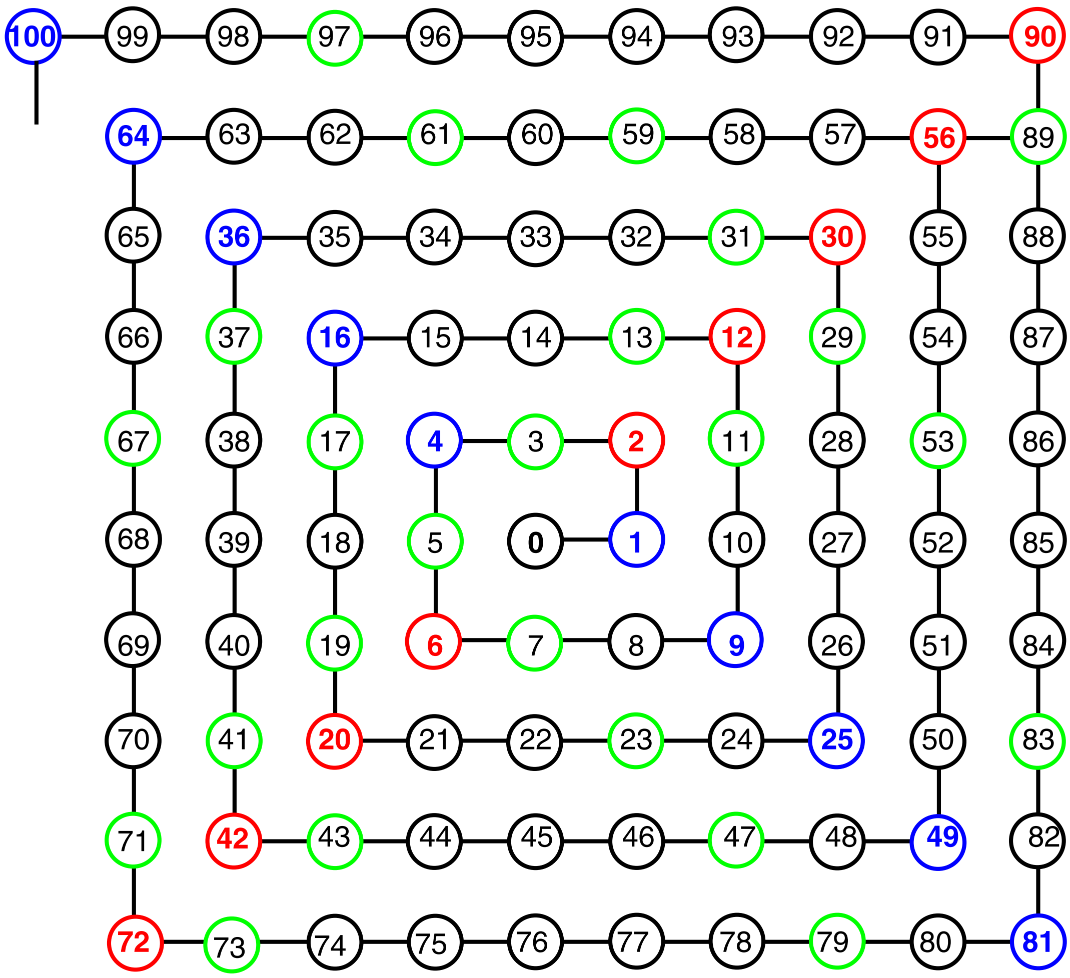
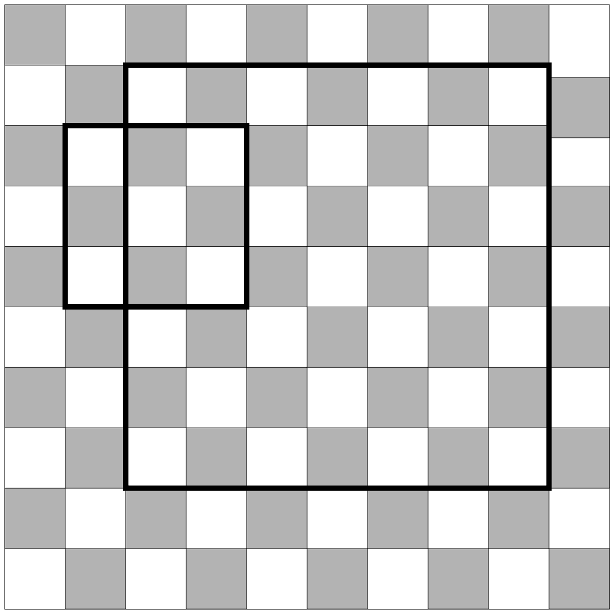
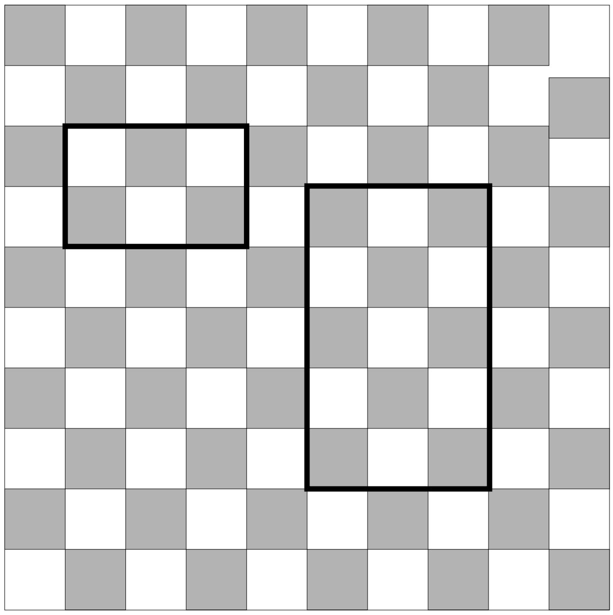
Sommen & Spiralen



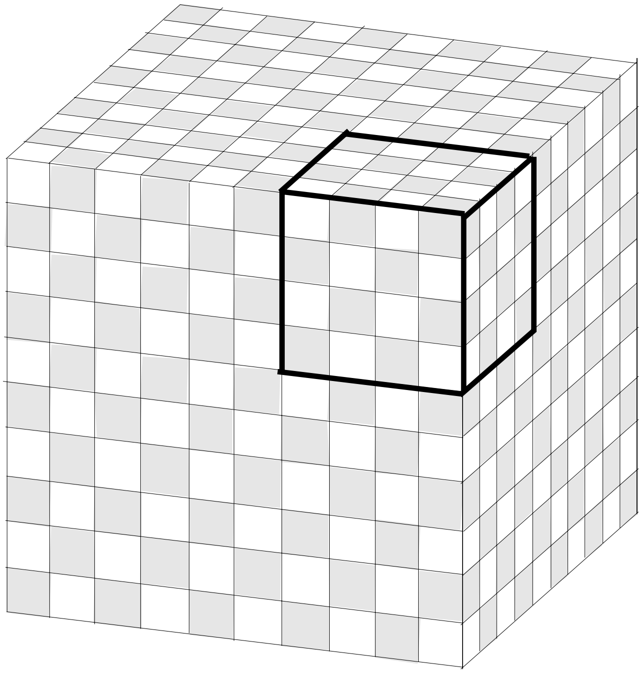
Hoeveel vierkanten bevat het dambord ?



