L. Streefland

editor

REALISTIC MATHEMATICS EDUCATION IN PRIMARY SCHOOL

On the occasion of the opening of the Freudenthal Institute

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REALISTIC MATHEMATICS EDUCATION
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Freudenthal Institute
Research group on Mathematics Education
Center for Science and Mathematics Education
Utrecht University, The Netherlands

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Introduction

L. Streefland

As an homage to the founder of realistic mathematics education in The Netherlands, the Research Group on Mathematics Education (OW&OC) has adopted the new name of 'Freudenthal Institute'. There are at least two additional reasons for this choice. First, in celebration of the Group's first decade of existence and second, the posthumous publication of Freudenthal's last book, 'Revisiting Mathematics Education'. The book can be regarded as the first milestone of the post-Freudenthal era. It is the presentation of his educational heritage. This current publication is intended as the next step.

The array of contributors is headed by Treffers in double respect. First he gives an historical overview. It deals with the silent revolution that occurred in mathematics at primary level in The Netherlands during the nineteen eighties. It concerns the shift from mechanistic to realistic textbooks. Treffers explains why this considerable change could occur and proceed, despite the fact that the policy of the Dutch government with respect to the so called 'Educational Provider Structure' was more counterproductive than supportive of this innovation during that period. The closing thoughts of the article are dedicated to Freudenthal and to the development of realistic mathematics education from idealism to realism in the last decade.

Treffers' second contribution unveils the main learning strands in realistic mathematics education at primary level, embedded in the instruction-theoretical frame work of it. With an emphasis on the vertical component of mathematization, learning strands for counting, memorising of addition and subtraction tables to ten and twenty, addition and subtraction to one hundred, ..., ratio and fractions, all pass the review. To conclude somewhat closer to classroom reality an example math lesson is given. Its description focuses on the horizontal component of mathematization.

Gravemeijer takes over the discourse by reflecting on the use of manipulatives in an instruction-theoretical manner. A concept at the heart of his train of thought is 'isomorphism', which stems from the action-psychological theory of Gal'perin. This concerns the isomorphism of material actions and the pursued mental actions. The author considers this problem from various instruction-theoretical points of view. One can learn from his contribution that the recent trend in educational psychology, the so-called 'mapping theory' is too simple a solution for theorising the connection between observed material actions in the behaviour of children and the subsequent easily concluded achievements of learning.

Next, two examples of (outcomes of) developmental research. This research in action aims at adapting prototypes of courses under development to individual learning processes under study provoked by them, and vice versa.

Van den Brink considers realistic math education for young children in a comparative study and Streefland looks at fractions from an integrated perspective. Van den Brink's research shows the benefits of the realistic approach to addition and subtraction by exploiting children's playing and creative potential within the context of buses at bus stops and passengers getting off and on. The children learned with greater insight, in less time than the control group, which was taught in a traditional manner, i.e. on the basis of a mechanistic textbook. There was no question of these 'bus children' failing to master the required skills.

Streefland reports his research on fractions, carried out from 1983 to 1986. An impression is given of this paradigm of developmental research which was also published in its entirety elsewhere. First it describes the observation of a long term individual learning process. Then special attention is devoted to the comparative part of the research, which was not evaluative in the usual sense. On the contrary, it too aimed at the improvement of the developed prototype of a course on fractions. Nevertheless the results were most promising from an evaluative point of view and very much in agreement with Van den Brink's findings. The data of a nationwide sample which was incorporated in the final assessment for primary schools by the National Institute for Educational Achievement (CITO), were also included in the comparison.

From an historical point of view geometry is both interesting and exceptional in primary curricula in our country. In the nineteenth century a type of geometry, called 'Vormleer' could be found in our schools, and was made compulsory by legislation for some thirty years. This is one of the interesting findings of De Moor in search of the historical roots of geometry in primary programs.

Next he analyses realistic geometry by distinguishing, labelling and describing its various aspects such as: sighting and projecting, locating and orientating, and so on. The discourse is rich in examples. These call for work to be done in groups, for investigation, experiments, discussion, and reflection. Geometry should constitute a part of primary programs, if only for its pedagogical and practical value.

Realistic mathematics education holds in it the potential of freedom to learn to certain levels. This characteristic must also be reflected by the various kinds of assessments. Among these the paper-and-pencil tests can continue to play a prominent role, provided that a number of measures are taken to make them more informative. Van den Heuvel-Panhuizen and Gravemeijer analyse a number of these measures, illustrating them by a rich variety of test items from the MORE-project. Paraphrasing their title one can indeed conclude that 'such tests aren't bad at all'. And instead of thwarting innovations, the authors conclude, they can contribute to the improvement thereof.

The first author goes even one step further. She deals with ratio in special education by means of similar tests. She reveals the abilities and skills of children with learning disabilities and of mentally retarded children in this regard. However, ratio is not a part of their regular programs! Despite that the results are not only encouraging but also refute the predictions of their teachers and of experts. Will the boundaries of 'normal' and 'special' schools indeed shift in the near future and will realistic ratios become part of the special program?

Special abilities and skills are required to teach mathematics according to the approach pleaded for in this book. These must be acquired either pre- or in-service. Van Galen and Feijs bring up the rear considering a new tool for teacher training. They describe their investigations on the part interactive video could play in an in-service course. The videos contain recordings of individual pupils and of lesson situations and do indeed give added value to in-service training and have, among other, changed the role of the instructor.

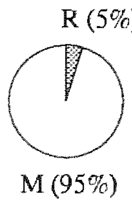
Finally, the reader needs to know why only one side of the total picture of the Freudenthal Institute is shown here. Would it be too bold to predict that the other side, that of secondary education, will be presented in the near future? I believe not. The blueprints on the drawing boards and the work done this past decade are sure signs of things to come.

Realistic mathematics education in The Netherlands 1980-1990

A.Treffers

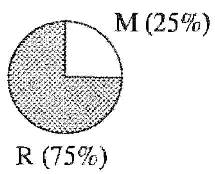
1 realism in textbooks 1980-1990

A revolution in mathematics education, that is what occurred in The Netherlands in the period 1980-1990. But a silent revolution it was. Not a single innovation expert is heard speaking or writing about it. And hardly a whisper in the media. That is fine: apparently nothing went spectacularly wrong. But the revolution is spectacular, nonetheless. What has taken place? The distribution of textbooks for mathematics in primary schools is the following in 1980:



M stands for mechanistic textbooks
R stands for realistic textbooks

In 1990 the situation shows a marked change:



Realistic textbooks went from 5% to a market share of no less than 75%!

To be able to make a proper assessment of this change it is of course essential to know the general difference between mechanistic and realistic. In short, what it boils down to is that the various textbooks are different in regard to the learning material, the composition of the learning strands and the place, the nature and the extent of applications or context problems. Without doubt differences of a fundamental nature. Because it really does make a difference when in the new textbook geometry suddenly appears, or when column arithmetic is taught differently, and much more attention is devoted to problems which are rich in context, both at the start of the learning strand as well as half way through or at the end. In short, the whole didac-

A third example: One closer to home. But it does concern that same Educational Provider Act. According to the draft of that Act it is not possible to finance developmental research (through the appropriate channels). But this now is precisely the research that must supervise the new realistic program development. But that is not allowed, because developmental research does contain a component of development. And that must be done elsewhere, by curriculum development, according to the interpretation given in the Educational Provider Act.

This curtailment of developmental research touches on the heart of academic educational research and has far reaching influence, as far as the area of educational theory and instruction. The content of subject didactics is suffocated as a result, of course not only in the area of mathematics. In consequence, subject specific didactics cannot develop sufficiently, certainly not if we measure this in terms of what this could mean for educational practice. One and other of course also puts restraints on the Freudenthal Institute.

Let us not end in a minor key after such a promising start. Just look at the developments which have currently been initiated in the area of teacher training (Pabo) for instance in regard to an in-service course for new teachers. Or at the efforts which are being made at local level to improve the quality of the subject of mathematics and didactics, supported by a national approach. But that takes us back to the informal sector, because all of this occurs from the Pabo-world itself, supported by the NVORWO, although with absolutely insufficient facilities.

Although an exception must be made for the Panama project of the HMN, Middle Netherlands College. This is a formal project, yet one which contributes much to the informal circuit. But the same can in certain respects also be said for the national curriculum institute (SLO) and the national test institute (CITO). In short, the formal and the informal are more interwoven with each other in the educational provider system than might appear from the foregoing. All the more reason to involve the informal NVORWO more in formal decision policy of the government, I would say. Because in my opinion, that is where the solution lies to better attune the development of textbooks and the required information, in-service training, teacher training, counselling, development and research. As long as this has not been achieved the rewards of the revolution in the textbook market are perhaps less than they could be.

5 conclusion

Has the concept about realistic mathematics education changed in the eighties? Is the concretising in the textbooks and in the 'Specimen of a National Curriculum' in line with Freudenthal's ideas?

Freudenthal is the founder of realistic mathematics education. He was the one to put Wiskobas on the right track: away from formalistic New Math, directed at reality. His didactical realism is coloured by idealism. His ideas emphasize rich thematic contexts, integration of mathematics with other subjects and areas of reality, differentiation within individual learning processes and the importance of working together in heterogeneous groups.

In the eighties it was notably also by the OW&OC that emphasis shifted to the importance of elementary context problems, to the alignment of learning strands and the steering task of teachers – issues for which Freudenthal had less of an eye.

This shift in emphasis in the didactical realism of the post IOWO period (from idealistic to realistic realism) is among other expressed in the attention given to basic skills (arithmetic rack, empty number line, mental arithmetic and estimation).

Freudenthal was very much aware of this change of course around 1989-1990. Also in the last long discussion that I had with him on Monday, 8 October 1990, he again raised with me this variation in the nature and richness of context problems, and the degree of steering and alignment.

According to him this diversity made it possible to link up realism with the different concepts about instruction and the various styles of teaching. A year prior to this, in an emotional discussion, he even expressed his doubts about whether his influence had not been too idealistically tinted and had not asked too much of educators: 'a person has his doubts' he said then.

What ever the case, from the shift in emphasis one can get an idea of the direction of developmental research that the OW&OC followed and that the Freudenthal Institute will pursue, at least as far as mathematics instruction for primary school is concerned. With the added remark that this will be in co-operation with other agencies – national and international. Or in other words: together within the mentioned informal provider structures of the subject area for mathematics under the umbrella of the NVORWO (the Dutch NCTM so to speak).

And if this co-operation and infrastructure did not exist, for secondary education as well I should add, then the name Freudenthal Institute would not have been chosen. Because isolated research, not related to development, training, assessment, could never be allowed to be associated with Freudenthal because he considered that to be fruitless. The official name of the research group of the Freudenthal Institute is: Developmental Research for Mathematics and Informatics Education, with the Dutch acronym OWIO. That abbreviation rings a bell, somehow.

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Didactical background of a mathematics program for primary education

A. Treffers

1 Introduction

This contribution outlines an instruction-theoretical framework of realistic mathematics education for primary school that was developed in The Netherlands in the period between 1970 and 1990. The outline is placed against the background of three directions in mathematics education, which in the seventies, and still today, in part determine the international image of mathematics. Determine it in terms of educational practice but also of psychological educational research. At the outset a learning-instruction structure is described. Next it is illustrated how this structure is intertwined in the various learning strands. And in conclusion, how these strands of learning are given concrete shape at lesson level. The developmental research that has led up to this realistic mathematics program is the fruit of ten years of IOWO and ten years of OW&OC effort, in co-operation with so many in the field of education. It has been especially the textbook authors who contributed to making this realistic mathematics program concrete. Today in 1991, approximately three-quarters of the primary schools in The Netherlands use a realistic mathematics textbook series. Which of course does not mean to imply that in practice the principles which lie at the basis of this realistic mathematics education, are also always served. Let us outline what those principles are.

2 Instruction-theoretical framework of realistic mathematics education

The instruction-theoretical framework is outlined on the basis of column arithmetic for division. A realistic course for 'long division' might, for instance start off with the following problem:

> The PTA meeting at our school will be attended by 81 parents. Six parents can be seated at one table. How many tables will we need?

The solution methods in a class of seventeen grade three pupils appear to be the following:

- Seven pupils add up '6 + 6 + 6 ...' or 6, 12, 18, ...' or recite the table of multiplication: '1 × 6, 2 × 6, 3 × 6; ...'

- Six pupils make an arithmetic short cut: they first take 10×6 and from sixty they continue, sometimes by adding, sometimes by multiplying.
- One pupil knows $6 \times 6 = 36$, doubles this, $12 \times 6 = 72$ and adds two more tables.
- For three pupils it could not be deduced how the answer was calculated.

In the discussion afterwards the teacher dwells on the three solutions mentioned above. The children themselves determine that the ten-times method was very efficient.

The second part of the lesson deals with a similar problem:

> How many pots of coffee will have to be made for the parents? One pot serves seven cups of coffee, and each parent will be offered one cup.

What have the children learned from the first part of the lesson, will they choose the efficient method to find the answer? This indeed appears to be the case.

- This time the step-by-step method is only used by one pupil (seven did so the first time).
- Thirteen pupils go straight for $10 \times 7 = 70$ (six did so before)
- None of the pupils do 7×7 (one the first time).
- And again for three pupils it cannot be determined how they worked.

The considerable shift to the ten-strategy is among other expressed in the work done by Linda (fig.1).

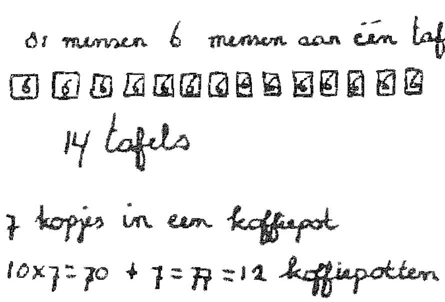


figure 1: Linda's work

A long division scheme is introduced in a next phase.

6/81\	
60	10 tables
21	
18	3 tables
3	
3	(1 table)
0	
	14 tables

Of course this scheme is not only suited for ratio division but also for distribution division. For example to divide 81 objects fairly among 6 people. Only then instead of 'tables' the indication is 'per person'. And... the answer is different this time.
At a somewhat later stage large numbers are worked out.

> 1128 soldiers are transported on buses that have 36 seats.
> How many buses are needed?

Now there is differentiation of solution methods (fig.2).

```
36/1128\
   360   10 bussen
   ---
   768
   360   10 bussen
   ---
   408
   360   10 bussen
   ---
    48
    36    1 bus
   ---
    12   (1 bus)

(a.)
```

```
36/1128\
   720   20
   ---
   408
   360   10
   ---
    48
    36    1
   ---
    12   (1)

(b.)
```

```
36/1128\
   1080   30
   ----
     48
     36    1
   ----
     12   (1)

(c.)
```

figure 2: solution levels for 1128 : 36

These solutions illustrate the abc of the entire course. The idea is for every pupil to eventually reach level c. Only, the one pupil will arrive at the abbreviated arithmetic method sooner than the other. But even if a child does not progress beyond level b, he or she can still do long division. And of course straight sums in long division are also practised in this course. But an item like the problem with the buses will continue to be present in the background. All the while doing arithmetic the children can imagine a transportation situation like this for themselves. The action of the procedure acquires a concrete background as it were, and a check on and steering of arithmetic work remains possible. The children are also given the opportunity to come up with problems themselves: straight sums but also application problems, so-called free productions.

At an even higher level the structure of long division can be made the subject of investigation. For example:

> Calculate 1128 : 36 with the aid of the calculator and then the remainder of the long division.

To be able to solve a problem of this kind the structure of the scheme for long division must be dissected, something which not every pupil will be capable of. What is attainable however, is to learn to carry out the long division procedure in one or other abbreviated form, and that is the point of the exercise.

The five major learning and teaching principles that lie at the basis of such 'realistic' courses will be described in the context of the previously mentioned learning strand for long division. The indication 'realistic' means to say that in instruction of this kind application situations fulfil an important function – but more about this later. For each of the five described pairs of principles the learning principle will be discussed first, followed by the instruction principle.

2.1 constructing and concretising

The first learning principle is that first and foremost learning mathematics is a constructive activity. Something which contradicts the idea of learning as absorbing knowledge which is presented or transmitted. This construction characteristic is clearly visible in the outlined course: the pupils discover the division procedure for themselves. That becomes possible because at the start of the learning strand a concrete orientation basis is laid for the skill to be learned – the first instruction principle. Upon introduction of the item about the buses the children come to realize what the arithmetic operations are leading up to.

2.2 levels and models

The second learning principle is: the learning of a mathematical concept or skill is a process which is often stretched out over the long term and which moves at various levels of abstraction. This level-characteristic can be seen in solution methods a,b and c of figure 2. First the notations refer to the division situation with the buses: the arithmetic is still context-bound there. Later this reference can be recalled but the manner of notation no longer refers to it – see solutions b and c of figure 2. At the end the procedure-actions are understood completely within the formal number system. By then the structure of the algorithm can be recognized and the applicability of division has been markedly broadened. To be able to achieve this raising in level from informal context-bound arithmetic to formal arithmetic the pupil must have at his disposal the tools to help bridge the gap between the concrete and the abstract.

Materials, visual models, model situations, schemes, diagrams and symbols serve this purpose. The terms 'concrete' and 'abstract' do not so much refer to absolute level indications, by the way. They are sooner relative terms: what was still abstract in the initial instruction, a bare number for instance, can be completely concrete by the upper grades of primary school. Which also tells us that a concrete orientation basis (see the first instruction principle) need not per se lie in the material atmosphere or in realistic context situations. And also schematising, models or the subject system itself can at a certain moment constitute a concrete basis for the learning process, for instance the arithmetic system as the basis for algebra – i.e. the shifting perspective of 'reality' as it is meant in realistic mathematics education. But more about that later.

2.3 reflection and special assignments

The learning of mathematics and in particular the raising of the level of the learning process is promoted through reflection, therefore by considering own thought process that of others – this is the third principle of learning. An example of this is the earlier mentioned bus item with the 'remainder'. But also finding the remainder at the end of the long division sum with the aid of the calculator, whereby the relation between the remainder number and decimal part of the quotient must be made. The most important category however is the assignment to produce items of one's own. This brings us to the third instruction principle: the pupils must constantly have the opportunity and be stimulated at important junctions in the course, to reflect on learning strands that have already been encountered and to anticipate on what lies ahead. Important assignments through which one and other can be achieved are the previously mentioned free productions and the conflict problems. In the case of the long division these free productions render information about (a) the magnitude of the numbers with which children can or dare work, (b) the level of schematising and abbreviation at which they calculate, (c) possible systematic errors and (d) the application problems, in this case, the kinds of divisions (ratio division, distribution division, inverse multiplications).

The following is an example of a conflict problem: 'If each of two divisions gives the result '31 remainder 12' are the original divisions then necessarily equal or equivalent?' On the basis of the customary manner of reasoning and notation we would tend to answer this question affirmatively. But after trial and analysis we discover this is not true. Because with a divisor of 36 the result '31 remainder 12' is actually $31\frac{1}{3}$, but if the divisor is 24 this is $31\frac{1}{2}$! Via the conflict situation reflection leads us to further consideration of the structure of long division.

2.4 social context and interaction

Learning is not merely a solo activity but something that occurs in a society and is directed and stimulated by that socio-cultural context – the fourth principle of learning. In learning long division a lively exchange of ideas takes place from the very start. The first discovery which is discussed is the rule about straight away placing ten tables or deploying ten buses, a solution which is quickly understood and adopted by fellow pupils. Further short cuts are discussed, evaluated and (often) adopted. Also the free productions of straight sums and dressed up problem situations (consider '1128 : 36' in changing contexts with varied outcomes, 31 or 32. Or 31, 33 or $31\frac{1}{2}$, ...) are discussed in the group and can give an impulse to efficient or less efficient solution methods. Besides individual work the group also acts as a booster for learning. In consequence the fourth instruction principle says that mathematics education should by nature be interactive. I.e. that besides room for individual work, it must also offer opportunity for the exchange of ideas, the rebuttal of arguments, and so forth.

2.5 structuring and interweaving

We close the circle of principles by connecting to the first principle. Learning mathematics does not consist of absorbing a collection of unrelated knowledge and skill elements, but is the construction of knowledge and skills to a structured entity. New concepts and mental objects are fit into the existing knowledge base or ensure that this structure of knowledge is modified to a greater or lesser degree. For instance, division can be connected with the three other basic operations via the bus item. Initially it is solved by repeated addition, repeated subtraction or supplementary multiplication. Later, division becomes more of an independent operation with a structure of its own. But the connection with the other operations continues to exist. Doing arithmetic sums, mental arithmetic, long and short arithmetic procedures, and applications in division thus constitute a structured entity. For the fifth instruction principle this means that learning strands must, where possible, be intertwined with each other. And at the same time that pure arithmetic and the making of applications must from the very start be connected with each other. Subsequently, reality is both the source and the application area of mathematical concepts and structures, hence the term realistic mathematics education.

2.6 the structure of the learning-instruction principles

In the foregoing the following learning principles and instruction principles were connected:

- the concept of learning as construction (L.1) with the laying of a concrete orientation basis (I.1);
- the level-character of learning seen on the long term (L.2) with the provision of models, schemes and symbols (I.2);
- the reflective aspect of learning (L.3) with the assigning of special tasks, with as main categories free production items and conflict problems (I.3);
- learning as a social activity (L.4) through interactive instruction (I.4);
- and the structural or schematic character of learning (L.5) with the intertwining of learning strands both mutually as well as per pupil, with reality (I.5).

The connections made above are arbitrary however, because every learning principle can basically be connected to each instruction principle. A random example is L.2 – I.4. On the basis of the long division it is clear that in interactive instruction the different levels in the learning process emerge in the more or less abbreviated solutions and methods of notation, plus the context -bound execution of the mental arithmetic procedures.

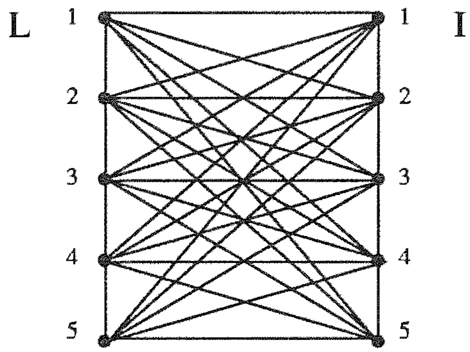


figure 3: educational learning structure

Or take L.3 and I.5: through the intertwining of the learning strand on division with other courses reflection on different solution methods is possible. The learning instruction principles form a structure in which all fundamental aspects of education and learning theories are locked up (figure 3). The specifics of this 5×5 structure can however only be made clear if they are connected to concrete instruction contents, such as long division for instance. It is not a formally, but rather a structurally determined structure. Yet the contents is still without much relief, because not a single alternative has been described for the course: the contrasting background has not yet been illustrated. That will be done in the next part by way of the description of four directions in mathematics education.

3 four directions in mathematics education

Besides the realistic direction in mathematics education there are three others that can be distinguished, namely: the empiristic, the mechanistic and the structuralistic. A brief characterisation will follow for each on the basis of the long division problem, followed by a more general description via mathematising.

3.1 the empiristic approach to long division

We can be brief about the empiristic approach to long division because, in general, it is one that is not taught. Instead, the short cut of informal arithmetic methods is pursued, in part via mental arithmetic. Context problems of division will be selected mainly from the realm of measurement. In the case where the purpose is to teach long division, the same global approach is chosen as the mentioned realistic method of progressive schematising and abbreviation on the basis of an elementary context problem, however without the strong insistence to achieve the standard level.

3.2 the mechanistic approach to long division

What the methodology of long division consists of is something many of us know from own experience. In tens of lessons the division algorithm is practised in case after case, with increasing complexity, starting with small sums and followed by more complex cases (figure 4).

The degree of complexity is especially determined by the magnitude of the numbers (dividend and divisor). Then there are also the necessary regrouping actions for multiplications and subtractions, plus the 'bothersome' zeroes which are found in the quotient (see the last and by far the most difficult case of figure 4).

Divisions with remainders do not appear until near the end of the course, and then only in the appearance of bare sums. Problems like '1128 soldiers must be transported in buses with 36 seats, how many buses will be needed?' are lacking totally. With the result that only one in three pupils succeeds in solving a problem like this by the end of primary school, therefore answers 32 and not 31 remainder 12, or 31, or 31.33 or something else. And in the event the children are allowed to use calculators the correct-score falls to below ten percent!

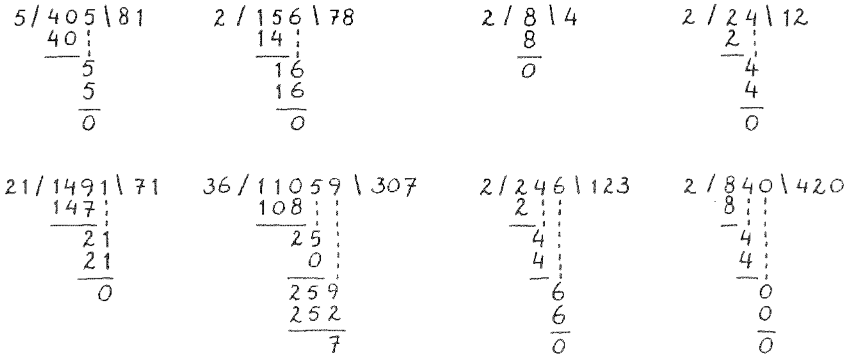


figure 4: some problems of long division

3.3 the structuralistic approach to long division

In the structuralistic approach the emphasis in teaching long division lies very much on the place value system. The numbers are unfolded in units represented by cubes, in tens with bars of ten, in hundreds by squares with ten bars etc. In the case of the long division problem this structural material is divided fairly, and the result thereof written down in a form which refers to this kind of division (figure 5).

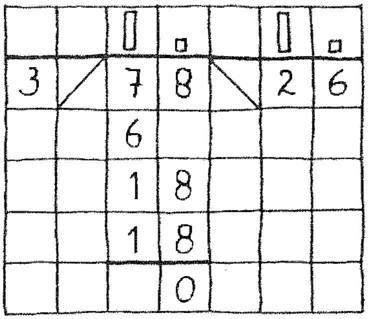


figure 5: long division with material

From the very start one is steered towards the standard algorithm by the most abbreviated arithmetic method. Gradually the numbers to be divided become larger, manipulation with blocks disappears more to the background and in time so does the reference to these blocks in the notation scheme. Both in the step-by-step forming of mental action (Gal'perin) as well as in the 'mapping theory' (Resnick) such a learning strand is built up as follows:

- first divide the position blocks;
- then connect this to the notation of the division in a scheme;
- next do the division and write down without manipulation, but by looking and expressing the actions of division;
- and finally, solely mental arithmetic and the writing down of the long division problem without the pre-printed scheme.

In short, the objections to such a structuralistic construction are the following:

The method is strongly prescriptive and in a certain sense unnatural. In our example of 78 : 3 it is natural for the children to first arrive at a division of two bars and two cubes, then to divide the remaining twelve (one bar, two cubes). But because they are steered almost directly towards the most abbreviated arithmetic method of the standard algorithm, there is barely room for such informal strategies. What weighs even heavier perhaps is that the making of application problems becomes more difficult by this structuralistic method. Take the previously mentioned problem '81 : 6' of the PTA-meeting and '1128 : 36' of the buses. For '81 : 6' we first get one bar in the division scheme. What we do next is however completely obscure in that context. Apparently we have first divided the parents into eight groups of ten. Division into or by six should have resulted in one group of ten people. But no, all of a sudden there are ten tables! In short, the procedural actions with the blocks do not fit the pertaining application problem. And that applies for the bus problem as well.

Naturally one can use blocks for both cases – for instance place blocks on a table or in a bus – but the actions which are then carried out will in that event correspond with those of the earlier outlined realistic approach and not with the structuralistic set up mentioned here. The result is that in the very first phase of structuralistic instruction either no applications are given or only one-sided problems are presented. In short, the problem with the structuralistic set up of the learning strand for long division is that the algorithm is taught primarily at the formal arithmetic level. However, this arithmetic procedure only displays indirect similarities with the arithmetic methods which the children employ in the solving of context problems at the concrete level. And the consequence of this in turn is that it must then be attempted to close the gap between pure arithmetic and applied arithmetic by teaching the children all kinds of (division) schemes which are directed at recognizing division in application situations. One will, as mentioned, be forced to follow this instruction strategy because context-bound arithmetic and formal arithmetic (in casu the algorithm) do not connect.

In the realistic set up on the other hand, the subject-systematic final algorithm is not pursued in a direct manner, but developed gradually via context problems and context restricted arithmetic. Pure arithmetic and applications go hand in hand in this course and are not separated from the very outset as in the structuralistic approach. The solving of application problems does not primarily concern the recognition of the operation. It is sooner the opposite: the operation is taught by the applications and these give meaning to the manner of calculation.

3.4 directions in the 5 by 5 learning-instruction structure

Let us take a closer look at the three directions we have discussed in the foregoing, now against the realistic background of the 5 by 5 educational structure. Looking at the empiristic approach what stands out especially is the difference in regard to the use of the model in order to break away from informal, context bound arithmetic to reach formal arithmetic. Empiricists largely lack such models, as a result of which the transition from the concrete to the more abstract level will mainly have to take place in an inductive manner via many context problems – an issue we will discuss in greater detail further on. Of course that also gives different content to learning-instruction principles 1, 3, 4 and 5 in comparison with the realists, as we shall see. But regarded formally, the similarities with realism outweigh the differences.

In the mechanistic approach the differences outweigh the similarities however. We will point them out with key words:

1 Learning is not considered as construction but as reproduction, instruction is not based on a concrete orientation but starts each time on the formal arithmetic level.

2 This entails that no levels are distinguished in the learning process and that subsequently no intermediary is put between the informal, context restricted work and the formal, subject restricted arithmetic.
3 No attention is devoted to reflection and the problems that are presented are stereotype: pure sums and worded problems, and no free productions, no conflict problems and no problems in which one must provide information by one's self.
4 Instruction is strongly individualised or rather, individual. I.e the pupils progress on the learning route individually: no social context, no interaction.
5 The teaching methods are not mutually related and there is barely any connection between the subject system, and reality as application area.

In short, a background which is in stark contrast with the outlined realistic learning-instruction structure.

The structuralistic method does not fit into the learning-instruction structure well either. At least, the content that is given to the mentioned concepts differs greatly.

For instance, the construction principle can only be applied to a limited degree because a solid concrete orientation basis is lacking. The approach with the blocks is in fact just as abstract as is working with symbols. The reason for this, as we have seen, is that the material actions do not connect naturally to the mental operations with the symbols in the long division procedure. And what is more, they sometimes even conflict with the informal methods which are used in the solving of context problems. This also means that the levels in the learning process cannot on the longer term be progressed through successively. Context bound arithmetic on the concrete level is not used as the basis for formal pure computation and the broad applications thereof at the abstract level. Instead, a level indication from material to mental action is set which is not attuned either to pure arithmetic nor to the making of applications. In other words: no adequate aids in the form of materials, models, schemes and symbols are provided, either for pure arithmetic or for the making of applications. In consequence there is insufficient reflection and not enough attention for the social element. The children are simply not given enough opportunity to develop their own methods. The learning strand for long division, mental arithmetic and the making of applications are not intertwined, but isolated from each other. The structure of knowledge that results is in principle a rather meagre one.

Observe the contrasting background of the realistic 5×5 education-learning structure, in which the mentioned principles have more relief. We made this comparison using the example of long division, but we could as easily have taken addition and subtraction under a hundred as a paradigm, or calculating with percentages, or fractions, or area. One can argue that the mentioned directions do too little justice to the domain-specific character of mathematics. It is a characteristic of mathematical activity, according to the realistic view, that the build up of elementary skills can take place via a process of reinvention or independent construction, yes, must imperatively take place if the mathematical rules and structures are to be widely applicable. Or to be more precise, that the mathematical learning fits into the 5 × 5-framework of progressive mathematisation. The rule of thumb for the development of mathematics instruction is that the developer in first instance searches predominately for those context problems which have model-characteristics, therefore problems which can serve as a model for gradually learning to work at the formal level. Pure arithmetic and the making of applications are connected from the very outset.

3.5 summary and generalisation

The four mentioned directions in mathematics education can be distinguished according to the presence or absence of the components of horizontal and vertical mathematisation – see figure 6.

	horizontal	vertical
mechanistic	–	–
empiristic	+	–
structuralist	–	+
realistic	+	+

figure 6: mathematising and directions

Horizontal mathematising is the modelling of problem situations thus that these can be approached with mathematical means. Or in other words: it leads from the perceived world to the world of symbols. And in the description of long division we have already observed that this 'concrete' perceived world is not an absolute level indication but a relative one. I.e that also parts of the world of symbols can become part of the perceived world, the personal reality. Vertical mathematising is directed at the perceived building and expansion of knowledge and skills within the subject system, the world of symbols.

Generally speaking both components of mathematising are missing in mechanistic education. Which is to say that neither the source nor the application area of arithmetic are sought in the perceived world. The start is made at the formal level of the world of symbols. Vertical mathematising is impeded and for many pupils even blocked because the perception principle is missing – with the result that the instruction becomes the presentation and drill of rules and regulations, in short, becomes algorithmic mathematics education. The formal straitjacket leaves room only for rule-directed education.

Characteristic for the structuralistic approach of education is that there insight is pursued. This is done in a specific manner. Here also the start is on the formal arithmetic level. Only, the structuralists concretize the operations, structures and such with the aid of structured material in order to represent the subject system concretely and perceptibly. For instance with structure material in the form of MAB – or Dienes blocks as the embodiment of the decimal place value system. Vertical mathematising takes place with this structural material, by means of visual representations thereof, to operations with symbols – 'enactive, iconic, symbolic' as Bruner calls the three, and Resnick refers to 'mapping instruction'. Real problems play no essential part in the learning of arithmetic in the initial phase.

Applications do not appear until after learning how to operate with pure numbers. This choice for the subject-structural point of departure also implies that it is not possible to emphatically build further on the natural, informal working methods of children. In short, the horizontal component of mathematising is missing. In empiristic education this component is clearly manifested. Informal, context-bound arithmetic is the basis of instruction here. This educational design departs from the premise that all the while working with a great diversity of realistic problems the pupils will themselves be able to make the leap to the level of formal arithmetic. I.e without the support of intermediary models, schemes and such – in any case instruction does not tend towards this. In other words, no express vertical component..

In the opinion of the realists, on the other hand, it is especially by means of strong models that children are given the opportunity to bridge the gap between informal, context-bound work and the formal, standardised manner of operation, through the constructive contribution of the children themselves. In short, they take the position that assistance can be given via the presentation of fitting models and schemes, that direction can be given to learning, that something essential can be offered from 'outside', that it is worthwhile to pursue concrete learning objectives, that a line can (must) be drawn which runs from the mentioned concrete, informal point of departure, that levels can be distinguished in the learning process, that ....
Many 'thats' which empiricists will not underwrite in this strict form.

Informal solution methods of children form the pretext here, to arrive at arithmetic procedures via a gradual process of schematisation, abbreviation and generalisation. Context-bound mathematics is made subservient to formal arithmetic; models act as intermediaries. These models should however meet very special specifications in order to be able to fulfil the bridging function between informal context-bound arithmetic and formal arithmetic. They must be 'models of' context problems which will serve as 'models for' the pure subject restricted and applied arithmetic in the pertaining area – multifaceted, in other words. They must moreover be strong in the sense that they can be deployed in all phases of the learning-instruction process in the intended area, therefore in the first phase close to informal, context-restricted arithmetic and in the last phase close to formal standardised operating, as well as in the extensive area in between. And thirdly, these models must connect naturally to the working methods of children and naturally, or with some stimulation, lead to schematising and abbreviation of pure arithmetic and to the generalisation of the applicability thereof. In short, model situations with corresponding models must be sought of a kind where the children who work on them and with them will themselves indicate and negotiate the learning route to formal arithmetic, be it under the proper guidance of the teacher and the group (figure 7).

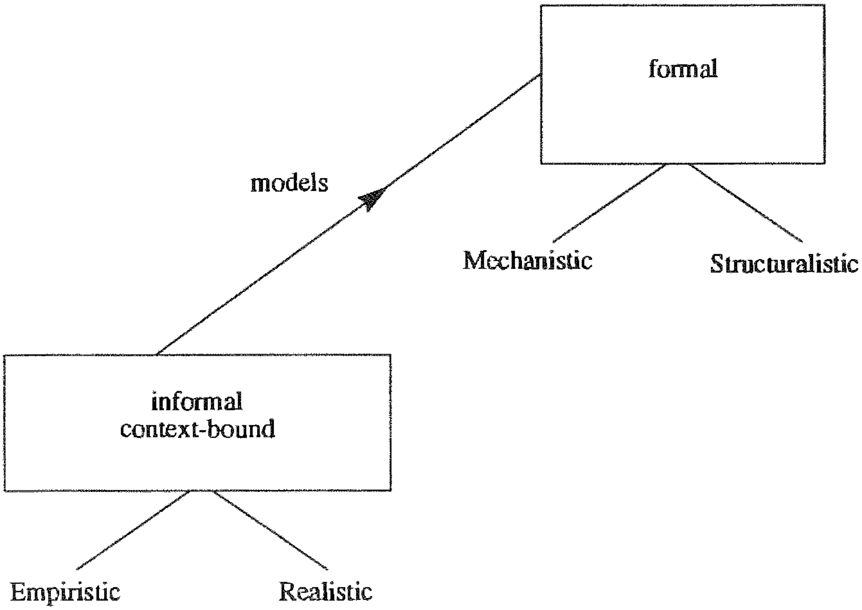


figure 7: four directions

4 examples of realistic learning strands with an emphasis on vertical mathematising

Described in the following are a number of examples of learning strands in primary school which fit into the outlined realistic structure.
These are:

1 counting
2 memorising of addition and subtraction tables to ten and twenty
3 addition and subtraction to one hundred
4 multiplication and division tables
5 mental arithmetic
6 column arithmetic
7 ratio
8 fractions

Where necessary and illustrative, a few remarks will be made about alternative courses of a mechanistic, empiristic and structuralistic nature, notably also there where this can throw light on the genesis of the particular learning strands. Can realists substantiate their educational claim, namely that, starting from the informal, context-bound level, children can in a constructive manner be brought to the formal level with the aid of suitable model situations, models, schemes and symbols? – that is the question at issue.

4.1 counting

The realistic approach to counting which will be described first is a reaction to the structuralistic concept in which counting is forced to the background in favour of practising Piaget phenomena, such as corresponding, seriating, classifying and conserving. In short, the development of the number concept by way of practising logical forms of reasoning at the expense of counting activities.
There are various forms and functions of counting, namely:

- acoustic counting, reciting counting rows;
- (a)synchronous counting, counting in accompaniment of (rhythmic) movements;
- resultative counting, counting amounts or else determining amounts;
- abbreviated counting on the basis of taking small amounts or larger structured amounts, also in application situations with ordered and unordered sets of objects, and with visible or partly invisible objects.

Elementary arithmetic is founded on counting and the accompanying movement, in short synchronous counting. Often this is done via the one-at-a-time touching of objects to be counted, initially that is. And often this fails initially – so to speak. Children recite the counting row incorrectly, do not count synchronously, do not count everything, or count things twice, keep starting all over after the 'how many' question, etc.

In short, in those cases the children are not yet able to count resultatively. The ability to count acoustically and synchronously are conditional for resultative counting. But that does not mean to say that if children have mastered these forms of counting they can then also determine amounts; they will not necessarily have a concept of numbers.

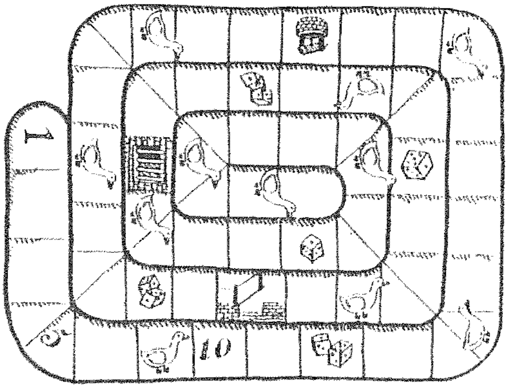
Numbers can refer to a name-number (sandwich number one) or to a number of measure (I am four years old) or amounts (I have three sandwiches) – initially children have difficulty understanding these aspects of number.

The question we are asking ourselves here is directed at one aspect of this, namely the how many-number: How can we help promote acoustic, synchronous and resultative counting in instruction to young (five to six year old) children? The dilemma is evident: the how many-question has no meaning if the child has no understanding of number, but one will get no where by not asking the question. Or to be more precise, then one will have to wait until the child has developed more and appears to be ready for it at a certain point. That is one extreme. The other lies in directed practising of the skills of the counting forms mentioned above. Maturing versus regulating.

There is a third educational path however and that is playing the game of counting and doing arithmetic by 'rolling the dice'. The route of the game of goose, ludo, parcheesi, etc.

Each subskill is not taught separately, but everything at once, implicitly and in a meaningful situation. You are told off if you do not count the number of dots on the dice correctly, do not make synchronous moves, or skip a step, because that would be unfair. The corrections take place in the natural game-situation. Imitation soon leads to marked progress. The number of how many is not just asked out of the blue, but the number thrown serves to 'get ahead' by way of synchronous counting-moves. And all of this under the guise of 'playing a fair game'. But there is more going on... Who is ahead? Who is behind? First the dots of the throw are counted, later the children see at a glance how many they have thrown.

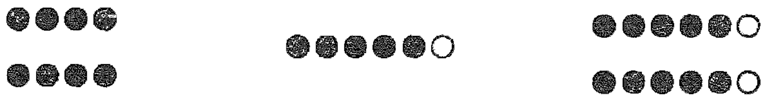
And if two dice are used there are again other phases that can be distinguished. First counting the separate numbers one after the other, and later determining the sum of both and counting them out one by one.



At a given moment the children know many or even all of the possible number combinations by heart. They then already have several splits and additions up to twelve on hand. Just see what they can learn at play and so unemphatically by playing with two dice. One can proceed even further and make up riddles: 'Someone threw seven, you can see one die with so many dots, how many dots were thrown with the other die?'

'seven'  ?

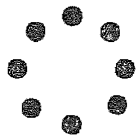
Instead of dice, dominoes can also be used in games. Or cards which feature numbers: doubles and five-structures. These cards must be taken from a stack in turns.



eight six twelve

But now we are anticipating on working with five-structures and doubling via string of beads and arithmetic rack to practice addition and subtraction up to ten, or even easy and fast arithmetic and the practising of arithmetic to twenty.

So what is so new about these old games? And why are such games necessarily better than other counting exercises?
This question can best be answered with a dot card on which tokens are placed in a circle.



If children are asked how many tokens there are, some young children will count them a-synchronously or keep on counting, precisely because they have no notion of number. But in the situation of the game of goose or the numbers race this will often become much clearer to the child. The mentioned 'miscounts' will not occur so easily, because that would be 'unfair'. And if they are made anyhow, they will be corrected in a very meaningful and natural manner. Imitation will soon lead to adequate action which will benefit the child now and later in resultative counting. In this manner counting and determining amounts have a clear meaning and function for the child.

4.2 automatising and memorising of addition and subtraction up to twenty

Working with dice has helped many children automatise and even memorise ever so many elementary additions, subtractions and 'split-ups' to twelve. And dot charts with structured patterns of doubles and with the five-structure have contributed as well. Expansion and completion of that process of automatising and memorising to twenty demands a new model situation however – but more about that further on.

Generally the instruction rule of splitting up is applied when ten is exceeded. According to this rule no informal working methods are allowed.

An instruction observation of such a case:

T: 'How much is 6 + 6?'
C: '12.'
T: 'Are you sure?'
C: 'Yes.'
T: 'Try to do it by way of ten.'
'6 + . = 10.'
C: '6 + 4 = 10.'
T: 'How many more do you need?'
C: '2.'
T: 'Why?'
C: 'Because 6 + 6 = 12.'

It appears that children employ the following informal working methods in counting:
- doubling (6 + 6 = 12), almost doubling (6 + 7 = 12 + 1 = 13)
- working with the five-structure (6 + 7 = 5 + 1 + 5 + 2 = 10 + 3 = 13)
- dividing at ten (9 + 7 = 10 + 6 = 16)
- counting on (9 + 2 = 9 + 1 + 1 = 11)
- dividing fairly (6 + 8 = 7 + 7 =14)
- and combinations.

The variation for subtraction is even greater because there two basic strategies are used, namely of counting back and adding to.

But as mentioned earlier, contemporary education does not connect to these informal solution methods. And for weaker pupils it is often observed that they continue to count because they do not know how to split up numbers to ten by heart, or because their working memory does not function properly, or because they feel insecure with abbreviated methods of counting.

Realistic didactics is directed at connecting to the informal working methods of the children. But not only that: it is also attempted to make these arithmetic methods accessible to weaker pupils who persist in counting (and partly due to this will irrevocably stay behind in advanced arithmetic). The models that are chosen:
- a string of beads with a five-structure for arithmetic to ten (and also to twenty): an ordinal model;
- the arithmetic rack with five-structure for arithmetic to ten and especially twenty: a cardinal model.

In the following we will give examples of how these models can be used.
To begin with the string of beads and the representations thereof for numbers to ten.

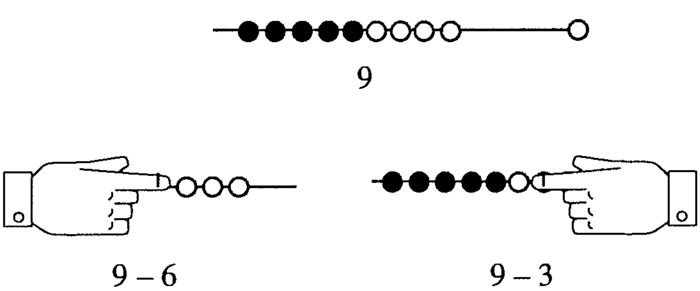


figure 8

Illustrations (1) are used, but (2) it is also demonstrated that it is permitted to remove beads from either side and that the result remains the same or is conserved. The splitting up of the numbers to ten (important for future arithmetic – especially for subtraction) can be learned relatively easily in this manner – counting is not suppressed, but mastered. The arithmetic rack can also be used for arithmetic to ten (see figure 9).

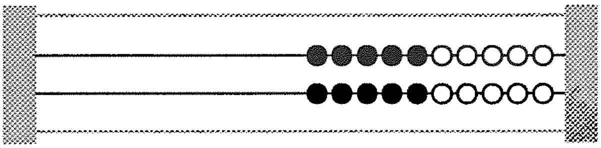


figure 9

Doubles as well as five-structures can be employed. Two examples are illustrated in figure 10.

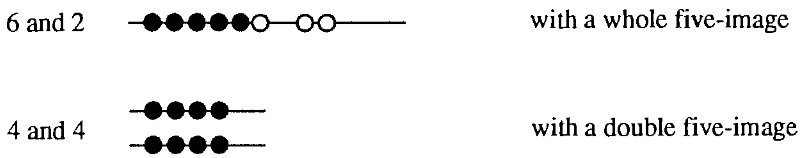
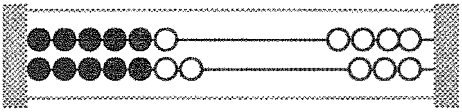
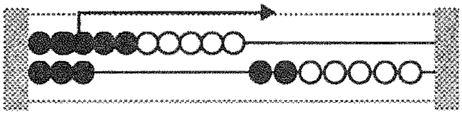


figure 10

In short, an ordinal and a cardinal model can also be used for addition, subtraction and division up to ten, or else the string of beads with the five-structure and the arithmetic rack with five and ten structure.
The arithmetic rack is especially effective for addition and subtraction. Figure 11 shows two examples.



6 + 7 on the arithmetic rack



13 – 8 on the arithmetic rack

figure 11

This is not the place to elaborate on the arithmetic rack and to show examples of how it can be used as a double-decker bus, or how number pictures are put into words (6, I see 5 and 1) and later used in notation. What does deserve mention is the informal claim, namely that it is possible to utilise the informal working methods of the children (doubling, five-structure, splitting up at the ten,...) to arrive at formal arithmetic, in this case from efficient and fast arithmetic to memorising. And especially that a model can give learning support here, namely the model of the double number line.

Keywords: context problems, counting, 'two-sided' subtracting, exploiting informal strategies, cardinal and ordinal methods, no strict rules, no suppression of counting, no laissez-faire in regard to memorising, no premature emphasis on the place value system. Of course realism could be stressed even more strongly by involving context problems (double-deckers) and development of arrow language and own productions and such. But precisely taking a rather sparse subject such as the memorising of elementary additions and subtractions up to twenty and showing on the basis thereof that the realistic method also works here, may convince.

