

DEFINING DERIVATIVES, INTEGRALS AND CONTINUITY IN SECONDARY SCHOOL

A phased approach inspired by history

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ABSTRACT

Historically, the concepts of analysis have been developed in the opposite order compared to the deductive order. In the 17th century, Newton and Leibniz introduced derivatives and integrals in order to solve physical and geometrical problems. In the 19th and 20th century, mathematicians have provided foundations for this theory, by giving a precise definition of the limit concept. Not only were foundation problems typical of the 19th century, but also these precise definitions had become necessary by the more general concept of a function. ‘Strange’ functions had appeared, whose continuity, differentiability and integrability could not be decided without refined definitions.

In many courses and textbooks, formal definitions are introduced from the beginning. For pupils, these definitions seem to complicate things unnecessarily. Later on, in these courses and textbooks, one makes “cheese with holes” (by replacing too difficult proofs by the words “one can prove that...”). Our proposal is to use “visual” concepts of derivative (slope of the tangent), integral (area) and continuity (connected curve) in a consequent way, at least in a first phase. At the end of the course, we propose to confront pupils with some oscillating functions and functions with an infinite number of discontinuities, in order to motivate refined definitions. The last phase, only for the most mathematically oriented pupils, consists of using these refined definitions in a few examples of proofs. For this proposal, we used historical inspiration without copying history.

1 Introduction

1.1 Cheese with holes

In many textbooks for secondary schools the concepts of calculus have at one hand complicated definitions and at the other hand graphical interpretations, which are much easier. The pupils get the impression that a mathematical definition should be complicated and formal in order to merit the status of a mathematical definition. For them, a derivative “is” simply the slope of the tangent and an integral “is” simply an area; limits of differential quotients and Riemann sums are in their view a way of saying the same thing in a needlessly complicated way. Similarly, a continuous function is for the pupils an uninterrupted graph and a limit is nothing more than the value obtained by going “closer and closer” to a point (or further and further away to infinity); they do not understand the usefulness of expressing this with ϵ and δ .

Is the rest of the calculus course based upon the “visual” ideas or upon the formal definitions? This is not always clear. For example, there is the *mean value theorem*, saying that the integral of a continuous function f in an interval $[a, b]$ is equal to $(b - a)f(c)$ for a c in $[a, b]$. From a visual point of view, this theorem is obvious (you “see” it in figure 1!). It does not need to be formulated nor proved.

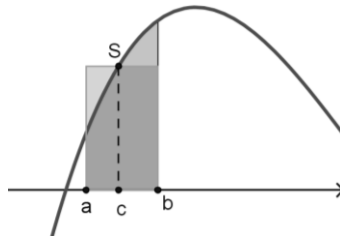


Figure 1

However, the textbook provides a proof, based upon the “intermediate values theorem” (a continuous function in $[a, b]$ attains all values between $f(a)$ and $f(b)$) and Weierstrass’ theorem (a continuous function in $[a, b]$ attains a minimum and a maximum). These two theorems, visually equally obvious as the mean value theorem itself, are not proved. The textbook says that *one can prove them* but that the proof is too difficult. This is cheese with holes! The pupil feels disoriented in this world the rules of which he does not understand.

1.2 Phases

We propose to replace the situation described in 1.1 by a phased approach. In a first *visual phase*, we start with derivatives and integrals, motivated by applications and defined using the visual ideas of tangent lines and areas, and using an intuitive idea of limit. A large part of the calculus course in secondary school can be done in this visual phase. In a second phase, we confront the pupils with some strange functions for which the visual definitions do not work anymore. This establishes a motivation to look for more refined mathematical definitions in a third phase. The fourth and last phase consists of using the refined definitions in proofs.

Our approach inverts the logical order, in which firstly the limit concept is defined and then derivatives and integrals are defined as limits. We rather follow the historical evolution: derivatives and integrals are from the 17th century (Newton, Leibniz) whereas the formal definition of a limit only appeared in the 19th century (Cauchy, Weierstrass), when the concepts of numbers and functions had evolved and “strange” functions had become possible, for which the visual ideas did not work anymore.

2 Visual phase

2.1 Derivative

We suggest introducing the derivative by using several contexts: skis following the slope of a (two dimensional) mountain, a tangent to a circle and the instantaneous velocity. In these contexts, pupils get familiarized at the same time with the notion of a tangent line and of a derivative. The common idea is a chord connecting two points $P(a, f(a))$ and $Q(x, f(x))$, which becomes a tangent “in the limit” when Q approaches P more and more. This can be shown with dynamic geometry software such as GeoGebra (figures 2a-d). The point Q can approach the point P from both sides, but when it coincides with P , there is no line PQ any more. So, the points have to get nearer and nearer to each other without ever coinciding. The limit idea is introduced here in an informal way; there has been no preliminary course about limits.