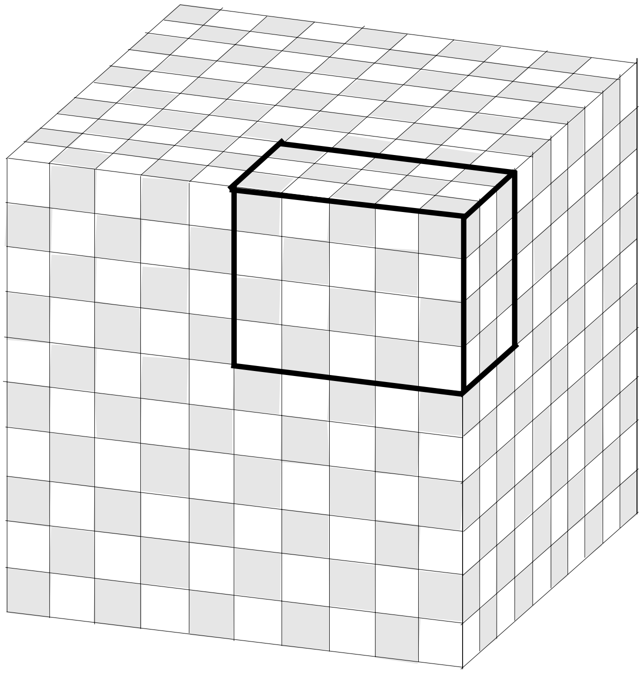
Hoeveel rechthoeken bevat het dambord?



Hoeveel 'deelkubussen'?



Hoeveel 'deelblokken'?



Dambord

vierkanten: $1^2 + 2^2 + ... + 10^2 = 385$

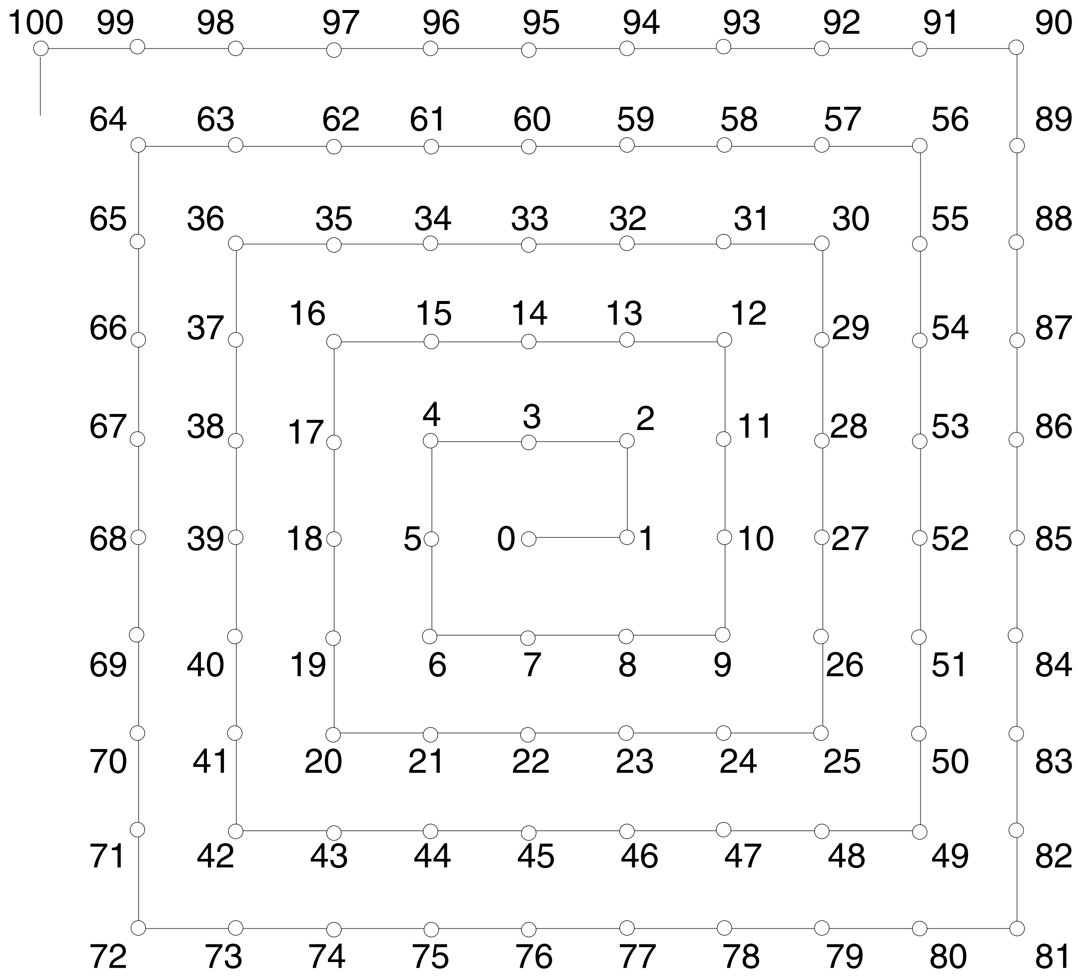
rechthoeken: $(1 + 2 + ... + 10)^2 = 3025$

TOEVAL?

