BRIDGING BETWEEN ARITHMETIC AND ALGEBRA: USING PATTERNS TO PROMOTE ALGEBRAIC THINKING

A THESIS

Submitted as a Partial Fulfillment of the Requirements for the Degree of Master of Science (M.Sc.)

in

International Master Program on Mathematics Education (IMPoME)

Faculty of Teacher Training and Education Sriwijaya University

(In collaboration between Sriwijaya University and Utrecht University)

By:
Ratih Ayu Apsari
NIM 06022681318077

FACULTY OF TEACHER TRAINING AND EDUCATION

SRIWIJAYA UNIVERSITY

JULY 2015

APPROVAL PAGE

Research Title	:	Bridging Between Arithmetic and Algebra: Using Patterns to Promote Algebraic Thinking
Student Name	:	Ratih Ayu Apsari
Student Number	:	06022681318077
Study Program	:	Mathematics Education

Approved by:

Prof. Dr. Ratu Ilma Indra Putri, M.Si. Supervisor I	Dr. Darmawijoyo, M.Sc., M.Si. Supervisor II
Dean of Faculty of Teacher Training and Education	Head of Mathematics Education Department
Prof. Sofendi, M.A., Ph.D. NIP 19600907 198703 1 002	Prof. Dr. Ratu Ilma Indra Putri, M.Si. NIP 19690814 199302 2 001

Date of Approval: July 2015

BRIDGING BETWEEN ARITHMETIC AND ALGEBRA:
USING PATTERNS TO PROMOTE ALGEBRAIC THINKING

A THESIS

Submitted as a Partial Fulfillment of the Requirements for the Degree of Master of Science (M.Sc.)
in
International Master Program on Mathematics Education (IMPoME)
Faculty of Teacher Training and Education Sriwijaya University
(In collaboration between Sriwijaya University and Utrecht University)

By:
Ratih Ayu Apsari
NIM 06022681318077

Approved by Examination Committee	**Signature**
Prof. Dr. Ratu Ilma Indra Putri, M.Si. **Sriwijaya University**	------------------
Dr. Darmawijoyo, M.Sc., M.Si. **Sriwijaya University**	------------------
Prof. Dr. Zulkardi, M.I.Komp., M.Sc. **Sriwijaya University**	------------------
Dr. Yusuf Hartono **Sriwijaya University**	------------------
Dr. Somakim, M.Pd. **Sriwijaya University**	------------------

STATEMENT PAGE

I hereby:

Name	: Ratih Ayu Apsari
Place of birth	: Denpasar
Date of birth	: April 22, 1991
Academic Major	: Mathematics Education

State that:

1. All the data, information, analyses, and the statements in analyses and conclusions that presented in this thesis, except from reference sources, are the results of my observations, researches, analyses, and views with the guidance of my supervisors.
2. The thesis that I had made is original of my mind and has never been presented and proposed to get any other degree from Sriwijaya University or other Universities.

This statement was truly made and if in other time that found any fouls in my statement above, I am ready to get any academic sanctions such as, cancelation of my degree that I have got through this thesis.

Palembang, July 2015
The one with the statement

Ratih Ayu Apsari
NIM 06022681318077

ABSTRACT

The preliminary research by Jupri, Drijvers, & van den Heuvel-Panhuizen (2014) found that the gap between arithmetic in primary level and algebra learning in higher level is contribute to a number of difficulties on learning algebra in Indonesia. Hence, the aim of the present study is to construct a smoother bridge to support the students' transition from arithmetic to algebra, through an early algebraic lesson. The aforementioned purpose is actualized by designing a Hypothetical Learning Trajectory (HLT) focused on patterns related activities for the fifth grader students (10-12 years old) in order to gain an insight of how the students develop their algebraic thinking. Reflect to the aim, design research was deliberately chosen as the approach to conduct this study. The designed instructional activities are guided by the design heuristic and the tenets of Pendidikan Matematika Realistik Indonesia (PMRI) which is adopted from Realistic Mathematics Education (RME) approach. The local context of Indonesian traditional dancing is chosen as the general theme of the investigational problems due to its familiarity to the Indonesian students' experiences. The data were gathered from video registrations, students' written works, field notes, observations and interviews from the implementation of two cycles teaching experiment which was involving the students of VA and VB class of MIN 2 Palembang academic year 2014/2015 and their mathematics teacher. The data were analyzed qualitatively using constant comparative method. The result showed that exploration in patterns can support the students in developing their sense of structure which is remarkably influence the students' algebraic thinking. Reflect from that, the study recommends the use of patterns activities based on the PMRI approach to be implemented in the early algebraic lesson in Indonesia's primary school.

Keywords: algebra, early algebra, algebraic thinking, patterns, PMRI, RME, Hypothetical Learning Trajectory, design research

ABSTRAK

Penelitian pendahulu yang dilakukan oleh Jupri, Drijvers, & van den Heuvel-Panhuizen (2014) menemukan bahwa kesulitan siswa Indonesia dalam belajar aljabar disebabkan oleh ketidakterkaitan antara aritmatika yang dibelajarkan di sekolah dasar dan aljabar yang dibelajarkan di sekolah menengah. Oleh karenanya, tujuan dari penelitian ini adalah untuk membentuk suatu jembatan penghubung yang dapat membantu transisi siswa dari aritmatika ke aljabar, melalui suatu pembelajaran pra-aljabar. Tujuan tersebut diaktualisasi dengan mendesain suatu trajektori pembelajaran yang disebut dengan dugaan lintasan belajar (Hypothetical Learning Trajectory) yang fokus pada kegiatan penelusuran pola untuk siswa kelas V SD (usia 10-12 tahun) untuk dapat memahami bagaimana siswa mengembangkan pola pikir aljabar mereka. Merefleksi tujuan tersebut, penelitian ini menggunakan design research sebagai metode penelitian. Adapun lintasan pembelajaran yang didesain menggunakan prinsip dari Pendidikan Matematika Realistik Indonesia (PMRI) yang diadopsi dari pendekatan Realistic Mathematics Education (RME). Dalam penelitian ini digunakan konteks lokal Indonesia, yakni formasi tari tradisional yang dekat dengan keseharian siswa. Data yang diperlukan dikumpulkan melalui video, lembar jawaban siswa, catatan lapangan, lembar observasi dan wawancara selama implementasi teaching experiment yang dilaksanakan dalam dua siklus. Penelitian ini melibatkan siswa kelas VA dan VB MIN 2 Palembang tahun ajaran 2014/2015, beserta dengan guru matematikanya. Data yang diperoleh dianalisis secara kualitatif dengan menggunakan metode konstan komparatif. Hasil analisis menunjukkan bahwa kegiatan penelusuran pola dapat mendukung siswa dalam mengembangkan kepekaan terhadap struktur (structure sense) yang mempengaruhi kemampuan berpikir aljabar siswa. Merefleksi temuan tersebut, penelitian ini merekomendasikan implementasi kegiatan penelusuran pola berbasis PMRI pada jenjang sekolah dasar di Indonesia.

