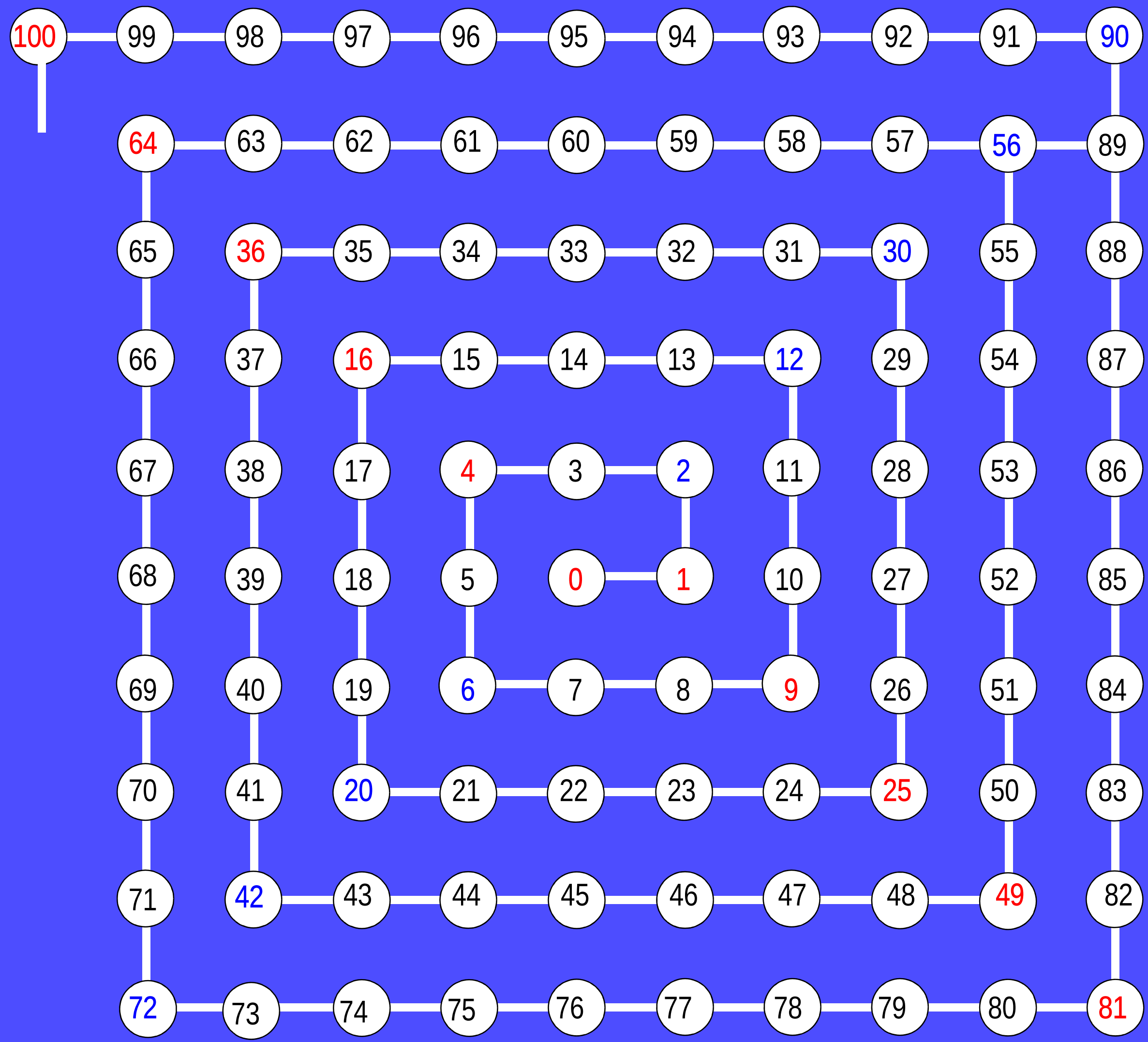
A Natural Way To Algebra



Martin Kindt
Freudenthal Institute

Algebra

the art of 'thingification'

And now is a good moment to open up Pandora's box and explain one of the most powerful general weapons in the mathematician's armory, which we might call the 'thingification of processes'.

Ian Stewart in *Nature's Numbers*

Algebra at school

RESTRICTIONS

- equations
- inequalities
- *linear programming*

PROCESSES CHANGE

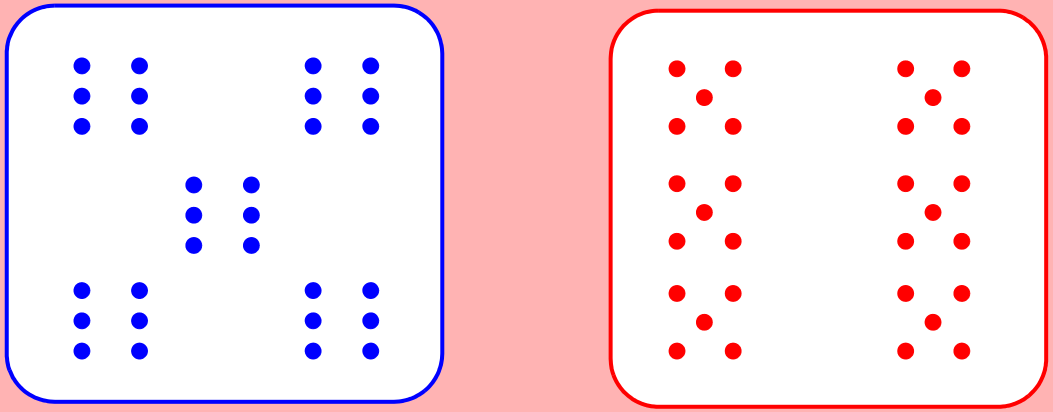
- operations
- functions
- graphs

PATTERNS & FIGURES

- *sequences*
- *figurate numbers*

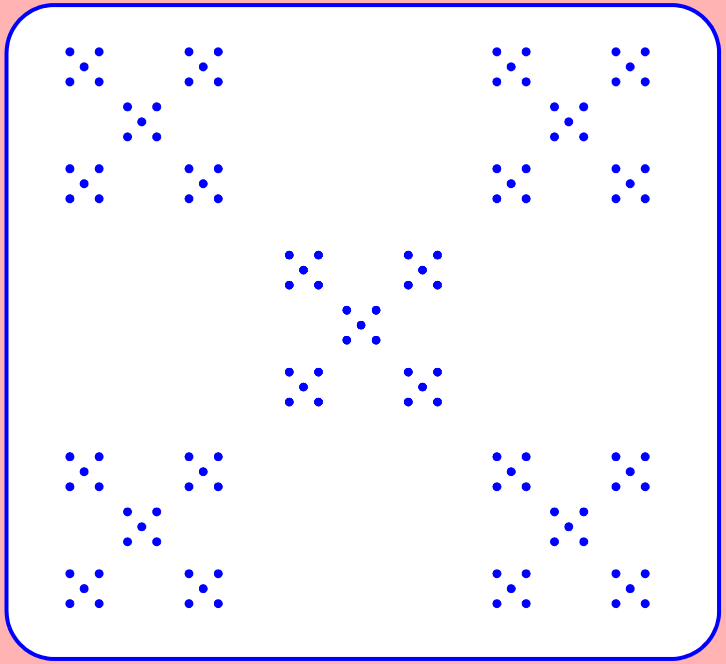
PATTERNS
&
FIGURES

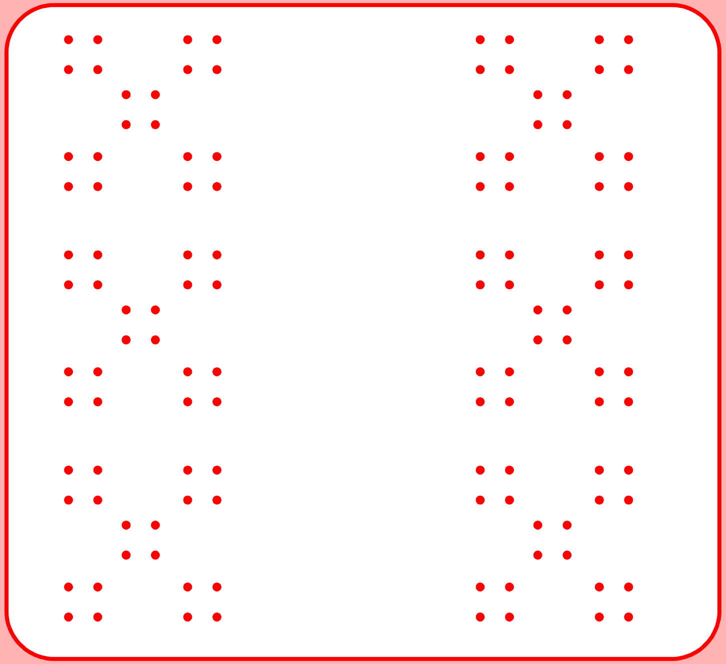
A *natural number* is an idea that has long ago been thingified so thoroughly that everybody thinks of it as a thing.



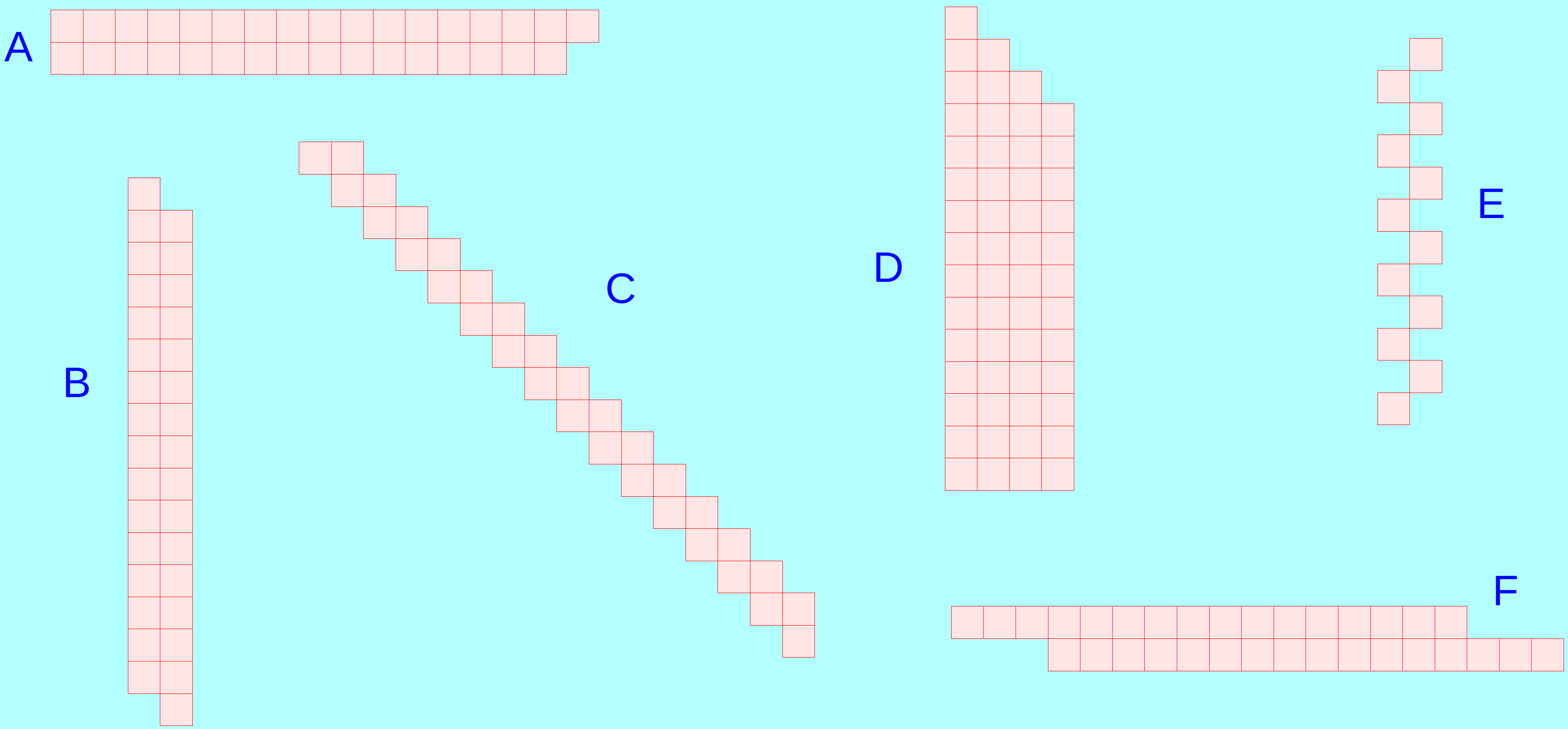
Which pattern has the biggest number of dots?

Same question ....