4.3 addition and subtraction to a hundred

Current instruction also employs strict rules for a problem like 87 – 39.
If the numbers are placed one after the other, this is done as follows:

$$87 \xrightarrow{-30} 57 \xrightarrow{-9} 48$$

And written down in columns:

$$\begin{array}{r} {}^{7}\cancel{8}7 \\ 39\,- \\ \hline 48 \end{array}$$

(There are also other standard methods for arithmetic in one line; we have only given this one example).

One method is prescribed for each case – supported by position material or not. Studies show that children often solve subtraction problems very differently.

A book has 64 pages.
I have read 37 pages.
How many pages are left to read?

Children of about eight years old appear to use scores of different methods – one per child of course, but with a great variety per group. In realistic didactics this diversity of solution methods is utilised to induce children to employ abbreviated, efficient arithmetic or mental arithmetic. Place value material is used for that purpose, among which money, but also the so called empty number line (figure 12).

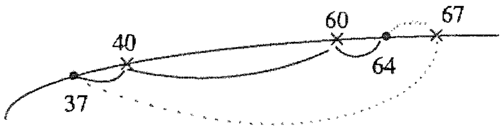


figure 12

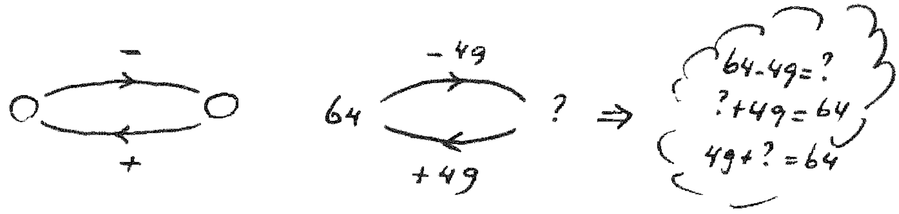
Long or shorter arithmetic methods can be illustrated on this empty number line. For example: counting, abbreviated counting via 37 (+3, +20, +4 or via efficient arithmetic +30 -3.) The model can also be used to further shorten the arithmetic. In doing column arithmetic realists do not aim directly for written arithmetic. That is to say that the following informal solutions also pass the review as predecessors to standard column arithmetic:

$$\begin{array}{c} 64 \\ 37 - \\ \hline 34 - 7 = 27 \end{array} \quad \text{or} \quad \begin{array}{c} 64 \\ 37 \\ \hline 30 - 3 = 27 \end{array} \quad \text{or} \quad \begin{array}{c} 64 = 50 + 14 \\ 37 = 30 + 7 \\ \hline 20 + 7 = 27 \end{array}$$

One and other is supported with positional material and all sorts of other methods (for example the bridging of the distance between 37 and 64 with the empty number line). Therefore both this ordinal as well as the cardinal aspect of the number is provided.

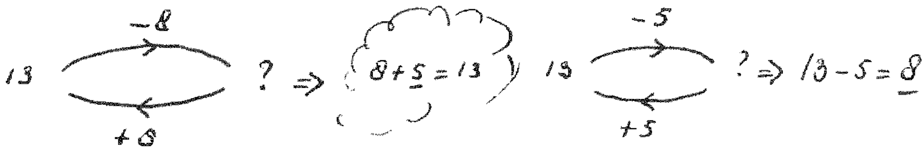
In the realistic viewpoint addition and subtraction to a hundred is therefore not (immediately) algorithmised, but there is room for all sorts of varied strategies of efficient (mental) arithmetic. First of all, to follow up on counting, the arithmetic is ordinal, and somewhat later cardinal, although not the standard algorithm. Notably the ordinal manner is of great importance for flexible mental arithmetic later on.

In conclusion a remark about the difference in level which pertains directly to the preceding learning strands and also to the following. This concerns the relation between the operations of addition and subtraction (and later between multiplication and division). Pupils who can picture the following scheme are capable of modifying their arithmetic strategies, notably in regard to subtraction, to the particulars of the numbers.

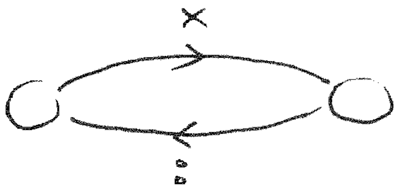


64 – 49 is at this formal level solved by adding to, that is to say the pupil that works at this level is capable of doing this and will probably apply that strategy for 105 – 88. However, for 64 – 15 a different strategy, namely the one of counting back, will be followed.

One and other has marked consequences for automatising and memorising. Again here pupils, working at the formal level, will use corresponding subtraction or addition strategies. For 13 – 8 probably adding, for 13 – 5 subtraction.



And that can facilitate memorisation. The same applies to the multiplication and division tables – about which more in the following.



4.4 learning the tables

In general terms there are two methods by which to learn tables: reproduction and reconstruction didactics. Reproduction didactics is first and foremost directed at being able to reproduce the tables which are dealt with successively. The method that is followed is the same for each table. Starting with repeated addition and thereafter connecting the multiplication sign (figure 13).

2	1 × 2 = 2
2 + 2 =	2 × 2 = 4
2 + 2 + 2 =	3 × 2 = 6
2 + 2 + 2 + 2 =	4 × 2 = 8
2 + 2 + 2 + 2 + 2 =	5 × 2 = 10
2 + 2 + 2 + 2 + 2 + 2 =	6 × 2 = 12
2 + 2 + 2 + 2 + 2 + 2 + 2 =	7 × 2 = 14
2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 =	8 × 2 = 16
2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 =	9 × 2 = 18
2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 =	10 × 2 = 20

figure 13

Practising the tables occurs through self-instruction, with games. The knowledge is remembered by way of repeated practice of the preceding tables. Informal strategies of efficient arithmetic are not stimulated because these are not considered as beneficial to direct drill. Actual applications will at best be presented later on in the learning-instruction process.

In contrast to didactics from the mechanistic direction stands the reconstruction didactics, which is supported by the realists. This method does not aim solely and directly for reproduction of memorised knowledge, but tries also to achieve this objective through a process of reconstruction, of knowledge building via skilled arithmetic, connecting to the informal working methods of children and in support thereof with suitable models.

In figure 14 the most common methods are indicated for the table of seven, departing from the question of how many days there are in so many weeks.

1 × 7	you know
2 × 7	you soon know from 7 + 7
3 × 7	via (2 + 7) + 7, one time more
4 × 7	double 2 × 7
5 × 7	half of 10 × 7, half of 70
6 × 7	via (5 × 7) + 7, you already know
7 × 7	varied, one you will soon know
8 × 7	(7 × 7) + 7, most difficult in all of the tables
9 × 7	(10 × 7) – 7, one time less
10 × 7	you know
12 × 7	a problem to examine

figure 14: how many days in so many weeks?

Figure 15 shows the two models that are best suited to support the learning of the tables: the double number line and the rectangle model. Both are 'models of' the context situation which will later serve as 'models for' multiplication in general. The double number line here is a representation of the time line, and the rectangle model a representation of a kind of calendar.

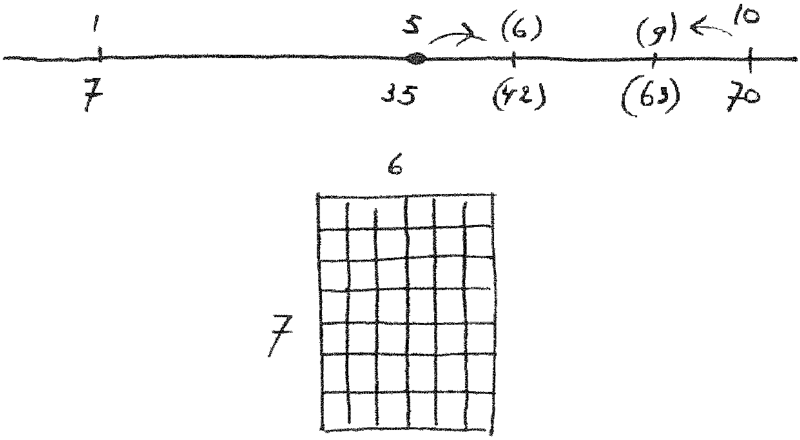
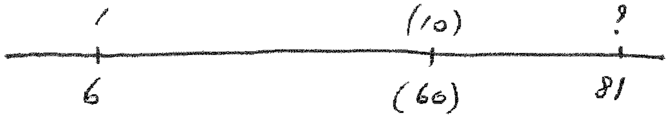
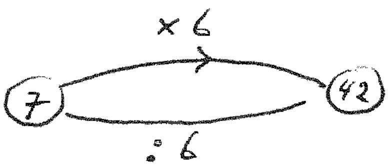


figure 15: double number line and rectangle model

The commutative and distributive properties are clearly illustrated from the rectangle model ($6 \times 7 = 7 \times 6$ and $6 \times 7 = 5 \times 7 + 1 \times 7$). On the double number line the aspect of ratio is clearly expressed as are all relevant properties, with the exception of commutativity. Notably also the relationship between multiplication and division which we referred to earlier, is clearly visible, something of great importance for learning to apply the tables of multiplication in division sums. Consider in this case the problem about the PTA meeting at the start of this article.



To make pupils conscious of the relationship between multiplication and division we can have pupils make free productions using the basic diagram.



Then there is yet another type of problem that is important, namely the question in which tables a certain number (say 24) appears.

At the end of this learning phase there must be practice and drill of the tables. This can best be achieved through short oral sessions with table dictation: the teacher reads the questions and the pupils write down the answer – the work is done at speed. For the free productions we can easily ask the pupils to write down easy and difficult sums, or else sums they know by heart, sums they can calculate quickly or sums they still have difficulty with – reflection on one's own learning. Besides the practice and drill with pure arithmetic sums, application sums for both multiplication and division are also given. For multiplication for instance, sums about wrapping things up and for division problems about both distribution and ratio. For one and other the ten can also be exceeded as was the case for the PTA meeting ... and for the problem directly below about mental arithmetic.

4.5 mental arithmetic and column arithmetic

Mental arithmetic is considered here as doing arithmetic mentally, therefore in contrast to column arithmetic, and not versus written arithmetic which can apply as the mechanistic interpretation of mental arithmetic. Various forms of mental arithmetic are distinguished, namely:

- estimating mental arithmetic;
- varied mental arithmetic;
- standard mental arithmetic.

We will illustrate each on the basis of four different solutions of one open arithmetic sum in the form of a newspaper clipping (figure 16).

THE POLISH KNOWN AS HARD WORKERS
(from our reporter)

Every year tens of thousands of Poles come to The Netherlands to work in the flower bulb industry for a few months.
(...)
This is the fourth time for Zygmunt. He has worked both in the fields and in greenhouses. Currently he is working in the transportation department of a company at the flower auction. "I load up the trucks, that is heavy work. On average I work 220 hours per week. That's no problem, because that's the way to make money" says Zygmunt.

figure 16: a newspaper clipping

- Estimation – see figure 17

Is that possible, a 220 hour work week?

Write down why you think yes or no?

Nee, want 10 werkdagen
is al 240 uur, dus
220 werkuren zou
dan 9 dagen +4 uur
zijn

No, because 10 working days would already be 240 hours,
so you would already need 9 days + 4 hours for 220 working hours.

figure 17: estimating 10×24

Arithmetic is done by estimation, or at least an impulse is given in that direction on the basis of an arithmetic fact that the pupil comes up with by himself (one day is 24 hours). This kind of problem is not found in the mechanistic nor in the structuralistic direction: little or no estimation, rounding off and approximation is found there. And problems in which information must be provided or used are lacking entirely.

- Varied mental arithmetic – see figure 18.

Is that possible, a 220 hour work week?

Write down why you think yes or no?

nee

een dag heeft 24 uur dan
doe ik 25×7= 175 uur−7=
168 uur per week

No, a day has 24 hours so I take
$25 \times 7 = 175$ hours $- 7 = 168$ hours per week

figure 18: 7×24 calculated from 7×25

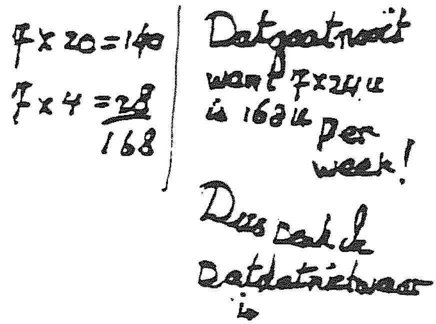
You can easily figure out that 7 quarters is 175 cents and with the distribution property 7×24 is calculated as: $175 - 7 = 168$.

- Standard mental arithmetic – see figure 19.

Here the standard method is followed in a fixed procedure. This method of arithmetic is related to column arithmetic that is also executed according to a standard procedure. Often this stylised mental arithmetic is pushed aside by the algorithm that in terms of procedure is more efficient and can partially be done mentally as well.

Is that possible, a 220 hour work week?

Write down why you think yes or no?



That's impossible because
7×24 hours is 168 hours per week.
So I think it is not true.

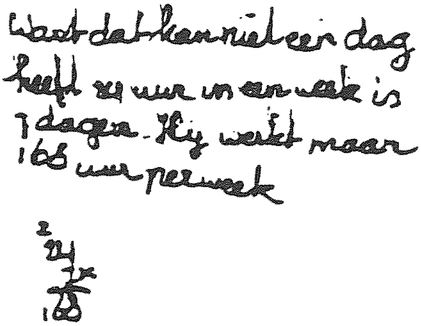
figure 19: 7×24 via $7 \times 20 + 7 \times 4$

4.6 column arithmetic

In mental arithmetic – see figure 20 – the numbers 7 and 24 are placed under each other, and the standard procedure is carried out mentally, the calculation is made in the head. It can be considered as an abbreviation of the previously described form of standard mental arithmetic. The ability to carry out this calculation also forms the basis for column multiplication as a whole.

Is that possible, a 220 hour work week?

Write down why you think yes or no?



That's impossible a day has 24 hours and a week is 7 days. He only works 168 hours per week.

figure 20: column multiplication

Characteristic for column arithmetic is that here the arithmetic is done with the individual digits, while with mental arithmetic the numbers that are operated with retain their own 'value'. The learning of the algorithm for column multiplication occurs in the same manner as outlined earlier for division, namely through progressive schematisation and abbreviation, departing from a context problem, for instance about how many hours there are in a week, or later in a month or a year. In the last case it is essential that pupils have mastered the zero rule. ($10 \times 24 = 240$; $100 \times 24 = 2400$ etc.). If such is the case then it is in principle no more difficult to calculate 30×24 than 7×24. And although 365×24 is more complicated it is in principle not more difficult.

In closing, a remark about the relationship between mental and column arithmetic in a realistic mathematics program. In grades 1,2 and 3 there is no room for the standard algorithm. Mental arithmetic must be developed first, according to the realistic idea. If the algorithms are introduced in grades 2 or 3 mental arithmetic does not stand a chance, certainly not for weaker pupils, and arithmetic threatens to deteriorate to blind manipulation with numerical symbols – this at the expense of both pure arithmetic as well as the ability to apply it. Mechanists and structuralists assign a more prominent position to the algorithms in respect to mental arithmetic than realists do, and empiricists on the other hand leave out the standard procedures altogether.

4.7 ratio

The most characteristic feature of the realistic approach of ratio is that it does not steer directly to the so called rule of three in working out the fourth term ($a : b = c : ? \rightarrow ? = \frac{bc}{a}$).

First, ratios are approached visually and connected to the visual world. Scale drawings, models, in short all kinds of representations of ratio are used to present problems. In first instance the point of concern is solely visual comparisons without numbers (see figure 22).

This picture is from 'Alice in Wonderland'. In the book Alice's size changes constantly.

a. Estimate Alice's length in this picture of her and the dog.

Alice was so small in comparison to the dog that it looked as if she was playing with a cart-horse and was afraid of being trampled under foot any minute.

b. Is this a correct comparison between man and cart-horse?

figure 21: Alice in Wonderland

If numbers are used in ratios then the double number line can help in the solving of four categories of ratio problems (according to Van den Heuvel-Panhuizen), namely:

- determining the ratio relationship ? : ?
- comparing equivalent ratios: $a : b$? $c : d$
- making equivalent ratios $a : b = ? : ?$
- determining the fourth proportional $a : b = c : ?$

The double number line is in principle presented empty, so that the pupils can fill it in themselves, first extensively, later only marginally because abbreviated methods are employed. In other words: the model of the empty double number line functions both close to the informal, context-restricted level as well as later close to formal, subject restricted arithmetic; it can be utilised more concretely and also more abstractly. In short, gradual algorithmising can be brought about in this manner.

How this is achieved exactly can be surmised from the following problem:

'a wheel covers a distance of 5 meters in 4 revolutions; how many meters will the wheel cover in 30 revolutions?'

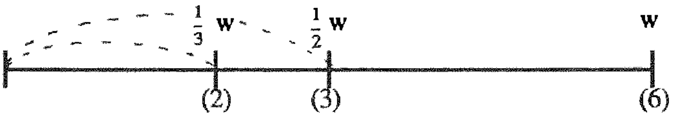
The synchronous 'rolling out' of the tables of 4 and 5, step-by-step is one possibility; or abbreviated calculation via 7 times 4 and 5, and via 8 times 4 and 5; or directly via 7 times 4 and 5, or by determining the relationship between the numbers above and below as $\frac{5}{4}x$; or by returning to 1 at the top and then making 30 steps; or ... a clear progression to the application of the rule-of-three has been revealed. With the empty double number line working with the ratio table and 'function machines' can be connected.

All in all what we see here once more is a gradual algorithmising on the basis of a strong and versatile model – first a 'model of' later a 'model for' abbreviations in the form of the empty double number line. This model is of course also suitable for the standardised ratio-measure of percentage.

4.8 fractions

At a certain point the double number line can also be employed for operations with fractions. First for fractions with concrete numbers and magnitudes, which is also expressed in the notation: $\frac{1}{2}$m, $\frac{1}{2}$d, $\frac{3}{4}$b, $\frac{2}{3}$p, in which the letters stand for metre, day, bar and price respectively. The comparison with fractions can now take place by switching to a different unit of measure. For example $\frac{2}{3}$h is smaller than $\frac{7}{10}$h (because 40 minutes is less than 42 minutes). One and other can be depicted on the double number line.

The empty number line can also be used for the addition and subtraction of fractions. For example: $\frac{1}{2}$w + $\frac{1}{3}$w and $\frac{1}{2}$w – $\frac{1}{3}$w.



Choose a fitting length for the line so that $\frac{1}{2}$w and $\frac{1}{3}$w 'work out' nicely, or in other words, divide w in a fitting number of equal parts – in this case six. If the calculation of the fractions is made above the line, the result can, as it were, be read from below the line.

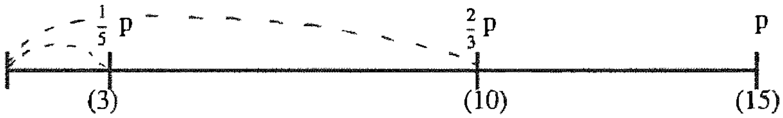
$\frac{1}{2}w + \frac{1}{3}w = \dots$

(3) + (2) = (5), whereby (1) later becomes $\frac{1}{6}w$

$\frac{1}{2}w - \frac{1}{3}w = \dots$

(3) − (2) = (1), (1) is $\frac{1}{6}w$

A much more difficult case is:
$\frac{2}{3}p + \frac{1}{5}p$ or $\frac{2}{3}p - \frac{1}{5}p$
Choose a fitting division of p below the line so that $\frac{2}{3}p$ and $\frac{1}{5}p$ work out nicely – therefore 15 parts, whereby (1) is therefore $\frac{1}{15}p$.



$\frac{2}{3}p + \frac{1}{5}p = \dots$

(10) + (3) = (13), (1) = $\frac{1}{15}p$

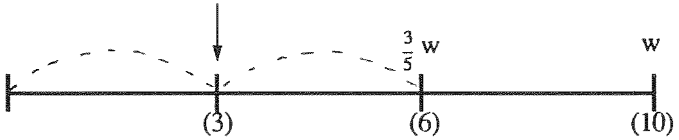
$\frac{2}{3}p - \frac{1}{5}p = \dots$

(10) − (3) = (7), (1) = $\frac{1}{15}p$

We have progressed rather far into the course, because first natural magnitudes are practised, therefore with metres, kilogrammes, hours, days, dozen etc., where the conversion is predetermined. Only later do we work with boxes into which changing amounts are put, prices which can vary, roads with different lengths, etc.

As mentioned, fractions are initially given as concrete numbers. The intention thereof is partly to make the specific distinction between the operations, notably between addition and multiplication.

Initially multiplication is not mentioned and written as such, but is introduced as: $\frac{1}{2}$ part of $\frac{3}{5}$m, or $\frac{1}{4}$ part of $\frac{2}{5}$h. Only later in sums like $2\frac{1}{2}$ times $\frac{3}{5}$m, or $2\frac{1}{2} \times \frac{3}{5}$m is it made explicit that $\frac{1}{2}$ part of $\frac{3}{5}$m is therefore $\frac{1}{2} \times \frac{3}{5}$m, but by then we have already progressed much further. (Note the distinction in addition, not $\frac{1}{2}\text{m} \times \frac{3}{5}\text{m}$ but $\frac{1}{2} \times \frac{3}{5}$m, the first number is an operator!). Let us start at the beginning, or better yet, slightly further on than the outlined start with natural magnitudes: $\frac{1}{2}$ part of $\frac{3}{5}$w. Find a division that fits for w so that $\frac{3}{5}$w and then $\frac{1}{2}$ part work out nicely – therefore ten.



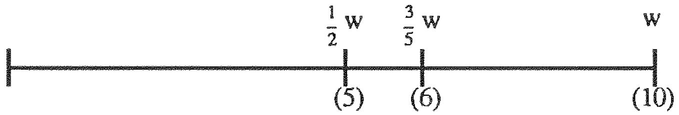
We now take $\frac{3}{5}$w and then $\frac{1}{2}$ part – determine under the line where we are, via (6) we have ended up at (3). Therefore $\frac{1}{2}$ part of $\frac{3}{5}\text{w} = \frac{3}{10}\text{w}$.

Because (1) is $\frac{1}{10}$w

$\frac{1}{2}$ part of $\frac{3}{5}$w = ...

$\frac{1}{2}$ part of (6) = (3) , $(1) = \frac{1}{10}\text{w}$

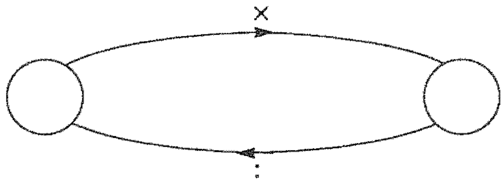
In closing, one more remark about division of fractions.



$\frac{1}{2}\text{w} : \frac{3}{5}\text{w} = ...$

$(5) : (6) = \frac{5}{6}$

Proportional division can be made visible in this manner. Generally speaking however, by the time pupils are introduced to division with fractions, they will already have reached such a level of formal, subject restricted operation that this operation can also best be taught at this level, departing from the basic relationship between multiplication and division we have mentioned earlier.



This illustrates how much realistic didactics can do justice to changing levels of concreteness, indeed justice must be done if one means to provide vertical mathematising. This will not happen until the concrete foundation has been laid and sufficient models have been presented to bridge the distance to the formal level, whereby again here the empty double number line fulfilled such an important bridging function.

5 an example of realistic mathematics education at classroom level with an emphasis on horizontal mathematising

The description of the learning strands was not complete and was placed at fair distance from teaching practice. The subject of measurement for instance, was hardly addressed at all. For this reason the description of a number of lessons will be devoted to this subject – especially to area. The starting problem for pupils of grade six of primary school is the following:

> So and so says that he has read in the Larousse encyclopaedia that the area of The Netherlands is 36,842 square meters. Do you think this is true?

5.1 lesson 1

The pupils go to work. The teacher goes around, offering assistance to groups of pupils and heads the discussion afterwards.

First some fragments from the brief discussion the teacher has at the start of the lesson with Margaret.

Mar: Then I would first have to know what a square meter is. I know that a football field is one hectare. A square meter might be half of that.
Teach: How tall are you Margaret?
Mar: One meter seventy
Teach: And now one square meter. Explain the word 'square'.
Mar: Oh, that is four times one meter (draws a square in the air).
Teach: Is this desk about one square meter? (It measures 1.30 m by 1.70 m).
Mar: No, it isn't square, so it is not a square meter.

Further explanation follows. Margaret makes steady progress, but new obstacles keep popping up. For example, when the area of The Netherlands must be determined via a rectangular model of 200 km by 300 km. By the way, on the basis of the excellent estimation of 200 by 300 that is made by Margaret herself! Margaret calculates 200×300 as column arithmetic 'the one under the other'! These two consecutive exclamation marks indicate the extremes between which the mathematical activities of Margaret lie: considered estimation and automatised arithmetic.

There are however also pupils who are well aware of the size of a square meter and who also have a fair concept of area, but who lack the mathematical attitude, departing from 36,842 square meters, to try some multiplications, or inversely on the basis of available experience information about the size of The Netherlands, to try to make their own estimate of the area. In passing the teacher hears: 'that number is so big that you can't picture it, so you can't say whether it's right or not'. A small clue just to try something – the area of a garden or something like that – soon puts them on the right track. After some time everyone is focused on the problem whether 36,842 square meters is the possible area of the Netherlands. Below a brief summary of the discussion which followed afterwards.

- If it is right, hundreds of people would have to live on one square meter because millions of people would live on those 36,842 square meters and that is impossible.
- 36,842 square meters is something like a strip of 36 kilometres long and 1 meter wide, a path in Holland.
- A rectangle of 200 meters by 180 meters, say two football fields, gives that area of 36,842 square meters.
- The Netherlands has the approximate area of a rectangle of 200 km by 300 km (among other Margaret's estimate), so those 36,842 square meters (note: meters) can never be right.

All of these comments are discussed. Almost everyone can grasp the various arguments and weigh them against the own found solution. The last solution mentioned is cause for the teacher to ask for the exact origin of the error. The whole group agrees that the answer is not in square meters but in square kilometres.

5.2 lesson 2

The teacher returns to the problem: How is it possible to arrive at such an exact answer as those 36,842 square kilometres. The thoughts on this are inventoried in a class discussion. What do you do with rivers, lakes, hills? Are they part of the area of 36,842 square kilometres? And what about high and low tide: is The Netherlands much bigger at low tide than at high tide? How much could that difference be? Shouldn't the area be variable?

Finally the key question: 'Is that possible, such an exact number, if we were to depart from a fixed model of The Netherlands?' A question which is largely answered by the teacher. The 'fixed' low tide line and the 'fixed' map model determine the model calculation. A model that in parts differs from reality of course...

> Write something about the original problem and make it clear to the person who made this statement that it is not possible. Explain what the correct answer is and explain it in greater detail, for example explain about high and low tide.

This assignment for homework concludes the two lessons about the area of The Netherlands.

6 conclusion

One could relate the described series of lessons which illustrates horizontal mathematising or the vertical learning strands to the learning-instruction structure which was outlined in the first part.

One could also investigate how the (empty, double) number line fulfils the bridging function in these courses between the informal context-bound level and the formal arithmetic level.

Or make a detailed analysis of the issues and indicate their function in the learning-instruction structure.

We will do none of this and end as we began, namely with long division (see figure 22). What formalistic instruction can lead to.

$$\begin{array}{r} 4\,/\,36\,\backslash\,18 \\ 4 \\ \hline 32 \\ 32 \\ \hline 0 \end{array} \qquad \begin{array}{r} \text{controle I} \\ 18 \\ 4\ \times \\ \hline 32 \\ 4 \\ \hline 36 \end{array} \qquad \begin{array}{r} \text{controle II} \\ 18 \\ 18 \\ 18 \\ 18 \\ \hline 32 \\ 4\ + \\ \hline 36 \end{array}$$

figure 22: a joke

But seriously, with which other numbers can this joke be applied?
Perhaps a problem that can be translated to instruction... Because a puzzle like this also fits into realistic mathematics education – working at the formal level. But not until such time as when the pupil is ready for it, or can even get that far.

An instruction-theoretical reflection on the use of manipulatives

K.P.E. Gravemeijer

1 introduction

From earliest times the use of manipulative material has played an important part in theories regarding mathematics instruction. In each theory the interpretation of exactly how such manipulatives ought to be used often varies. And each time there is the endeavour to give the theoretical grounds for the use of such manipulative material. Notably cognitive psychology and action theory are focused on an instruction-theoretical foundation, while constructivist psychologists tend to approach the effect of manipulatives from an epistemological point of view. In realistic instruction-theory the use of manipulatives does not hold a very prominent position. Yet here also can we indicate an explicit view of the function of manipulatives. In analysing the various theories we are trying to find an answer to the question of if and how manipulative material can be deployed in a significant manner.

2 action psychology

From action psychology we know Gal'perin's theory of the stepwise formation of mental actions (see Van Parreren & Carpay; 1972). The major difference in regard to the use of manipulatives is the notion of a complete orientation basis, the principle of shortening the action and making the distinction of different parameters in the development of the action. It should moreover be mentioned that for Gal'perin the manipulative action is not necessarily carried out with manipulative material; symbolic representation can also be employed (also referred to as materialised action).
Characteristic for the action psychology is the attention which is devoted to mental activity. Mastering the action is defined as 'internalising'. The aim of the Gal'perin procedure is the forming of well formed mental action. To achieve this it is essential that the manipulative action is isomorphous with the pursued mental action. This precondition gives us a criterion by which to judge the use of manipulatives: is the manipulative action isomorphous with the intended mental activity?

Working with manipulatives does not automatically fulfil this requirement. In developing the textbook series 'Rekenen & Wiskunde' we analysed the use of the abacus as a concrete preparation for column addition and subtraction. A discrepancy appeared to exist between the manipulative action and the intended mental action.

figure 1: 684 on an abacus with 2 x 10 beads per rod

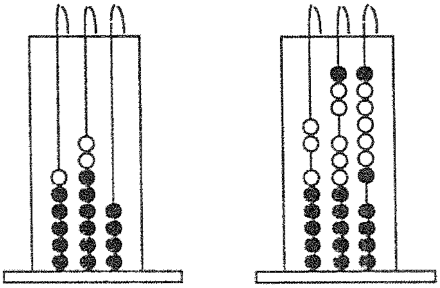


figure 2: 684 and 684 + 237 respectively on an abacus with 4 x 5 beads per rod

Because of the large number of similarly coloured beads on each rod the pupils have to repeatedly count the beads one by one (fig. 1). In consequence the basic facts for addition and subtraction under twenty are not used. However, the written algorithm, for which the abacus is a preparation, rests entirely on the use of this basic automatism. There are no beads left to count. The manipulative action was therefore not isomorphous with the intended mental action. To overcome this problem we divided the beads on the abacus into groups of five.

This allowed the pupils to 'read off' the numbers and set them up without counting (fig. 2). Practice however proved that again here the pupils developed strategies that were specific to the device (Van Galen, no year). The basic facts were again not used, but now the quinary structure was employed to facilitate calculations.

If the pupil has four beads and needs to add seven, he sees that he needs one more of the same colour, then a group of five of the other colour – that already makes six – so that only one more bead is needed to make seven (see fig. 3).

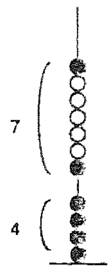


figure 3: 4 + 7 as 4 + [1 + 5 + 1]

No complete isomorphism was therefore reached between the abacus action and final action. It is for this reason that the period of time in which the abacus is used in 'Rekenen & Wiskunde' has been kept short. The abacus serves primarily as a thinking model. The use of the abacus as a working model might accustom the child to action structures which do not correspond with the mental action that must be conducted when doing written arithmetic. To make this clearer we will first briefly explain the concepts 'thinking model' and 'working model'. In speaking about a working model we refer to a model that can be used to arrive at the answer to a calculation. In this sense manipulative material often serves as a working model. Pupils use the material to find the answer to a calculation, but that does not mean to say that they are aware of what they are doing. The material serves as a primitive calculator as it were. Notably the use of MAB-materials can work this way, as is evidenced by the observation of Resnick & Omanson (1987) for instance, that pupils did not find it obvious that the calculation with blocks and the calculation on paper would produce the same answer. Manipulation with material is sometimes a rather insignificant procedure.

The problem of the difference between the material (or materialised) action and the mental action is a problem that arises in more places. We also encountered this problem when using the number line as an aid in support of addition and subtraction under twenty. When adding 5 + 4 the pupils were able to find five on the number line, count four positions further, and read out the answer. The problem however is that on the number line the pupils count out 'one, two, three, four' and read out 'nine' while if they have to solve the problem mentally they have to count 'six, seven, eight, nine' and at the same time keep up with the steps (6 → 1; 7 → 2; 8 → 3; 9 → 4;). This is an entirely different action!

Through analyses like these action psychology can help us to considerably improve the effectiveness of working with manipulative material. We are however left with another problem, the problem that is as it were illuminated by action psychology itself: what exactly is the isomorphism between the material action and the (full) mental action? Should we imagine that in his or her mind the pupil is manipulating with concrete material? That would not seem very efficient nor very flexible mental action. In an application situation the pupils must first translate the problem to the manipulative material, then carry out the operation in their mind with this concrete material, finally to interpret the solution back to the original context.

In his 'building block model' Van Parreren (1981) shows that something other than strict isomorphism between the actions at the beginning and at the end of a learning process is possible. He sees different actions as building blocks, as separate entities, which can be used in various 'constructions'. The integration of a number of sub actions into a new action which can be called upon as one, he calls a short cut. Van Parreren distinguishes three types of short cuts:

- the forming of perceptive actions;
- the automation of motor skills;
- the restructuring of a task.

It is this last type of short cut that we are interested in. Restructuring means that in the course of the learning process the pupil switches from the one to the other action. At a certain moment the pupil discovers that you can replace one action (re-counting for example) by another action that gives the same result (counting on, for example, the use of a property, or of a memorised fact), and the pupil dares to trust in this at a certain stage. Thus Van Parreren gives further substance to the idea of the short cut of the action as we know this from Gal'perin. And this gives us a different, that is clearer, image of the mental act which is ultimately formed.

With Gal'perin it seems that the pupil keeps on thinking about concrete material. Van Parreren shows us that ultimately the pupil can let go of every reference to the material source. The building block model offers the pupils precisely the possibility to call up and make a complete 'construction' as a whole, without consciously having to execute the various sub actions. The building block model also indicates that a number relationship or an operation with numbers can ultimately be set free of thinking about concrete quantities. This does not however explain how this step is actually achieved.

3 cognitive psychology

It is precisely this problem of transition that American cognitive psychology is running up against. We will elaborate on this in the following, but first let us discuss this cognitive approach.

Cognitive psychology is characterised by the conception that knowledge is stored away in the memory as an organised entity of elements of knowledge; usually indicated as the schema, or as the cognitive structure. Learning is considered as an active process:

- whereby expansion of knowledge generally takes place by fitting in new elements of knowledge into an existing cognitive structure (assimilation);
- but whereby the cognitive structure must sometimes be completely reorganised to make room for the new knowledge (accommodation).

Cognitive psychology describes acquisition of knowledge as information processing. The cognitive structures appear to play an important role in the interpretation of new information, in remembering and recalling information. The cognitive structures of experts and beginners are analysed to be able to give direction to the learning process. It is therefore not surprising that within cognitive psychology there is an important movement which is involved with an advanced form of task analysis (see Schoenfeld (1987) for example). The task analytical approach as we know it from Gagné (1977) has been stripped of its behavioural traits because one no longer stops at making an analysis of the externally perceptible behaviour. However, further refinement of task analysis does in this manner lead to very complex models. Models which it is hoped can be tested notably through computer simulation. Aside from this, one is of course experimenting with this task analytic approach in education.

What has not changed in comparison with the old behavioural task analysis is the top-down strategy that is followed. The pursued action, the expert behaviour, forms the starting point for the analysis. This focus on the pursued action makes that the expert model and the procedures to be learned are so much at the centre of things that the aim towards acting with understanding suffers in consequence. The computer metaphor is so dominant that it seems as if the only question that is being asked is how to get pupils so far that they will exhibit the discovered model behaviour, without asking oneself if the pupils understand what they are doing. Illustrative is the multiplication model that Greeno (1987) used to solve the following problem:

> Dr. Wizard has discovered a group of monsters living in a dark cave in South America. He has counted seven monsters, and there are eight fingers on each monster. If there are four fingers on each monster hand, how many monster hands did he find.

The solution of the problem is outlined by Shalin (Greeno, 1987) in the following manner (see fig. 4)

The focus on the general solution model causes him to overlook the simplest solution: from the number of fingers you deduce that there are twice as many hands as there are monsters, hence $2 \times 7 = 14$ monster hands.

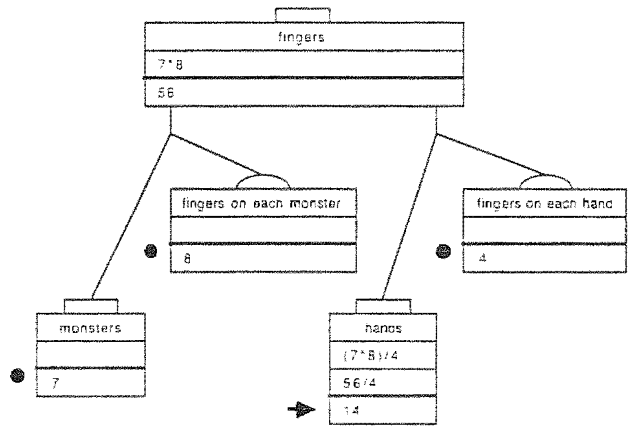


figure 4: Shalin's outline

It is noteworthy that in cognitive psychology the approach to arithmetic is much the same as the one arrived at by Van Erp (1988) and Gal'perin (1989) from action theory. Among other this is evident from the work of Resnick & Omanson (1987).

The study by Resnick & Omanson was directed at the potential cause of the so-called 'buggy algorithms', systematic deviations from the standard algorithm that result in a recognisable pattern of errors. They based themselves on Brown & Van Lehn in this regard, who suggest that the pupil reverts to 'repairing algorithms' when stuck. The repair algorithms can for instance comprise:

> (...) performing the action in a different column, skipping the action, swapping top and bottom numbers in a column, and substituting an operation (such as incrementing for decrementing).
> (Resnick & Omanson; 1987; 45)

The result of actions like these are then assessed by the pupil on the basis of a kind of self-constructed list of criteria which the answer to this type of problem should meet. For example, that there must be something in each column and that no two numbers are allowed in the same column. The supposition of the researchers now is that instruction that is directed at the basic principles which lie behind arithmetic and the applications thereof, solves the problems, or at least diminishes them. In an exploratory study it has been established that the cause of the problems should not be sought in the lack of the prerequisite knowledge of the relevant principles, but in an inadequate connection of these principles to the symbols and the syntax of doing arithmetic on paper. In experimental instruction it must be attempted to bring about this connection by means of so-called 'mapping instruction':

Mapping instruction requires the child to do subtraction problems both with the blocks and in writing, maintaining a step-by-step correspondence between the blocks and written symbols throughout the problem.
(Resnick & Omanson; 1987; 71)

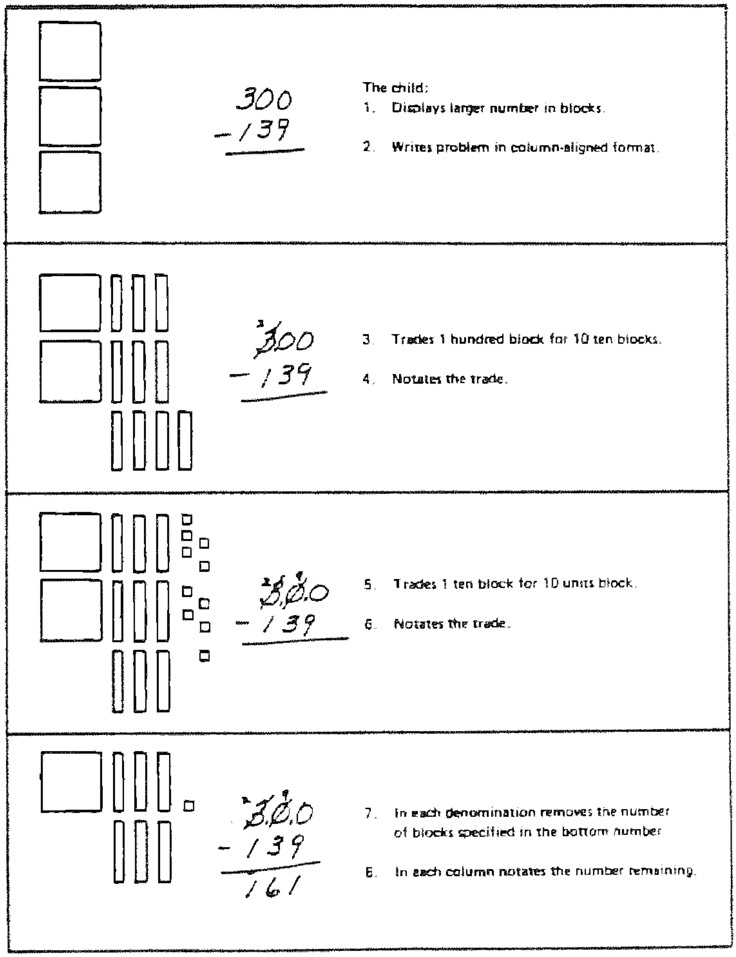


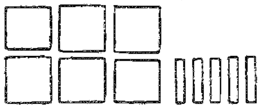
figure 5: mapping instruction (Resnick; 1987)

The blocks referred to are Dienes-blocks with which the calculation can be made concrete. The mathematical relationships are thus embedded in manipulative materials. And the connection between the mathematical principles and doing column arithmetic is in this set up replaced by a connection between working with manipulative material and working on a symbolic level (see fig. 5). Characteristic here is that the blocks must be handled according to rules set by the researchers. The small blocks stand for the units, the bars for the tens and the squares for the hundreds. The compensation principle must be observed when changing the blocks: one bar is changed for ten small blocks and one square for ten bars.

Subtraction is done in columns, from right to left. I.e. that first it is tried to take away the correct number of small blocks, then the bars and finally the squares. The children often become confused. That already starts when determining the number of blocks. (see fig. 6).

b) Jane

E: Good. So how much do you think this would be?



S: (Touching the hundreds blocks) 100, 200, 300, 400, 500, 600 . . . (touching the tens blocks) 700, 800, 900, ten hundred, eleven hundred.
E: Are these (tens) worth 100?
S: I count them all together.
E: But these (tens) aren't hundreds.
S: I am counting these like tens.
E: OK. But how much would these (tens) be worth then?
S: Oh. 10, 20, 30, 40, 50 . . . 50 dollars.
E: How much would this (entire display) be worth altogether?
S: 600 . . . wait! It's 5 and 6.
E: But how much is it altogether? This (hundred) is 6, right?
S: Eleven hundred.

figure 6: examples of difficulties children encounter (Resnick; 1987)

This, in our view, is where the price is paid for the fact that no distinction has been made between that which one counts and the representation of the number. The blocks are both the objects to be counted as well as the representation of the result of that count. As a result the differences between the mathematical concept 'ten' and 'hundred' and the visual representations of them become unclear here. The results of the experimental program are disappointing. Only two of the nine pupils did the borrowing correctly on the test which was given immediately after the instruction. On the basis of (otherwise debatable) data analysis which shows that the nature of the verbal interaction is important, Resnick & Omanson then also arrive at the conclusion that what is needed is a learning process of some other order (page 90):

> Instead of attention to the blocks as such, it seems to be attention to the quantities that are manipulated in both blocks and writing that produces learning.
> (...)Perhaps any discussion of the quantities manipulated in written arithmetic, without any reference to the blocks analog, would be just as successful in teaching the principles that underlie written subtraction.

Nevertheless the conviction remains that working with the blocks is extremely worthwhile:

> We believe, however, that mapping between blocks and writing may play an important role in learning by helping children to develop an abstraction – a higher level of representation – that encompasses both blocks and writing.

Just like Van Erp (1988), Resnick & Omanson share the conviction that the connection between working with manipulatives and doing written arithmetic must bring the solution of the problems. Resnick & Omanson do however realize that raising of level is essential:

> If the analogy between blocks and writing is clear, as it is likely to be when a step-by-step mapping is required, then a condition is created in which it is reasonable to construct a new cognitive entity that is neither blocks nor writing, but could be used to characterize both.

4 criticism on the task-analytical approach

Cobb (1987) directs his criticism on the data processing approach in cognitive psychology precisely at the forming of abstract mathematical objects. According to him this task- analytical approach falls short there.

> (...) the lack of an appropriate explanatory construct to account for the transition from concrete action to abstract, conceptual knowledge such as an objectified part-whole structure is apparent. In lieu of an explanation, it is implied that students *will come to 'see'* various abstract, arithmetical relationships.
> (Cobb; 1987;18)

Notably the arithmetic teaching method of Resnick & Omanson falls short according to him. The analogy between working with the blocks and doing written arithmetic 'is spelled out in detail' as Greeno calls this. But this analogy is only clear to the designer, because he created the units of ten or a hundred as mathematical objects. For the pupil, who does not yet have this mathematical knowledge, there is nothing to see!

Characteristic is the fact that the denary structure is not respected. Exactly the same problem that Labinowitz (1985) also observes. When Dienes-blocks are used the children often count the small blocks as tens while another time it is the bars that are counted as units (see the example in fig. 7)

It is presumed that the pupils will immediately recognise the bars as 'tens' but that appears not to be so easy. According to Cobb (1987) this is because the mathematical concept 'ten' is not such a simple concept for children. He refers to Steffe and Von Glaserfeld who, in a long and detailed observation study, have identified six levels in the construction of ten as a mathematical object that can be both one ten as well as ten ones. From there it can be derived that the distance between the lowest level 'ten as a perceptive unit' and the highest (abstract) level is not easily bridged. The children that do 'see' the relationships between the tens and the ones in concrete material are, according to Cobb, the children that have already construed 'ten' as an abstract object. Or in other words (Cobb; 1987;19) 'those that have got it get it'.

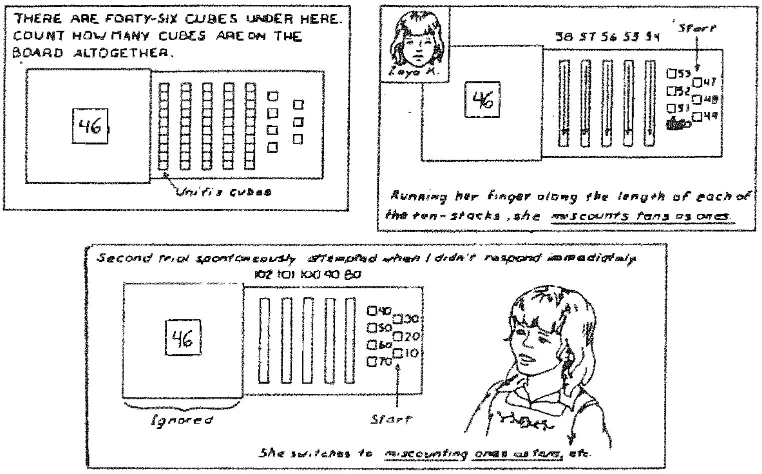


figure 7: Zoya's confusion between tens and units (Labinowitz 1985)

According to Cobb, distinction must be made between an 'actor's point of view' and the 'observer's point of view'. One should be able to look at the world through the eyes of the pupil in order to judge the significance of learning material. The misunderstandings surrounding the concept 'ten' are also caused because we already have the abstract mathematical object and therefore do not see the problems of the children who do not have it.

The problems become clearer when we regard working with concrete representations in the same manner as children do, namely as working with concrete material and not as working with the incorporated mathematical relationships. Then we see that this will lead to a mental action which consists of imagining a material act.

Cobb (1987) points out that the word representation can be confusing here. Representation can stand for a mental representation in the mind of the child ('Vorstellung') and for a didactical representation in the form of concrete material ('Darstellung'). The fact that no clear distinction is generally made here goes back to the 'observer's' standpoint. For the adult the mental representation is already there and the person 'sees' it in the material as well.

To illustrate this Cobb brings up the experience of Holt, who in first instance is most enthusiastic about the Cuisenaire material. The relationships between the material and the world of numbers are so evident that it would appear that working with this material would afford the pupils a wonderful entry into the world of numbers. But:

> The trouble with this theory was that Bill and I already knew that the world of numbers worked. We could say, 'oh, the rods behave just the way numbers do.' But if we hadn't known how number behaved, would looking at the rods have enabled us to find out?

By not making a clear distinction between internal and external representation it goes unnoticed that one is mixing up the time order: the pupil needs the mental representation which he or she must construe to be able to interpret the concrete representation!