Kata-kata kunci: aljabar, pra-aljabar, berpikir aljabar, pola, PMRI, RME, dugaan lintasan belajar, design research

INTERNATIONAL MASTER PROGRAM ON MATHEMATICS EDUCATION
FACULTY OF TEACHER TRAINING AND EDUCATION
SRIWIJAYA UNIVERSITY
A Thesis
July 2015

Ratih Ayu Apsari

Bridging Between Arithmetic and Algebra:
Using Patterns to Promote Algebraic Thinking

SUMMARY

Algebra is acknowledged as a gatekeeper of all high school mathematics (Brawner, 2012). Unfortunately, teaching and learning algebra in many countries, including Indonesia, are struggle due to a number of difficulties. Egodawatte (2011) pointed the problems as: lack understanding of the meaning of variables, making sense of the relation between words problems and algebraic expressions and incorrectly processing the elements in algebraic equations. Jupri, Drijvers, & van den Heuvel-Panhuizen (2014) observed the same phenomenon in Indonesia and specifically mentioned that the difficulties is caused by the gap between arithmetic and algebraic lesson. Hence, the aim of the present study is to provide a smoother bridge to support the students' transition from arithmetic to algebraic thinking by designing a learning trajectory in early algebra. This study also want to get an insight of how the students develop their algebraic thinking during the learning process using the designed instruction.

Early algebra is algebra which is taught in early ages, which aims is to provide experience in working with the number structures. In this study we employed a patterns activity as the main topic of learning exploration. Patterns is suitable choice of pre-algebraic activity due to its "dynamical representation of variables" (Lanin, 2005; p.233). In addition, the patterns we used in this study is embodied in the visual representation.

In order to accomplish the goals, a Hypothetical Learning Trajectory (HLT) on patterns activity was designed. The activities in HLT were designed based on the design heuristics and tenets of Pendidikan Matematika Realistik Indonesia (PMRI) which is adapted from Realistic Mathematics Education (RME) approach. Moreover, since the purpose of the present study is to *design* a local instruction theory in early algebra and to get an insight of *how* the students develop their algebraic thinking, design research was deliberately chosen as the method of the study. Hence, we employed three steps of design research, namely: preparation and design, teaching experiment and retrospective analysis. The research was conducted by involving the fifth grader students of MIN 2 Palembang and their mathematics teacher. The data of teaching experiment were gathered through video registrations, students' written works, field notes, observations and interviews. The

gathered data were qualitatively analyzed in retrospective analysis phase, by employing constant comparative method.

Based on the data, we found that the organization of the task and the choice of the number are powerful to encourage the students' shifting from basic strategy to more advance one. The basic strategies are indicating the dominant of arithmetical thinking, while when the students are able to think out of the "merely continuing the sequences", the algebraic thinking started to be built on them. From the analysis, we found that the way the students perceive the structure of the given visual representations contributes to their way in generalizing the pattern itself. Different point of views may lead the students to different levels of making the "general" statement. The general strategies of doing generalization used by the students in this study can be classified into: drawing, listing, constructing a recursive formula and constructing a general formula. For some students the four aforementioned general strategies are come in the sequential order.

Reflect to the result of the study, we concluded that the patterns activity are able to promote the students' development of algebraic thinking by encourage them to observe the structure of the number. This is a step further than a usual arithmetical task that commonly learnt by the students. In the same time, it is close as well with the field of algebra. Hence, the patterns activity is a powerful bridge to support the students' transition from arithmetic to algebra.

INTERNATIONAL MASTER PROGRAM ON MATHEMATICS EDUCATION
FAKULTAS KEGURUAN DAN ILMU PENDIDIKAN
UNIVERSITAS SRIWIJAYA
Tesis
Juli 2015

Ratih Ayu Apsari

Bridging Between Arithmetic and Algebra:
Using Patterns to Promote Algebraic Thinking

RINGKASAN

Aljabar dikenal sebagai pintu gerbang dari semua cabang matematika yang lebih tinggi (Brawner, 2012). Akan tetapi pada kenyataannya pembelajaran aljabar di banyak Negara di dunia, termasuk di Indonesia masih menemui sejumlah kendala. Sebagaimana yang disebutkan oleh Edogawatte (2011), masalah dalam pembelajaran aljabar dapat dikelompokkan atas: kurangnya pemahaman akan makna variabel, mengaitkan antara kalimat sehari-hari dan representasi aljabarnya, serta ketidaktepatan dalam memperlakukan elemen-elemen dalam suatu persamaan aljabar. Terkait hal tersebut, penelitian yang dilakukan oleh Jupri, Drijvers, & van den Heuvel-Panhuizen (2014), menemukan hasil serupa di Indonesia dan menyimpulkan bahwa hal tersebut disebabkan oleh kesenjangan antara pembelajaran aritmatika dan aljabar. Oleh karenanta penelitian ini bertujuan untuk menghasilan suatu jembatan transisi yang dapat membantu menghubungkan pembelajaran aritmatika dan aljabar. Hal ini penulis lakukan dengan mendesain suatu lintasan belajar pada cabang pra-aljabar. Sejalan dengan itu, penelitian ini juga bertujuan untuk mengetahui bagaimana siswa mengembangkan pola pikir aljabarnya ketika mengikuti pembelajaran sesuai dengan lintasan yang dirancang.

Pra-aljabar merupakan aljabar yang dibelajarkan di kelas awal, yang bertujuan untuk memberikan pengalaman belajar dengan struktur bilangan. Dalam penelitian ini kami menggunakan aktivitas penulusuran pola. Penelusuran pola dipilih atas kemampuan dinamisnya dalam merepresentasi variable (Lanin, 2005). Lebih lanjut, pola-pola yang digunakan diilustrasikan dalam suatu representasi visual berbentuk bangun geometri untuk memberikan lebih banyak ruang dan bantuan sehingga siswa dapat mengeksplorasi struktur yang terkandung dalam pola tersebut.

Untuk mencapai tujuan yang diharapkan, dalam penelitian ini disusun suatu hipotesis penelitian yang disebut *Hypothetical Learning Trajectory* (HLT). Hipotesis ini bersifat fleksibel, yakni bisa mengalami penyesuaian selama kegiatan penelitian. Aktivitas yang didesain berangkat dari prinsip-prinsip Pendidikan Matematika Realistik Indonesia (PMRI) yang diadopsi dari pendekatan *Realistic Mathematics Education* (RME). Design research dipilih sebagai metode penelitian mengingat tujuan dari studi ini adalah untuk mendesain suatu lintasan belajar dan mengetahui bagaimana siswa membangun kemampuan berpikir aljabarnya. Dengan

demikian, tiga tahapan design research yang meliputi: persiapan, implementasi pembelajaran dan analisis retrospektif diterapkan pula dalam penelitian ini. Tahap implemetasi dilakukan dalam dua siklus yang melibatkan siswa kelas V MIN 2 Palembang dan guru matematika di kelas tersebut. Data dikumpulkan melalui video registrasi, lembar jawaban siswa, catatan lapangan dan observasi serta wawancara yang dilakukan selama implementasi. Data tersebut kemudian dianalisis secara kualitatif dengan menggunakan metoda perbadingan konstan.

Berdasarkan data yang diperoleh, kami melihat bahwa pengorganisasian tugas dan pemilihan bilangan berkontribusi dalam merangsang siswa untuk berpindah dari strategi dasar dan sederhana ke strategi dengan tingkat pemikiran yang lebih tinggi. Strategi yang sederhana menandai kecenderungan siswa dalam menggunakan pola pikir aritmatika. Apabila pada suatu fase siswa berhenti dalam melanjutkan pola dengan menghitung satu per satu dan kemudian berusaha mencari bentuk umumnya, saat itulah pola pikir aljabar mulai terbentuk. Dari analisis yang dilakukan, ditemukan bahwa cara pandang siswa terhadap struktur dari representasi visual yang diberikan berdampak pada level generalisasi yang dihasilkan. Secara umum, siswa menggunakan strategi berikut: menggambar, mendaftar, membuat rumus rekursif dan membuat rumus umum, dalam menyelesaikan masalah yang berkaitan dengan pola. Pada beberapa siswa strategi tersebut muncul berurutan sebagai fase perkembangannya.