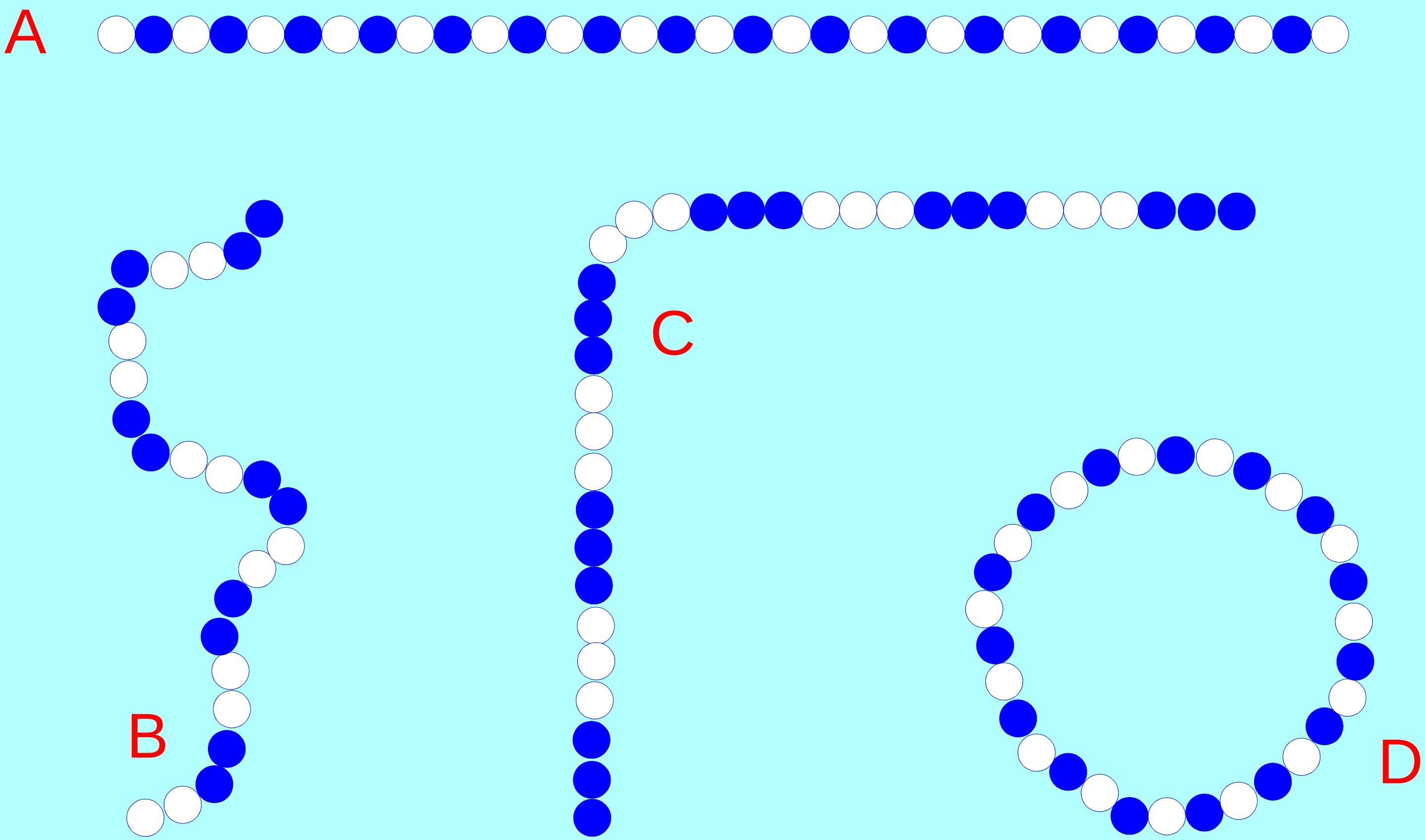




Even & Odd



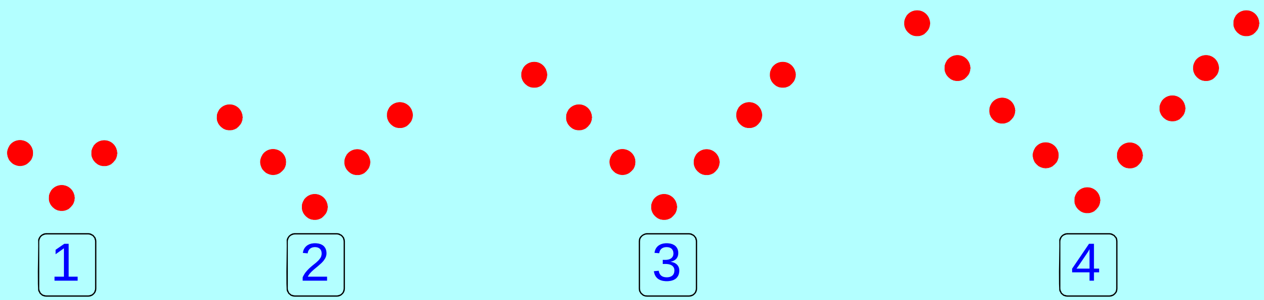
Even or Odd? Explain your strategy!



Without counting the beads: *even* or *odd*?

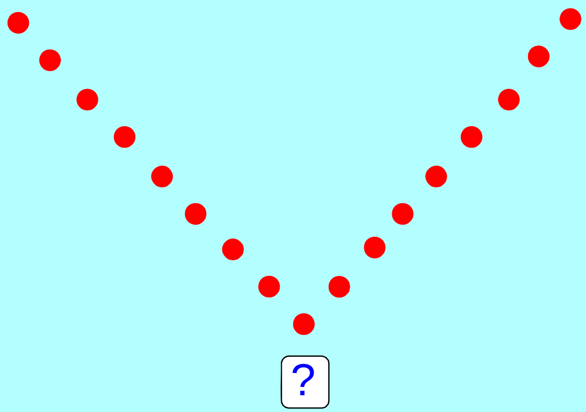
Groups of birds sometimes fly in a ***V-pattern***

Sequence of V-patterns using dots:



Exercises

*



* *How many dots has the* *V-pattern* *with* 85 *?*

* *Two groups of geese are flying above the IJsselmeer,*
  both in perfect *V-pattern*.
  Before going to the South they join.
  Can the total group form a perfect *V-pattern* *?*

Some of the answers of young students (11 years)

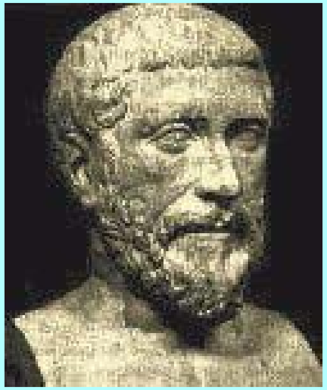
* *You don't know if you don't know the numbers*

* *You cannot know, because you don't know how they are going to fly*

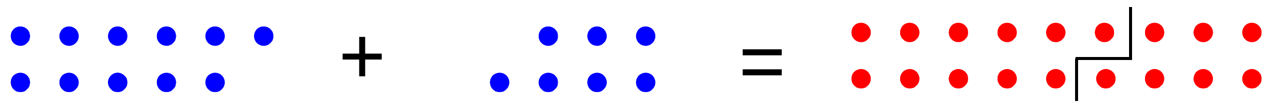
* *No, for there are two leaders now*

* *No, because together it makes even*

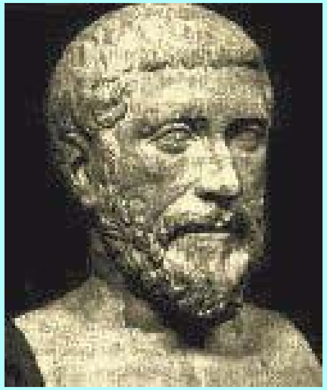
Pythagoras



Odd + *Odd* = *Even*



Pythagoras



Odd + *Odd* = *Even*

In formal language:

$$[2a + 1] + [2b + 1] = 2(a + b + 1)$$

And if 3 groups in perfect V-pattern join?

..

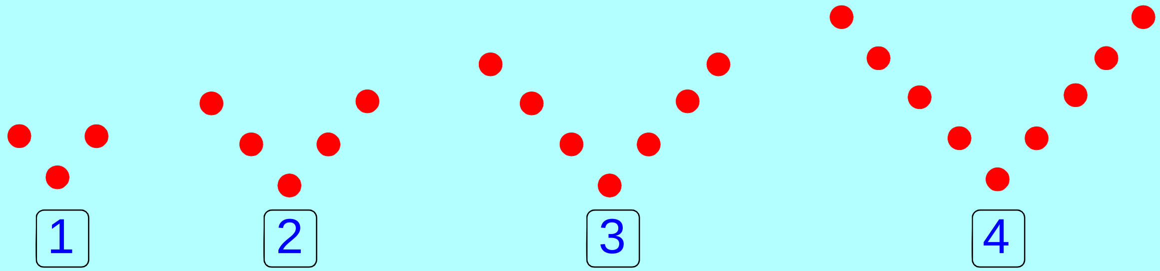
And 4? And 5 ?

...

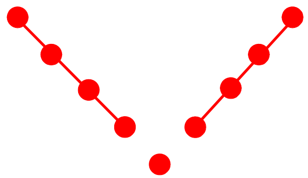
Generalization

...

Formal expressions



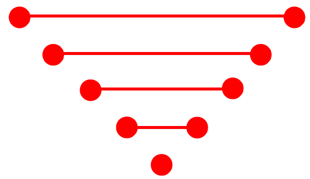
V-number = 1 + 2 × *pattern-number*



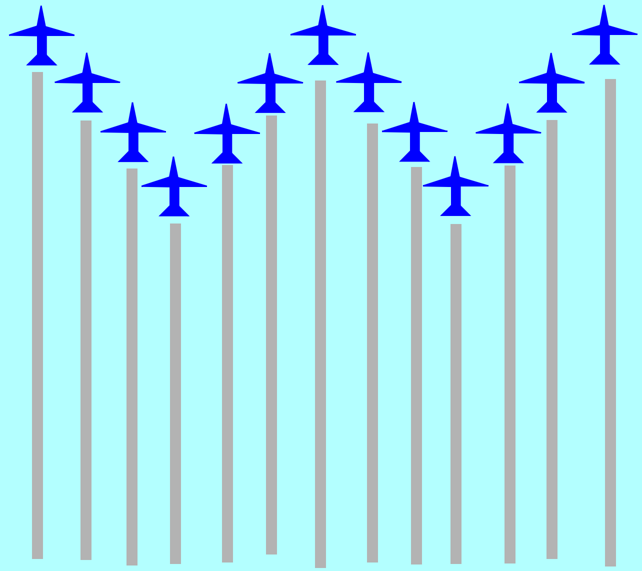
$V = n + 1 + n$

$V = 2 \times n + 1$

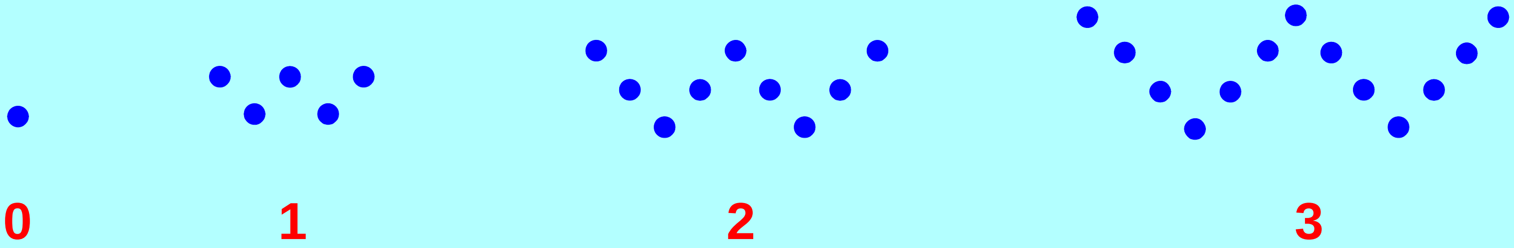
$V = 1 + n \times 2$



During a show a squadron of airplanes flied in a W-formation

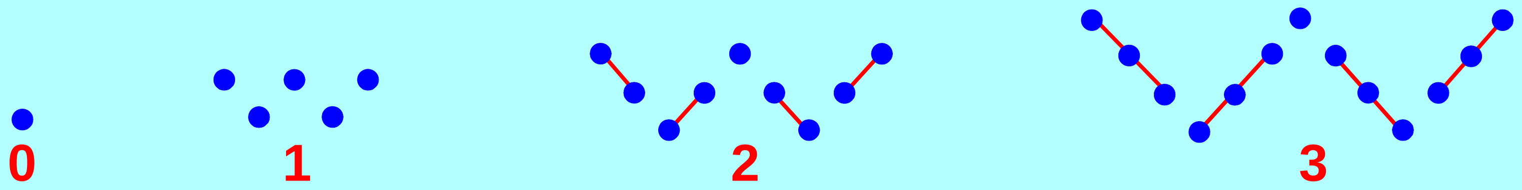


W-numbers

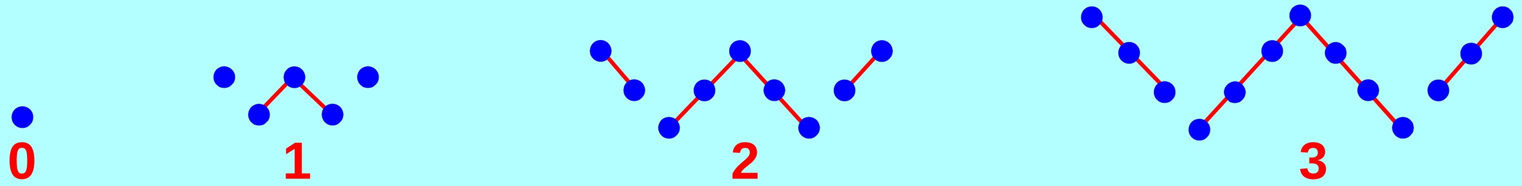


pattern-number	0	1	2	3	4	5	6
number of dots	1	5	...	...	...	...	...

Expression?