This issue reflects the same communication mix up between teacher and pupil that Van Hiele (1973) observed in secondary education. He explains the problem in a discussion about the geometric concept 'rhombus'. The pupils only recognise a rhombus by its shape, not by its properties. A square is not recognised as a rhombus, unless you place the square on its tip (fig. 8).

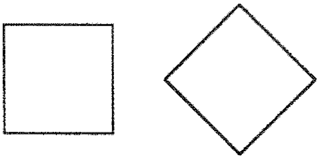


figure 8: recognition of a square as a rhombus

For the teacher the rhombus is a collection of properties: an equilateral parallelogram, perpendicular diagonals etc.

For the teacher it is evident that a square is a rhombus. He 'sees' that from the properties of the square. But an explanation at that level will not be understood by the pupils as long as they do not have the mathematical object of the rhombus.

After an analysis of this matter Van Hiele arrives at a distinction in three levels of concept forming: the ground level, where the concepts are still bound to concrete objects; the first level, where the concepts act as junctions in a network of relations; and the second level, where the relations between relationships are the subject of investigation. These levels are subject or domain specific and Treffers (1987) characterises these levels very adequately as concrete level, descriptive level and the level of the subject systematics.

According to Van Hiele, instruction must start at the ground level. By experimenting at ground level the pupil can discover relations and in that way the pupil will build up the relation network himself. That according to Van Hiele is also the only way: the pupil must build up the relation network by him or herself, no teacher can talk him or her into this knowledge. Working at a concrete level – and in the case of plane geometry this also means working with concrete material – therefore also for Van Hiele forms the basis for understanding, just as for Resnick, Gal'perin, Van Erp and others.

Cobb (1987; 14) even says:

> sensory-motor action is a primary source of mathematical knowledge.

But how does one avoid the problems that are encountered precisely in doing so?

Van Hiele has really already given the answer to that: by placing the initiative with the pupil.

Formally that is also one of the points of departure of cognitive psychology. And also Gal'perin pursues this in his proposal for a complete orientation basis. In practice these good intentions are often not realized because the designers are insufficiently aware that they are taking the 'observers' point of view.

5 constructivism

We can guard against this, according to Cobb, by adopting a constructivistic standpoint. Constructivism departs from the idea that there is no strict logical way to know 'objective reality'. Radical constructivism purports that you cannot even know if there is an objective reality. Radical constructivists call the reference to 'genuine reality' metaphysical realism. It is precisely the reference to the reality 'out there' that causes the misunderstanding. One must continually keep in mind that one is talking about constructions and that these constructions are idiosyncratic. Only through social interaction, through consultation and negotiation, can one try to attune the various constructions as much as possible.

In education one must provide pupils with the opportunity to build up their own knowledge by themselves. According to constructivists, every individual will try to build a theory of reality that is acceptable to him or her, and children try this as well. Constructivists find proof for this in the so-called 'misconceptions' (or 'alternative conceptions').

Examples of misconceptions are also found in optical illusion (fig. 9), naive expectations in physics (fig. 10a, b) and in the own solution strategies of young children.

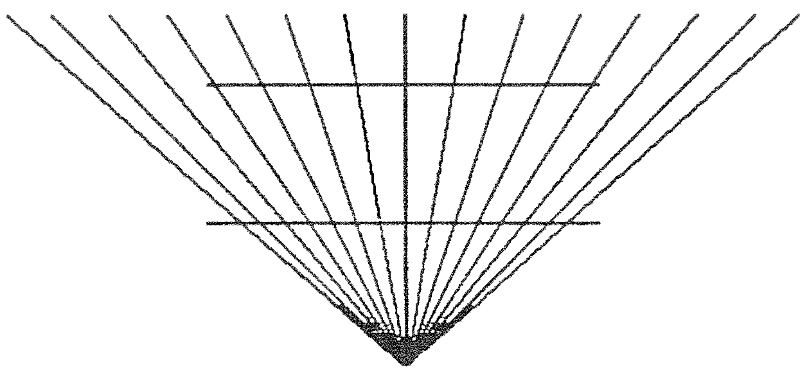


figure 9: optical illusion

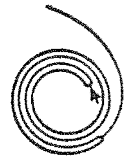


figure 10a: naive expectation of the path of a bullet as it is shot into a spiral shaped tube at great speed

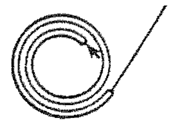


figure 10b: actual path

If this interpretation is correct this means that children will try to interpret their experiences in a logical manner. In education we can make good use of this, although we must be aware that the logic children discover will differ from ours.
Learning-theoretical concepts corresponding with this approach are widely adopted in cognitive psychology, in following of Piaget, whereby the focus is on concepts such as assimilation and accommodation.

This process of acquiring knowledge displays a clear similarity to the development of scientific knowledge, as described by Kuhn and Lakatos. Main elements are consistency and the not immediate rejection of an accepted 'theory' when unexpected results are encountered. Scientists do not give up their theories so easily. According to Kuhn no less than crisis and scientific revolution are necessary to make that happen. And also in our everyday life we do not give up our theories about reality so easily, that much is proved by the existence of stubborn preconceptions.

6 realistic instruction theory

According to Jan van den Brink (1981) one can induce children to discuss their theories by creating conflict situations.
For example, a conflict can be created by comparing the number of boys and the number of girls in a class, with the help of the conflicting graph (fig. 11).

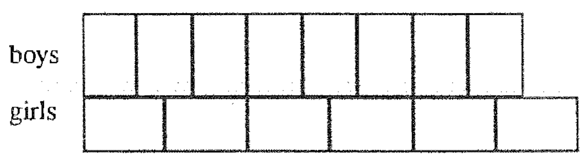


figure 11: graph

From the graph it looks as if there are more girls, while actually there are more boys. For a good understanding of the conflict situation it is necessary that we are aware that the conflict situation does not arise because one can prove by counting that there are more boys. For the pupils that does not have to lead to a conflict. The counted number is yet too far removed from the quantity number and the perception of quantity. The solidification of the knowledge about the number of boys and girls in the own world of the children causes the conflict: the pupils know – from P.E. class for example, from games – that there are more boys than girls. The result of the graph conflicts with this self developed knowledge.

We can ask ourselves however whether conflict situations are really necessary for learning the algorithms for column addition and subtraction. It remains to be seen whether the detected misconceptions constitute a necessary phase in the learning process. It would sooner appear that the misconceptions have been caused by education itself. Or in other words that the misconceptions are the result of the 'observers' expectation that the material will make clear the mathematical structure.

In realistic mathematics education it is attempted to prevent this by following the reinvention principle (Freudenthal; 1973). Here the pupil follows a learning route that takes its inspiration from the history of mathematics. In this case the symbolical representation of large numbers in a denary structure, the positional notation and the use of the abacus are important milestones. If the pupil discovers the meaning of these aids by him or herself in problem oriented instruction, the occurrence of misconceptions can be avoided.

To develop the denary system with the children an apocryphal shepherd appears on the scene who keeps track of the number of sheep he has by putting aside a stone for each sheep. At a certain point the shepherd has so many sheep however that the sack of stones is becoming a burden to him. The problem of the shepherd is made the problem of the children. How does he solve it? When the solution of the shepherd is finally presented, it is also experienced as a genuine solution for a real problem. If the number of stones becomes too great the shepherd changes ten stones (as many as he has fingers) for one coloured stone. This process of making groups of ten and the representation and interpretation thereof is re-enacted with tokens. Here the function of the material is different from the Dienes blocks for example. In a (too limited) introduction of the blocks the agreement about grouping on the basis of ten is communicated by the material. In the case of the shepherd a very conscious agreement is made to solve a certain problem. In that phase the work is still with unstructured material. Later on the position system is construed in a similar manner. Only later are materials and contexts introduced where the denarity has been solidified, such as working with money or with the decimal system.

The introduction of the abacus follows largely the same method as the case of the shepherd: concrete material is used to symbolise quantities situated in a context. The idea of repeated grouping on the basis of ten is again picked up in the story of the sultan. When ever the fancy strikes him the sultan wants to know how many gold pieces he owns, and to make counting them easier the coins are grouped as stacks of ten and bundles of a hundred (fig. 12).

figure 12: the sultan's pieces of gold

In class this story is retold with checkers, and changing and grouping is practised with drawings. Later the abacus is introduced, a device that also played a major role in the history of the algorithm. The beads on the different rods refer to loose pieces of gold, the stacks of ten and the bundles of a hundred. The principles of changing, borrowing and carrying are developed here against the background of packing and unwrapping pieces of gold. Only then is there the transition to something like the written algorithm, which first leaves ample opportunity for writing down the interim steps or interim scores and which is only later abbreviated to the standard algorithm. The principle difference with the Dienes blocks or other base ten material is again the relative unstructured character of the material. Concepts such as tens and hundreds are not illustrated by the material. The context provides a situation model in the story of the sultan and the method of wrapping the gold pieces. Insofar as the material is structured, the structurization is directed at eliciting certain mental activities. For that reason the quinary structure is put on the abacus as a visual support for the setting up and reading off of the number of beads per rod. While the bars in turn are an aid to differentiate between the units, tens and hundreds.

Naturally there is also the danger here of trickery action. First of all the handing in of one bar can be done without thinking about the context or meaning. We have already seen as well that pupils develop own solution methods which are not isomorphous with the actions needed to do column arithmetic on paper. A reason why actions with concrete material must in our view be regarded especially as a transition phase. The manipulations on the abacus, together with the sultan context must together offer a framework of reference for arithmetic on paper. Purpose then is that the pupil in thinking about the abacus, has the global structure in mind. It is not the intention that the pupil add or take away beads in his mind on the basis of the denary structure. What must be prevented therefore, is that these kind of actions become habit.

As an alternative precisely the opposite route might be followed whereby the informal arithmetic methods of the pupils are taken as point of departure. Especially when the quinary structured abacus is used long in advance of doing arithmetic under twenty, the basic facts can possibly be developed from informal arithmetic methods of the children. If the sultan's story and the 'arithmetic' on the abacus is not introduced until thereafter, the chance of too strong a binding with concrete manipulation is much smaller. With such a prominent position for the self thought out arithmetic methods of children we add a new element to the realistic approach.

In the 'traditional' realistic approach the path along which the algorithms are developed are to a great extent, predetermined. The instruction is designed thus that the pupil makes discoveries himself, but what is discovered and in what order, has been determined in advance by the constructor of the course of instruction. He tries to achieve this by way of a didactical series of problems and by eliciting the corresponding discussion and reflection.

Meanwhile, (realistic) developmental research, such as that by Ter Heege (1983) and Streefland (1988) has been the cause for an awareness that the children themselves invent alternative solution procedures which are as good, or even better suited to lining out the course of instruction. In this respect Streefland mentions solutions which 'anticipate' and which act as 'road signs' for the developer.

Designing a possible learning route on the basis of the own solutions of children can be regarded as a further refinement of the re-invention principle. The re-invention principle does not only praise the history of mathematics as heuristic, it also refers to a certain manner of learning: the pupil who globally follows the historical course of instruction, reconstructs the thus discovered mathematics.

This idea of the self (re)construction of mathematical knowledge is much more fundamental than the historical aspect. Freudenthal chooses for the re-invention principle from his idea about how one, as a mathematician adopts new mathematical knowledge. The history of mathematics can help one to find a fitting course of learning, but as it appears, so can the own solutions of the children. Recently Treffers (Treffers c.s.; 1988) also applied this principle to the basic facts and column addition and subtraction.

Research shows that children can spontaneously come up with a number of informal strategies to arrive at the basic facts for addition and subtraction (see Groenewegen & Gravemeijer; 1988, for example). First, most answers are still found by counting. Then the children develop ways of counting and calculating efficiently to shorten the counting activity. At the same time it appears that the doubles (ties) and the quinary/denary structure are often used as points of reference. Of course not every child will develop efficient strategies with the same ease. Hence the search for concrete material that will elicit the development of such habits. Suitable aids here would appear to be the arithmetic rack and the bead string (see fig. 13a and b).

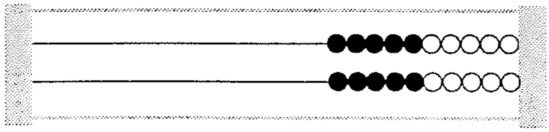


figure 13a: arithmetic rack

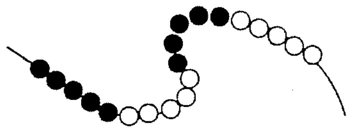


figure 13b: bead string

We will not elaborate on both learning aids here, but restrict ourselves to a brief description of working with the arithmetic rack. For a detailed description we refer to Treffers c.s. (1988). On the arithmetic rack the beads on the left count as the numbers that are being worked with, the beads that have been moved to the right do not count (any longer). The quinary structure in first instance only offers visual support in quickly overseeing the numbers. As such this manner of structure also provides support to the discovery and remembering device-restricted number relationships such as 'five is three plus two', 'five and two is seven' and 'six and six is twelve'. These are precisely the number relationships which the quick pupil will spontaneously use as points of reference. Also the use of these anchoring points is facilitated by the device. In this way you can read $6 + 7 = 6 + 6 + 1$, or $6 + 7 = 5 + 5 + 1 + 2$ at a glance (fig. 14).

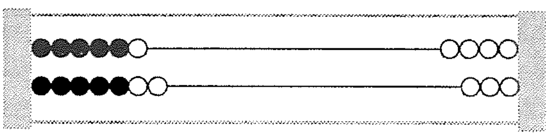


figure 14: six plus seven

Finally, the device provides opportunity for different strategies. The sum $13 - 7$ for instance, can be solved in various manners (see figure 15).

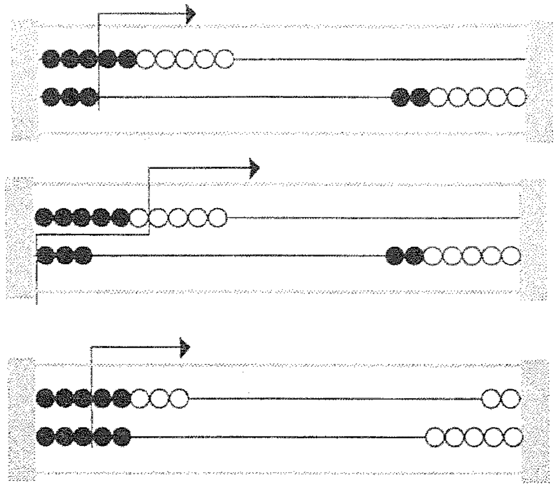


figure 15: various solutions to 13 – 7

So much for the informal strategies for doing arithmetic under twenty.
The arithmetic methods invented by children themselves can also be used to outline column arithmetic as we mentioned earlier.

Treffers (Treffers c.s. 1988) shows that many pupils develop an informal procedure in which subtraction is done in columns from left to right. Contrary to the standard procedure the children consciously employ the position values of the numbers: in 675 – 482, 7 – 8 is read as 70 – 80. The problem is solved by first subtracting the hundreds: 600 – 400 = 200. Then the column of the tens follows. Here the pupils end up ten short.

This can be settled straight away: 200 – 10 = 190 (some pupils use an intermediary notation and calculate this later). The units are next, 5 – 2 = 3 and the answer is compiled: 190 + 3 = 193.

These informal solution methods can be seen as an intermediary form between mental arithmetic and column arithmetic and can ultimately be abbreviated to an arithmetic method.

Turning back to the fundamental criticism by Cobb (1987) of the implicit 'observers' standpoint that is often taken, we can establish that recent developments in realistic mathematics education place the 'actor' even more than before, at centre stage. If one wants to adopt the viewpoint of the pupil one will have to take the solution procedures of the children very seriously. That would seem the best route to follow to avoid misunderstandings. Although some kind of tension will continue to exist between the following and guiding of children. In this sense the realistic approach also differs from constructivism.

Constructivism is still primarily a research approach that is directed at analysing micro didactical situations and the actual theories of children. Realistic instruction theory is directed on long term learning processes and tries therein to do justice to the own contribution of the pupils.

7 conclusion

Both from action psychology as well as from cognitive psychology it can be understood that the danger exists that working with manipulative material does not prepare for working without manipulatives.

Action psychology makes us aware of the possible differences between the external action, the mental action and the pursued structure of action. On the one hand the danger lurks here of a manipulative action without insight that does result in the requested action result. On the other hand there is the problem of the transition from thinking about material to thinking in terms of mathematical relationships and concepts. Cognitive psychology makes us aware of the fact that pupils interpret new information, therefore also the use of manipulatives, from their own knowledge. Cobb points out that the consequence then is that manipulative material must be regarded from the standpoint of the pupil. The pupil only sees the manipulative material and not the mathematical relationships which adults recognise in it. In this connection he refers to the mixing of the intended internal (mental) representation and the actual external (concrete) representation.

We see that for the Dienes blocks where ten, a hundred and a thousand are concretely present as perceptive units, while it is expected that the pupils are using mental mathematical objects.

Cobb's distinction between the point of view of the child (actor) and that of the outsider (observer) is induced by his theory of constructing knowledge. The conception that everyone forms his own image, his own theory about reality makes Cobb realise that the reality of the pupil is a different reality than that of the developer/researcher. Children construct their own theories about reality and will in general tend to hold on to these theories.

In this sense realistic instruction theory can be regarded as in concordance with the constructivistic approach. The reference to conflict situations as a means to further learning already points in this direction. But especially the idea of re-construction of knowledge relates closely to constructivism. In realistic education theory, two sources are tapped in designing instruction courses which are meant to elicit this reconstruction process: the history of mathematics and the spontaneous, self thought up arithmetic methods of children.

The use of manipulatives is thus placed in a different perspective. It is not the material that transmits certain knowledge. In the 'historical' elaboration of realistic instruction theory; material is only an aid to solve certain practical problems in a certain context. There, understanding and insight are supported by the context, which can serve as a situation model. In the 'informal solution' variant the material is used to elicit (mental) arithmetic actions which other children have previously developed themselves. Close study of the actual occurrence of such acts is necessary.

In a general sense we can draw the conclusion that it must not too readily be assumed that instruction activities and visible learning behaviour will lead to the intended learning result. And even though the realistic approach seems to offer solutions to prevent discrepancies, here also a study of the actual solution process of the children and of the actual forming of mathematical concepts and relationships, remains of the essence.

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Realistic arithmetic education for young children

F.J. van den Brink

1 introduction

Would it not be better, when teaching six to seven year old children arithmetic, to start from playing activities familiar to the children from kindergarten? When, in The Netherlands, kindergarten and primary school were put under one roof, the influence of primary school on kindergarten was clearly stimulated. Would it also not be possible for primary school arithmetic education to 'learn' how instruction is given in kindergarten? In other words, how can math and the playing potential of children be brought closer together?

In the nineteen-seventies the IOWO (Institute for the Development of Mathematics Education) began a project to develop a new mathematics program for primary school under the name 'Wiskobas'. The curriculum for teaching addition and subtraction to six years olds was built around a workbook entitled 'The Bus'. This workbook appeals to the creative potential of the children. It was subjected to four different research projects.

2 four research projects

a The starting point of the first project was a list of the presumed goals of 'The Bus' workbook, drawn up by external researchers. Emphasis was laid on product goals and how the children went about solving the problems was not taken into account. Some goals were viewed as minimum levels, with the result that too much was expected of the children. Nor did this research project take into account the type of education present at the time of testing. It was this project that revealed the need for research to follow in the footsteps of education.

b In the following project, 'Evaluating Education at the Drees School', the teachers who used The Bus workbook at this design school had their say. It was obvious that these teachers had been using the workbook in very different ways. Additions had been made to it, and topics had been rejected, emphasized or altered. During this project it became evident that the teachers held different opinions on how to use the workbook.

c In the third project – an hypotheses developing research – the children had the opportunity to demonstrate their own ideas regarding the bus. 'Mutual observation' was used here, a research method that enables one to discover many of the children's hidden ideas. This research method developed from the researcher's habit of taking detailed notes on all sorts of things (behaviour, outward appearances, actions and so forth) during the discussions. This took a great deal of time. The children often had to wait quite long, which was nu fun. To shorten the waiting period and to maintain a relaxed atmosphere, the researcher read aloud what he was writing.

results

- First of all, a relaxed atmosphere was indeed maintained.
- Moreover, the children became aware (some to their astonishment) that the researcher's notes were about them. Many children had evidently never realized this. And they were proud that everything they said or did was being written down.
- The children would correct the researcher if he wrote down something wrong.
- And, in order to help him with the notes, they became aware of what they had or had not said or done.
- They were thus reflecting on their own thoughts and came to be, as it were, observers along side the researcher.
- In this way the children were able to understand what the researcher was trying to find out from them. Questions like 'Why did you think that?' and 'How did you think?' were not clear to them. They would often simply answer, 'Just because!' Evidently, the child must be aware of the researcher's intention of finding out how the child thinks. This method of mutual observation involves the subject in the setting of the actual research work, making him or her better aware of the intentions.

methods and techniques, inspired by education

In our research project, which developed in close proximity to educational practice, it became clear that education can serve as a source for new research methods and techniques. During the discussions with pupils, for instance, a number of social techniques were found which could be used for mutual observation and which had been inspired by didactical procedures in education.
For example:

- Reading aloud the notes immediately or else somewhat later.
- Making use of conflicts and surprises:
  - the researcher feigns ignorance;
  - the researcher makes intentional mistakes in the notes and then reads these to the pupils.

- Creating game situations:
  - having the pupil read aloud what the researcher has written about her or him;
  - pretending that the researcher knows what the subject was thinking by 'bluffing' or by making a game of it: 'I know it – and you have to guess what I'll think about your answer';
  - having the researcher and the subject exchange roles.

discussion

Mutual observation is an improvement on the clinical interview. Characteristic of improvements in the clinical interview (such as introspection, retrospection, active participatory research) was the fact that, until now, only the subject's views were at issue (Ginsburg, 1981). The researcher's interpretations of these views were never questioned. As an observer, he remained outside the events, and simply directed the subject's behaviour by means of questions (Giddens, 1979). He observed, yet remained invisible himself (sometimes literally so behind a one-way mirror). The observer felt no need to inform his subjects of his opinions, nor even of what he thought he had seen or heard. But isn't the criterion for evaluating research results, for determining a researcher's interpretations, the degree in which the subjects can recognize themselves? (Stokking, 1984). The method of mutual observation, on the contrary, makes use of the fact that the pupil not only observes herself or himself, but that she or he is also in a position to observe the interpretations of the researcher – as long as these are announced. The subject is deemed competent to evaluate the observations and interpretations of the researcher. This method offers the subject the role of observer, recognizes the subject's thoughts and activities as divergent from the ideas of the researcher, and attempts to unearth these divergences by means of all sorts of social techniques (conflicts and surprises, making jokes, game situations).

Mutual observation is objective in the sense that it does justice to the object of the research and that the researcher is the one who determines the degree of role exchange. The method forms a new perspective for observation in educational research.

This third research project revealed that young children do not think about buses and other contexts as we adults do. They also have entirely other ideas regarding counting and numbers.

d The fourth project is a comparative study between realistic education, in which The Bus plays a major role, and traditional, mechanistic education. The research was carried out at the above-mentioned Drees school in Arnhem and at the Nieuwland school (now called 'De Oversteek') in Dieren.

The two types of education in the schools mentioned above differed primarily on three points: contexts, arithmetic languages, and exercises.

The goal of the research project was to shed light on these differences: in which ways did they differ and where did one function better than the other.

3 three differences in education

3.1 contexts in education

figure 1: bus driver with cap

In playing math drama and illustrative stories about people getting on and off a bus, the children had to imagine the situation and thereby 'realize' their own ideas, experiences and fantasies on the bus. We therefore called this type of mathematics education 'realistic'. Sometimes the child played the role of actor or storyteller, at times that of observer or listener. In this way the pupils made their own contribution towards determining what took place.

This was not the case in traditional mathematics education. There, an attempt was made to imitate addition and subtraction with numbers by using drawings and objects. Certain activities, called 'arithmetic manipulations', were learned using rods, but no attention was paid to what the pupil already knew about numbers. These arithmetic manipulations were practiced step by step, over a long period of time. The exercises were made increasingly difficult and mistakes were avoided as much as possible. This mechanistic gradualness was another aspect which differed from the realistic arithmetic education. The pupils' 'realization' of their own ideas primarily took place in conflict with the ideas of others or with mathematical properties as yet unfamiliar to the pupil. Therefore conflict situations as educational contexts were actually created in the realistic education.

3.2 arithmetic languages in education

arrow language

A special aspect of The Bus workbook is that it is written in a different language. Where traditional sums use the 'is' sign, this new language uses arrows for writing a sum. The sums are presented in the drawings of buses.

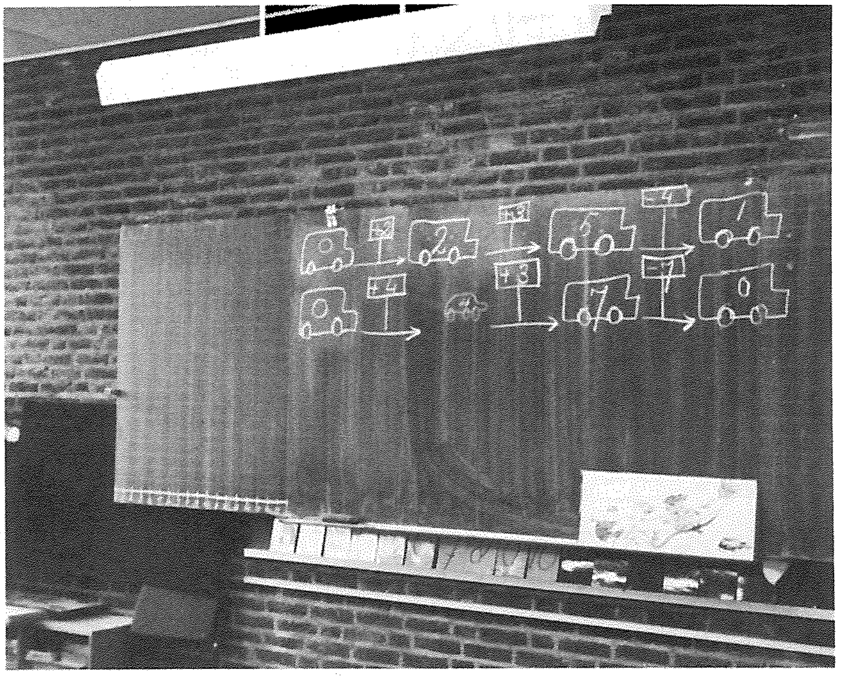


figure 2: blackboard with bus arrows

A number is written on the side of the bus indicating how many people are to indicate on the bus. An arrow indicates direction. When the bus comes to a bus stop, the sign at the bus stop shows how many people get on or off. At first the arrows are provided with decorations pertaining to the bus, but the children soon think up all sorts of other situations for the arrows, such as a post office or shop, at the dentist's waiting room, a game of skittles or shuffleboard. The children then decorate the arrows accordingly.

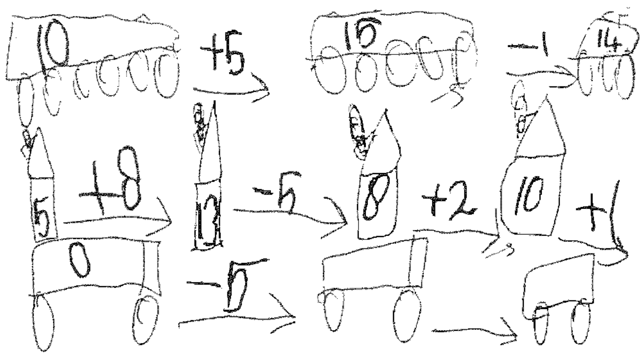


figure 3: Manfred drew a number of decorations

The decorations are invented by the children and later on the pupils omitted them without the arrows losing their meaning. This is how the 'bare' (formal) arrow sums are generalised: by inventing all kinds of decorations.

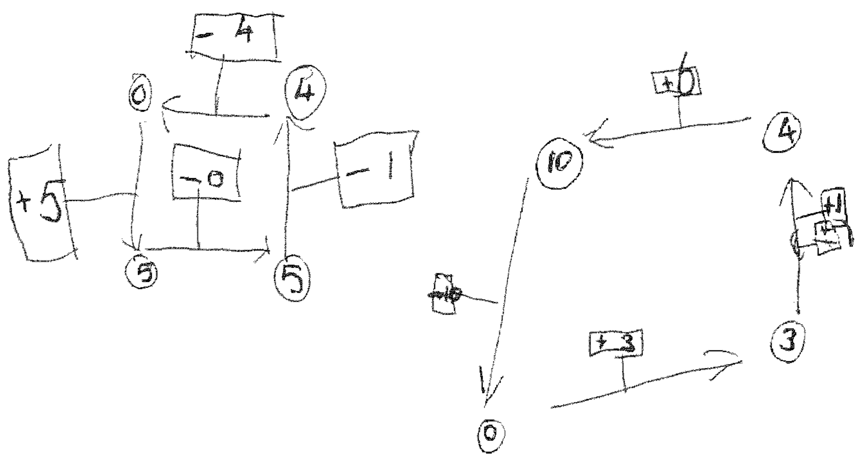


figure 4: Nolly makes cycles of the length 4

four linguistic characteristics of the arrow language

Arrow language meets the requirements of four general linguistic phenomena that are of importance for education:

- the gradual fading of the original meaning of symbols;
- the transparency of symbols;
- new things stated in old terms;
- alternate use of the real world and the world of symbols.

fading of the meaning of symbols

A well-known linguistic phenomenon is that, after a while, the original signification of meaningful symbols begins to fade. One needs to be aware of this in education. Adults use language as if the meanings are already present, or as if the symbols on the whole have no meaning at all (Cobb, 1987). When asked, for instance, to do an errand in the neighbourhood, nobody will still think up exactly how to go about this (Von Glaserfeld & Steffe, 1986). The used symbols take the place of the imagined performance of the activity. A detailed diagram of what route one will take is not necessary. If required, this can still be provided from memory.

The gradual fading of the meaning of symbols supports the idea of vertical mathematisation, that is, the transfer of the illustrative contexts (such as the bus) to the arithmetical context of bare sums. This takes place by means of the bare arrows which were already hidden in the material (as contextually embedded arrows).

transparency of symbols

Alongside the fading of the meanings of symbols lies another linguistic phenomenon: the transparency of symbols (Polyani, 1962). In a discussion or written text we are only subliminally aware of the actual words. We allow the meaning to shimmer through the symbols. Our ideas regarding the bare, but meaningful arrow language fit in well here.

old and new

People continually attempt to understand new things in already familiar terms (Glaser, 1984). This is connected to what Vygotsky (1964) described as 'shift': the transfer of familiar, reliable knowledge to new arithmetic topics, whereby the pupil becomes aware of the existence of a newer, more general knowledge.

Lehrer (1974) discovered that, with children, this 'carry-over of terms' is not limited to single words, but that the entire familiar structure held by a term in the old context is transferred to the new context.

real world and world of symbols

Freudenthal (1987) regards children's mathematisation as activities in which they move back and forth between the real world and the world of symbols. Thus one world can expand at the expense of the other and also contract to the other's benefit. Arrow language, decorated with a fringe of context, supports this linguistic phenomenon.

traditional is-language

Some researchers are of the opinion that the cause of arithmetic difficulties in traditional arithmetic language lies in the 'is equal to' sign, which is not sufficiently mastered by the children (Van Erp, 1983).

Various studies have shown that, when faced with reverse or complex equations, children will ascribe other meanings to the equal-sign than is intended (De Corte & Verschaffel, 1980; Van Eerde, 1981). It is generally recognized (Davis, 1975; Ginsberg, 1977; Van Eerde, 1981; Vergnaud, 1982; Van Erp, 1983) that children first use the is-sign as a resultative (operative) sign and not as a relative sign. Children do not regard the is-sign in isolation, but rather connected to one operation or another (addition, subtraction, multiplication or division). Some children understand the operation, but are at a loss once the is-sign appears (Grazer, 1933). In the resultative view, the is-sign joins an operation (3 + 5 or 11 – 3, for instance) with a number (8) as the result of the operation, and not a number with a number, as is the case where the is-sign is a relative sign (Davis, 1975).

Moreover, the is-sign is constant for all operations, and yet one does not know whether the sign has the same meaning for addition as, for example, after a multiplication. In any case, the resultative is-sign contains specific properties that diverge from the is-sign as a relative sign: the resultative is-sign is not symmetrical, it is sequential (children tend to draw rows of these signs), and it is sub-reflective (the reflection only pertains to parts of the whole equation). Confusion as to the two significations of the is-sign is the cause of much arithmetical difficulty. An arrow does not provoke such confusion because it is purely a resultative sign.

arrow language versus is-language

There are a number of important differences between arrow language and is-language:

- Some constructions in arrow language

  $$5 \xrightarrow{+3} 8 \xrightarrow{-2} 6$$

  are not possible in is-language.
  On the other hand, every notation using an is-sign can be transferred to arrow notation. Arrow notation therefore has a broader area of application than does the is-sign.
- Through the decorations, the arrow language is more closely related to the (bus) play context than is the is-language, with its arithmetic rods and thereby acquired arithmetical manipulations. Very intensive practice is needed with the is-language and arithmetic rods before the link can be seen (Resnick, 1983; Greeno, 1983).
- Arrow language is a 'rich bare context' (Treffers, 1987): arrow-puzzles provide material for reasoned arithmetic and one's own constructions. Equations in is-language that are related to such arrow-puzzles, on the contrary, form insurmountable obstacles.

3.3 practising in arithmetic education

Characteristic of arithmetic practice in traditional education is the intention to avoid mistakes, exclusive use of bare sums and the provision of ready-made sums.

In realistic education, on the other hand, the children's own, free productions are also valued, as well as incorrect solutions to equations and attempts at reasoned arithmetic in arrow puzzles, in which strategic 'mistakes' have been placed on purpose. Application of sums by means of arithmetic languages which refer to these in familiar contexts is also valued, as is inventing ones own problems within a meaningful framework, such as making an arithmetic workbook for other children. The idea behind this is that the children are the ones who must eventually learn to do arithmetic and that they can learn from their mistakes as well.

4 comparative research

design

Throughout one school-year the researcher regularly held discussions which lasted about ten minutes with each pupil, using the method of mutual observation. In addition, at both schools – the Nieuwland school (N) and the Drees school (D) – the respectively traditional and realistic arithmetic education was notated in detail. Journals, keeping track of the time allotted to instruction and preparation, and counting the number of presented and completed arithmetic tasks allowed us to explain the performances of the pupils from an educational perspective.

Three criteria were used to test all pupils with regard to their learning achievements:

- general arithmetic skills
- equations
- the arithmetic workbooks made by the children themselves for the children who will enter first grade the following year.

results of the three differences in education: contexts, arithmetic languages and types of practice

What differences did we see in the achievements of the children from the two schools in question?

contexts

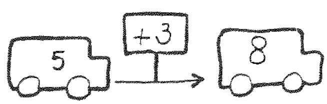
- At D (the realistic school), a minimum of instructional time (two days) was needed to make the children aware of both addition and subtraction, with the help of playing the bus context.

  The traditional approach, whether or not with illustrative objects, took seven weeks at N. Only addition was learned during this time, after which the children

could start with subtraction.
- After six weeks of preparation at N, the children were able to do problems independently, using the rods of the arithmetic box.
  The children at D were doing arrow problems independently during the first week after they were introduced.
- In the arithmetic workbook made by the D children for pupils the following year, 21.3% of the arithmetic games were from outside of school: hopscotch, skittles, marbles, skipping rope. This category was absent from the workbook at N. There, only games taken from arithmetic were mentioned: lotto, arithmetic box, abacus.

arithmetic languages

The most important characteristic of arrow language was clearly the fact that the arrows can be decorated. Not only with wheels and bus stop signs, for instance, to indicate the bus context, but also and especially with decorations indicating contexts invented by the children themselves. In this way the arrows were 'applicably abstracted'. Not by disregarding the significations (as is often the definition of 'abstraction'), but by having the children think up new meanings for the arrows. Bare arrows began to appear, which could be easily applied. The is-sign, too, was used as a decoration for the arrows, allowing the transition to and from problems in traditional arithmetic language (is-language). Arithmetically weak children in particular were able to profit from this.



$$5 \xrightarrow{+3} = 8$$

$$5 + 3 = 8$$

figure 5: from, via, to

On the other hand, the problems stated in traditional arithmetic language at D were decorated with arrows for about two months in order to attach signification to these bare sums.

The individual productions, too, revealed the equivalence of the two arithmetic languages. More than half of the D children designed both bare and decorated arrow problems in their individual productions.

The traditional arithmetic language – the 'is-language' with is-sign – as used from the very start at N, proved, on the contrary, to be applicable only in problems with rows of identical figures, arithmetic using rods, and in bare sums.

Arrow language turned out to be quite useful to the children. The language made them aware of the hidden structure within a complicated problem (which could not even be described in is-language). All of the children were able to verbalize the problem correctly and more than half of them could solve it correctly.

'Tell the bus story again', we asked the D and the N children on November 2:
'There are three people on the bus. It passes two bus stops and at the second one four people get on. Now there are five people on the bus.'
Unfortunately, no one was able to repeat the story, neither at D nor at N.

We told the story again, and this time the children were allowed to write down the numbers mentioned as a help when repeating the story. Later, on November 15, the children had to repeat the story themselves, but this time were allowed to use a chain of arrows.

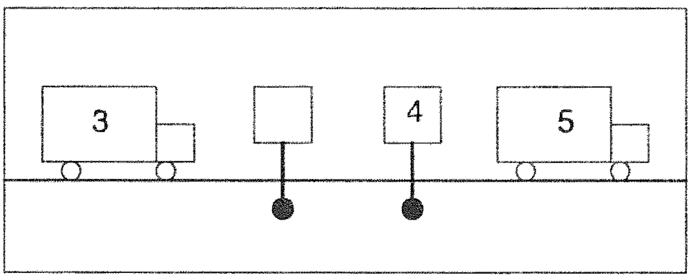


figure 6: chain of arrows

On December 14, again only the numbers were permitted.
The results of the three discussions are presented in figure 7.

A situation can evidently be reproduced more precisely using arrow language than with just bare numbers. After introducing The Bus workbook at D, the story was again told to 11 children there. All of them described the situation correctly using arrow language and, moreover, 7 of the 11 solved the problem. Apparently, arrow language helped make the structure more transparent.

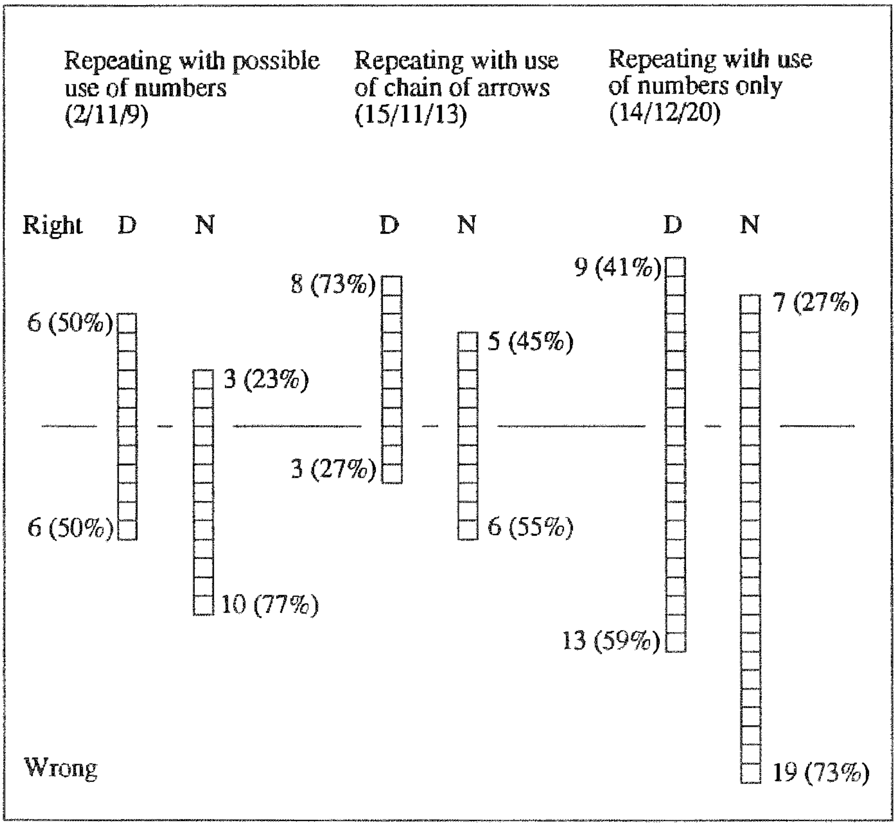


figure 7: comparison of the results of the three test discussions

practising

- In spite of the fact that, on the average, the N children did more than three times as many sums as the D children, there were no noticeable differences between them in their general arithmetical skills.

  The longer a given type of problem was practised, the more mistakes that were made. This was true for both schools.

  The results to the sums 1 + 5 – 3 = . and 3 + . + 3 = 8 are quite striking.

  On June 23, more mistakes were made in the sum 1 + 5 – 3 = . at N than at D and on March 23. And the children at D made more mistakes in the sum 3 + . + 3 = 8 on June 23 than they did on March 23 and those at N. Here the cause lies in the instruction. On March 23, the mistakes in the two sums were made primarily because both D and N children had only regarded a part of the sum and did not use the solution to that part in order to solve the entire sum.

– The sum 3 + . + 3 = 8 was answered correctly on March 23 by more D children than N children. This was due to the successful application of 'central sums' (3 x 3 = 9, 2 x 3 = 6, etc.).

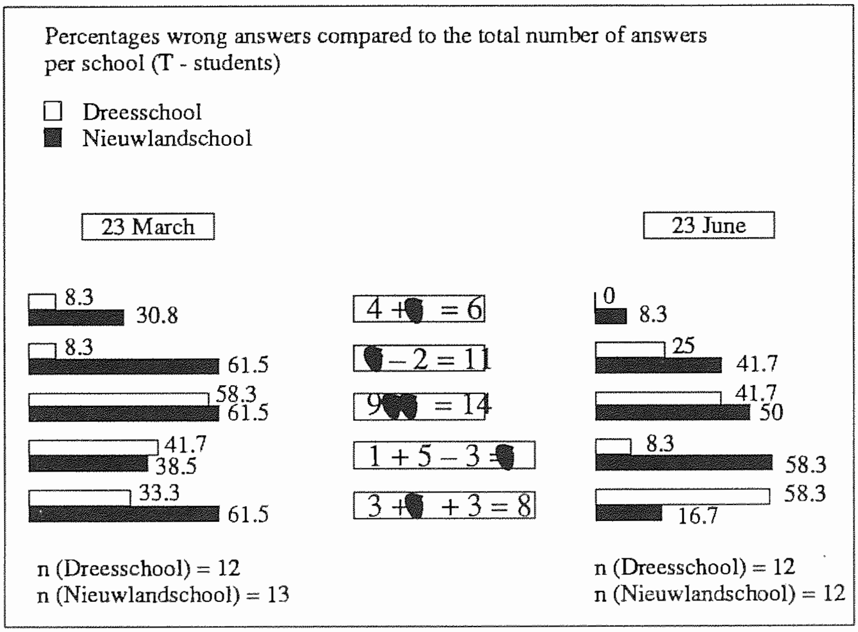


figure 8: spot sums

However, after that the instruction at D turned primarily to chains of arrow-sums, whereby 1 + 5 – 3 = . was solved effortlessly, but where more mistakes were made in 3 + . + 3 = 8 than in March.

From March to June, instruction at N emphasized central sums with two terms. This explains why, in June, mistakes were made in solving 1 + 5 – 3 = . and why they did better at 3 + . + 3 = 8.

– We have ascertained that emphasis on one type of bare sum was to the detriment of other types of problems. This was the case during each instructional period at both D and N.
– The 'fill-in-the-blank' equations were not as problematic at D as at N. At D the children could actually act out the situation and connect it to an illustrative context with the aid of arrow language.
– A warning sign was that almost none of this type of sums appeared in the own productions of the pupils.
– The play-acting remained in the children's memories for at least three to five months. They could later use this as an area of application.

– The assignment to make an arithmetic workbook for next year's pupils was clearly a positive influence on all the children's arithmetic skills.
The average percentage of mistakes in the workbooks made by the children during the period from April 25 to May 30 was about 2.7% for D and 3.4% for N for all the sums per pupil. During the same period, 4.4% of the official arithmetic assignments were done incorrectly at D and 4.5% at N.
So, fewer mistakes were made in the problems thought up by the children for the workbooks than in the arithmetic tasks that were officially intended for practice purposes.
– The expectation that more practice sums would appear in the N workbook than the D workbook, because more than three times as many sums were done at N, proved unfounded. The total number of sums done per pupil per page lay between 15.3 and 24 in the D workbook and 10.1 and 26.8 in the N workbook; not an appreciable difference. The children at N apparently found the large number of practice sums of less import than did the author of their mechanistic textbook.
– The children's workbooks revealed a clear difference between the work of children who had had lessons in arrow language and those who had worked exclusively with is-language. There was more variety in the type of sums created by the D children (see fig. 9). They invented arrow puzzles, worked earlier and more often with large numbers (see fig. 10), and applied their knowledge of arithmetic to all sorts of situations, both outside and inside school.

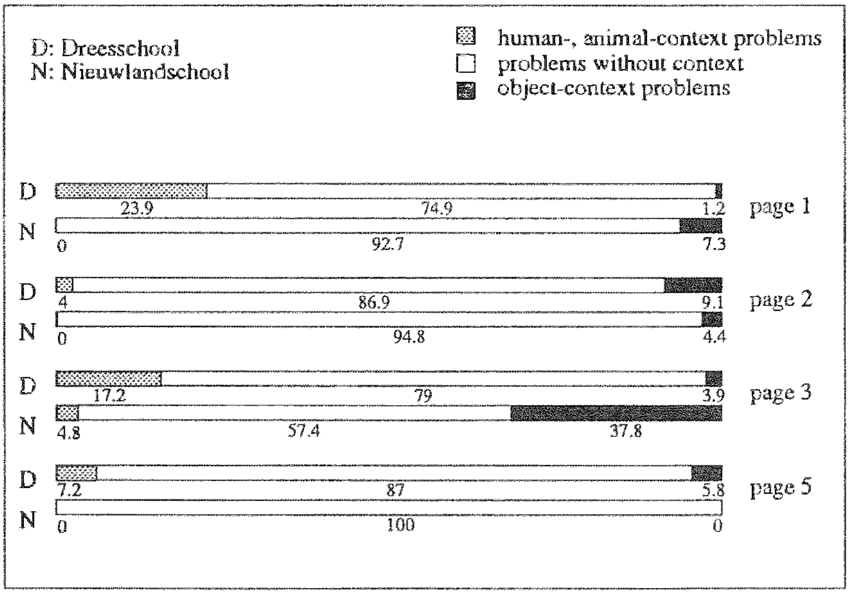
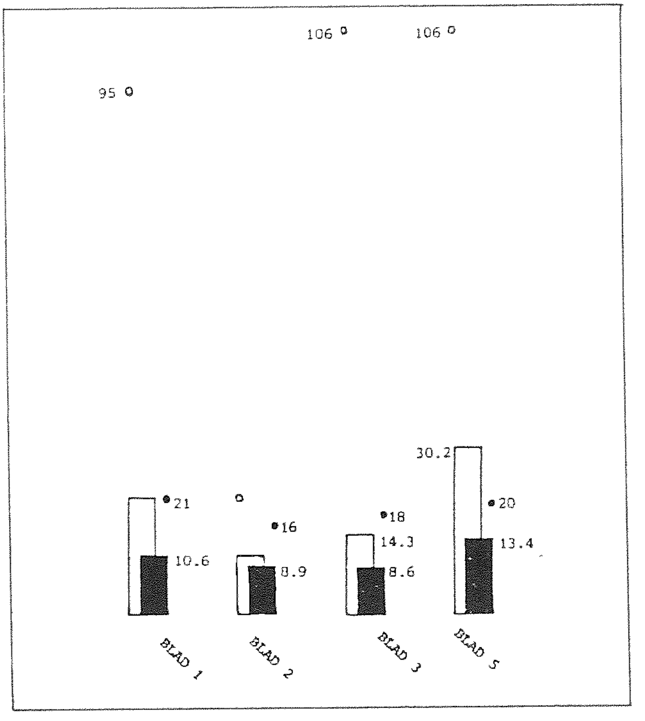


figure 9: percentages of different types of problems (averages)



□ Dreesschool ○ maximum on Dreesschool
■ Nieuwlandschool ● maximum on Nieuwlandschool

figure 10: the largest number used

5 General conclusion

The research (Van den Brink, 1989) showed that, by taking children's playing and creative potential into account and making use of it in education, learning addition and subtraction can take place faster and with more insight. Only half the number of practice sums are needed without losing mastery of the skills, and the arrow language can be abstracted during application. Arithmetic, in this design, need not be a dry, isolated activity, but can become an integrated part of the children's lives.