Merefleksi hasil dari penelitian ini, penulis menyimpulkan bahwa kegiatan penelusuran pola dapat membantu siswa dalam mengembangkan kemampuan berpikir aljabar dengan cara menimbulkan rasa ingin tahu mereka untuk mencari struktur dari suatu bilangan. Hal ini selangkah lebih maju dibandingkan dengan tugas aritmatika biasa yang umum dipelajari siswa sekolah dasar. Di satu sisi, kegiatan generalisasi pada penelusuran pola ini juga dekat dengan topik aljabar, Dengan demikian, kegiatan penelusuran pola merupakan suatu alternatif untuk menjembatani proses transisi siswa dari aritmatika ke aljabar.

Dedicated for my parents,

Ni Ketut Suasih & Marwan Luthfy

and my sisters,

Kartika Ayu Permata Sari & Surya Ayu Audina

for their unconditional love and supports

PREFACE

First and foremost, I praise the Almighty God, for blessing and providing me the opportunities to do all the requirements of my study.

Two years have been passed since the first time I came to Palembang and start a new chapter in my life. With the guidance and support from many people, I had a chance to study aboard in Utrecht and finally return here to finish my thesis. This thesis is submitted as the partial fulfilment of the requirements for a master's degree on mathematics education in collaboration between Sriwijaya University and Utrecht University. Therefore, in this occasion I would like to express my sincerely thanks to the people who highly contribute to make this thesis possible and generally for those who make my learning journey become so unforgettable.

First of all for my Dutch supervisors: Mieke Abels, Maarten Dolk, and Dolly van Eerde for the continuous support during my study and research. Thank you for always try to understand my complicated way of thinking.

I also would like to express my gratitude to my Indonesian supervisors: Prof. Dr. Ratu Ilma Indra Putri, M.Si., and Dr. Darmawijoyo for a meaningful discussion and encouragement to finish my study on time. Besides of my advisors, a sincere thanks also addressed to the rest of my thesis defense committee, Prof. Dr. Zulkardi, M.I.Komp., M.Sc., Dr. Yusuf Hartono, and Dr. Somakim, M.Pd., for an insightful comments to enhance the quality of this thesis.

My special gratitude goes to all lecturers and staffs of Freudenthal Institute for Science and Mathematics Education (FIsme), Utrecht University. Because of your kindness and very warm welcome, the whole year I spent in hard work turn into a lovely memories. Especially for Martin Kindt, Frans van Galen, and Monica Wijers for the inspiration, advices, and encouragement during the preparation of the research. For Mark Uwland, who patiently checking my English and help me with the administrational issues.

I am indebted to the PMRI team, DIKTI, and Nuffic Neso for the chance and support given to me to continue my study. Thank you for Prof. R.K. Sembiring for the inspiration and the chance given to me to be selected as the awardee of the IMPoME scholarship. My sincere gratitude goes to my lecturer in Universitas Pendidikan Ganesha: Dr. I Wayan Sadra, M.Ed., Prof. Dr. I Nyoman Sudiana M.Pd., and Prof. Dr. Drs. I Wayan Muderawan, M.S., for the recommendations, supports and opportunities to me to follow the selection of the scholarship.

I also want to thank Prof. Dr. Badia Perizade, MBA., as the Rector of Sriwijaya University; Prof. Sofendi, M.A., Ph.D., as the Dean of the Teacher Training and Education Faculty of Sriwijaya University; and Prof. Dr. Hilda Zulkifli, M.Si., DEA., as the Director of Graduate Program Sriwijaya University. For all lecturers

and staffs of the mathematics education department. For Mbak Tessy who gave many supports during my study.

My sincere gratitude to the headmaster of MIN 2 Palembang, Budiman, S.Pd., M.M.Pd., and all teachers, staffs and students who friendly welcome me during my research. Especially for Istiarti Sri Sa'diah, S.Pd.I., the mathematics teacher who participate in my study. Thank you for the great cooperation.

My study live will never complete without the existence of a super amazing friends of IMPoME and BIMPoME Batch V. Thank you for being so helpful during our togetherness. The days will be different without you all.

And of course, for my sister-friends: Ni Putu Desi Selviana, Made Widya Suryaprani, Luh Putu Risa Prabandari, and Komang Sukraniasih, who always support me even when we are separated a thousand miles.

Finally, I take this opportunities to express my profound gratitude for my beloved parents and sisters for the encouragement and all hard work you have done until I can focus to pursuit my dreams. Sorry for not being home so long. Thank you for all supports and prays. Thank you for the motivation and the advices. Also for my grandmother, my late grandfathers and grandmother, my uncles and aunties, especially Bi Darsih, Bi Luh, who never stop express their love and supports to me and my sisters.

I am fully aware that I cannot mention all parties who contribute to the accomplishment of my study. From the deep of my heart I would say thank you for all kindness.

Palembang, July 2015

Ratih Ayu Apsari

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF FRAGMENTS

CHAPTER 1
INTRODUCTION

Algebra is a compulsory branch in secondary school mathematics. Even though it has no exact definition, school algebra has a wide scope including tasks with patterns, functions, equations, symbols and variables as described in Fosnot & Jacob (2007). In line with the topic discussed in algebra, Drijvers, Goddijn, & Kindt (2011) emphasize the importance of learning algebra as a base for the students to learn other subjects, higher education or professional career requirements and to solve related problems in daily life. Despite of the significant role of algebra in pupils' development, algebra is considered a difficult subject to learn and to teach (Jupri, Drijvers, & van den Heuvel-Panhuizen 2014; Capraro & Joffrion, 2006).

In many years the difficulties of learning algebra in secondary school are mostly blamed on the gap between what is learned in elementary and higher education (Dekker & Dolk, 2011). It occurs worldwide, including in Indonesia, where algebra has been taught as a 'new branch' of mathematics which the students encounter merely if they enter higher education. As a consequence, most students learn algebra in a rigid way and hardly see the relation between algebra and their previous learning in another mathematical domains.

In answer to the demand to reform the school mathematics to be more meaningful for the students, educational practitioners pay special attention to the paradigm in teaching algebra. First, even though most teachers believe that algebra should be postponed until higher education, a number of recent studies have showed that the contrary is true. Starting an algebraic class in the primary school was studied by Freudenthal (1974), the results of which strongly support the integration of algebra in the arithmetic classroom. Similar results are also pointed out by Kieran (2004) and Carraher, Schliemann, Brizuela, & Earnest. (2006).

A second concern is given to the common difficulties in students' algebraic development, which Lee & Wheeler (1987) stated as the inability to see algebra as the generalization of arithmetic. In the school level, algebra can be seen as the generalization of arithmetic. However, since in the algebra lesson the relation between algebra and arithmetic is mostly ignored, most of the students failed to acknowledge that fact.

As a solution to bridge the movement from arithmetic to algebra and at the same time to start algebraic lessons in early grades, National Council of Teachers of Mathematics (1989) recommends the use of pattern investigation and generalization. The pattern investigation could be designed in a different form, for instance pictorial or geometrical representations, number patterns, patterns in computational procedures, linear and quadratic patterns and repeating patterns (Zaskis & Liljedahi, 2002). One example of the implementation of pattern investigation activities can be studied in Lannin (2005) which uses a pictorial representation of the arithmetical sequence to support students' movement to generalize the number in algebraic form. The study found that the students are successful in providing a verbal generalization.

