seem to flourish (Nijhof, 1993; Romiszowski, 1981). The meager success of the design models made researchers wary and the conviction took root that design is an opaque, chaotic creative process that is inaccessible to science. In recent years, however, design has again begun to attract interest. Only now it is no longer a direct attempt to find prescriptive models for improving the quality of the design process but has become design as a research method. More and more researchers are setting out along the path of constructional research (Brown, 1992; Cobb, Perlwitz and Underwood, 1992; Lampert, 1990; Steffe, 1983). Whereas, in the 1960s and '70s, the emphasis lay on scientific knowledge as input for design activity, now attention has shifted to scientific knowledge as output.

So the yield after many years of developmental research in the area of realistic mathematics education is more than merely a collection of realistic math units. The result of all this research also includes a domain-specific educational theory. The question is, however, how does one legitimize this theory? Is the developmental research upon which it rests sufficient to justify the theory, or is something else necessary? One can, of course, always attempt to test the educational theory in evaluation research. But in evaluation research one mainly tests the practical significance of the theory: what are the results of the curricula based on this domain-specific instruction theory?

Developmental research should be able to provide a more direct foundation. If developmental research is to fulfill this function, then both the substance of said research and its product must be clear. This demands reflection upon and reconstruction of the practice of developmental research. Developmental research is not, after all, a strictly regulated methodology[1] but, rather, a manner of working that has grown through being put into practice. Only by reflecting on such practice can it take shape as a method.[2] This process demonstrates a general principle of realistic theory forming: choose a starting point that lies in practice and in theories of practice. This is what occurred in the first chapter, where the developmental practice of the 'Rekenen & Wiskunde' textbook series was taken as a starting point for an analysis. This led to a characterization of developmental work as theory-guided bricolage in chapter 4.

Furthermore theory-guided bricolage was used to characterize developmental research in practice. Actual practice as a starting point was also the foundation for Treffers' (1987) reconstruction of practice theories as a domain-specific instruction theory. Theory development with respect to in-service teacher training has taken place in the same manner. By experimenting with in-service teacher training courses, a solid foundation was laid for reflection on in-service teacher training (De Moor, 1980; Gravemeijer, 1987; Gravemeijer and De Moor, 1988; Van Galen et al., 1990, 1991, 1992). In the meantime, moreover, it has become evident that one must first know more about educational practice before providing implementation support (see Van den Heuvel-Panhuizen, 1993).

In the end, the basic philosophy of mathematics as a human activity also has its roots in actual practice. Not, it is true, in experiences with this type of education (the idea was there before the education), but in personal experience. Freudenthal (1973) justifies his view of mathematics and mathematics education by referring to his own manner of doing mathematics. He defends the reinvention principle with, among other things, the claim that this is the way he himself learns mathematics. This is the way in which he becomes familiar with other people's mathematics proofs. He uses clues in the proof to reinvent how he might have discovered it himself. The central principle of realistic mathematics education – *'mathematics can and should be learned on one's own authority, through one's own mental activities'* – stems from reflecting on his own learning process. The conviction that, not only is this way possible but, in fact, necessary is based on an analysis of existing educational practice.

In this chapter I will endeavor to elaborate further upon the concept of developmental research. The manner in which I do so is in keeping with what I have described above. I have chosen my starting point in the actual practice of developmental research. From there I will go in search of precise functions and characteristics, after which I will examine how this research approach can be developed further. The guideline for this exercise is the role developmental research can play in legitimizing the educational theory produced by the research. Keeping the potential of legitimizing the realistic instruction theory in mind, the following section will first discuss the possibility of it being tested in evaluation research.

6.1 evaluation research

Two decades of developmental research whose guideline was the principle of realistic mathematics education have eventually led to the development of a domain-specific theory for realistic mathematics education. But what can now be said concerning the validity of this theory? In other words: researchers from the realistic camp may well be convinced of the correctness of this theory, but how can this realistic educational theory be legitimized externally?

By external legitimization I mean legitimization to outsiders, as distinguished from legitimization within the realistic camp, or internal legitimization. External legitimization can take place by:
- testing the theory in evaluation research, and by
- using the internal legitimization for this purpose.

The first manner will be discussed in this section. The realistic instruction theory can, for example, be evaluated by examining the results achieved using realistic mathematics textbooks. The results of this type of research, however, give rise to some debate, as described in chapter 5. The debate here touches the weak spot of this type of evaluation research. There is always room for criticism where the results of such research are concerned. The research situation is so complex that it is impossible to keep track of all the potentially significant variables. The result is, therefore, that the choices made and actual operationalization are open to debate.[3] And this only has to do with research into curricula. When the theory behind the mathematics textbook series is at issue, then the discussion becomes even more complex. The theory, after all, is tested very indirectly. All in all, the result is that an isolated research project is not very persuasive. This type of research only becomes convincing when a series of research projects point in the same direction. Ideally, each successive research project should try to take into account the criticism leveled at previous projects. By this means, a sequence of research projects would emerge which could subsequently address all possible criticisms.

My suspicions are that the realistic curricula in The Netherlands will surpass their competitors in the area of learning results. The findings of the PPON-research (Wijnstra, 1988) are quite convincing. It would seem that the realistic approach is a qualified success. The primary objective for me, however, is the reform and revision of the education itself. The MORE-project, discussed in chapter 5, demonstrates how difficult it is to put the realistic theory into practice. The MORE-project also shows the direct influence exerted by subject matter content and structure. The positive PPON results can be explained by the revisions made in subject matter content and structure. Evidently, revision of the subject matter has a positive influence on the learning results, even when the realistic instruction theory is only put into practice to a limited degree.

That is, if we take the MORE data as a yardstick. The question is, however, whether this data can be generalized so simply. There are two obvious limitations. In the first place, the MORE-project involved a specific textbook and, in the second place, specific grades – first, second and third. Whether the realistic theory would be expressed better in higher grades cannot be directly predicted. But it does seem reasonable to assume that the textbook is of influence. The textbook in question had a limited teacher's guide. Moreover, the theoretical elaboration was not optimal (Feijs, De Jong, De Moor, Streefland and Treffers, 1987; Van den Heuvel, 1991). In general, it may be assumed that a more complete and better structured guide will lead to a better implementation (Van den Akker, 1988). On this assumption, a better implementation may be expected with a textbook series such as 'Rekenen & Wiskunde'. Preliminary results of a research project that compares the use of the textbooks, 'Wereld in Getallen' and 'Reken & Wiskunde' seem to affirm this hypothesis (De Vos, 1994). Anecdotal support for this idea can be found in journalistic observations in a national Dutch newspaper.

> 'The text in the arithmetic book looks familiar:
> Margriet is bicycling from Hilversum to Arnhem. She leaves at eight o'clock. After one and a half hours she sees a road sign, which shows that she still has 45 of the 75 kilometers to go.
> 'I'm making good progress', she thinks.
> But the question, 'what time will she be in Arnhem?' is missing.
> Nico Schilder, a teacher at the 'Zuidwester' school in Volendam asks the class something else instead: 'Why does Margriet think she's making progress?'
> It is the beginning of an educative discussion.
> Jaap knows: 'Because she's bicycling fast.'
> 'How fast?'
> 'Twenty kilometers per hour.'
> 'Which of you thinks that's fast?'
> Most of the class does. It depends whether Margriet has the wind with her or against her. Arnold doesn't think it's so fast; he says he usually bicycles thirty kilometers per hour. Thirty. This is disputed by his classmates. That's as fast as a moped, they say. How fast are the bicyclists in the Tour de France? How can you measure your speed yourself? Without realizing it, the students are involved with the connection between time and distance.
> How late Margriet gets to Arnhem is mentioned in passing. The answer is less important than the arithmetic itself.'
>
> (Paul Stapel, 1989)

This is not only an example of good implementation, it also shows the kind of support a textbook can offer. The teacher's guide to the 'Rekenen & Wiskunde' textbook, in which this lesson appears, contains an extensive description of this activity (Gravemeijer et al., 1987).

> Here, among other things, are responses the students might give to the question 'Why does Margriet think she's making progress?' The guide also deals with the question of whether Margriet is bicycling fast or slowly. In addition, the guide suggests paying attention to the fact that the concept of 'average speed' suggests steady speed, although this need not be the case. Finally, the guide indicates the possibility of working with ratios: 30 km in $1\frac{1}{2}$ hours, so 15 km in $\frac{3}{4}$ hour and, therefore, 45 km in $(1\frac{1}{2} + \frac{3}{4})$ hour.

It is clear that the textbook can play an important role (see also Meeuwisse, 1985; Kraemer, 1988). And if we assume that the quality of the teacher's guide will, indeed, be decisive, then we may ask ourselves whether this 'direction by the guide' might not be expanded further. This would certainly seem possible if one chose a more directed form of realistic mathematics education. The emphasis should then be shifted from the form of open discussion to a form of interaction where the teacher is more dominant. Treffers commented informally on this, calling it 'explanation on a variety of levels'. The teacher's explanation should provide footholds for all students by, for instance, discussing a variety of solution strategies and then relating them to one another. The students not only would be able to recognize their own solution strategy, but would be able to progress a step further if they saw the relation to other strategies. Such prepared explanations could be included as examples in the teacher's guide to a textbook. The teacher could use such an explanation not only as an example, but also as background information.

If evaluation research repeatedly reveals a lag between the intentions and the implementation of realistic mathematics education, the realistic educational theory will lose much of its practical significance. The question of the scientific legitimization of the educational theory will no longer be as relevant if it appears that the theory is impracticable for the average teacher. In other words, evaluation research may well offer an indirect assessment of the instruction theory, but this does not make the results any less important. Any indication of problems surrounding the implementation should be taken seriously. For the time being, two paths are available for improving the implementation:
- directly influence the teachers' views, knowledge, insight and skills (as argued in chapter 5);
- choose a more directed form of realistic mathematics education and adapt the textbooks accordingly.