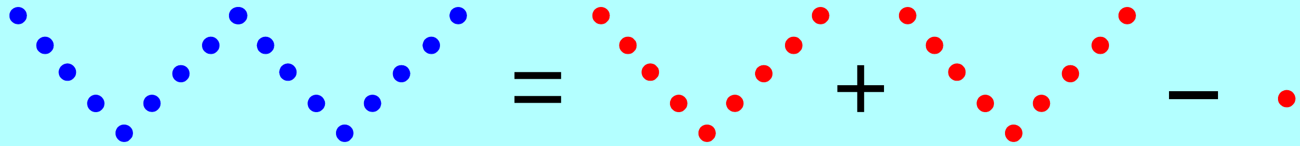


$$W = 4 \times n + 1$$

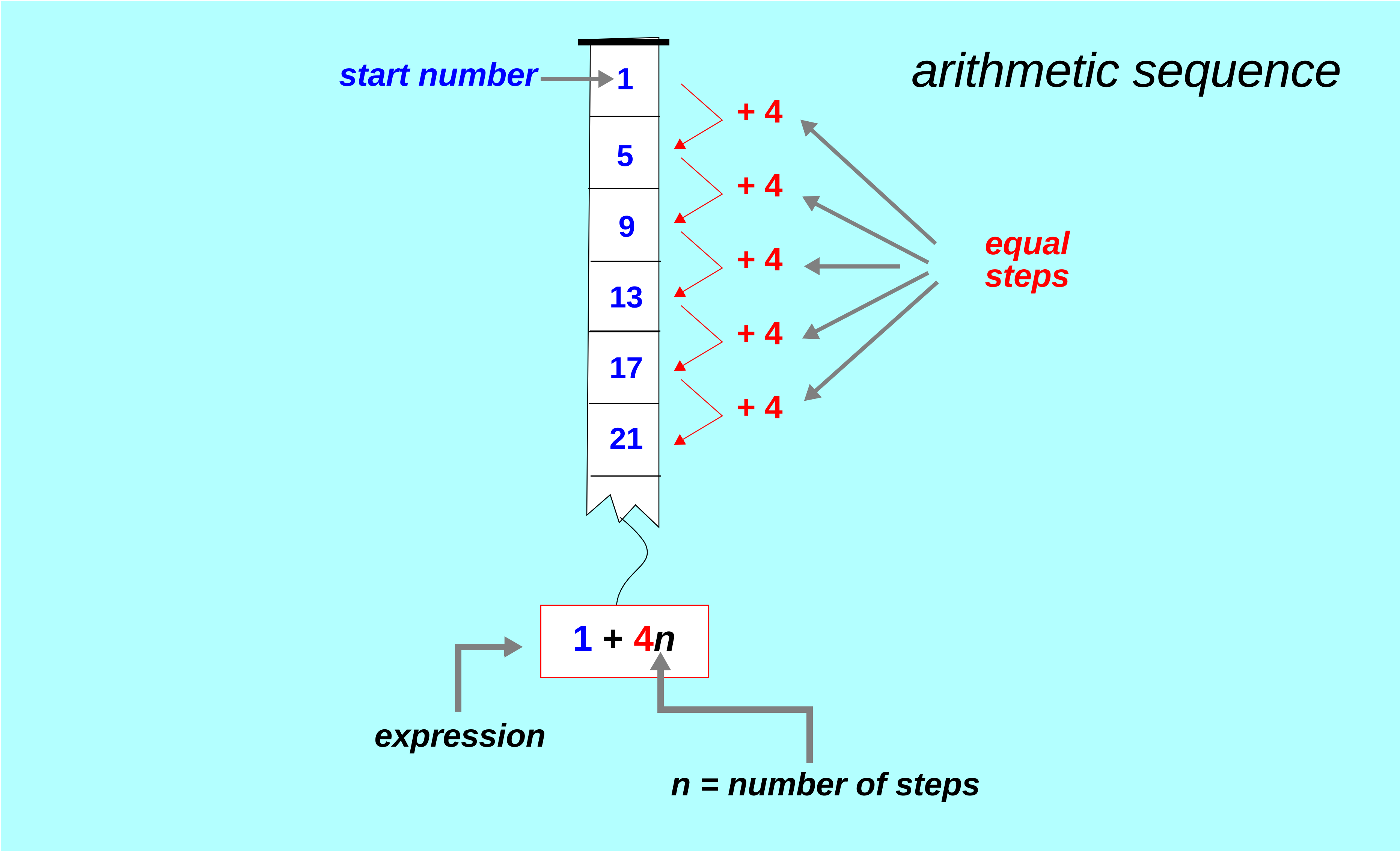


$$W = n + V + n = n + [2n + 1] + n = 4n + 1$$

W = double V – 1



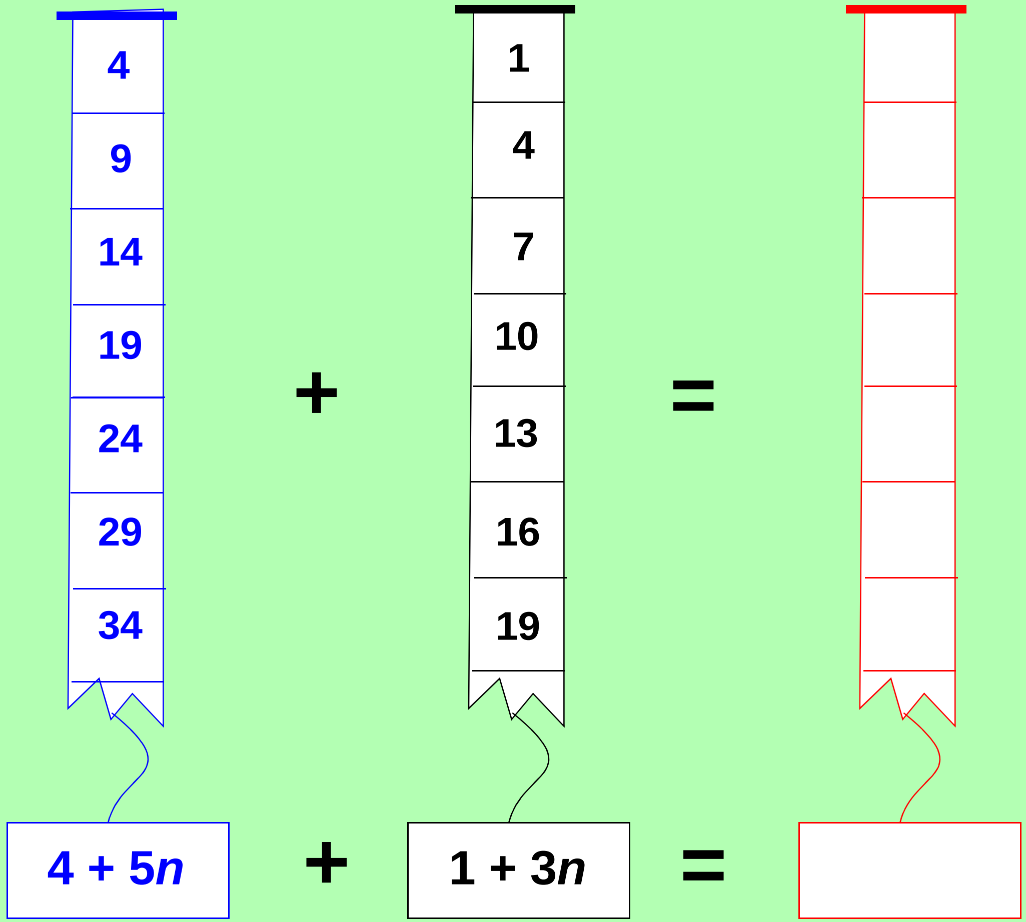
$W = (1 + 2n) + (1 + 2n) - 1 = 1 + 4n$

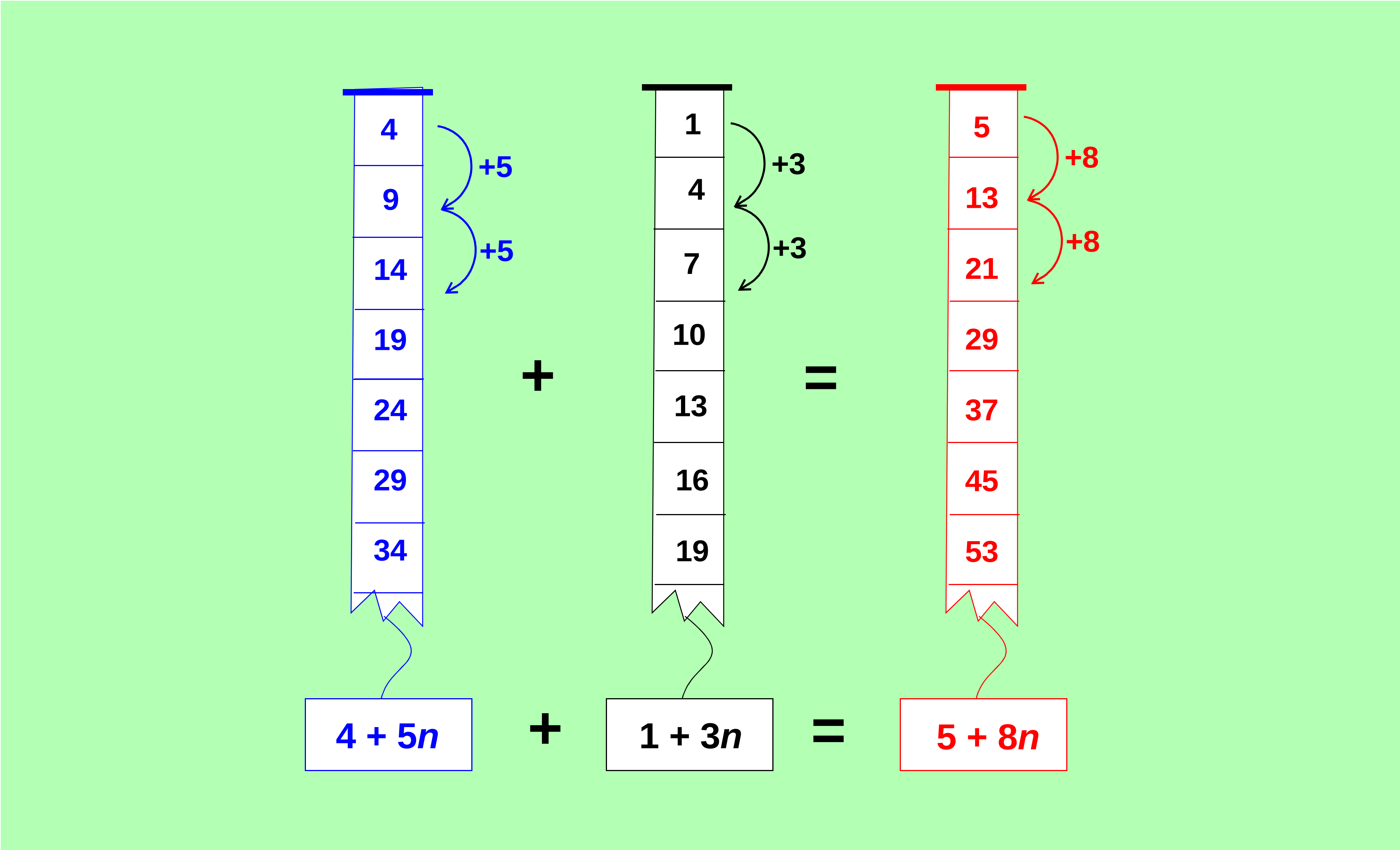


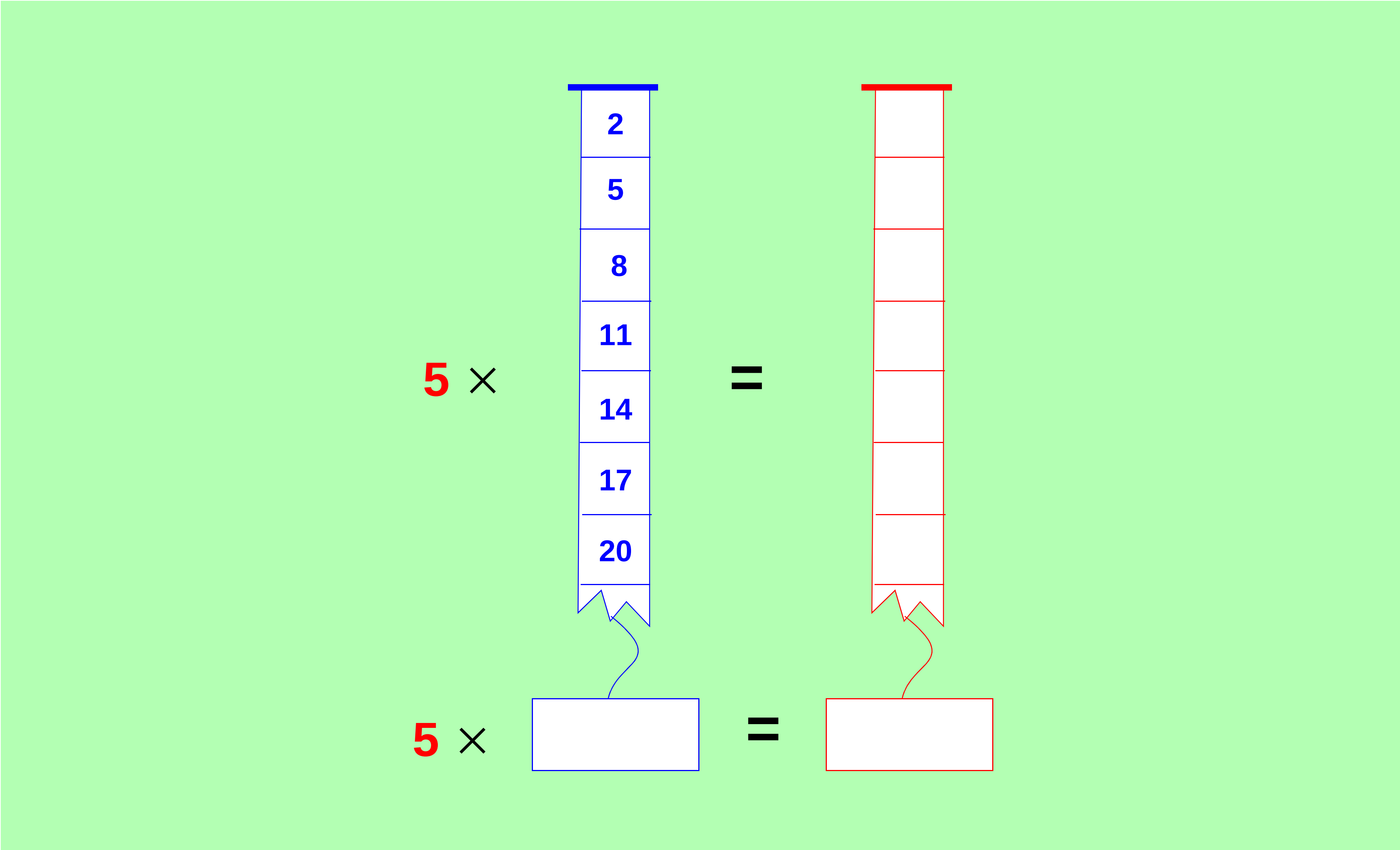
arithmetic sequence
start number
1
5
9
13
17
21
+ 4
+ 4
+ 4
+ 4
+ 4
equal
steps
1 + 4n
expression
n = number of steps