Even though a number of studies pointed out the usefulness of pattern investigation activities as a connection to the preliminary algebraic lesson, the study of this field is not done yet. As is pointed out by Quinlan (2001), the generalization from geometric pattern to algebraic symbolization is too hard for many students.

The other reason to conduct this study in Indonesia is that the educational stakeholders are not familiar with the preliminary algebra lesson. Hence, the curriculum still separates algebra and arithmetic in such a way, and the secondary school students in Indonesia should learn algebra in a formal way (see Kementerian Pendidikan dan Kebudayaan, 2006; 2013). This condition causes the students to assume that algebra has no intertwinement with other branches of mathematics and impacts the students' low performances in algebra (as indicated in TIMSS 1999, TIMSS 2007, PISA 2009 and PISA 2013). Jupri, et al. (2014) predict that the discrepancy between the ideal condition and the students' achievement in algebra is a "consequence of the directly formal and traditional algebra teaching which is still prevalent in Indonesia".

To deal with the aforementioned condition, the aim of this study is to contribute to an empirical grounded instructional theory to support the students to shift from arithmetic to algebraic thinking. The focus is seeing algebra as the generalization of arithmetic which involves the use of variables as the notion for the “range of value” as well as their function as the representation of “unknown”. Here, the pattern activities will be used as the starting point and the problems will be designed to encourage the need to generalize the pattern into algebraic expression. The research question addressed here is: “*how can patterns support students’ algebraic thinking*?”

CHAPTER 2
THEORETICAL BACKGROUND

The aim of this study is to contribute to the local instruction theory in teaching and learning early algebra to bridge the shift from arithmetic to algebra. For that purpose, a set of algebraic learning activities through pattern investigation will be designed for fifth grade elementary school students. This chapter provides the underpinning theories used as the starting point of this study. First we will formulate the definition of algebra and school algebra used in this study and then difficulties encountered by the students in learning algebra. Then we will explain what is viewed as early algebra, followed by the approaches used in this study to introduce algebra in earlier grades by using pattern activity. The goal of this study, the development of algebraic thinking, will have its independent subsection in this chapter, where we will discuss three main points of focus: the generalization of arithmetic, the relation between the numbers and the language of algebra. The discussion will be followed by the theory of Realistic Mathematics Education (RME) which is employed as the grounded theory of this study. In the last section, there will be a short introduction of the Indonesian classroom in terms of algebraic lesson and the position of this current study.

2.1 Established the Core of School Algebra

2.1.1 What is Algebra?

The question 'what is algebra?' will be answered differently by different people. The students, due to the traditional teaching method, might see algebra as a branch of mathematics in which you have to find the real value of *x* or *y*. A mathematician might immediately emphasize advanced algebraic topics like rings and groups. A professional worker, like a technician who uses algebra in his/her work, might see algebra as a complicated expression which involves large numbers with exponents. But which of those definitions is the "real" definition of algebra?

Generally speaking, the term algebra has different interpretations based on the point of view of the subject. Back to the origin of algebra, no one can dismiss the role of the Arabic mathematicians called Abu Ja'far Muhammad Ibnu Musa al-

Khwarizmi as the author of *Hisab al-jabr w'al-mugabala*. In his book, al-Khwarizmi introduced algebra as "the whole discipline dealing with 'equations'" (Kvasz, 2006, p. 291-292). As mathematics developed, the scope of algebra has become wider and the people have started to classify algebra as abstract algebra and school algebra.

2.1.2 Algebra in School

The main difference between what is meant by algebra in general and algebra known in the school is in the level of flexibility to work in the abstract (Renze and Weisstein in MathWorld). However, it still has the same core: it's all about generalization.

Drijvers, et al. (2011) emphasized the role of school algebra as a starting point to develop algebraic thinking which becomes the base for advance algebra. Even more, Brawner (2012) stated that "algebra is recognized as the "gatekeeper" of all high school mathematics; as a prerequisite for all advanced mathematics" (p.6) Hence, the gap between school algebra and abstract algebra should be bridged during the algebraic lesson in primary and secondary school. Briefly, they distinguished the strands in school algebra in four main points: patterns and formula, restriction, function and language. In line with that, Usiskin (1988) proposed four conceptions of school algebra, including algebra as the generalization of arithmetic, procedures to solve problems, relationships among quantities and the study of structures.

In spite of its importance for students' further studies and careers, the teaching and learning algebra still encounters a number of problems. Egodawatte (2011) refers to the four conceptual components of algebra which are the most problematic areas, including variables, algebraic expressions, algebraic equations and word problems. In the field of variables, it is noticed that the students still don't perceive a clear image about the use of certain letters and signs in different situations. Specifically, the students hardly recognize the importance of it as the representation of the changeable unknown which refers to the range of a value of a function. The students mostly failed to understand the meaning of symbols used in the equations and later incorrectly applying operation within the elements of them. Meanwhile in the case of word problems, the difficulty is to translate the information described in the words to the corresponding algebraic expression.

2.2 Early Algebra

In many countries, algebra is first introduced in secondary school level. This occurred because of a historical reason: algebra was invented long after the invention of arithmetic (Carraher et al. (2006). Hence, the traditional teaching and learning approaches assume that algebra should be learned after the students have sufficient understanding in arithmetic.

The assumption is theoretically correct, indeed arithmetic is the basic prerequisite knowledge needed to be able to learn algebra. However in practice, the teaching and learning of algebra tend to stand by its own without a sufficient relation with arithmetic (Jupri et al., 2014). The rigid separation between arithmetic and algebra leads to a discrepancy of seeing algebra as the generalization of arithmetic and having impact on a number of difficulties in learning algebra as mentioned in the previous chapter. Carraher et al. (2006) argue that this was caused by the lack of opportunities given to the students to see the relation between those topics. Therefore, a smoother bridge is needed to support the students' transition from arithmetic to algebra.

One strategy to bridge the gap between arithmetic and algebra is known as early algebra. This is algebra which is taught in early ages, in this case in the elementary school level, the aim of which is to provide experience in working with the number structures. Nonetheless, this statement is not aimed at transfer the high school curriculum to the elementary one, but to create a smoother shift from "learning rules for symbol manipulation toward developing algebraic reasoning" (Jacobs, Franke, Carpenter, Levi, & Battey, 2007; p.259).

As the students in earlier grades in primary school are already engaged in a number of arithmetic related topics, the arithmetical thinking already builds on their cognitive schema. Hence, the activities which focused on the transition from the typical arithmetical problems to the algebraic problems are needed. For instance, before they are in arithmetic class, the students learn that 1 is an odd number and 2 is an even number. Later, in early algebraic class they will identify which numbers are characterized as odd and which are even.

The importance of early algebra is acknowledged due to its contribution to prepare the pupils for flexible working with structures (Ball, 2003) and "to facilitate students' transition to the formal study of algebra" (Jacobs et al., 2007; p.261). In line with that, Tall & Thomas (1991) even stated, "there is a stage in the curriculum when the introduction of algebra may take simple things hard, but not teaching algebra will soon render it impossible to make hard things simple" (p. 128).

2.3 The Development of Algebraic Thinking

Algebraic thinking is defined by Kieran (2004) as the ability to focus on relations between the numbers. It consists of the "generalized arithmetic (using literal symbols), the development of mathematical models and the development of the language of algebra" (Dekker & Dolk, 2011, p.70). In line with that, they established five mathematical activities which are crucial for the development of algebraic thinking: (1) generalize and reason within algebraic structures, (2) develop mental models which refer to the pupils' construction of a possible strategy to work within algebra related tasks, (3) construct fundamental algebraic ideas, (4) observe, formulate and visualize patterns and (5) solve algebra related problems.