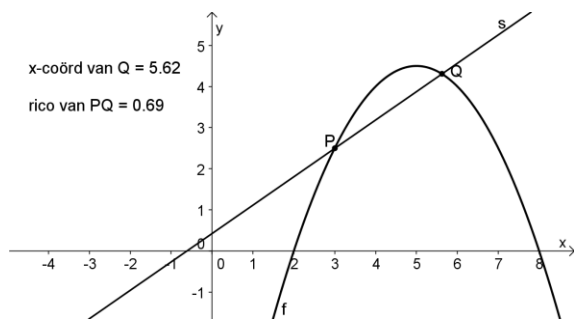


Figure 2a

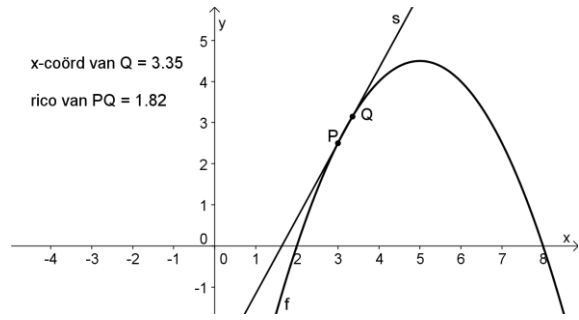


Figure 2b

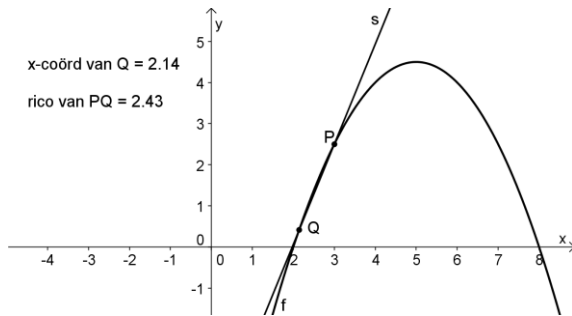


Figure 2c

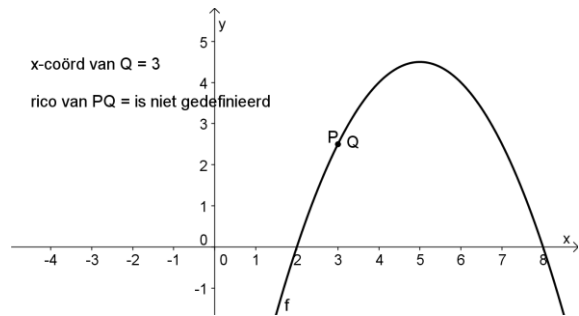


Figure 2d

The slope of the tangent line to the graph of f in the point $P(a, f(a))$ is called the *derivative* of f at a and is written down $f'(a)$. In the visual phase, this is the *definition* of the derivative and not its graphical interpretation.

In order to calculate the derivative by means of the formula of the function f , we calculate the slope $\frac{f(x) - f(a)}{x - a}$ of the chord PQ and then we determine the *limit* value when x approaches

a more and more (without coinciding with a). This limit is written $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$. This is a way of calculating the derivative, not its definition!

In the example of the derivative of $f(x) = x^2$ at 3, we have $\frac{x^2 - 9}{x - 3} = x + 3$ (for $x \neq 3$). When x approaches 3, $x + 3$ approaches $3 + 3 = 6$, of course. We write:

$$f'(3) = \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} (x + 3) = 6.$$

The non-differentiability cases at a point where there are two semi-tangents (figure 3a) or a vertical tangent (figure 3b), are also dealt with in this visual phase, as well as derivative functions, rules to calculate derivatives, asymptotes, limit calculations, l'Hospital's rule, the study of functions and several applications.

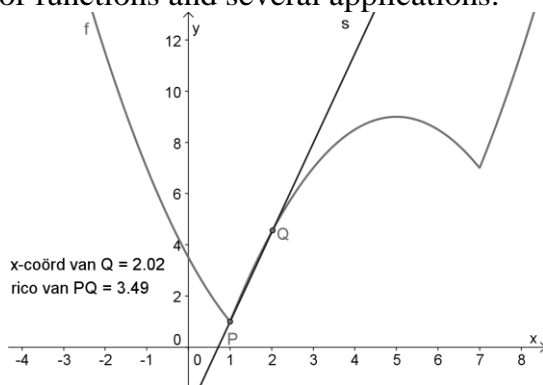


Figure 3a

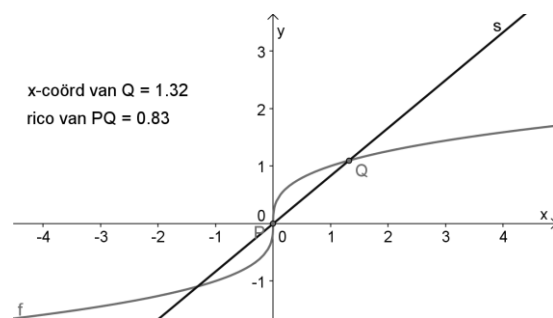


Figure 3b

2.2 Integral

Integrals are also introduced by way of contexts: the distance moved from the speed, the water volume in a bath from the flow rate... These contexts show that the oriented area between the graph and the horizontal axis can have different meanings depending upon the quantities represented on the axes. The integral of a function f from a to b is defined as the oriented area between the graph and the horizontal axis from $x = a$ to $x = b$. In the visual phase, this is the *definition* of the integral and not its geometrical interpretation. Riemann sums (lower sums, upper sums and others) are a method to calculate the integral, not its definition.