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Fractions, an integrated perspective

L. Streefland

1 introduction

From 1983 to 1986 developmental research on fractions was carried out at OW&OC (Streefland, 1991[a]). This research aimed at the development of a prototype of a course on fractions and the foundation of a realistic theory on the teaching and learning of fractions (Treffers, 1987). The study of long-term individual learning processes lies at the heart of such research.

By means of a teaching experiment and individual interviews it has been attempted to tie in the course and the teaching process with the individual learning processes as much as possible, or, perhaps better yet to interweave both course and learning processes.

In fact, the pupils were enabled to contribute to the course under development by means of their individual constructions and free productions. At the end of the fraction-project a comparative study was carried out in order to trace possible omissions in the course (and teaching experiment).

This contribution offers an impression of what was done with respect to this in the course of a few years. In particular the comparative part of the research will be emphasized here (section 3).

In order to shed some light on the course as developed and on the individual learning processes as they took place, one of the participating pupils will be followed during her long term learning process (section 2).
A brief discussion will close the discourse (section 4).

1.1 the course reflected by an individual learning process

In order to illuminate what was transpiring in the developmental research regarding the progress of individual pupils and – derived from that – regarding the content of the teaching experiment, the individual learning process of one pupil, Clara, will be described here in detail.
Five indicators were used to describe the quality of the learning process, namely:

1 Concept acquisition (fractions, ratio) and $\mathbb{N}$-distractor errors. Such errors are a case of failing to regard fractions as numbers which describe a ratio (or at least a ratio value); incorrect additive proportional reasoning is lack of insight in the ratio of pairs of numbers intended for construction or comparison (Streefland, 1991[a], 1991[b]).

The temptation to use arithmetic rules inherent in natural numbers when dealing with fractions or ratios is an indication that these concepts are not yet present to a sufficient degree.

For example: when faced with a portion of '$\frac{1}{3} + \frac{1}{4}$' a pupil reasons that there were three and four participants in the original sharing situation, or, if a shade of purple is created by combining two units of red with four units of blue, a pupil upon being asked to create the same shade based on four units of red, decides on six units of blue, arguing that 'this is also two more'.

2 Progression in schematising. This means the application of abbreviations in the schemes, like the ratio-table, with increasing efficiency. In the description of the individual learning process this will be elaborated further.

The following indicators already have meaning by their wording. Moreover, the pupil's learning portrait contains a further explanation.

3 Flexible use of visual models and applications of diagrams and schemes, flexible application of basic computational skills included.

4 The ability to construct (mental) images of formally stated problems, i.e. in a formally symbolised way.

5 Individual constructions and productions of fractional material on a symbolic level.

2 the case of Clara (9;4) - (11;8)

2.1 general remarks

Clara contributed regularly to the teaching process; mostly in a reproductive sense, but occasionally constructively as well. Clara was a rather friendly little girl, whose cognitive potential was not easy to gauge. She and her girlfriend often approached the teacher – nearly always for help.

Was it insecurity? Or was it based on getting attention? It could well have been her girlfriend's insecurity that was the deciding factor in such cases. Whatever the cause, Clara succeeded in making an ambivalent impression. On the one hand she was a good pupil, particularly in her ability to verbalize certain matters; on the other, when it came to inverting or simplifying problems, inventing examples herself and such, she confirmed the opinion of her various teachers: 'doesn't have much pluck'. Clara, was also inclined for some time to hang on to concrete representations. Her attitude when an initiative for something new was required, was one of wait-and-see. She worked rather slowly. In terms of pace she brought up the rear, along with others. When circumstances forced Clara to take sides, for instance when an $\mathbb{N}$-distractor dilemma put her on the spot, she would occasionally yield to these errors. This was probably due to insecurity, and not because her concept of fractions was not equal to the situation. And we should regard her need for attention as an explanation for her hesitation and insecurity. Clara played the role of the chambermaid in the school musical, a role which suited her well. Her score (in percentiles) for arithmetic on the final CITO test for primary school education, 1986, was 100, and 77 for language and processing of information. She answered all 27 fraction and ratio problems correctly. She went on to a comprehensive school for havo and vwo (these are mainstreams which prepare for higher vocational education and university respectively). Time will tell whether her capacities were equal to this choice.

2.2 specific information

indicator 1: concept acquisition and N-distractors

Clara played a constructive role in the first efforts at starting the mathematization process. Her opinion regarding fraction-producing distributions such as 'divide three bars among four children' was that they should be characterized by 'the same amount for everyone', and not based on taste. She contributed to the abbreviation of distribution results, for instance by replacing $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$, provoked by dividing one bar after another, with $\frac{1}{2} + \frac{1}{4}$. Clara encountered certain difficulties during this early stage of the learning process. For instance, she had trouble accepting that a divided unit (equivalent to a whole) could be represented by 1. She was not alone in this. She preferred to stick to notations such as $\frac{8}{8}$ or $\frac{4}{4}$ because they reflected the actual division of the unit. Clara quite quickly knew how to manage repeated halving and was later able to apply it accurately, as can be seen in her work on the description of revolutions on a compass-card (fig. 1).

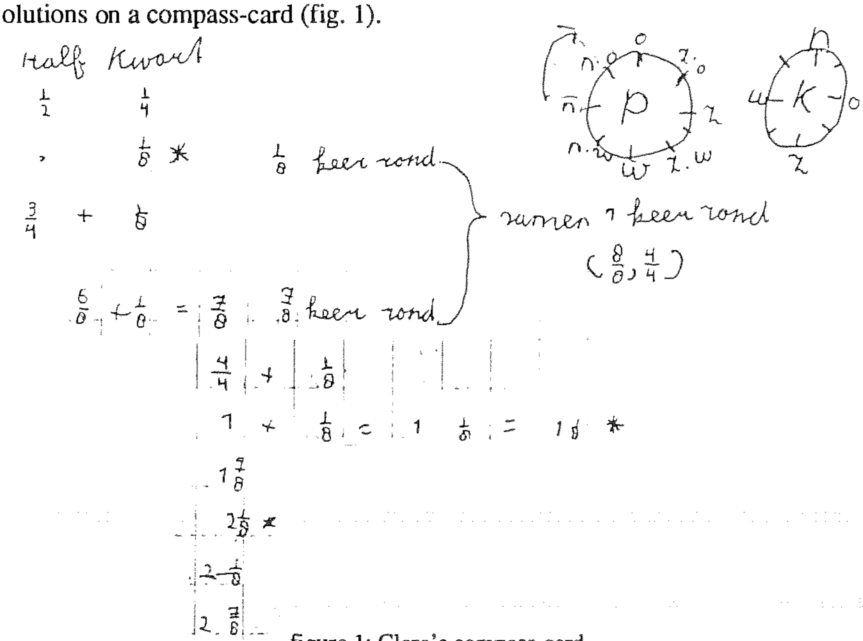

figure 1: Clara's compass-card

For a long time Clara used the terminology 'halves, quarters and half-quarters' in order to insightfully support her work with eighths. She knew how to visualize distribution situations and found handy ways of performing the distributions involved.

In the case where eight children were each to receive $\frac{3}{4}$, Clara suggested giving them all a one and then having them all return $\frac{1}{4}$. This was her way of arriving at 'six for eight'. And so it continued. Clara was quite quick to connect the situation, (which had a symbol of its own $\frac{3}{4}$, which meant four people at a table and three treats on it) and the results of distributing, for instance $\frac{3}{4} \leftrightarrow \frac{3}{4}$, without resorting to intermediate steps. When situations had to be compared, she made use of the mediating support of drawing the divisions concerned. Clara repeatedly demonstrated that she was indeed able to *follow* what was going on.

This did not, however, prevent her from making some IN-distractor errors. But she would then immediately come in conflict with her own standpoint when her alternative solution procedure produced a result which conflicted with the original solution.

When asked whether someone receiving $\frac{1}{2} + \frac{1}{3}$ could be seated at $\frac{2}{5}$ Clara at first said yes, because $1 + 1 = 2$ and $2 + 3 = 5$. But she then corrected herself, saying, 'No, because $\frac{2}{5}$ one will receive $\frac{2}{5}$ and $\frac{2}{5}$ is less than $\frac{1}{2}$.'

While reconstructing the distribution story in which each person received $\frac{1}{2} + \frac{1}{5}$, Clara at first said that there were 7 people sharing $(2 + 5)$, an answer which can also be regarded as an IN-distractor error. But upon constructing a seating arrangement diagram for this situation she realized that there were ten people rather than seven, namely (fig. 2).

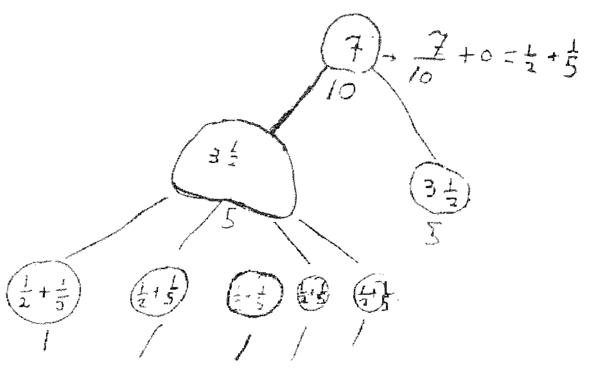

figure 2: seating arrangement

For the rest, Clara was fairly resistant to IN-distractor errors. Results for her were usually not dependent on the method. Her written work was free of such mistakes right from the beginning. Something similar can be said of her work in comparison situations. This agrees with the fact that her written work was consistently of a higher quality than her oral reactions. She was quite hesitant when it came to speaking out, perhaps due to fear of failure.

At first Clara made drawings of the distribution in order to determine the result. She first distributed $\underset{5}{\textcircled{3}}$ and $\underset{8}{\textcircled{5}}$ globally – first everybody gets $\frac{1}{2}$, etc. According to Clara, $\frac{1}{2} + \frac{1}{8}$ was bigger than $\frac{1}{2} + \frac{1}{10}$, because $\frac{1}{8}$ indicated fewer sharers per unit than $\frac{1}{10}$.

Qualitative comparison of matters of time-distance or coffee strength at first proceeded rather haphazardly. Clara sometimes had trouble with the relative standpoint and confused the numbers in the ratios. She initially interpreted 'x times as weak or strong as ...' additively.

She was quite good at producing new situations from given material under certain restrictions and, in the case of coffee strength, even attempted to change both the number of spoonfuls and cups simultaneously. At first Clara appeared to avoid time-distance problems (because she could make no drawings of distributions here?). Later, when it was a matter of determining the difference as well as the order, she sometimes limited herself to qualitative solutions, for example for the comparison of coffee strength, in the cases of $\underset{6}{\textcircled{16}}$ and $\underset{4}{\textcircled{10}}$ she wrote:

> 16 spoonfuls for 6 cups is stronger because you put 6 spoonfuls more and 2 cups more so 16 with 6 is stronger.

Somewhat later Clara applied ratio tables as well. In her conclusions she expressed the differences in absolute terms or limited herself to qualitative statements.

Regarding the situation as a whole we can conclude that, with the exception of a few weaker moments in the first part of her learning process, Clara knew how to relatively interpret situations and their results in the correct manner. Her concept of fractions and ratios was pretty much a match for the distractors.

Indicator 2: progression in schematising

Clara had some difficulty in getting started with the schematisation for the seating arrangement tree. She had once eaten at the restaurant in question so she drew her tables as rectangles and placed them apart. She quickly negotiated this obstacle, however, and was soon drawing actual diagrams, in which her method of abbreviation was to remove or to leave out superfluous branches (see previous section). About three weeks after the class had invented this symbol and diagram, Clara applied another type of abbreviation, namely that of leaving blank those tables she did not need (fig. 3).

Her tables were still rectangular here, but at the next opportunity she exchanged them for round ones. After this, Clara did not come up with new means of abbreviation. She removed branches systematically and found a way of using this diagram for comparing situations. Her need to concretize situations by drawing the distribution become less evident in consequence.

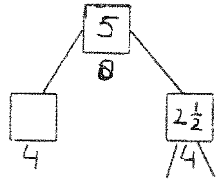

figure 3: blank table

When she got stuck in the diagram due to an inconvenient choice of portions and found herself facing insurmountable numerical obstacles she did not give up, but looked for other possibilities and tried them out (also see indicator 3). The computer program on progression in schematising provided no new insights for Clara; she had the following to say about it:

'I learned on the computer how you can distribute easily.'

In schematisations with the ratio table Clara at first tended to generate pairs through repeated doubling, although not exclusively. Sometimes she added an extra intermediate step. For coffee strength 4S ÷ 3C, for instance, she constructed the following table (fig. 4).

4	8	16	20	32
3	6	12	15	24

figure 4

At this stage Clara was using both the tree-diagram and the ratio-table. It depended on whether a situation had to be reduced to lower terms or extended to higher ones.

Three months later, when comparing coffee strengths $\frac{18}{8}$ and $\frac{14}{6}$, Clara constructed tables until the same 'denominator' appeared.

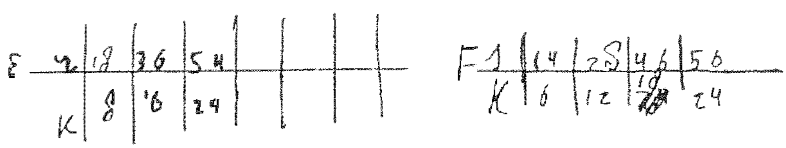

figure 5: making comparable

This phenomenon then continued to dominate Clara's schematisations with ratio tables – she would stop the moment she found an absolute difference.

On a few occasions these comparisons occurred without the aid of any means of schematisation whatsoever. Compared to the learning process of the group as a whole, Clara stayed well within the extremes.

indicator 3: flexible use of models, application of schemes and diagrams (accompanied by flexible calculations)

Clara's schematisation process was not unusual, as we have seen above. Her treatment of the seating arrangement diagram – except for the abbreviations – displayed a certain amount of flexibility. She was repeatedly able to free herself from an impasse by switching to partitions more suited to the numbers involved.

For example: in a retention test after the summer holidays Clara drew the following diagram to compare $\frac{13}{15}$ with $\frac{4}{5}$, after she had first tried to divide by two and (unsuccessfully) by three (fig. 6).

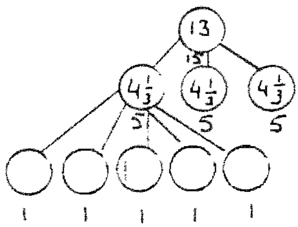

figure 6: reducing to lower terms

She also supported her activities on occasion by making drawings of the distributions. She was quite good at making rough estimations, for example for $\frac{5}{16}$ (fig. 7).

figure 7

She divided the last unit no further than in eighths and then decided: everyone gets more than $\frac{1}{4}$. When using the number-line, Clara appeared to prefer to use line-segments as a means of visualization. Four months after the previous activities, when dealing with traffic lights, she worked the multiplication table for $1\frac{3}{4}$ onto the number-line. She only recorded $\frac{3}{4}$, $\frac{2}{4}$, etc. while ignoring the whole numbers. This is a sign of her tendency to regard the number-line as a series of line-segments. Again a number of months later, at the end of the second research year, Clara estimated the following results for divisions (fig. 8).

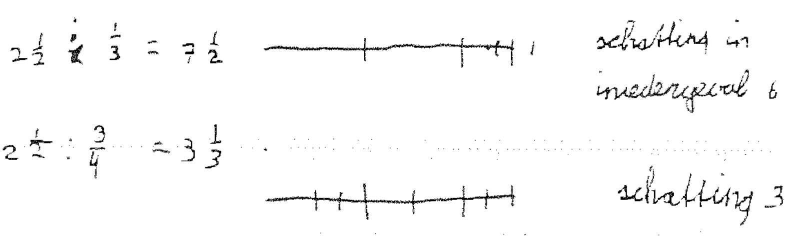

figure 8: estimations

This is clearly more a case of connected line-segments than application of the number-line.

With the exception of the number-line, Clara was able to deal quite efficiently with the resources available, without, however, there being much question of particular flexibility in the use of the models and application of the diagrams and schemes.

indicator 4: ability to build images of formal problems

For a long while Clara acted on her need to concretise more or less formally stated problems. Take, for instance, how she dealt with the computer program. While constructing a diagram for $\frac{45}{27}$, after having divided twice by three, Clara was faced with $\frac{5}{3}$ ○ ○. In order to take the next step in the diagram, she visualized the 'tables' as pizzas. Because of the three tables and three sharers, she then got the numbers mixed up and proceeded to divide by five. Her awareness of what the situation actually meant had retreated to the background because of her need of a representation, of something concrete. Besides visualizing, Clara was also able to verbally describe a problem. Her distribution story for $1\frac{1}{6}$ was as follows:

> 6 children divide 7 pancakes the first one is served, each person gets $\frac{1}{6}$; the second one is served, they get the same again; the same thing happens with the other 5 too; eventually they've each eaten $1\frac{1}{6}$.

With respect to comparing situations or altering them under restrictions, Clara's capabilities were of average quality. Clara was also able to form a clear image of the consequences and results of altering situations, particularly in a qualitative sense. We must mention here that the degree of formality in such matters is debatable, due to the symbols in question, $\frac{x}{y}$ and $\frac{p}{q}$. There was no question of functioning higher than on the intermediate level between the concrete and the formal level. For divisions with (prescribed) application of the number-line, Clara again visualized in terms of line-segments. However, she avoided, problems such as: '$\frac{3}{4} \div \frac{1}{8}$ =; think up a story'.

All in all, Clara's work showed a definite need for mental images. Visualization and verbalization were the appropriate means for her. That suited her, and agreed with the opinions of her various teachers through the years that she was 'quite good' at verbalizing. Sometimes she needed outside help, as in the case of '$\frac{1}{2} + \frac{1}{3}$ and $\frac{2}{5}$', where her first reaction was purely formal.

indicator 5: the pupil's own constructions and productions on the symbolic level

For the previous indicator we discussed Clara's ability to construct through concretization, schematisation and symbolization. In this section our attention is primarily directed towards productions on a symbolic level.

Particularly towards the end of the research project Clara contributed a significant amount in this area. She was aware, for instance, of the potential of pseudonyms, which is the children's term for equivalent fractions (this term 'schuilnaam' sounds less learned in Dutch than in English) as a productional method for monographs. By a monograph we mean a series of fractional number sentences fitting to a single distribution in which the fair sharing takes place in different ways. For instance the monograph '$\frac{3}{4}$':

$\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$
$3 \times \frac{1}{4}$
$\frac{1}{2} + \frac{1}{4}$
$1 - \frac{1}{4}$

fits to 'three bars and four children'. Afterwards it can be extended by means of pseudonyms as a productional method. For the rest, however, her applications had a rather ad hoc character and little line could be found in her work.

After $3\frac{1}{2}$ months: free productions
In terms of quantity, Clara was one of the least productive, but her eight sums were of high quality.
For instance, $8 \times \frac{2}{4} = \frac{16}{4} = 4$; $2 \times \frac{5}{8} = \frac{5}{4}$; $\frac{4}{4} - \frac{1}{8} - \frac{4}{8} = \frac{3}{8}$.
Her work was characterized by:
- application of the operations of addition, subtraction and multiplication;
- use of halves, quarters and eighths;
- application of mutual equivalence.

Two months later: monograph for $\frac{5}{8}$
Clara first divided globally and then by unit. She erred systematically while writing down the intermediate stages.
After $2p = \frac{1}{8} \times \frac{1}{8} = \frac{2}{8}$ up to after $5p = \frac{1}{8} \times \frac{4}{8} = \frac{5}{8}$
For the rest, one of the most remarkable features of her work was the use she made of a previous result.

One month later: monograph of order, $\frac{5}{4}$ and $\frac{3}{4}$
For $\frac{5}{4}$ Clara constructed a tree-diagram and placed the result for each person on the number-line. She determined each person's portion of $\frac{3}{4}$ without resorting to any extra aids. Her work was topped off by the following monograph:

$1\frac{1}{4}$ is more than $\frac{3}{4}$	$1\frac{1}{4} - \frac{1}{2} = \frac{3}{4}$
$1\frac{1}{4} > \frac{3}{4}$	$1\frac{1}{4} - \frac{3}{4} = \frac{1}{2}$
$1\frac{1}{4}$ is $\frac{2}{4}$ more than $\frac{3}{4}$	$\frac{3}{4} + \frac{2}{4} = 1\frac{1}{4}$ (making equal)

One month later: another monograph of order, $\frac{5}{4}$ and $\frac{6}{8}$
It was the first problem, so she took her time with it. This was evident from her work. Compared with the time before, a new element appeared here, namely, the inclusion of inequalities in two directions, with all its monographic consequences:

$\frac{5}{4}$ is more than $\frac{6}{8}$ $\frac{6}{8}$ is less than $\frac{5}{4}$
$\frac{5}{4} > \frac{6}{8}$ $\frac{6}{8} < \frac{5}{4}$

Five months later: constructive filling-in of the number-line.
The point of departure was to 'divide three licorice whips among four children and, later, among eight children'. Clara needed no extra supports on the number-line. The monographic material she produced showed the same features as earlier productions: use of multiplicative abbreviations and previous results and hardly any transpositions to equivalent fractions. She put together the following constructions, in which the corner-stone $\frac{3}{4}$ was not always clearly recognizable:

$2 - \frac{1}{2} = \frac{6}{4}$ (for $\frac{3}{4} + \frac{3}{4}$ and $2 \times \frac{3}{4}$) or $3 - \frac{3}{4} = \frac{9}{4}$ (for $3 \times \frac{3}{4}$).

Her material for $\frac{3}{8}$ showed similar features, albeit that $\frac{3}{8}$ remained visible as a corner-stone.

Four months later: traffic lights
Clara constructed a multiplication table for the following traffic light:
$\frac{2}{3}$ of a minute green, $\frac{1}{4}$ yellow, $\frac{3}{4}$ red (fig. 9).

$\frac{2}{3} + \frac{1}{4} + \frac{3}{4} = 1\frac{2}{3}$

$1 \times 1\frac{2}{3} = 1\frac{2}{3}$
$2 \times 1\frac{2}{3} = 3\frac{1}{3}$
$3 \times 1\frac{2}{3} = 5$
$4 \times 1\frac{2}{3} = 6\frac{2}{3}$
$5 \times 1\frac{2}{3} = 8\frac{1}{3}$

$1\frac{2}{3} \times 6 = 10$
$1\frac{2}{3} \times 7 = 11\frac{2}{3}$
$1\frac{2}{3} \times 8 = 13\frac{1}{3}$
$1\frac{2}{3} \times 9 = 15$
$1\frac{2}{3} \times 10 = 16\frac{2}{3}$

figure 9: table for green, yellow, red

From one month later: monographs
Clara's work was based on the following production methods:

- variation in operations; she applied all the main operations except for division and made combinations as well, for example $1\frac{1}{4} = 6 \times \frac{2}{4} - 1\frac{3}{4}$;
- bilateral use of pseudonyms, yet not systematically as was the case, for instance, with her girlfriend;
- application of properties such as commutative and halving-doubling in a product, for example $\frac{6}{8} = 6 \times \frac{1}{8}$ and $\frac{6}{8} = 3 \times \frac{1}{4}$ in the monograph for $\frac{6}{8}$.

Not much system could be seen here either. Sometimes, as in the example above, she proceeded successively while, at other times, no particular order could be discerned at all.

Six months later: Land of Together (LOT) test

In LOT – as the name says – the inhabitants do a lot, if not everything, together. This is so important to the Lotters, that even their numbers reflect this quality. When, for instance, someone has two apples because he has shared four apples with someone else, in LOT they call this $\frac{4}{2}$ (four for the two of us).

LOT-children learn arithmetic at school with these numbers. The question was whether 'our' children could determine 'their' arithmetic rules and whether 'they' would be able to compute with 'our' fractions.

At the end of this topic the children in the research produced a test for LOT. Clara's test – due to her sluggish pace – was of modest proportions but quite varied. Moreover, her last item reflected a promising continuation, yet would never come to fruition. It involved making rules for addition.

The characteristics of Clara's test were:

- a relatively large number of assignments for practising the main LOT operations, operations which Clara solved correctly herself, including the division sums, for instance $\frac{40}{5} \div \frac{20}{5} = \frac{2}{1}$; $\frac{16}{4} \div \frac{8}{4} = \frac{2}{1}$, etc. (they dealt each time with equivalent cases);
- translation assignments into LOT language (addition only);
- a few application assignments, with the view on the definition of a LOT number;

  16 pancakes are served and there are 4 of you, how many would you get in the Land of Together?

- a few assignments for construction and free production, for instance, constructing a table which ends in $\frac{40}{20}$, or adding together two equivalent numbers under the restriction that the result must also be expressed with a different denominator.

Clara did not, moreover, exceed the boundaries of LOT, that is, she did not design real fractional problems for het test. To a certain extent she was able to develop her own method for an operation, multiplication for instance. This went no further, however, than operating insightfully while applying the distributive property. She developed no method for division other than that of calculating and measuring with line-segments. By the end of the research period Clara was also doing fairly well in formal operations with fractions. She occasionally made mistakes due to holding on mentally to intermediate results rather than writing them down.

In the comparative test at the end of the research-project she calculated $\frac{3}{4} \div \frac{1}{2} = \frac{1\frac{1}{2}}{1} = \frac{3}{2}$ and $1\frac{1}{4} \div \frac{6}{12} = \frac{2\frac{1}{2}}{1}$ correctly. But she went astray with $2\frac{3}{4} \div 1\frac{2}{3} =$ because of the above-mentioned problem.

summary

Clara felt most at home close to reality, and she preferred to stay there throughout a good deal of her mathematization process. She made a number of constructive contributions to first-level mathematization, that is, moving from the concrete into mathematics, yet she was inclined to follow rather than to lead. A few times she gave in to IN-distractors and such, but it did not cost her much effort to extricate herself from the first level in this respect. Her written work never exhibited this sort of error. Also the second level that is, between the concrete and the formal, symbolic level, Clara for a long time needed to maintain links with reality. She did this by visualizing comparison problems or by verbally anchoring standard procedures in a real context, for instance by means of French division which means unit by unit. Or else with language. While applying fractions for repeated halving on the second level, she made the connection between quarters and eighths by describing one-eighth as 'half a quarter'.

Her ability to rise in level by means of schematisation was fair. Seating arrangements were initially closely attached to contexts. Eventually she systematically applied abbreviations by cutting away superfluous branches.

That she increasingly made use of relations instead of concretizations in her activities can be gathered from the fact that her tendency to draw the divisions gradually receded into the background. An indication of this can also be seen in the way she reacted when arriving at an impasse during schematisation. She would then seek a means of escape by finding other numerical relationships which she then, indeed, applied.

Schematisations using the ratio table reflect something similar. The table served to generate the first ratios with a common denominator, which simplified determining the difference. So schematisation took place primarily on the second level with some initial links to the first level through the applied concretizations. Formally stated problems, such as bare division sums, were at first materialized using line-segments. Occasionally there was a question of an ad hoc use of models, for instance when incorporating the computer program.

Clara skipped concretizations of formal problems such as '$\frac{3}{4} \div \frac{1}{8}$, think up a story for this', or so it seemed. Her learning process had certainly not yet reached a conclusive phase. Clara was unable to sufficiently support formal problems using the existing concrete basis, which does not alter the fact that she solved all but one application problem on the final test correctly. Operating formally also proceeded virtually trouble-free.

The nature and quality of Clara's learning process towards the end of the research project contrasts sharply with what took place earlier on. By the end she seemed to have entered well onto the third level with its formal relations. She was aware of these relations and was able to apply them as well. This may have helped to increase her self-confidence.

3 the comparative part of the research.

3.1 introduction

The research was concluded with an external or summative evaluation, in which the effects of the prototype were compared with those of other programs. However, I use these terms reluctantly. Indeed one should recognise all the difficulties one meets when comparing programs with completely different objectives and theoretical frameworks in a mathematical-didactical sense. For this reason it was tried to construct an *average test*, that aimed at objectives also present in the other courses (the test consisted of 32 scorable parts divided in 22 'bare' numerical questions and 10 scorable parts of text-problems). Moreover, our evaluation in the first place aimed at the improvement of our prototype and therefore at eliminating possible omissions in or defects of the course. Our second point of interest – self-evident in developmental research – was contributing to the theory of teaching and learning fractions.

In the analysis of the data special attention was paid to problem-solving process variables such as the application of (visual) models, schemes, clever calculation, algorithms and the presence of ℕ-distractor errors and whether the problems of the test were solved directly, that is, without using any mediating, mathematical tool except for the nation of the outcome.

For this reason the size of the so-called control group (some two hundred pupils) had to be much larger than that of the experimental group (thirteen pupils). Moreover, a representative variety of programs needed to be available. Only then, after all, can particular qualities in the solution procedures and results produced by the experimental group be considered to be effects of the program in question. A few items from the final test in the research will be selected and discussed, together with the data, they rendered.

Before doing so, however, first the 'representative variety of programs' needs to be characterized further.

3.2 programs with which to compare the research-prototype

'participating' textbooks

The following textbooks participated in the comparative part of the research and will be briefly characterized here with regard to the section fractions.

'The Individual Program for Arithmetic' (NCR) and 'Towards Individual Arithmetic' (NZR) can be characterized by a mechanistic approach.

'Arithmetic Work' (RW), 'World in Numbers' (WIG) and 'Number Language' (TT) underwent the influence of the innovations by IOWO and OW&OC. In consequence the sections on fractions have realistic features.

table 1

Textbook Series or Program	n =
Arithmetic Work (RW)*	62 (3)**
World in Numbers (WIG)	15 (1)
Number Language (TT)	76 (3)
Individual Program for Arithmetic (NCR)	27 (1)
Towards Individual Arithmetic (NZR)	22 (1)
Experimental Program (EP)	13 (1)

* between brackets the abbreviations in Dutch
** the number of groups in parentheses

NCR and NZR's mechanistic approach

The most fundamental characteristics of the approach of fractions in these two textbooks, which, at the same time, are the differences between this approach and the realistic one:

- there is a lack of situations taken from a child's reality; for example, exploring and manipulating actual distribution situations, performing these divisions and describing the divisions and their results.
- the impression is raised that calculating with fractions is just like calculating with the whole numbers mentioned, for example, 3 fourths + 2 fourths = ... by analogy with 3 km + 2 km = ...; this strongly undermines the fact that a fraction serves to describe a relation and the anchoring of this idea in particular in the teaching process; working with unchangeable units in the fraction material also contributes to this; stated in the extreme: from such programs, children will not learn what a fraction is.
- a prevailing emphasis is laid on practising the rules for operations, case after case, row after row, section after section, with only one goal in mind: the isolated, rule- oriented reproduction of the system.

Such fraction courses in terms of numbers of pages, comprise about 30% of the whole program available for the three upper grades of primary school in The Netherlands (10-12 years).

the realistic traits of RW, WIG and TT

WIG, RW and TT contain realistic features in the sense that actual distribution problems serve as a source for fractions and as a domain of application. Not only that, but measuring is used as a source as well, both in TT and in RW. On the other hand, it must be stated that both textbooks jump too quickly to levels that are too formal. Other similarities between these programs and the developed prototype are, for instance, the manifold use of diagrams and visual models. In all three of the above-

mentioned programs, the ratio table appears often and diverse visual models are frequently applied, for instance, the number-line and the model of surface area.
Similar features are also found in the pursuance of internal relations, for instance, the relation between fractions and decimals or fractions and percentages; albeit that these relations are often stated on the basis of structural and formal considerations.
A difference exists between these textbooks in the final goals with respect to the four main operations. RW explores the four main operations unrestrainedly. TT does not proceed quite as far, regarding only whole number divisions. WIG is another step behind and only presents those cases in which a division of two small whole numbers is transposed into a fraction, for example: $2 : 3 = \frac{2}{3}$, $5 : 2 = \frac{5}{2} = 2\frac{1}{2}$.
WIG also lays restrictions on multiplication, by only multiplying a fraction or mixed numbers by whole numbers. The amount of attention paid to fractions in the three textbooks described here is comparable to the experimental program. This subject is dealt with explicitly on the average of once every ten lessons, about half of what is found in mechanistic programs.
In TT the activities are grouped in clusters of lessons. This is to a certain extent also the case in WIG and RW, albeit that the clusters are smaller. The experimental program could only be conducted in individual lessons, the interval between them varied from one week to three months.

3.3 the final test

the items

The final test, as said before average in character with respect to the goals reflected by it, consisted of 32 scorable elements, 22 of which were bare sums and 10 consisted of (parts of) textual problems.
In order to offer an impression to the reader of the context of the test, some of the items will be selected, their goals discussed and the results presented both in a qualitative and in a quantitative manner.
The discussion of the items' goals reflects the arguments why these items are considered to be suitable for an average test.
Moreover, as far as the expected solution-procedures are concerned, the goal description is connected with the indicators serving as a framework for the description of an individual learning process.

some items and their results

1 *First example:* Item 6
Fill in: 'more than', 'less than' or 'equal to'.
Write down why you think so.
$\frac{4}{10} + \frac{1}{10}$ is .... $\frac{1}{2}$, because ....
$\frac{1}{5} + \frac{1}{5}$ is .... $\frac{1}{2}$, because ....

$\frac{2}{5} + \frac{7}{10}$ is .... $\frac{1}{2}$, because ....
$1\frac{1}{4} + \frac{8}{12}$ is .... 2, because ....
$4\frac{3}{8} + \frac{7}{8}$ is .... 6, because ....

Goal
Depending on where the emphasis was laid in the teaching process, various matters might be involved in these problems. On the one hand, it may have to do with whether the fraction concept is operational on an insightful basis in certain comparison matters where addition holds a dominant position. And, on the other hand, whether operating is more directed by rules ('making them equal', for instance). This is related to the indicators 1 and 5, whereby the first is of overriding importance due to the opportunity for making IN-distractor errors.

Other indicators might also be involved in cases where pupils apply other resources for their solutions.

Results
The results, quantitatively summarized, boiled down to (table 2).

table 2: compounded results

characteristics of the problem solving	percentage of pupils		
	experimental	realistic	mechanistic
(visual) model	8 (15)	1	2
scheme (table)	(15)	–	–
clever calculation	69	2 (3)	–
direct outcome	–	38	35
algorithm (rule)	23	56	63
not done	–	2	–
total	100	99	100
IN-distractorfailure	–	8	18
outcome correct	87	79	59

Qualitative examination of the results.

Experimental group
Clever calculation stands out here most and will be analysed in more detail. Of these nine pupils, a few still sought support in a visual model or diagram, but with the emphasis laid on clever calculation; this is why the number in parentheses for these characteristics is quite substantial.

clever calculation

This was evident primarily in the flexible interpretation of both the value and the appearance of a fraction. For example:
'$\frac{1}{5} + \frac{1}{5}$ is less than $\frac{1}{2}$, because $\frac{2}{5}$ is almost half, just a little less and $\frac{3}{5}$ is too much', according to Margreet, or: '$\frac{1}{5} + \frac{1}{5}$ is less than $\frac{1}{2}$, because $\frac{1}{5} + \frac{1}{5} = \frac{2}{5}$ and $\frac{2\frac{1}{2}}{5}$ is half', stated Clara, or: '$\frac{2}{5} + \frac{7}{10}$ is more than 1,because $\frac{2}{5} + \frac{7}{10}$ $(3\frac{1}{2}) = \frac{5\frac{1}{2}}{5}$ and $\frac{5}{5}$ is one', was Kevin's reasoning.

Six of the nine children who calculated cleverly reasoned in this manner.

This attests to the ability to flexibly deal with equivalences in order to perform the comparison in question, without rigid imitation of the rules dictating the solution procedure. No N-distractor errors were made.

Control group

The algorithmic approach and direct solutions were the most apparent (table 2). With the exception of one TT group, all groups made N-distractor errors, albeit on a fairly limited scale. 'Simplifications' such as $\frac{2}{5} = \frac{1}{10}$ were made twice in both the realistic group and the mechanistic group but are not counted here as N-distractors. They may, however, be judged to be so, due to the misconception that the product of numerators and denominators is constant in equivalent fractions.

algorithm (rule)

The following collection of comments reflects the algorithmic orientation of a portion of the realistic group, which prevailed regardless of the course that was followed. These pupils' view of fractions is related to this orientation.

'$1\frac{1}{4} + \frac{8}{12}$ is less than 2, because $1\frac{1}{4} = 1\frac{3}{12}$; $1\frac{3}{12} + \frac{8}{12} = 1\frac{11}{12}$' may be said to be characteristic of the correct rule-oriented approach.
'$\frac{1}{5} + \frac{1}{5}$ is $\frac{2}{5}$, less than $\frac{1}{2}$, because $\frac{2}{5} + \frac{2}{5} = \frac{4}{5}$ and you can't divide that.'

'there aren't any halves of $\frac{5}{5}$.'
'you can't divide a 5th by two.'
'you can't divide $\frac{2}{5}$ by two.'
'because 5 can't be divided with fractions.'
'in decimal numbers $\frac{1}{2}$ is 0.50 and $\frac{2}{5}$ is 0.40.'
There were some less 'efficient' approaches as well.
'$\frac{2}{5} + \frac{7}{10}$ is more than 1, because $\frac{55}{50} = 1\frac{5}{50}$ and not 1.'
'$1\frac{1}{4} + \frac{8}{12}$ is $\frac{12}{46} + \frac{32}{46} = \frac{44}{46} < 2$.' (error in calculation)
'$4\frac{3}{8} + \frac{7}{8}$ is more than 6, because $\frac{56}{64} + \frac{24}{64} = \frac{80}{64}$, 80 - 64 = 16 whole ones = $\frac{20}{64}$.'
'$\frac{4}{10} + \frac{1}{10}$ is equal to $\frac{1}{2}$, because 10 fits $1 \times$ in 10 so you have to also do 1×4 and 1×1.'

A few comments contained premature generalizations with regard to the fraction concept, for instance, '$\frac{1}{5} + \frac{1}{5} < \frac{1}{2}$, because the smaller the fraction, the larger the value, so $\frac{1}{2}$ is larger.'

A line of reasoning, in which a fraction such as $\frac{2\frac{1}{2}}{5}$ appeared, for instance, occurred in four cases.

On the whole, the results of this item reflect the fact that the majority of the pupils from the realistic group regarded fractions as nothing more than compiled numbers – some of which consist of disconnected parts – which can and should be dealt with algorithmically. The main source of error, besides N-distractors, was the incorrect application of rules (errors in calculation).

The results of the mechanistic group concur with the above and reflect, if possible, an even stronger focus on rules and algorithms. This is attested to by the relatively larger number of N-distractor errors as well as by the following errors of transposition: $\frac{2}{5} = \frac{1}{10}$ and $\frac{7}{10} = \frac{14}{5}$.
'$\frac{1}{5} + \frac{1}{5}$ is less than $\frac{1}{2}$, because you have to have $2\frac{1}{2}$ 5ths to make a half', which attests to a more flexible view of fractions, appeared twice.

Occasionally, the pupil answered correctly, but without knowing why, for instance: '$\frac{2}{5} + \frac{7}{10}$ is $\frac{4}{10} + \frac{7}{10} = \frac{11}{10}$....1, because I don't know, sorry, I can't help it, at least I'm trying.'

Other students escaped by means of the following logic: '$\frac{2}{5} + \frac{7}{10}$ is more than 1, because 1 is less.'

These results lead to the conclusion that – with a few exceptions – fractions have acquired absolutely no significance either on the concrete level or on the intermediate level.

direct

For this characteristic we shall only regard the quality of the argumentation underlying the determined order.

Of the 60 realistic group pupils 27% who reasoned directly used the wrong arguments, some of which were quoted above. In the mechanistic group this was 44%. Even more striking was the number of students who partially or entirely omitted any argumentation. The mechanistic groups and the RW schools were particularly conspicuous in this respect. Argumentation was missing in 18% and 7% of the cases respectively, while only a very few students avoided the problem entirely.

These results form an indication that introduction on a too formal level (mechanistic group), or advancing too quickly to this level (RW) leads to negative effects for at least some of the students.

2 *Second example:* Item 8

3 peppermint sticks are divided fairly among a group of 4 children.

7 peppermint sticks are divided fairly among a group of 10 children.

\> Compare: in which group will a child get more, in the group with 4 children or in the group with 10 children? How much more?

Goal

Comparison problems did not appear in all the textbooks under consideration. With respect to the experimental program, this problem refers to indicator 1, in connection with the possibility of not taking proportional reasoning into account (IN-distractor errors), and the opportunity for introducing visual models.

Indicator 2 may also be involved when the ratio table is used as a means of schematisation. In cases where this last method is applied in an extremely abbreviated form, this may also signify the presence of a rule or algorithm (lcd-method), therefore indicator 5 as well.

Similar considerations may also hold for some of the realistic programs (RW, WIG). One could state in general that this problem, offers two distribution situations, which must be dealt with separately before comparing the results.

Moreover, the point here is to set a standard for situations, keeping comparison in mind; setting standards appears in all programs with regard, at any rate, to percentages, scale concept and addition and subtraction of dissimilar fractions. For programs, therefore, where few or no comparison problems appear, this item can still be linked to suitable goals.

Note:

Among other, keeping the formulated intentions in mind, a quite varied impression in terms of the mathematical resources used can be gained from the solutions offered.

Results

The results, again quantitatively summarized, are shown below (table 3).

table 3

characteristics of the problem solving	**percentage of pupils**		
	experimental	**realistic**	**mechanistic**
(visual) model	15 (31)	7	14 (16)
scheme (table)	62	3	–
clever calculation	(8)	4	–
direct outcome	15	41	43
algorithm (rule)	8	35	33
not done	–	9	10
total	100	99	100
IN-distractor failure	–	3	2
outcome correct	85	33	6

Qualitative examination of the results.

Experimental group
Application of visual models and schemes predominated here while, in the control group, the preference for direct solutions and application of algorithms is more apparent. These aspects will be examined in more detail.

visual models
Four pupils approached this problem by regression to the unit or through global distributory. Two of them described the process using equivalent fractions and two switched to the ratio table after having carried out the division.

ratio table
Three pupils worked towards $\frac{15}{20}$ and $\frac{14}{20}$, while one student extended the table for $\frac{7}{10}$ by one more ratio. One pupil sufficed with a table for $\frac{3}{4}$ and used this in a clever way (fig. 10) to arrive at:

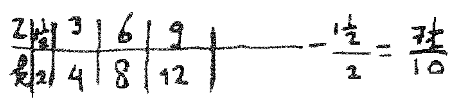

figure 10: clever application of ratio table

One pupil made use of abbreviated tables ('leaps') (fig. 11).
Two pupils applied extremely abbreviated tables, working directly towards situations with common denominators. The solution scored as direct was similar to the above, but involved no schematisation with the ratio table.

(Here you get $\frac{2}{40}$ more)!

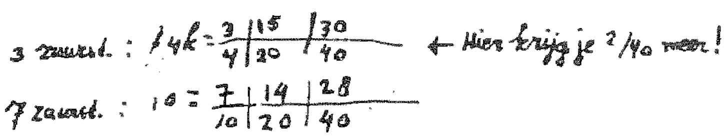

figure 11: 'leaps'

As mentioned above, the direct and the algorithmically determined solutions draw attention here (table 3). These and other aspects will be examined in more detail.

direct
This item was apparently less suitable for direct solutions. All of the pupils from the mechanistic group who dealt with it in this way came up with an incorrect answer; of the realistic group pupils, 25% had the correct answer.

In the ten cases where $\frac{\bullet}{3}$, $\frac{\bullet}{7}$ and $\frac{\bullet}{21}$ appeared in the results, this construction of incorrect operational relations betrayed its basis of an incorrect relative standpoint.

algorithm (rule)

This approach was nearly as well-represented as the direct one.
Within the mechanistic group, there was evidence of a preference for setting a standard of 100 or for transposing into percentages. Half of the rule-oriented pupils dealt with the problem in this manner.

In the realistic group, finding equivalences predominated (one-third of the pupils), followed by the transposition into decimal numbers (one-quarter of the pupils). A common derailment in both groups (which was also observed in the direct solutions) was the placing of the number in question in an incorrect context and then determining the result by means of long division and decimal numbers or by normal fractions (fig. 12).

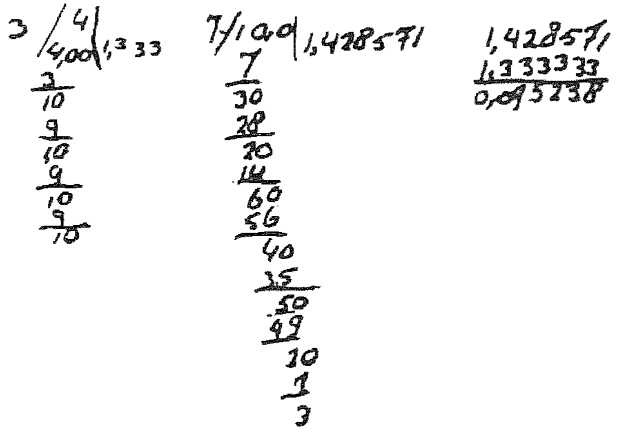

figure 12: derailment

This phenomenon appeared in one-third of the algorithmically tinted solutions of both the mechanistic and the realistic groups. If we include the direct solutions, then more than 15% of the control group pupils acted in this fashion (23 mechanistic and 10 realistic).

This approach, notwithstanding the algorithmic orientation of the pupils concerned, attests to an extremely narrow notion of division, namely, 'the smaller number has to go on top of the larger one, because division makes things smaller'.

Other aspects

For an impression of the other characteristics of the solution procedure, we will now present one example of each, all from the realistic group.

visual model

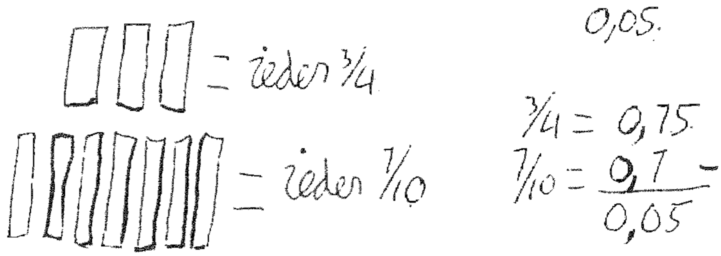

figure 13

This solution (fig. 13) shows that the pupil only needed the visual material to get started. It was not necessary to perform the division.
Awareness of the situation was sufficient to then determine the solution on the second level by setting a standard of 100 through decimal numbers.

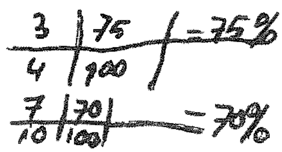

figure 14: extremely abbreviated ratio table

Another form of setting a standard of 100, applied by a few pupils from the same RW school, was organized in a ratio table (fig. 14) and, moreover, nicely put into words by one pupil:

> (If there had been 100 children in *a*, then you would have needed 75 peppermint sticks. If there had been 100 children in *b*, then you would have needed 70 peppermint sticks. So one child gets 0.75 peppermint stick in *a*, and one child gets 0.70 peppermint stick in *b*, so you get more in *a*).

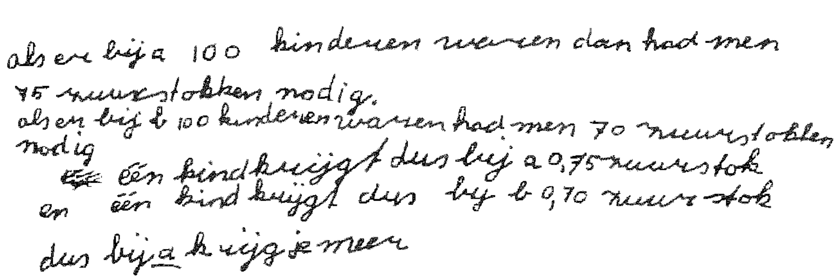

figure 15: solution verbalized

In addition to the decimal numbers, percentages were also used in this case for determining the result (fig. 15).

Imagined model (global distribution)

Here the pupil was able to do without visual material and described quite well how the division process can take place.

The last solution (fig. 16) reflects a fraction concept that evidently functions excellently on the concrete level and on the intermediate level between the concrete and the formal level of subject systematics.