Kubus

deelkubussen: $1^3 + 2^3 + ... + 10^3 = 3025$

deelblokken: $(1 + 2 + ... + 10)^3 = 166375$


100 99 98 97 96 95 94 93 92 91 90
64 63 62 61 60 59 58 57 56 89
65 36 35 34 33 32 31 30 55 88
66 37 16 15 14 13 12 29 54 87
67 38 17 4 3 2 11 28 53 86
68 39 18 5 0 1 10 27 52 85
69 40 19 6 7 8 9 26 51 84
70 41 20 21 22 23 24 25 50 83
71 42 43 44 45 46 47 48 49 82
72 73 74 75 76 77 78 79 80 81

1

3

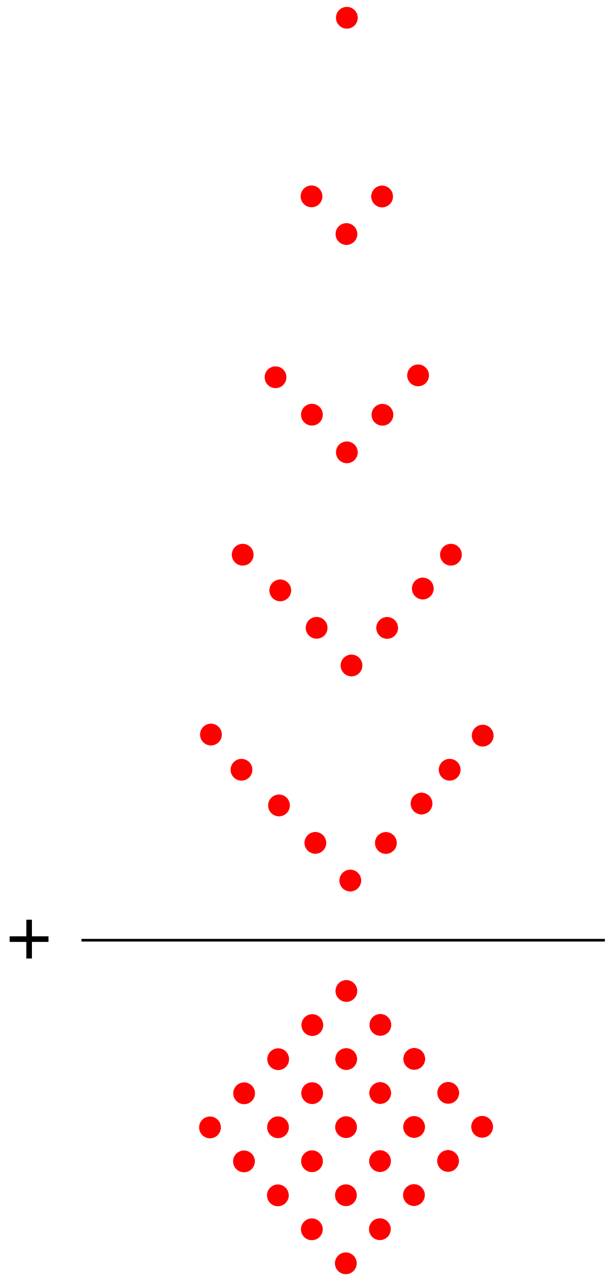
5

7

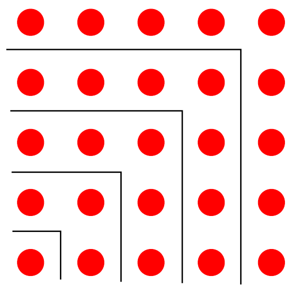
9

\+ ______

25

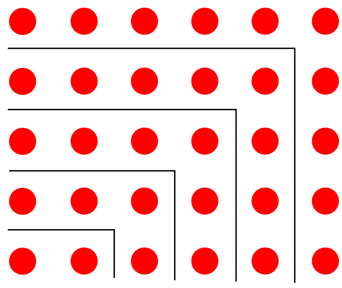


$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$



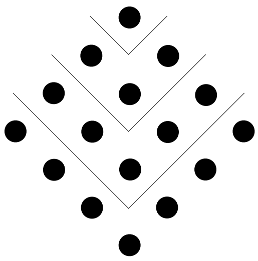
$$\sum_{k=1}^{n} (2k - 1) = n^2$$

$$2 + 4 + 6 + \ldots + 2n = n^2 + n$$

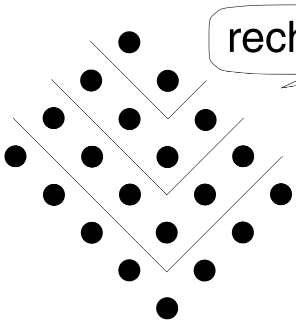


$$\sum_{k=1}^{n} 2k = n^2 + n$$

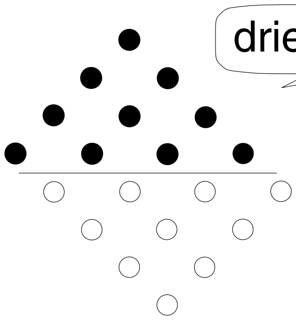
Nikomachos van Gerasa (ca. 100 na Chr.)



$$\mathbf{1 + 3 + 5 + \ldots + (2\boldsymbol{n} - 1) = \boldsymbol{n}^2}$$



$$\mathbf{2 + 4 + 6 + \ldots + 2\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{n} + 1)}$$



$$\mathbf{1 + 2 + 3 + \ldots + \boldsymbol{n} = \tfrac{1}{2}\boldsymbol{n}(\boldsymbol{n} + 1)}$$

Voorbeelden van onderzoekjes:

de som van twee opvolgende driehoeksgetallen is een kwadraat

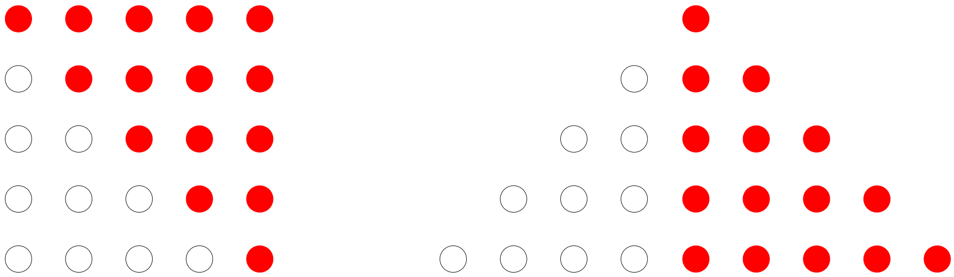
4 × rechthoeksgetal **+ 1** = kwadraat

produkt van twee opvolgende rechthoeksgetallen is weer een rechthoeksgetal

Nikomachos intoduceerde ook vijfhoeksgetallen

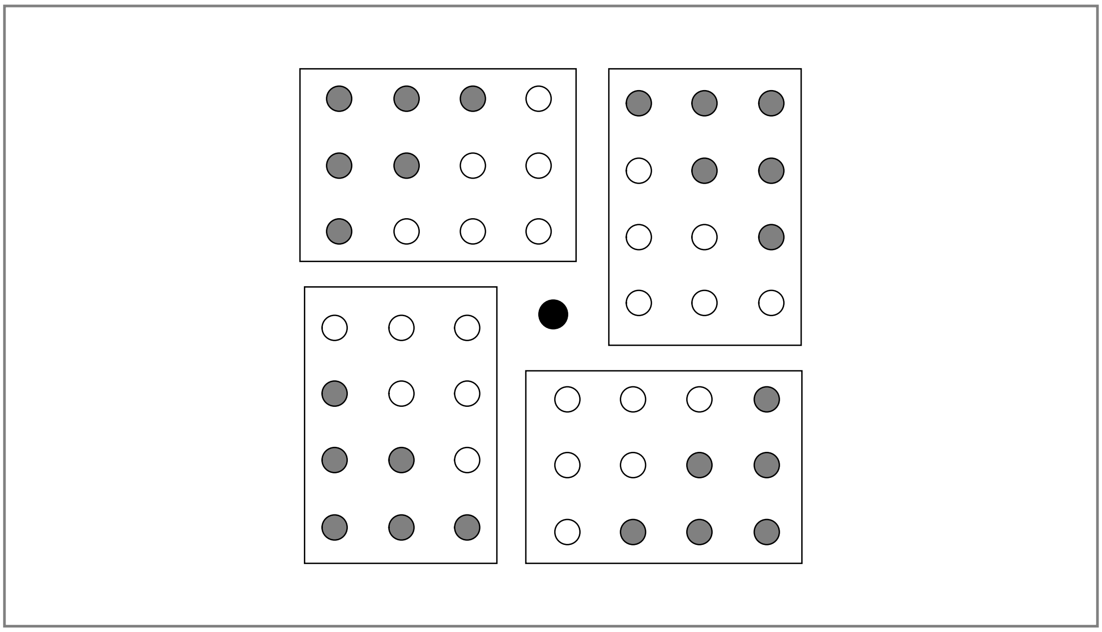
5, 12, 22, 35, 51, etc.

Hoe zou je die door stippenpatronen voorstellen?
Welke formule past daarbij?



$$\frac{1}{2}n(n-1)+\frac{1}{2}n(n+1) = n^2$$

8 × driehoeksgetal + 1 = 4 × rechthoeksgetal + 1 = kwadraat



Met algebra:

$$8\times\frac{1}{2}n(n+1)+1 = 4n^2+4n+1 = (2n+1)^2$$

$$\underbrace{1 + 3 + 5 + 7}_{4^2} + \underbrace{9}_{3^2} = 5^2$$

$$\underbrace{1 + 3 + 5 + 7 + 9 + 11 + \ldots + 23}_{12^2} + \underbrace{25}_{5^2} = 13^2$$

$$\underbrace{1 + 3 + 5 + 7 + 9 + 11 + \ldots + 25 + \ldots. + 47}_{24^2} + \underbrace{49}_{7^2} = 25^2$$

etc.

Pythagorese drietallen

3 , 4 , 5

5 , 12 , 13

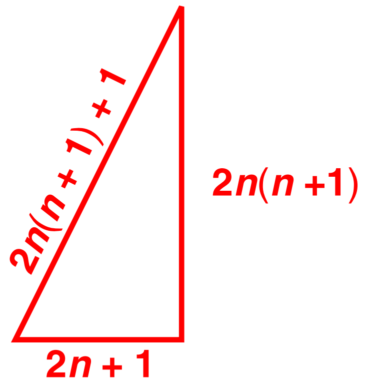
7 , 24 , 25

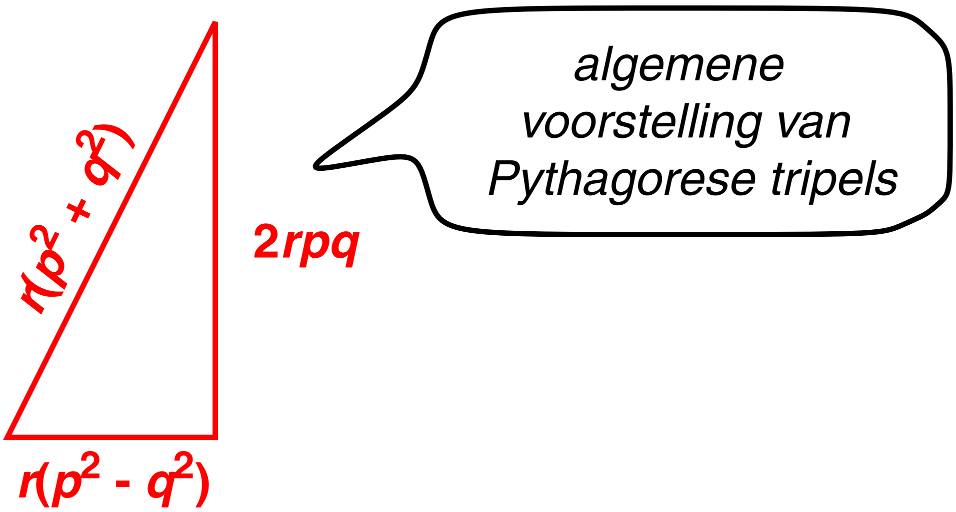
9 , 40 , 41

11 , 60 , 61

13 , 84 , 85

etc.





$$p = n + 1$$
$$q = n$$
$$r = 1$$

$$p^2 - q^2 = 2n + 1$$
$$2pq = 2n(n + 1)$$
$$p^2 + q^2 = 2n(n + 1) + 1$$

Nikomachos

$$1 = 1^3$$

$$3 + 5 = 8 = 2^3$$

$$7 + 9 + 11 = 27 = 3^3$$

$$13 + 15 + 17 + 19 = 64 = 4^3$$

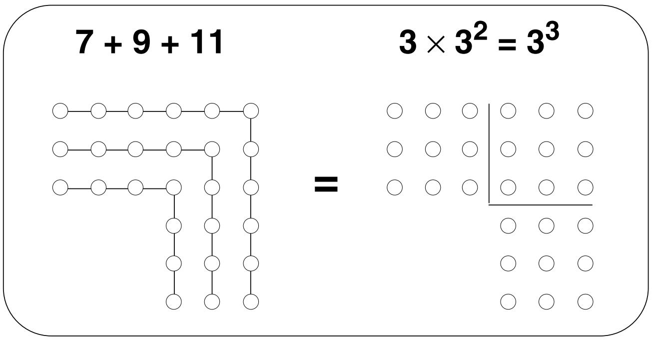
etc.