Let us see how we can prove the fundamental theorem in this visual phase. We define the *integral function* $F_a(x) = \int_a^x f(t)dt$. We want to prove that $F_a'(x) = f(x)$. We have

$$F_a'(x) = \lim_{\Delta x \rightarrow 0} \frac{F_a(x + \Delta x) - F_a(x)}{\Delta x}.$$

The numerator is the area from a to $x + \Delta x$ minus the area from a to x , hence the area from x to $x + \Delta x$ (figure 4a). In order to divide this area by Δx , we can transform it into the area of a rectangle with basis Δx (figure 4b). The upper side of this rectangle cuts the graph in (at least) a point $(c, f(c))$, and $f(c)$ is the height of this rectangle (figure 4c). Because the number c is “squeezed” between x and $x + \Delta x$, it will approach x more and more when Δx diminishes towards 0. In formulae:

$$F_a'(x) = \lim_{\Delta x \rightarrow 0} \frac{F_a(x + \Delta x) - F_a(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(c)\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} f(c) = f(x).$$

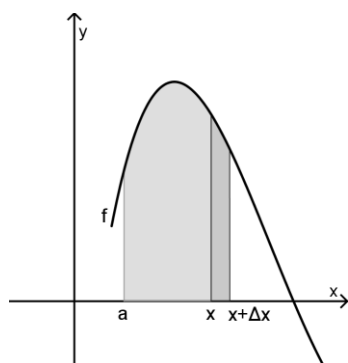


Figure 4a

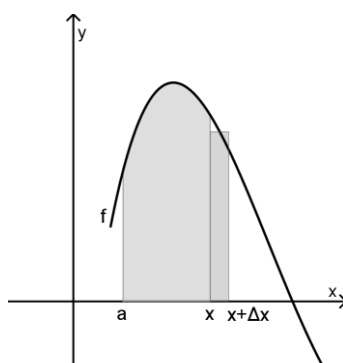


Figure 4b

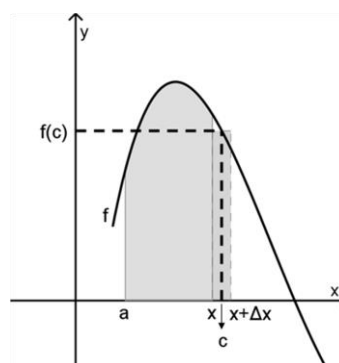


Figure 4c

We could specify that we supposed f to be continuous by saying that the upper side of the rectangle cuts the graph. However, a “visual” idea of continuity (an uninterrupted graph) is sufficient. We could also verify that the proof remains valid when f or Δx are negative. But in spite of this, the proof is satisfying in this visual phase.

The continuation is well known: primitive functions, integral calculation by means of primitives, integral calculation techniques, geometrical applications, applications in physics and economy. All this is situated in the visual phase.

3 Problems with the visual definitions

In a second phase, we confront the pupils with some *strange* functions. These functions show that the *visual* definitions of the concepts do not suffice any more. Moreover, these functions will cast doubt on certain properties that were “obvious” in the visual phase. For example, in the visual phase the pupils were convinced that a function is always piecewise continuous (continuous except at the points where the graph makes a “jump”), that the only non-

differentiability cases were those of the figures 3a and 3b (different left and right derivatives; vertical tangent) and that the integral of a bounded function always exists.

We show the following graphs to the pupils, preferably with computer software as GeoGebra so that they can zoom in on interesting parts (figure 5a-c).

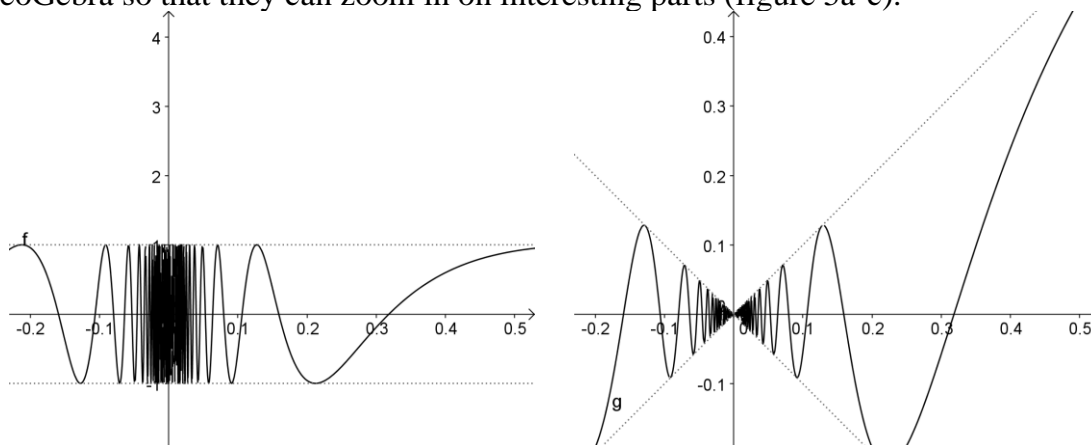
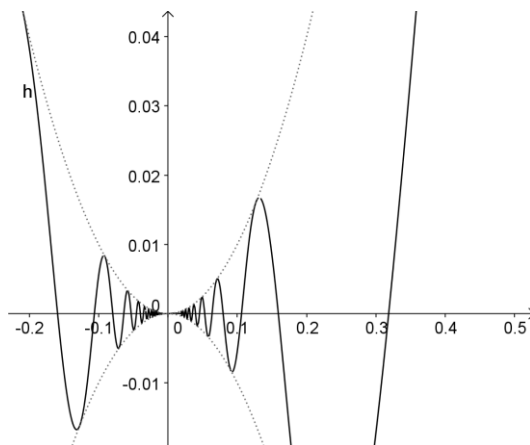


