3d-Objects and Mathematical Equations

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Intro

MAKE YOUR OWN CURVED SURFACE!!!

Families of Graphs of Cubic Polynomials

- SHOW THIS in a dynamic geometry software. E.g., $f_t(x) = x^3 + tx$

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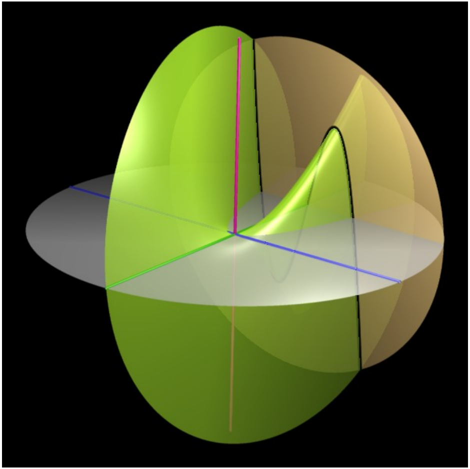
- All these curves in a single picture: use surfex.

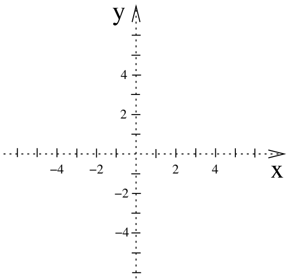
Families of Graphs of Cubic Polynomials

- SHOW THIS in a dynamic geometry software. E.g., $f_t(x) = x^3 + tx$
- All these curves in a single picture: use surfex.
- **Parametrization:** $z = f(x, y)$, e.g.: $f(x, y) = x^3 + yx$.



The Geometry of Factorization in 2d; age 13/14

Problem: In the following coordinate system, draw all points, for which $x^2 - y^2 = 0$ holds:



$a \cdot b = 0$ if and only if $a = 0$ or $b = 0$: becomes relevant!

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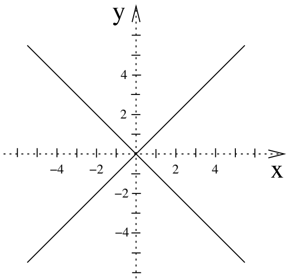

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The Geometry of Factorization in 2d

Problem: Factorize $f(x, y) = 3x^2 + 2xy - 2x^2 - 3xy$ (from a school-book for 13-year-olds). *Addition:* In a coordinate system, draw all points for which $f(x, y) = 0$.

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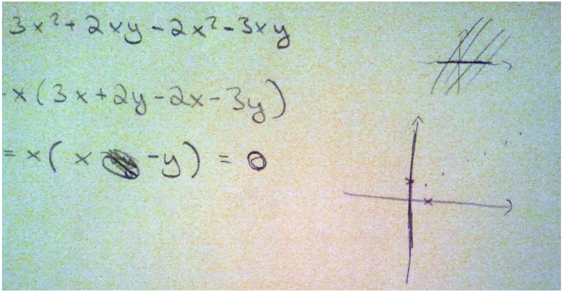
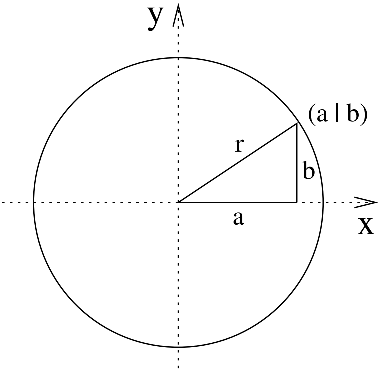


MindCurver

Problem: Solve all problems of the MindCurver game:
http://oliverlabs.net/MindCurver/

Circle Equation



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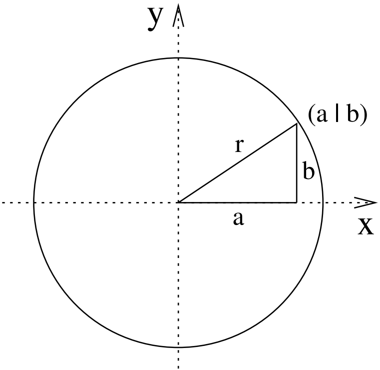
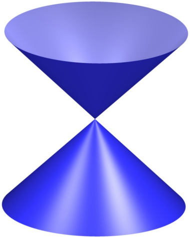


$$x^2 + y^2 = r^2$$

Double Cone Equation



1. What is the equation satisfied by all points (x, y, z) of a double cone with opening angle 45°?
2. Can you adapt the equation to a hyperboloid of one sheet?
3. …of two sheets?