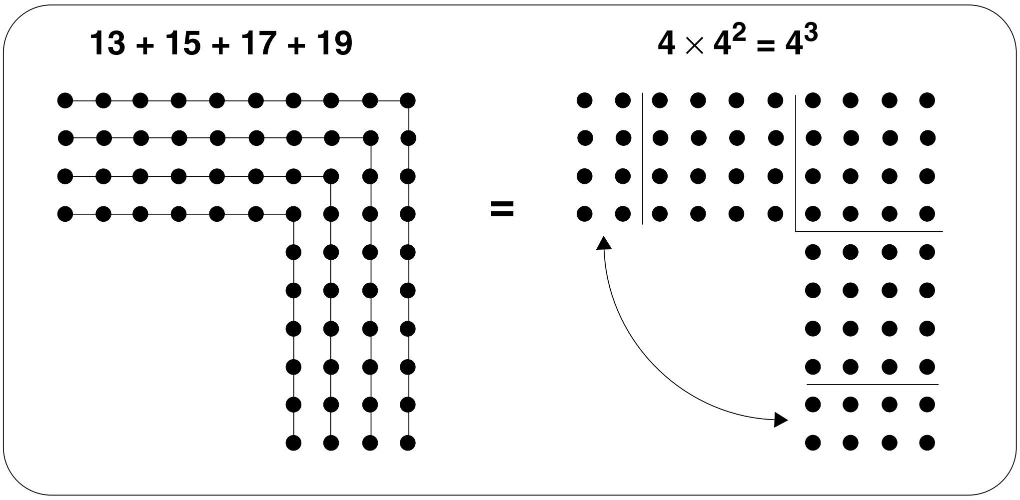
$$3 \overset{\textcircled{4}}{+} 5 = 2 \times 4 = 2^3$$

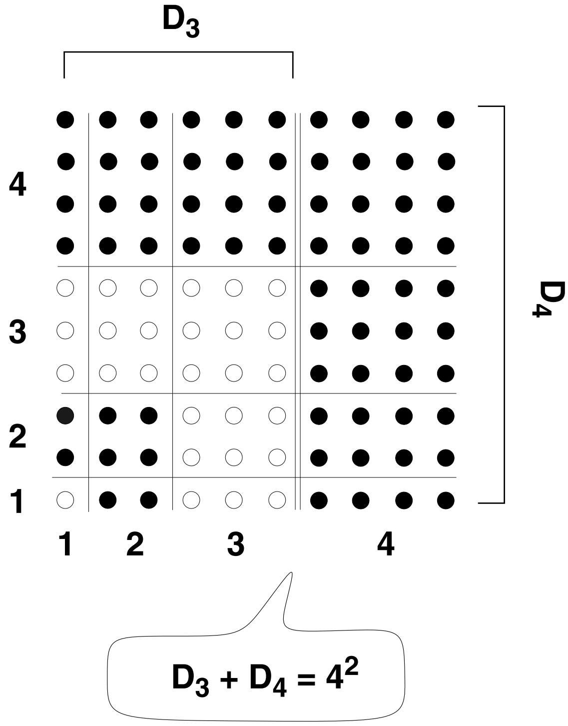
$$7 + \overset{\textcircled{9}}{9} + 11 = 3 \times 9 = 3^3$$

$$13 + 15 \overset{\textcircled{16}}{+} 17 + 19 = 4 \times 16 = 4^3$$

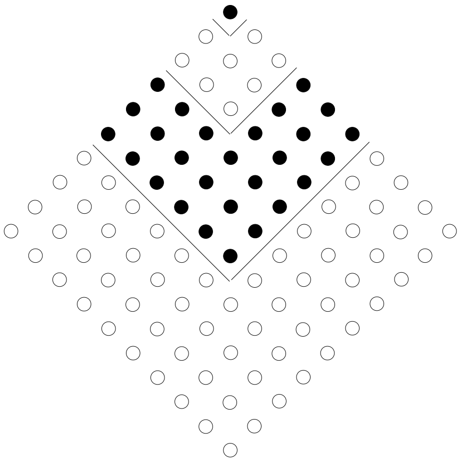
$$21 + 23 + \overset{\textcircled{25}}{25} + 27 + 29 = 5 \times 25 = 5^3$$