(3 peppermint sticks 4 children
first $\frac{1}{2}$, 1 remains, then $\frac{1}{4}$ they get $\frac{3}{4}$ peppermint stick
7 peppermint sticks 10 children
first $\frac{1}{2}$, 2 remain, then $\frac{1}{5}$ they get $\frac{7}{10}$ peppermint stick)

3 zuurstokken 4 kinderen
eerst $\frac{1}{2}$ blijft er nog 1 over dan $\frac{1}{4}$
krijgen ze $\frac{3}{4}$ zuurstok
7 zuurstokken 10 kinderen
eerst $\frac{1}{2}$ blijf over: 2 dan een $\frac{1}{5}$
krijgen ze $\frac{7}{10}$ zuurstok

figure 16: solution story

Here is how the final tests was considered in the research.

Table 4 offers an overall impression of the outcomes, which are rather in favour of the experimental course.

table 4: characteristics of solutions

characteristics of the problem solving process	average application per pupil	
	experimental group	control group
(visual) model	2,2	0,5
scheme (table)	1,3	–
clever calculation	3	0,6
direct outcome	3	4,8
algorithm (rule)	1,9	3,5
outcome correct	22,3 (70%)	18,2 (57%)
IN-distractor failure	0,5	0,7

It should also be stated, however, that the evaluation revealed some defects of the prototype, such as paying too little attention to applications of surface area related to the multiplication of fractions.

One of the most striking conclusions might be the following. When traditional education is coupled to the use of a traditional (that is a mechanistic) textbook, the effects of such education are really poor, even in the areas where a great deal of time, effort and energy has been invested, that is, the 'bare' sums. This is indicated by the test results of the mechanistic subgroup of the control group. The average number of correct results was 13.5 (42%), a much poorer result when compared with the subgroup of the control group using textbooks with realistic traits – 19.7 (62%) – and the experimental group – 22.3 (70%).

This occurred despite of the fact that mechanistic textbooks included twice as much material for fractions as the others. This means that much of this time-consuming work can be scrapped without detriment to the results.

In turn this conclusion is affirmed by the results of the final CITO-test for primary school, 1986. These results reveal how the control group's results on mathematics lagged behind those of the experimental group (tables 5 and 6).

table 5: CITO-test 1986

Arithmetic 1986	**Average number of correct results (max. 60)**
Experimental group	50 (84%)
Control group	38 (64%)
'Realistic' subgroup within control group	40 (67%)
'Mechanistic' subgroup within control group	33 (55%)
Nationwide sampling n = 2240	43 (72%)

If we regard within these results the cluster of 27 problems involving fractions, ratios and percentages, then the following results can be seen:

table 6: subtest on fractions etc.

Arithmetic 1986 subtest on fractions etc.	**Average number of correct results (max. 27)**
Experimental group	23,5 (87%)
Control group	16 (59%)
'Realistic' subgroup within control group	16,7 (62%)
'Mechanistic' subgroup within control group	13,7 (51%)
Nationwide sampling n = 2240	18,7 (69%)

4 discussion

With regard to the attitude developed in the teaching process, the control group could be seen as an entity, considering its powerful focus on rules and algorithms. Nevertheless, there was a fundamental difference between the realistic and the mechanistic sub groups. In all cases, the realistic group scored higher than the mechanistic, whenever the problems were solved without any observable mathematical or visual aids. This indicates that constant confrontation with problems situated in a context will have an effect in the long run, regardless of the manner in which the teaching process is otherwise furnished. The same is true to the same degree of bare fractional arithmetic, taking into account the considerable difference between the realistic and mechanistic subgroups in how they dealt with the operation of multiplication.

The control group lagged dramatically behind the experimental group in the area of clever calculation. On the other hand, the algoritmic orientation of the control group and its predominating tendency to apply rules did not give them any advantage with respect to the experimental group.

This is an indication as well that the teacher's behaviour is of great influence since, in the realistic programs, attention is certainly paid to clever calculation. Indications for the influence of the teacher's behaviour are formed, among other things, by the following observations:

- in the WIG group, not one pupil applied the ratio table, even though this means of schematisation forms a structural part of the course for fractions, ratios and percentages in these textbooks.
- in the RW group, which consisted of sixth grade classes from three different schools, one group distinguished itself where the use of visual models and the ratio table was concerned; on the basis of the program, this should have been the case for all three groups.
- in the TT group, which also consisted of sixth grade classes from three schools, one group distinguished itself by the powerful domination of rule-orientation in items 6, and 10, the last item consisting of 12 bare problems concerning the main operations.

  In this group, 40 pupils strong, half the pupils did not realize, for instance, that $\frac{1}{2}$ goes into $\frac{3}{4}$ one-and-a-half times. They calculated as follows:
  $\frac{3}{4} : \frac{1}{2} = \frac{3}{{}_{2}\cancel{4}} \times \frac{\cancel{2}^{1}}{1} = \frac{3}{2} = 1\frac{1}{2}$.

One and other signifies that we must state – with caution – that even the realistic textbooks are used for mechanistic instruction or, at any rate, education which is strongly rule and algorithm oriented.

On the other hand, we should mention that the instructional behaviour of the teachers who participated this development research, and who did during the course of the project develop the necessary attitude for realistic mathematics education, also influenced the results of this research.

IN-distractor errors are unavoidable, even when one counteracts them during the educational process. Their appearance indicates that the ties with the sources of insight have been forcibly broken, should these sources have been used to begin with. Surprisingly enough, in the experimental group two pupils who, during their learning process, had apparently built up enough resistance to these distractors made such an error on the test. In connection with the above it may be stated that an approach on too formal a level will result in the blind performance of rules, particularly for the weaker pupils. This will then stifle original insights (in repeated halving, for instance) and attitudes (the spontaneous tendency to visualize).
Finally, the most important conclusion, the research resulted in an outline of a framework for courses on fractions, which permits many different elaborations but at the same time exploits the main outcomes of this research. It offers an integrated perspective on fractions. But that is a different story (cf. Streefland, 1991[a]).

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Geometry instruction in The Netherlands (ages 4 -14) – the realistic approach –

E. de Moor

1 short history

nineteenth century

The first legislative acts which deal with education in The Netherlands date back to the early nineteenth century. These acts also stipulated the contents of the curriculum. It will not come as a surprise that reading, writing and arithmetic were designated as the most important subjects of instruction for primary school. But already in the nineteenth century we also find a type of geometry in the curriculum, for some thirty years even made compulsory by legislation. Officially the subject was referred to as 'Vormleer'. (In German: Formen und Grössenlehre), literally meaning 'theory of shapes'. Several interpretations of this subject matter were prevalent at various times in The Netherlands.

'Vormleer' is the brainchild of the renown Swiss educationalist Pestalozzi (1746-1826), whose name is strongly connected with the idea of 'Anschauung'. According to his ideas about teaching and learning, practical experience and observation should always precede the abstract word.[1]
His educational theories and methods have had a major influence on the development of education in Western Europe, particularly in Germany and from there also in The Netherlands.

In our country this 'Vormleer' caught on during the first half of the nineteenth century as 'a method for the continuing development of one's intellectual faculties' as Van Dapperen describes it in the most prominent textbook of that era.[2] This textbook contains a large number of think-activities that can best be described in terms of ordering and working systematically.

There were many at the time who believed strongly in the formal value of mathematical activities such as these. In this way people would become better thinkers and hence the quality of society would improve. Here is one problem item from the book in question (fig. 1): 'If 4 lines are joined in 1 point, how many angles can be made?'[3]

figure 1: Vormleer (1820)

The only thing that 'Vormleer' had in common with geometry was the fact that the reasoning had to do with points, lines and other geometric concepts. Real (Euclidean) geometry was not introduced until secondary education (at age 12). Another interpretation of 'Vormleer' was the following one. Pupils should closely observe real objects in their surroundings in order to be able to grasp elementary geometric concepts such as rectangle, square and so on. In the meantime they could acquire some skill in drawing these figures.

From the mid-nineteenth century the emphasis of formal value was stressed less. From then on more attention was devoted to the applicability of mathematics. Taking a look at the 'Vormleer' textbooks of that period we find problems about measuring and calculating length, area and volume and problems dealing with geometric constructions. But all of these problems to be carried out from the principles of observation. So for instance, the area of a parallelogram was to be 'seen' as the result of a (mental) dissection and re-arrangement of the parts (fig. 2).

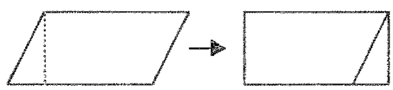


figure 2: cut and glue

Dutch textbooks from the second half of the nineteenth century and discussions about 'Vormleer' in educational journals make it clear that 'Vormleer' had extended well beyond the level of primary education. But the schoolmasters of that time were not trained to teach this subject matter. And so 'Vormleer' was deleted from the official curriculum for primary schools in 1889.

From the mid-nineteenth century education for the very youngest pupils (age 4-6) received a status of its own. Instead of the nursery schools, the idea of 'Kindergarten' was now introduced. These became schools with a special and adequate program.

Fröbel (1782-1852), who was once himself a student of Pestalozzi and also taught at his institute for a number of years, did remarkable work in the field of education for this age group (4-6).

In regard to the subject at issue here, geometry, it is especially Fröbel's ideas about the development of spatial orientation that are of interest. In a different manner than Pestalozzi, who started with the 2-dimensional plane, Fröbel began with exploration of our 3-dimensional space. Famous are his so called 'gifts' by which children, while playing, could become acquainted with elementary geometric concepts (fig. 3). He thought that concrete experience was more important than observation purely with the eye. In Pestalozzi's view the manner of instruction was to have a synthetic character. But in Fröbel's view the method of instruction is based on the analytical principle. Building with blocks, construction with sticks (3-D and 2-D), cutting and pasting, drawing, weaving and mosaics are still regarded as the fundamental activities of geometry instruction.[4]

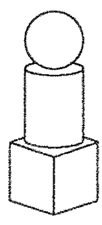


figure 3: sphere, cube and cylinder, Fröbel's 'second gift'

In The Netherlands Fröbel's ideas – and later those of Montessori – have had noticeable impact on the curriculum for infant schools (ages 4-6).

But because infant school (4-6) and primary school (6-12) were strictly separated, the geometric activities remained limited to infant school. Because since the abolishment of 'vormleer' at primary level, the contents of the mathematics curriculum was reduced to arithmetic only. Since 1985 these two schools have been joined to form one primary school for ages 4 to 12. Whether from now on geometry will constitute part of the curriculum and how it will then be taught is something we shall discuss in the following.

1900 – 1950

At secondary school level, which prepared for University and higher vocational training, the geometry curriculum initially started from the centuries old tradition of Euclid. It was around 1920 when for the first time a didactical discussion arose about this logic-deductive approach. Among others it was T. Ehrenfesst-Afanassjewa who argued in favour of a preparatory course in geometry. Such a course was to have an 'intuitive' character and be based on 'empirical' geometric experience.[5]

But obviously the time had still not come for these ideas to be accepted. And so it was that introduction to geometry in secondary education during the first half of the twentieth century continued to have a logic-deductive character.

During the nineteen-fifties the Van Hiele's published their dissertations on the introduction to geometry. Dieke van Hiele-Geldhof with an intuitive introductory course[6], Pierre van Hiele with his famous level-theory about the geometric learning process.[7]

Then in the sixties another interesting experiment was carried out in which traditional plane geometry was composed according to the principles of the geometry of transformations.[8] But this program was never officially introduced either. In consequence of a radical change of the Education Act in 1968, and also influenced by the New Math movement, a new curriculum for mathematics in secondary education was introduced. In this curriculum algebra and geometry no longer existed as separate subjects. In fact, one could say that at that moment 'traditional' geometry disappeared from the curriculum. And so Euclid was replaced by a short informal introduction of transformation geometry, quickly followed by vector geometry (linear algebra). This program was maintained for some twenty years. Currently there is attention once more for solid geometry.

1970 – present

In the meantime, in the early seventies, the Institute for Development of Mathematics Education (IOWO) was founded. Other publications deal with the total work of the IOWO.[9] We will restrict ourselves here to the development work that was done in the field of initial geometry (age 4-14). Under the inspiring leadership of Freudenthal (1905-1990) a new impulse was given to this subject.

The essence of the new approach – from here on to be called the *realistic approach* – was based on the following starting points:

- besides the study of number, mathematics also deals with the study of space;
- geometry instruction must start with and relate to real phenomena of the space that surrounds us;
- pupils of ages 4-12 are also entitled to geometry;
- the method for the introduction to geometry must have an intuitive character;
- formalisation will constitute the end point of a vertically planned curriculum.

We will not dwell on the process of development here. We will restrict ourselves to the results of some fifteen years of study and (experimental) research, that still have not come to an end. One of the problems that remains is to construct a satisfactory sequence for the various geometry topics for the primary level.

2 realistic geometry

Below we will first describe the various aspects of realistic geometry, which we have labelled as follows:
Sighting and *projecting*, *locating* and *orientating*, *spatial reasoning*, *transforming*, *drawing* and *constructing*, *measuring* and *calculating*.[10]

We will give a brief description of each of these aspects that will be demonstrated further by a number of geometry problems.

sighting and projecting

With regard to this aspect we list the following activities: Looking at (observing), perceiving, representing and explaining spatial objects and spatial phenomena, in which the idea of the straight line plays a central role. First as a 'line you are looking along' (sighting on), but also as a 'ray of light' (projecting). Some derived concepts will also be raised such as: point, direction, angle, distance, parallelism, intersecting and non-intersecting lines in space, planes, ..., and elementary relations between these concepts.

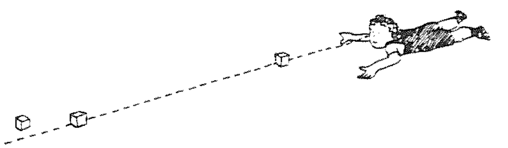


figure 4: what is a straight line?

From the very outset space ought to be explored on this aspect, working from everyday experiences and simple experiments. The activities ought to have an informal character so that intuitive notions of these elementary concepts of geometry can be raised. That means starting from the real world. Various everyday phenomena prove to be a rich source of geometric activities.
For the very youngest pupils (4-6):
- hide and seek
- constructing peep-boxes
- far-near experiments
- making shadows
- looking through a cylinder
- ...

For ages 6-10 many of the same activities can be explored again. But now with more attention for explanations through *visualisation*, as shown in figures 5a and 5b for instance.

To mention a number of other activities:
- hold the thumb in front of the eyes and alternately close one eye and then the other, why does the thumb jump from right to left and vice versa?
- walking in the sun your shadow always has the same length. Why?
- why does the moon keep pace with you when you are out walking at night?
- what happens with the projected image when you place the overhead projector at various distances from the projection screen?
- what kind of figures can you make when holding a wire-model of a rectangle with its diagonals in the sunlight?
- and what happens if you do the same using a light bulb?

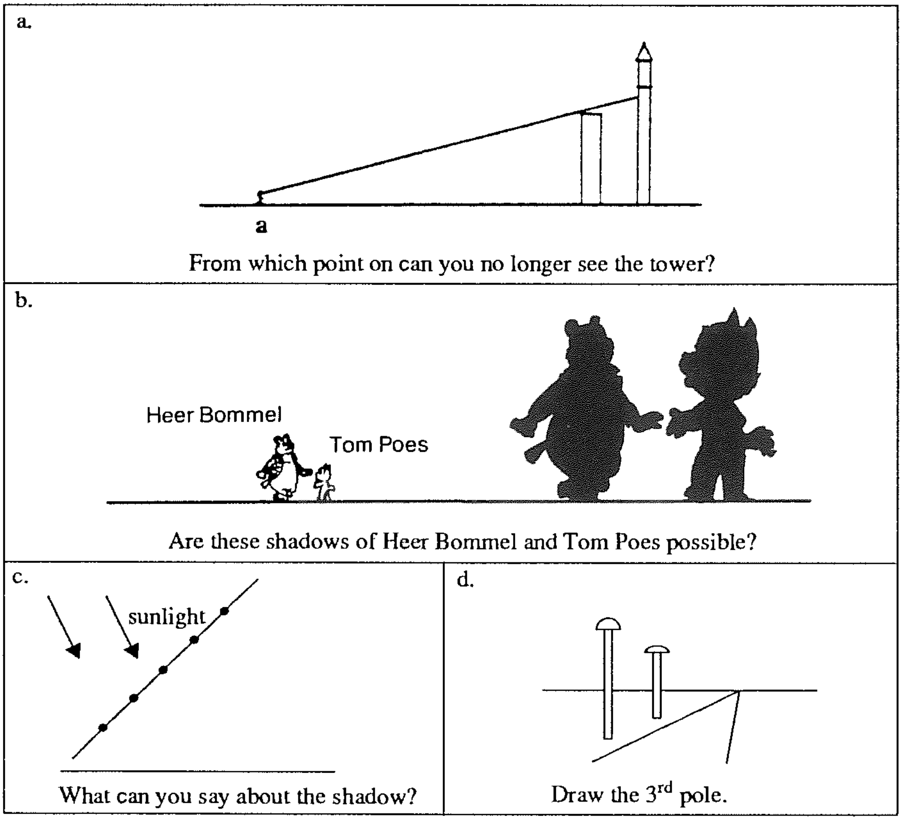


figure 5

For ages 10-14 it is possible to delve further. But by that time one may expect that the pupils are capable of (drawing) certain constructions and that they are able to give simple explanations for the phenomena they have seen before (fig. 5c and 5d).

The cartoon of figure 5b illustrates the essence of central projection. In the perspective problem (fig. 5d), the pupil must know that the point of intersection of the diagonals of the image of a rectangle (the middle) will be mapped on the point of intersection of the diagonals of the image (a trapezoid). A theorem that one can 'see' so easily from the light-shadow experiment. And in problem 5c we can recognize the main theorem about the invariance of ratio in affine projection.[11]

orientating and locating

Orientating in everyday life simply means that one knows where one is in the surrounding space and that one knows how to get from one point to another. Both in the immediate surroundings (a room, house, neighbourhood) as well as in the more extended surroundings (city, country, world, ...). Locating has less to do with the individual himself, but is more related to the descriptive aspect of orientation. Locating is defining the (relative) position (and sometimes the time) of an object in a given space. Such a description can be made by means of language, drawing, symbols and/or formulas. Speaking in more concrete terms, locating has to do with descriptions of routes, maps, blueprints, graphs and (spatial) models.

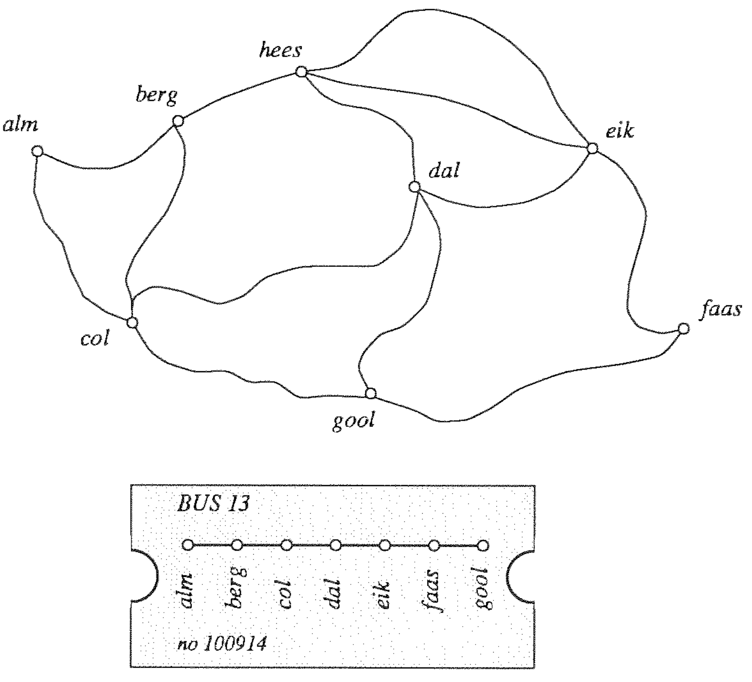


figure 6: map and bus ticket

Traditionally, orientating has been regarded as important in Kindergarten. And indeed, there are possibilities to acquire concepts such as 'in front of/behind, up/down, far/near, ...' and so on, through games both at school and at home. It is not long ago that this was the only thing undertaken in this regard.

Currently there are possibilities to develop a more structured curriculum, also in this respect. Take for instance the problem where one has to describe by telephone the exact shape of a certain construction of blocks. The levels of description can vary from self-thought up descriptions to certain codes which can ultimately result in the use of coordinates (fig. 7).

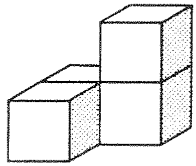


figure 7: explain by telephone how to build this

Special attention should be devoted to the idea of time in relation to the activities of orientating and locating. In regard to this aspect, vertical planning of the curriculum, in which so called destination-time graphs play an important role, must also be included (fig. 8).

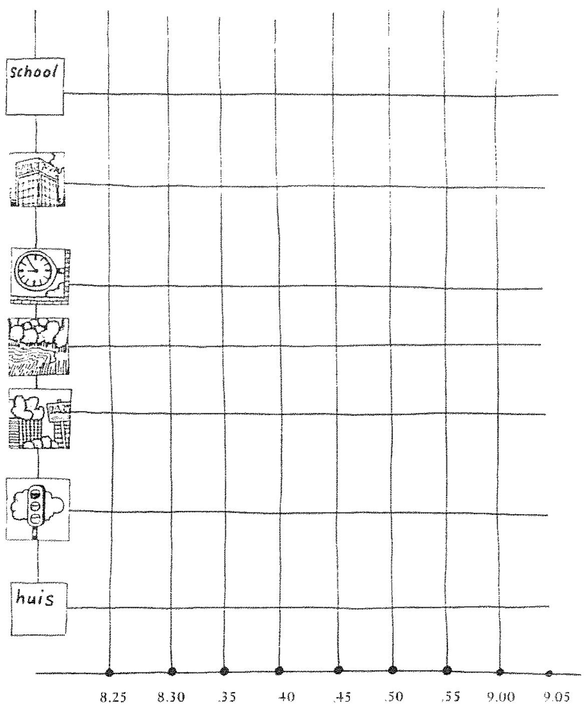


figure 8: tell the story

spatial reasoning

In former times reasoning was one of the topics of Euclidean geometry. Starting from a number of axioms the theorems were deduced logically the one from the other. The most important aid for doing that was traditional proposition logic.

However, it is also possible to reason logically (use common sense) without the explicit knowledge of formal logic. In our case this concerns reasoning that stems from a geometric problem or has geometric aspects. We will demonstrate this once more with the problem of the block construction (fig. 9).

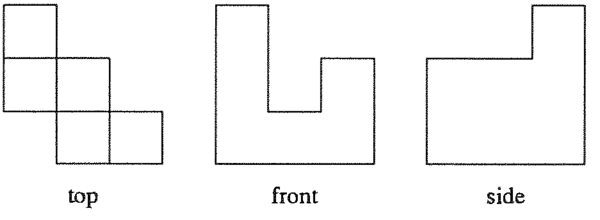


figure 9: how was this constructed?

The question is to determine the exact composition of the construction from the given top, front and side view. In this item the aspect of reasoning has to do with the activity of combining certain facts. And if we ask ourselves whether it is a right or a left side view we are dealing with, we notice that we have to apply 'if-then' logic. In this manner children are given the opportunity to use a typical mathematical (scientific) method at a level of their own. Namely, posing hypotheses, trying them out, refuting them and proving them. In more specific terms we can speak here about the following mathematical activities:

- inductive reasoning
- reasoning by analogies
- (local)-deductive reasoning
- generalising
- recognising and using isomorphism
- ordering and working systematically
- developing and applying methods of visualisation

Of course the aspect of reasoning is not restricted to the idea of block constructions. It constitutes a major part of all other aspects yet to be discussed, as it does in other parts of mathematics in general. To conclude this point, a number of problem items that can be employed in primary education.

- How many essentially different 4-cube houses can you build (fig. 10)?

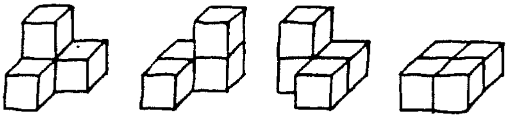
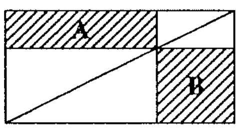


figure 10: 4-cube houses

- How many lines connect 5 points, 6 points, ...?
- Take a look at the soccer ball in figure 11. How many pentagons, how many hexagons, how many edges (seams), how many vertices are there?

figure 11

- Why are the areas of A and B equal in figure 12?



A = B

figure 12

transforming

Since the sixties there is growing interest in elementary transformations such as reflection, rotation, translation and the relations between them, not only for secondary but also for elementary level. Special attention is devoted to symmetry in line and plane. Already for the very young (4-6) there are numerous meaningful activities available such as folding, translating and the use of a mirror (fig. 13).

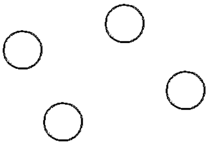


figure 13: using a mirror, make 8, 7 or 6 dots

Symmetry was already considered to be a meaningful topic by Pestalozzi, Fröbel and Montessori. And the realistic approach in geometry also devotes attention to this subject. Any kind of formalism is rejected in this regard however.

Skills and concepts are not taught directly or defined explicitly. But they are derived from the reality of our perception by means of adequate contexts and in an informal manner. The transformations we spoke of, are the so called congruency transformations, since the element of size remains unchanged in consequence. But there are many transformations that do not fit this characteristic. When figures change through transformation thus that they retain their shape we call this a similarity transformation. Now in everyday life we call this enlargement or reduction. Enlargement and reduction are of great importance for initial geometry instruction because the youngest of pupils come across transformations of this kind.

When, for instance, the teacher makes a drawing on the blackboard which the children must copy, they think it completely normal that the size of their copies is ten times smaller than the original drawing. The image must be a good look-alike, have the same shape. In our terms this means that ratios within the perceived object (original) must not be altered in the corresponding image. The children are already familiar with this phenomenon from photographs, drawings, slides, images from the overhead projector, models, and so on. The concept of similarity can also be illustrated by non-examples such as positioning the projection screen at an oblique angle.

One of the main goals of enlargement and reduction is drawing on scale and the ability to measure and calculate with scale maps.

Especially important is the development of adequate use of language. The overhead projector and photocopier can serve as an excellent aid here. It also provides the opportunity to introduce the concept of the scale line in a natural manner (fig. 14).

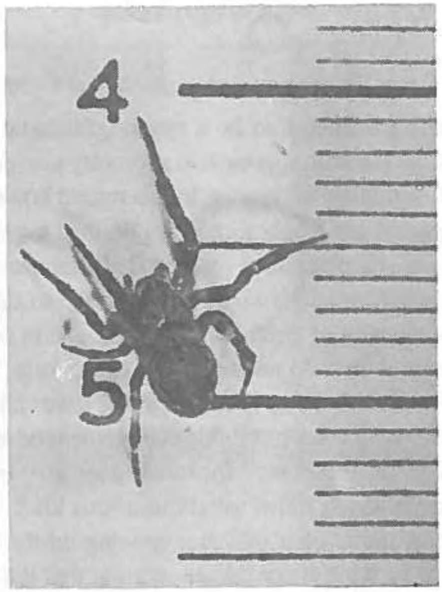


figure 14: what is the size of this common Dutch spider?

Together with activities of enlargement and reduction of 2- and 3-dimensional shapes the effects on area and volume can be studied. But always in contexts and at a level that is fitting for the children (fig. 15).

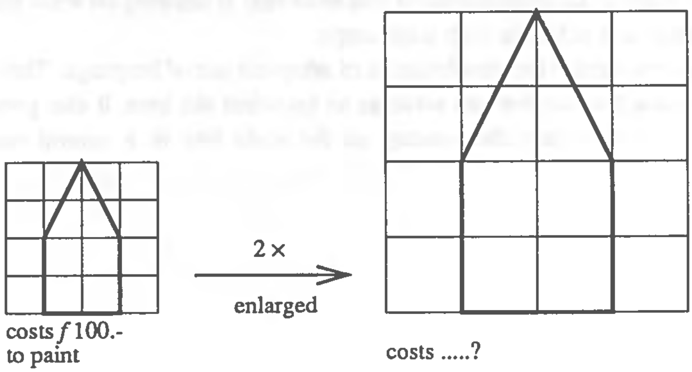


figure 15

When doing calculations with scale a great deal of attention is devoted to the use of the factor of enlargement or reduction. Sometimes it is necessary to find this factor first. Let us take, for instance, the well known problem of finding the height of a tree by using the shadow of the sun (fig. 16).

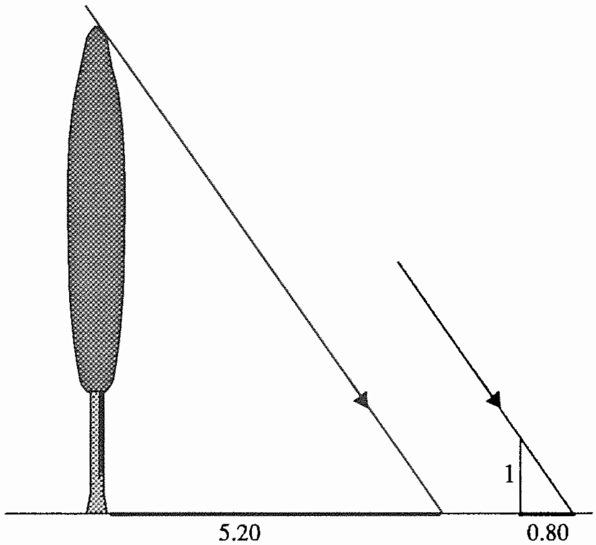


figure 16

The shadow of the tree is 5.20 meters. The shadow of the stick of 1 meter is 0.80 meters. The sides of the large triangle are $6\frac{1}{2}$ times (520 : 80) as large as the sides of the small triangle. Therefore the tree is $6\frac{1}{2} \times 1 = 6.50$ m high. Working in a formal manner one could have written $80 : 520 = 100 : x$. We worked with external ratios. But we could also have worked with internal ratios, these are the ratios of the sides of each triangle. The vertical lengths are $1\frac{1}{4}$ times (100 : 80) as large as the horizontal ones. Therefore the tree is $1\frac{1}{4} \times 5.20 = 6.50$ m high.

The algebraic notation for this calculation is $80 : 100 = 520 : x$. We must be careful not to use this formal notation too early on in the learning process.

constructing and drawing

The term constructing is as loaded as the term proving. In the Euclidean tradition constructing means the drawing of figures with a ruler (with no scale) and a pair of compasses only. We want to take a broader view of constructing and start from its natural meaning. That is fitting together two and three dimensional figures under certain conditions. At the outset this can be done with concrete material (blocks, meccano strips), but later on this will progress to activities of a (semi)mental nature.

During the period of concrete activity we construct with blocks and lego, work with cut-outs, mosaics, tangram and pegboards. Paper folding, using (transparent) mirrors and constructing models.

Also taking into account the aspect of drawing, we have drawing on scale, designing pavements, drawing three dimensional figures and finding locuses.

This last aspect is illustrated in figure 17. The problem is to draw the fishing zone of several miles around a quadrilateral 'island'.

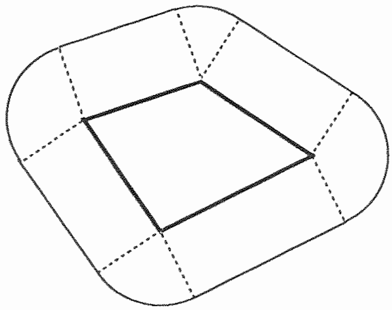


figure 17: island and fishing zone

The problem provides the opportunity to suggest other interesting geometric investigations:

- Try to invent a device which will draw such a zone.
- Give the exact description of such a zone.
- Compare islands that have a different convex shape but the same perimeter (!).

In regard to the question of finding a practical method to establish the fishing zone (e.g. a revolving cardboard circle with a pencil point through the centre) we see that the invention and construction of such (primitive) devices are also part of the aspect of constructing.

For this problem we could also have used a pair of compasses and a ruler. Although we can utilise all kinds of aids (scissors, string, templates) at some point certain skill must be acquired with the well known instruments such as the ruler, compasses, drawing triangle, protractor, transparent mirror and so on. Drawing therefore also has a technical side. But also the art of sketching is important, especially when used to visualise (geometric) problems. Visualisations (rough sketch, a scheme, a diagram, ...) which give support to (semi) mental construction are indispensable to geometry. But they can also serve us well in other areas of mathematics. Consider for example the making of views of a block construction or the visualisation of a route to be travelled. The reverse, namely the interpreting of drawings and visualisations, is strongly connected with think-activities. In short, constructing is not only about empirical activities, but one of its main purposes is the development of spatial reasoning. Which we want to illustrate again from the items of figures 18a, b, c, d.

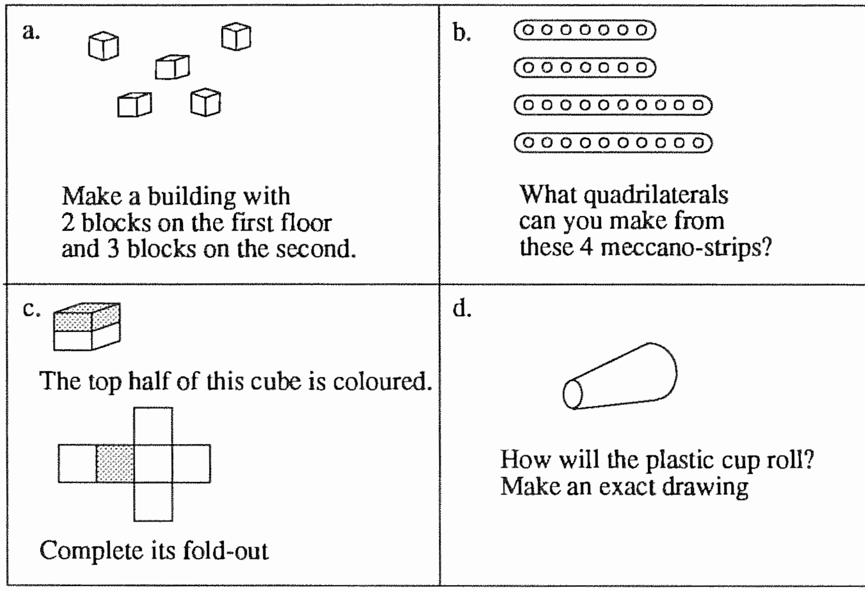


figure 18

measuring and calculating

The Dutch word 'meetkunde' literally means the ability ('kunde') to measure ('meten'). The more international term 'geometry' points to the original Greek meaning 'ge' (earth) and 'metrein' (measure). And indeed, the practical measuring of lengths, areas and volumes. has been the basis for the origin of geometry.

Therefore the Greeks would have solved the problem in figure 19a in a manner as is demonstrated in figure 19b.

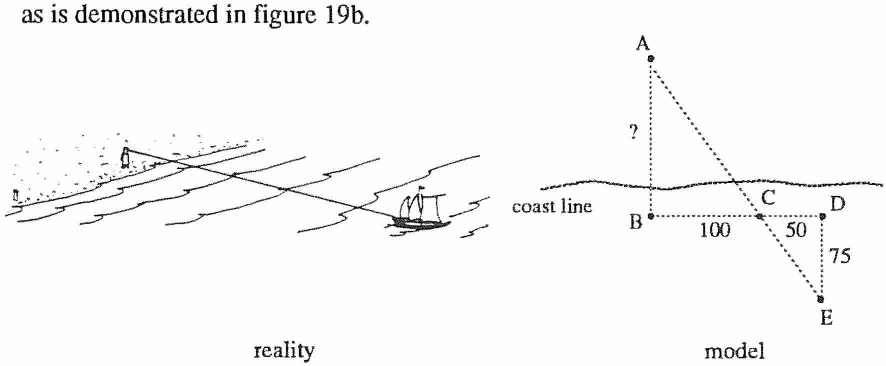


figure 19a and b

We are in B and we see the ship (A) in a direction perpendicular to the coast. We walk 100 steps along the coastline and plant a stick in C. Then 50 steps further to D and after that a quarter turn and then as far inland until C and A are seen in one line.

The number of steps from D to E proves to be 75. From the similarity of the triangles ABC and EDC we can now calculate that BA is 2 × 75 steps, which is about 150 metres. Once the geometric model is found the problem is reduced to choosing a unit of measure and doing the calculation. The essence of measuring strategies like these (compare this to the problem of finding the height of the tree in figure 16) often has to do with a geometric model. Thus by the activity of measuring a connection is made between geometry and calculation.

We will not go into the specific elements of measuring here, nor how the concepts of measure are acquired. Here we will restrict ourselves to one aspect of measuring that is specific to the subject of geometry. And that is the way of finding the areas of quadrilaterals and triangles. An important principle in that respect is the ability to restructure figures in such a manner that children become aware of the idea of conservation of area (fig. 20).

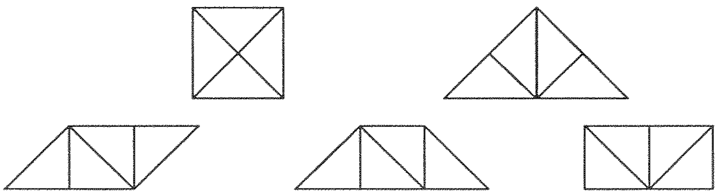


figure 20: conservation of area

A second principle is the replacing of a figure with a certain measure of area with a figure of 2 times, 3 times, ... its area (calculating with area).

The third principle deals with the completion of plane figures to a rectangle (fig. 21).

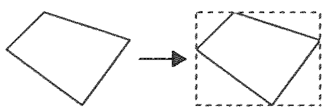


figure 21

If the area of a rectangle is known to be the product of length and width, the area of any triangle (and quadrilateral) can be illustrated (fig. 22) thereby.

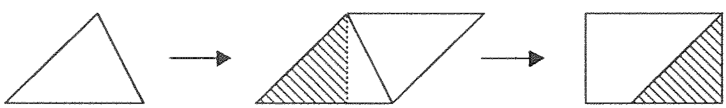


figure 22: triangle is half the rectangle

A vertically planned curriculum on area should extend over the entire period of age

4-14. Initially there will be emphasis on concrete activities, later on the visual and mental activities. The topic can be concluded with a more formal and systematic approach.

3 geometry in the curriculum

It will be evident from the given examples that the various aspects should not be regarded as subjects on their own. And therefore not be taught as isolated topics. On the contrary, the various aspects are closely related and this intertwining constitutes the whole: realistic geometry.

The realistic interpretation of geometry, namely as the investigation of the space in which we live and of the phenomena that occur in it, make it possible to broaden our view of geometry compared with the traditional formalistic view.

Upon introducing geometry in primary education we are not only dealing with a new subject but as it turns out, the ideas that teachers have about the subject are totally different from how they remember geometry from their school days or studies.

In conclusion, realistic geometry does not resemble individual paper and pencil work, nor is it a matter of the teacher doing the explaining and the pupil imitating the activity. Instruction in realistic geometry calls for work to be done in groups, where investigation, experimentation, discussion and reflection are the core of the teaching- learning process.

The aspects we have discussed in the foregoing can help us to describe the outlines of the subject and its didactics. In most current Dutch textbooks realistic geometry has become a part of the math curriculum. Some textbook writers regard geometry as a subject that can bind together the other subjects of the math curriculum. From this viewpoint geometry is tied in to the idea of mathematizing. It is recommended for its possibilities to visualise problems through use of number lines, graphs, and so on. But also the applicability in other fields, such as geography, is exploited. In all of the textbooks we find indications that geometry can also be treated and studied as a subject on its own.

The fact that geometry has been included in the textbooks does not guarantee that systematic attention will also be paid to the subject in class. We are not all that optimistic in this regard. And one of the main reasons for this scepticism is that in The Netherlands there is not much of a tradition in this area. And this means that teachers have no experience with the subject and that it will be very difficult for them to recognise the significance of the subject. But more than anything else the strong emphasis on arithmetic means that little time is left for geometric activities. In the draft standards drawn up at the instruction of the government, geometry is however now listed as a specific subject. These standards are and will be worked out in a didactical manner in the 'Specimen of a National Curriculum'. And our national institute for assessments (CITO) is planning to publish tests for geometry. So, perhaps after some one hundred years there will once again be an opportunity to bring back geometry to the curriculum for primary education.

why?

The argument of the formal value, specifically with respect to geometry, has not been used since the sixties. Justification for geometry is now based on *applicability*, *preparatory value*, (*subject specific*) *value* and *personal value*.[12]

The argument of applicability was also used during the nineteenth century. But today this is interpreted in a broader sense, as we have seen in the foregoing. The most striking difference with former times can be seen in the following. Applications are not now the only end point of the learning process, but reality is used as the very source to explain geometric phenomena and to develop geometric concepts. As an example we recall once more the investigations of the effects of the shadows of the sun, in which at a higher level the ideas of parallel projection can be seen (fig. 23).

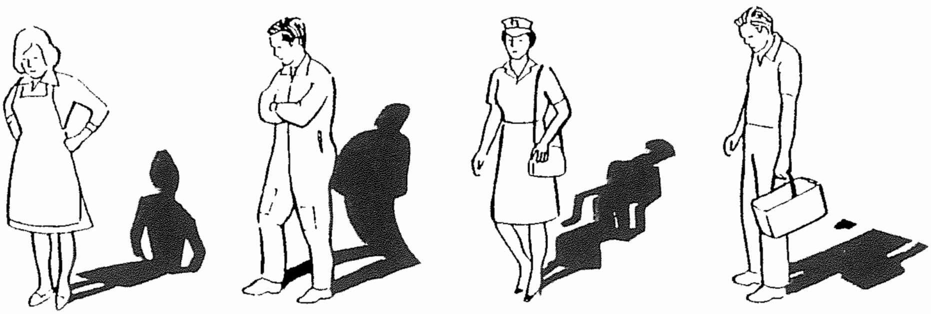


figure 23: what is that?

The argument of preparatory value is also worded differently. During the last century the argument was not a very valid one anyway because only a small percentage of the children went to secondary school at all. Later, during the first half of the twentieth century, a preparatory informal approach to geometry was not appreciated by secondary educators either. It was thought that geometry was and should remain a purely deductive system which should not be violated in any manner, not even in regard to the didactics.

Now in the realistic approach there is a way by which it is possible – working from intuitive notions – to develop a system of concepts for the pupil to such a level that it is possible to make a first ordering of geometric reality. On a concrete level and later at a more schematic one, even the very youngest pupils can already grasp elementary concepts, begin to discover relationships and are able to make simple reasonings. In this manner a broad basis of orientation is laid which can be later be worked out more systematically. Since we now have textbooks that take this approach and since the didactical and psychological ideas have been accepted, the argument of preparatory value is more to the point than ever before.

The subject-specific value of geometry illustrates that mathematics is more than working with numbers and symbols. Without the use of geometric models one can barely give a description of the real world.

Geometric problems, also the most simple ones, often have a motivating character. While solving them one often employs a mixture of intuition, trial, drawing and reasoning. Geometry cannot be forced into the straight jacket of algorithms. In solving problems our reasoning is always supported by visual aids.

Geometric patterns are not only a rich source for explorations, but working with the subject children can also be attracted to the esthetic aspect of geometry. The visual power of some geometric theorems, such as the theorem of Pythagoras (fig. 24), can be so overwhelming that it should not be withheld from any pupil.

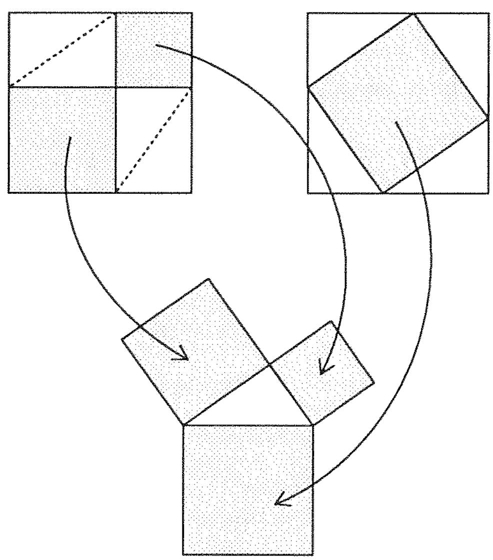


figure 24: how beautiful!

With respect to the realistic approach to mathematics in primary school we distinguish six subjects: basic facts, algorithms, ratio and percentage, fractions and decimals, measuring and geometry.

One of the particular aspects of geometry is the motivating character of the problems, as we have mentioned earlier. But it is also different from the other subjects in regard to the many sidedness of the mathematical aspects (visualising, modelling, reasoning, reflecting and application). Hence it would appear that – besides measuring – geometry is best suited to promote a mathematical attitude. And therefore it is also of great personal value.

In conclusion, from school practice it turns out that certain pupils have a special aptitude for geometry. Not seldom has this been observed in pupils who do poorly in arithmetic. A geometry curriculum as we intend it, can be found in 'Rekenen en Wiskunde' a complete series for realistic mathematics curriculum for ages 6-12. Originally this program was designed for the less proficient pupils.[13]

One of the authors of the series, Jean Marie Kraemer, co-developer of the geometry part, has written a report about this process of development. He too is searching for the justification for bringing geometry into the curriculum of the primary school. He also refers to the important aspects of personal value and describes his views in the following conclusion:

> 'In short, the world-orienting character of geometry for primary school, does in principle create the possibility to take every child seriously and to challenge the child continuously to make good use of own abilities to further personal development.'[14]

geometry, why not?

Indeed why not? What would our idea of mathematics be like without it? And what would we be keeping from our children?

Notes

1 Delekat, F. (1968). *Johan Heinrich Pestalozzi.* Heidelberg: Quelle & Meyer.
2 Dapperen, D. van (1820). *Vormleer.* Amsterdam: Johannes van der Hey.
3 Lines were segments; only angles smaller than 90° were regarded.
4 Leeb-Lundberg, K. (1970). *Kindergarten Mathematics Laboratory – nineteenth Century Fashion.* The Arithmetic Teacher, 17, (5), 372-386.
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6 Hiele-Geldof, D. van (1957). *De didaktiek van de Meetkunde in de eerste klasse van het VHMO* (doctoral dissertation). Utrecht.
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Tests are not all bad

An attempt to change the appearance of written tests in mathematics instruction at primary school level

M. van den Heuvel-Panhuizen and K.P.E. Gravemeijer

summary

Mathematics tests are often associated with written tests consisting of sequences of bare sums of increasing difficulty. An important drawback of these kinds of tests is that they give insufficient information on children's abilities and strategies. This article describes a number of measures which can be taken to make tests more informative.

1 introduction

To be a good teacher one requires continuous information on the progress of one's pupils. What have they picked up from the lessons and what can you expect from them in the future? This information should of course be collected as efficiently as possible. Teachers have much more to do than to test children.

It is no wonder that class-administered written tests – for which Thorndike laid the basis with his first achievement tests which date back to 1910 – occupy an important place in instruction. At one go and within a short time one can investigate in an entire class whether certain abilities have been mastered or not. In mathematics mostly written tests are used, which consist of sequences of bare sums of increasing difficulty.

Although it is hardly possible to imagine school without written tests, they are met with some ambivalence. On the one hand they are considered an indispensable tool, on the other one shudders to think about the harm they can do. It depends on one's general ideas on education how this is viewed.

2 drawbacks of the usual tests

As long as there have been class-administered written tests, objections have been raised against this method of gathering information on children's learning progress. Earlier generations of teachers already had the opportunity to read about incorrect conclusions brought on by unwarranted trust of test results (see e.g. Weaver's article in a 1955 issue of 'Arithmetic Teacher').

In general it can be stated that the usual class-administered written tests reveal only the bare results and tell nothing about the children's strategies. This lack of information on children's strategies does not only have the consequences that incorrect conclusions are likely to be drawn from the children's performance. Another consequence is that too little information can be obtained about the progress of instruction; for instance, nothing is learned about the children's informal knowledge and solving methods (cf. e.g. Ginsburg, 1975). And a last consequence is that it is almost impossible to diagnose the children's mathematical problems; any error analysis that depends solely on the results can never suffice to discover the children's problems and misconceptions.

Another drawback of these tests is that they are too narrow, both with regard to the subject matter and the children. Usually the tests are restricted to such subject matter as can easily be tested and do not allow the children to show what they are capable of optimally; perhaps lacking abilities are balanced by others that are not given a chance.

3 alternatives

As an alternative for failing written tests one nowadays regularly pleads for individual observation and interviews as the only sound way to investigate children's knowledge and abilities. Though true, this is only half the truth. In the package of means of evaluation available to the teacher, written tests cannot be excluded. Indeed, they allow the teacher to screen a whole class at once. Tests, rather than rejected, should be altered.