Parabolas in 3d

A parabolic antenna

Picture: Wikipedia

- How does it work?
- Where is the receiver placed?
- An equation for the parabolic antenna:

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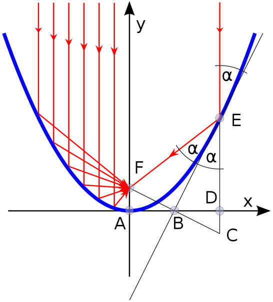

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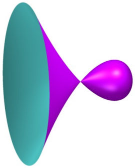
Higher Surfaces of revolution



$$x^2+y^2+z^3-z^2=0$$

- The shape of so-called **algebraic surfaces** is completely understandable, even for school, kids, e.g. for surfaces of revolution.
- E.g., through a reduction to the curve-case, by cutting with a suitable plane
- and factorization: $x^2 + y^2 = z^2 \cdot (1 - z)$.
- $z = 0$ and $z = 1$ yield the radius 0.
- $z < 0$ and $0 < z < 1$: radius > 0,
- otherwise ($z > 1$): no solution at all.

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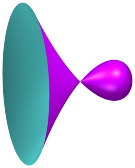

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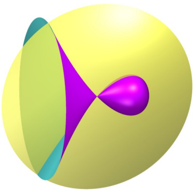

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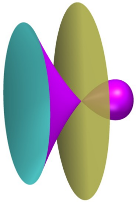

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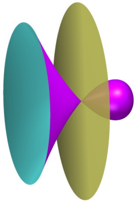

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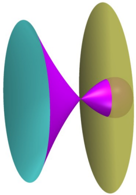

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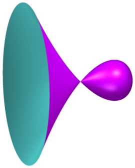

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Surfaces of Revolution

How do the following surfaces look like approximately?

1. $x^2 + y^2 = z^4$
2. $x^2 + y^2 = z^3$
3. $x^2 + y^2 = z^3 + 0.5$
4. $x^2 + y^2 = z^3 - 0.5$
5. $x^2 + y^2 = z^5$

Can you write down equations for other surfaces of revolution?

1. a lemon,
2. a sand clock,
3. ...

Surfaces of Revolution

When at home, continue thinking about this, e.g. by producing your own surfaces of revolution: **JSurfer** or Surfer.

Applications of Algebraic Surfaces: Parabolas in 3d

A curved roof construction

Picture: Wikipedia

- How does the construction work?
- What is the relation to parabolas?
- Equations for the roof construction:

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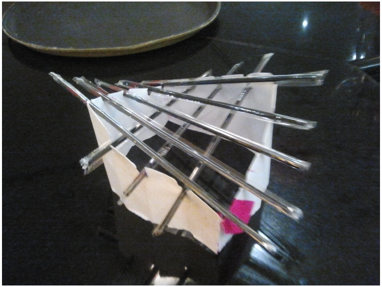

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- Equations for the roof construction: $z = x^2 - y^2$!!!

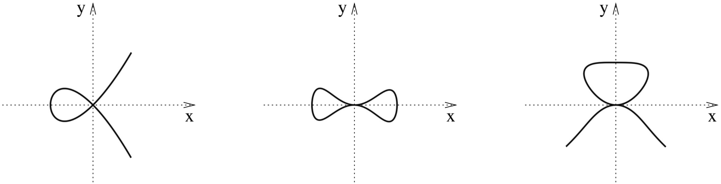
Lines on the Hyperbolic Paraboloid

Problem:

1. Find two planes cutting $z = x^2 - y^2$ in a line.
2. Take planes parallel to those: What is there intersection with the hyperbolic paraboloid?
3. Take other plane cuts of the object, e.g. horizontal ones ($z = a$) or vertical ones such as $x = a$ or $y = a$.
4. Describe the geometry of the surface as good as you can.

$x^2 = (-x)^2$. Symmetry by a line

Problem: Which of the following curves belongs to which equation?



(1) (2) (3)

$$
\begin{aligned}
f: \quad 0 &= x^4 - y^2 - x^6 \\
g: \quad 0 &= x^4 - y^2 - y^5 \\
h: \quad 0 &= x^2 - y^2 + x^3
\end{aligned}
$$

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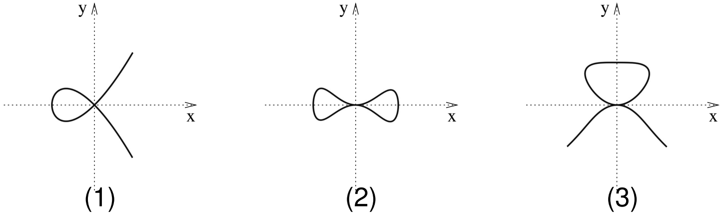

$$\begin{aligned} f(x,y) &= x^4 - y^2 - x^6 \\ f(-x,y) &= (-x)^4 - y^2 - (-x)^6 = x^4 - y^2 - x^6 = f(x,y) \\ f(x,-y) &= x^4 - (-y)^2 - x^6 = x^4 - y^2 - x^6 = f(x,y) \end{aligned}$$