Figure 5a:

$$f(x) = \begin{cases} \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$$\text{Figure 5b: } g(x) = \begin{cases} x \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$



$$\text{Figure 5c: } h(x) = \begin{cases} x^2 \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

Are these functions continuous? In a way, none of these graphs are interrupted, but they are very different from graphs drawn “in one stroke”. Does the function f have a limit in 0? When x goes to 0, $f(x)$ continues to oscillate between -1 and 1 ; should we say therefore that there is an infinity of limits at 0, i.e. all the numbers of the interval $[-1, 1]$? Do these functions have a derivative at 0? It is very difficult to imagine skis following the slope of the “mountain” towards $x = 0$. What could be the value of the integral, e.g. from -1 to 1 ? The oriented area is composed of an infinite number of pieces above and below the horizontal axis; the integral would thus be an infinite series... The pupils realize that the visual definitions do not suffice in order to decide on these questions.

It can be even worse. The Dirichlet function (figure 6) seems to have as a graph a set of two uninterrupted lines, but we could also say that the function jumps “all the time”. Is it a function? Newton would have said *no*; the functions in the 17th century were determined by

algebraic formulae or power series. But in recent mathematics, it suffices that each number x in the domain has a unique image in order to speak about a function¹.

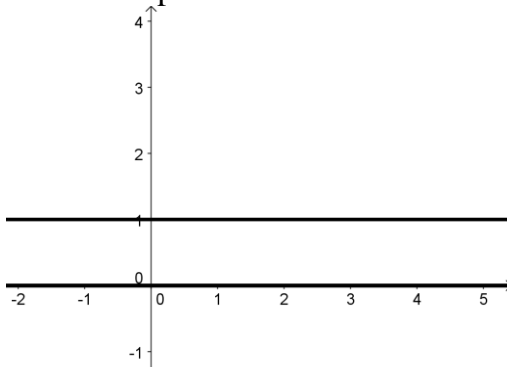


Figure 6: $d(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

The last function, our favourite, is the Thomae function (figure 7). Does it have limits? Is it continuous? Integrable? This function will bring us some surprises. Surely, we are very far away from the “visual” concepts of paragraph 2.

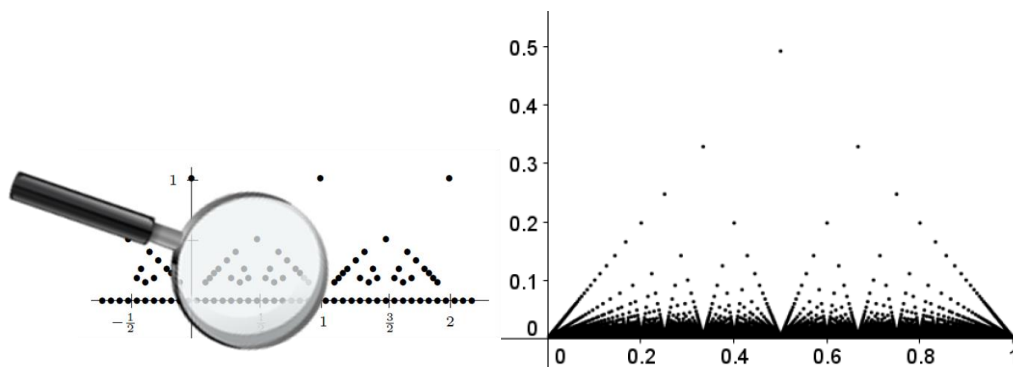


Figure 7: $t(x) = \begin{cases} \frac{1}{q} & \text{if } x \text{ is rational and equals } \frac{p}{q} \text{ after simplification} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

4 Refined definitions

4.1 Limit and continuity

The refined definitions of derivatives and integrals will be based upon the limit concept. Hence, we should first of all find a definition of limit that is essentially not based upon visual ideas. Of course, in order to find this definition we will accept inspiration from visual ideas.

Let us restrict ourselves to the case of a limit in a number equal to a number. In an analogous way, we can treat the cases of a limit at infinity and/or equal to infinity, as well as left or right limits and limits of sequences.

Compare the graphs of the functions f_1 and f_2 (figure 8). The first one is the graph of the function f of paragraph 3 (figure 5a) translated by the vector $(3, 0)$. The graph of f_2 is the graph

¹ The formulae fans will be pleased to know that there is indeed a formula for the Dirichlet function:

$$d(x) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \cos^{2n}(m! \pi x).$$

of the same function but “squeezed” between the lines $y = \pm 2(x - 3)$. (The numbers 3 and 2 have been inserted merely in order to create a more general example.)

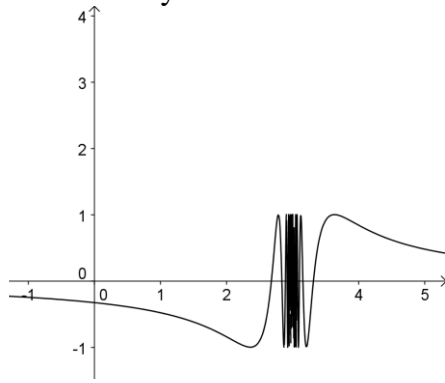


Figure 8a: $f_1(x) = \sin \frac{1}{x-3}$

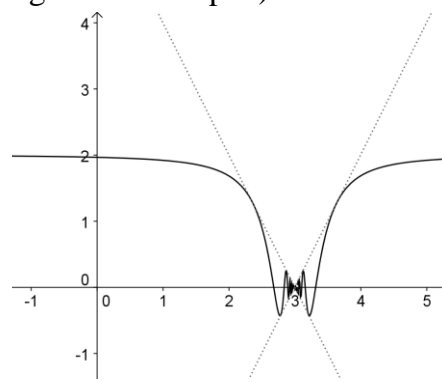


Figure 8b: $f_2(x) = 2(x-3) \sin \frac{1}{x-3}$

These functions have an interesting behaviour around $x = 3$. Let us have a closer look (figure 9).

