Kantonsschule Rämibühl Zürich
Mathematisch Naturwissenschaftliches Gymnasium

Maturität 2008
Class 4a Bio/Ch

Mathematics

Media:

- A handwritten summary of two A4 pages (i.e. four A4 sides).
- Calculator TI-89

Note:

- You have four hours for this test. Start with reading through all the exercises.
- Use a **separate** sheet of paper for each one of the six exercises.
- Only when the calculator is explicitly mentioned, you are allowed to omit steps. You should document how you have used the calculator in this case e.g. **solve**($x^2 + x = 0$ **,** x).

  In all other cases points are deducted for missing steps in the calculations.
- You should leave roots, fractions, logarithms, π, e, ... (unless mentioned otherwise, numerical results are not marked).
- Every question gives the same number of points. You do not need to solve every single question – approximately 80% correctly solved will give you the mark 6.
- Do not give more than one answer to a question. Cross out the answers which you do not want to be marked.
- Two points are given for exposition and formalism.

Good luck!!!!

Dr.M.Akveld & Dr.B.Roost

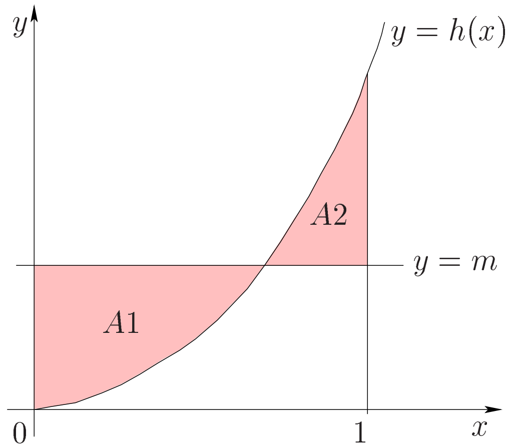
1. Consider the functions f, g and h given by

$$f(x) = x - \sqrt{x}, \qquad g(x) = -x^3 - x^2 + 2x \qquad \text{and} \qquad h(x) = x^3$$

   (a) Calculate the zeroes and the extrema of f and g. Sketch the graphs of f and g in the same coordinate system – the coordinate system should be at least quarter of an A4 page.

   (b) Consider the tangent to the graph of f in the point $P = (1, 0)$. This tangent together with the graph of f and the y-axis bound a finite area. Rotate this area about the x-axis and calculate the volume of the corresponding solid of revolution.

   (c) The horizontal line $y = m$ and the graph of h determine two areas A_1 and A_2 over the intervall $[0, 1]$ – see figure below. For what value of m is the sum of the areas $A_1 + A_2$ an extremum? Is this a maximum or a minimum?



2. (a) Consider the graph of $y = f(x)$ in figure 1. Carefully sketch the graph of the derivative $y = f'(x)$ into figure 1.