Since, in general, one has been inclined to look outside the scope of formal tests for alternative means of evaluation, less attention has been paid – certainly at primary level – to alternatives within the range of class-administered written tests. With regard to secondary education this is different. See De Lange Jzn's (1987) research on the 'two-stage task', the 'take-home task' and the 'essay task'.

In this article we will draw attention to an alternative for the usual written tests at primary school level. We employ a variety of possibilities to meet such objections as regard the lack of information about children's strategies. A few examples will be shown, taken from test series (Van den Heuvel-Panhuizen and Gravemeijer, 1990) developed in The Netherlands in the MORE-project.[1] With one exception the examples are related to the third grade (eight year olds).

The tests are meant to be administered by the teacher to the whole class. Each pupil is given a test booklet and the teacher gives a brief oral explanation of each item.

4 tests with a different appearance

The development of the tests was no isolated activity, but one closely related to a new approach to teaching mathematics, the realistic approach. This approach was strongly influenced by the ideas of Freudenthal (1978, 1983). Treffers (1987) made an a posteriori description of the theoretical framework of realistic mathematics instruction. He gave the following characteristics of the realistic approach: the phenomenological exploration by means of contexts which serve both as source and as field of application, the broad attention paid to models which have a vertical bridging function, the utilization of children's own productions and constructions, the interactive character of the learning process and the intertwining of learning strands (cf. also Treffers and Goffree, 1985).

Related to this new didactics it has been our intention to produce tests which provide for a maximum of information about children's knowledge and abilities, covering the whole breadth and depth of the mathematical area. It is also a major point that children are allowed to show what they are capable of. At the same time the tests must be as easy to administer as the usual written tests which consist solely of bare sums. In other words, the tests are to be applied in the classroom situation with a minimum of explanation – at any rate, with no extensive oral or written instructions, which would result in a reading or listening comprehension test rather than in a mathematics test. For this reason we have looked for tasks which are accessible to each child and require no additional information beyond that minimum of instruction that is needed to get the intention across.

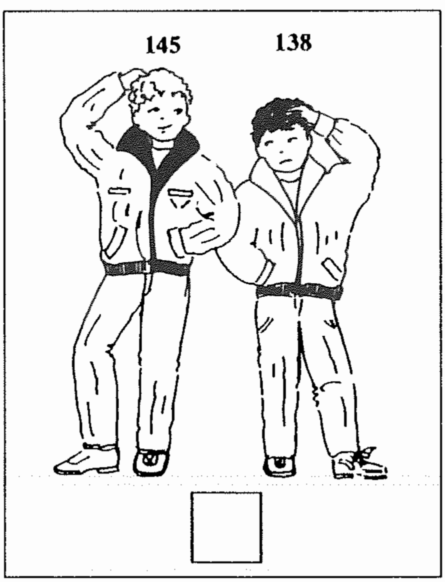

figure 1

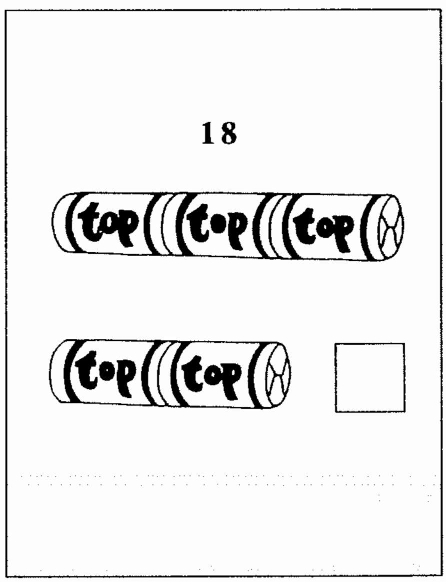

figure 2

The figures[2] 1 to 3 – which are related to the subjects 'addition and subtraction', 'ratio' and 'measurement and geometry', respectively – convey an idea of the kind of items which have resulted. In brief: not much text, but pictures which are self-explanatory and related to meaningful situations.

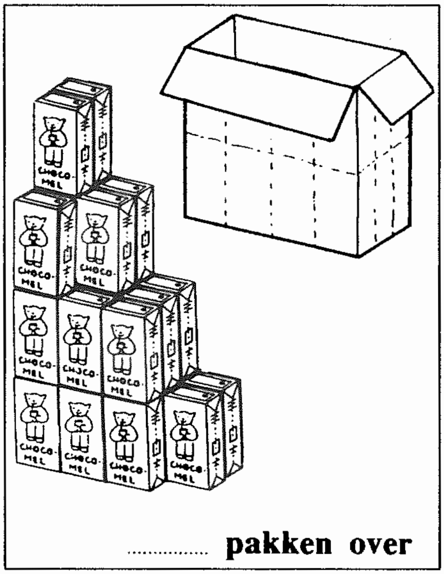

figure 3

The item of figure 1 shows two children comparing their height. The obvious question is: how much is the difference? The question of the item of figure 2 is: how many peppermints are there in the roll? In figure 3 the word 'over' means left and the question is: how many packages of chocolate milk are left when the box is full.

The true-to-life contexts do not only help the children to immediately grasp the situation of the items but also offer the opportunity to sound out the children's abilities while avoiding obstructions which are caused by formal notation. So by means of the tests, matters which have not yet been taught, can be tackled to provide important information for instruction. If not to invent new educational activities, this information about children's informal knowledge and solutions can be used to attune the teaching to the children's previous knowledge. The 'ratio' and 'measurement and geometry' tasks of figures 2 and 3 are good examples of items which anticipate on instruction, but the same applies to the item of figure 1. By means of the context children may be able to answer it even before they are capable of doing sums like 145-138.

Moreover, this item shows what context can contribute to the accessibility of test problems: the context can suggest clever solution strategies which make them easier.

The picture of the two children comparing their height elicits a complementary strategy for the subtraction task.

In addition to giving an impression of what the test items look like, the three examples together also indicate that they cover the whole mathematical area.

The following examples will show that extensions are possible not only in breadth but also in depth. The test items need not be restricted to flat one-step tasks which can be performed directly, without preceding analysis, but they can as well aim at more complex activities, of a problem-solving character.

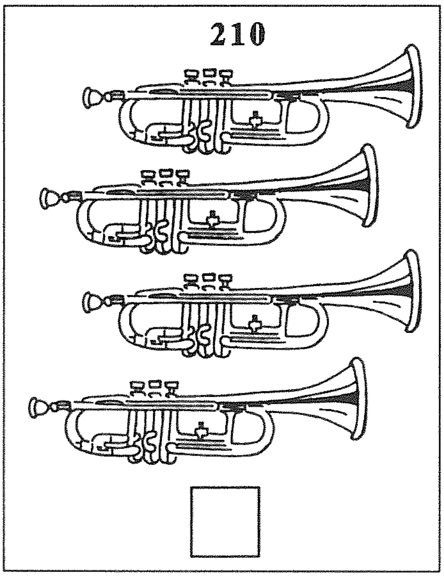

figure 4

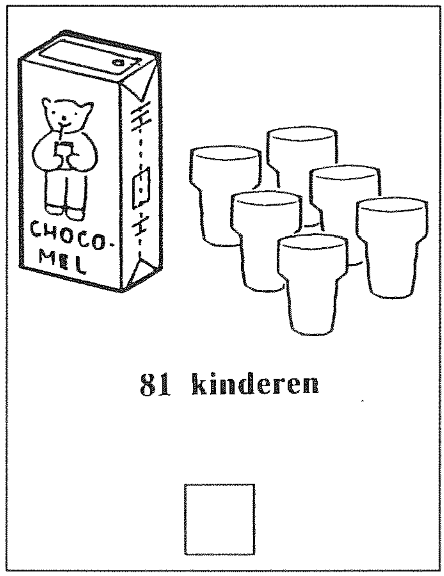

figure 5

In figure 4 an example of a one-step task is shown: the youth brass band buys four trumpets; they are second-hand and cost 210 florins per piece; how much do they cost together? Here the question itself suggests addition, or perhaps multiplication. Compared with this, the item of figure 5 is considerably less flat. The question here is: how many packages of chocolate milk are needed for 81 children. First of all, the arithmetic operation is not given straight away; moreover even a correct calculation of 81 : 6 does not directly yield an adequate answer. In the case of figure 6 where the children are asked to estimate the height of the neon letters on the building, besides indications on the operation to be used, data needed for the calculation are lacking, and, in order to arrive at a solution, the children must appeal to their own knowledge of measures (in this connection also see the item of figure 22 to 24). Yet one more example of a higher order problem is the item about the small train in figure 7. A short ride takes ten minutes and the question is: how much time will the long trajectory take. Although a simple question in itself, it gives no indication of the operation which must be carried out. On the contrary, a thorough analysis of the problem is required.

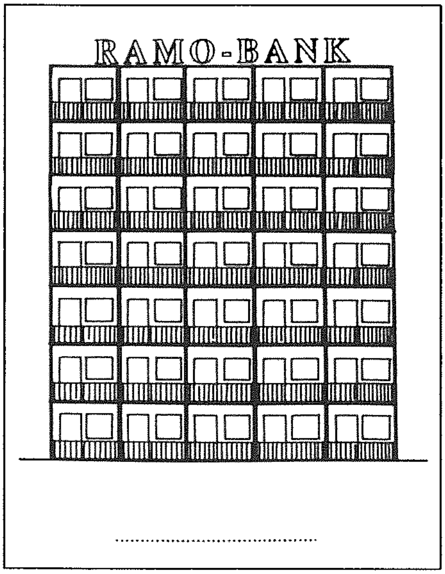

figure 6

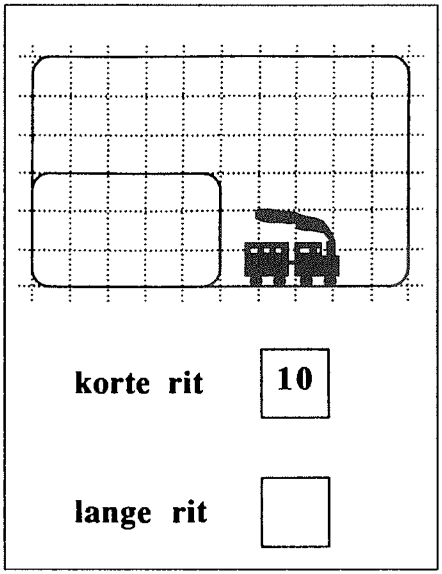

figure 7

Another feature of the last item is a certain built-in stratification. The way the item is presented allows for solutions on several levels. In addition to the use of motivating and supporting contexts, it provides opportunities for the children to show what they are capable of. For instance, children who have no idea yet of what happens to the perimeter when length and width are doubled, can, indeed, operate on a lower level and arrive at a solution.

One more example of stratification is offered by the item where children must find out how many small triangles are used to pave a certain square. This also can be done by strategies of quite a different level based on using the illustration differently. If one wants to know how the children arrived at the solution, one can get clues by looking for traces on the test page. For instance, it is clear that the pupil who completed the testsheet of figure 8 simply counted the triangles, while the one of figure 9 almost certainly used the '25 times 4' structure. If, however, rather than relying on incidental traces, one prefers a more goal-directed manner of investigating how the children proceeded, then other methods will have to be applied. We are now ready for the important question: how can written tests yield more information on what children know and are capable of?

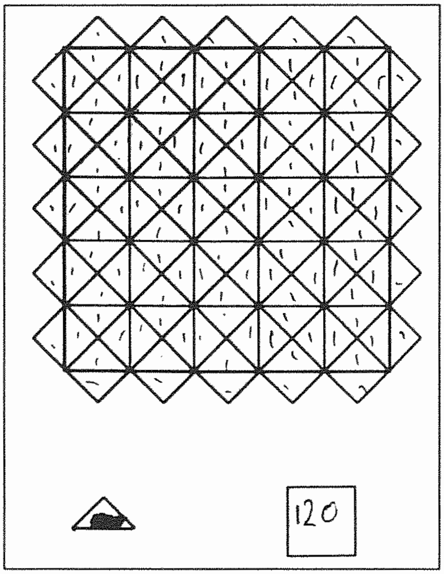

figure 8

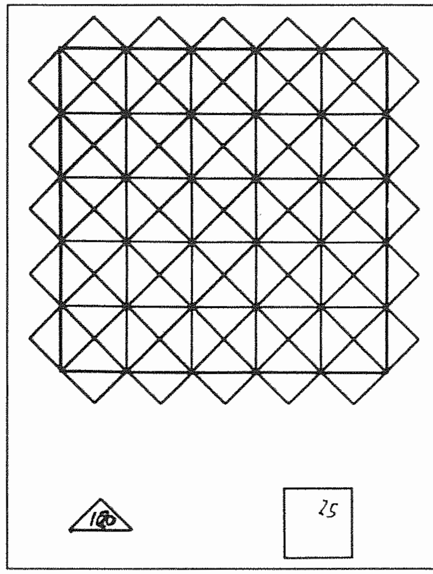

figure 9

5 more information

Generally speaking, we can proceed in two manners. We can take measures concerning both the presentation of the question and the way the children must answer. We will discuss three variants of both, starting with the question side.

5.1 choices

The first measure that can be taken in this regard is to construct test items which present the opportunity to choose. What is normal in an interview, that a subject who cannot solve a problem will be given another, to determinate what she or he knows precisely, can in a way be formalised in a test. An example is shown in figure 10. With this item (which is part of a test for grade 1) the children themselves choose what to buy. Since there are several degrees of difficulty here, the choice creates indications of what the children are capable of. Of course preferences for a certain object can play a part, but experience has shown that quite a few children make numerically similar choices on tests of this kind which follow one another. Notwithstanding the fact that by the choice the subject shows to what degree of difficulty sums are mastered, a correct answer does not yet disclose anything about insight in properties of operations and the ability to apply them.

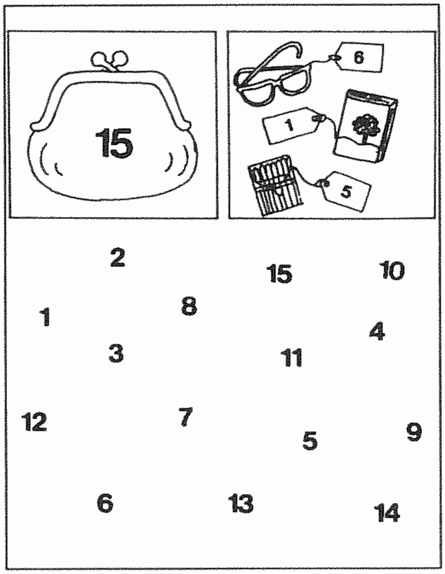

figure 10

5.2 auxiliary sums

One way to investigate insight into properties of operations and the ability to apply them is to use auxiliary sums. The children must get used to the fact that one sum on the testsheet has already been completed for them (see figure 11 and 12).

86+57=143

86 + 56 = 144

57 + 86 = 143

860 + 570 = 1430

85 + 57 = 142

143 - 86 = 57

86 + 86 + 57 + 57 = 286

85 + 58 = 143

figure 11

86+57=143

86 + 56 =

57 + 86 =

860 + 570 =

85 + 57 = 137

143 - 86 =

86 + 86 + 57 + 57 =

85 + 58 =

figure 12

It looks so easy, even though these are sums with large numbers which have not yet been dealt with in the lessons. But it is only easy if there is insight into the properties of numbers and operations, which evidently is the case with the pupil of figure 11, yet not with the one of figure 12, who obviously did not make use of the auxiliary sum. It is revealing that later on in the test, when the sums 85 + 58 and 143 – 86 are given once more without the auxiliary sum, the pupil of figure 11 made mistakes. This more or less confirms the surmise that properties of operations were used in the first case.

5.3 changing the presentation

By the foregoing an example is also given of the next measure which can be taken to make tests more informative: presenting the same problem in various manners. Besides presentation with and without an auxiliary sum there is the choice of presentation with and without context. Figures 13 and 14 give an example.

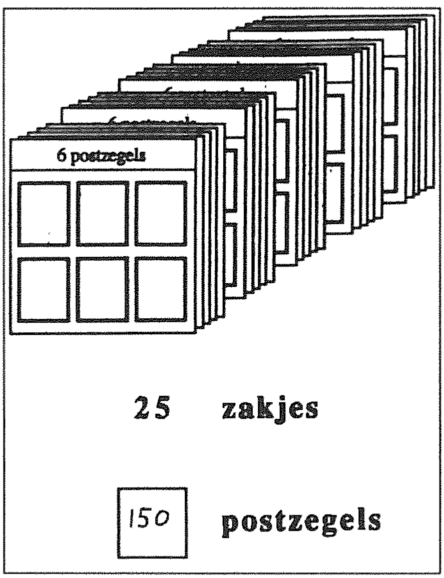

figure 13

25 x 6 =120

4 x 15 =420

figure 14

The context problem about 25 envelopes with stamps is easier for this pupil than the bare sum of 25 × 6. There are, however, pupils for whom it is exactly the other way around. Another pupil solved the bare sum correctly but failed the context sum. Did the item of figure 15 in between prove to be a push in the right direction?

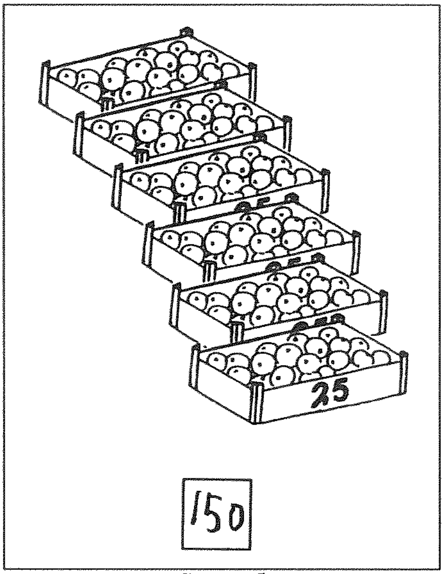

figure 15

5.4 more than one correct answer

Measures can also be taken on the answer side. A first example is to abandon the idea that is often wrongly associated with written tests that answers are unique for each item. By admitting items with several correct answers the children are given the latitude to come up with solutions.

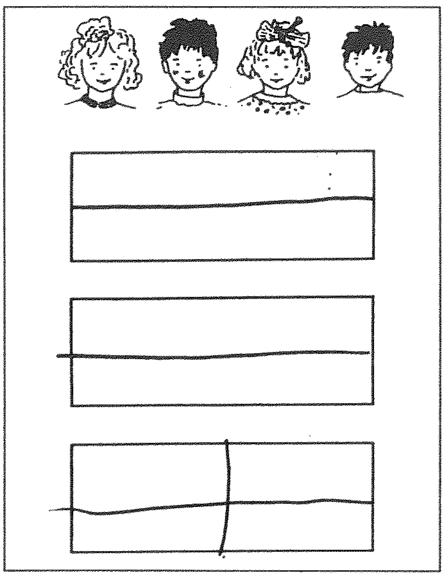

figure 16

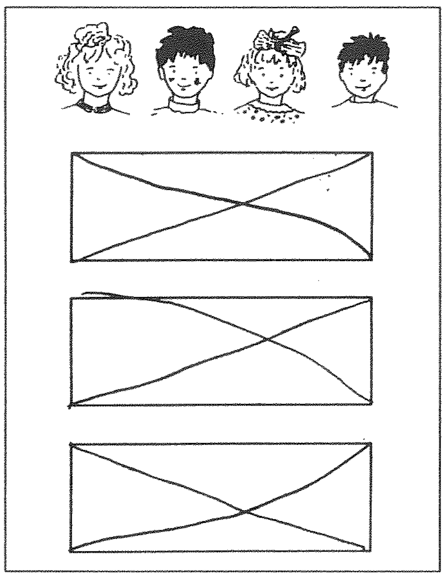

figure 17

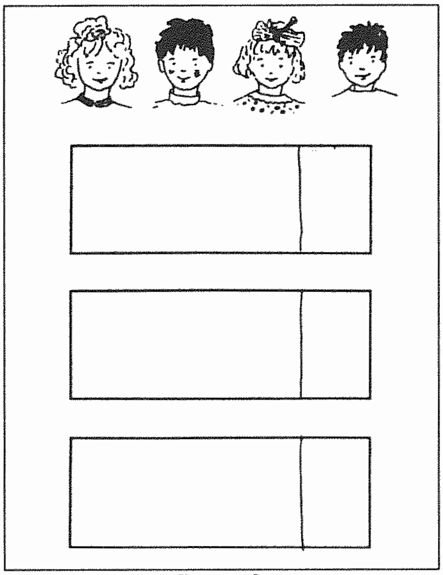

figure 18

This not only contributes to the opportunities children get to show what they know but at the same time makes tests more informative due to the answers the children give on their own level. Consider in figures 16 to 18, for instance, the diversity in solving the problem to divide three bars of chocolate among four children (see also Streefland, 1987)!

A great variety of answers is also possible on an item about a family; mother, father and two children, go to the circus and pay fifty florins admission for the four of them. The question is to specify the admission charges. The tickets the children filled in differ greatly. There are pupils who have three persons pay ten florins and the fourth twenty. Others distinguish between adults and children and arrive at thirteen and twelve, or fifteen and ten florins, respectively. Still others divide the total equally over four persons to get $12\frac{1}{2}$ or 12.50 and that at a moment when decimal and other fractions have not yet been taught.

5.5 scrap paper

When considering the variety of results the main question is, of course, how the children arrived at the answers. This can also be answered by written tests, namely by means of a piece of scrap paper pictured on the answer page. The children can use this piece of scrap paper when solving the problem to write down the results of intermediate calculations. With regard to the preceding item figures 19 to 21 show the following strategies: the pupil of figure 19 first tries to approximate the total amount of fifty as closely as possible while the others immediately apply a formal division or less formal distribution strategy.

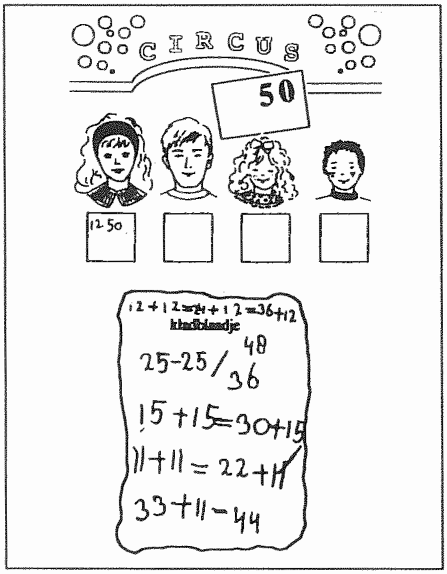

figure 19

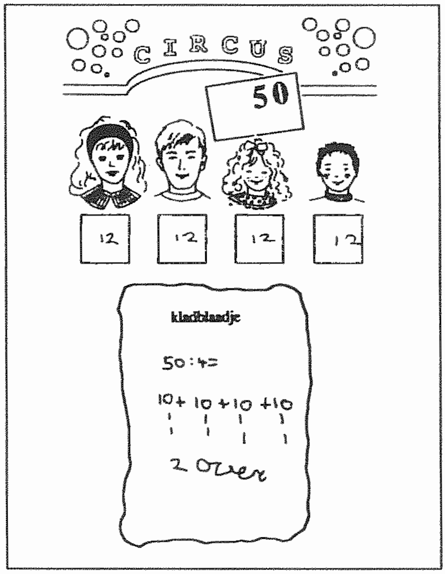

figure 20

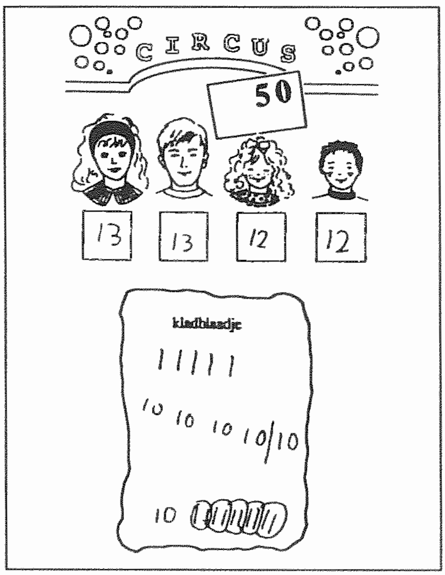

figure 21

Another item with a piece of scrap paper on the test page is that where it is asked how many children together weigh as much as the bear. Not unlike the item shown earlier about the height of the neon letters (figure 6) it appeals to the children's knowledge of measures.

Only the weight of the polar bear is given. It is left to the pupils to determine how much a child generally weighs. Some, like the pupil of figure 22, stick to their own weight; others, such as those of figures 23 and 24 prefer a round number, or they weigh precisely 30 or 25 kg.

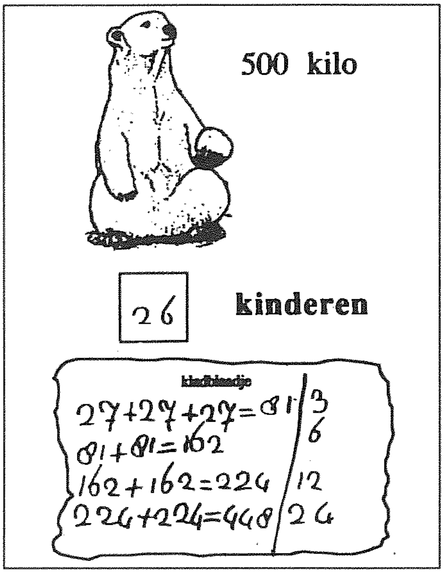

figure 22

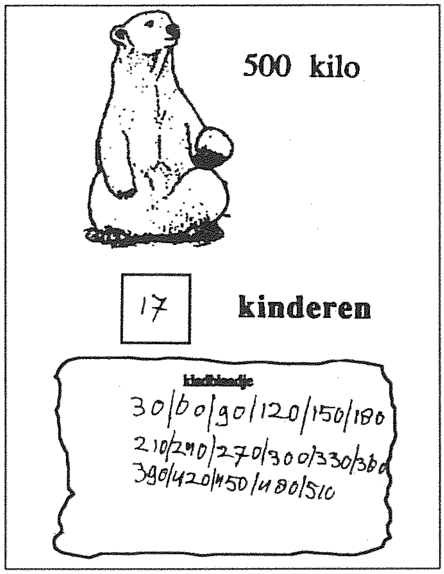

figure 23

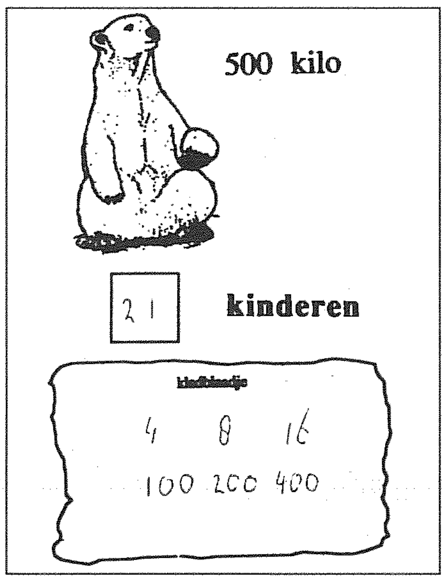

figure 24

In addition to the choice of weight on which the calculation is based the children disclose a number of interesting strategies and notations for strategies. The pupil of figure 22 follows a doubling strategy, whereas the scrap paper of figure 24 shows a ratio table.

5.6 children's own productions

The most open way to uncover what children are capable of is to elicit own productions. Then the child is asked to think up rather than to solve sums (cf. also Van den Brink, 1987 and Streefland, 1990). A simple way to estimate the scope of children's abilities is the task to produce an easy and a difficult sum. Thanks to the latitude children are given in own productions it not only reveals what children are capable of but also what their manner of working is.

In the next example, the children are asked to come up with as many sums as they can with the result of one hundred. There is not only a variety of numbers and kinds of sums but also of working behaviour. Some record only isolated sums (figure 25) whereas others proceed systematically, for instance, by always changing the first term by one unit (figure 26) or by applying commutativity.

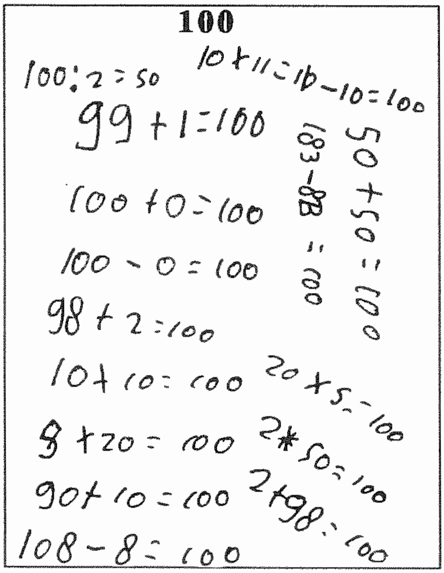

figure 25

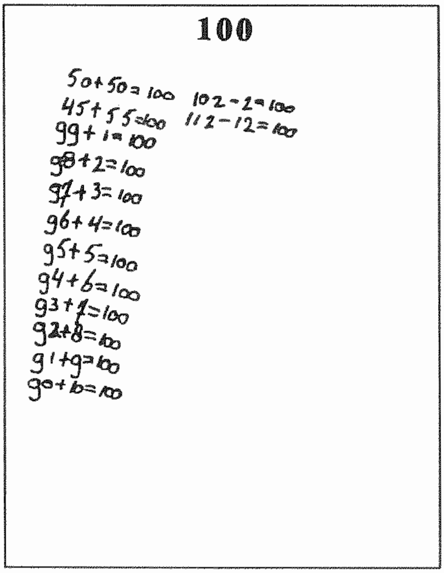

figure 26

Besides sums, other types of problems lend themselves to own productions. An example is the item of figure 27 where the children are asked to make a program for a birthday party, or rather, complete it, as the starting time and the activities are already given. It is left to the children to determine how long each activity will take. The only thing that is predetermined is 45 minutes for the movie. Like most open items this one allows a great many observation-points. There must be progression of time and durations must be in tune with the activities; and finally the digital notation of time must have at least been understood.

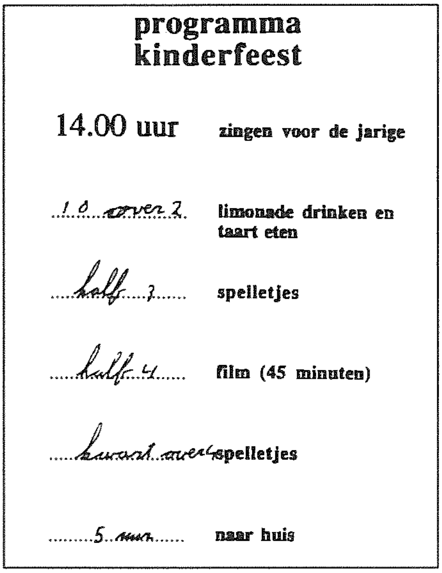

figure 27

Let us close the series of measures to make tests more informative here. Perhaps 'more informative' is not strong enough. This can be illustrated by the experience with a similar test for grade 1 (Van den Heuvel-Panhuizen, 1990). The test was administered after three weeks in the first grade and was also presented to a panel of four heterogeneous groups consisting of four or five persons, among whom primary school teachers, educational counsellors, teacher trainers and researchers. These experts were asked to make estimates of the presumed scores of the children. The comparison of the estimated and the found scores revealed that the numerical knowledge and abilities of the children starting grade 1 have been greatly underestimated. The children had far greater capabilities than presumed. Discovering this by a class-administered written test proves that tests can indeed be an important tool for teachers – and of course the same holds for researchers – for gathering information about children's knowledge and abilities, even that of young children. (A second example of the possibilities of these kinds of tests is given in another article by Van den Heuvel-Panhuizen in this book. This article concerns a ratio test administered to mentally retarded children.)

6 scoring

A question that certainly arises after the foregoing examples of test items and the reactions of the children is that of the consequences for scoring. Indeed, this can be a problem but it need not be if one distinguishes clearly between two kinds of information provided by the test, namely quantitative and qualitative information.

The quantitative information is the number of correct answers which can be used as an indicator for the level of performance of the subject. This quantitative feature extends to the own productions, provided a certain criterion has been agreed upon. For the last item for instance, this could be the duration of the movie. In this respect open tasks need not endanger objectivity. Qualitative information is not accessible to objective scoring. Often it cannot be settled whether a strategy is right or wrong. But this does not matter as these data have quite a different function. They allow teachers to make decisions on the didactics to be applied. Moreover this kind of items also provides for excellent instructional material. As an example take the strategies recorded on the pieces of scrap paper (figure 19 to 24) and the dividing of chocolate bars (figure 16 to 18). The test pages can be transformed into worksheets for the next lesson.

7 better tests – better instruction

This implies that tests do not function only in the margin of instruction but are integrated in the instructional process. The tests have lessons, not only for the pupils who can hit upon ideas of clever strategies, but also for teachers who can get a better understanding of what children are capable of and be put on the track of richer didactics. This involves a reversal of the part played by tests in innovation. Rather than thwart innovations, tests would contribute towards improving education.

Notes

1 The MORE-project is a collective research project of the research group OW&OC and the Department of Educational Research of the Interdisciplinary Social Science Research Institute (ISOR), both of the State University of Utrecht. It is supported by a grant from the Dutch Foundation for Educational Research (SVO-6010). The project is an investigation into the implementation and the effects of a new didactics if compared with the traditional kind.

2 The test items are produced in reduction. The true size is twelve by seventeen cm.

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Ratio in special education

A pilot study on the possibilities of shifting the boundaries

M. van den Heuvel-Panhuizen

summary

There is a wide gap in The Netherlands between mathematics instruction in regular primary schools and that in schools for special education. The innovation of mathematics instruction as it occurred in the field of regular education did not find acceptance in special education. In contrast to the realistic approach to mathematics in regular primary schools, the approach in schools for special education is a very mechanistic one and is restricted almost exclusively to only doing bare sums. The major argument for the latter approach is that the realistic alternative is too difficult for children in special education: it cannot be helped, but the capabilities of these children are indeed limited. This article gives an account of an attempt to shift the assumed boundaries. The attempt was made with the aid of a written test on ratio. This subject for instance, is not one that is included in the math program of schools for mentally retarded children. The test was administered to 61 children in the upper grades of two schools for such children. The results of this test indicate that there is at least reason to subject the mathematics program of special education and the didactics involved, to reflection.

1 introduction

Except for regular schools for primary education in The Netherlands, there are also schools for special education. Some 5% of children of primary school age attend such schools.[1] The system of special education comprises fourteen different kinds of schools. Among them are schools for children with learning disabilities and schools for mentally retarded children, schools for deaf children, schools for the blind and visually handicapped and schools for children with severe emotional problems.

This article concerns the first two kinds of schools: schools for children with learning disabilities and schools for mentally retarded children. Some three quarters of the total group of children in special education attend one of these two types of schools.

Although the pupils of both types of schools have much in common, there is a marked difference between them in respect to the level of ability. While for a child with a learning disability it might be possible to achieve the objectives of regular primary school, for a mentally retarded child this is almost out of the question.

In respect to mathematics the ability level at the end of primary school for special education is assumed – more precise data are not available – to be the following (cf. Luit (ed.) et al., 1989 and Damen, 1990): children with learning disabilities eventually attain a level of ability which lies between the middle of grade 3 and the end of grade 6 of regular primary school; mentally retarded children attain a level of ability which lies between the end of grade 1 or the beginning of grade 2 and the end of grade 4. It happens sometimes that a mentally retarded child attains the level of the end of grade 5 of regular primary school.

In order to illustrate the level of ability in mathematics at the end of primary school for mentally retarded children, the worksheets of two children are shown in figures 1 and 2. Both are in grade 6 of their school. The work was done halfway through the year.

The work in figure 1 was done by Martijn. He is eleven years and ten months old and his mathematical ability is above the average of his class. Because he is fairly young it is being considered to keep him at the school for an additional year.

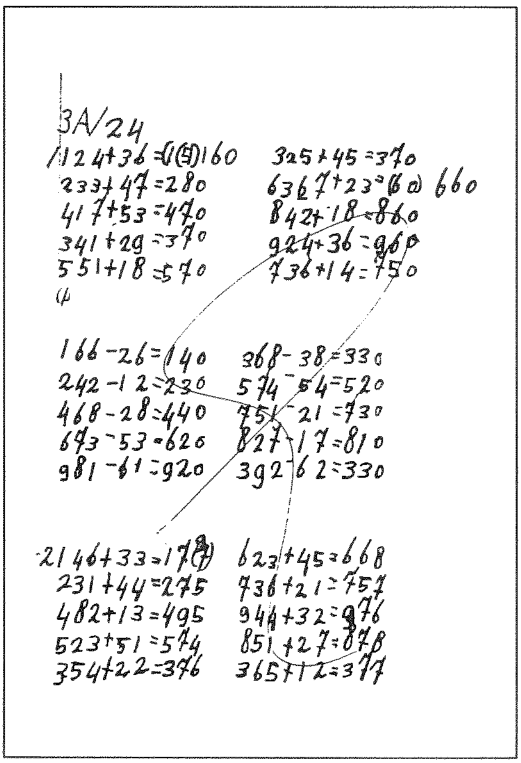
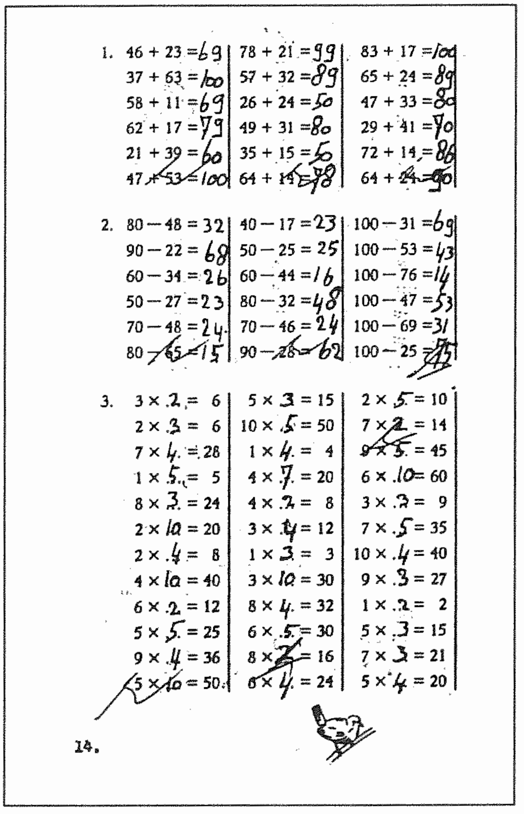


figure 1: worksheet completed by Martijn



1.	46 + 23 = 69	78 + 21 = 99	83 + 17 = 100
	37 + 63 = 100	57 + 32 = 89	65 + 24 = 89
	58 + 11 = 69	26 + 24 = 50	47 + 33 = 80
	62 + 17 = 79	49 + 31 = 80	29 + 41 = 70
	21 + 39 = 60	35 + 15 = 50	72 + 14 = 86
	47 + 53 = 100	64 + 14 = 78	64 + 24 = 90
2.	80 − 48 = 32	40 − 17 = 23	100 − 31 = 69
	90 − 22 = 68	50 − 25 = 25	100 − 53 = 43
	60 − 34 = 26	60 − 44 = 16	100 − 76 = 14
	50 − 27 = 23	80 − 32 = 48	100 − 47 = 53
	70 − 48 = 24	70 − 46 = 24	100 − 69 = 31
	80 − 65 = 15	90 − 28 = 62	100 − 25 = 75
3.	3 × 2 = 6	5 × 3 = 15	2 × 5 = 10
	2 × 3 = 6	10 × 5 = 50	7 × 2 = 14
	7 × 4 = 28	1 × 4 = 4	9 × 5 = 45
	1 × 5 = 5	4 × 7 = 20	6 × 10 = 60
	8 × 3 = 24	4 × 2 = 8	3 × 3 = 9
	2 × 10 = 20	3 × 4 = 12	7 × 5 = 35
	2 × 4 = 8	1 × 3 = 3	10 × 4 = 40
	4 × 10 = 40	3 × 10 = 30	9 × 3 = 27
	6 × 2 = 12	8 × 4 = 32	1 × 2 = 2
	5 × 5 = 25	6 × 5 = 30	5 × 3 = 15
	9 × 4 = 36	8 × 2 = 16	7 × 3 = 21
	5 × 10 = 50	6 × 4 = 24	5 × 4 = 20

14.

figure 2: worksheet completed by Harm

Harm, the boy who completed the work of figure 2, attends grade 6 of another school for mentally retarded children. He is twelve years and nine months old. With respect to mathematics he is one of the weakest pupils in his class. He will leave the school at the end of the year to attend a junior secondary vocational school.

2 a gap between two approaches to mathematics instruction

The given examples of written work by pupils do not only illustrate the ability level at the end of primary school for mentally retarded children, they also give an indication of the kind of mathematics instruction that is given in special education. The traditional mechanistic approach prevails here. Both at Martijn's school and at the school of Harm the textbooks that are used are of the mechanistic type.[2]

In special education the mathematics program is more often than not restricted to the four main operations, extended with word problems, tasks dealing with measurement, money, time and the calendar. The structure of instruction can be characterized as sparse, strict and step-by-step. Whereby the children are offered fixed solution procedures whenever possible. Because there is fear for confusion, there is no room for different strategies.

In other words, special education has hardly been influenced by the innovation of mathematics education described earlier on in this book (cf. for instance the articles by Treffers).

Developments towards the realistic approach to mathematics instruction took place almost solely in regular education. In consequence there is a wide gap between regular primary schools and schools for special education in The Netherlands with respect to mathematics instruction.

Arguments which plead for a more realistic approach of mathematics in special education (Van den Heuvel-Panhuizen, 1986; Ter Heege, 1988) have had little effect until now. This comes as no great surprise because until the present there are few research data to substantiate these arguments.

However, the arguments mentioned above have not been completely without effect. Currently, various efforts are being made in special education to move in the direction of a realistic approach to mathematics. One example is a program[3] for children with math problems which is derived from a series of realistic textbooks. Another example is a realistic last chapter added to an otherwise rather mechanistic textbook series, developed specifically for special education.[4] A last clear example is the recent endeavour to implement realistic mathematics in the education of deaf children.[5]

In general however, both teachers and psychologists in special education remain most reluctant to shift from the traditional to a more realistic approach. Besides uncertainty which results from a lack of research data, there are also other objections. In particular these objections arise in respect to group instruction, the building on informal knowledge of the children and its connected variety of solution strategies, and because of the interaction in the classroom and the complex factor of starting from contexts (Van Luit, 1987 and 1988; Damen, 1990). What these objections actually boil down to is that this kind of instruction demands too much of children in special education.

School practice confirms these objections time and again because of the constant confrontation with the low ability level of the children. In one way or another each test which is administered increases the distance from the realistic approach. If the children are not capable of doing their math in the usual manner, what to do in the case where a context has been added and the children have to come up with solution strategies on their own?

Related to this conclusion it is important to bear in mind that the abilities of the children must not be considered apart from the applied instruction. Perhaps an other manner of instruction would lead to different learning results. The problem is that actual experiences hardly present any reasons to move towards realistic instruction.

3 breaking the vicious circle

In the following a report is given of an attempt to break this vicious circle. The best way to do this is probably to carry out a teaching experiment with an experimental and a control group in which the first group is taught in the realistic manner. In the research under review here, a choice was made for a different approach. This was done mainly for practical reasons, but also because there is a less complicated alternative on hand which – as likely as not – also has some cogency. The attempt to break the vicious circle was in fact carried out without involving any kind of realistic instruction – or to put it more provocatively – without involving any instruction at all. It will be tried to prove the attainability of realistic mathematics with the aid of the achievements of the children.

At first glance this seems to be very contradictory to the remarks made in the foregoing in regard to the achievements of the children. There is a major difference however: the manner in which the abilities of the children are assessed. In this current research a type of testing is employed (cf. Van den Heuvel-Panhuizen, 1990a; Van den Heuvel-Panhuizen and Gravemeijer, 1990; also see the article 'Tests are not all bad' in this book) that offers the children some help. In consequence the children can better show what they are capable of. To achieve this the tasks that are employed must be very accessible. Tasks, the intention of which the children will grasp immediately and which do not require any prior knowledge of procedures and notations. In other words, tasks that make it possible to investigate the abilities of the children without the impediment caused by formal notation.

Empirical evidence exists that this manner of testing is most revealing. This appears for example from a test which contains tasks which entail the features mentioned above, a test which was administered in the first grade after three weeks of instruction. The test revealed that the children were capable of much more than presumed (Van den Heuvel-Panhuizen, 1990[a]). What was the most remarkable however was that this finding – which was already known from individual interviews with children (cf. for instance, Ginsburg, 1975) was discovered by means of a class administered written test.

In order to prove the attainability of realistic mathematics to be extra powerful, the research in question is directed at mentally retarded children who are without doubt the weaker pupils within the group of special education children – certainly when compared to children with learning disabilities.

Moreover, the subject which was chosen is not one which is regularly included in the mathematics program of schools for mentally retarded children. An important characteristic of the current manner of working on mathematics in special education is that the subject matter is outlined in the following manner: first small numbers are processed, then larger ones; easy operations such as addition are dealt with before the more difficult operations like subtraction; bare sums are performed before applications.

The consequence of this is that some pupils might not have even had the opportunity to become acquainted with certain aspects of the subject matter. And this happens not only because of the difficulty of these aspects, but also because they are planned at the end of the program and the children could have been held back by some of the obstacles on the way. In consequence there are children for instance who never reach doing sums with guilders because they never succeeded in doing sums up to one hundred, while it is very possible to do calculations with guilders without being very skilled in arithmetic up to one hundred.

Except for missing certain parts of a subject it is also possible that children will not encounter some subject matter components as a whole.

A recent study involving 82 pupils with learning disabilities and 78 mentally retarded pupils from six different schools for special education shows, for instance, that at the end of grade 6 neither the pupils with learning disabilities nor the mentally retarded pupils had reached even an introduction to the subject of ratio (Damen, 1990). According to a program review for mentally retarded children by Thorton et al. (1983) this is not only true for The Netherlands, but also for the United States.

The question is whether it can be justified to exclude the subject of ratio from the mathematics program of special education. In order to answer this question a test on ratio was administered to a number of pupils in the higher grades of a school for mentally retarded children.

4 the topic of ratio

The topic of ratio involves operating with numbers which express a relation. This relation can concern all measurable characteristics possible, such as number, length, area, volume, weight, duration, price, etcetera. These measurable characteristics – also called magnitudes – can be described in a relative manner by means of ratios. The one is expressed in relation to the other.

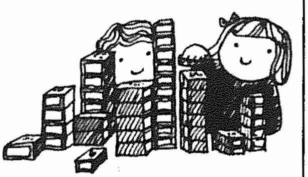
There are several ways of expressing the relation of the one to the other. One manner is to express the length of something in relation to the length of something else. Another possibility is to make a comparison within one object, for instance, by comparing its complete length to the length of a part of it, or by making a comparison of length in time or between two different situations.

Except to one magnitude – whether or not it concerns one single object – the comparison can also refer to different magnitudes. It is possible to express the relation between the length of a certain route and the time it takes to cover this distance, or between the length of something and its price, or between the area of a country and the number of inhabitants. As a matter of fact, by relating different magnitudes to one another, new compound magnitudes are created: velocity, price per meter, density of population.

The ratio problems the children can deal with are different in nature due to the varying mathematical structures behind the problems. In consequence, different kinds of ratio problems can be distinguished, namely finding the ratio (? : ?) or comparing ratios ($x : y \; ? \; a : b$), or producing equivalent ratios ($x : y = ? : ?$) and finally, finding the fourth proportional ($x : y = a : ?$).

It is not surprising that the subject of ratio has not been included in the mathematics program of schools for special education. Ratio is a rather difficult topic. This is caused by leaving natural numbers referring to concrete quantities in favour of relational numbers. However, the special feature of the subject of ratio is that it is quite accessible in spite of this difficulty. The easy part about ratio is that it has strong informal roots based on visual perception. Long before its numerical approach and its formal notation children are already able to see ratio. A toy car looks the same as a real car. It was only made on a smaller scale. The reduction can indeed have been made on different scales and hence there are toy cars of different sizes.

Addie and Peter each have a collection of different match boxes.
Addie has 120 and Peter has 180.
They arrange the boxes in stacks of 10.
Addie then has 12 stacks and Peter has 18.
They can also make stacks of 20.
Then Addie will have 6 and Peter 9.
They can also make even fewer stacks.
Addie can make 2 and Peter 3.
Then each stack will have 60 match boxes.



The *ratio* between the number of boxes that Addie and Peter have, is as 2 stands to 3.
Or to put it shorter: Addie : Peter = 2 : 3.
The *ratio figures* are made as small as possible.