The focal point of this study is the development of students' algebraic thinking, focusing on the three important characteristics mentioned above: generalization, structure of the numbers and the language of algebra.

2.4 Pattern Activities

In order to introduce algebra in earlier grades, the classroom activities should promote the students' reinvention of the concept of generalization. Lannin (2005) summarized some recommendations from a number of studies (read Mason, Graham, Pimm, & Gower, 1985; NCTM, 2000; Stacey and McGregor, 2001), one of which is to use pattern investigation as the center of early algebra activities due to its "dynamical representation of variables" (p.233). An algebraic model of a pattern is the beginning of formula building, with attention to the investigation, identification and the relation between its structures algebraically (Drijvers, Dekker, &Wijers, 2011).

One strategy to use pattern activities is to use number patterns which are embodied in visual representations. A number of previous studies which use this approach (see Ma, 2007; Vogel, 2005; Herbert & Brown, 2000; Carraher, Martinez, & Schliermann, 2008) pointed out that pattern investigation can be helpful for the algebraic activity and therefore recommend this to start algebraic lessons in earlier grades.

Visual support of mathematical objects can be helpful to promote the students' construction of structural awareness (Rivera, 2011) which will be helpful to develop personal inferences while looking at a particular pattern. For example, consider the following square pattern in Figure 2.1 which takes form as (a) a series of numbers and (b) visual representations.

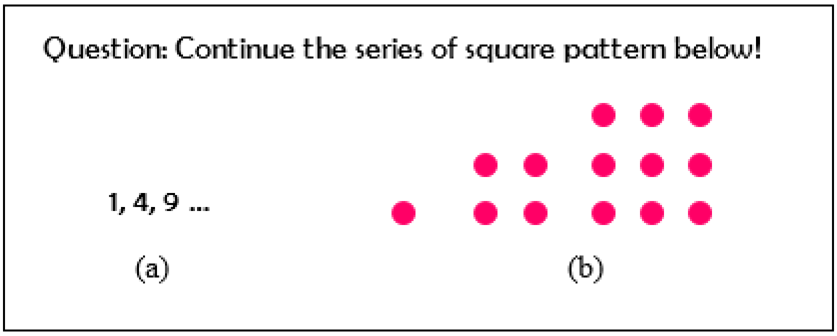


Figure 2.1. Square pattern problem

The number sequence in (a) directly forces the students to see the relation between numbers without any support. If we give this kind of task to fifth grade students, they might feel lost and refuse to investigate further. On the other hand, the second representation of number sequence by using dot patterns which shows the shape of a "square", gives the students a different point of view.

Some predictions may apply here, for example:

- Prediction 1 : The students directly notice that the number that will fit the series is the one which can construct a square.
- Prediction 2 : The students notice the increase of dots in each figure.
- Prediction 3 : The students notice the relation between the number of the figure and the number of dots in a row and a column.

Despite the advantages of using patterns as an activity to start algebra in earlier ages, the implementation still encounters a number of obstacles, especially to support the students' movement from pattern to algebraic notation (Quinlan, 2001; Lannin, 2005). With regard to the aforementioned problem, this study attempts to provide a local instructional theory in the field of early algebra by combining the importance of generalization as the heart of algebraic activity with the development of the students' algebraic language.

2.5 Realistic Mathematics Education (RME)

As briefly mentioned in the beginning of this chapter, this study is aimed at *designing* a series of learning activities in early algebra by using pattern activities. In order to support the students to learn in a meaningful way, our design will be based on the domain specific theory called Realistic Mathematics Education (RME).

RME is an approach in mathematics education which was originally formulated by a Dutch mathematician and educator called Hans Freudenthal. He defined mathematics as a human activity and therefore it should be connected to reality, close to people's experiences and has a contribution to the human being and civilization (van den Heuvel-Panhuizen, 2000). Furthermore, van den Heuvel-Panhuizen stressed the position of RME which is to reflect on how the students should learn mathematics and how mathematics should be taught in the classroom.

In conjunction with the purpose of this study and the choice of RME as the approach to achieve the intended product, in order to design learning sequences this study will be based on the principles of the RME. In the next discussion, the five tenets and three heuristic for design in RME will be zoomed in.

2.5.1 Five Tenets of RME

Treffers (1987) described the following five tenets in RME which contribute to the construction of mathematical instruction.

a. Phenomenological exploration

On 1987, Freudenthal proclaimed for the common sense mathematics which refer to how *realistic* the mathematical ideas can be conceptualized by the learners. To illustrate the notion, Gravemeijer (2011) distinguishes two types of question: (1)

"how much is 4+4?" (p.7) and (2) "how much will the total amount of apples be if 4 apples are combined with another 4 apples?" The first question is likely to be understood by most adults. However, it will not make sense for young children who have not yet learned about the relation between numbers and about addition symbols. On the other hand, the second problem provides them with more insight since they can make a relation between the problem and their daily experiences. So mathematics in the RME point of view "will certainly not have to put their relevant knowledge and experience aside" (Treffers, 1987, p.255). In this study we use a local Indonesian context as the starting point. The context is about dancing, since Indonesia has numerous dances with different characteristics which can be a suitable context to start the investigation of pattern. One example used in this study is the formation of dancers in a certain dance which represents the square numbers.

b. The support of vertical instruments

A vertical instrument is a bridge to relate the phenomenological case given in the beginning of a learning activity to the intended mathematical concepts. It can be models, schemas, diagrams and symbols. Treffers argues that models are the handiest support to bridge the students' conceptualization from "reality" in the real world to the "abstract world" in mathematics, since it gives a sense of visualization and a close representation to the actual condition in phenomena. The use of models in this study will support the translation between the context of dance and the mathematical concept of regularities. For instance, from the example given in the phenomenological exploration above, the dots model which imitates the position of the dancers can give a meaningful support for students to see the characteristics of the number of dancers that can perform the square formation.

c. Students' own constructions and productions

RME values the students as young mathematicians who should be guided to reinvent the mathematical concepts and doing mathematization. In line with that, the students' own contribution in constructing and producing mathematical knowledge is highly appreciated. Treffers defines the term construction as the students' action which can be found in their written work, while production refers to the students' reflection which can be inferred from their discussion especially in discussing different ideas for the same problems or transferring ideas to other related problems. The teacher is the main actor who can create a rich mathematical environment which enhances the students' learning. The choice of context, the activities and the problems should be thoroughly considered. In addition, when the students get stuck during their work and reflections, the teacher is allowed to give limited support, for example using questions as a scaffolding strategy.

d. Interactivity

The activities by the teacher and the students during learning are regulated in the interactivity instruction. Treffers highlighted that the instruction should give the students "opportunity to consult, to participate, to negotiate, to cooperate, with review afterwards" (p.261). In addition, he emphasized the teacher should be able to hold him/herself to give explanations. As an important aspect of this study, we quote the view from van den Heuvel-Panhuizen (2000) that since learning mathematics is a social activity, the learners should share with and listen to each other. Hence, the activities will focus on students' interaction in pairs, groups and as a class community, in which they can ask, argue, propose, agree or disagree. The role of the teacher as a part of the learning process is mainly in three aspects: to manage the discussion, to select the topic to be discussed and to build the classroom norms together with the students.

e. Intertwinement with the learning strands

The intertwinement principle is related to the global connection between various domains of knowledge. In this study, we use the concept of intertwinement in at least three mathematical aspects: algebra, arithmetic and geometry. This is possible because the pattern activities which are aimed to introducing algebra as the generalization of arithmetic use the visual representation of number sequences which adopt the geometrical representation.