My preference as yet is for the first option, even though it is much more difficult to put into practice, and probably not feasible in the short term. On the other hand, a more directed approach might provide a good basis for growing towards a more open form of realistic mathematics education.

6.2 internal legitimization

Being a combination of development and research, developmental research has a dual function that I would like to refer to as production and justification. Here the emphasis is on the legitimization aspect. What is it that you justify in developmental research? A course or, better, the choices upon which the course is based. In development, one is constantly making choices. Not every choice is equally important; numerous details also demand attention, of course. But some of the choices are crucial. These are the choices that reflect fundamental ideas regarding mathematics education. Among these are, for instance: the choice of the initial contexts, the choice of contexts for application, the choice of models, the form of notation, planning, and so on. Moments of reflection and increasing awareness belong on this list, too, just as do the central points for a class discussion and for the individual productions.

The result of developmental work is a prototypical course. The result of research is a description of the course on a meta-level (a local instruction theory) and a justification. The justification goes something like: 'This course satisfies the basic principles of realistic mathematics education, because ...' This is followed by a line of reasoning supported by a theoretical analysis, empirical data and the interpretation of this data. A justification of this nature usually contains: an analysis of the area of subject matter, an intrinsically substantiated characterization of the structure and content of the course, paradigmatic examples (of student work and interaction) and a reflection on the realistic caliber of the whole. A balanced reflection will mention any negative characteristics in addition to positive ones.

If the developer fails to note any negative aspects, it is likely that others will do so. At an early stage, the prototype designs will come to the attention of interested parties. The idea will be taken in hand and many will get right to work. This will significantly expand the subjective research experience of the developmental researcher. In general, the researcher will not have taken elaborate methodological measures to ensure theoretical 'reproducibility' of the research results. Instead, however, there will be *actual repetitions of the experiment conducted by others.* Each person will repeat the experiment in his or her own way and under different circumstances, which will serve to broaden the empirical basis.

The new prototype will become the subject of discussion among the experts in the domain of mathematics education. This group consists not only of developers and researchers but, also, of school counsellors, textbook authors, teacher trainers and teachers. Discussion among such people will include topics such as: effectivity, feasibility, theoretical quality in the light of the realistic objectives, etc. Eventually, an inter-subjective agreement regarding the value of the prototype will come forth. This process strongly resembles the one described by Ernest (1991) in his analysis of how new insights into mathematics acquire a scientific status.

A characteristic and essential facet of this process is a certain homogeneity of the group in question. In order to carry on a respectable discussion and to arrive at a consensus, it is necessary to have a communal frame of reference. This also demonstrates at the same time the limitations of the value judgments and claims. Because of the communal frame of reference, the various members of the group will interpret the same phenomenon in (more or less) the same way, giving rise to the danger of systematic errors. Not that this is anything new – every research group must contend with this problem. Take, for example, Piaget's conservation experiments. The tests have been repeated numerous times and, for many years, this only led to confirmation of the earlier results. Eventually, however, Margareth Donaldson arrived at a different explanation, one she saw confirmed by experiments (Donaldson, 1978).

In my opinion, there are no objective facts as such that can be attached to unequivocal conclusions; an interpretative theory is always necessary. Let me offer a more extensive example of the connection between theory and empiricism. The following example from Gould (1992) has to do with research in paleontology.

> For many years there has been discussion about the cause of the extinction of the dinosaurs. Was it the result of a catastrophe or was it, instead, a gradual process? Until recently, it was indisputably accepted that the cause was a gradual process. This hypothesis was confirmed by geological research. No dinosaur bones – or the bones of other animal species that became extinct during that period – could be found in the earth layers dating from the period before the presumed meteor impact that caused the extinction. The time of extinction, moreover, varied depending upon the species.
> Powerful geological arguments, however, for the occurrence of an extremely large meteor impact made the researchers begin to doubt the correctness of their findings. The accepted empirical data became, once again, the focal point of discussion and gave rise to an alternative interpretation.
> Uncommon species and badly preserved fossils do not appear in every stratum; once in every 100 feet is no exception. So it is quite possible to find fossils only in strata dating from far before the actual extinction. Indeed, in the case of a catastrophe, the strata will vary in which the most recent fossils are found. The most uncommon species and badly preserved fossils lie the lowest and the more common species and well preserved fossils the highest.
> This gives the impression of a gradual process of extinction, which is why the empirical evidence that no catastrophe had taken place held up for years. New research was begun. If the impact theory was correct, then fossil remains should indeed be found somewhere in the strata in question. The new assignment was akin to searching for a needle in a haystack. The search was almost immediately successful, and the evidence in favor of the catastrophe theory continues to grow. Only now can one see what was not seen before, thanks to the willingness to view things from a different perspective.

This example shows that building a research community carries with it the danger of one-sidedness. Blind spots may appear but remain unnoticed by the members of the research community. On the other hand, it is also true that such a research community is the basis for growth. This is, after all, the way in which research programs work (Kuhn, 1970; Lakatos, 1978). Keeping this danger of bias in mind, now return to the progress of the research in the realistic research community.

As time goes by, more and more prototypes are developed, improved, or even replaced. Within the group, a theory of practice gradually forms regarding the characteristics of curricula that fulfill the realistic demands. The next step in the process is to reflect on this knowledge. This is how Treffers (1987) constructed his domain-specific instruction theory. What are the claims of this theory? In principle, the theory only says: this is what we do. It is, after all, a description of curricula characteristics developed within the realistic movement. Implicitly, the theory also professes: there is a system to what we do, and what we are doing makes sense. Or, to put it more pointedly: the realistic theory of education claims that it describes education that fulfills the central realistic idea of 'learning mathematics on one's own authority, through one's own mental activities'.

How is the educational theory legitimized? In the first place, through the collected developmental research. It is a generalization of the choices on which the prototypes are based. In addition, the communal learning process of the realistic community provides a second legitimization. It should be taken for granted that the experiments with and discussions regarding the prototypes will lead to inter-subjective agreement. Moreover, those involved must feel at home under the umbrella instruction theory. After all, if the realistic instruction theory does, indeed, actually characterize the experts' theory in practice, then these experts should be able to identify themselves within it. This does not mean that the instruction theory has to be a perfect mirror image. There are, of course, individual differences. Moreover, organizing the theory and embedding it in a larger theoretical framework also involves adding something to the theories of the experts.

If we take a step back then we must, of course, acknowledge that the instruction theory does have more pretentions after all. The underlying idea is that the realistic instruction theory can serve as an alternative for other instruction theories and is (according to its proponents), in fact, better. At first glance, such a belief would seem easy to maintain inside the realistic camp, but this is not as self-evident as one would imagine. It requires – at the very least – *internal consistency, usefulness and progression*. The success of the realistic approach in The Netherlands is partly due to the fact that, inside the realistic camp, the developments of the past twenty years have been seen as consistent and useful, and that the new developments are viewed as progress.

6.3 developmental research clarified

As mentioned at the beginning of this chapter, the methods for externally legitimizing the results of developmental research are evaluational research and expansion of the internal legitimization. In my opinion, the latter is the correct manner. Curriculum evaluation can be very useful, but is primarily of significance – its very title says it all – for evaluating curricula. In any case, theory evaluation only takes place indirectly.

In principle, developmental research offers the potential for a much more direct manner of evaluation. There is room within the broad concept of educational development for different types of research: developmental research, implementation research and curriculum evaluation. Each can have its own function and its own research design and methods. In this way, developmental research occupies an independent place alongside other foundational research, such as psychological research on learning, child development, and social interaction. Not that I wish to isolate developmental research. Interaction with psychological research, implementation research and evaluation research is, in fact, essential for productive developmental research.[4]

The question is now: how can we strengthen the role of developmental research as a foundational research? The first step, I would think, is to clarify the method, which is my objective in this section. For outsiders, developmental research is an impenetrable process. Evidently, something is going on as theory-guided bricolage, but certain questions still remain: [5]

- how does the developer get her or his ideas?
- what procedures are used?
- which criteria does the researcher use to make adjustments?
- how does the researcher evaluate his or her own standpoints?

I will now discuss these questions one by one. First of all, where do the ideas come from?

heuristics and design principles

Developmental research is a creative process in which implicit knowledge plays an important role. This does not mean that ideas simply fall out of the sky, nor that how they arose needs no comment. I have already indicated the role of domain-specific knowledge (e.g. in chapters 1 and 3). The researcher may be expected to have come prepared and, therefore, to be in possession of thorough domain-specific knowledge.

Before the researcher dives into the cyclical process of invention, experimentation and reflection, he or she will make an analysis of the situation: Why are the existing curricula unsatisfactory? In this analysis, the demands which the new course is expected to meet will become visible. After all, the new course must give consolation where the old ones were remiss.

Alongside this analysis, a general concept of a course must develop before the actual experiments can begin. Both the problem analysis and the design of a general course design rest to a great degree on the domain knowledge of the researcher. The constitution of a general course design does not, however, take place undirected. The researcher can use the theory of realistic mathematics education by applying the central principles mentioned in chapter 3 heuristically. These are: the reinvention principle, the didactic phenomenology and the mediating models.

The guideline of the *reinvention principle* is: 'think how you might have figured it out yourself'.

Reinvention can be viewed ontologically, taking the history of mathematics as a basis. But there is another access route. Children's spontaneous solution strategies can also be used to put the developer on the track of a possible reinvention-route (Streefland, 1985; Gravemeijer, 1991).

Didactic phenomenology points to applications as a possible source. Following on the idea that mathematics developed as increasing mathematisation of what were originally solutions to practical problems, it may be concluded that the starting points for the reinvention process can be found in current applications. The developer should therefore analyze application situations with an eye to their didactic use in the reinvention process. Streefland (1993) adds to this the principle of exploiting the context by different variations of shifting. In this way, the links between the various learning strands can also come to light, enabling the developer to make use of the intertwined learning strands.