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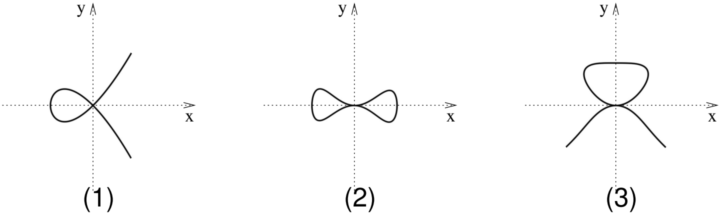

$f(-x, y) = f(x, y) = f(x, -y)$.
$g(-x, y) = g(x, y)$.
$h(x, -y) = h(x, y)$.

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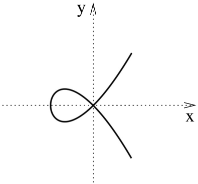

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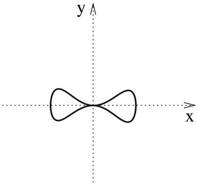
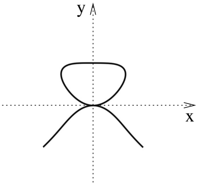


(2)



(3)

$f(-x, y) = f(x, y) = f(x, -y)$. lines of symmetry: y, x-Axis
$g(-x, y) = g(x, y)$. line of symmetry: y-axis
$h(x, -y) = h(x, y)$. line of symmetry: x-axis

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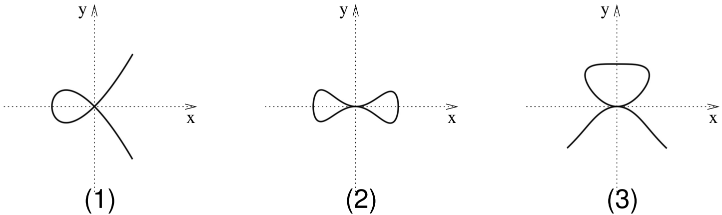
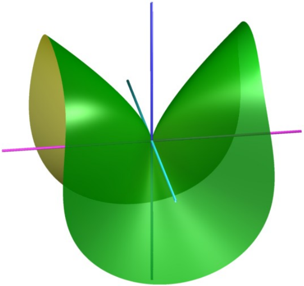


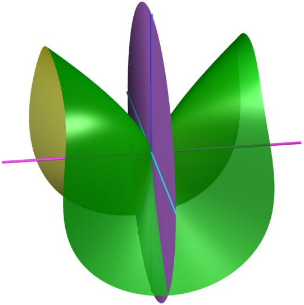
f: $0 = x^4 - y^2 - x^6$. Symm.: x-,y-axis, so: (2)
g: $0 = x^4 - y^2 - y^5$. Symm.: y-axis, so: (3)
h: $0 = x^2 - y^2 + x^3$. Symm.: x-axis, so: (1)

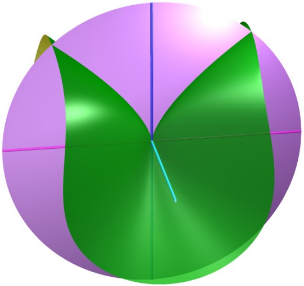
$x^2 = (-x)^2$. Symmetry-Planes



Problem: Which of the axes in the picture of $x^2 - y^2 - z^3 = 0$ is the z-axis?

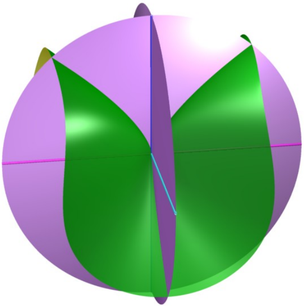
$x^2 = (-x)^2$. Symmetry-Planes

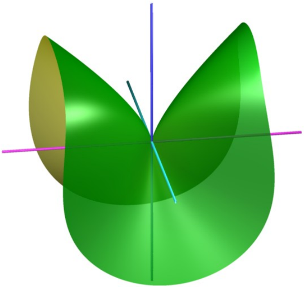




$x^2 - y^2 - z^3 = 0$ has symmetries: $x \mapsto -x$, $y \mapsto -y$, because: $(-x)^2 - y^2 - z^3 = x^2 - y^2 - z^3$ etc.

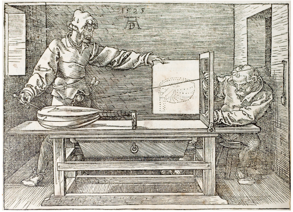
$x^2 = (-x)^2$. Symmetry-Planes





So: The coordinate plane without symmetries is the $z = 0$-plane. Thus, the vertical axis has to be the z-axis.