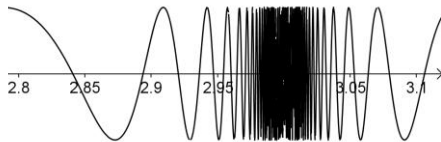


Figure 9a: $f_1(x) = \sin \frac{1}{x-3}$

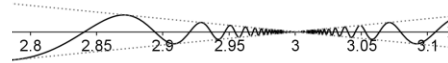


Figure 9b: $f_2(x) = 2(x-3) \sin \frac{1}{x-3}$

In each interval around $x = 3$, f_1 continues to oscillate between -1 and 1 . It does not “go to a limit”. We say that the limit of f_1 at 3 does not exist. However, f_2 becomes “very small” when x approaches 3 . The function “goes to 0 ”, even if it continues to oscillate and does not get nearer to 0 in a monotonous way. The difference between $f_2(x)$ and 0 can be made “as small as we want”. For this purpose, it is sufficient to take x “close enough” to 3 . If we decide, for instance, that the difference between $f_2(x)$ and 0 should be smaller than 0.1 , it suffices to take x at a distance 0.05 to 3 . Indeed (figure 10), if $|x - 3| < 0.05$, then

$$|f_2(x) - 0| = |2(x-3) \sin \frac{1}{x-3}| = 2|x-3| \cdot |\sin \frac{1}{x-3}| < 2 \cdot 0.05 = 0.1.$$

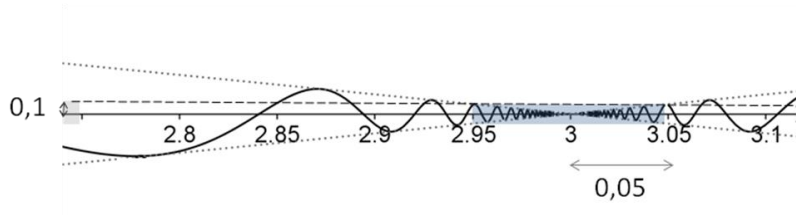


Figure 10

After another example where the limit is not 0 , we can generalize this idea in a definition:

$\lim_{x \rightarrow a} f(x) = b$ means:

$$\forall \varepsilon > 0: \quad \exists \delta > 0: \quad 0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - b| < \varepsilon$$

For ε as small
as you want,

if you take x close enough
to a (but not equal to a),

$f(x)$ will be in the interval
 $]b - \varepsilon, b + \varepsilon[$.

Hence, we adapt δ to ε . As our function f_2 is squeezed between two lines with slopes ± 2 , it suffices to take δ equal to the half of ε .

For the function f_1 , the definition is fulfilled for no value of b . As soon as we take ε equal to or smaller than 1, there will always be some $f(x)$ outside the interval $]b - \varepsilon, b + \varepsilon[$, whatever close x is taken to 3. Hence there is no limit at 3.

Let us specify that one does never take limits at points that are not accessible within the domain of the function. The point has to be an *accumulation point* of the domain, i.e. each interval $]a - \delta, a + \delta[$ (for any δ) should contain points of the domain unequal to a itself.

Essentially, the definition of limit is based merely upon numbers, inequalities and logic. The visual and dynamic idea (x getting closer and closer to a) has been eliminated indeed.

A limit is not very interesting at a point where the function is continuous. There, the limit is, of course, equal to the function value. This gives us the possibility to *define* continuity now that we dispose of a refined definition of limit: f is *continuous* at a if $\lim_{x \rightarrow a} f(x) = f(a)$.

Let us verify that the new definitions of limit and continuity enable us to decide on the continuity of the functions of paragraph 3.

The functions f and g (figure 5) are analogous to the functions f_1 and f_2 used here to introduce the refined definition of limit. So the function f (figure 5a) has no limit at 0 and is therefore not continuous at 0. The limit of the function g (figure 5b) is 0 and coincides with $g(0)$, so g is continuous at 0. The function h is also continuous at 0.

For the function d of Dirichlet (figure 6), no limits exist. Indeed, whatever a and b , for $\varepsilon = 0.5$ (or smaller) the images of numbers in an interval $]a - \delta, a + \delta[$ are both 0 and 1, so they cannot be all in $]b - \varepsilon, b + \varepsilon[$. Hence the function is nowhere continuous.

The Thomae function (figure 7) is even more startling. Take a number a between 0 and 1, rational or not. We will prove that the limit at a is 0. Take any $\varepsilon > 0$. Except a finite number of points, all points of the graph between $x = 0$ and $x = 1$ lie under the line $y = \varepsilon$ (figure 11). The set of the points u in $[0, 1]$ such that $t(u) \geq \varepsilon$ is finite. Take δ equal to the distance between a and the element of this set closest to a . If $0 < |x - a| < \delta$, then we have $t(x) < \varepsilon$ and thus $|t(x) - 0| < \varepsilon$.