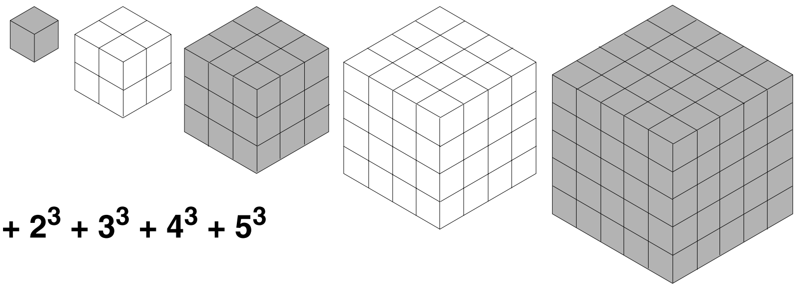

7 + 9 + 11
3 × 3² = 3³
=


13 + 15 + 17 + 19
4 × 4² = 4³
=

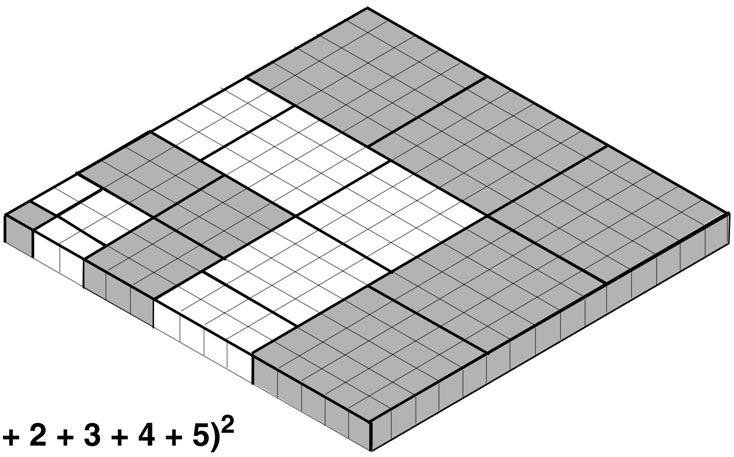

D3
D4
4
3
2
1
1
2
3
4
D3 + D4 = 4²

$$1^3 + 2^3 + \dots + n^3 = (1 + 2 + \dots n)^2$$





$1^3 + 2^3 + 3^3 + 4^3 + 5^3$



$(1 + 2 + 3 + 4 + 5)^2$

$$1^2 + 2^2 + \ldots + n^2 = \tfrac{1}{6} n(n+1)(2n+1)$$

↓

$n(n+1)(2n+1)$ deelbaar door **6**

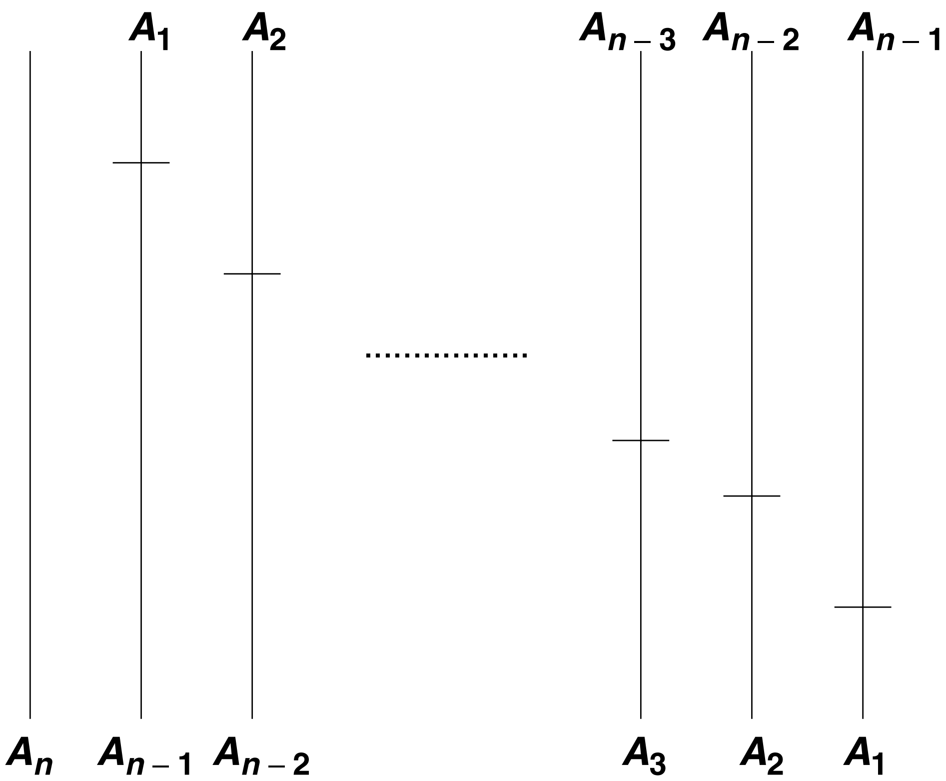
$n(n+1)$	$2n+1$
2	3
6	5
12	7
20	9
30	11
42	13
56	15
72	17
90	19
110	21

Archimedes:

Laat $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, $\boldsymbol{A_3}$, ... , $\boldsymbol{A_n}$ lijnen zijn in een rekenkundige rij, waarvan het constante verschil gelijk is aan de eerste term, dan:

$$(n + 1) \times \mathrm{Kw}(A_n) + A_1 \times (A_1 + A_2 + A_3 + \ldots + A_n) =$$

$$3 \times [\mathrm{Kw}(A_1) + \mathrm{Kw}(A_2) + \mathrm{Kw}(A_3) + \ldots + \mathrm{Kw}(A_n)]$$



$$A_1 = a,\ A_2 = 2a,\ \dots,\ A_n = na \text{ en } \mathrm{Kw}(A_k) = (ka)^2$$

$a = 1$

$$(n+1)n^2 + (1 + 2 + 3 + \dots + n) =$$

$$3(1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$(n+1)n^2 = 2n(1 + 2 + 3 + \dots + n)$$

$$(n+1)n^2 + (1 + 2 + 3 + \dots + n) =$$

$$(2n+1)(1 + 2 + 3 + \dots + n)$$

Twee tussenstappen (in bewijs van Archimedes):