Mediating models are deployed in realistic mathematics education in order to connect informal and formal knowledge with one another. Or, in other words, to serve as a catalyst for a growth process in which the formal knowledge evolves from the informal knowledge. The developer must therefore search for ways to model the students' informal strategies so that models, diagrams, manners of notation, and suchlike evolve which can then be used to generalize and formalize the informal knowledge and strategies. In this plan, the models are first linked to contexts. Referring to the context gives them meaning. Providing variety in the kinds of problems will aid independence and the ability to generalize, so that the same models can then function as a basis for further formalization. In this way, the reinvention process is structured along four levels, which have to do not only with actual models but also with concepts, strategies and manners of notation. These levels may be outlined as follows:

1. the level of the situations, where domain-specific, situational knowledge and strategies are used within the context of the situation
2. a referential level, where models and strategies refer to the situation described in the problem
3. a general level, where a mathematical focus on strategies dominates the reference to the context
4. the level of formal arithmetic, where one works with conventional procedures and notations.

The concrete level requires some clarification. What do we precisely mean by 'reality'? The reality outside school can hardly be brought into the classroom. An attempt is made, however, to make this reality as authentic as possible. Bus rides, for instance, can first be performed as arithmetic play-acting (Van den Brink, 1989). The reality of a realistic context should not, by the way, be identified with everyday reality outside school. Realistic mathematics education has to do with situations that are 'experientially real' to the students. These may be everyday situations, but they may also be fantasy worlds in which the students can immerse themselves. And – last but not least – it may be the mathematics itself that is experientially real (see also Davis and Hersh, 1981). The objective of realistic mathematics education is that the mathematics developed by the students themselves be experientially real. Freudenthal (1991) spoke in this context of a developing 'common sense'. The mathematics developed in realistic mathematics education should be experienced by the students as common sense.[6]

Use of the above-mentioned heuristics means that preparatory research must be conducted. The didactical-phenomenology heuristic entails, for instance, presenting selected application problems to (a few) students in order to see whether their solutions are insightful. The usefulness of the chosen models can also be tested beforehand. Moreover, in accordance with the principle of theory-guided bricolage used by the researcher in making a global design, all available designs and experiences that are useful can be utilized. Two principles of a different kind can be borrowed from the analysis of developmental work found in chapter 1.

The first principle involves the *planning of long-term learning processes* (see also Streefland, 1985). The focus of realistic mathematics education is not simply on local success with small units of subject matter. Guided reinvention entails a gradual structuring of the subject matter. Reconstruction of the development of the unit for arithmetic up to twenty in chapter 1 showed how thinking in terms of learning strands guided the developmental work.

The second principle that emerged in chapter 1 was the idea of the *phased structuring of a relational network.* In this particular case, the construction of a network of numerical relationships was phased and structured by linking the research on the numbers up to twenty with strategies for deriving numerical relationships (cf. also Greeno, 1991; McIntosh, Reys and Reys, 1992).

theory-guided bricolage

The second question was: What procedures does the researcher use in developmental research? The answer is roughly: various forms of 'theory-guided bricolage'. This concept was introduced in chapter 4 as a description of professional developmental work in the early days of the realistic textbooks. The emphasis at that time was strongly on fitting and adapting whatever was available in educational designs.

In the same chapter, theory-guided bricolage was also used to describe developmental research. It was noted here that, in developmental research, this has to do with a different modality. In developmental research, the theoretical charge is more important and the emphasis lies more on growth: growth of knowledge in an iterative process of theory-guided adaptation, improvement and expansion. This section will deal with three aspects: the long-term perspective for theory development in developmental research, the theoretically based construction of a preliminary design, and the cyclical process of invention, experimentation and reflection on a micro-level.

Theory-guided bricolage as 'A Never Ending Story'. The goal of developmental research is to develop a domain-specific instruction theory, but without a deadline involved. Not only is theory development through developmental research a long-term process, but it knows no time limitations. Theory development is seen as 'A Never Ending Story', not because realistic mathematics education is an unattainable utopia, but because it is a living thing. Objectives of mathematics education evolve, while the conditions under which the education is given also change. As far as the objectives are concerned, at this moment we can see a shift in the direction of 'mathematical literacy' as an educational objective. Education itself is directly influenced by social change, which then influences the expectations and behavior of the students. In addition, the availability of new technologies is of increasing influence.

The realistic educational theory is rooted in concrete educational activities and, although the theory will become more and more absorbed as time passes, it will also have to be adapted to new developments. It is this unlimited perspective that attracted Hoeben (1994) to use the term 'relaxed developmental research'. As far as theory development is concerned, developmental research is, on the whole, indeed relaxed. There are no unrealistic expectations and, each research project produces its own modest contribution. But if we look at the actual circumstances surrounding most developmental research, then the term 'relaxed' is misplaced. With a few exceptions,[7] developmental research in The Netherlands is a by-product of developmental work. Developmental research, on the whole, is not recognized as a subsidizable form of research. The primary objective, for instance, of many of the research projects conducted by the Freudenthal Institute is the development of educational material or curricula. In practice, this is always accompanied by one form or another of developmental research. No status quo developmental work is conducted; it is always a matter of innovative, ground-breaking developmental work – and that requires developmental research. The nature of the work is such that the time available for actual realization of the research component is often limited. The fact that this research still yields so much is due to the coherence of the various research activities. There is a community of developers and researchers who let themselves be guided by the same theoretical starting points. And, thanks to a continuous discourse, each new project can be built on previously acquired knowledge. It is the combination of continuity and graduation (together with the productivity of the guiding theory) that makes theory development through developmental research a success.

Theory-guided bricolage as the basis for a theoretically based preliminary design. As the theory development progresses, the theory as a means of guiding assumes more significance, particularly for developmental research. This means that the constitution of the preliminary design begins to carry more and more weight. Whereas the selection and adaptation of available instructional activities played an important role in the early phase of the textbook development, now it is the overall design of a course that acquires a pronounced constructional character in developmental research. In a focused search process resting on general educational and psychological knowledge, knowledge of research results, theoretical knowledge of the subject, and above-mentioned heuristics and development principles, a course design can be invented that is truly new. Development of the arithmetic-rack can serve as an example of this kind of focused design activity.

> The objective was to support the use of strategies for deriving facts when learning the basic facts for arithmetic up to 20. This was also one of the objectives of the first grade course in the 'Rekenen & Wiskunde' textbook, discussed in chapter 1. Construction of the differentiated strategies was mainly of a rational nature; the strategies were de-

veloped from the observer's point of view. Use of doubling was an exception. Later, the five referenced strategies were added when, upon investigation, these appeared as spontaneous solution strategies (informal comments by Van den Berg and Van Eerde, pers. comm.). Some of the strategies mentioned above were found to conform with the informal strategies seen in reaction time analysis and in research based on clinical interviews (Groenewegen and Gravemeijer, 1988). This was not found to be true, however, of the use of the inverse relation when solving subtraction problems; the spontaneous tendency of the students was to 'count on' strategy. Another strategy, which received no attention in 'Rekenen & Wiskunde', was that of compensation; for instance, one can calculate $7 + 5$ by means of $6 + 6$, which is found via $(7 - 1) + (5 + 1)$.
Research into the support of informal strategies for learning the basic automatisms received a new impulse in the project 'Nieuwe Media' (Van Galen et al., 1991). At that time the research of Hatano (1982) was generating a great deal of attention. This research introduced manipulatives based on the five-structure. Simultaneously, attention was drawn to the natural character of the five-structure. Apparently, the five-structure could be found in the number words of many African languages (Zaslovsky, 1984). The Japanese approach did not, however, fit the realistic principles. Working with Hatano's 5-tiles was based too much on a fixed methodology. The answer to each problem had to be found via a translation into a five-structure. The same drawback was true of Fletcher's 5-frames (Fletcher, 1988). Within and around the Nieuwe Media group arose discussion regarding the use of manipulatives to help the weaker students especially. The possible alternatives were:

- the 10-boxes used in second grade 'Rekenen & Wiskunde', which are similar to Wirtz' (1980) '10-frames';
- the numerical images in older arithmetic approaches (Radatz and Schipper, 1983);
- the idea of using a string of beads or an abacus.

The basic concept was that appropriate numerical images could support the mental structuring of numbers. Take the 10-boxes as an example. In each box one can fit ten blocks, in two rows of five. The structure of the box makes it possible, in principle, to determine how many blocks are in the box without counting. Because exactly five blocks fit in lengthwise, one can tell by the proportionate length between the empty and full sections how many blocks there are in a row. This can then be used to determine the total number of blocks (fig. 6.1).

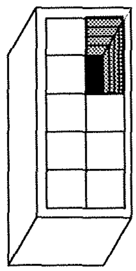


figure 6.1: 10-box

A drawback to the boxes was that little support was given to 'bridging' ten. A string of beads arranged in groups of five would appear to be more appropriate. This idea meshed with experiences involving the abacus. As mentioned in chapter three, a special abacus was developed for the third grade of 'Rekenen & Wiskunde'. This abacus consisted of twenty beads per rod with alternating colors every five beads: light, dark, light, dark. The point of these colors was that the students would easily be able to read the amounts, eliminating the need to count individual beads (fig. 6.2).

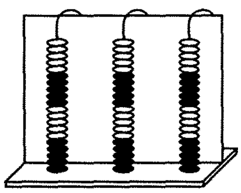


figure 6.2: abacus with five-structure

The students would then be able to apply their knowledge of the basic facts. In practice, however, the students developed strategies suited specifically to the five-structure (Van Galen, 1983). For this reason the abacus was introduced in the second grade as an aid in 'bridging' ten. Only the rightmost rod was used for this purpose. By using a string of beads instead, a more manageable aid would be available. Moreover, subtraction could then be done in two ways: the number to be subtracted could either be slid to the left or the right (fig. 6.3).

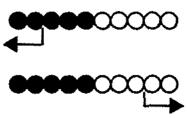


figure 6.3: take away from the beginning or the end

The 'take away from the beginning' meshed nicely with the informal strategy of counting for solving subtraction problems. The disadvantage of the string of beads, however, was that doubling could not be represented. This was possible, however, on the 5-boxes or 10-boxes, by arranging them strategically. But then manipulation did acquire a somewhat forced character.