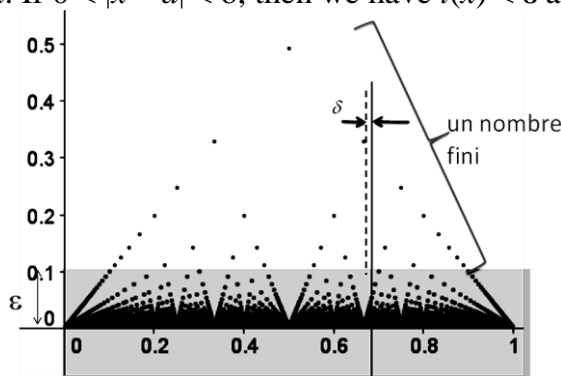


Figure 11

As the function is periodical, this result is also valid for a elsewhere than between 0 and 1.

At the rational numbers, the limit is 0 and not equal to the image. At the irrational numbers, the limit is 0 and equal to the image. So the function is continuous at each irrational number and discontinuous at each rational number! The visual idea of continuity is far away!

4.2 Derivative

In the visual phase, the derivative was defined as the slope of the tangent; the limit of the differential quotient was not the definition but merely a method to calculate the derivative from the formula of the function. Now that we have at our disposal a precise definition of limit, we can inverse things. We define the derivative as the limit of the differential quotient: if

$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ exists and equals a number, we call f *differentiable* at a and we call that number the *derivative* of f at a , notated $f'(a)$. The *tangent* at the point $(a, f(a))$ is, by definition, the line $y - f(a) = f'(a)(x - a)$.

With this definition, we can verify, for instance, that the oscillating function g (figure 5b) is not differentiable at 0 and that the function h (figure 5c) is indeed differentiable at 0.

$$\begin{aligned} g'(0) &= \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x \sin \frac{1}{x}}{x} \\ &= \lim_{x \rightarrow 0} \sin \frac{1}{x} \end{aligned} \qquad \begin{aligned} h'(0) &= \lim_{x \rightarrow 0} \frac{h(x) - h(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{x^2 \sin \frac{1}{x}}{x} \\ &= \lim_{x \rightarrow 0} x \sin \frac{1}{x} \end{aligned}$$

As we saw in paragraph 4.1, the first limit does not exist and the second one is equal to 0. So $g'(0)$ does not exist, whereas $h'(0) = 0$.

The sign of the derivative of h changes an infinite number of times in each interval around 0. The sign of h' around 0 does not correspond to one of the “known” cases of the visual phase ($\boxed{+0-}$, $\boxed{-0+}$, $\boxed{+0+}$, $\boxed{-0-}$).

The pupils can construct their own “beautiful” oscillating functions with another derivative as 0. The graph of figure 12, for instance.

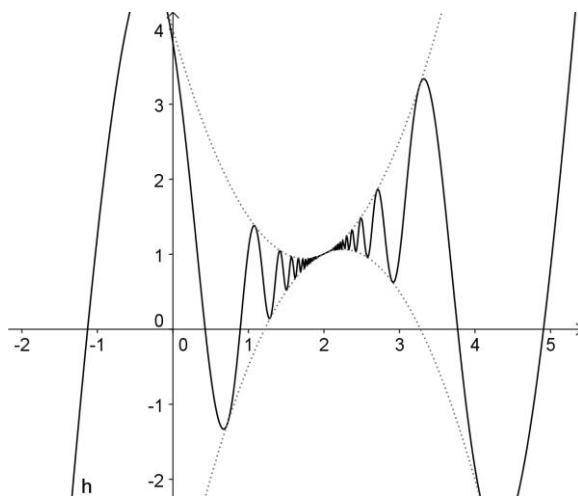


Figure 12: $y = (x - 2)^2 \sin \frac{10}{x - 2} + \frac{x}{2}$

The numerator 10 instead of 1 makes the oscillations more visible; the substitution of x by $x - 2$ translates the interesting point to the right and the addition of the term $\frac{x}{2}$ makes the derivative at 2 to be $\frac{1}{2}$ instead of 0. Note that the derivative at 2 is positive but that there is no interval around 2 wherein the function is increasing.

4.3 Integral

As for the derivative, the calculation method of the visual phase, based upon limits of Riemann sums, becomes the definition.

Given a bounded function f and $[a, b]$ an interval in the domain of f . We divide $[a, b]$ into n equal parts $[x_{k-1}, x_k]$ ($k = 1, \dots, n$) of width $\Delta x = \frac{b-a}{n}$. We consider the *lower sum* $s_n =$

$\sum_{k=1}^n m_k \Delta x$ and the *upper sum* $S_n = \sum_{k=1}^n M_k \Delta x$, wherein $m_k = \inf f[x_{k-1}, x_k]$ and $M_k = \sup f[x_{k-1}, x_k]$. We say that f is *integrable* if $\lim s_n$ and $\lim S_n$ are equal to the same number. This number is *the integral* $\int_a^b f(x) dx$.

This definition is equivalent to the definition of Riemann (19th century), who divided the interval into parts that were not necessarily equal.

The definition is once again based upon the concept of a limit (of sequences). In this “refined” world, we define the area by means of an integral and not the other way round.