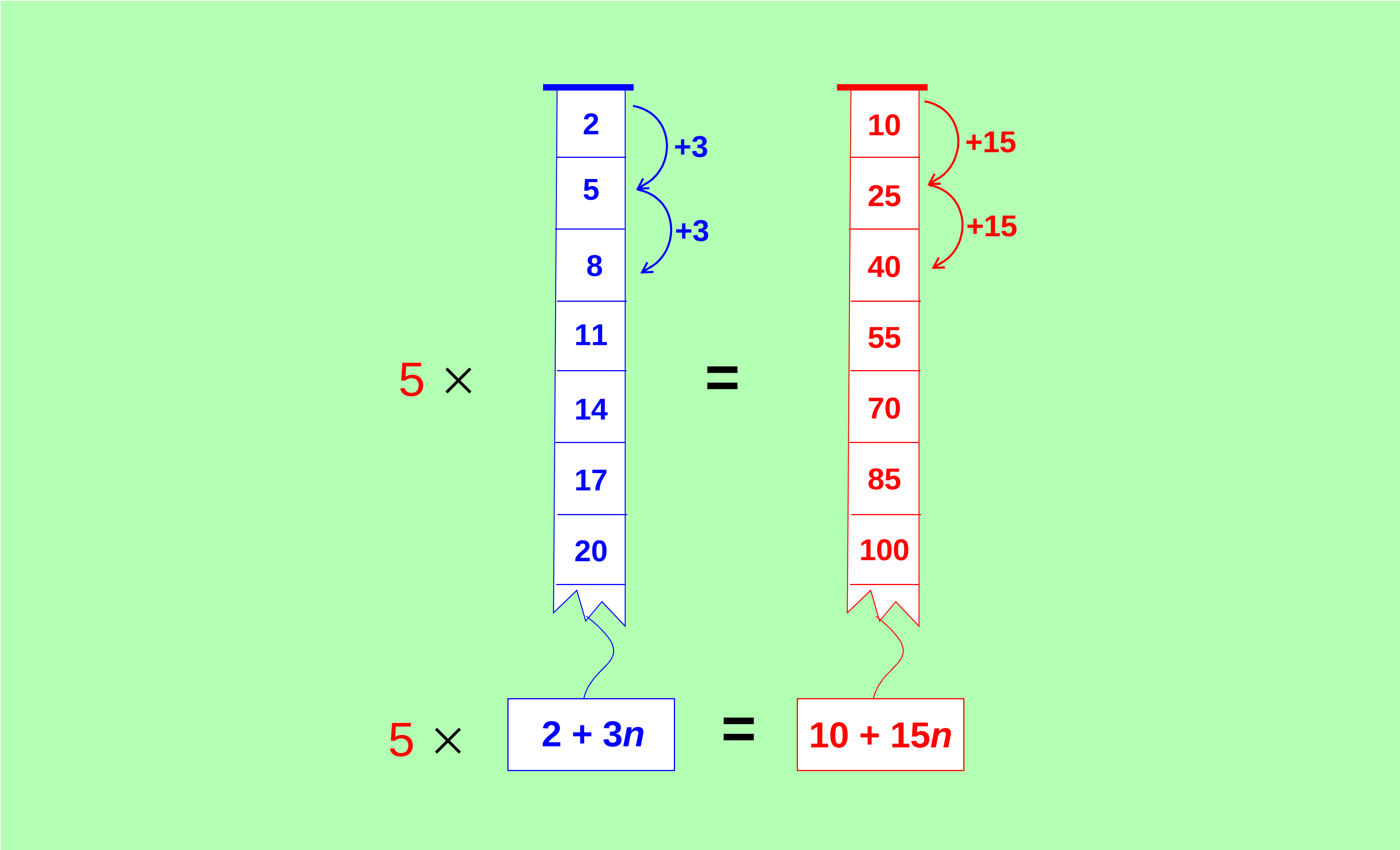
Exercise:

adding sequences

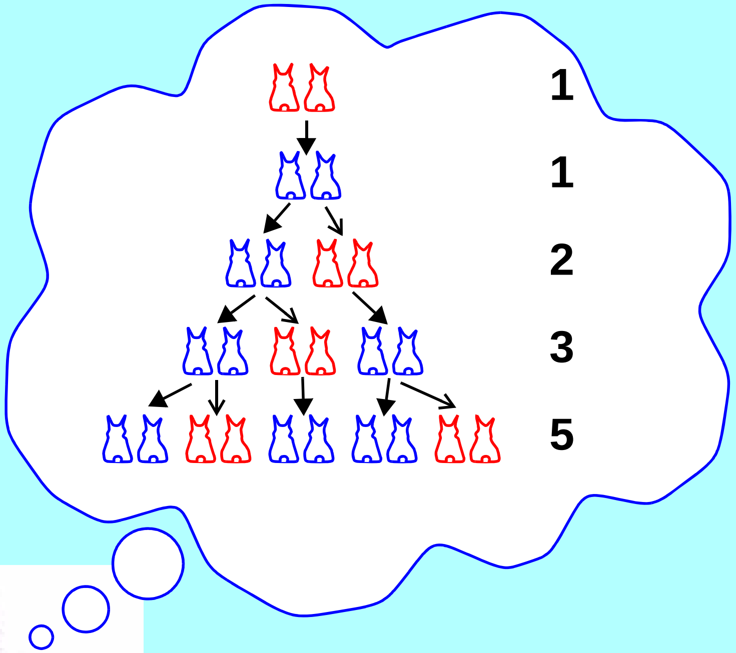



4
9
14
19
24
29
34
+5
+5
+
1
4
7
10
13
16
19
+3
+3
=
5
13
21
29
37
45
53
+8
+8
4 + 5n
+
1 + 3n
=
5 + 8n

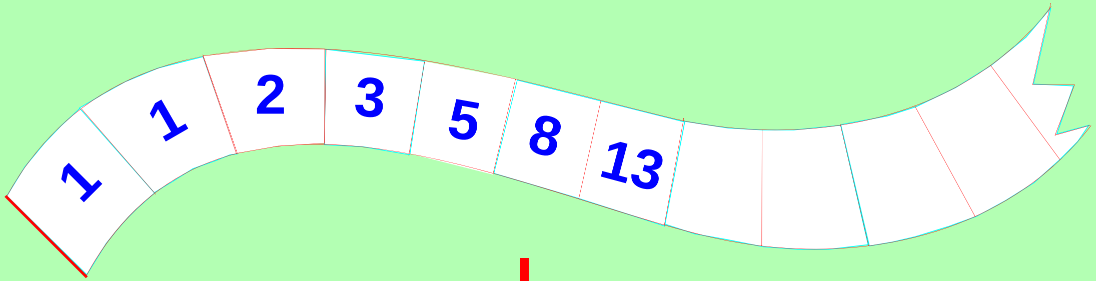

5 ×
2
5
8
11
14
17
20
=
5 ×
=


5 ×
2
5
8
11
14
17
20
+3
+3
=
10
25
40
55
70
85
100
+15
+15
5 ×
2 + 3n
=
10 + 15n

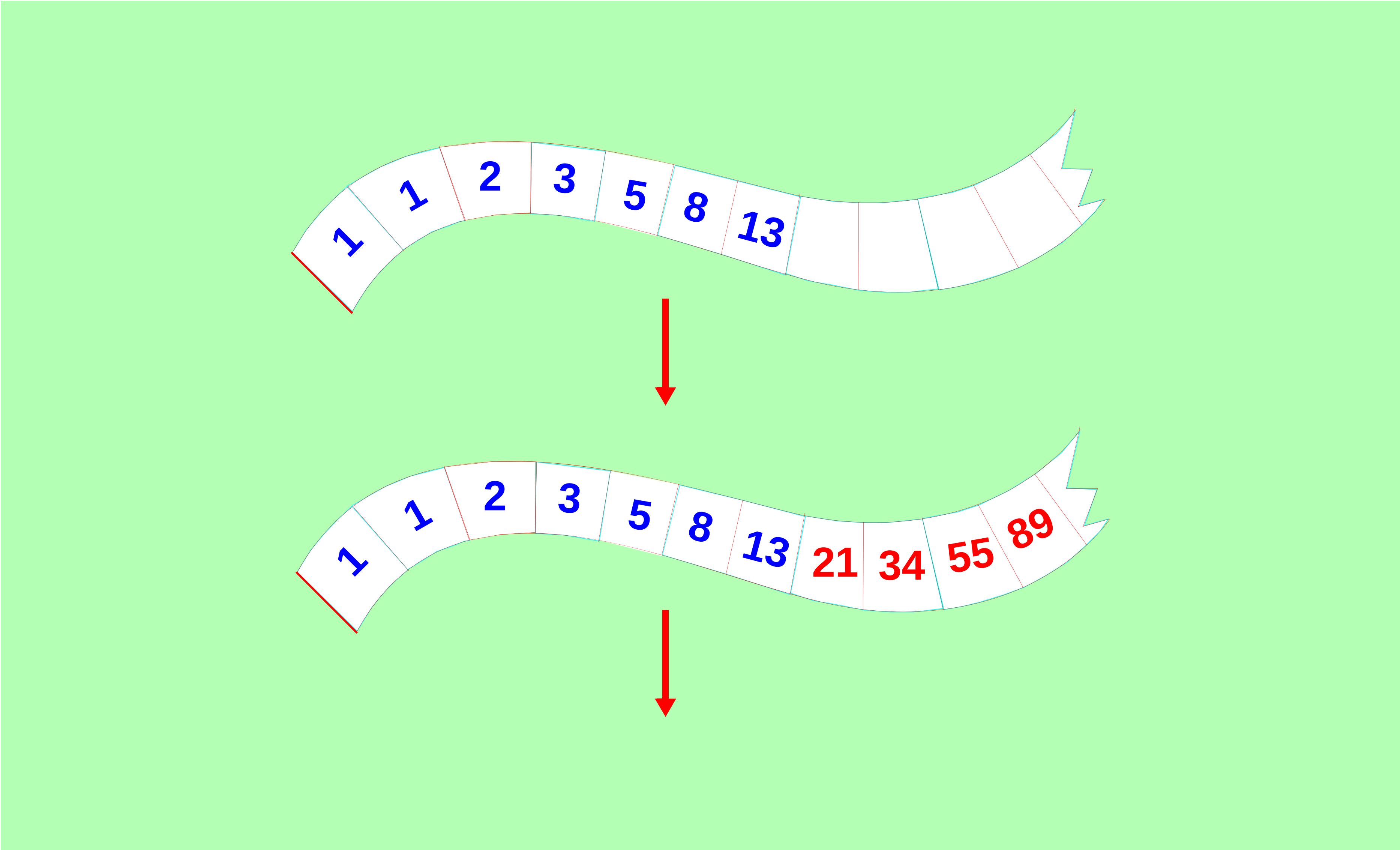
A famous sequence



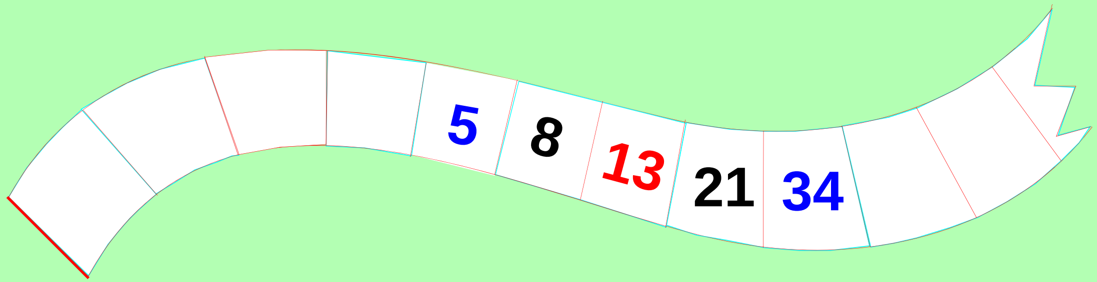
Fibonacci
Liber Abaci (1202)



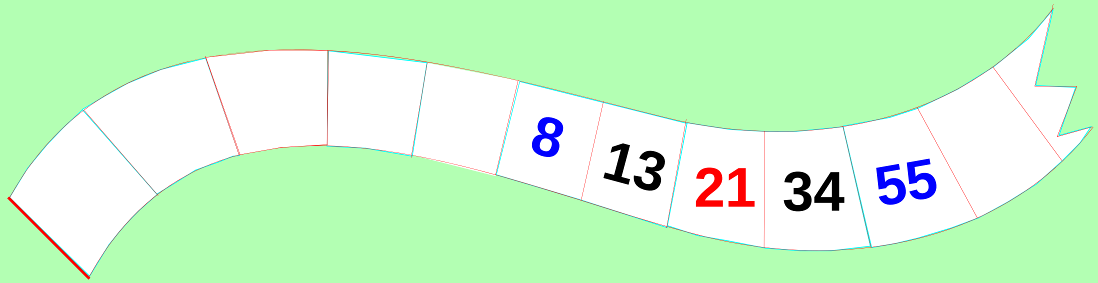
Continuing the sequence ...