2.5.2 Design Heuristic

Gravemeijer and Bakker (2006) define three instructional design heuristics of RME as discussed in the following section.

a. Guided Reinvention

Guided reinvention is closely related to Freudenthal's argument of the students' role as young mathematicians who actively construct their own conceptual structure. The teacher is supposed to be a guide, who supports the students in inventing mathematical concepts on their own.

b. Didactical Phenomenology

Didactical phenomenology is the analysis of how the students' learning path will be taken into account in the classroom. When designing activities, we need to look at the possible reinvention route by the students and what circumstances can encourage its performance.

c. Emergent Modelling

As mentioned in the five tenets in RME, the use of models as a bridge which connect the phenomena and the mathematical concepts is important. Hereby, activities are needed that can develop the use of model from the very informal (model *of*) to the formal level (model *for*). In this study, the problem itself is the model, since it is represented in a mathematical model, for instance strips, dots, symbols and table. The emergent modelling takes into account that the students themselves develop their reasoning of using the model. For instance, in this study the model will firstly be used as the representation of the original situation, before the students use it to think about the relation between the numbers and finally they can construct a general rule to represent a particular number pattern without using the model.

2.6 Classroom Norms

As mentioned above, this study will employ the RME approach in order to design a set of learning sequences. The whole design will not only produce the learning materials, but also the instructional settings which focus on the teacher's and students' behaviors and the interaction between them.

The role of the teacher and students based on the RME point of view is different compared to the role of the teacher in a traditional classroom. In the traditional setting, the teacher is the center of learning activities who has to do knowledge transmission to the students (and therefore is also assumed to be the one who has the right to determine what is right and wrong). In contrast, the teacher in an RME classroom is the one who facilitates the students who are the center of learning.

As result of that, the teacher plays a significant role to create the intended learning environment to support the students' conceptual construction. Hence, this subchapter will discuss the social norms and socio-mathematical norms which will guide the teacher in managing the learning activities.

2.6.1 Social Norms

Social norms are the pattern of social interaction which is followed by a certain group of people, in this case, the teacher and students in the classroom (Yackel & Cobb, 1996). We value social norms in the classroom as an essential part of this study. In the most traditional teaching method, the teacher is the center of knowledge who will transfer the mathematical concepts to the students (Kieran, 2004). Even if there is a chance for the students to work in groups, the students still feel insecure and keep asking the teacher for agreement, which direct them to follow the teacher's solution (Desforges & Cockburn, 1987). This type of teaching is clearly not expected to occur in the implementation of this study.

In general, there will be three kinds of discussions during the learning activities: in pairs, in a group and a whole class discussion. In those three kinds of discussion, the students will be the center of learning. The pair and group works are arranged alternately to give the students a chance to express their ideas, while the class discussion will focus on the crucial topic of the day. Based on the result of the studies which pointed out a lack of group working in the learning activities (Sharma, 2014), there are always be the silence, dominance and off-task-talk students. To anticipate this unintended condition of group work, we have come up with the idea of pair discussion. Pair discussion can support the students who are not brave enough to share their idea directly in front of a lot of people (Sampsel, 2013).

In addition, to prevent the tendency of asking judgement for each step they take, we offer the teacher a strategy to make an explicit agreement before the class discussion which mentions the role of teacher and the role of students. The teacher will only help by giving a scaffolding when the discussion gets stuck, while the decision in solving the problems will be the responsibility of the group's members.

2.6.2 Socio-mathematical Norms

Yackel & Cobb (1996) define socio-mathematical norms as the norms which focus on the mathematical activities of the teacher and students in the classroom and which are constructed through the agreement between them. The agreement here focuses on the mathematical qualities of certain solutions, for instance in terms of their efficiency, sophistication and effectiveness.

In this study, during the discussions attention is given to socio-mathematical norms. The students will discuss by comparing, debating, arguing or supporting certain strategies used by themselves or another students' work which they consider to be a more effective, more efficient or more sophisticated way to solve the problem. The students will also judge whether some strategies are completely similar to or different from the others.

In this study, we emphasize the role of teacher as the manager of the class discussion, by which the teacher should be aware of "unique" strategies used by the students and keep the class discussion in "good order". What we valued as unique strategies are the students' solutions which can provide a rich discussion. It is not necessarily the best answer, but it can encourage the other students to participate in the discussion. The order of the presented works should be carefully managed. If in the beginning the perfect answer is already given, then this will stop the discussion. The teacher should pick, for instance, a less sophisticated strategy and move smoothly to a sophisticated one.

2.7 Algebra in Indonesian Classroom

Indonesian students start learning algebra in the first grade of lower secondary school. The text book initially discusses about variables and algebraic expressions, mostly linear equations with one variable. The focus of learning is on the procedures of solving linear equations. This causes the students to see a variable merely as the representation of an unknown, which means anytime they see that, they will be looking for a single number as an answer. In fact variables can be used not only as the representation of unknown, but also as the representation of a range of values, and. The second use of a variable is mostly forgotten in Indonesian classroom. Lee and Wheeler (1987) acknowledge it as the cause of students' inability in seeing algebra as a generalization of arithmetic.

Jupri et al., (2014) identified a number of difficulties Indonesian students have in learning algebra, which mostly focus on these aspects: (1) mathematization, (2) interpretation of the algebraic expressions, (3) application of the arithmetical operations, (4) conceptualizing the use of the equal sign and (5) the use of variables. The study criticizes the formal way of teaching and learning algebra of which the center of learning is to remember formulas, and the study indicates this as the source of the aforementioned difficulties.

2.8 The Position of This Study

In order to support the development of the algebraic lesson in Indonesia and to prevent the aforementioned obstacles, the intention of this study is to design learning activities which offer a new approach to how algebra should be learned in the Indonesian classroom. Questioning the traditional algebraic lesson, which is formally given to secondary school students, we used arithmetic as a first step in a big leap to formal algebra. Pattern investigation is used as the main activity of the study, due to its visual support for the students. Furthermore, the focus of the study will be on the development of algebraic thinking, especially in terms of the ability to generalize arithmetic, seeing the number structure and use the appropriate algebraic language.

The designed learning activities are based on the tenets and principals of RME which will be tested to fifth grade elementary school students. As is stated before, the aim of this study is to contribute to a local instructional theory in early algebra to help the development of the students' algebraic thinking. In order to accomplish the goal, we attempt to answer this following research question: "*how can patterns support students' algebraic thinking?*"

CHAPTER 3
METHODOLOGY

The following chapter will discuss about the research approach, characteristic, phase, data collection and analysis method which is chosen to conduct the study in order to answer the formulated research question and achieve the intended goals of this study. The chapter will be closed by a brief discussion about research subject and location of this study.

3.1 Design Research

Theoretically, the purpose of this study is to contribute to the local instructional theory to support the students' transition from arithmetic to algebra by investigating number sequences which is visualized through geometrical representation. Practically, this study is conducted to design a learning trajectory involving pattern investigation and how it can be used in Indonesian local context as a support to develop students' algebraic thinking. In general, this study can be simplified in two main actions: create a series of learning activities to support the intended learning goals and in one hand investigate how the design will be implemented. Hence, both of design and research activities are used in this study.

Reflect to the aforementioned aim which is to *design* and answering the question of *how* the design will work, the appropriate approach which function is correspond to this study is Design Research (Plomp, 2007). Besides considering the design aspect as the crucial part of this study, design research is chosen as the method also due to its advisory aim. The advisory aim means that the study can be used to give a theoretical insight to the researcher about how to innovatively promote the teaching and learning process (van Eerde, 2013).