Albrecht Dürer (1471-1528) to Ray Tracing



- Object, given by a set of equations $f(x, y, z) = 0$, e.g. a polynomial of degree d.
- An eye looks through a point on the canvas (or screen): $g(t) = (x(t), y(t), z(t))$ (parametrization of a line)
- Substituting: $f(g(t))$ yields a polynomial in 1 variable t, degree d.
- Zeros of $f(g(t))$ yield points on the object. Still to compute:

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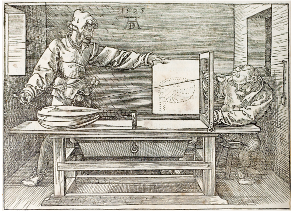

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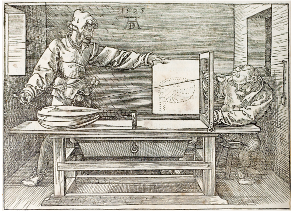

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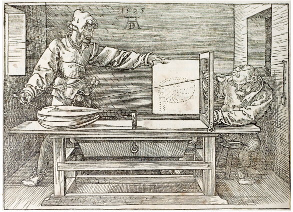

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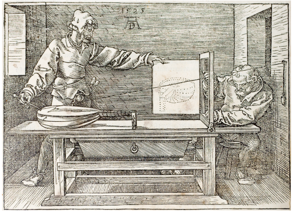

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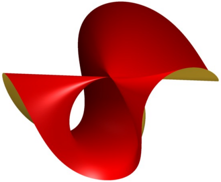
Substituting in Surfaces

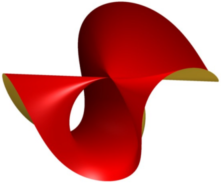


- Show: All points of the form $T = (-1, t, 0)$, $t \in \mathbb{R}$, satisfy the equation:

$$x^3 + y^2 z + xz^2 + x^2 = 0.$$

- A line is the intersection of two planes. What are two other lines on the surface?

Substituting in Surfaces



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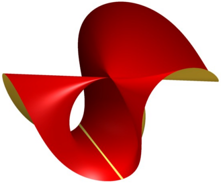

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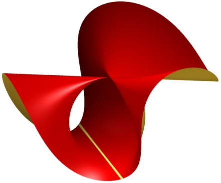
Substituting in Surfaces



- Show: All points of the form $T = (-1, t, 0)$, $t \in \mathbb{R}$, satisfy the equation:

$$x^3 + y^2z + xz^2 + x^2 = 0.$$

- A line is the intersection of two planes. What are two other lines on the surface?

CAD

An important application of the previous techniques:

- Objects: Surfaces allowing implicit and parametrized descriptions.
- Inside-/Outside-Test is very difficult to compute without an implicit equation,
- Intersections of surfaces are easy to compute if both descriptions are known,
- For many operations surfaces of higher degree are even necessary, because it would be too slow if one would only use approximations by triangles, to make a curved shape appear smooth!).

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Some examples

Explain the geometry and history of some higher degree surfaces, e.g.:

- surfaces with a singularity at the origin and simple equations,
- cubic surfaces and their lines,
- world record surfaces.

3d Printing Models

- geometries are easy to produce which would be almost impossible by using other production methods
- very precise (depending on the 3d-printing method)
- stability issues still exist. . . workaround: laser-in-glass objects

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Producing 3d-data for a 3d-printer

- bound the volume by a set of triangles
- neighbouring triangles meet exactly in one of their edges
- no overlapping or intersecting triangles

Problem: Write down a set of triangles for a simple 3d-object. Use netfabb to check its correctness.

STL File Format

ASCII STL [edit]

An ASCII STL file begins with the line

```
solid name
```

where *name* is an optional string (though if *name* is omitted there must still be a space after solid). The file continues with any number of triangles, each represented as follows:

```
facet normal ni nj nk
    outer loop
        vertex v1x v1y v1z
        vertex v2x v2y v2z
        vertex v3x v3y v3z
    endloop
endfacet
```

where each *n* or *v* is a floating-point number in sign-mantissa-"e"-sign-exponent format, e.g., "2.648000e-002" (noting that each *v* must be non-negative). The file concludes with

```
endsolid name
```

The structure of the format suggests that other possibilities exist (e.g., facets with more than one "loop", or loops with more than three vertices). In practice, however, all facets are simple triangles.

White space (spaces, tabs, newlines) may be used anywhere in the file except within numbers or words. The spaces between "facet" and "normal" and between "outer" and "loop" are required.[5]

Thank you

very much for your attention.

Oliver Labs

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