Let us examine whether the function d of Dirichlet (figure 6) is integrable, for instance in the interval $[0, 1]$. It is clear that for all n , the lower sum s_n is 0 and the upper sum S_n is 1. So $\lim s_n = 0 \neq 1 = \lim S_n$. The function d is thus not (Riemann-)integrable.

What can be said about the function t of Thomae (figure 7), for instance in the interval $[0, 1]$? Again, all the lower sums s_n are 0 and thus $\lim s_n = 0$. We will prove that $\lim S_n = 0$, so that the integral exists and equals 0.

In order to prove that $\lim S_n = 0$, we have to establish that

$$\forall \varepsilon > 0: \exists p: n > p \Rightarrow S_n < \varepsilon.$$

Suppose $\varepsilon > 0$ is given, as small as you want. Draw the line $y = \frac{\varepsilon}{2}$. For each upper sum S_n , the sum of the areas of the rectangles that remain below this line is certainly less than $\frac{\varepsilon}{2}$, the area of the rectangle of length 1 and height $\frac{\varepsilon}{2}$ (figure 13). (More exactly, the sum of the terms $M_k \Delta x$ for which $M_k < \frac{\varepsilon}{2}$ is less than $\frac{\varepsilon}{2}$.)

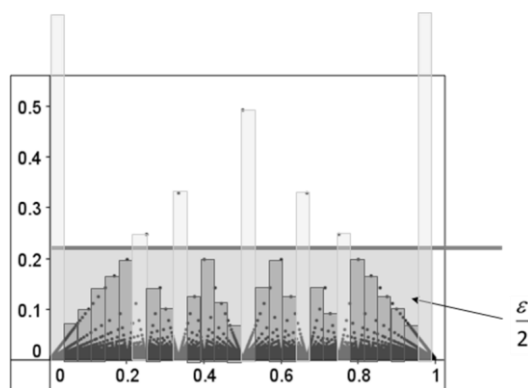


Figure 13

We can take n such that the sum of the areas of the other rectangles, those exceeding the line $y = \frac{\varepsilon}{2}$, is also less than $\frac{\varepsilon}{2}$. Indeed, there is only a finite number of such rectangles, say m (figure 14). The sum of the areas of the rectangles that exceed the line $y = \frac{\varepsilon}{2}$ is less than $m \cdot \frac{1}{n}$. Therefore, it suffices to divide the interval $[0, 1]$ into a number n of parts such that $m \cdot \frac{1}{n} < \frac{\varepsilon}{2}$, in other words: to take $n > \frac{2m}{\varepsilon}$. For $p = \frac{2m}{\varepsilon}$, we have $n > p \Rightarrow S_n < \varepsilon$, what we had to prove.

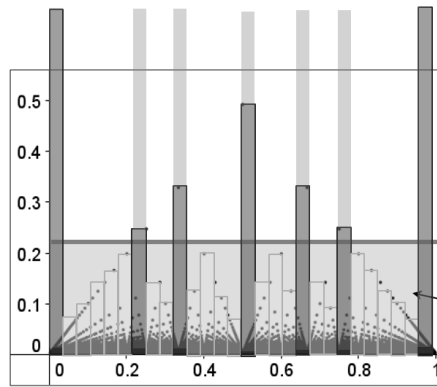


Figure 14

5 Proofs with the refined definitions

What remains valid of the visual phase? In principle, everything should be done again! Of course, this is not what we propose for the secondary school. Only to some mathematically gifted pupils, we could show one or two examples of proofs in this “new world” governed by the refined definitions. This would prepare these pupils for a university analysis course, which starts immediately with the fine definitions. In this text, we restrict ourselves to one example of a demonstration (paragraph 5.1) and a few descriptions of results (paragraphs 5.2-4). For more details, we refer the reader to our article in *Uitwiskeling* (Eggermont & Roelens, 2009).

5.1 The limit of a sum

The theorem “the limit of a sum is the sum of the limits” is an example of a property that did not need a proof in the visual phase: if $f(x)$ “goes to” b and $g(x)$ “goes to” c , then of course $f(x) + g(x)$ goes to $b + c$. However, in the world of the fine definitions, a limit replaces a complicated expression with quantifiers, an implication and inequalities. As we cannot trust visual intuition any more, we need to give a proof by ε - δ .

It is given that $\lim_{x \rightarrow a} f(x) = b$ et $\lim_{x \rightarrow a} g(x) = c$. We have to prove that $\lim_{x \rightarrow a} (f(x) + g(x)) = b + c$. This means:

$$\forall \varepsilon > 0 : \exists \delta > 0 : 0 < |x - a| < \delta \Rightarrow |f(x) + g(x) - b - c| < \varepsilon.$$

Take an arbitrary $\varepsilon > 0$. We have to find a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) + g(x) - b - c| < \varepsilon.$$

The given limits can be written like this:

$$\forall \varepsilon_1 > 0 : \exists \delta_1 > 0 : 0 < |x - a| < \delta_1 \Rightarrow |f(x) - b| < \varepsilon_1.$$

$$\forall \varepsilon_2 > 0 : \exists \delta_2 > 0 : 0 < |x - a| < \delta_2 \Rightarrow |g(x) - c| < \varepsilon_2.$$