1. Make the ratio figures as small as possible.

12 : 16 = . . . : . . .	75 : 15 =
24 : 18 = . . . : . . .	49 : 21 =
18 : 12 = . . . : . . .	12 : 8 =
21 : 28 = . . . : . . .	24 : 30 =

2. Write down the ratios.

48 crates	: 36 crates	= 4	× 12	crates	: 3	× 12 crates	= 4 : 3
13 sacks	: 52 sacks	= 1	× . . .	sacks	: . . .	× . . . sacks	=
15 bottles	: 75 bottles	= . . .	× . . .	bottles	: . . .	× . . . bottles	=
15 pencils	: 25 pencils	= . . .	× . . .	pencils	: . . .	× . . . pencils	=

3. Marja and Jos are asked to hand out construction paper. There are 40 children in the class. Marja hands out 24 sheets of paper and Jos . . .
The ratio Marja : Jos = . . . : . . .

figure 3: an example of a mechanistic introduction to ratio

It should be observed that the latter view on ratio – the non-numerical roots of it – is neglected in mechanistic textbooks entirely. In these textbooks ratio is introduced on a numerical level. One often starts with teaching the formal notation. An example[6] of such an introduction is shown in figure 3.

This mechanistic approach contrasts markedly with the realistic introduction to ratio as shown in the example[7] given in figure 4.

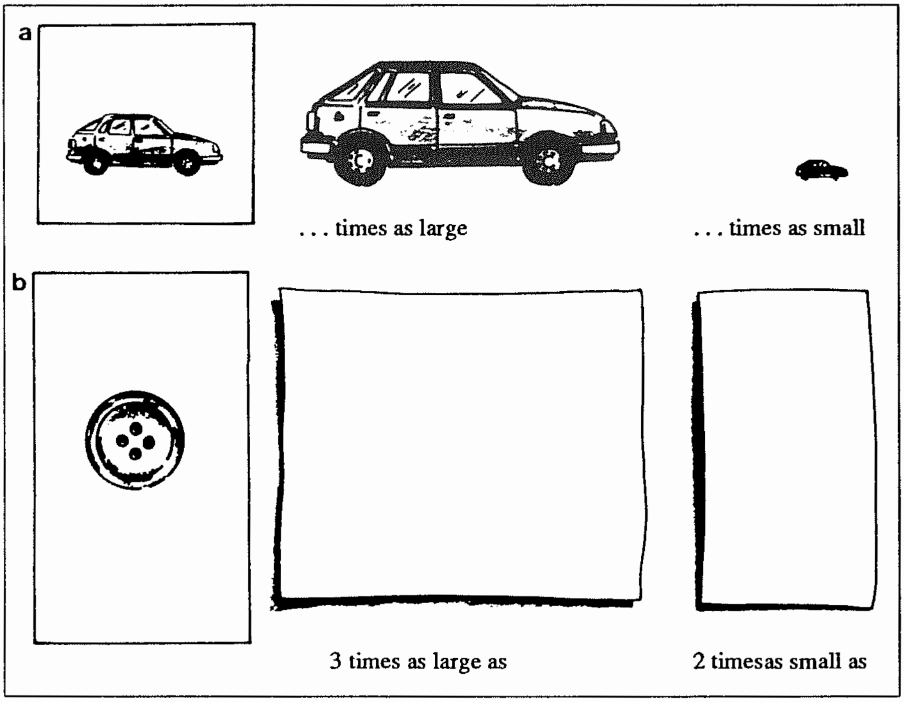


figure 4: an example of a realistic introduction of ratio

5 the test on ratio

The test on ratio which was developed for the study under consideration was devised in accordance with the principles of the MORE-tests (Van den Heuvel-Panhuizen, 1990[a]; Van den Heuvel-Panhuizen and Gravemeijer, 1990; also see the article 'Tests are not all bad' in this book) mentioned earlier on in section three. This means that what was looked for was tasks which are expressed by pictures which, where possible, are self-evident, which refer to meaningful situations and, if possible, present ideas to arrive at a solution. In order to prevent the test from becoming a reading comprehension test rather than a mathematics test, the instructions are given orally and the test sheets contain only the most important textual information.

In regard to content, it has been attempted to design a test which contains a variety of different situations in which children encounter ratio and situations that they are familiar with somehow, through their experiences in everyday life (cf. Van den Brink and Streefland, 1979). In addition it has been attempted to contrive tasks which correspond with the different kinds of ratio problems as distinguished in section four: finding the ratio, comparing ratios, producing equivalent ratios and finding the fourth proportional.

With respect to each kind of problem there has been an attempt to include some difference in level by inserting non-numerical or qualitative tasks besides numerical tasks (cf. Van den Brink and Streefland, 1979; Streefland, 1984; Van den Heuvel-Panhuizen, 1990[b]). Numerical tasks are considered as tasks which contain numerical information and to which the solutions can be found by means of a calculation. The non-numerical tasks on the other hand are tasks in which some numerics can be involved, but for which no numerical information is given on the test sheet. It would be reasonable to assume that these problems will not be solved by calculating, but mainly by trial and measuring and by reasoning.

It should be noted that the tasks which represent a certain kind of ratio problem as distinguished above, do not only differ in respect to the feature non-numerical/numerical. The framework of the test is not that strict either.[8]

The ratio test as a whole consists of sixteen items.[9] To give an impression of the test, some of these items are shown in figure 5. These items have been reproduced at a reduced[10] size.

The text in italics reflects a summary of the oral instructions that are given. These instructions are not printed on the test sheets.

Contrary to the reproductions in figure 5 the real test sheets – as far as the numerical tasks are concerned – feature a scrap paper. The children can use this scrap paper to write or draw something on which will help them solve the problems.

The oral instruction that is given will not mention the use of measuring aids. This means that children are not stimulated to use measuring aids. However, if they do so spontaneously, this is allowed.

Although the test is meant to be administered to entire classes, no time limit has been set for each item. Within reasonable limits the children are allowed to work on the items for as long as they want. In consequence, the faster workers will have to wait a while after most items. However, this will not take that long because the items are not complex and do not require complicated calculations or reasoning.

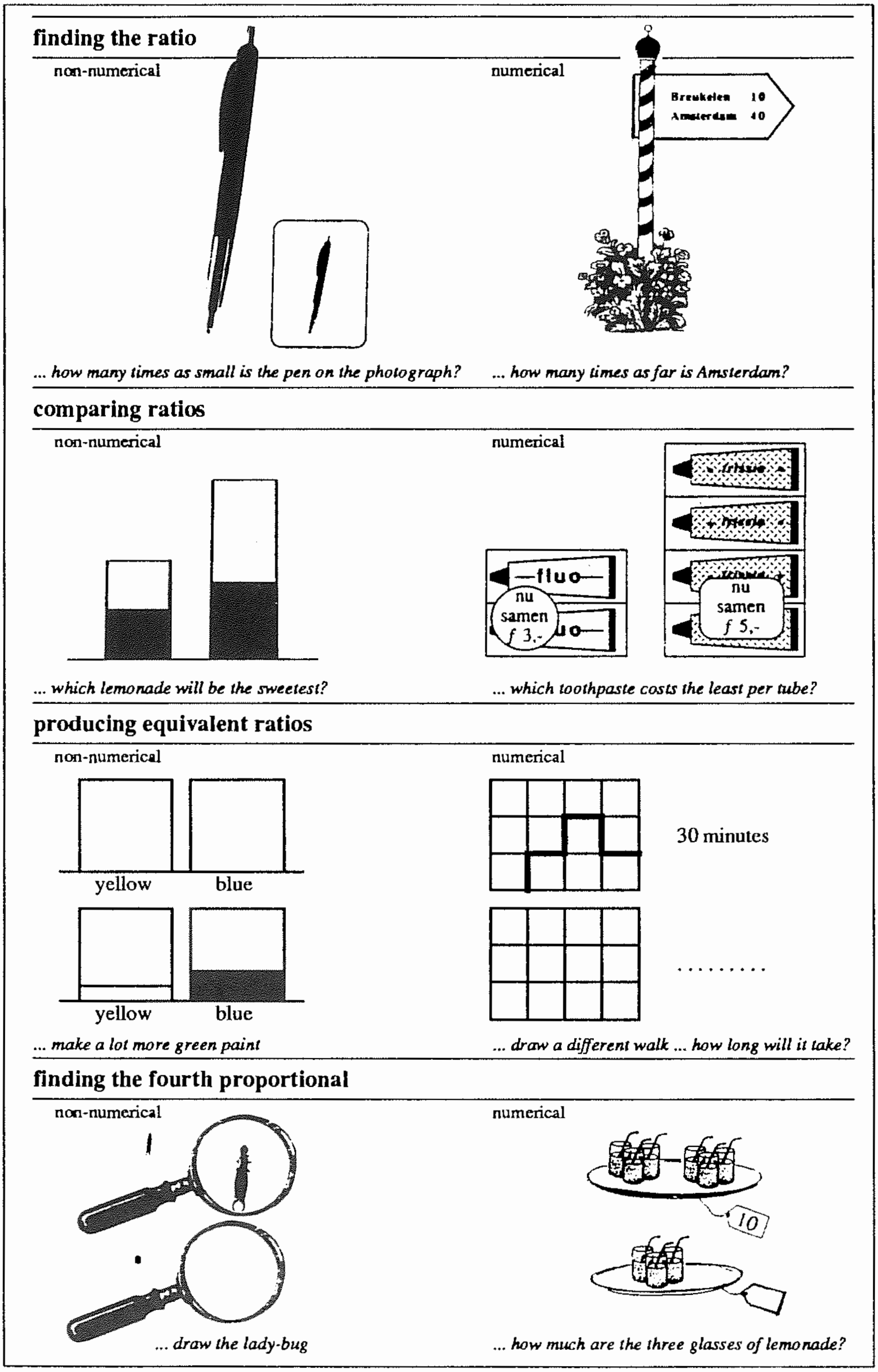


figure 5: examples of test items on the ratio test

6 the design of the research

Because the research involves a pilot study, the test group is restricted to two schools for mentally retarded children. The two schools are located in a town and a city in the Southeastern part of the province of North Brabant. The schools were selected at random. The participants of the study are pupils from the upper two grades of these two schools. The total number of children that participated is 61, of whom 32 from grade 6 (from each school 16 children), and 29 from grade 5 (14 from one school and 15 from the other).

Besides that it is investigated by way of a written test held in class whether the pupils are capable of solving ratio problems, some other pupil data are collected: their age and sex, whether or not the pupil will be leaving school at the end of the year and their math level. The math level was determined by arranging the pupils per class in order from good to poor. Point of departure for this was the progress which the pupils had made in the mathematics program that is used in class.

An inventory was also made in each class of the math topics that had been dealt with this year or before.

To assess what the teachers in the four classes under study think about the feasibility of the topic of ratio in special education, they were asked to make an estimate in advance per test item of the number of pupils that would do the pertaining test item correctly. The teachers made the estimate on the basis of the test booklet and the corresponding test instructions. The teachers were not told that the topic of the test was ratio. This information was withheld intentionally in order not to scare off the teachers too much in the event that the topic of ratio did not constitute part of their program. And which would therefore mean that something was going to be tested that had not yet been taught.

Besides the teachers, two inspectors and two educational psychologists for special education were also asked for their estimates. They were asked to estimate the percentage of pupils that would prove capable of solving the test items by the end of primary school for mentally retarded children. The estimates were made on the basis of the same information that was given to the teachers, with this difference that the test instructions and the manner of scoring was discussed with these educational psychologists. However, here again no information was supplied about the backgrounds and intentions of the test.

7 research results

7.1 the testing

Testing took approximately three quarters of an hour. The children understood the tasks well. Barely any explanation was necessary. Now and again part of the instructions were repeated. The questions which the children asked were not only in regard to the phrasing of the question and the corresponding information, but also moved in the direction of the solution strategies to be employed. This was true especially for pupils in the upper grade. One pupil in grade 6, for example, laughingly asked about the walk (see figure 5) whether the other walk could take equally long. In short, the reactions of the children were such that it was sometimes difficult to follow the test instructions very closely and not to start teaching.

7.2 scoring

It applies for most of the test items that they can be scored unambiguously because the answer can easily be counted correct or incorrect. But there are also items for which the difference between correct and incorrect is not all that clear. These are notably the items for which the answer must be drawn[11], as is the case for the item about paint (also see figure 5). For this type of item the children's drawings were measured and a certain margin was allowed in the assessment within which the 'answer' had to lie to be considered correct. For the item on paint the ratio between yellow and blue paint (1 : 2) had to lie between $1 : 1\frac{2}{3}$ and $1 : 2\frac{1}{2}$.

The items that posed the most problems in terms of psychometrics are the two-choice items in regard to the comparison of ratios (see figure 5). Both an expansion of the choice options per item as well as an expansion of the number of items were not among the possibilities. The first would make the items too difficult and the latter would make the test too long. Attempts were made to reduce the chance factor of the items in some other manner, namely by incorporating strong distractors which refer to the incorrect answer. For instance, the lemonade in the glass with the most syrup will not be the sweetest, even if it does have the most syrup in it.

7.3 psychometric data

Before announcing the results, first some psychometric data about the test itself. These data refer to the now conducted testing of 61 pupils in the two upper grades of two schools for mentally retarded children. The age of the pupils ranges from ten and a half to thirteen years of age and the average age at the moment of the test lies at twelve years and one month. In view of the size of the group tested the test has a reasonable internal homogeneity. The alpha-value of the test is 0.61 and is not sensitive to the leaving out of certain items (there is only a slight increase if the calculation is only conducted for the grade 6 pupils; then the alpha becomes 0.64).

For the majority of the items there is evidence of a significant correlation with the total score. These correlations run from 0.22 to 0.57. Between the items mutually no real clusters of related items can be distinguished. There are a few item-pairs which show a significant correlation, but there is no evidence of a particular pattern. The framework which was in a certain way at the basis of the development of the test (different types of problems and numerical and non numerical tasks for each type) is in no way reflected in the results of the pupils. Something which was not to be expected either because the items differ from each other in many more aspects than in regard to the nature of the problem and the numerical or non-numerical presentation. The frequency distribution of the pupils' total scores of the test reveals a rather normal distribution without top and bottom effect (see figure 6).

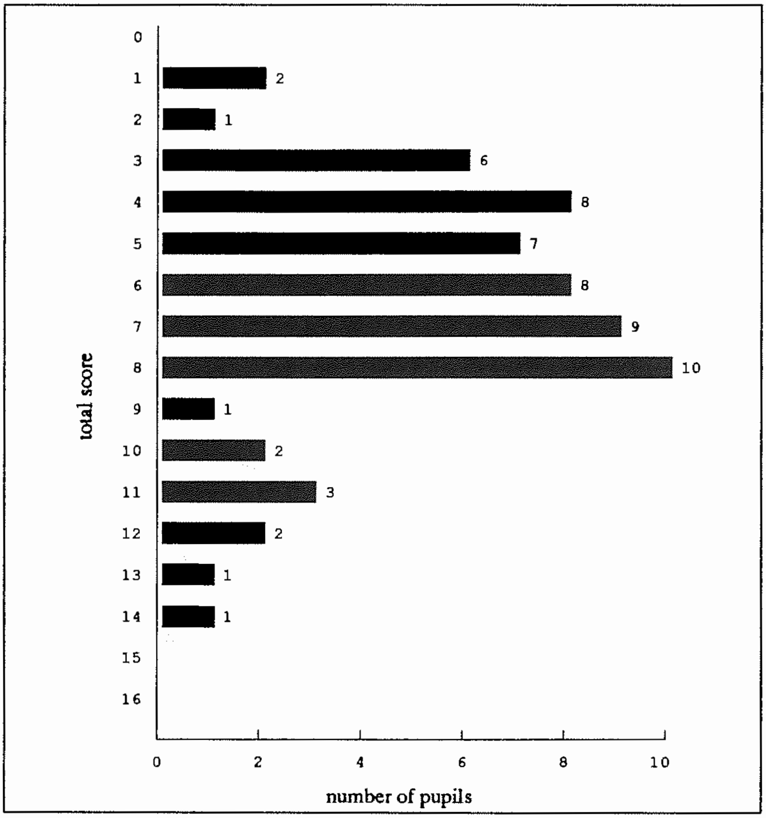


figure 6: frequency distribution of the pupils' total score on the ratio test.

7.4 test results

As can be seen from the frequency distribution of the total scores, the lowest total score was 1 and the highest score 14. The average total score is 6.4 and the standard deviation 2.9.

The percentages correct of the items lie (calculated over the entire group tested) between 13% and 64%. Of the sixteen items, six have a percentage correct of between 40% and 60%. Table 1 gives the percentage correct of each item.

These percentages correct will be explained briefly in the following. In a few instances certain incorrect answers given by the children will also be dealt with, as will what the children wrote on the scrap paper.

table 1: percentage correct per item (total group tested)

	%		%
1. pen	39 (+10)	8. paint	43
2. paper clip	28 (+23)	9. coin dispencer	44
3. road sign	13 (+48)	10. walk	38
4. tree	57	11. lady-bug	64
5. lemonade	54	12. string of beads	51
6. toothpaste	30	13. steps	38
7. swimming	44	14. glasses	64
		15. newspapers	26
		16. lpg	13

Items 1, 2 and 3 are about finding the ratio, or more precisely, determining the reduction or enlargement factor. Besides the problem of discovering the size of this factor, these items deal with the matter of formulation. A number of children gave answers that would suggest that an additive solution was used there instead of a multiplicative solution ('the pen is twice as small' instead of 'the pen is three times as small'). This is a phenomenon known from other research (Hart, 1988; Küchemann, 1989), one which will not be elaborated at this time. The percentage mentioned in the table between brackets refers to the percentage of children that gave an 'additive' answer. If these answers would also be counted as being correct, the percentage correct would rise considerably. This applies particularly to the item about the road sign (see figure 5). For this item almost half of the children indicated the difference in kilometres. However here the additive answer differs somewhat from the additive answer mentioned earlier.

Items 4 through 7 call for the comparison of ratios. Note that here the numerical items (6 and 7) were completed less well than the non-numerical items (4 and 5) and that the item about toothpaste was done less well than the item about the fastest swimmer. During the test there was also greater involvement on the part of the children for this last item. The toothpaste item did not appeal as much and moreover has an insufficient degree of implicit incentive to make them compare the price per tube of toothpaste.

Items 8 through 10 deal with producing equivalent ratios. Although the instructions for these items seem rather complicated at first glance, they tested well and the correct-score was fairly high. Around 40% of the children solved these items correctly. That even applies to the item about the walk (item 10). The work shown on the left in figure 7 serves as an example, it is Harm's solution, the pupil from figure 2! If this item had been marked less strictly the percentage correct would have been even higher. In scoring as we did, the solution on the right in figure 7 was marked incorrect, even though probably only a small error was made in calculating the time that one part of the walk takes. In total, 8% of the children gave an answer whereby it took six minutes to do one part of the walk, instead of five minutes. Another 16% of the children gave an answer whereby it took four to six minutes to walk one part. In other words, answers which although certainly realistic, were nevertheless marked incorrect.

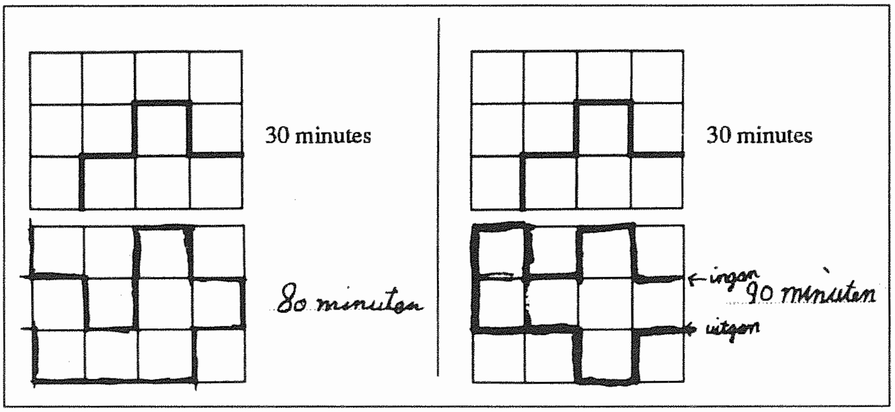


figure 7: two examples of pupils' answers to test item 10

For the remaining six items the children had to find the fourth proportional in each case. Of this series only item 11, the item dealing with the lady-bug, is non-numerical. This item, together with the numerical item about the price of the three glasses of lemonade, were the two that were answered best.

The most difficult items (apart from the item about the road sign) are 15 and 16, whereby the price of twelve kilos of newspaper and the price of forty litres of lpg, respectively, had to be calculated. No wonder, because that is asking rather a lot from a pupil at a school for mentally retarded children. Yet there were pupils who solved these items correctly.

Figure 8 shows the answers of two pupils to item 16. Notice that the answer on the right, although reasonably realistic, is marked incorrect by this current analysis. A total of four children gave a similar solution. The two answers which were encountered most often for this problem, are the answers Dfl 25.– and Dfl 16.–. Both were given by thirteen children (= 21%). These answers also point to an additive solution: the number of litres has increased by ten or by one ten and therefore the price was also raised by ten or by one.

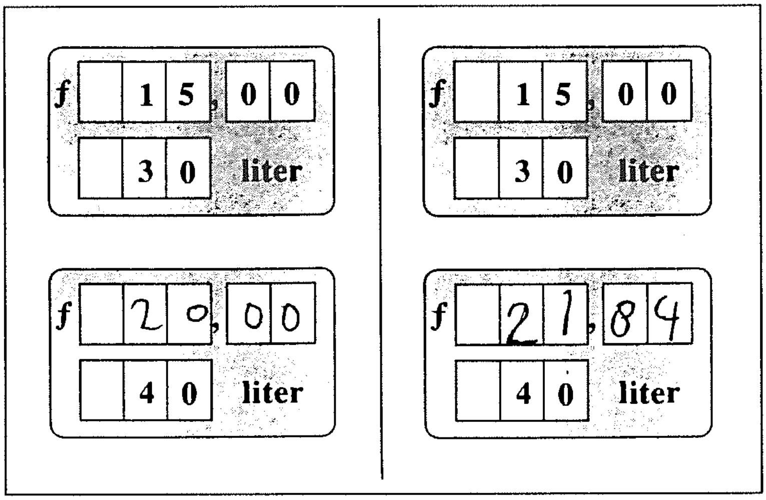


figure 8 : two examples of scrap papers with solutions to test item 16

Figure 9, which refers to item 15, illustrates that the children are not only capable of solving the problem, but that they are also able to indicate how they arrived at this answer. One child (see scrap paper on the left) arrived at the answer by means of three times four kilos. The other (see scrap paper on the right) first calculated the price per kilo.

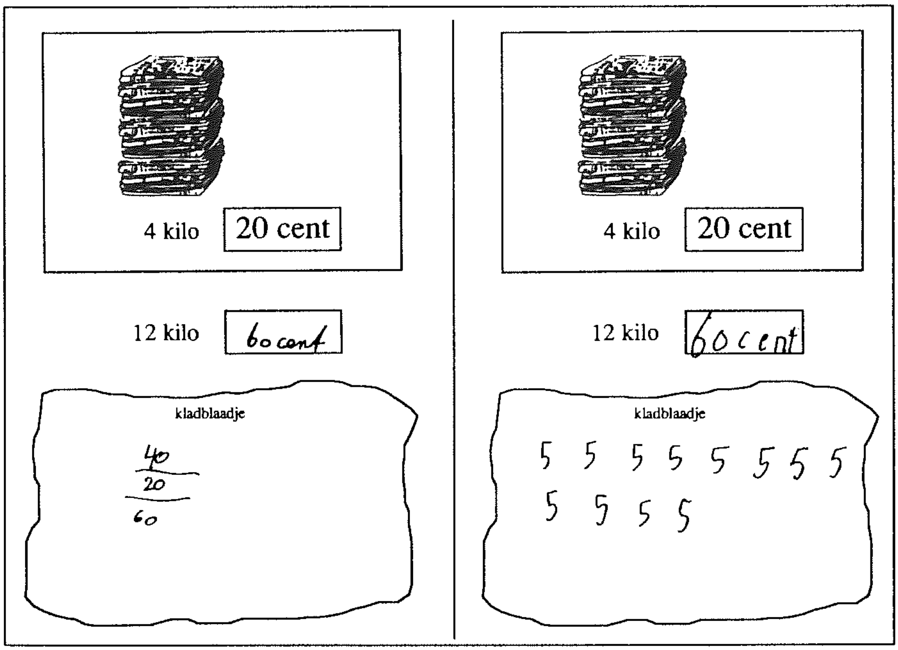


figure 9: four examples of scrap papers with solutions to test item 15

On the whole the frequency in which scrap paper was used is not high: 41% of the children used it once or more. It should be mentioned however, that they were not explicitly asked to work out their answers on the scrap paper. Only if the children wanted to were they allowed to use the scrap paper. Nevertheless this did result in a number of interesting scrap papers. Scrap papers which illustrate that reflection about solution strategies is anything but impossible for pupils at a school for mentally retarded children. Figures 10 through 12 show the scrap papers for three test items.

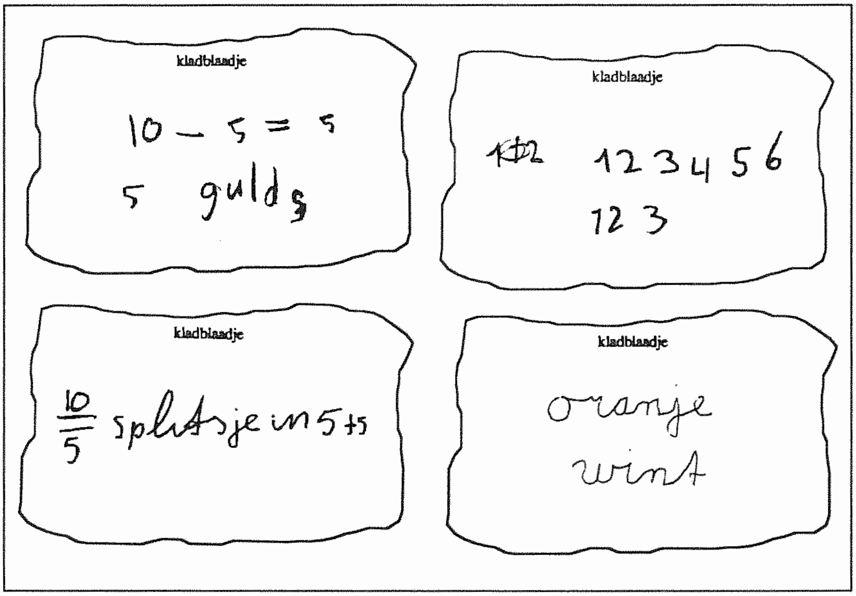


figure 10: three examples of scrap papers with solutions to test item 14

The scrap papers in figure 10 pertain to test item 14 about the price of the three glasses of lemonade (see figure 5). To arrive at the solution to this problem it is important to see that the number of glasses has been reduced by half. In consequence the amount in money must also be halved. The scrap paper on the lower left shows this strategy in the most direct manner: '10 split up into 5 + 5'. The scrap paper on the upper left shows a post-confirmation of the halving process while the scrap paper on the upper right reveals something about an earlier stage of the solution process. By means of the two rows the child could have discovered the halving process. As can be seen on the scrap paper on the lower right, not every scrap paper gives all that much information about the strategy that was applied.[12]

Given in test item 12, figure 11, is the number of white beads and the children are asked how many black beads the chain has. The problem is that the children cannot see all of the beads. The corresponding pieces of scrap paper show the use of a model at three different levels to arrive at the solution.

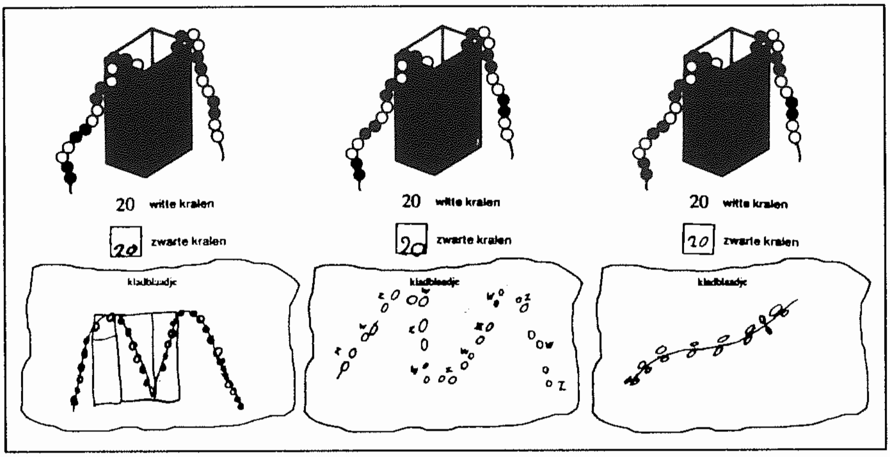


figure 11: three examples of scrap papers with solutions to test item 12

The most concrete one is the one on the left, the most abstract the one on the right. In the latter case both the number of beads and the specific pattern of two black beads followed by two white beads are no longer important. The only thing that counts is that there is an equivalent relationship between the black and the white beads.

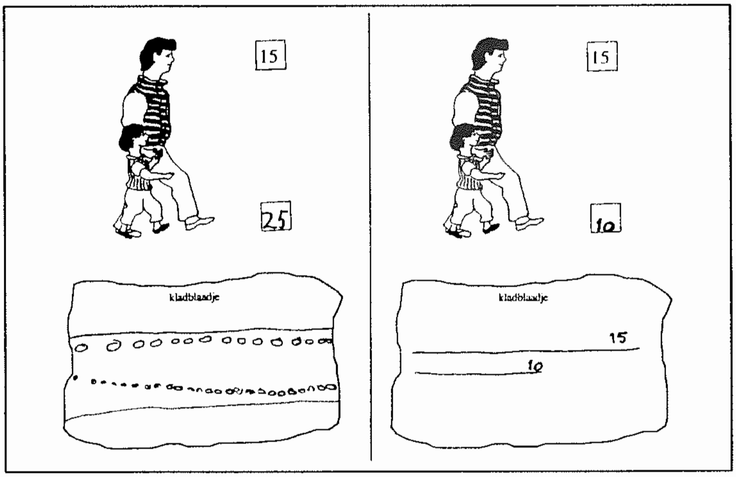


figure 12: two examples of scrap papers with solutions to test item 13

In item 13, figure 12, a father and his son are measuring up the garden together. The father measures fifteen steps and the question is how many steps does the son count. The piece of scrap paper on the left again shows the use of a concrete model, making it possible to find the answer by counting. Sometimes pieces of scrap paper clearly illustrate where the pupil went wrong. An example of this is the scrap paper on the right. Instead of the number of steps, this pupil probably focused on the distance covered after a certain number of steps.

7.5 mathematics program

How good are the results that were found to this ratio-test? Without other references that is hard to say, but in any event the results contrast sharply with the results of the inventory of the contents of the mathematics program. As indicated in table 2 none of the pupils tested had ever been taught about ratio. The results of the ratio test were therefore attained without any explicit instruction in the area of ratio taking place. That offers perspectives.

table 2: inventory of subject matter components that were dealt with this year or before

	class 1.5[1]	class 1.6	class 2.5	class 2.6
(mental) arithmetic to 20	x	x	x	x
(mental) arithmetic to 100	x	x	x	x
column addition/subtraction	x	x	x	x
column multiplication	x[2]	x		
column division	x[2]	x		
fractions				x[6]
percentages				
decimal numbers				
ratio				
geometry				
measuring	x[3]	x[5]	x	x[7]
metric system	x[3]	x[5]		
arithmetic with money	x[4]	x[5]	x	x[8]
other				

1) school 1, grade 5
2) not all of the children
3) m, cm, mm
4) with amounts of money to Dfl 1.-
5) grade 4 level primary school
6) only the concept $\frac{1}{2}$ and $\frac{1}{4}$
7) m, cm, dm
8) assigning names to coins, assigning values and changing the amount of Dfl. 2,50

7.6 expectations

In spite of the given that the topic of ratio does not constitute part of the mathematics program at the school for mentally retarded children, the expectations about the number of pupils with the correct answers, are not that low at all. This probably has something to do with the fact that when taking inventory of the expectations it was in no way indicated that the test was about ratio. Also a remark made by the educational psychologists for special education about the difficulty factor of the contextual use did not result in low estimated percentages correct.[13] Familiarity on the part of these educational psychologists with the backgrounds of the test and the surprising differences between the estimated and the resulting percentages correct on similar tests, may have contributed to the fact that the estimates were not on the low side.

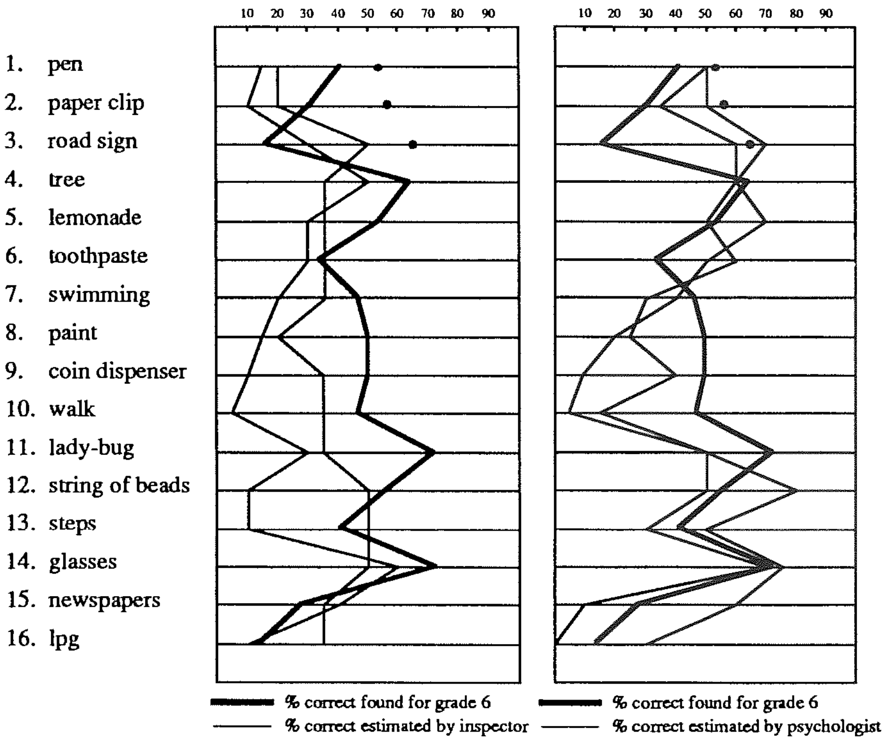


figure 13: percentages correct[14] found in grade 6 and the percentages correct estimated by two inspectors and two educational psychologists

Nevertheless, for the estimates in total, it applies that the skills of the children at the end of primary school for mentally retarded children were underestimated on a fair number of points. For the inspectors this applied more than for the educational psychologists for special education (see figure 13) and for the two grade 6 teachers (see figure 14) this is only the case for one of the two. For the other teacher the estimate corresponds to a large degree with the percentages correct that were found.

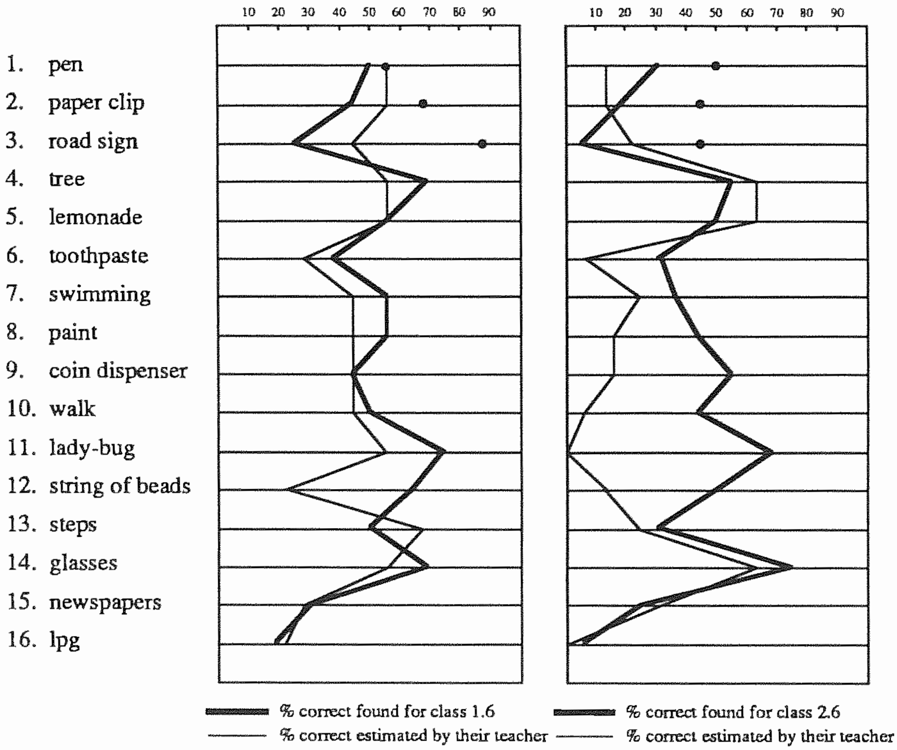


figure 14: percentage correct[14] in two grade 6 groups as these were found and as estimated by teacher

7.7 relationship with certain pupil characteristics

Along with the analysis of the test results it was also investigated whether the total score is related to certain pupil characteristics.

By way of a variance analysis it was investigated whether there are significant differences in regard to the total score between boys and girls, between grade 5 and grade 6 and between the two schools which the children attend.

Two regression analyses were conducted to investigate the relationship with the age of the children and the math level in the class. Of the five investigated relationships, only the connection between the math level in the class and the total score on the ratio test appear to be significant ($F(1,59) = 9.14$; $p < 0.001$). The correlation between these two variables is 0.37 ($p < 0.01$) (see figure 15).

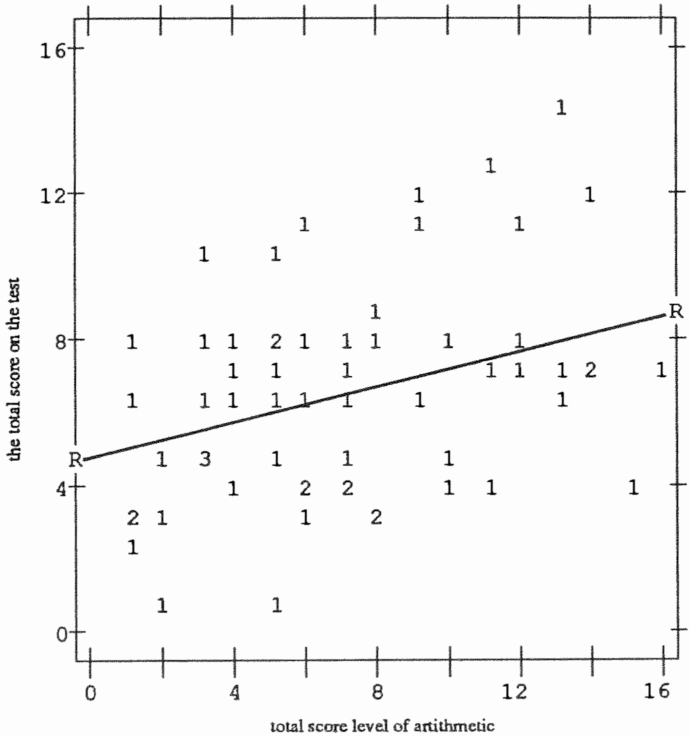


figure 15: relationship between the math level in the class and the total score on the test

8 conclusions

Although no general conclusions can really be made on the basis of this limited study, the encountered test results and the experiences in giving the test do lend some support to the idea that a topic such as that of ratio, has unjustly been left out of the mathematics program in special education.

Another conclusion, which must also be regarded with equal caution but for which the experiences gained in giving the test and the evidence on the scrap papers do provide some grounds, concerns the making aware and talking about strategies. If it appears that children are spontaneously capable of writing down a strategy they have chosen themselves, talking about it together could also very well lie within their reach.

The third careful conclusion concerns working with contexts. It can be derived from the test results that this does not have to be the limiting factor it is so often thought to be. Much depends on the choice of the contexts however and how these are presented. It is essential that the contexts lead to involvement of the pupils and that they induce strategies.

In short, although the study does not provide the certainty that is wanted in special education, there is at the very least reason to subject the mathematics program of special education and the didactics involved, to reflection. This study points out that there are indeed possibilities to cross the didactical demarcation line between regular and special education and to shift the assumed boundaries of children attending special education. This does not mean to say that everything that is possible in regular education can also be realized in special education. Already earlier on it was emphasised (Van den Heuvel-Panhuizen, 1987) that a realistic approach to teaching requires modification when it concerns pupils who are less than proficient in mathematics.

Notes

1 In The Netherlands the percentage of children that attend a school for special education varies from 8% to 3% (cf. Meijer, Pijl and Kramer, 1989). The percentage depends on how the calculations were made. For six to thirteen year olds this percentage is almost 5%.

2 The school Martijn attends uses the mechanistic textbooks 'Niveaucursus Rekenen' with additional material from 'Remelka' and 'Zo reken ik ook' (cf. notes 3 and 4). The school Harm attends uses a series of own workbooks based on the mechanistic textbook 'Steeds Verder'. The series of workbooks contains sixty booklets which have to be worked through successively. Each booklet covers a particular type of calculations. Harm's work (figure 2) is from booklet thirteen on column arithmetic. The number of booklets on this subject is thirty. Besides the booklets on column arithmetic Harm has already completed some booklets on word problems, measurement, money, time and the calendar.

3 The 'Remelka' program for children who have problems in doing mathematics. It is related to the realistic textbooks 'De Wereld in Getallen'.

4 The textbook series 'Zo reken ik ook'. Until now these math textbooks are the only ones that have been developed specifically for special education.

5 From the school year 1988/1989 on the Institute for the Deaf in Sint-Michielsgestel has been working on the implementation of realistic mathematics education. The implementation is based on the realistic textbook series 'Rekenen & Wiskunde'.

6 This example is taken from the grade 6 booklet of the textbook series 'Niveaucursus Rekenen'.

7 This example is taken from 'Pluspunt' (Scholten and Ter Heege, 1983; Ter Heege, Van den Heuvel-Panhuizen and Scholten, 1983). A series that was developed to offer Dutch teachers the opportunity to encounter examples of realistic mathematics education. It is related to a television program that was broadcast on Dutch television. The series covers only grades 2 to 5 and consists of 24 worksheets for each grade. The picture is taken from the grade 5 booklet.

8 Taking into account the aim of the test in this study, this is not necessary. However it certainly is essential in the case where the aim is to investigate what exactly determines the level of difficulty of a ratio item.
Apart from the feature non-numerical/numerical for each kind of ratio problem many other features can be distinguished, like:
- ratios within or between magnitudes;
- ratios concerning one object or more than one;
- ratios concerning simple magnitudes or compound magnitudes;
- ratios concerning one-, two- or three-dimensional magnitudes;

- ratios which imply an increase or a decrease of the value of something;
- ratio problems whereby the standard of one is given or not, or can be calculated or not;
- ratio problems which can be solved by means of internal comparison or external comparison, or by means of both;
- ratio problems which do or do not need the contribution of own knowledge of measures;
- ratio problems whereby a formal ratio language is or is not the case;
- ratio problems in which something must be precisely calculated or which can be solved by means of an estimation (also see Van den Heuvel-Panhuizen, 1990b).

9 The test items which concern the finding of the fourth proportional all except one belong to the MORE-tests.

10 The true size of the test sheets is twelve by seventeen cm.

11 Besides the test items, for which the answer must be drawn, some other test items were also such that a range of answers could be considered as being correct. This is for instance the case for the test items about the pen (see figure 5). Here the answer of $3\frac{1}{2}$ could also have been considered as correct; something which was not done in the current analysis. Another example is the item about the lpg-pump (see figure 8), where, instead of Dfl 20.–, amounts in the vicinity could also be marked as correct. The last example is the item about the father and son who are measuring up the length of the garden (see figure 12). Here any answer between 25–35 was accepted as correct.

12 Here the scrap paper only contains the words 'orange is the winner', where 'orange' refers to the Dutch national football team.
Indeed, the items in a bare form were assumed to be easier than in their contextual version.

13 The following remark was made in this regard: 'Actually the items should also be administered in their bare form.' As a matter of fact, bare sums were assumed to be easier than tasks presented in a context.

14 The dots in the diagram for the first three test items show their percentage correct in the case where both the multiplicative and the additive answers were considered correct.

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A mathematics lesson on videodisc

F. van Galen and E. Feijs

1 introduction

In 1988 the project 'New media in in-service education on mathematics' was started by OW&OC at the request of the Ministry of Education and Science. Its purpose: to investigate the role that interactive video could play in in-service teacher education. In the framework of that project a course was developed on basic skills in mathematics education, around a videodisc on that same topic. In the in-service education course the videodisc provides the 'homework' for the participants. Working with the videodisc is a partial substitute for practical training assignments and the reading of literature.

Under discussion in this article is how participants are given the opportunity to study a lesson which deals with division, by way of interactive video. This lesson was chosen for the videodisc because it, in our opinion, is characteristic for realistic instruction of mathematics.

2 in-service education

The principles of realistic mathematics education are found in other contributions to this book (Treffers and Gravemeijer). Teaching according to these principles demands a different kind of attitude on the part of the teacher: not teaching by demonstration, but stimulating children to arrive at solutions of their own. The actual classroom situation is not like that: there is a preponderance of routine exercises, pencil and paper work, and teacher-dominated conversation.[1] Although practical constraints of classroom management are at least in part responsible for this[2], it goes without saying that in-service education can play an important role in the realisation of changes. Until the present, the possibilities of video have not yet (at least in The Netherlands) been utilised to the full.

In the course 'Basic math skills' video recordings play a leading role. A part of these recordings have been put on videodisc, to be studied by the participants of the course as homework, another part is recorded on video tapes that are played during the sessions of the course. Video provides important advantages for in-service education:

- Video helps to bridge the gap between theory and practice. What teachers see on video elicits more discussion than an article or a lecture by the instructor. It is

easier to make the connection to math instruction in the own classroom.
- Video recordings can be viewed more than once. During the course sessions the instructor can play back the tape if there is reason to do so; one press of the button is all it takes for the videodisc.
- In discussions the participants have the same point of reference, because they have all watched the same video recordings. The conclusions which they draw can differ, however. That makes it easier to discover different points of view.
- The task of the participants of the course is restricted to observation and analysis. Interference between roles, such as occur doing practical training assignments, is thus avoided.
- The video teacher or the interviewer can serve as a model for own performance.
- Participants have the opportunity to observe pupils whose ages differ from their own pupils.

The listed advantages apply to both video recordings on tape (linear video) as well as for interactive video. This article will deal with the possible role of interactive video in in-service education. Not instead of, but alongside linear video.

3 interactive video

Two kinds of video recordings are used in the course: interviews with children and recordings of lessons. Prominent in the interviews is the personal arithmetic activity of the children. In a certain sense the lessons go beyond that: they show that children use all kinds of arithmetic strategies of their own, but at the same time they show how a teacher can work with these differences between children. Part of the recordings are on videodisc and constitute the 'homework' that the participants study between sessions. This is done in small groups of two or three.

Subjects which are dealt with on the videodisc are: counting and the concept of number, addition and subtraction to twenty, multiplication and division.[3] On one side of the videodisc there are interviews with young children. The reverse side deals mostly with recordings of two math lessons. The important advantage of the videodisc is that the participants of the course are forced to form opinions. In group sessions they can wait for someone else to speak up. One of the participants formulated this as follows:

> 'The advantage of the homework is that you can take your time and have another look at what was not clear the first time, or something you missed, or something about which you cannot yet formulate an opinion. For me it clearly supports the sessions because beforehand everyone present has seen the material that is going to be discussed, instead of a few people having their say while the rest sit there nodding in approval.'

A second important advantage of the videodisc is that participants have the opportunity to study the video recordings at their own pace. During sessions recordings are usually studied rather globally. Not everyone is interested in the details of the recordings and the instructor has to maintain a certain pace in the course.

Videodisc and video tapes have separate functions in the course. The videodisc ensures that the participants of the course form an opinion about the topic, while the video tapes serve as a stimulus at the sessions for group discussions. The function of interactive video can be illustrated on the basis of the videodisc lesson discussed in this article, a lesson about arithmetic problems surrounding a PTA meeting.[4]

This article will first give a description of the lesson and will explain precisely why this lesson was chosen for videodisc. Next, it is shown how the participants of the course studied the videodisc lesson in small groups and the important function of the group sessions is described. Finally, a number of conclusions are formulated about the role of video and interactive video in in-service education.