1
1
2
3
5
8
13
1
1
2
3
5
8
13
21
34
55
89

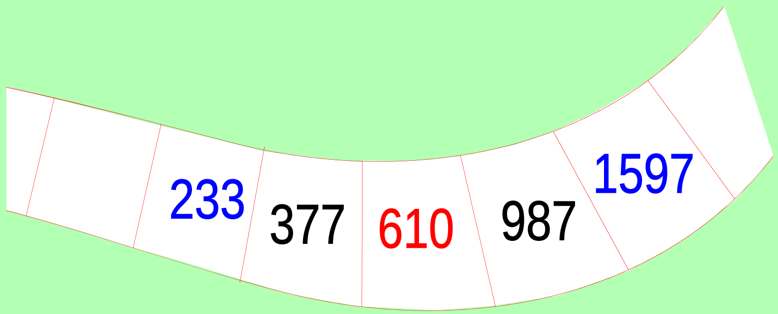
five consecutive Fibonacci-numbers ....



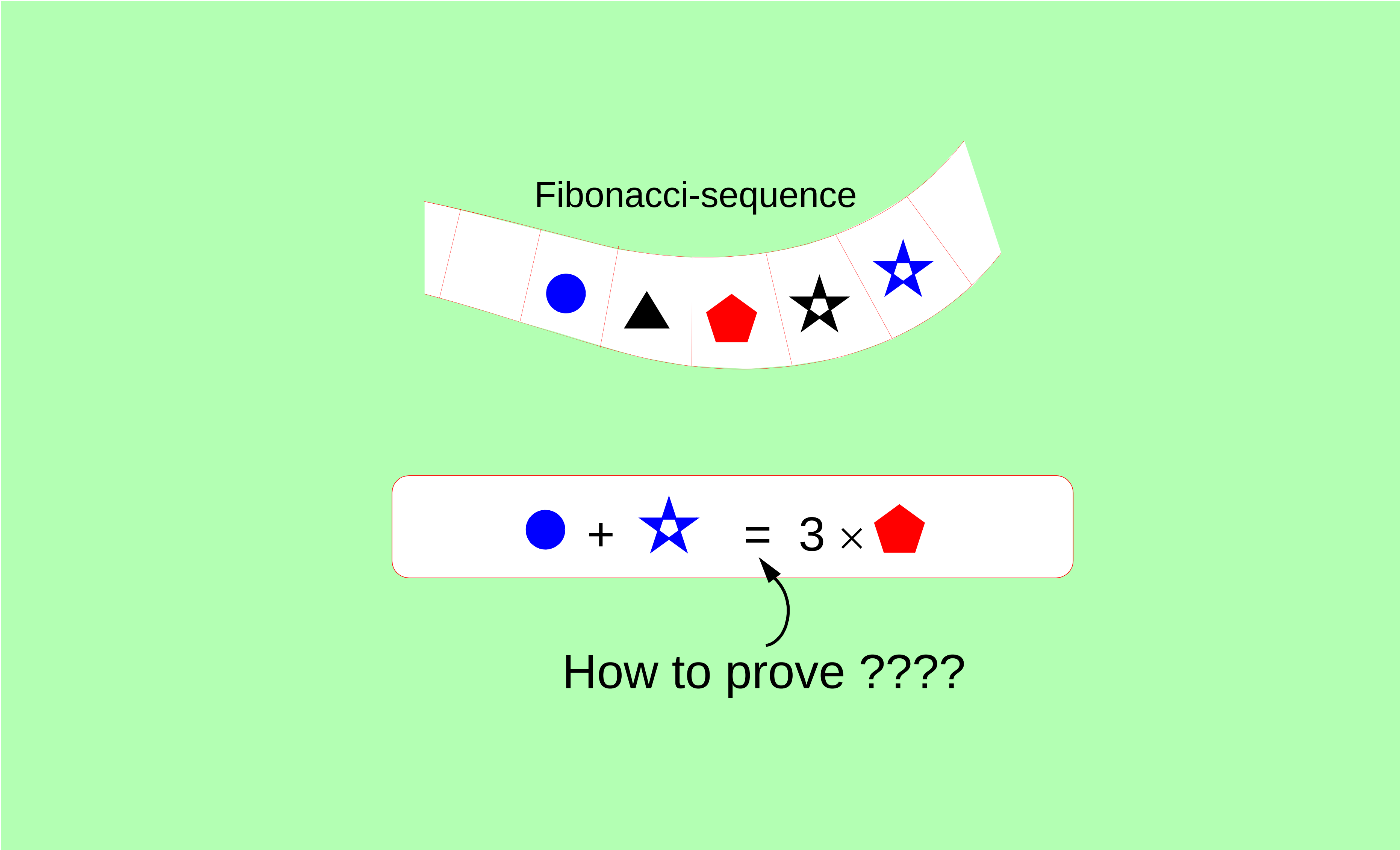
$$5 + 34 = 39 = 3 \times 13$$

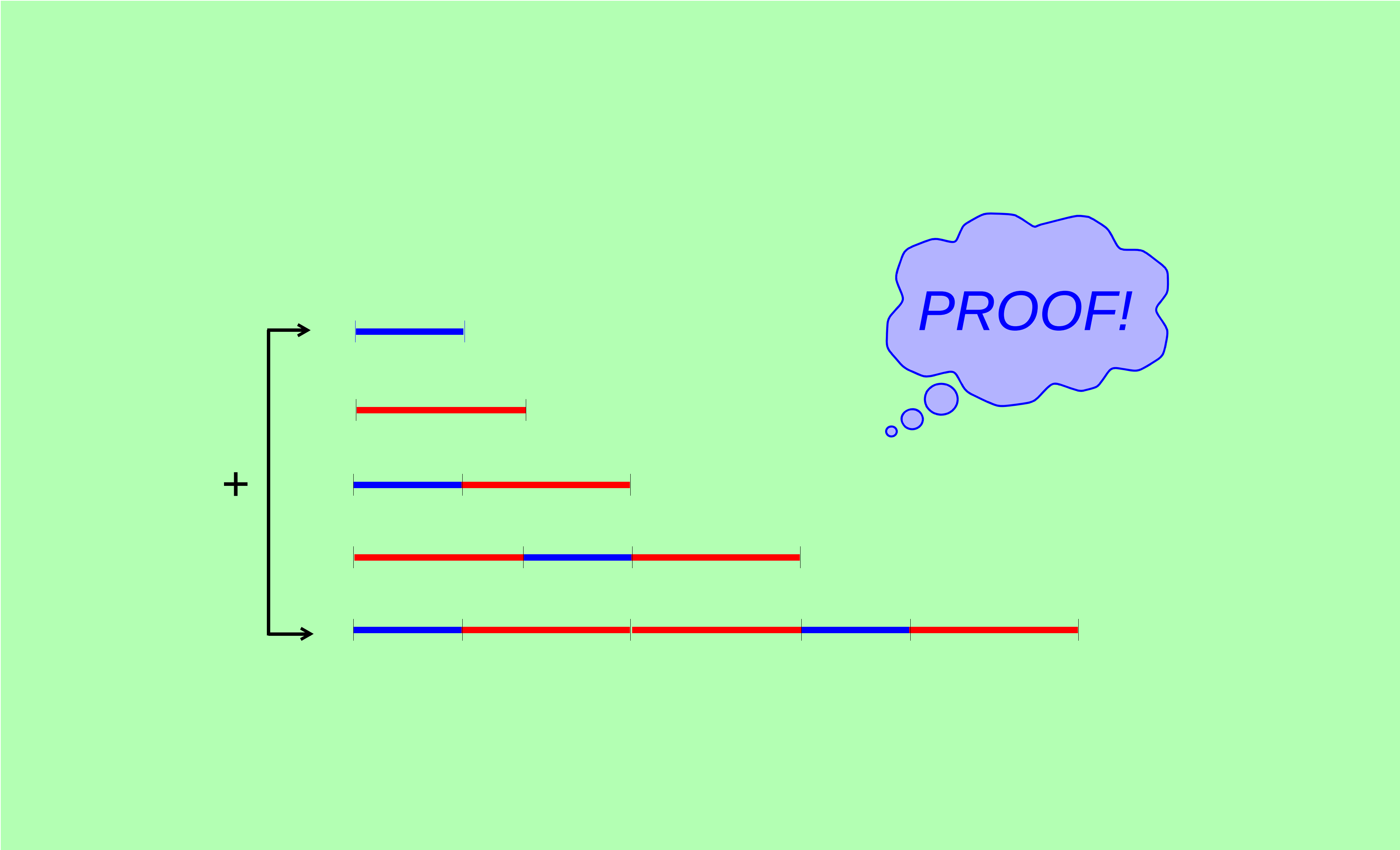


$$8 + 55 = 63 = 3 \times 21$$



$$233 + 1597 = 1830 = 3 \times 610$$


Fibonacci-sequence
+ = 3 ×
How to prove ????


PROOF!
+

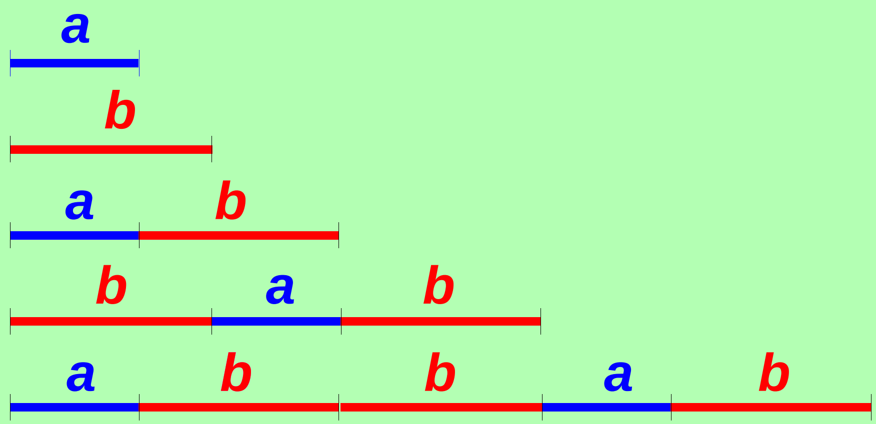
a

b

$+ \quad a + b$

$b + a + b$

$a + b + b + a + b$



$$a + (a + b + b + a + b) = (a + b) + (a + b) + (a + b)$$

$$a$$

$$b$$

$$+\quad \boxed{a + b}$$

$$a + 2b$$

$$2a + 3b$$

$$a + (2a + 3b) = 3 \times (a + b)$$

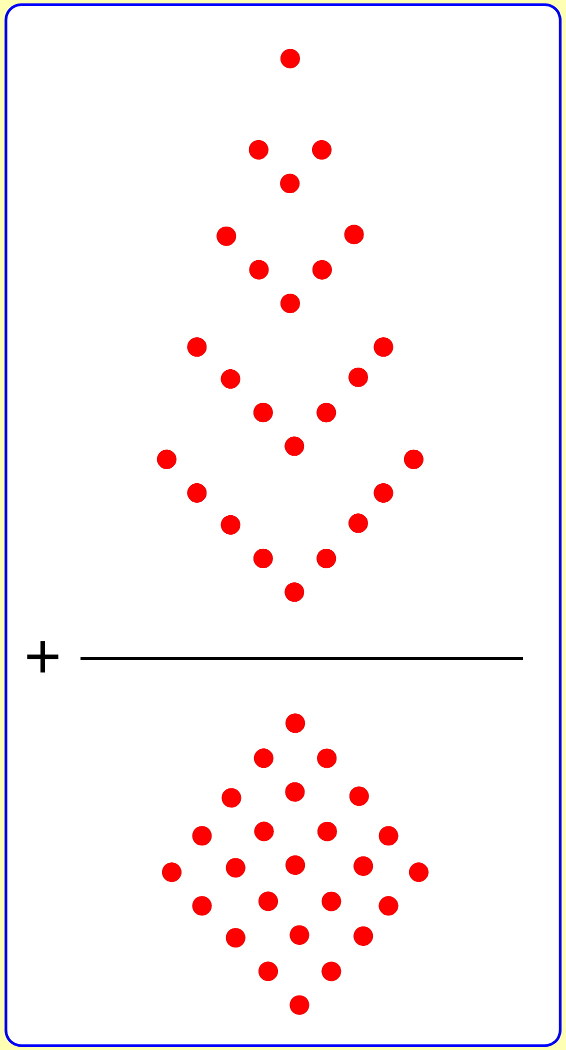
More ‘Fibonacci-exercises’

- Take any subsequence of nine consecutive numbers. Then the sum of the first and the ninth number equals 7 times the number in the middle.

- The sum of any six consecutive numbers in the sequence is exactly 4 times the fifth one.

- Design your own Fibonacci-exercise.