The triangle inequality provides a relationship between $|f(x) + g(x) - b - c|$, $|f(x) - b|$ and $|g(x) - c|$:

$$|f(x) + g(x) - b - c| \leq |f(x) - b| + |g(x) - c|.$$

If we can prove that $|f(x) - b| + |g(x) - c| < \varepsilon$, we will have attained our goal. Let us apply the given expressions with $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$:

$$\exists \delta_1 > 0 : 0 < |x - a| < \delta_1 \Rightarrow |f(x) - b| < \frac{\varepsilon}{2}.$$

$$\exists \delta_2 > 0 : 0 < |x - a| < \delta_2 \Rightarrow |g(x) - c| < \frac{\varepsilon}{2}.$$

In order to have at the same time $|f(x) - b| < \frac{\varepsilon}{2}$ and $|g(x) - c| < \frac{\varepsilon}{2}$, we should take x close enough to a so that, at the same time, $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$. Therefore, take $\delta = \min \{ \delta_1, \delta_2 \}$. Then we have:

$$0 < |x - a| < \delta \Rightarrow |f(x) + g(x) - b - c| \leq |f(x) - b| + |g(x) - c| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

what we had to prove.

5.2 Continuity

In the visual phase, continuity was a global property of the graph in an interval. However, the fine definition is about continuity at individual points. In the examples of strange functions, we met a function that is continuous in certain points without being continuous in an interval (the Thomae function in the irrational points). But if a function is continuous in an interval (that is: at each point of the interval), does it indeed have an uninterrupted graph in that interval? We should specify the idea of “uninterrupted”. In fact, the following theorems express the intuitive idea of an uninterrupted curve.

Intermediate values theorem: a continuous function f in an interval $[a, b]$ attains all values between $f(a)$ and $f(b)$.

Theorem of Weierstrass: a continuous function f in an interval $[a, b]$ attains an absolute minimum and an absolute maximum in this interval, in other words:

$$\exists c, d \in [a, b] : \forall x \in [a, b] : f(c) \leq f(x) \leq f(d).$$

The proofs of these theorems are not easy. The first one is proved in Eggermont & Roelens, 2009.

5.3 Derivative

The strange functions have broken some simple ideas about derivatives. Some properties that were considered obvious in the visual phase are true, other are false. We specify that a function is called *increasing* in an interval if for all x_1, x_2 in that interval $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$, in other words if a point of the graph that is more “to the right” than another point, is always “higher” than that other point.

- If the derivative at a of a function is positive, f is increasing in an interval around a . This is *false*. A counterexample has been given in figure 12 (paragraph 4.2).
- If the derivative at a of a function f is positive, there is an interval around a where $f(x) < f(a)$ for $x < a$ in this interval and $f(x) > f(a)$ for $x > a$ in this interval. This is *true*. The difference with “increasing” is that here we do not compare two arbitrary points but each time one arbitrary point with the fixed point $(a, f(a))$.
- If a function f has a positive derivative everywhere in an interval $[a, b]$, it is increasing in $[a, b]$. This is *true*, but not easy to prove. The difficulty is that the derivative is defined in a point whereas the definition of “increasing” involves the comparison of two points. The proof goes through the theorems of Rolle and Lagrange. We think this is rather university material.
- If the sign of the derivative around a is $\boxed{+ 0 -}$, then the function has a maximum at a . This is *true*.

- If a function has a maximum at a , the sign of the derivative around a is $\boxed{+ 0 -}$. This is *false*. A counterexample can be constructed by squeezing an oscillating function between two parabolas, as in figure 15.

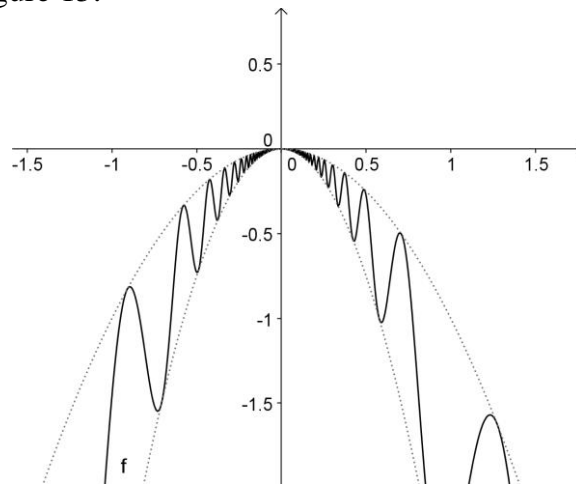


Figure 15: $y = x^2 \sin \frac{10}{x} - 2x^2$

5.4 Integral

We could discuss the proof of the fundamental theorem (paragraph 2) again. When the integral from x to $x + \Delta x$ has been replaced by $f(c) \Delta x$ (the area of a rectangle), we made use of the *mean value theorem*, which did not need to be formulated nor proved in the visual phase. Let us recall this theorem: “the integral of a continuous function f on an interval $[a, b]$ is equal to $(b - a) f(c)$, for a c in $[a, b]$ ”.

If we decide to prove this theorem in the refined phase, we have to use Weierstrass’ theorem (see 5.1), which gives us two numbers m and M so that $m \leq f(x) \leq M$ for all x in $[a, b]$. This implies, by the use of Riemann sums,

$$m(b - a) \leq \int_a^b f(x) dx \leq M(b - a),$$

or, after dividing by $b - a$:

$$m \leq \frac{1}{b - a} \int_a^b f(x) dx \leq M.$$

By the intermediate values theorem, this implies that there is a c in $[a, b]$ such that

$$\frac{1}{b - a} \int_a^b f(x) dx = f(c),$$

which proves the mean value theorem.

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