So the search was for material that would combine the advantages of the string of beads with the opportunity to use doubles as reference points. This led to the invention of the arithmetic rack (Treffers, 1991). The string of beads was, as it were, cut into two strings of ten beads, one of which was then placed under the other (fig. 6.4). Instead of two separate strings, two rods of beads were used, making a kind of elongated abacus.

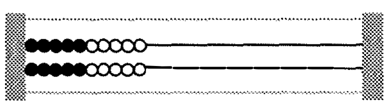


figure 6.4: the arithmetic rack

The principle of 'those shifted to the left count' is now true of both rods. This means that one can work in two different ways:

1 first use all the beads on the upper rod and then continue with the beads on the

lower rod, or
2 use beads on both the upper and lower rods.

In the first case we carry on just as with a string of twenty beads. This supports the 'bridging' ten. The 'take away from the beginning' cannot actually be carried out, but one can cover a number of beads at the left (fig. 6.5).

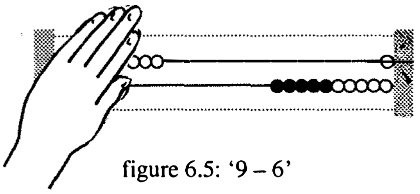


figure 6.5: '9 – 6'

The second manner is good for representing doubles and almost doubles. This representation is also suitable for the informal strategy of compensation (fig. 6.6).

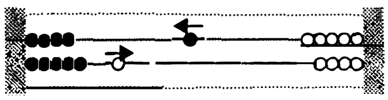


figure 6.6: '4 + 6 = 5 + 5'

Small experiments with individual students revealed the practical value of the arithmetic rack. The researchers, however, were not satisfied until the use of the arithmetic rack could also be theoretically integrated into the realistic approach. The model-heuristic sent the researchers in search of situations that could be modeled using the arithmetic rack. An initial situation was found in Van den Brink's double-decker bus context (Van den Brink, 1989). Passengers on a double-decker bus can be arranged in various ways on the two decks. The number of passengers above and below can be illustrated on the rods of the arithmetic rack. Note that the relation between the beads and the passengers is of a fairly formal nature; the beads represent the number of passengers, not the passengers themselves. So it doesn't matter which bead is shifted when a passenger gets off the bus, as long as the total decreases by one.
On the whole, we can characterize the place of the arithmetic rack as follows. With the introduction of the arithmetic rack the students are (or become) familiar with the idea that you can find the answer to an arithmetic problem by using concrete objects. So the arithmetic rack provides a standard set of countable objects. As the students become more familiar with the numerical relationships connecting the numbers under twenty, the bead patterns begin to acquire an independent significance. The first seven beads on a rod now represent seven as 'five plus two'. Using this knowledge, the answer to 7 + 6 can easily be found via 'five plus two plus five plus one is ten plus three.'
In other words, a shift occurs from manipulating beads on the arithmetic rack as a representation of addition and subtraction of amounts to (thinking of) manipulation on the arithmetic rack as a model for formal arithmetic.

This example shows how significant the initial phase of developmental research can be if plenty of knowledge is present. This does not mean that an educational experiment is redundant in such a case. Practical elaboration and testing remain necessary. An effective execution is only possible in conjunction with classroom experiments. Moreover, the overall design has the character of a collection of hypotheses that are tested in the educational experiment. The thorough process of deliberation ensures that the researchers will begin the educational experiment with a favorable and well-founded overall design. In contrast, Lampert's educational experiments (Lampert, 1986, 1989, 1990) would seem to have a more exploratory character.

Theory-guided bricolage on a micro-level. Whenever various versions of a unit are tried out one after the other, this may be called a cyclical process on a meso-level. The testing of the unit, however, consists itself of a collection of mini-cycles. Although the experiment does start with an overall preliminary design, this is expanded and adapted in a cyclical process of inventing, testing, and reflecting on educational activities. One might call this a small-scale empirical cycle. Freudenthal (1988) speaks in this context of 'thought-experiment' and 'educational-experiment'. That which is thought up behind the desk is then tried out and adapted in the classroom. The researcher goes in search of signs that confirm the expectations of the thought-experiment, as well as signs pointing to the contrary. Moreover, the researcher keeps his or her eyes open for new possibilities. The short cycles lead to what in chapter one was called 'feed-forward'. Whenever the development of educational material and its testing in the classroom follow upon one another's heels, it is possible to react immediately to the classroom experiences.[8] In the material still to be developed, one can take previous successes and failures into account. In my opinion, a central principle in the guiding of the bricolage process on a micro-level is the micro-didactic deliberation on the learning process. As noted in chapter 1, the discussions on didactic topics in the 'Rekenen & Wiskunde' development group were carried out on the basis of 'micro-theories about the mechanism of the proposed instructional activities'. And this is how it ought to be. Terms such as 'educational development' and 'instruction-theory' could leave the impression that education is viewed solely from the supply side. Nothing is less true. The whole point is the student's own mental activities. These are central to the developmental work.

criteria

The third question was: Which criteria does the researcher use to make adjustments? The criteria used by the researchers to make assessments and to carry out adjustments are taken from the theory for realistic mathematics education. In practice, they flow from the heuristics outlined above. The heuristics lead to a preliminary furnishing of the unit, which is based on the expectations derived from the heuristics.

The *reinvention principle* assumes that the students' own solutions will pave the way towards reinvention. That is, the students' solutions must actually express a variety of solution levels. These solution levels must, moreover, provide a good reflection of the learning path to be followed. The learning path must, as it were, be visible in the students' solutions. This requirement of a dispersion of solution levels is not only significant as a formal characteristic of the reinvention concept. Longitudinal dispersion (in solution levels within the planned learning path) and transversal dispersion (over students) are also necessary in order to offer the teacher a chance of success. Guided reinvention assumes that the teacher will find a modus operandi for reconciling guidance and self-reliant invention with one another. A productive classroom discussion is only possible when there is a difference both in the solutions of various students and in solution level. If the students all use the same solution procedure, then there can be no discussion. If all solutions are on the same level, then the teacher has no other choice than to exert a strong guiding influence.

A related criterion is that the reinvention path not only be traveled upwards, but also downwards. When the students are faced with a new problem that presents difficulties, they should spontaneously take a step backwards in their own learning history. They can then solve the problem on a lower level. This should not be seen, by the way, as a conscious step backwards. It is expected of the students that they approach new problems as situations to be mathematised. This means that, in such a case, they will choose a 'bottom-up' approach, beginning by getting a sense of the problem and then seeing what they can do with their already acquired knowledge. Nor is it true, however, that the student approaches a problem as a blank slate. When structuring the problem, the student is also guided by strategies and techniques with which she or he is already familiar. The student shuttles back and forth, as it were, between the problem and the available knowledge (see also chapter 3). On which level the connection will eventually be made depends upon the complexity of the problem and the familiarity of the student with the solution procedure.

The *didactical phenomenology* requires the researcher to go in search of appropriate context problems. The suitability of the context problems is thereby automatically a criterion. This has to do primarily with the relation between the context and the students' spontaneous solution strategies. Do the students indeed make use of the footholds offered by the context? Do they apply their own domain-specific knowledge? And, at least as important, do the solutions they come up with offer possibilities for vertical mathematization? This last question, by the way, can be answered empirically as well.

Another aspect of the phenomenology involves the applicability. Solutions will, in the first phase, be local. Each problem is approached as a new problem and the solutions will contain clear context-specific elements. After a time, the communal must begin to prevail and a more broadly applicable piece of mathematics must develop. Whether the knowledge developed in this manner can truly be broadly applied is an important criterion for the researcher. The didactical phenomenology outlines the area of application to be considered for research.

The level structure and the related role of *models* also provide the necessary criteria. The criteria for mediating models was mentioned in chapter three as: 'naturalness', 'vertical power' and 'breadth of application'. Characteristic of the bottom-up approach to models is that the models spring from the students' own activities. It is in this sense that there must be naturalness. This can become visible in various ways.

The best, of course, is when the students (re)invent the models on their own. In cases where the model is presented, one requirement is that it fit in with the informal strategies demonstrated by the students. Another indication of naturalness would be when the student easily adopts the model and spontaneously applies it to new situations.

In my opinion, it is essential that the model offers the students the opportunity to be faithful to their own solution procedures. In other words: the model must not dictate to the students how they should proceed but, instead, must follow the students' thought processes (Gravemeijer, 1993). This should become evident through, among other things, flexible use of the models. 'Vertical power' may be deemed to be present if the students abbreviate and schematize their way of working on their own. The level shift from 'referenced' to 'general' is crucial here. This can be seen from the (lack of) connection between the structure of the problems and the structure of the solution strategies. Another important indication of vertical power is when the students themselves bring up the matter of the efficiency of an approach. The matter of applicability was already raised by the didactic phenomenology. I used the term 'breadth of application' specifically for models in order to indicate that students were able to deal with a model in all sorts of application situations. In closing, the following is a summary of the criteria mentioned above:

- reflection of the learning path in the students' solutions
- longitudinal and transversal dispersion of solutions
- bottom-up problem solving
- use of footholds offered by the context
- situation-specific solutions with vertical perspective
- applicability
- naturalness, vertical power and breadth of application of models
- spontaneous abbreviations
- shift from context-bound to solution-focused
- flexibility.