Back to dot-patterns

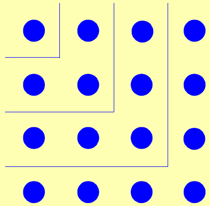


1
3
5
7
9
+
25

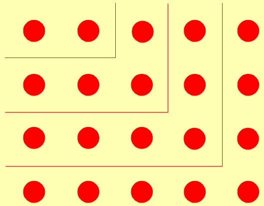
Nikomachos of Gerasa

(ca. 100 AD)

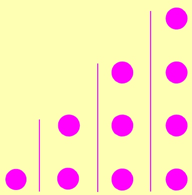
‘figurate numbers’



sum of ‘odds’ = square number



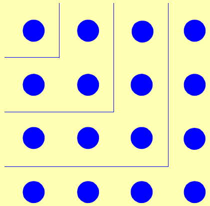
sum of ‘evens’ = oblong number



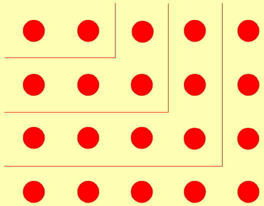
sum of consecutive numbers

=

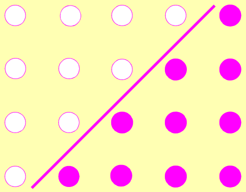
triangular number



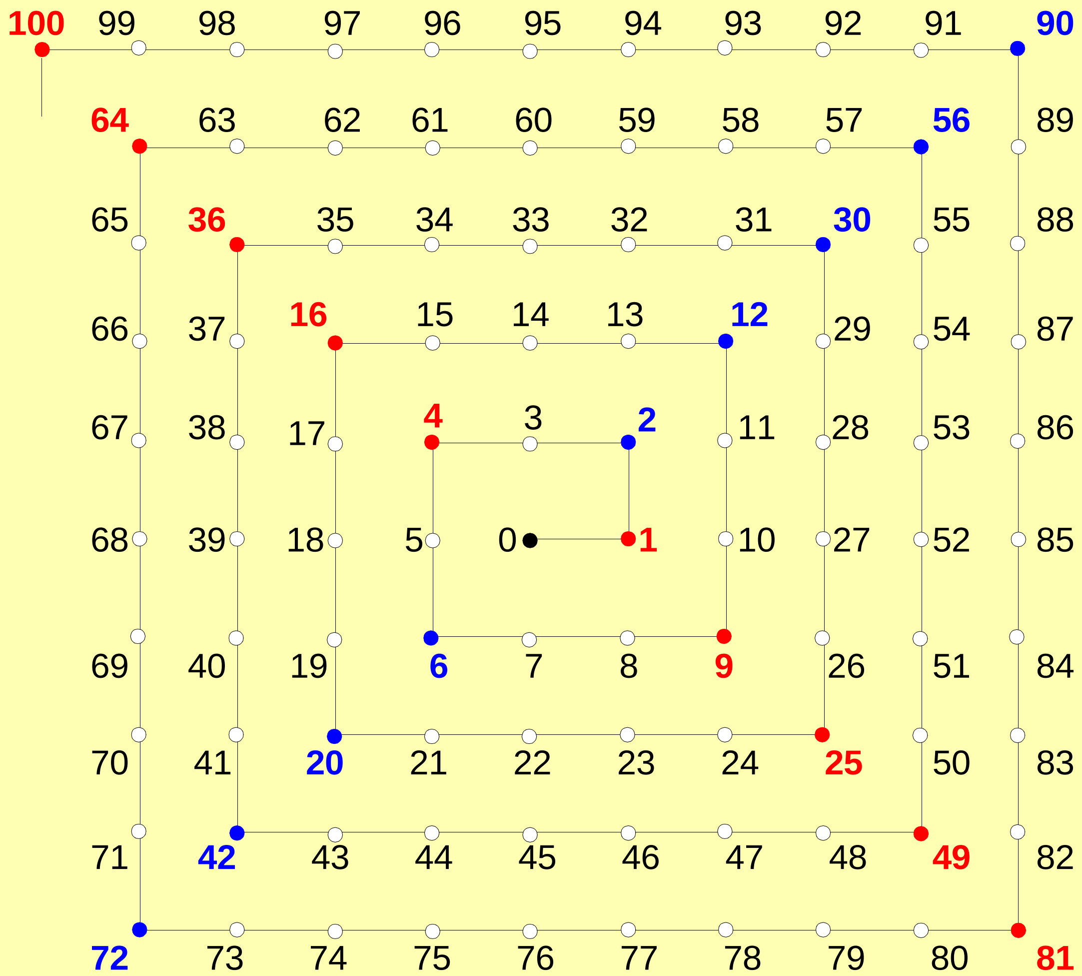
square number → n^2



oblong number → $n \times (n + 1)$

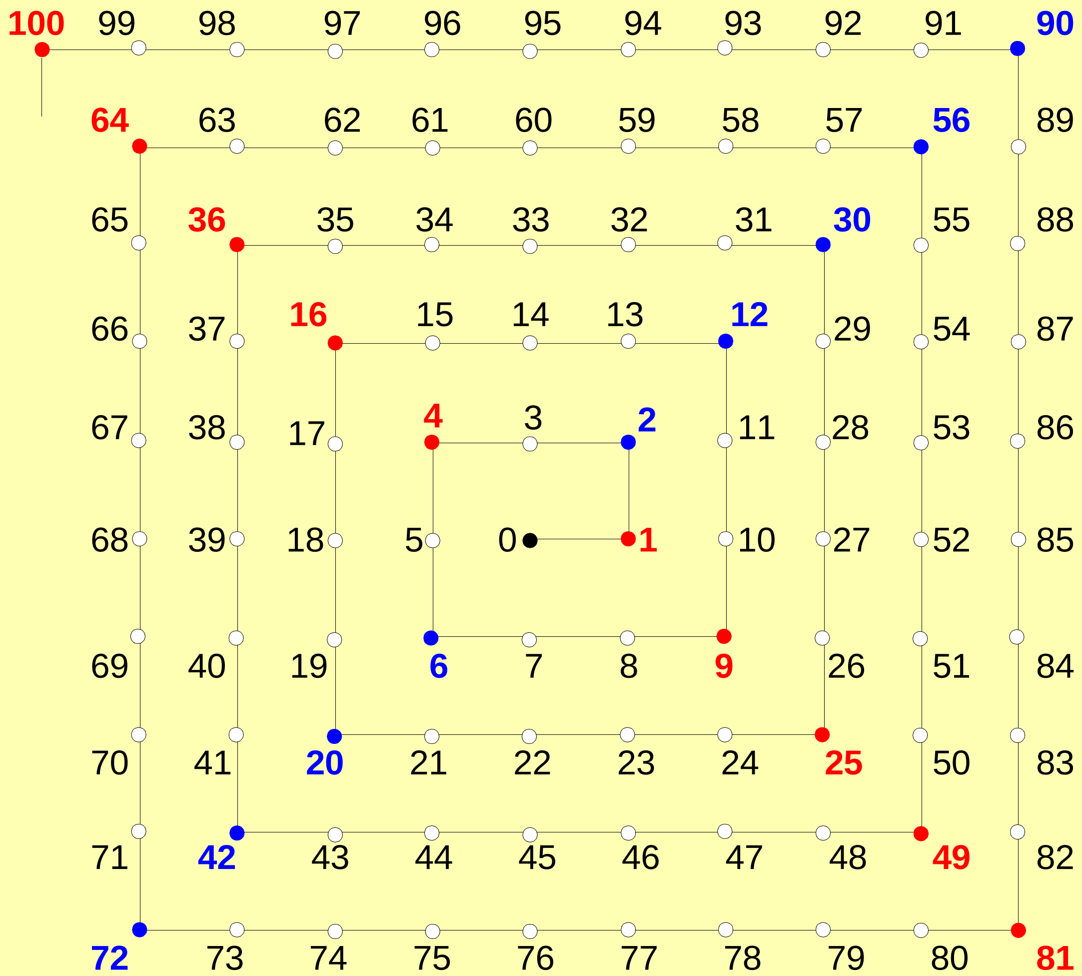


triangular number → $\frac{1}{2}n \times (n + 1)$



Continue the spiral with two more full laps.

What would be the corner numbers then?



** In the spiral you may see that every square is exactly in the middle of two oblong numbers*

How to explain this?

(representative) example:

8×8 is exactly in the middle of 7×8 and 8×9

General:

$(n-1) \times n$	$n \times n$	$n \times (n+1)$
↓	↓	↓
$n^2 - n$	n^2	$n^2 + n$

Some investigations with figurate numbers

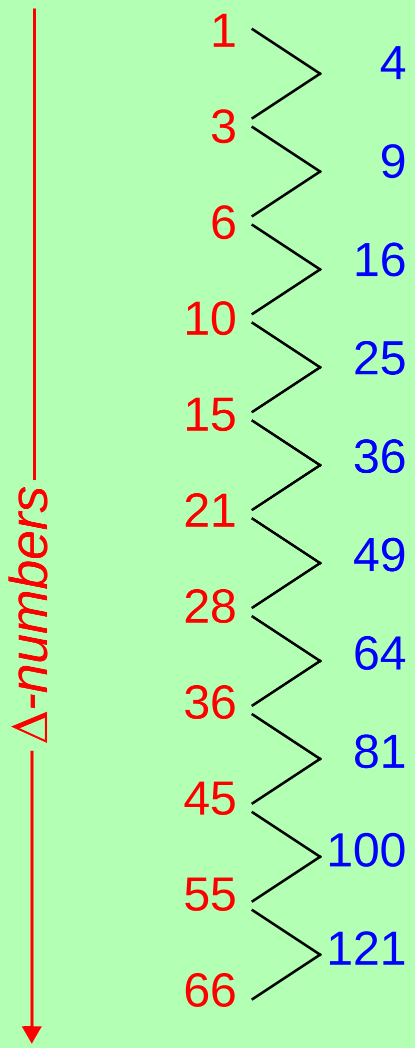
Repeated add
two subsequent
triangular numbers.
Which familiar
sequence
do you get?

Which W-numbers
are
square numbers?

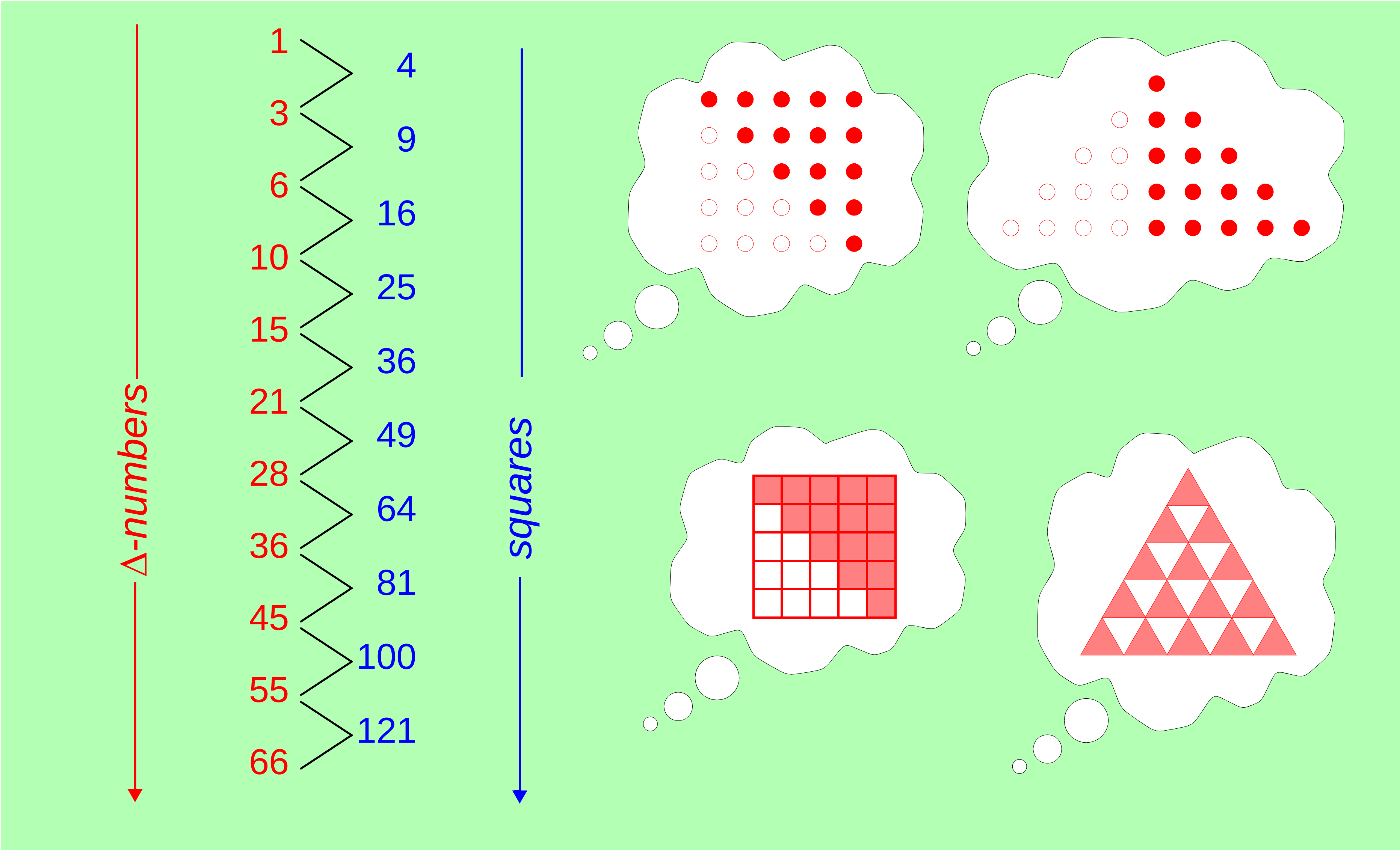
Nicomachos
introduced
pentagonal numbers:

5, 12, 22, 35, 51, ...