According to Barab & Squire (2004), design research is "a series of approaches, with the intent of producing new theories, artifacts, and practices that account for and potentially impact learning and teaching in naturalistic settings" (p.2). The natural setting emphasized in the last sentence plays a significant role which makes design research more powerful compared with the traditional design approach. As stated in Anderson & Shattuck (2012), the result of the design research can be generalized in the common teaching and learning situation rather than if it is done in clinical study, because it was tested in the real classroom. Van den Akker, Gravemeijer, McKenney, & Nieveen (2006) described it as utility-oriented, which is mean it should be practically work in reality. Hence, it can be used to answer the complex condition of the real educational setting (Plomp, 2007), since in this case the subject is the students, who cannot be controlled as the other experiments' specimens.

3.2 Characteristic of Design Research

There are five characteristics of design research which is applied in this study, as summarized in Bakker & van Eerde (2015).

1. Developmental purpose, which means that the design research is aimed to develop the theories about teaching and learning. In this study, the theory in early algebra.
2. Interventionist nature, which refer to the setting of the implementation of design research in the real classroom.
3. Prospective and reflective components, which refer to the intertwinement between hypotheses of the study (prospective part) and the real classroom situation (reflective part).
4. Cyclic nature, which means the iterative process of the invention and revision of the study. The iterative characteristic of design research is aimed to develop the local instructional theory (Gravemeijer, 2004). In order to accomplish the goal, he continues, the instructional activities which is designed based on the theory (also called thought experiment) should be "tried, revised, and designed on daily basis during the teaching experiment" (p.110). Furthermore, Drijvers (2003) distinguished the micro- and macro-cycles to establish the cyclic character of design research. Micro-cycles is the subsequent lessons while the macro-cycles is the global level of the teaching experiments. A macro-cycle consist of three phases: preparation and design phase, teaching experiment and retrospective analysis (Bakker & van Eerde, 2015). The following Figure 3.1 illustrated the relation between theory and reflection from the instructional experiment or implementation's result.

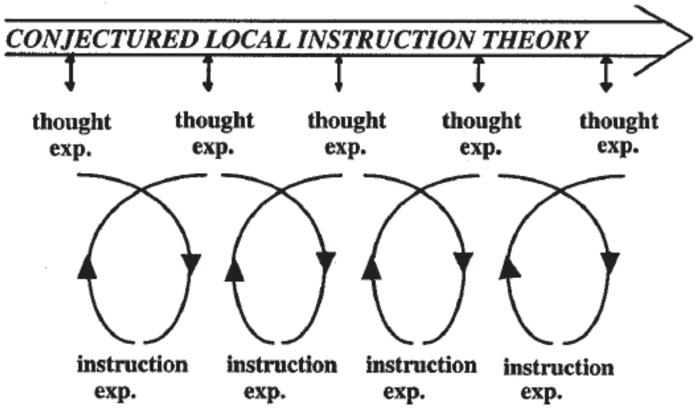


Figure 3.1. Cyclic process in developing local instructional theory (Gravemeijer, 2004, p.112)

5. General applicability of the theory. The last characteristic of the design research is the transferability of the study if it is implemented in different classroom.

3.3 Hypothetical Learning Trajectory

The hypothesis used in the design research is quite different with the common hypothesis used in the experiments study. The hypotheses in the experiments "are well-defined before data collection starts and not changed anymore during the rest of the study" (van Eerde, 2013, p.2). Meanwhile in design research, the hypotheses are tested continually and any result of the implementation, for example the observed learning is different with the expected one, can contribute to the change of hypotheses.

The hypotheses on design research is called Hypothetical Learning Trajectory (HLT). Simon (1995) formulate HLT as follows:

> The hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process—a prediction of how the students' thinking and understanding will evolve in the context of the learning activities. (p.136)

As briefly mentioned in the fourth characteristic of design research, there are three phases in design research. In each phase, HLT has different important role as is summarized in the following Table 3.1.

Table 3.1

The Position of HLT in Each Phase of Design

Phase	The Position of HLT
Preparation and Design	The HLT is started to be formulated and it's become the guideline of the instructional materials which will be designed for the study.
Teaching Experiment	The HLT is used as the guideline for the teacher to conduct the lessons and also functioned for the researcher as the focus point of observe or interview.
Retrospective Analysis	The HLT is applied as the guideline to do analysis, by compare it with the actual learning process in the classroom. After the global analysis, HLT will be revised and reformulated. The new version of HLT is used for the next cycle.

Further discussion related to the HLT of this study can be seen in the Chapter IV.

3.4 Phase in Design Research

3.4.1 Preparation and Design

The purpose of the initial step in design research the preparation of the research instrument, in this case the HLT and the classroom materials, including activities, students' worksheets, teacher guides and lesson plans. To be able to do that, the researcher should collects the preliminary data about the characteristic and prior knowledge of the subject of study related to the chosen mathematical topic. The process can be done by study literature, classroom observation and interview. Table 3.2 shows how this study will gather the intended information.

Table 3.2

Data Collection in the Preparation Phase

Data	Aim	Method
Curriculum and Syllabus	To ensure the appropriateness and the applicability of the chosen topic to the Indonesian curriculum.	Document analysis.
Teaching Experience and View on Mathematics	To get insight how the teacher usually conduct the lessons, her point of view toward the students' position in learning and how the chosen topic generally taught in her class.	Semi-structured interview with the teacher.
Teacher's Knowledge about PMRI	To get insight about the teacher's point of view, general knowledge and experience with PMRI.	Semi-structured interview with the teacher, classroom observation.
Classroom Norms	To get insight how the teacher manage the students and learning environment, also the social norms agreed in the class.	
Socio-mathematical Norms	To get insight how the decision making process regarding the mathematical content in	

Data	Aim	Method
	the lesson, i.e.: discussing which responses are valued as the more sophisticated, more efficient and more effective solutions.	
Students' Characteristics	To get general knowledge about the students' motivation and participation in learning process.	
Students' Proficiency	To get general overview whether the students mastered the prior knowledge needed to participate in this study.	

(For the interview and observation schema, see Appendix 1)

3.4.2 Teaching Experiment

The term 'experiment' in the teaching experiment is not refer to the comparison between control and experimental group, but to the experimental classroom in which the designed activities according to the HLT are applied. The purpose of this phase is to improve the initial design based on the actual students' responses in the classroom.

Bakker (2004) describes the activities and the data collection method in the teaching experiment, which will includes: (1) students' works on worksheets, (2) students' works in pre- and posttest, (3) pair/group/whole class discussion through the observations, field notes and video recordings and (4) audio-recording of the interview with the teacher and students. In this study, the teaching experiments will be conducted into two cycles as follows.

a. First Cycle

The first cycle of this teaching experiment is conducted as a pilot study to observe the responses of the students regarding to the designed activities and materials in early algebra. This phase merely involves a small group of students and the researcher will be the teacher. The teaching experiment is started by the pre-test (see teacher guide), in which the students will work to a set of problems about arithmetic and early algebra. The questions about arithmetic is aimed to check the students' prerequisite knowledge towards: order of natural numbers, applying basic operations like addition, subtraction, multiplication and division and the multiplicative numbers. The second part of the test is aimed to check the students' pre-conception toward the topic of the study, which is early algebra. We reflect to the students' responses while doing the pre-test to adjust the initial HLT. After that, the lessons will be conducted and the teaching experiment will be closed by post-test (see Teacher Guide). The post-test consists of early algebra related problems. Some of them are similar with the item in the pre-test, as is used to check the students' development after followed the designed instructional series. We also include different problem as we want to see whether the students can apply it in new problems. The data will be gathered by collecting students' written works and video observations during the lesson. The data will be analyzed to revise and further develop the first HLT.

b. Second Cycle

The revised version of the HLT from the first cycle is used as the guideline to conduct the second teaching experiment. In this phase, the study will involve the real classroom situation, with all students in the class and the teacher of that class will be the teacher in the experiment. As a note, the students in the second cycles are not the same as the students in the first cycle. In one hand, in the second lesson we will focus only to a focus group. This focus group will consist of 4 students who works in the same group. The chosen group doesn't have to be the brightest group since the choice of it is based on the students' willingness to cooperate with the study. The teacher will be involved to give some considerations to choose the focus group. The researcher will collects the data from classroom observation, students' works on pre- and post-test, students' worksheets, and interview with the teacher and the students in the same way as in the first teaching experiment.