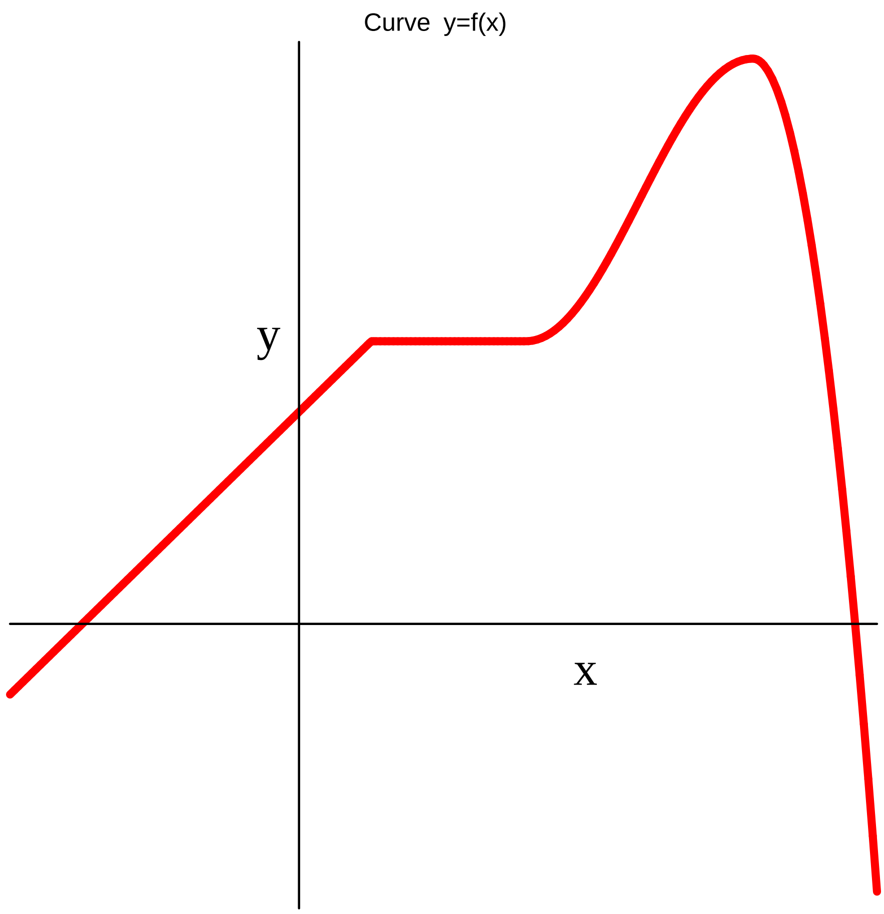


Figure 1: Given $y = f(x)$, sketch $y = f'(x)$

(b) Consider again the graph of $y = f(x)$. Carefully sketch the graph of the anti-derivative $y = F(x)$ into figure 2 such that $F(0) = 0$.

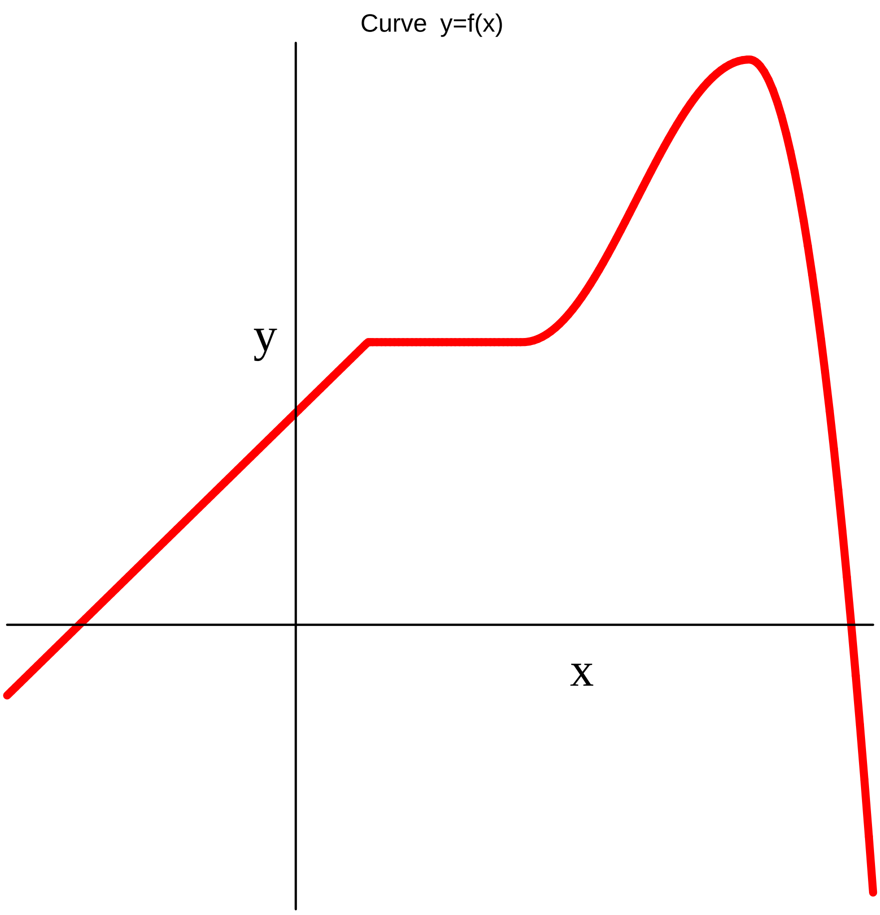


Figure 2: Given $y = f(x)$, sketch $y = F(x)$ with $F(0) = 0$

(c) Calculate the derivative of f from first principles, where f is given by

$$f(x) = \frac{1}{(2-x)^2} \quad \text{with} \quad D_f = \mathbb{R} \setminus \{2\}$$