This list is, of course, incomplete in the sense that it is not exhaustive. Moreover, the categories are not mutually exclusive. And yet, it does offer a representative picture of the criteria used by developmental researchers in the field of mathematics education. For the sake of completeness I should like to point out that these criteria not only serve as a gauge of the expectations arising from the thought-experiment. They also make up the searchlight for discovering students' insightful approaches.

evaluation

The fourth question was: How does the researcher evaluate his or her own standpoints? This question requires a differentiated answer. Some standpoints, namely, are not to be discussed. One of these is the realistic starting point that mathematics can be learned on one's own authority through one's own mental activity. This standpoint is not open to discussion; it functions, as it were, as the core of a research program. The objective of realistic mathematics education is to develop education that enables students to learn mathematics in this way. In this sense, developmental research is related to physics research as viewed by Barnes (1982). In a period of 'normal science' (Kuhn, 1970), the work of a physics researcher consists of reconciling the theory and the phenomena being investigated with one another. The point is to expand the theory's area of application. This means that the phenomena must be interpreted in such a way that they lie within the theory. If the researcher does not succeed in doing this, then it is the researcher who has failed, not the theory. This is at any rate the case as long as 'normal science' is involved, according to Barnes.

No mechanistic situation will, therefore, ever be considered in developmental research in the area of realistic mathematics education. This kind of irrefutable core is both the strength and the weakness of every research program. Obviously, the operationalization of this axiom does leave some room for interpretation. The situation is not as rigid as all that. In general, for instance, it is not assumed that training and imprinting are forbidden, as long as an insightful basis is first laid and 'the sources of the insight are held open'. The realistic core is translated into criteria such as those listed above. During the development of a prototype, the findings are evaluated against these criteria. Starting points that are open to discussion involve the concrete choices made in the prototype. These are the choices that are evaluated against the above-mentioned criteria. The example of the development of a unit for multiplication, which is discussed in the following section, demonstrates that developmental researchers are, indeed, willing in actual practice to retreat from earlier standpoints.

6.4 external persuasiveness

Developmental research can be seen as the researchers' learning process. This is also why it is so difficult to transfer the yield of developmental research. The researchers must describe their learning process in such a way that it can be traced by outsiders. In ethnographic research this is called 'trackability' (see chapter 4). The outsider must be able to trace the train of thought.[9] Making the aspects of developmental research outlined above more explicit will certainly be of help here. In addition, it is, of course, important that the researchers make their own learning process conscious. This can be done by taking logbook notes in which an attempt is made to record 'reflection-in-action' (Goffree, 1986). Eventually, the researchers need to ask themselves: What have I learned? and: Why do I believe this to be true?

In answering the first question, it may be helpful for the developers to describe their 'starting theory' from the beginning of the research (Wijers, 1989). Where the second question is concerned, if we succeeded in objectifying the answer to it, developmental research could be considerably reinforced. Another way to increase the persuasiveness of developmental research is to broaden the theoretical base. These two ways of increasing the persuasiveness of developmental research will be elaborated upon in the following section.

objectification

I will first discuss the various ways of objectifying the answer. Due to practical considerations, no distinction will be made here between measures that are already present in examples of developmental research and new suggestions. At first glance, the sole point of objectification would appear to be the reinforcement of the empirical base. At least as important, however, is the objectification of the interpretation (or analysis) of the empirical data. In developmental research we find, namely, crucial moments which the researchers experience as 'Aha-Erlebnissen'. This Aha-experience has a great deal to do with the reference framework of the person involved. Often, it is possible to share this insight with those who are like-minded. Outsiders, however, find it generally incomprehensible why so much significance is attached to a specific observation. In order to make the significance of this observation accessible to others, the researcher must endeavor to make the theory underlying this observation as explicit as possible. Let me take the following example as an illustration.

> During a lecture at a symposium on developmental research, Treffers (1993) described 'Els' error' as a crucial moment in the development of a unit on multiplication. The unit in question is based upon the so-called 'intersection model'. In this model a number is illustrated by a corresponding number of parallel lines. The numbers which are to be multiplied are illustrated by two groups of perpendicularly intersecting lines.

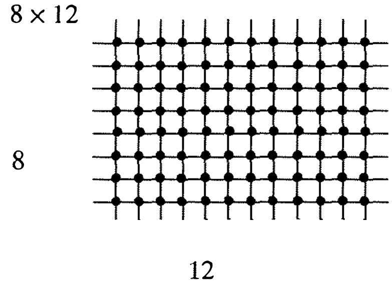
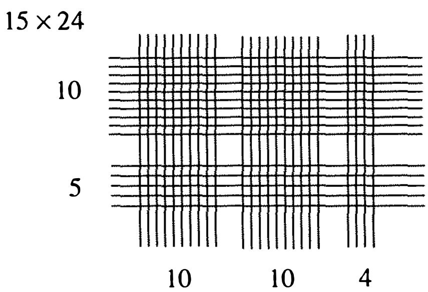


figure 6.7: intersection model for multiplication

> The product of the two numbers corresponds with the number of intersections formed in this way (fig. 6.7). This intersection model was introduced using a simple context. Large numbers soon have to be dealt with, which compels the student to find a systematic approach. A smart approach is to use the decimal structure of the numbers to

group the lines in bunches of ten (fig. 6.8a).



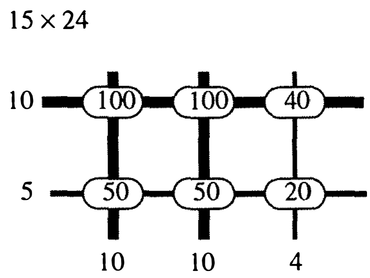


figure 6.8a: decimal structure

figure 6.8b: schematization

Once you know that there are ten lines in a bunch, then you no longer need to draw all ten. You can then replace them with a thicker line, which stands for ten lines. In this way, the illustration acquires a structure. Four sub-areas are created that correspond with four sub-products (see fig. 6.8b). The standard algorithm begins to come into view.

The developmental research was progressing nicely and schematization of the lines was occurring quite spontaneously. The researchers were satisfied ... until Els' error occurred. The students had been given a simple application problem, but Els did not use the intersection model to solve it. Instead, she chose an entirely different solution strategy that was, moreover, incorrect.

The problem is:

> Next door lives a family consisting of a father, a mother and a son. The son is 14 years old. The father is four times as old as his son.
> >> How old is the father?

Els draws an intersection model that corresponds to 14 × 14 (fig. 6.9a).

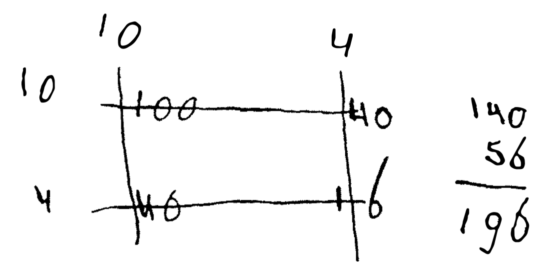


figure 6.9a: Els' intersection model

She says: Now it is added two times.

Then she adds 196 and 196 (fig. 6.9b) and she says: Now it is added four times.

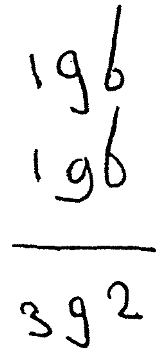


figure 6.9b: Els' addition

But one week later she had mastered her own solution strategy, which worked well for her (fig. 6.10).

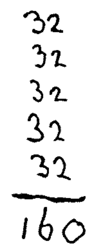


figure 6.10: Els' strategy

The researchers realized with a shock that they were on the wrong track. The result was that they abandoned the intersection model and switched to repeated addition. Treffers (1993) used this example in order to demonstrate to relative outsiders that crucial empirical facts are of an entirely different nature in developmental research than in conventional curriculum research. The example failed to persuade researchers from outside the circle of arithmetic experts (Kanselaar, 1993; Elbers, 1993). How can you derive such sweeping conclusions from one incident, was their reaction.

I will attempt to reconstruct why this incident could, indeed, acquire such significance. The researcher's own domain-specific knowledge offers the first foothold. The strategies used by Els are very natural ones; the doubling present in her first solution was used by the ancient Egyptians. Doubling and repeated addition are informal strategies known to occur quite often. Evidently, the intersection model lacks the naturalness present in the other strategies (criterion of naturalness). The realistic theory of education also plays a role in this interpretation. This course was based on the reinvention principle, developed in a form of progressive mathematization. The idea behind this is that one links up with the student's own level and that the student regards the progressive steps of mathematization as a natural expansion of her/his domain. The student should also be able to retrace her/his steps when necessary. This was mentioned earlier as one of the criteria for the researcher. One expects the student to change level spontaneously when confronted with an unsolvable problem. In the in-

> mentioned earlier as one of the criteria for the researcher. One expects the student to change level spontaneously when confronted with an unsolvable problem. In the intersection unit, this means that the student steps back to the drawing of separate lines. Els, however, did not use this strategy.
>
> Another aspect of the realistic standpoints concerns applicability. The phenomenological analysis indicates which areas of application must be taken into account. The intersection model is a didactic elaboration of the area model that plays an important role in countless application problems. This was one of the reasons that this model was expected to guarantee a broad applicability. But this did not turn out to be the case. The criterion of breadth of applicability was not fulfilled. This can be understood if we lay the context problem and the model next to each other. The phenomenological structure of the context problem is so different from the structure of the model that the link can only be made on a formal level. If we want the students to have a model at their disposal that can also be used in such application situations, then the intersection model is not the right one.[10] By contrast, repeated addition can be used to model a multiplication situation that is characterized by an area model. Moreover, as mentioned before, repeated addition is a natural strategy. New research for this unit was then begun from this standpoint (Treffers, 1979).

Alongside the objectification of the analysis or interpretation of data, I also mentioned the objectification of the empirical observations themselves. A few simple measures can easily be taken. For instance, the researcher can collect information on specific characteristics of the research population and the research situation. A test can be administered at the beginning and at the end of the project. The experimental lessons can be recorded on video or audio cassette, or else drawn up in lesson protocols. But the problem lies in the developmental research itself. Due to the complexity of this evolutionary development and research process, it is simply not possible to report exhaustively on each and every experience, deliberation and adjustment. This problem, however, does not only occur in realistic circles. Verschaffel (1993) points out the existence of an international community of researchers who are grappling with the same problem. The study of learning and teaching mathematics has, namely, grown to become an internationally recognized scientific terrain (see also Kilpatrick, 1992; Wittman, 1994).[11] The development of this research community was accompanied by an increasing diversification and liberalization of the methods of research and analysis (see also Walker, 1992; Romberg, 1992; De Corte, Greer and Verschaffel, in press). This does not, however, mean that the methodological problems have thereby disappeared.