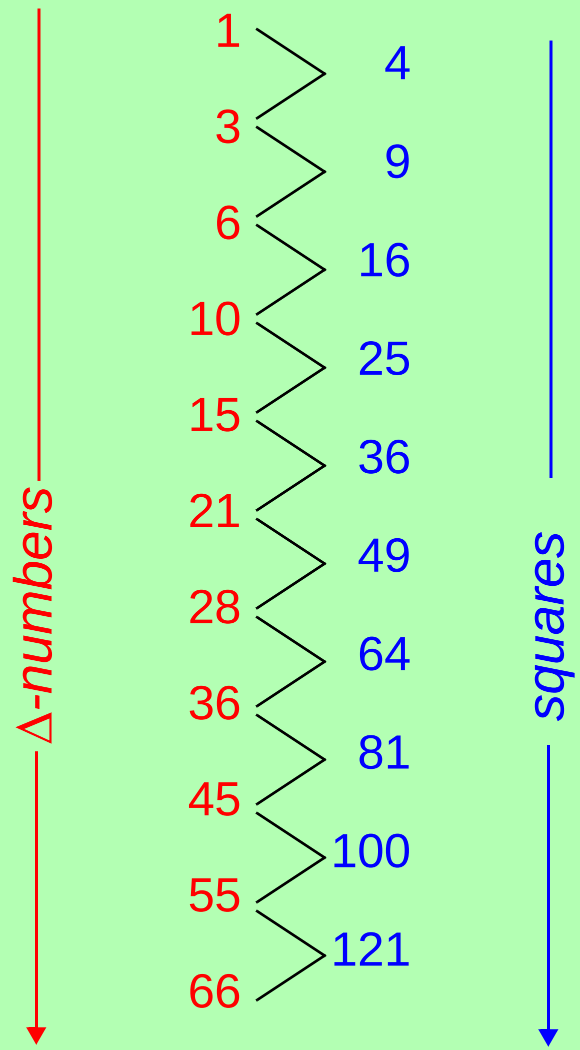
Design corresponding
dot patterns.
Which formula fits
the sequence?



Repeated add two subsequent triangular numbers .....

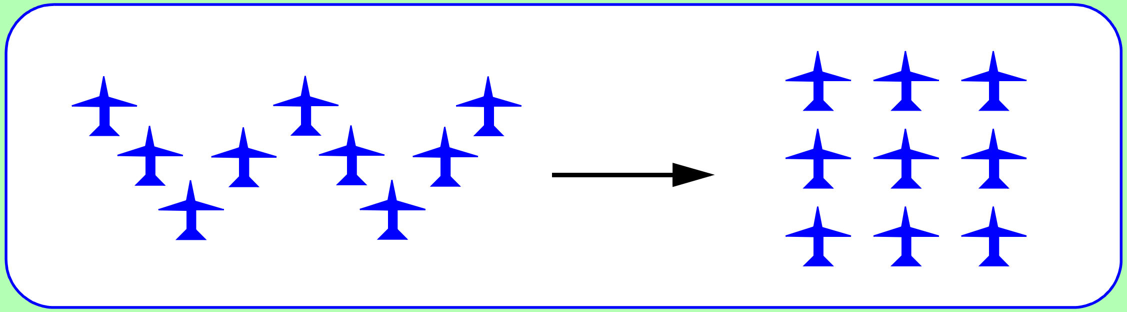


Δ-numbers
1
3
6
10
15
21
28
36
45
55
66
4
9
16
25
36
49
64
81
100
121
squares



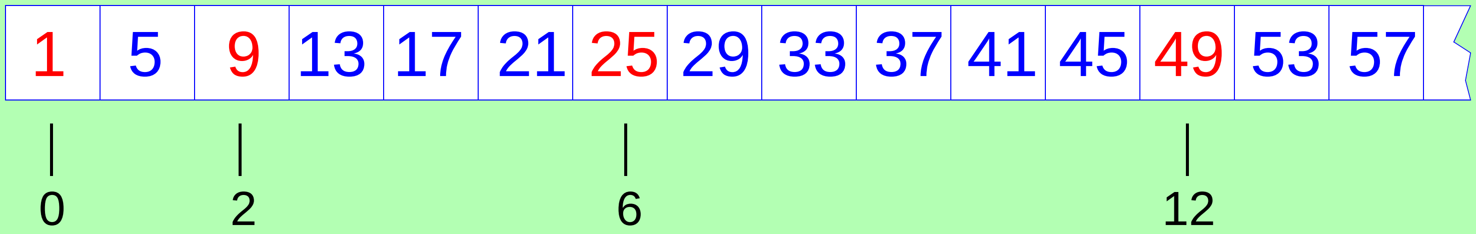
formal proof

$$\frac{1}{2}n(n-1) + \frac{1}{2}n(n+1) = n^2$$



Which W-squadrons are square(dron)s ?

Sequence of W-numbers



Sequence of W-numbers

1	5	9	13	17	21	25	29	33	37	41	45	49	53	57	

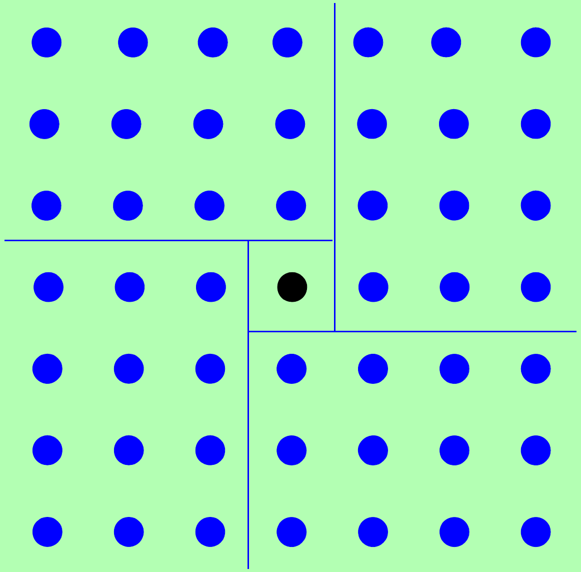
Is every odd square a W-number ?

$$
\begin{array}{r l}
2n + 1 & \\
2n + 1 & \\
\hline
 & \times \\
2n + 1 & \\
4n^2 + 2n & \\
\hline
 & + \\
\end{array}
$$

$$4n^2 + 4n + 1 = 4(n^2 + n) + 1$$

$(n^2 + n)$ → oblong number

$4 \times$ oblong number $+ 1 =$ square number

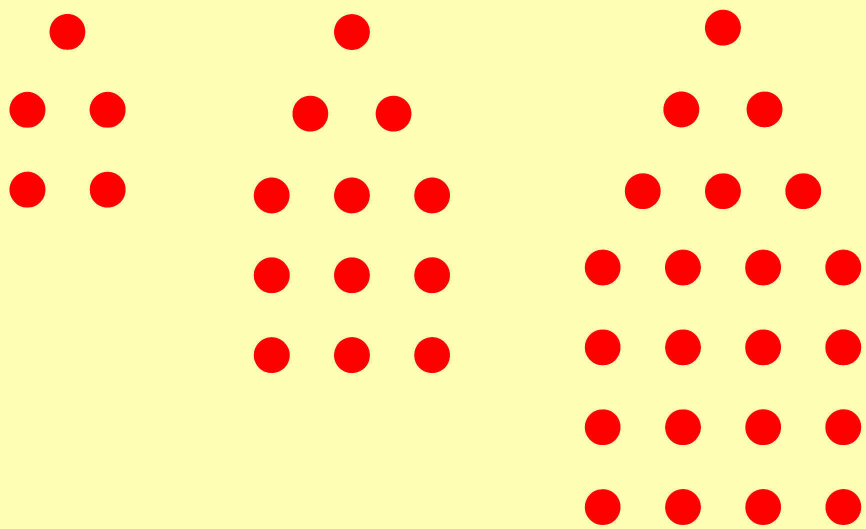


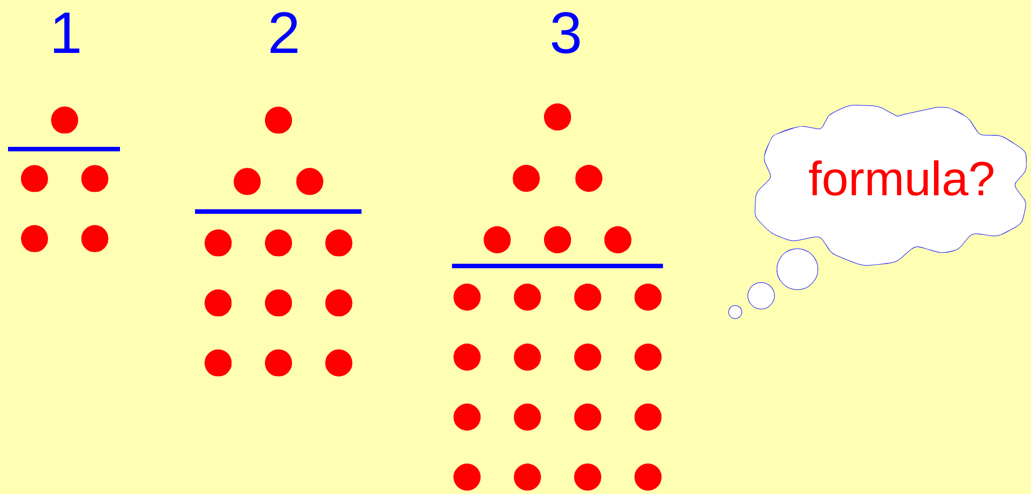
$$4 \cdot n(n+1) + 1 = (2n+1)^2$$

formula?

Pentagonal numbers

5, 12, 22, 35, 51, etc.



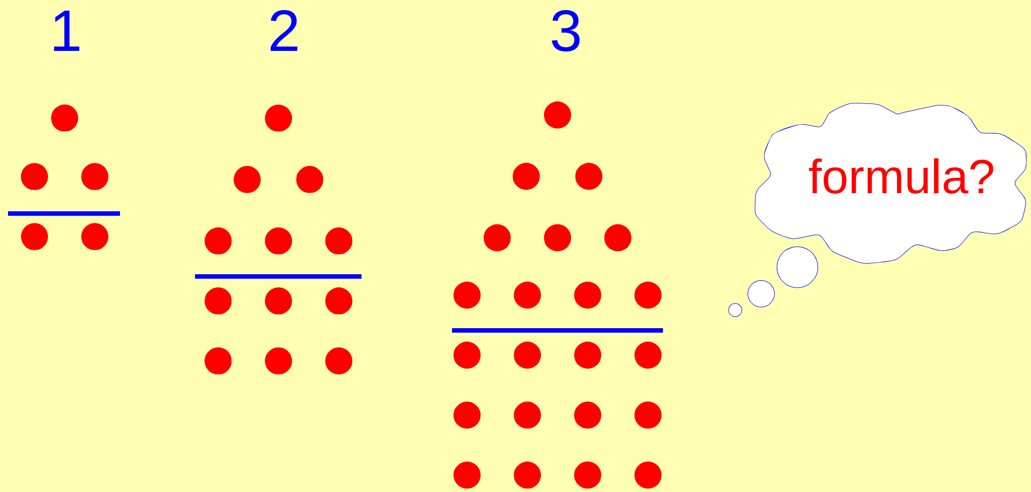


$$(n+1)^2 + \tfrac{1}{2}n(n+1)$$

$$\downarrow$$

$$\tfrac{1}{2}(n+1)(3n+2)$$

or



$$(n+1)^2 + \tfrac{1}{2}n(n+1)$$

$$\downarrow$$

$$\tfrac{1}{2}(n+1)(3n+2)$$

or

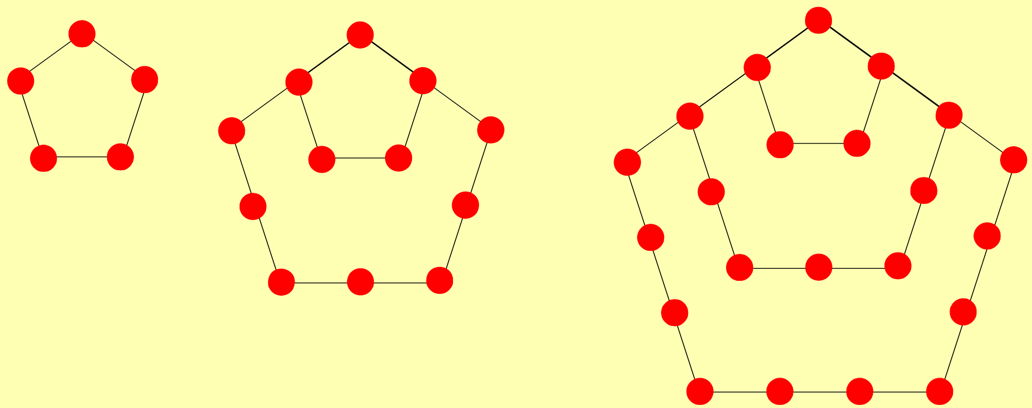
$$n(n+1) + \tfrac{1}{2}(n+1)(n+2)$$

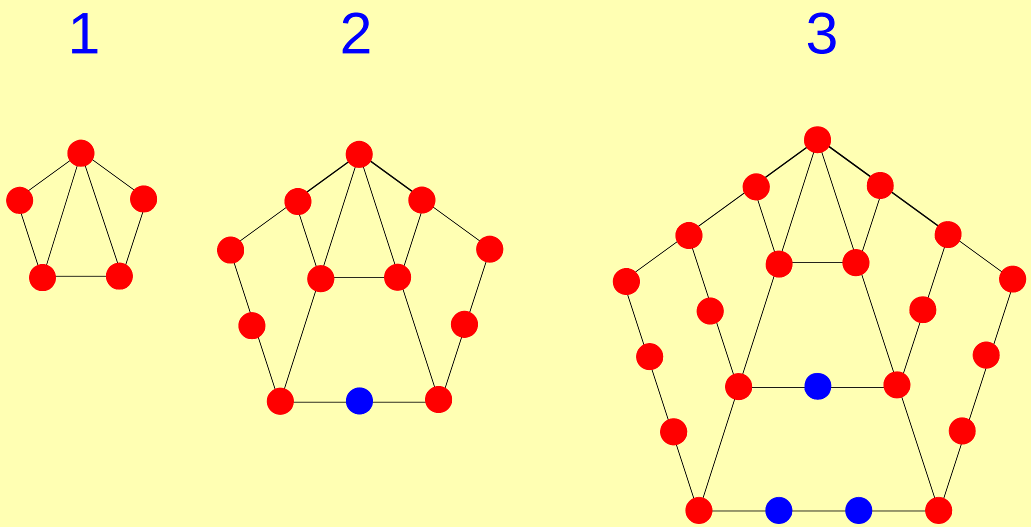
$$\downarrow$$

$$\tfrac{1}{2}(n+1)(3n+2)$$

Pentagonal numbers

5, 12, 22, 35, 51, etc.