4 the PTA meeting

The lesson is given in grade three. The lesson consists of two main problems:

- Eighty-one parents will be attending the PTA meeting and tables will have to be arranged in the auditorium for them. Six parents can be seated at each table. How many tables will be needed all together?
- Coffee will have to be made for the eighty-one parents. A big pot serves seven cups of coffee. How many pots of coffee will have to be made?

The two items can be described as problems of division, namely 81 : 6 and 81 : 7. Notation of division has not yet been introduced in this class, however, and the pupils will therefore have to look for other suitable solutions. The numbers pose two problems for the pupils:

- What it concerns here is multiplication above ten times. By now, the pupils know the tables of multiplication; these have been practiced extensively. Pupils can put their knowledge of the tables to good use, for instance by reasoning onward from $10 \times 6 = 60$ or $10 \times 7 = 70$. Our expectations beforehand – on the basis of a number of pilot lessons – was that no more than a few children would go to work at such a high level straight away. The lesson is constructed thus that other children are given the opportunity to adopt that approach;
- The sums do not 'work out nicely'. Eighty-one parents are expected and each table can seat six people, but $13 \times 6 = 78$ and $14 \times 6 = 84$. How many tables will you need?

  The same problem arises for the coffee pots. In arithmetic problems children often tend to look only at the numbers, in consequence of which they can have difficulty interpreting a possible remainder, for instance. As long as they bear in mind what the numbers represent this remainder will not be a problem, however.

The lesson was recorded in a studio, but that does not show in the reactions from the pupils. The lesson begins with a short discussion about PTA meetings at the school. The teacher tells about another teacher who has to organise such a meeting and knows that eighty-one parents will be attending. She knows this because the parents have received a letter with a return slip which they had to complete and which their children handed in at school.

> 'She figured, the meeting with the eighty-one parents will have to be held in the auditorium. And we will have to put chairs out because you can hardly expect eighty-one people to stand all evening. So the teacher figured: I need eighty-one chairs. And when she thought about arranging them she wondered whether to put them in rows or in groups. Then she thought, you know what, I'll place them around tables. And if I take large tables ...'
>
> In the meantime the teacher draws a table with chairs around it on the blackboard (fig. 1).

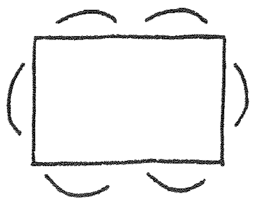

figure 1: table with chairs

> '...Big tables, not the small ones like we have here, and we'll seat ... how many shall we seat at each table?'
> Pupils call out: 'four', 'six', 'five'.
> 'The teacher thought, let's try it out ... like this ... six to one table. Alright? I'll just draw another table here.'
> One pupil answers: 'Twelve'.
>
> The teacher draws the second table more schematically (fig. 2).

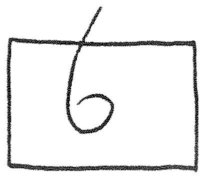

figure 2: six to one table

> 'Okay. I'll just write down a six, six chairs to this table. Then the teacher wondered: how many more tables am I going to need?'

The pupils are given a piece of paper and they go to work on their own.

how many tables are needed?

It appears that the children have followed different strategies. Anita draws all of the tables and chairs on her paper (fig. 3).

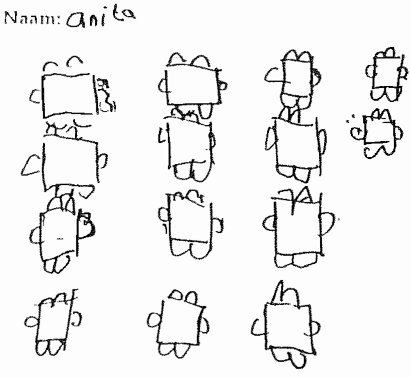

figure 3: worksheet Anita

Osman starts off by drawing tables and chairs, but after a time he leaves away the chairs. He keeps track of the score on the tables themselves (fig. 4).

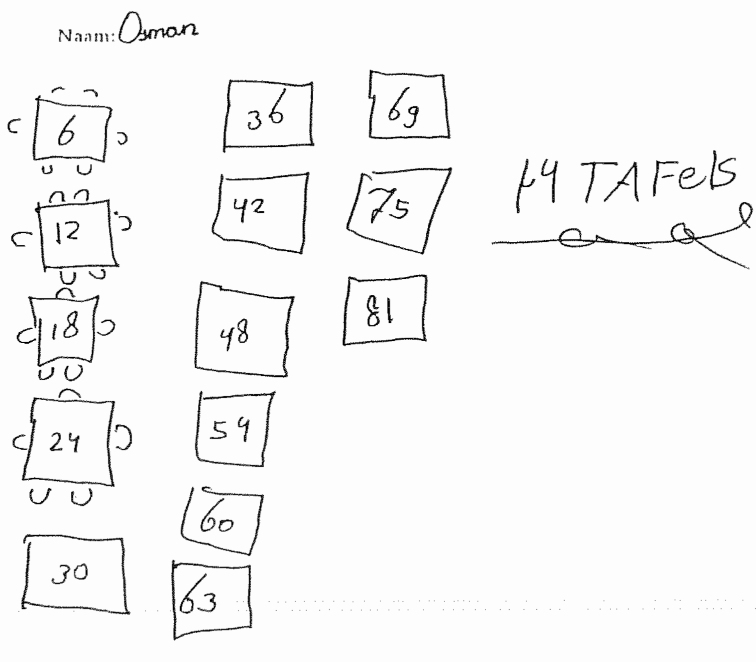

figure 4: worksheet Osman

After 60 he makes the jump to 63 and ends up at 81 exactly. 'Tafels' means tables.

Fatiha starts by drawing the tables, but after a few she switches to numbers (fig. 5).

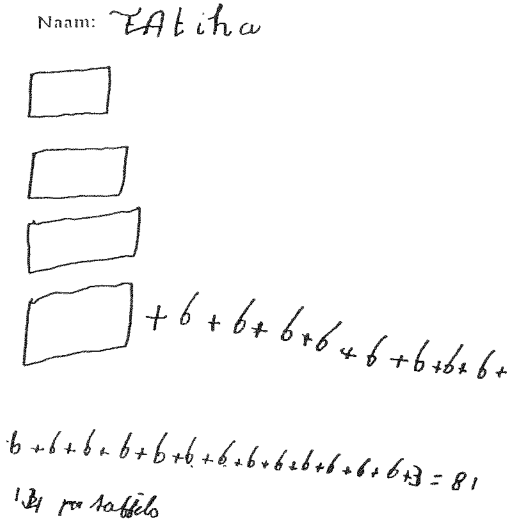

figure 5: worksheet Fatiha

Later on, she starts again. She has probably found the answer by repeated addition, therefore '6, 12, 18, ...'
Noura approaches the problem in the most efficient manner (fig. 6).

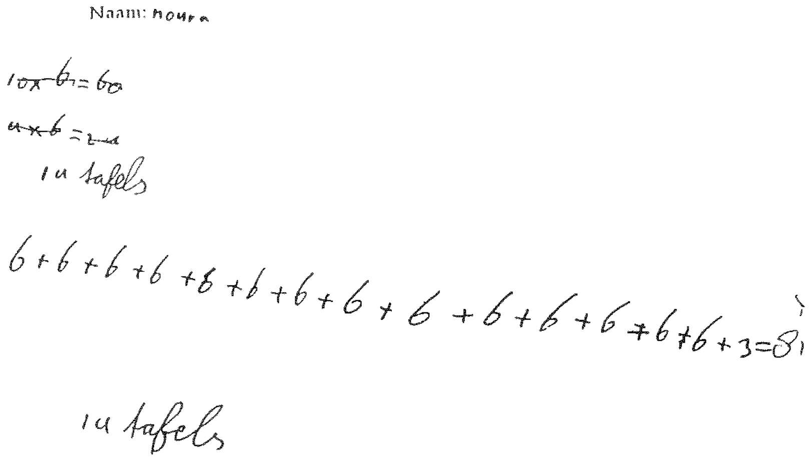

figure 6: worksheet Noura

She writes down '$10 \times 6 = 60$' and '$4 \times 6 = 24$' and arrives at the answer of '14 tables'. For reasons that are not clear she crosses out this solution, however, and proceeds to make a long addition. Perhaps it was the fact that by multiplying she arrived at 84 and not at the required 81 that bothered her. In that case, the long addition sum is a justification in second instance. She had already found the solution.

The class discussion of the problem takes three minutes. It is not Noura but Wendy who is the first to put the method of calculating via '$10\times$ ' into words.

Teacher:	'Wendy did it differently. She didn't draw any tables at all.'
Wendy:	'I just ... you just ... ten, and that is sixty, and then you keep adding six until you get to eighty-one. And for the last one you add three, otherwise you will have too many.'
Teacher:	'Right. Did you all hear what Wendy did? What she says is that you don't have to draw tables at all. If I start by taking ten tables, and what did you say then? Ten tables is how many?'
Wendy:	'Sixty, and then you add ...'
Teacher:	'Six and that gives you ...'
Wendy:	'That gives you 66, and then eh ...'
Teacher:	'Another six'
Wendy:	'is 72'
Teacher:	'And another six'
Wendy:	'is 78.'
Teacher:	'Have we seated everyone now?'
Wendy:	'No.'
Teacher:	'No, so you add another table.'
Wendy:	'That makes fourteen tables.'
Teacher:	'And that makes fourteen tables. And now there are ...'
Wendy:	'You don't add another six, you need three more.'
Teacher:	'Yes, right. Why?'
Wendy:	'Otherwise you will have too many.'
Teacher:	'Yes, otherwise there will be empty seats. But it's better to have a few seats too many than not enough seats.'

Wendy's paper is chaotic because she apparently started over a number of times (fig. 7). Her explanation is perfectly clear, however. Wendy and Noura appear to be the only pupils to have worked with $10 \times 6 = 60$.

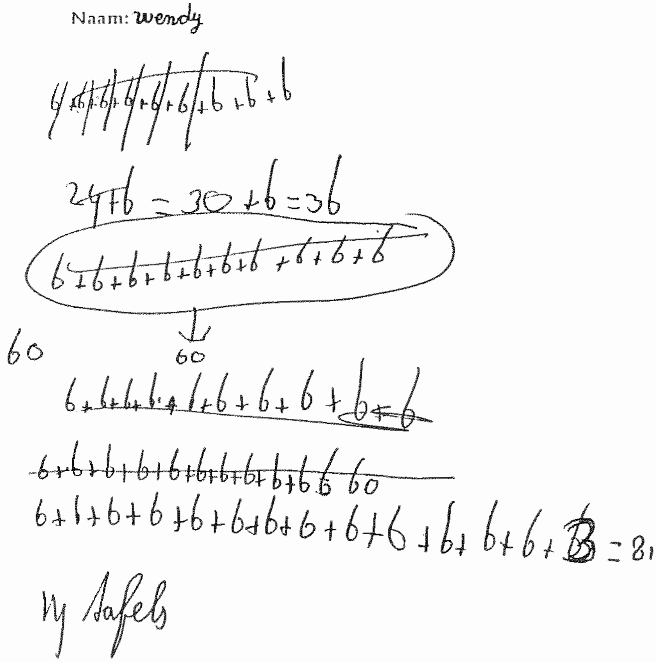

figure 7: worksheet Wendy

how many pots of coffee?

When the discussion about the tables is finished the second problem about the pots of coffee is introduced. The teacher draws a coffee pot on the board and tells the class that it was found out that one pot serves seven cups of coffee. So he writes a '7' on the pot. Again, the children are given paper and told to calculate how many pots the teacher will need. The problem is almost identical to the previous one and the teacher hopes, of course, that pupils will choose a more efficient method this time. This is why extra attention was devoted to Wendy's method at the end of the discussion about the previous problem.

We can now see whether the children will indeed employ more efficient methods. One pupil who does follow the ten times method is Anita. Compare her approach of the first problem (fig. 3) to what she writes on the second paper (fig. 8).

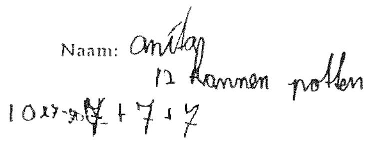

figure 8: second worksheet Anita

Fadoua has also picked up the '10 × ' method, although this is not evident from her paper (fig. 9).

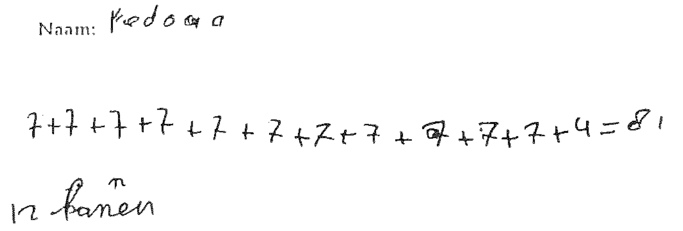

figure 9: worksheet Fadoua

During the discussion afterwards she says:

Teacher:	'Fadoua, you had twelve pots. How did you do that?'
Fadoua:	'Ten times seven is seventy and then ...'
Teacher:	'Yes, pay attention boys.'
Fadoua:	'Plus seven is 77. And then you subtract 3 from the 7.'
Teacher:	'Right, but how many pots of coffee do you need?'
Fadoua:	'Twelve.'
Teacher	'Twelve, but in the last pot, you said that too, ...' (points to other pupil)
Other pupil:	'Four ... four.'
Fadoua:	'Four cups.'
Teacher:	'Then you have four cups left.'

Valentina has also adopted the '10 × ' approach without this being evident from her paper. On her paper she has written out eleven sevens and one four, but during the discussion she says:

'First I did ten times seven, that makes seventy, and then I added another seven, that makes 77 and then another four.'

A number of children who did not use the ten times approach did employ other methods. Fatima, for example, who only drew chairs the first time, now calculates with steps of seven: 7 + 7 + 7 + ...

Two children again drew all of the coffee pots. One of then is Sanjay (fig. 10).

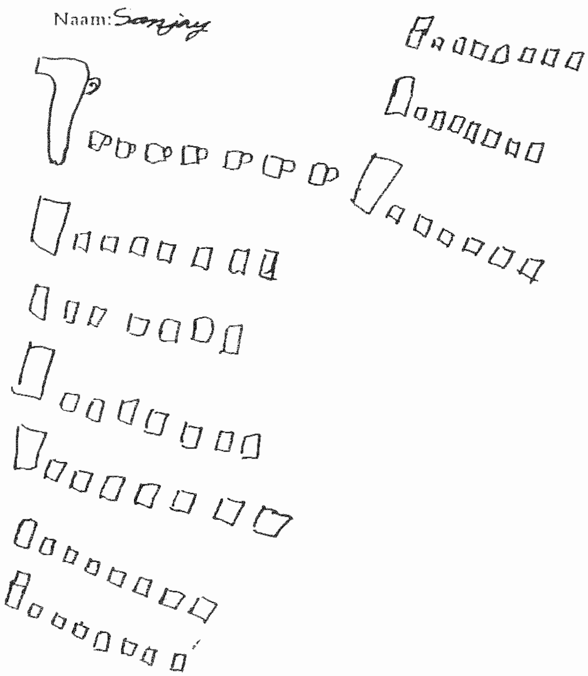

figure 10: worksheet Sanjay

5 the lesson as an example

The lesson about the PTA meeting was recorded on videodisc to illustrate that children are capable of solving rather tricky problems independently and that the solutions they come up with are divers in level. Accept the differences in level, is the message, and try to steer pupils towards an approach at a higher level. What was looked for was a lesson which would illustrate that children can learn from such an approach; hence, the chosen structure: item – discussion – second item. The lesson does not concern problem solving as an isolated activity. The lesson has a very specific place in the line of multiplication and division to column arithmetic and as such constitutes part of a (realistic) curriculum.

The lesson can be described from three different points of view: (1) the learning of the children, (2) the role of the teacher in that learning process and (3) the curriculum, i.e. education on the longer term.

the learning of the children

Mathematics is not passive absorption of knowledge which is presented by others; learning mathematics is to construct, or at least it should be. In the 'realistic' approach to mathematics education it is posed that context problems therefore play an essential part: problems which are so concrete that the children are prepared to look for an answer on the basis of the knowledge they already have.[5] In the lesson about the PTA meeting the context of the tables and coffee pots is realistic to such a degree that the pupils have the possibility of solving the two problems at a very concrete level – namely by drawing and counting. The context also helps the children to interpret the remainder. The choice was for two similar problems: what pupils pick up from the first discussion can be applied in solving the second problem.

For recording-technical reasons the group was kept small: thirteen pupils, about half the size of an average class. What the pupils wrote or drew on their papers is summarised in table 1.

table 1: the written work of the pupils for the first problem (how many tables are needed?) compared to their written work for the second problem (how many pots of coffee are needed?)

2nd problem / 1st problem	drew cups	only drew pots	addition or table of 7	'70'
drew chairs	2		2	1
only drew tables		2	1	
addition or table of 6			3	
'60' on their paper				2

What the children wrote down cannot as such serve to determine how they worked. In principle, there are four ways by which the solution was found:

- By counting; counting as many chairs or cups as it takes to reach the required number.
- Repeated addition; the pupil recites to himself: 'six, twelve, eighteen, ...' (for the coffee problem 'seven, fourteen, twenty-one, ...') until he comes close to the required number. The pupil keeps the score by drawing the tables, coffee pots respectively.
- Also repeated addition, but now the pupil keeps track by writing out the addition sum ('6 + 6 + 6 + ...', '7 + 7 + 7 + 7 + ...', respectively).
- Reasoning from a familiar product of the tables, like '$10 \times 6 = 60$'.

What is especially difficult is to determine whether the children counted or not. It is possible that a child draws chairs around the tables and still keeps track by adding instead of counting; the chairs are only drawn as a finishing touch. It is significant, in this connection, that the number of chairs or cups is incorrect on any of the papers!

On Anita's paper (fig. 3) there are 84 chairs – all of the tables are full – something that is also seen on the other papers. For the coffee pot problem Miriam draws thirteen times seven cups and Sanjay (fig. 10) besides pots with seven cups, also draws pots with six cups. The fact that not a single child draws exactly 81 chairs or 81 cups does not, however, mean that they used repeated addition; they might have counted as well as calculated. The discussions do not throw light on this point, because when pupils answered that they drew everything the teacher did not inquire further.

If we look only at the written work there seems to be relatively little progress. Nine of the thirteen children approached the second problem in the same manner as the first. Three children switched from drawing to writing out sums and therefore worked at a more symbolic level the second time. There is, however, only one child who draws for the first problem and writes down '$10 \times 7 = 70$' for the second item.

We can also take into account what the children contribute in the discussion afterwards. Table 2 compares the written work and the discussion on the point of reasoning further on 10×6 and 10×7.

table 2: use of 'ten times' or not for the solution to the second problem. The numbers in brackets indicate which conclusion might be drawn if one looks only at the written work of the children.

2nd problem / 1st problem	counting or repeated addition	$10 \times 7 = 70$
counting or repeated addition	(10) 8	(1) 3
$10 \times 6 = 60$		(2) 2

Now, we see that three of the children who solved the first problem by counting or repeated addition adopted 'Wendy's method' for the second problem, namely Fadoua, Anita and Valentina. What they did precisely was described in paragraph 4. Three children who adopted the ten times approach, is that many or few? We will come back to that when discussing the place of the lesson as part of the curriculum. In any case it is not so that all of the pupils immediately adopt the efficient approach. What one does as a teacher with pupils who do not choose the most efficient approach can be a point of discussion. Does one tell them what to do or wait patiently for it to happen?

the role of the teacher

Essential to realistic mathematics education is that pupils receive ample opportunity to solve problems at their own level. The two problems of the lesson are not all that easy, but almost all of the pupils proved capable of solving it because if need be, they could draw the situation. An important aspect is that the teacher stimulates the pupils to find a solution for themselves and not to wait until some brighter fellow pupil comes up with the correct answer. The atmosphere in the classroom must be such that the children dare come forward with their solution without being afraid of looking 'silly'. Moreover, the teacher must dare take the time for a lesson like the one described above. In the lesson on the videodisc a total of thirty minutes is devoted to only two 'sums', nevertheless intensive mathematics is being done during that time.

On the one hand the teacher must provide opportunity for solutions at personal level, but at the same time he must help pupils to solve problems at a higher level. Pupils must be stimulated to choose for a more efficient approach. To achieve this, the teacher must steer without being forceful. In the lesson on the videodisc the teacher steers by drawing the tables in two different manners: once as a table surrounded by six chairs (fig. 1) which indicates a solution via the counting of chairs and once as a rectangle with a '6' in it (fig. 2). He also steers by dealing explicitly with Wendy's handy solution to the first problem during the discussion. The fine art of teaching is finding the proper balance between latitude and steering. If in the discussion a teacher were to say very explicitly: 'See, so a problem like this is solved by way of $10 \times 6 = 60$', children will copy it as a trick, without adopting this approach for themselves. If a teacher does no steering whatsoever the chance is that the pupils will stick to a laborious approach. Inventory in class of solution methods plays a key part here. In the discussion all kinds of strategies come forward without the teacher having to make an explicit choice. The teacher can, as it were, hide behind what the pupils have done, how he himself might have done the problem is left aside.

learning on the longer term

Not every interesting problem is also a useful problem. In a good mathematics method each problem that is presented has a place in the learning process on the longer term. The lesson about the PTA meeting constitutes the beginning of a learning strand which ultimately leads to column division. Via context problems like the ones found in this lesson, it becomes clear how you can make efficient use of '10 × '. At a next phase, long division is then introduced (fig.11).

6 /81\	
60	10 tables
21	
18	3 tables
3	
3	1 table
0	
	14 tables

figure 11: long division scheme

This scheme cannot only be used for ratio division but also for distribution division, for example dividing eighty-one objects fairly among six people. Only then instead of 'tables' the indication will be 'per person'. Whether the result is fourteen, or thirteen, or perhaps $13\frac{1}{2}$, depends on the context. At a later stage, larger numbers are introduced.

> 1128 fans want to go watch Feijenoord play an away match. The treasurer of the supporters club has learned from a bus company that one bus can seat 36 passengers and that per ten buses the club will be eligible for an extra discount.

The information about the discount is intended to point out to the pupils the possibility of subtracting several times 10×36 from 1128. Pupils can solve the problem in the various manners indicated in figure 12.

36/1128\		36/1128\		36/1128\	
360	10 buses	720	20	1080	30
768		408		48	
360	10 buses	360	10	36	1
408		48		12	
360	10 buses	36	1		
48		12			
36	1 bus				
12					

figure 12: three levels of solution

The last solution is the end level that is the objective. A number of pupils reach this end level very quickly, others take longer or remain at the level of the first or second solution. Nevertheless, this does mean that these pupils are also capable of doing long division.

Although only indicated here by a few examples, it will be clear that a learning strand like this is far removed from mechanistic drilling of the algorithm. Pupils can reach a level that is comparable to the standard algorithm, but if they encounter difficulties they can always fall back on the more extensive form.

The lesson about the PTA meeting therefore has a specific place in the line to column division, but in the recording of this lesson that remains implicit. The ongoing line is made point of discussion at a session of the in-service education course, among other in the discussion about the results which were achieved in the lesson: three pupils who adopted the ten times method, is that many or few? Whether it is many or few depends on the question of how much one is in a hurry.

At the moment when the lesson is given, division notation has not yet been introduced. Nor have the division tables as such been practiced. The lesson therefore constitutes a preparation on division in general, whereby, by way of calculating with '$10 \times$', an advance is taken in the direction of column division. The first introduction of column division will not take place until almost a year later. In view of the place of the lesson in the complete learning strand the achieved result is certainly acceptable for the time being.

Summarising, the described lesson was recorded on videodisc because:

- the lesson shows that pupils are capable of tackling rather tricky arithmetic problems; on their own they come up with solutions at different levels;
- it shows the importance of the well balanced, well thought out steering; the teacher gives pupils opportunity to search for solutions independently;
- the lesson fits the learning on the longer term.

6 the PTA meeting lesson as interactive video

Interactive video supposes the link up of a videodisc player to a computer. The videodisc itself contains only video images and sound; questions about these video images are asked via the computer. The videodisc contains illustrations from a book as it were and the computer adds text to this. The latter is a fitting comparison particularly to the courseware developed for this project, because the layout of the texts and illustrations on the computer screen have the appearance of 'booklets' (fig. 13).

When the television symbol is selected with the mouse the pertaining video excerpt starts on the screen of the video monitor. The symbols at the bottom of the page provide the option to page to next pages (arrow to the right), to previous pages (arrow to the left), to skip to the survey of all available courseware booklets (cards), to the index of the booklet one is working on (the 'i'), or to extra explanation (the question mark). Answers to questions can be typed in on the dotted lines.

In the lesson about the PTA meeting a total of thirty minutes was devoted to the two described problems. The videodisc shows some ten minutes of video recordings. The videodisc also contains still pictures. As far as the lesson is concerned the videodisc contains:

- video fragments which show how the teacher introduces the problems;
- still pictures and short video clips which show the pupils trying to find solutions on their own;
- all of the worksheets of the pupils as still pictures;
- video recordings of the entire discussion of the table problem (about three minutes) and the coffee pot problem (a minute and a half). These discussions have been recorded in such a manner that they can also be viewed per pupil whose turn it is.

The idea is for the participants to review the courseware together with a colleague.

The courseware booklet 'The PTA meeting' globally follows the progress of the lesson as it was given. Participants first see how the teacher introduces the problem of the tables. His summary of the problem is the following:

> 'Right, what is the problem: we will have eighty-one people coming, six people to one table. The teacher wants to know: how many tables do I place. Draw it, calculate, do it anyway you want.'

The participants of the course are asked how they themselves would find the answer (fig. 13), and on the next page they are asked to predict how many pupils will succeed in finding the correct answer.

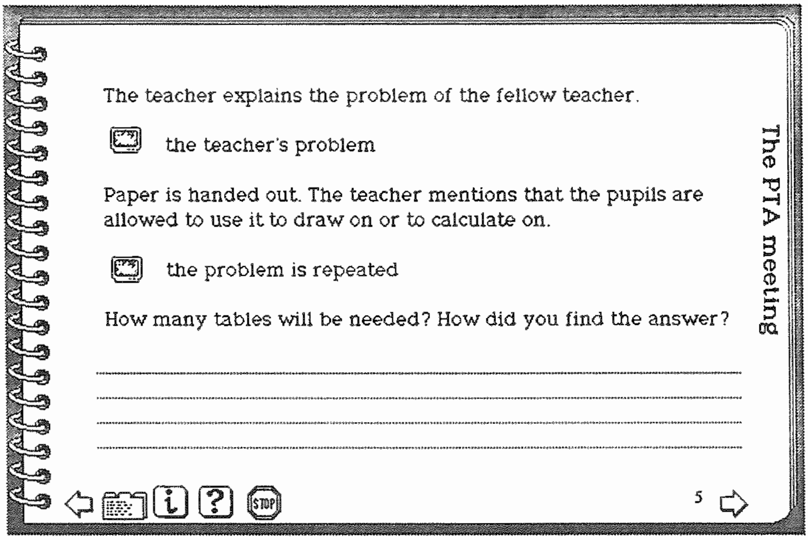

figure 13: page in 'The PTA meeting'

Once the participants have been shown some excerpts from the eight minutes the pupils were at work on their own, they get to see the pupils scrap papers. The question that is asked is 'Suppose you are the teacher and you are walking around in the classroom. Which pupils would you involve in the discussion afterwards? Explain why.' A seating plan of the classroom can be called upon. It is shown in figure 14. At the moment when the question concerning which pupils one would call upon is asked, the bottom three buttons of the row at the upper right are not visible; participants can only choose between 'photo' and 'worksheet 1'.

Clicking on one of the tables makes the scrap paper of the pertaining pupil appear on the television monitor. Later, the participants can watch the class discussion which was held afterwards, first in its entirety and later if they wish – via the seating plan – in short fragments. They are asked to make a classification of the various manners in which pupils attacked the problem.

Next, the participants are shown how the teacher introduces the coffee pot problem. The teacher explains that one pot serves seven cups of coffee and asks:

> 'Could you try to find out how many pots of coffee the teacher will have to prepare?'

The pattern more or less follows that of the first problem: first watching how the teacher introduces the problem, then studying the scrap papers of the pupils and then watching the class discussion that is held afterwards. One may expect that the participants will observe that the work the pupils do on paper does not always tell the whole story. At the end of the booklet 'the PTA meeting' a number of questions are asked about the lesson as a whole. We will discuss some of them later on.

As one progresses through the courseware booklet the number of buttons at the right above the seating plan becomes larger: the number of options increases.

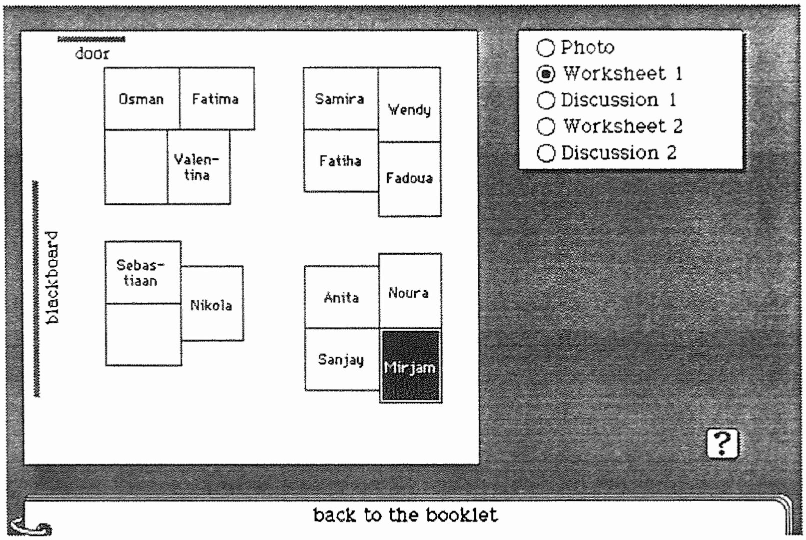

figure 14: seating plan in 'The PTA meeting'

Initially, the participant can only choose between 'photo' and 'worksheet 1', but ultimately there are five buttons in a row (fig. 14).

- *Photo*: a photograph of the pertaining child can be recalled to refresh the memory;
- *Worksheet 1*: the paper on which the pupil has solved the problem about the tables appears on the television monitor;
- *Discussion after problem 1*: if a pupil has been called upon in the discussion this particular fragment from the discussion can be recalled;
- *Worksheet 2*: the scrap paper for the coffee pot problem;
- *Discussion after problem 2*: video fragment from the second discussion.

Via the seating plan new information can be compared to what a pupil did earlier in the lesson. In reviewing the work sheets for the coffee pot problem, for example, the participant can ask for what the pupil wrote down and said for the table problem.

A linear description has been given in the foregoing of the contents of the booklet

'The PTA meeting'. That corresponds with the manner in which participants usually progress through the courseware booklet. The progress of the lesson is followed: how does the teacher introduce the problem, how do the pupils attack the problem, what happens when the next problem is introduced? The videodisc also makes it possible to examine per child – from the seating plan – what went on in the lesson. For example: how did Valentina approach problem one, what did she contribute to the discussion, how did she attack problem two, what did she say in the discussion of problem two? The seating plan offers the possibility, as it were, to make a cross section of the lesson per child.

For the courseware booklet 'The PTA meeting' the computer does not give feedback on what the participants type in.[6] The questions are intended to make the participants sit down and reflect; these are not questions to which only one answer is possible. When they are finished the participants can print out what they have typed in and bring the print out along to the next session.

7 reactions to courseware questions

The course was given in 1990 by different instructors to three school teams. Between sessions of the course the participants worked through the courseware. Working with the videodisc constituted the 'homework' as it were for the participants. The equipment – the videodisc player, monitor and computer – was available at that particular school for the duration of the course. One course consisted of eight or nine sessions and the courseware consisted of twelve sections. The section on 'The PTA meeting', which is discussed in this article, was studied at the end of the course, between the seventh and eighth sessions.

What the participants learned from working through the courseware booklet 'The PTA meeting' will be described on the basis of the answers that they typed in response to the questions in the booklet. This paragraph deals with the reactions on the part of the participants to the most important questions and we will follow the order of the questions as they appear in the courseware. In the following paragraph we will investigate whether what we intended did indeed come across.

As far as the section 'The PTA meeting' is concerned, we have at our disposal the answers of thirteen groups of participants. Nine groups consisted of two participants and three groups of three participants. On one occasion five teachers watched the PTA meeting-lesson together because the second videodisc player at the school was defective. The answers that the participants typed in were saved on the diskette of that group and were later printed out by us. In a number of instances supplementary information was available. A member of the project team was present as an observer in six of the groups and in three groups a tape recording of the discussions about the courseware was later typed out.

predictions

When the participants have seen how the teacher introduced the first item of the lesson – how many tables must be arranged to seat eighty-one parents? – they are asked to predict whether the pupils will prove capable of solving the problem. To allow good video recordings the 'class' was kept small.

> 'How many of the thirteen pupils do you expect will be able to find the correct answer?'

The estimates by the participants vary from 'five or six' to 'twelve'. We asked the question to make the participants aware of how important it is to allow the problem to be drawn; in principle that makes it possible for all of the children to find the answer. And that proves to be the case, because from the scrap papers of the pupils – which the participants are later allowed to look through – it appears that all thirteen of them found the correct answer. It should be mentioned in this respect that the papers do not show how much help they possibly had from the teacher while working on their own.

giving turns

Next, the participants have the opportunity to look at the pupils' work. The question that is asked about this work is:

> 'Suppose you are the teacher and you are walking around in the classroom. Which children would you give a turn in the discussion? Explain why.'

The intention of the question is that participants discover the different solution strategies. At the same time the issue that touches on the essence of interactive instruction is raised: which solutions must be dealt with in the discussion and why?

A few of the groups of participants indicate that they would ask pupils whose strategies are unclear for an explanation. One example:

> 'Sanjay, his work is difficult to read and we want to see whether he knows the tables. Osman, because after sixty he all of a sudden added three and then went on by adding six.'

These participants chose only the children whose answers they themselves do not understand. Most groups indicated, however, that they would give turns to pupils to present different types of solutions and their arguments are therefore of a didactical nature. One clear answer in this respect:

> 'Fatima, the lowest level. Samira, one step higher. Sanjay, 6, 12, 18, the table of 6 and the answers. Have Sebastiaan explain how he got 10×6 (60), and then have the class respond how to get 81.'[7]

All levels of solutions must pass the review, these participants feel. And rightly so, in our opinion, because in this manner the pupils are given the opportunity to progress to a higher level, without that necessarily having to be the highest level. In five of the thirteen answers explicit didactical considerations are indicated.

classification

Later, when they have also had an opportunity to watch the discussion held afterwords in class, the participants are asked to make a classification:

> 'There are children who approached the problem in more or less the same manner. Write down the categories of solutions that you have distinguished.'

From their answers it appears that the participants realize that there is a wide variety in approaches. What is of the most interest is to see whether the participants have discovered the two extremes: as the most concrete level the drawing and counting of the chairs and as the highest level calculating with '10×6'. The drawing or counting of chairs is explicitly mentioned in nine out of thirteen answers. Two groups gave a rather vague answer ('drawing', 'drawing of the tables') and two groups mentioned only the more efficient methods. Calculating via 10×6 is mentioned by seven groups. Two gave a vague answer ('use table of six', 'use of the tables') and four groups made no mention of calculating via 10×6. No far reaching conclusion needs to be related to the fact that '$10 \times$ ' was not mentioned in every case; the groups that did not mention '10×6' do refer to solutions with '$10 \times$ ' in their answers to other questions.

the second item solved differently?

After the participants of the course have watched the teacher introduce the coffee pot problem, they are given the opportunity to look through the new worksheets of the children. The question that is asked of them is:

> 'Are there pupils that solved the coffee pot problem in a different manner than the table problem? Explain your answer.'

The formulation of the question is open; it is not explicitly asked to indicate which pupils solved the second problem more efficiently. The participants appear to understand that this is the question at issue, however.

> 'There are no entirely new solution methods, but most of the children did calculate more abstractly, worked with the table of seven or did repeated addition.'

> 'A few pupils followed the 10×7 strategy this time. Other pupils who drew the first time, count on this time.'

the role of the teacher

At the end of the booklet 'The PTA meeting' a number of other questions are asked about the lesson as a whole. One of these questions concerns the role of the teacher:

> 'The teacher gives the pupils latitude to attack the problem in their own way. He does steer the learning process. Describe two moments during the lesson which clearly illustrate that the teacher is trying to steer the learning process.'

One group of participants skipped this question. Of the other twelve groups eleven mention the fact that the teacher emphasises 'Wendy's way' – calculating with $10 \times$.

Other instances of steering that are noted are:

- When introducing the table problem the teacher draws two tables on the board, first one with and then one without chairs (see fig. 1 and 2); he indicates that it is not necessary to draw the chairs. (Is mentioned twice).
- When the children are working on their own, the teacher remarks to Osman that it is not really necessary to draw all of the chairs. (Mentioned twice).
- Upon introducing the coffee pot problem Osman calls out spontaneously: 'The table of seven.' The teacher repeats: 'The table of seven, Osman says.' (Mentioned three times).
- The teacher emphasises the remainder, there is one left over. (Mentioned once).

Especially the manner in which the teacher draws the tables on the board is significant. He clearly emphasises that the pupils are allowed to draw the chairs, but at the same time shows the possibility of a short cut. This instance was however mentioned by only two of the twelve groups.

subject matter

In two questions the issue of the subject matter is raised.

> 'Describe the teacher's intentions with the lesson. What is he hoping the pupils will learn?'

> 'Can you indicate how the lesson fits into the learning of the children on the longer term?'

From the answers to these questions it is apparent that the intention of the lesson has been understood. The participants mention the preparation to division and the use of '10 × '. One example of an answer:

> 'Objective: introduction to division.
> Places emphasis on '10 × ', so the intention is also that the pupils pick up the structure of '10 × '.'

Not one group draws the explicit relationship with column multiplication and column division.

what did you learn?

The last question in the booklet 'The PTA meeting' reads:

> 'Do you think you have learned something from watching this lesson. If yes, what?'

Two groups did not answer the question. The observation report indicates that in one of these groups a teacher remarked that she has learned that they are in too much of a hurry and are working much too abstractly. The answer is, however, left open. No observation report is available for the other group.

Two other groups say that they have not learned anything:

> 'We do not.'
>
> 'Not really, but it was fun to see the situation in practice.'

From the observations it appears that these teachers feel that they themselves teach in much the same manner. The other nine groups responded affirmatively. In their description of what they have learned it is especially the objectives of the lesson and the role of the teacher that is emphasised:

> 'That children can learn from each other's explanations.'
> 'Devoting attention to different solutions.'
> 'That they can fall back on this lesson, therefore imagine concrete situations. And the clever way of using 10 × straight away.'
> 'Yes, making a problem visible by placing it in a certain context (making up a story), in other words, insight.'
> 'Working with multiplication tables in a more practice-oriented manner.'
> 'Yes, you can introduce division by means of a very practical situation close to the reality of the children. They must come to realize that it is good for something. Much attention for the solution strategy gives other children an idea of how to work more efficiently.'
> 'The teaching attitude.'
> 'Yes, the way in which problems can be presented.'
> 'Steering the problem in a playful manner.'
> 'Yes, we rediscovered the importance of self discovery of solution strategies. We must try to avoid to be too quick in prescribing certain strategies for the children.'

8 relationship between courseware and the course sessions

In the courseware the emphasis lies on the learning processes of individual pupils. Participants of the course are asked to examine how pupils solve problems, they compare solutions on paper with what is being said, and they examine whether the pupils learn anything from the lesson. The lesson has deliberately been chosen to demonstrate that children can learn from solving and discussing a problem. In the courseware the role of the teacher receives relatively less emphasis; only at the end of the booklet are there some questions about it. From the answers given by the participants it appears that they did not observe very well how the teacher was steering the learning process. About the place of the lesson in the whole of the curriculum – the third point we meant to raise by the lesson – there is also only one question in the courseware. The responses to this question were not clear.

Although the participants are positive about what they have learned from the lesson, there are a number of points which merit further attention. This is not a problem, as the courseware is only the homework for the course and not intended to stand on its own. At the following sessions of the course the lesson is topic of discussion again. The participants have their answers with them, printed out on paper.

It would take up too much space to describe the actual sessions in our experiment. Let us suffice by describing in general terms which points should be raised at the sessions.

As far as the learning of the pupils is concerned, the course session is not the place to handle in detail the solutions of the various children; the courseware offered opportunity enough for that. It is a good idea to discuss, as an introduction to the session, which various kinds of solutions were presented by the children.

More attention should be devoted to the role of the teacher. It should be discussed what the function of the class discussion in the lesson is, and also how such a discussion can be organized. It must become clear why the children are given a turn: not to get an explanation for one's self, but to inform the other pupils of the various possibilities. Questions which the instructor of the course could ask are: Which pupils must especially be given a turn? and: Is it a good idea to choose a certain order in giving turns? Many teachers tend to allow the 'brighter' pupils to come forward with their solutions. However, that means that the class discussion will remain restricted because the correct solution is given straight away. Moreover, this is not very motivating for pupils who have greater difficulty with the problems. The other extreme is that especially children with laborious or incorrect solutions are given a turn. The danger is great that the discussion will focus on the specific problems of that one pupil.

Besides that the teacher steers the learning process through the class discussion about the various approaches to solving the problem, the teacher also steers in other ways. In the answers to the question about this in the courseware most participants responded that this was evident when the teacher emphasised 'Wendy's way'; the more subtle instances of steering are mentioned infrequently. On the basis of the print out which the participants make of their answers, an inventory can be made at a following session about the ways in which the teacher tries to steer the learning process.

Finally, during the sessions attention should also be devoted to the place of the lesson in arithmetic education. Notably it should be illuminated how this lesson presents a preparation to arithmetic via '10 ×' in column arithmetic and the taking of 'chunks of ten' in column division. What participants see in this lesson are learning processes at a micro level. It should be made clear in the sessions of the course how the principles that lie at the basis of this lesson are extended to learning the algorithms.

After studying the courseware and after the session which follows, the participants should be asked to prepare and give a lesson of this kind as a practical training assignment. Only in this manner can they experience for themselves which didactical choices are made in the preparation of such a lesson – for instance in regard to the presentation of a problem and other explicit moments of steering – and what the implications thereof are for the progress of the lesson.

9 conclusions

In this article it has been demonstrated, on the basis of one of the topics of the videodisc 'Basic math skills' which opportunities interactive video can offer for in-service education of primary school teachers. Besides the described lesson the videodisc contains recordings of a lesson on multiplication plus a large number of fragments of pupil interviews. The experiences with the videodisc have led to conclusions which we will summarise briefly below. They very much resemble the conclusions drawn by Hansen[8], who experimented with interactive video for training interview skills.

1 Video recordings of pupils and lesson situations add a substantial element to in-service education.

Pupil interviews that have been recorded on video and recordings of lessons play a main part in the course 'Basic math skills'. Video recordings of this kind can ensure that in-service education is not theory only, but that it is applied by the participant teachers in their own instruction. Video recordings have important advantages: they can be watched more than once, participants have the opportunity to concentrate on observation and analysis, the recordings provide a common point of reference for the discussion and they make it possible to illustrate examples of good instruction. These points pertain both to video in the course sessions as well as for working with the videodisc.

2 Interactive video and linear video both fulfil an individual function in in-service education.

Interactive video is a new medium that has not yet been put to frequent use for in-service education. The experiences gained thus far in the course 'Basic math skills' are very positive. Interactive video does not make linear video (video tapes) redundant, however. Sooner the opposite. Interactive video and linear video can be employed in such a manner that they supplement each other in a meaningful fashion. The videodisc is notably suited to allow participants of in-service education courses to analyse video excerpts on their own and allow them to form their own opinions. Video tapes can be used at the group sessions as a point of departure for a discussion in the wider group.

3 Interactive video should facilitate reflection.

The tendency exists to regard the computer as a device that must do the teaching: knowledge which the pupil must acquire is stored in a computer program. Indeed the computer knows less than an instructor, but on the other hand (1) the computer has all the patience in the world and (2) it can be deployed on an individual basis. It fits this conception of the role of the computer to construct expert systems which can simulate the instructor to an even higher degree.

It is most doubtful whether the computer can ever play that kind of role in learning the complex skill that teaching is. In any case, the role of the computer in the course 'Basic math skills' is a totally different one. Interactive video affords the participants the means to study video recordings in detail. The questions in the courseware serve to set the participants to reflect, but the computer does not supply the answers. At best the participants can compare their own answers to those of the developers.

Reflection can be elicited by a confrontation with the ideas of others: the opinions of a colleague with whom the videodisc is studied, the opinions of others in the team, the ideas of the instructor. The question is therefore what the place of interactive video must be in the total learning environment. In the in-service experiment we have carried out, the choice was for a certain form. In other situations, for example in the initial education of teachers, the exact place of this medium must be reconsidered.

4 Learning experiences gained in using interactive video need to be cycled back into the course group.

Working in smaller groups with interactive video leads to discussions. There are, however, reasons to expand the discussion to the total group at the course sessions. The discussion within a small group will often be incomplete. The group as a whole is where a greater variety of viewpoints will emerge. Moreover, during the group discussion the course instructor will be present who can ensure that no essential issues are overlooked. Besides, discussion in the larger group is important for the motivation of the participants, because there is a need to discuss personal conclusions with those of others. This applies especially if the larger group consists of colleagues from the own school. The fact that the team has the opportunity to take part in the course by the participants has been valued as very positive.

5 Use of interactive video alters the role of the course instructor.

The use of interactive video in in-service education alters the role of the course instructor. When doing their homework the participants of the course have already seen and discussed a great deal and the instructor must be prepared not to repeat all that in the larger group. The discussion at the sessions must be directed at the main points at issue. That means that the instructor must have an idea of what the participants have done in self study. Use of interactive video can mean that fewer course sessions are required. The demands on the instructor are greater, however.

Notes

1 Romberg, T.A. and T.P. Carpenter (1986). Research on Teaching and Learning Mathematics: Two Disciplines of Scientific Inquiry. In: Merlin, C. (ed.), *Handbook of Research on Teaching*. New York: Macmillan Publishing Company, 850-874.
2 Desforges, C. and A. Cockburn (1987). *Understanding the Mathematics Teacher*. Lewes: The Falmer Press.
3 Galen, F. van, M. Dolk, E. Feijs, W. Uittenbogaard and V. Jonker (1989). *Videodisc 'Basic math skills'*. Utrecht: OW&OC.
4 For the description of a pilot lesson see: Dolk, M. and W. Uittenbogaard. De ouderavond. *Willem Bartjens, 9*, (1), 14-20.
The videodisc shows the definitive recordings, the same lesson, but at a different school.
5 Gravemeijer, K. and F. van Galen (1984). De betekenis van contexten. In E. Feijs and E. de Moor (eds.), *Panama Cursusboek 4*. Utrecht: SOL/OW&OC, 135-142.
6 In other courseware booklets that have been developed for the videodisc, this is the case.
7 The reference to Wendy should probably be Sebastiaan.
8 Hansen, E. (1990). The Role of Interactive Video Technology in Higher Education: Case Study and a Proposed Framework. *Educational Technology, 30*, (9), 13-21.

List of publications in English

In the past decade several books in English have been written by members of the Research Group on Mathematics Education OW&OC. Realisitc Mathematics Education is a theme in each of these publications.
The following list puts these publications in chronological order.

1 Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures.* Dordrecht: Reidel.
2 Lange, J. de (1987). *Mathematics, Insight and Meaning.Teaching, Learning and Testing of Mathematics for the Life and Social Sciences.* Utrecht: OW&OC.
3 Treffers, A. (1987). *Three Dimensions, a Model of Goal and Theory Description in Mathematics Instruction – The Wiskobas Project.* Dordrecht: Reidel.
4 Heuvel-Panhuizen, M. van den, K.P.E. Gravemeijer and L. Streefland (1990). *Contexts, Free Prodductions, Test and Geometry in Realistic Mathematics Education.* Utrecht: OW&OC.
5 Streefland, L. (1991). *Fractions in Realistic Mathematics Education. A Paradigm of Developmental Research.* Dordrecht: Kluwer Academic Publishers.
6 Freudenthal, H. (1991). *Revisiting Mathematics Education. China Lectures.* Dordrecht: Kluwer Academic Publishers.
7 Streefland, L. (editor) (1991). *Realistic Mathematrics Education in Primary School.* Utrecht: CD-β Press, Freudenthal Institute.

Since OW&OC has changed its name in Freudenthal Institute, the books that have been published by OW&OC will be available from now on at this Institute.