> 'Indeed, as soon as the researchers left the familiar, smooth methodological paths and took to new roads, they ran into all sorts of (unforeseen) problems in connection with the collection, analysis, interpretation and reporting of data, to which no answers could be found in the classical methodological guides, but for which they still hoped to find a scientifically satisfying solution. It is, therefore, not so surprising that researchers began to apply themselves more and more urgently to these methodological matters.'
>
> (Verschaffel, 1993, p. 11-12)

Verschaffel follows here in the footsteps of Lesh and Landau, who observe, in 1983:

> 'that formerly useful, borrowed methodologies are frequently inconsistent with the purposes and assumptions underlying the newly emerging theoretical perspectives'
>
> (Lesh and Landau, 1983, p. 1)

To which they add:

> 'Major mathematics education research projects ... have had to engage in the development of research methodologies as well as in the generation of knowledge related to the improvement of mathematics instruction.'

They mention, in this context, the use of standardized questions. Their argument is that the whole idea of a standardized question is useless if one is starting from a theory that assumes that:

> 'two students frequently interpret a single problem situation or stimulus in quite different ways' and that 'two responses that appear identical may be produced using completely different solution paths'
>
> (Lesh and Landau, 1983, p. 2)

Verschaffel refers to Schoenfeld, Brown and Saxe, who came to the conclusion that the arsenal of research methods and techniques that they had acquired during their training was not sufficient for studying the complex phenomena surrounding the learning and teaching of mathematics.

Schoenfeld (1992) endeavors to arrive at a methodology for analyzing videotapes by analyzing and systematizing his own method of working. Here we find ourselves on familiar ground: choose your starting point for theory forming in the practice of the activity itself. As examples of newly developed methods, Verschaffel (1993, p. 11 and 13) mentions clinical interviews, ethnographic methods, micro-genetic analyses and constructive research projects such as Steffe's (1991) teaching experiment, Lampert's (1986) education experiment and the developmental research at the Freudenthal Institute. In addition, Verschaffel mentions the Dutch contribution to specific research techniques, such as the technique of mutual observation (Van den Brink, 1981) and the technique of 'students as textbook authors' (Van den Brink, 1987). The latter is an example of the more general technique of using free productions (Van den Brink, 1989; Streefland, 1990).[12] In addition, I should like to point out the development of tests geared to realistic mathematics education (De Lange, in press; De Lange, Burrill and Van Reeuwijk, 1993; Van den Heuvel, 1990).

theoretical basis

The power of persuasiveness of developmental research will increase as the research results become more embedded in a broad theoretical framework. This idea is not new. It has to do, in fact, with what is a rather common manner of working in realistic circles (see for example Streefland, 1980; Treffers, 1987). I will, therefore, refer to this idea only briefly. Concretely, it means that the researcher demonstrates – particularly in the documentation – how the research results relate to generally accepted theories. This does not mean that the researcher's findings must be made to agree with these theories. An argument with a general theory may prove quite enlightening. Chapters 2 and 3, for instance, show that activity theory and information processing result in educational approaches that do not do justice to the central principle of realistic mathematics education. Constructivism, on the other hand, fits the realistic approach much better. The central principle of constructivism is that each person constructs his or her own knowledge, and that direct transfer of knowledge is not possible. This idea of independent construction of knowledge supports the central realistic principle.

The realistic instruction theory indicates how instruction can be developed that enables the independent construction of knowledge and focuses it as well. Socio-constructivism provides a frame of reference that enables an effective deliberation of realistic mathematics education. Points of interest here are the interpretation of instructional tasks and the social norms that guide the interaction between teacher and students. In addition to psychological research, curriculum research is, of course, important as well. Informal feedback and the feedback acquired from implementation and evaluation research were mentioned in chapter 4 as feedback-loops. Implementation and evaluation research from an innovation perspective can provide a particularly valuable contribution. The MORE research project can serve here as an example (see chapter 5). In contrast with the research of cognitive psychologists, the analyses of the learning processes in developmental research often remain rather superficial. Here, too, lies an interesting way of increasing the scientific yield of developmental research. In collaboration with other researchers, developmental research can be expanded by using cognitive psychological and social psychological research into the inherent learning and interactive processes. This type of research is now being conducted by Cobb, Yackel and Gravemeijer (Cobb and Yackel, 1993).[13] This particular research project takes a close look at the role played by models, such as the arithmetic rack and the empty number line, in the transition from 'model of' to 'model for'. With the support of the results from this type of research, the elucidatory power of developmental research can increase further. After all, this has to do with research into the 'mini-theories on the functioning of instructional activities.'

6.5 conclusion

The above is an outline of developmental research and of how it is related to other sorts of research. I have demonstrated how developmental research functions as an internal legitimization of local and domain-specific instruction theories within the circle of realistic oriented researchers. I have also endeavored to illuminate developmental research with an eye to the external legitimization – the legitimization for outsiders. Finally, I have discussed possibilities for reinforcing the external persuasiveness. This extensive analysis of developmental research can be justified by the practical and theoretical significance of such research. Developmental research is an elaborated example of the transformational research requested by the Research Advisory Committee NCTM (1988): the kind of research that is needed to bring about educational change in mathematics education.

Furthermore, the analysis shows the remarkable character of this type of research. In summary, we can say, it is evolutionary, stratified and reflexive.

It is *evolutionary* in the sense that theory development is gradual, iterative and cumulative.

It is *stratified* in the sense that theory development takes place at different levels:
- at the level of the instructional activities (micro theories)
- at the level of the course (local instruction theories)
- at the level of the domain-specific instruction theory.

It is *reflexive* in the sense that theory development is fostered by reflexive relations between the aforementioned levels.

The Dutch developmental research in the area of mathematics education has, after all, resulted in theory forming and instructional materials which have received international attention. This can be seen, for instance, in the participation by the Freudenthal Institute in three projects financed by the National Science Foundation of the United States. Verschaffel's analysis quoted above demonstrates that research related to developmental research is on the rise. It would seem that, nearly twenty-five years after the foundation of the Institute for Development of Mathematics Education (IOWO), developmental research is finally receiving the recognition it deserves.

notes

1 Mark that the label 'developmental research' does not have a singular meaning in the Netherlands. Already a few years ago Van Eerde (1988) listed various interpretations, and since then new approaches have emerged (e.g. Van den Akker, 1993). In this text 'developmental research' refers to the concept employed at the Freudenthal institute.

2 Other, kindred, researchers do, for that matter, follow the same approach in developing new research methods (Whitenack and Cobb, 1994; Schoenfeld, 1992).

3 Moreover, the statistical techniques used in this kind of research are criticized, too (De Leeuw, 1988).

4 An example of a fruitful interaction between psychological research and developmental research is shown in the work of Beishuizen. His research on mental strategies in the number domain 20-100 brought the N10 and 1010 strategies to the fore (Beishuizen, 1985). This influenced the way the hundred square was used in the textbook series 'Rekenen & Wiskunde'. That again led to new research (Beishuizen, 1993). The distinction between N10 and 1010 strategies was also integrated in the developmental research on the so called 'empty number line' (Treffers and De Moor, 1990). The idea of an empty number line was subsequently the core of a new research project by Beishuizen (Boekaerts and Beishuizen, 1991) that is being carried out in collaboration with Treffers.

5 These questions came to the fore in the contributions of Elbers (1993), Kanselaar (1993) and Koster (1993) to a symposium on developmental research.

6 Goffree (1993, 42) expresses Freudenthal's beliefs regarding the relation between mathematics and common sense as follows:

> By the activity of *mathematizing in realistic situations, interactively* with others and reflectively, common sense is continually brought to a higher state. This implies that it can be applied more intensely in more situations. Initially one has a lot of 'natural' common sense but when challenged by rich contexts, inspired by the opportunity to make inventions oneself and guided by someone who knows about mathematics when looking for more certainty, then common sense will be *enriched with mathematics.* The essential means to integration of common sense and mathematics or, if you like, to assimilation of the latter into the former, is *reflection.* Mind you, mathematizing can very well take place in a *mathematical context.* For mathematicians also use their common sense in particular when working on the boundaries of their science.

7 Examples of this type of exception are the dissertation researches of Streefland (1988) and Van den Brink (1989).

8 This manner of working close to the classroom also improves the practical quality of the product. By keeping an eye on manageability and implementability, future problems with implementation can be anticipated. Anticipating the implementation is, of course, particularly important in textbook development. And yet, this aspect should not be neglected in prototype development either. Moreover, anticipating implementation problems may, in fact, form an explicit area of attention in prototype development. One must keep in mind, however, that the involvement and expertise achieved by the teachers in such a situation cannot be transferred without further ado to new implementation situations.

9 Trackability in this way replaces reproducibility as the criterion for reliability. In the same way, the actual repetition of the research experiment, conducted by others under different conditions, can ensure external validity as a nuanced generalizability (see chapter 4).

10 Later on, research was found in the research literature in which the same conclusions are drawn. Students familiar with the intersection model did not apply it when asked to figure out how many coins cover a rectangular table (MacIntosh, 1979).

11 This development is also expressed in a great number of research journals, in which psychologists and educational researchers, as well as mathematicians and mathematics educators publish articles (e.g. Journal for Research in Mathematics Education, Educational Studies in Mathematics, Journal for Mathematical Behavior, For the Learning of Mathematics, Zeitschrift für Didaktik der Mathematik, Recherches en Didactique de Mathématiques, Tijdschrift voor Nascholing en Onderzoek van het Reken-Wiskundeonderwijs.)

12 In addition to using new techniques as such, a combination of techniques can also be considered, in accordance with the triangulation principle.