(d) Determine a function $y = g(x)$ which is non-constant and continuous such that the following equation is true:

$$\int_{-1}^{3} g(x)\, dx = 0$$

3. (a) Given the complex numbers $w = 6 + 2\sqrt{3}i$ and $z = -\sqrt{3} + 3i$. Compute

i. $|z + 2w|$

ii. $\dfrac{\bar{z}}{w}$

iii. $\mathrm{Im}(z \cdot w)$

(b) Solve $\omega^4 = -64$ and write the solutions in Cartesian form.

(c) The following equation describes a curve in the complex plane. Find the Cartesian equation of this curve and sketch the curve.

$$\mathrm{Re}(z) + 1 = |z - 1|$$

(d) Consider the function $f : \mathbb{C} \to \mathbb{C}; z \mapsto f(z) = z^3$.

i. Determine the fixed points of f.

ii. Determine the true period 2 points of f.

iii. Sketch the solutions of i) and ii) in the figure below (in different colours).

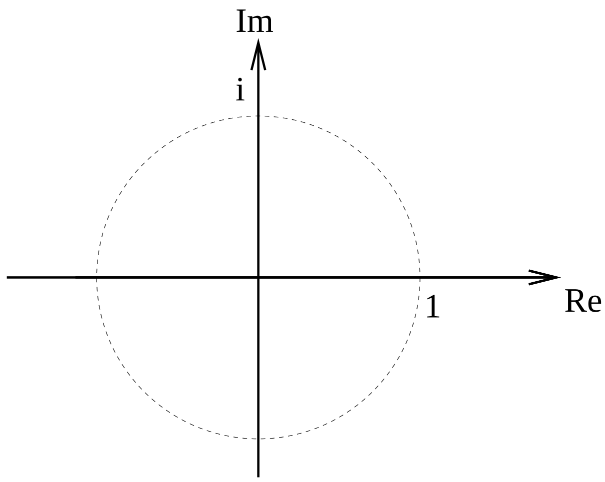


Figure 3: The complex plane

4. **[You can use the calulator in this exercise, but you should document when and how you use it.]**

   A florist sells tulip bulbs of which 15% are white (that is, they produce a white flower), 25% are yellow (that is, produce a yellow flower) and 60% are red (that is, produce a red flower). He sells the bulbs unsorted — this means one cannot tell what colour a bulb is.

   (a) Determine the probability that three randomly chosen tulip bulbs are three different colours.

   (b) Jeremy has bought three bulbs. Two have been in his living room and have already produced yellow flowers. Determine the probability that the third tulip – which has been kept in the bedroom (a bit colder) – also produces a yellow flower.

   (c) Suppose you buy 50 tulip bulbs to plant in your garden. How many different combinations of colours are possible? For example, one combination is 47 whites, 3 reds and no yellows; another combination is 10 white, 10 red and 30 yellow. (Note, you may assume that all bulbs will produce tulips).

   (d) How many bulbs does one have to buy to be more than 95% certain that at least one will be white?

   (e) Let X be the number of red tulip bulbs in a set of 50 bulbs. Calculate the mean, $\mu(X)$, and the standard deviation, $\sigma(X)$, of X.

   (f) Determine the probability that in your set of 50 bulbs at least 25 and at most 35 bulbs will be red.

5. Given the points $A = (2, 1, 0)$, $B = (2, 0, 1)$, $C = (0, 3, 2)$, the line

$$l : \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

and the family of planes

$$F_k : x + ky + z = 3 \quad \text{for} \quad k \in \mathbb{R}$$

(a) Let E be the plane through A, B and C. Determine the Cartesian equation of E.