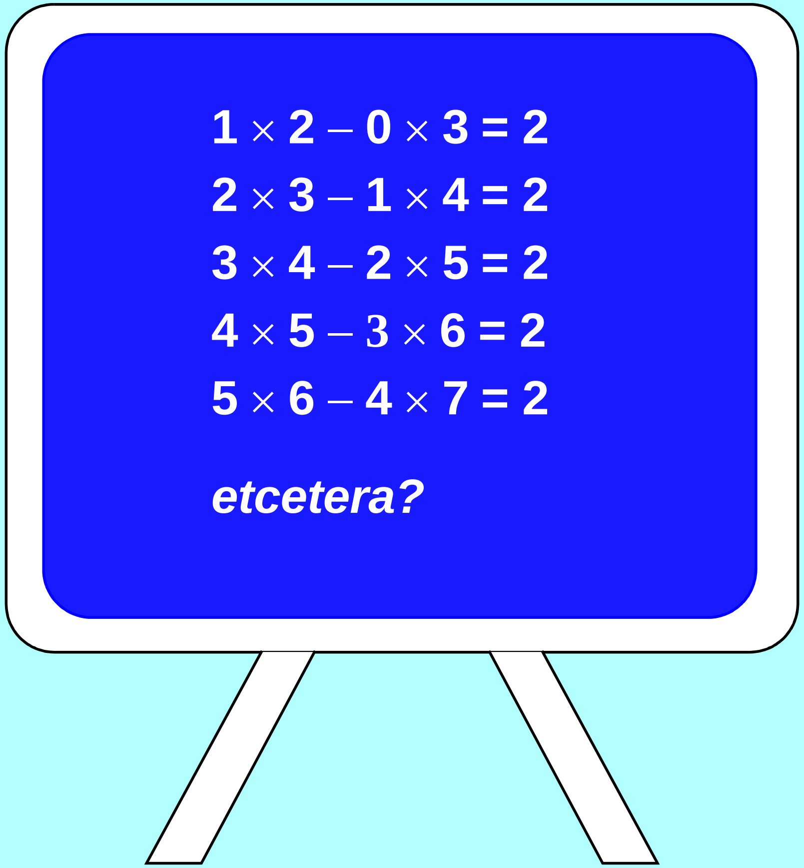


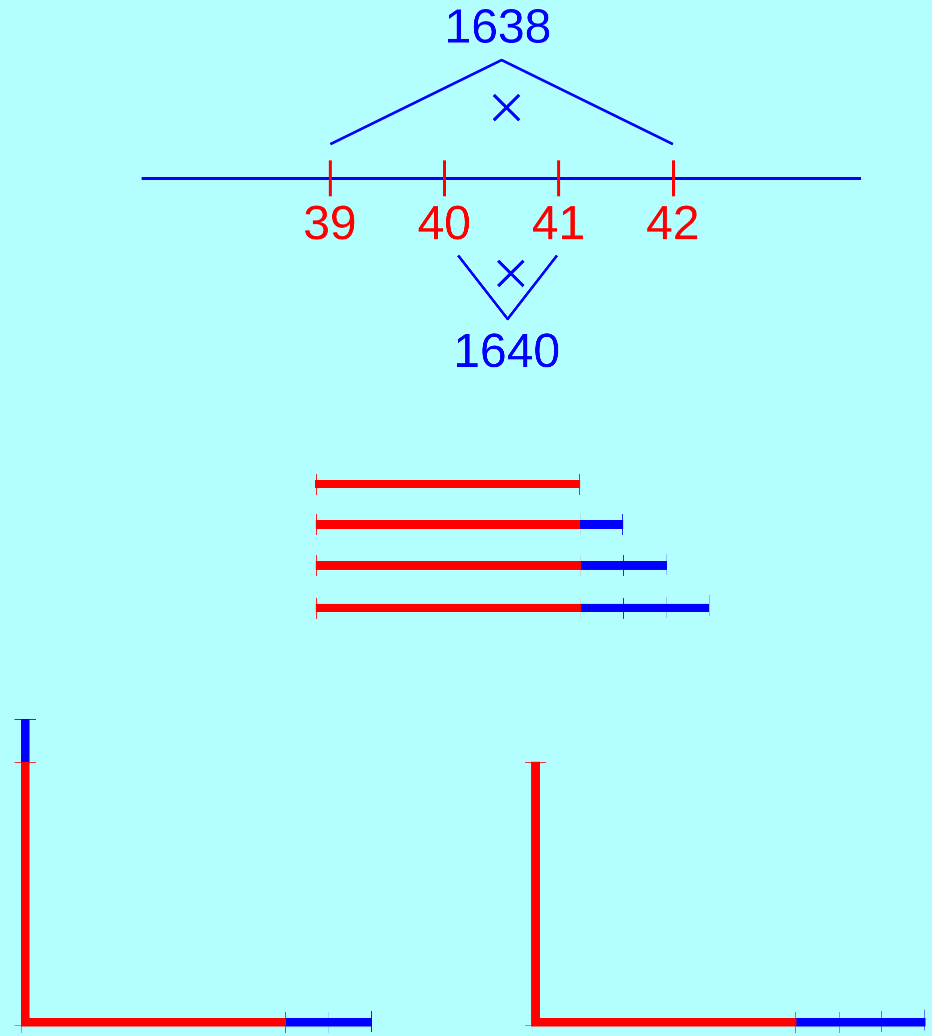


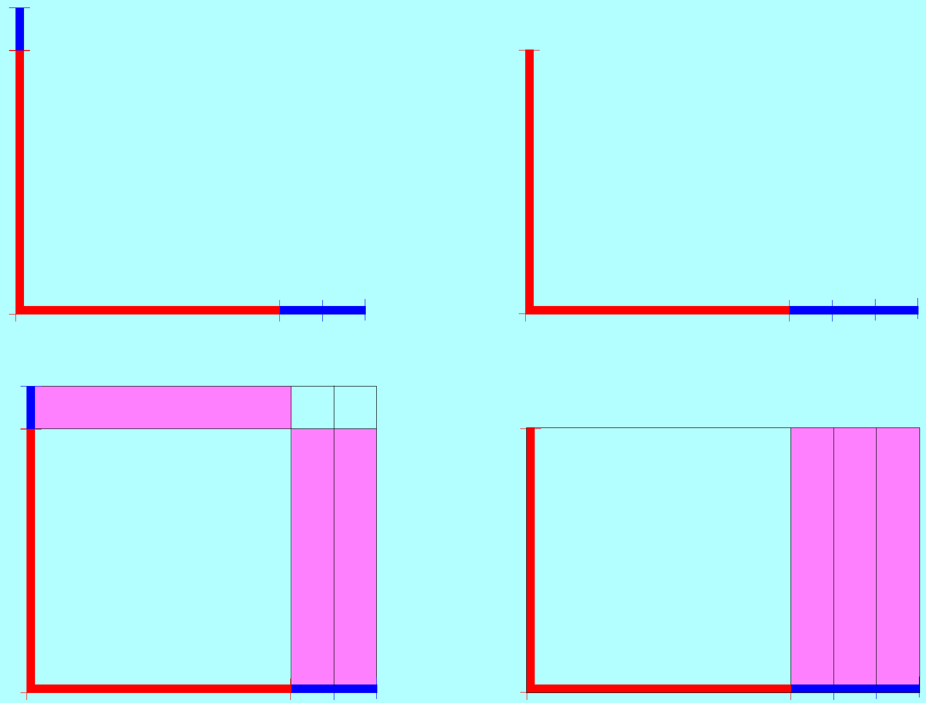
$$2 \cdot \frac{1}{2}(n+1)(n+2) - 1 + \frac{1}{2}(n-1)n$$

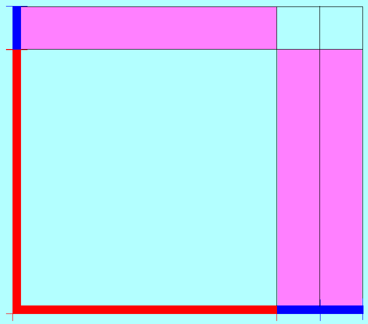
$$\downarrow ?$$

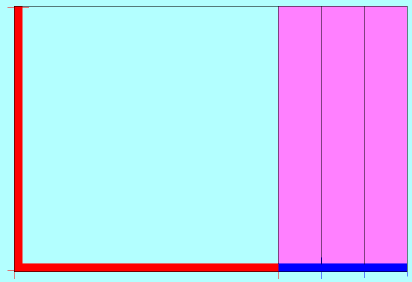
$$\frac{1}{2}(n+1)(3n+2)$$


$1 \times 2 - 0 \times 3 = 2$
$2 \times 3 - 1 \times 4 = 2$
$3 \times 4 - 2 \times 5 = 2$
$4 \times 5 - 3 \times 6 = 2$
$5 \times 6 - 4 \times 7 = 2$
etcetera?


1638
39
40
41
42
1640

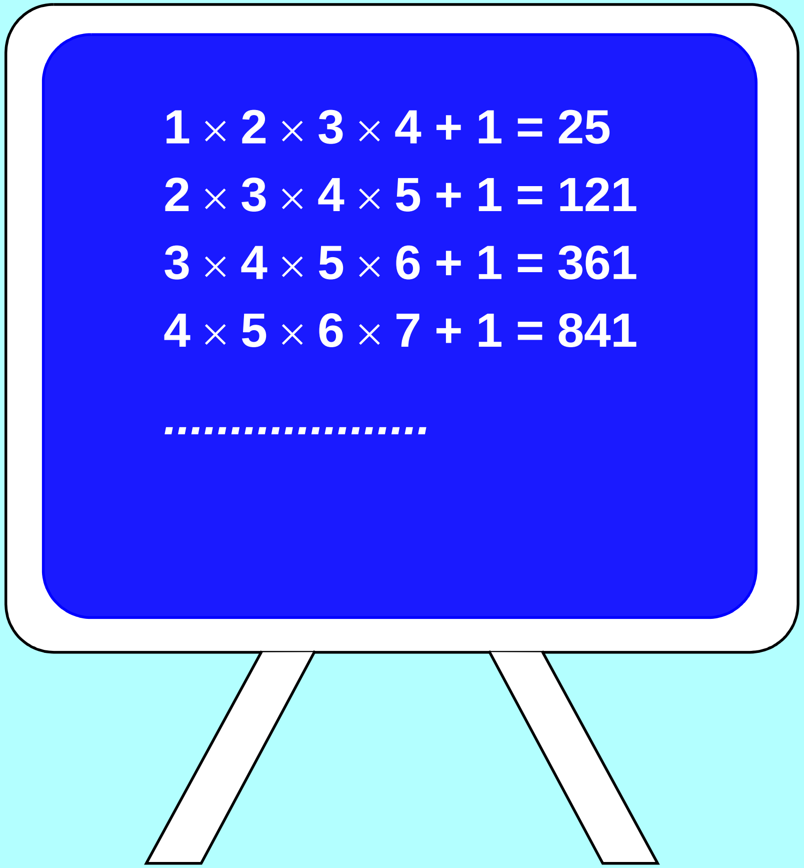






$$(n + 1)(n + 2) = n^2 + 3n + 2$$

$$n\,(n + 3) = n^2 + 3n$$



* Check that the results are squares.

* Give two lines more ...

** the product of any four consecutive numbers added to 1 seems to be a square ???*

??

$n(n+1)(n+2)(n+3)+1$ is a square

$$(n+1)\cdot(n+2)\cdot n\cdot(n+3)+1$$

$A+2$ $\quad$ A

$$A\,.\,(A+2)+1=(A+1)^2$$

$$n(n+1)(n+2)(n+3)+1=(n^2+3n+1)^2$$

It might be a good idea to do early algebra in the field of natural numbers

The mixing of algebra with fractions or negative numbers can be temporarily postponed due to their more abstract character and the resulting complications.

A teacher of mathematics has a great opportunity.
If he fills his allotted time with drilling his students in routine operations,
he kills their interest, hampers their intellectual development,
and misuses his opportunity.

But if he challenges the curiosity of his students by setting them
problems proportionate to their knowledge, and helps them to solve
their problems with stimulating questions, he may give them a
taste for, and some means of, independent thinking.

George Polya
1887-1985