13 Other examples are the collaboration between Beishuizen (Department of Educational Studies, RUL (University of Leiden)) and Treffers, and between Elbers (Department of General Social Sciences, RU (University of Utrecht)) and Streefland.

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Samenvatting

Deze studie vindt zijn oorsprong in de ontwikkeling van een onderwijsprogramma voor rekenen-wiskunde voor de basisschool. Dit ontwikkelwerk vond plaats in het project Onderwijs en Sociaal Milieu (OSM) en resulteerde in de methode 'Rekenen & Wiskunde'. De door OSM voorgestane onderwijstechnologische ontwikkelmethodiek bleek echter niet te passen bij de door de ontwikkelaars voorgestane vernieuwing van het reken-wiskundeonderwijs: de door het IOWO geëntameerde vernieuwing ('realistisch reken-wiskundeonderwijs').

De onderwijstechnologische ontwikkelmodellen waren te zeer op concrete operationele doelen gericht, en te weinig flexibel om dit sterk vernieuwende ontwikkelwerk te kunnen sturen. Dit conflict tussen 'onderwijstechnologie' en 'onderwijsvisie' vormde de basis voor de kernvraag van deze studie: 'Hoe ontwikkel je realistisch reken-wiskundeonderwijs?

Deze vraag wordt ruim opgevat; er wordt aandacht besteed aan curriculumontwikkeling, ontwikkelingsonderzoek, implementatie en evaluatie. Deze verschillende categorieën worden beschouwd binnen het ruimere kader van 'onderwijsontwikkeling'. Onderwijsontwikkeling is meer dan curriculumontwikkeling alleen. Enerzijds omvat het het complete proces van vernieuwingsidee tot en met de verandering van de onderwijspraktijk. Anderzijds omvat het ook al de activiteiten die ondernomen worden om de vernieuwing te realiseren: onderzoek, ontwikkelwerk, opleiding, nascholing, begeleiding en voorlichting. De motor van de onderwijsontwikkeling wordt gevormd door het ontwikkelingsonderzoek dat resulteert in prototypische leergangen en (lokale) onderwijstheorieën. Deze vormen een inspiratiebron voor schoolboekauteurs, opleiders, nascholers en schoolbegeleiders. Zo kunnen vernieuwende reken-wiskundemethoden tot stand komen die met de steun van de bovengenoemde groepen adequaat geïmplementeerd kunnen worden.

Dit boek is opgezet als een bundel van min-of-meer op zichzelf staande artikelen. Het eerste hoofdstuk beschrijft ontwikkelwerk binnen het OSM-project. Een reconstructie van het ontwikkelen van de leergang optellen en aftrekken onder de twintig van het eersteklas-deel van de methode 'Rekenen & Wiskunde' (Gravemeijer et al., 1983) dient als basis voor een beschrijving van de praktijk van het ontwikkelen van realistisch reken-wiskundeonderwijs. Deze analyse laat zien dat de eerste fase van het ontwikkelwerk voor een belangrijk deel bestaat uit het selecteren, inpassen en aanpassen van beschikbare onderwijsactiviteiten. Criterium voor dit selecteren, inpassen en aanpassen is de door de ontwikkelaars gekozen visie op reken-wiskundeonderwijs.

In navolging van Freudenthal (1971) is gekozen voor 'wiskunde als menselijke activiteit'. Deze onderwijsvisie wordt uitgewerkt in het reinvention-principe van Freudenthal (1973), relationeel leren (Skemp, 1976), de niveautheorie van Van Hiele – toegepast op getalbegrip (Van Hiele, 1973) – en Freudenthals (1973) fenomenologische analyse van getalbegrip. De uitwerking van het voorlopige ontwerp vindt feitelijk gedurende de proefinvoering plaats. Bij die uitwerking wordt het ontwerp ook gedurig aangepast. Daar het uitwerken van de onderwijsactiviteiten in de praktijk vrijwel gelijk opgaat met het uitproberen in de klas, kunnen de ervaringen in de klas direct worden verwerkt in de nog uit te werken onderwijsactiviteiten.

Al met al betekent dit dat de formatieve evaluatie een geheel andere invulling krijgt dan in de onderwijstechnologische ontwerpmodellen gebruikelijk is. Daar wordt in het algemeen het accent gelegd op produktevaluatie en formele – vooraf opgestelde – beslisregels. De formatieve evaluatie krijgt de rol van scheidsrechter toebedeeld en de beslisregels bepalen welke consequenties de gevonden data moeten hebben voor het curriculum. In de zojuist geschetste ontwikkelpraktijk is de invloed van empirische resultaten minder direct. Men gaat op zoek naar een verklaring voor de gevonden resultaten, om zo te komen tot argumenten voor aanpassingen. De empirische data vinden hun basis bovendien meer in ervaringen in de klas – in observaties van leerlingen en in oordelen van leerkrachten – dan in toetsresultaten.

Binnen het ontwikkelteam van 'Rekenen & Wiskunde' vonden de doordenking en rechtvaardiging van onderwijsactiviteiten steeds plaats op basis van micro-theorietjes omtrent hetgeen zich in de hoofden van de leerlingen afspeelt. Dit type overwegingen wordt in het tweede hoofdstuk geïllustreerd met een analyse van het gebruik van concreet materiaal. Concreet materiaal wordt bezien vanuit de handelingspsychologie, de informatieverwerkingstheorie, het constructivisme en het realistisch reken-wiskundeonderwijs. Opvallend genoeg neemt concreet materiaal zowel in op de handelingspsychologie, als in op de informatieverwerkingstheorie gebaseerde leergangen, een belangrijke plaats in. In beide gevallen ontbreekt het echter aan een uitgewerkte theorie over hoe de overgang van materiële handeling naar mentale handeling – om de handelingstheoretische terminologie te gebruiken – daadwerkelijk verloopt. De handelingspsychologie geeft ons wel aangrijpingspunten voor een analyse. Zo kun je je afvragen, of de mentale handeling die de leerling verricht bij het uitvoeren van een materiële handeling wel isomorf is met de mentale handeling die je nastreeft. Dit blijkt meestal niet het geval. Het werken met concreet materiaal leidt vaak tot, materiaalgebonden, tel- en afleesstrategieën, die niet model staan voor de mentale handelingen die de leerling zonder het materiaal zou moeten verrichten. Omgekeerd kan het ook niet de bedoeling zijn dat de mentale handeling van de leerling bestaat uit het in gedachten manipuleren met concreet materiaal.

Cobb (1987) bekijkt het gebruik van concreet materiaal door een 'constructivistische bril' en constateert dat men geen onderscheid maakt tussen een 'actor's point of view' en een 'observer's point of view'. De onderzoeker 'herkent' de wiskunde waar hij of zij zèlf over beschikt in het concrete materiaal, maar de leerling die nog niet over deze wiskunde beschikt 'ziet' deze ook niet in het materiaal. Concreet materiaal kan de mentale objecten waar de deskundige over beschikt niet overbrengen, de leerlingen zullen ze zelf moeten construeren. Wat we conform het constructivistische uitgangspunt van Cobb moeten doen, is proberen het constructieproces van de leerling te stimuleren en bij te sturen via sociale interactie.

De realistische benadering van het reken-wiskundeonderwijs sluit goed aan op dit constructivistische uitgangspunt. Hier wordt namelijk niet geprobeerd via gestructureerd materiaal informatie over te dragen. In plaats daarvan wordt de combinatie van ongestructureerd materiaal en een zinvolle vraagstelling gezien als een situatie waar de leerling zelf de wiskunde kan ontwikkelen die nodig is om de situatie te structureren.

Het derde hoofdstuk beschrijft de huidige stand van zaken ten aanzien van het theoretische referentiekader voor dit type ontwikkelwerk aan de hand van de vraag: hoe overbrug je de kloof tussen concreet en abstract? De eerder bekritiseerde 'informatieverwerkingsaanpak' wordt hier als contrasterende achtergrond gebruikt. Er zijn twee verschillen tussen beide aanpakken die daarbij in het oog springen.

Het eerste verschil betreft de invulling die wordt gegeven aan het begrip 'concreet'. In het ene geval wordt concreet gekoppeld aan concreet materiaal, in het andere geval wordt concreet gekoppeld aan 'vertrouwd', 'herkenbaar'.

Het tweede verschil betreft de tegenstelling tussen het top-down karakter van de ene benadering (modellen worden afgeleid uit de formele wiskunde) en het bottom-up karakter van de andere (modellen komen voort uit het werk van de leerlingen zelf). Met dit laaste verschil hangt samen dat de informatieverwerkingsbenadering mikt op 'transfer' van het geleerde en toepassingen achteraf. Terwijl in de realistische benaderingen de toepassingen voorop staan en de algemene wiskundige aanpakken worden ontwikkeld door te 'generaliseren'; dat wil zeggen door het gemeenschappelijke karakter in een reeks van oplossingen van toepassingsproblemen te isoleren. Bij de beantwoording van de vraag, 'hoe overbrug je de kloof tussen concreet en abstract?' worden drie kernpricipes van realistisch reken-wiskundeonderwijs toegelicht met het leren cijferend delen als voorbeeld.

Het eerste kernprincipe betreft het reinventionprincipe, dit geeft als richtlijn, bedenk hoe je het zelf uit had kunnen vinden. Hierbij kan de ontwikkelaar gebruik maken van kennis van de geschiedenis van de wiskunde model en kennis van het leren van kinderen. Met name kennis over spontane oplossingstrategieën van kinderen kan de ontwikkelaar helpen bij het ontwerpen van een reinvention-route.

Het tweede kernprincipe, de didactische fenomenologie, verwijst naar toepassingen als bron. Aansluitend bij de gedachte dat de wiskunde ontwikkeld is als het steeds verder mathematiseren van wat oorspronkelijk oplossingen van praktische problemen waren, wordt geconcludeerd dat de startpunten voor het reinventionproces gevonden kunnen worden in de huidige toepassingen. De ontwikkelaar dient daarom toepassingssituaties te analyseren met het oog op hun didactische gebruik in het reinventionproces.