I

$$2(1^2 + 2^2 + 3^2 + \ldots + n^2) =$$

$$(n+1)n^2 - 2[1(n-1) + 2(n-2) + \ldots + (n-1)1]$$

II

$$1^2 + 2^2 + 3^2 + \ldots + n^2 =$$

$$n + 3(n-1) + 5(n-2) + \ldots + (2n-1)$$

Optelling

$$3(1^2 + 2^2 + 3^2 + \ldots + n^2) =$$

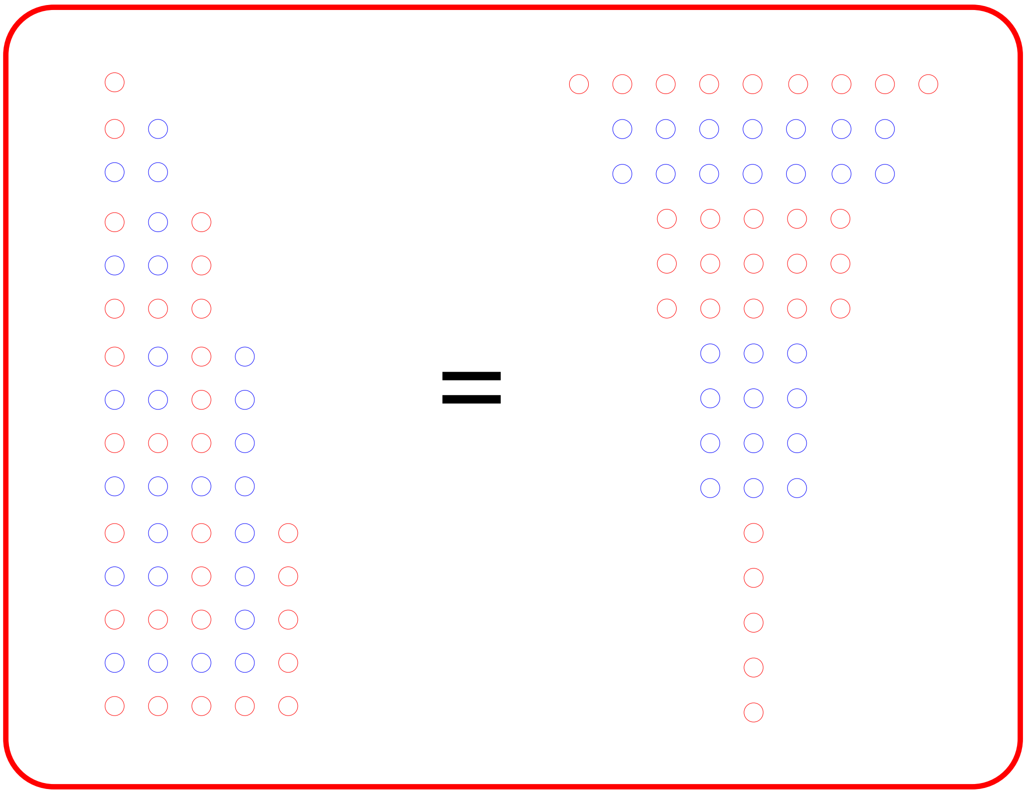
$$(n+1)n^2 + (1 + 2 + 3 + \ldots + n)$$

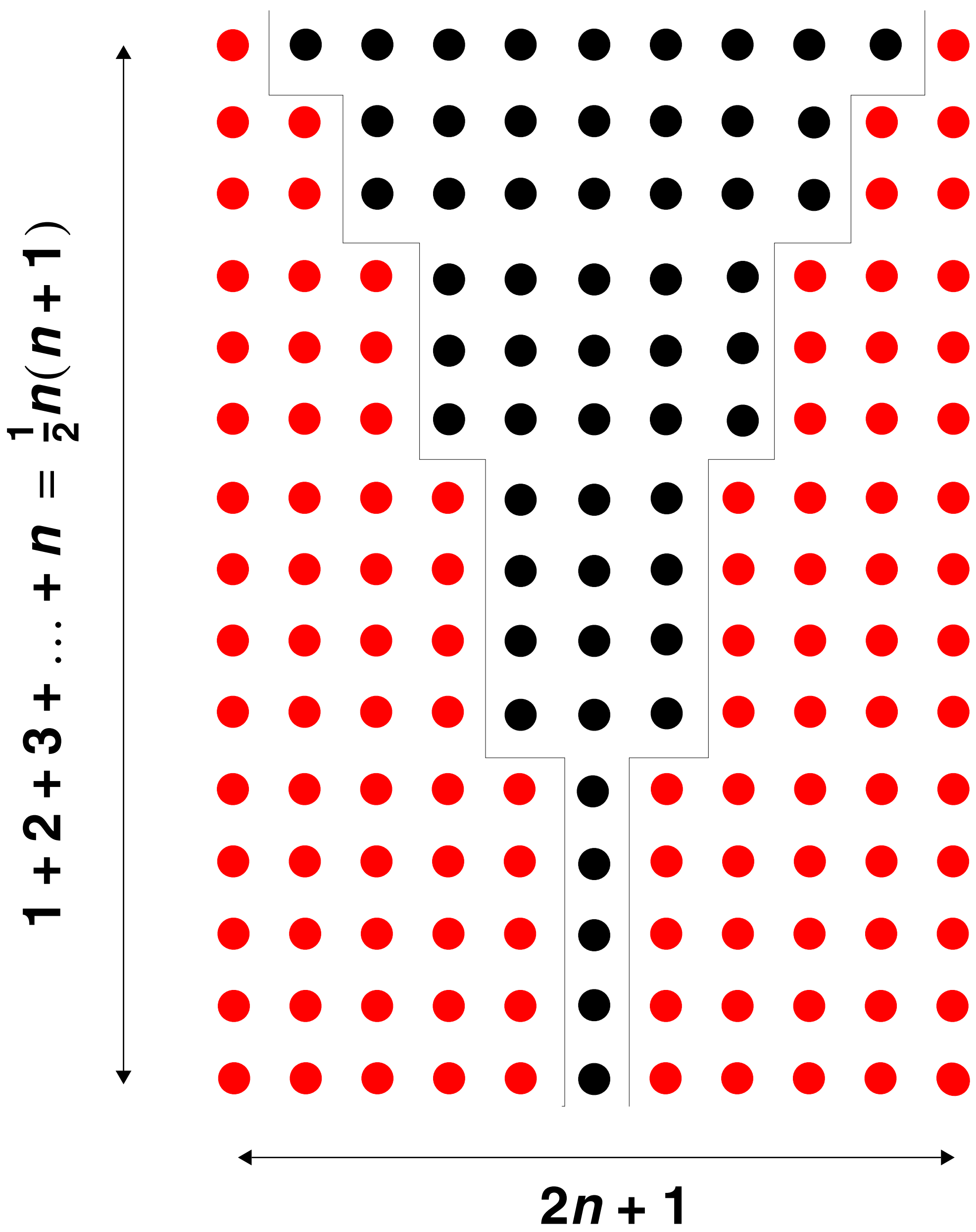
tweede stap

$$1^2 = 1$$
$$2^2 = 1 + 3$$
$$3^2 = 1 + 3 + 5$$
....
$$\boldsymbol{n}^2 = 1 + 3 + ... + (2\boldsymbol{n} - 1)$$

$$+$$

$$1^2 + 2^2 + 3^2 + ... + \boldsymbol{n}^2 = \boldsymbol{n}1 + (\boldsymbol{n} - 1)3 + (\boldsymbol{n} - 2)5 + ... + (2\boldsymbol{n} - 1)$$





$1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$
$2n + 1$

Start $$1^2 = \frac{1}{6} \cdot 1 \cdot 2 \cdot 3$$

Stel

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{1}{6}k(k+1)(2k+1)$$

Dan:

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (\overline{k+1})^2 =$$

$$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$

$$= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)]$$

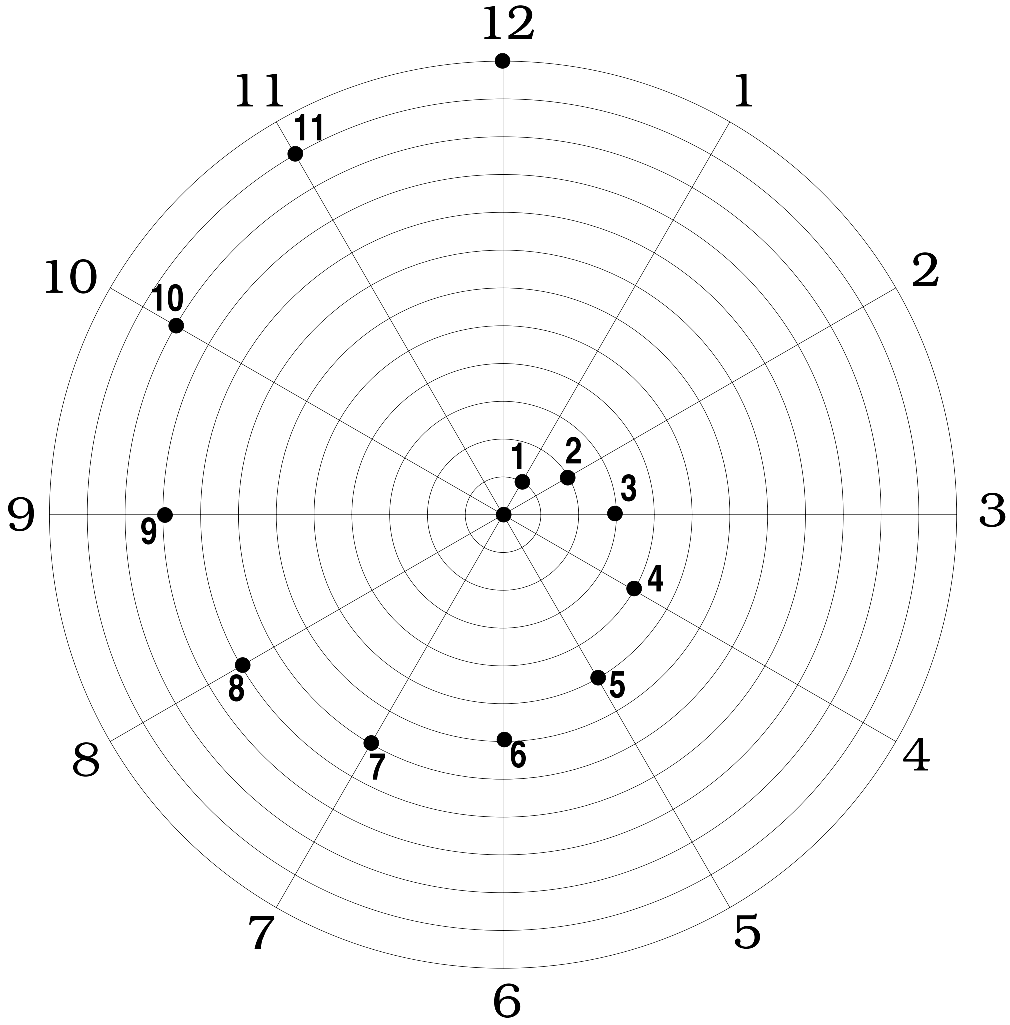
$$= \frac{1}{6}(k+1)[2k^2 + 7k + 6]$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

$$= \frac{1}{6}(\overline{\boldsymbol{k+1}})(\overline{\boldsymbol{k+1}} + 1)(2(\overline{\boldsymbol{k+1}}) + 1)$$

Q.E.D.

De spiraal van Archimedes



Ongelijkheid van Archimedes

$$1^2 + 2^2 + 3^2 + \ldots + n^2 =$$

$$\tfrac{1}{3}n(n+\tfrac{1}{2})(n+1) > \tfrac{1}{3}n^3$$

$$1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 =$$

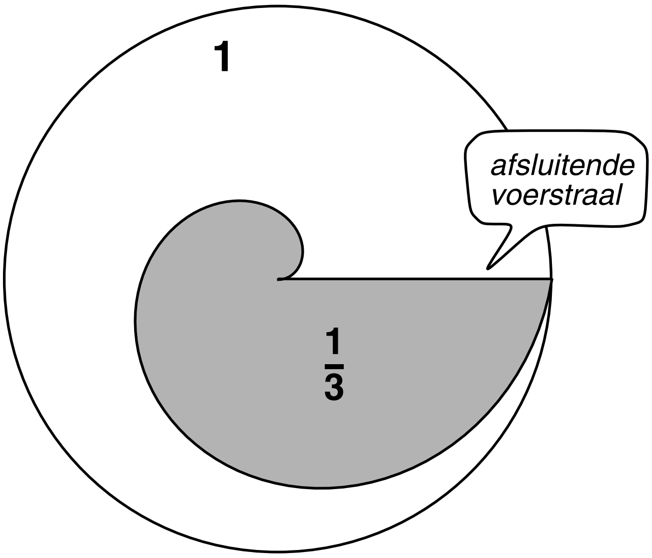
$$\tfrac{1}{3}(n-1)(n-\tfrac{1}{2})n < \tfrac{1}{3}n^3$$

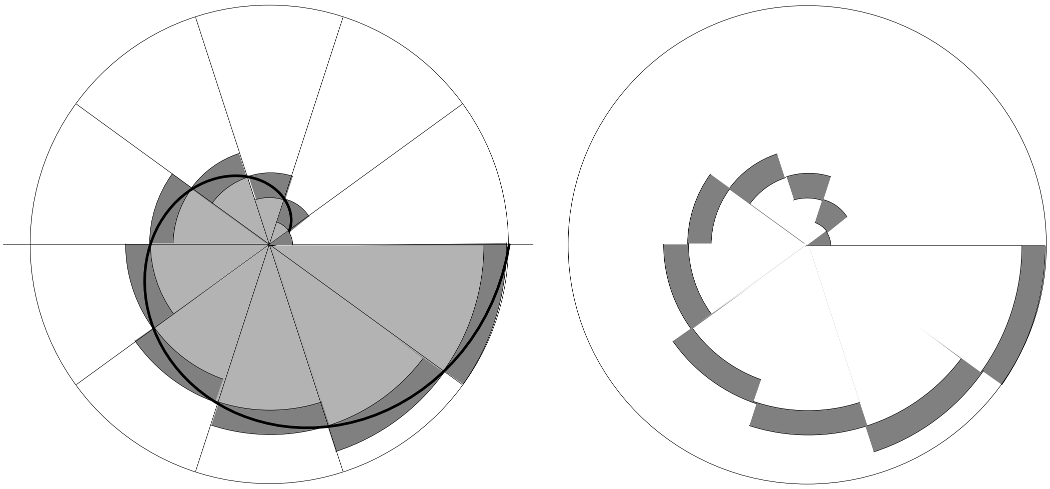
$$\Delta = \frac{1}{n}$$

$$\frac{1^2 + 2^2 + \ldots + (n-1)^2}{n^2} < \frac{1}{3}$$

en

$$\frac{1^2 + 2^2 + \ldots + n^2}{n^3} > \frac{1}{3}$$





$$\frac{1}{n}\times\left(0^2+\left(\frac{1}{n}\right)^2+\ldots+\left(\frac{n-1}{n}\right)^2\right) < \frac{\boldsymbol{S}}{\boldsymbol{C}} < \frac{1}{n}\times\left(\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\ldots+1^2\right)$$

$$\frac{1}{n^3}\times(1^2+2^2+\ldots+(n-1)^2) < \frac{\boldsymbol{S}}{\boldsymbol{C}} < \frac{1}{n^3}\times(1^2+2^2+\ldots+n^2)$$

$$\boldsymbol{S} = \frac{1}{3}\boldsymbol{C} = \frac{4}{3}\pi^3$$