(b) Show that $l \subset E$.

(c) Determine k such that $l \subset F_k$

(d) Determine p and q such that the following system of equations has infinitely many solutions

$$\begin{cases} 2x + y + z &= 5 \\ x - y + z &= 3 \\ -2x + py + 2z &= q \end{cases}$$

(e) Determine the coordinates of the point $S \in F_{-1}$ with z-coordinate equal to 4 such that the pyramid $ABCS$ has volume 4.

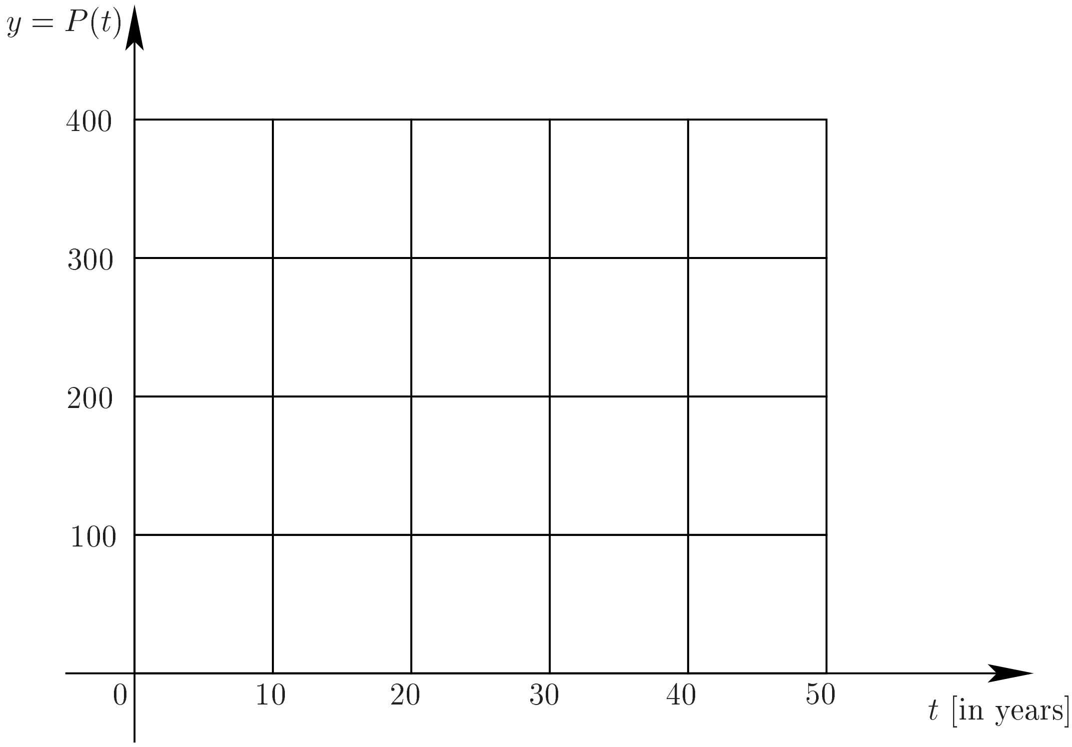
6. Consider a population $P(t)$ which grows under normal circumstances exponentially. Due to an increasing poisoning of the nutrition, the death rate increases proportional to time t. Therefore we can assume that the following ODE models the situation

$$P'(t) = a \cdot P(t) - b \cdot t \cdot P(t)\,,$$

here a is the growth rate and $b \cdot t$ with $b > 0$ denotes the death rate.

Let $a = 0.1$ and $b = 0.01$.

(a) Sketch the direction field in the grid given below.



(b) Sketch the solution curve with initial condition $P(0) = 200$ and discuss its asymptotic behaviour.

(c) Solve the differential equation with the initial condition $P(0) = 200$.

(d) When does the population achieve its maximum? And what is the value of this maximum?

(e) When is the population reduced to only half the size it started off with? You can use the **solve** command on the calculator.