Het derde principe behelst het gebruik van zelfontwikkelde modellen die worden benut als bemiddelende modellen om de kloof tussen de informele kennis van de leerling en de formele wiskunde te overbruggen. De ontwikkelaar moet hiertoe op zoek gaan naar manieren om de informele strategieën van de leerlingen zo te modelleren dat een voortschrijdend mathematiseringsproces in gang wordt gezet. In deze opzet zijn de modellen eerst gebonden aan contexten. Het model is een *model van* een situatie. Via variatie over opgaven wordt veralgemenisering en verzelfstandiging in de hand gewerkt, waardoor dezelfde modellen kunnen gaan fungeren als *model voor* formele wiskunde. Het reinvention-proces wordt zo gestructureerd in vier niveaus, die niet alleen betrekking hebben op echte modellen maar ook op concepten, strategieën en notatiewijzen: context-niveau, verwijzend niveau, algemeen niveau en formeel niveau.

In het vierde hoofdstuk wordt de overgang gemaakt van 'ontwikkelwerk' naar ontwikkelingsonderzoek' en wordt toegelicht wat onder 'ontwikkelingsonderzoek' wordt verstaan. De in het eerste hoofdstuk geschetste eerste fase van het ontwikkelwerk als 'selecteren, inpassen en aanpassen' op basis van een visie of een onderwijstheorie, vormt de aanleiding voor de introductie van de term 'theorie-geleide bricolage'. De pragmatische, creatieve en doelgerichte manier waarop een 'bricoleur' werkt, en die door Levi-Straus (geciteerd door Lawler, 1985) wordt opgevoerd als metafoor voor 'denken', wordt hier gebruikt om het ontwikkelwerk te typeren. De betekenis van deze metafoor wordt nog vergroot als blijkt dat Jacob (1982) dezelfde metafoor gebruikt om de evolutie te beschrijven als een proces van eindeloos bijschaven en aanpassen. Een dergelijke beschrijving past ook bij een aanpak van ontwikkelwerk waarbij de nieuwe leergang niet in één keer op papier wordt gezet maar geleidelijk aan gestalte krijgt in een proces van bijstellen en uitproberen. Nog beter past deze beschrijving bij ontwikkelingsonderzoek, omdat hier het lange-termijnperspectief overheerst. Daarbij richt ontwikkelingsonderzoek zich niet primair op produktontwikkeling maar op theorie-ontwikkeling.

Ontwikkelingsonderzoek kenmerkt zich door een cyclisch proces van doordenken en beproeven: een cyclische opeenvolging van gedachte-experiment en les-experiment. Het resultaat is een theoretisch en empirisch gefundeerde leergang, doordacht en beproefd. De theoretische opbrengst van het ontwikkelingsonderzoek vindt zijn beslag in het leerproces van de bij het onderzoek betrokken ontwikkelaars. Om het resultaat van het onderzoek overdraagbaar te maken, zal de ontwikkelaar/onderzoeker dit leerproces moeten expliciteren. Hij of zij zal dat bovendien op een zodanige wijze moeten doen dat dit leerproces 'navolgbaar' wordt voor buitenstaanders.

Freudenthal (1988) benadrukt dit aspect met het oog op het informeren van de gebruikers van de ontwikkelde produkten. Maar deze navolgbaarheid, of trackability, is ook essentieel voor de verantwoording naar een wetenschappelijk forum (Smaling, 1987). De complexiteit van dit type onderzoek en de bijbehorende verslaggeving wordt in dit hoofdstuk toegelicht aan de hand van het ontwikkelingsonderzoek met betrekking tot de lege getallenlijn.

Het vijfde hoofdstuk richt zich op onderzoek naar de implementatie en de effecten van realistisch reken-wiskundeonderwijs, met het MORE-onderzoek als voorbeeld (Methoden Onderzoek REkenen-wiskunde, Gravemeijer et al., 1993). In dit hoofdstuk wordt betoogd dat de evaluatie van realistisch reken-wiskundeonderwijs rekening moet houden met een aantal specifieke kenmerken van de vernieuwing. Deze hebben betrekking op onder meer de aard van de leerdoelen, de verandering van het onderwijs als innovatiedoel en het feit dat de vernieuwing zich voltrekt via de schoolboeken. Wanneer we het succes van deze vernieuwing willen vaststellen, moeten we rekening houden met deze specifieke kenmerken. Dit betekent onder meer dat de implementatie in de evaluatie van de vernieuwing moet worden betrokken. Voorts betekent dit dat er passende toetsen moeten worden ontwikkeld.

Het voorbeeld van het MORE-project laat zien dat zo'n evaluatie aan betekenis wint wanneer deze wordt geplaatst in het perspectief van de innovatie. In kwantitatief-constaterend evaluatie-onderzoek kan de mate van implementatie en de met de nieuwe onderwijsaanpak behaalde leerresultaten in principe objectief worden vastgesteld. In principe, omdat het vaststellen van de leerresultaten een voldoende implementatie van de vernieuwing veronderstelt. Dat bleek hier echter niet het geval; de implementatie van de vernieuwing bleef achter bij de bedoeling.

Kwalitatief-interpreterend evaluatie-onderzoek dat ook deel uitmaakte van het MORE-project bleek aangrijpingspunten voor een betere implementatie op te leveren. Bovendien konden aanwijzingen worden gegeven voor een verbetering van de manier waarop de basisautomatismen in de onderzochte methode 'De wereld in getallen' worden opgebouwd. Door de evaluatie vanuit innovatieperspectief te relateren aan de specifieke manier waarop de vernieuwing van het reken-wiskundeonderwijs plaatsvindt – via de schoolboeken – konden bovendien aanbevelingen worden gedaan voor implementatiesteun. In concreto komen deze neer op het stimuleren en steunen van leerkrachten bij het doorlopen van een leerproces dat zich richt op het verwerven van algemeen pedagogische en microdidactische kennis en vaardigheden.

In het zesde hoofdstuk wordt het concept ontwikkelingsonderzoek nogmaals onder de loep genomen. Nu met name met het oog op de 'externe' legitimeringsfunctie van dit type onderzoek. In de praktijk speelt de legitimering van (de opbrengst van) ontwikkelingsonderzoek zich voornamelijk af binnen de kring van 'realisten'. Om de legitimeringsfunctie voor buitenstaanders te versterken wordt allereerst getracht nader te expliciteren wat ontwikkelingsonderzoek precies inhoudt.

In het kort komt het hier op neer, dat ontwikkelingsonderzoek wordt gekarakteriseerd als 'evolutionair', 'gelaagd' en 'reflexief'. Het is *evolutionair* in de zin dat de theorie-ontwikkeling geleidelijk verloopt en iteratief en cumulatief van karakter is. Het is *gelaagd* in de zin dat theorie-ontwikkeling zich voltrekt op verschillende niveaus:

- op het niveau van de onderwijsactiviteiten (micro-theorietjes)
- op het niveau van de leergang (lokale onderwijstheorie)
- op het niveau van de domeinspecifieke onderwijstheorie.

Ontwikkelingsonderzoek is bovendien *reflexief* in de zin dat de theorie-ontwikkeling tot stand komt via een wisselwerking tussen de zojuist genoemde niveaus. Deze niveau-indeling wordt ook weerspiegeld in de beschrijving van de werkwijze van ontwikkelingsonderzoek als een combinatie van verschillende soorten theorie-geleide bricolage:

- theorie-geleide bricolage op microniveau
- theorie-geleide bricolage als basis voor een theoretisch gefundeerd voorontwerp
- theorie-geleide bricolage als een langlopend proces.

Wat het laatste punt betreft; elk op zichzelf staand ontwikkelingsonderzoek levert slechts een kleine bijdrage aan theorie-ontwikkeling. De kracht van het ontwikkelingsonderzoek zit hem in het cumulatieve effect van de opbrengst van tal van onderzoeken.

Ten aanzien van punt twee, het samenstellen van een theoretisch gefundeerd voorontwerp, geldt dat de ontwikkelingsonderzoeker het reinvention principe, de didactische fenomenologie en het principe van het gebruik van zelfontwikkelde, bemiddelende modellen kan inzetten. Aan deze heuristieken worden bovendien de criteria ontleend, die de onderzoekers gebruiken om hun onderwijsexperimenten te beoordelen.

Tenslotte worden nog twee andere manieren genoemd om de legitimeringsfunctie van ontwikkelingsonderzoek te versterken, namelijk, het objectiveren van de uitkomsten van ontwikkelingsonderzoek en het versterken van de theoretische basis. In verband met het laatste wordt gewezen op het belang van een verbinding van ontwikkelingsonderzoek met psychologisch onderzoek waardoor een versterking van de legitimeringsbasis verkregen kan worden. Een verbinding die inmiddels via samenwerking wordt gerealiseerd.

Objectiveren van ontwikkelingsonderzoek is echter problematisch, omdat het lastig is adequaat verslag te doen van alle facetten van dit complexe proces. Bovendien hebben cruciale momenten in het leerproces van de ontwikkelingsonderzoeker vaak het karakter van Aha-Erlebnissen. Om buitenstaanders de betekenis van zo'n Aha-erlebnis te laten inzien dient veel van het referentiekader van de ontwikkelingsonderzoeker te worden geëxpliciteerd. Dit principe wordt in dit hoofdstuk toegelicht met een Aha-Erlebnis die zich voordeed bij het ontwikkelen van een leergang cijferend vermenigvuldigen.

Verder wordt opgemerkt dat ontwikkelingsonderzoek zijn eigen methoden en technieken vraagt. Zo wordt Verschaffel (1993) geciteerd, die erop wijst dat er internationaal een gemeenschap van onderzoekers van wiskundeonderwijs is ontstaan die ook de noodzaak van het ontwikkelen van nieuwe methoden en technieken erkent. Bovendien is er sprake van een zodanige diversificatie en liberalisatie van methoden en technieken dat het concept 'ontwikkelingsonderzoek' daar ook binnen past.

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