3.4.3 Retrospective Analysis

After conducting the teaching experiments, the collected data which consist of various types of data including students' written works, field notes, video and audio registrations; will be analyzed through retrospective analysis phase. In this study we would like to apply qualitative analysis by using constant comparative method (Bakker, 2004). This method is useful when the aim of the research is "to gain more theoretical insights into learning process" (Bakker & van Eerde, 2015; p.443). The process can be describe as follows: (1) watches all of videotapes in order, (2) makes the general transcripts, (3) marks the interesting segment which is indicate the crucial moment of learning, (4) completes the transcript in detail, (5) looks for the confirmation or contradiction of the interesting segment (as mentioned in the third step) in other segments, (6) discusses some of the findings with colleagues to see whether they interpret the similar tendency from the observation and (7) draws the conclusions. In addition, the analysis of the second teaching experiment will be discussed with the teacher as the participant of this study to check his/her agreement toward the researcher's interpretations.

The analysis of the first teaching experiment are used to revise the HLT which will be employed as the guideline of the second teaching experiment. The analysis of the second teaching experiment will be used to get insight about how the students learning process was develop and whether it was assumed in the HLT. The results will be used to revise the HLT.

In addition, the pre- and post-test will be checked based on the students reasoning for each problem. The data from interview will also be used to confirm the written works of the students. The result of pre-test majorly used as the starting points of the teaching experiment. Together with the advices from the teacher, the result of the pre-test in the second teaching experiment also will be used to choose the focus group. In one hand, the result of post-test is mainly used to reflect of the students' way of thinking when encounter the algebra related problems. Furthermore, we also compare the result from pre-and post-test to obtain qualitative explanation of how teaching and learning process using designed materials can support the development of students' algebraic thinking. Qualitative analysis is chosen because the focus of this study is not merely to see how far the change of the students' score in the test before and after the lessons. Contrary, the comparison of the students' ability in answering pre- and post-test is used to support the observation of the teaching implementation.

3.5 Validity and Reliability

To enhance the quality of this study, we pay attention to the validity and reliability issues during the data collection and data analysis process. Validity refers to the accurateness of the researcher to measure what he/she intended to measure, while reliability focuses on the trustworthiness of the researcher (Denscombe, 2010).

3.5.1 Validity and Reliability in Data Collection

In gathering information during the preparation and implementation phases, we employ data triangulation, in which more than one method will be used to gathering the students' algebraic thinking. This method increases the internal validity of the study. In this study, we will use field notes and video registration from classroom observation related to the process of learning, teacher's and students' interview and the students' written works. Moreover, the use of electronic devices, in this case video and audio recorder, can improve the internal reliability of this study. In addition, the ecological validity in this study is warranted by its implementation in the real classroom.

3.5.2 Validity and Reliability in Data Analysis

The validity and reliability issues are not solely the concern of data collection method, but also play significant role in the data analysis. As describe in Bakker & van Eerde (2015) p.443, "the researchers want to analyze data in a reliable way and draw conclusions that are valid".

a. Internal Validity

Internal validity (which also mention as credibility in qualitative research) refers to the trustworthiness of the result for the others (Frambach, et al., 2013). To increase the internal validity of this study, the researcher will apply several strategies, including: (1) using data triangulation, (2) testing the data with the HLT and the improvement of the conjectures continuously, (3) looking for counterexamples to the conjecture during the retrospective analysis and (4) asking feedback from the participants to check the researcher's interpretation toward the learning process.

b. External Validity

External validity (known as generalizability and transferability as well) is described as the extent on which the finding can be useful in other context (Bakker & van Eerde, 2015). In this study, details descriptions and analysis of the subjects' condition, teaching and learning process and the students' participation, are used as to contribute the increase of external validity.

c. Internal Reliability

The degree of researcher's independence in analyzing the data is called internal reliability (Bakker & van Eerde, 2015). To ensure the internal reliability in this study, during the retrospective analysis we discuss the critical fragments with colleagues before drawing conclusions.

d. External Reliability

External reliability is the term for the confirmability of the result of the study (Frambach et al., 2013). This means, the interpretation of the data should be based on the actual condition of the participants, not the researcher's bias. To increase the external reliability of this study, the researcher will present the data as transparent as is can be, by: (1) make a documentation of the whole research (2) explain the how the study is done and (3) explain how the conclusion was drawn. Hence, the reader can follows "the trace of the learning process" (Bakker & van Eerde, 2015, p.445).

3.6 Research Subjects and Timeline of the Study

The research was conducted in an Indonesian Islamic primary school named MIN 2 Palembang. This school is one of the partners of Universitas Sriwijaya which already participates in the PMRI related developmental activities, for instance by attending workshop, seminars and as research partners. For the first cycle of the teaching experiment, four students of class VB of this school were chosen as the subject. Meanwhile in the second cycle, 32 students of VA class had been participated, also with their mathematics teacher called Istiarti Sri Sa'diah S.Pd.I. We summarized the timeline of teaching experiment of this study in Table 3.3.

Table 3.3

Timeline of the Study

	Date	Description
	Preparing the Study	
Preparation	August 2014 – January 2015	Studied literature and designed the initial HLT.
School Visit	February 10, 2015	Asked permission to do the study in the school partner communicated with headmaster and the teacher, observed general condition of the school, and discussed which class will be chosen as the subjects of

	Date	Description
		teaching experiment. We agreed to select four students from VB to be involved in the first cycle and then the whole class of VA for the second cycle.
	Teaching Experiment I (First Cycle)	
Observation	February 13, 2015	Observed the mathematics lesson in the grade VB and interviewed the mathematics teacher.
Pretest	February 18, 2015	Followed by 30 students of VB. The aim of pretest is to check the students' prior knowledge related to the designed materials.
Interview	February 20, 2015	Interviewed 6 students based on the pretest result and in the end 4 students was chosen based on the interview and further discussion with mathematics teacher.
First Meeting	February 23, 2015	Lesson 1: Color Costume
Second Meeting	February 24, 2015	Lesson 2: Dance Formation
Third Meeting	February 27, 2015	Lesson 3: Dance Formation
Fourth Meeting	March 2, 2015	Lesson 4: Board Sign
Fifth Meeting	March 3, 2015	Lesson 5: Crown Maker
Posttest	March 6, 2015	Followed by 4 focus students. It is aimed to confirm the students' progress and development of algebraic thinking.
Interview	March 6, 2015	Followed by 4 focus students. It is aimed to gain more understanding toward the students' written works.
	Teaching Experiment II (Second Cycle)	
Observation	March 6, 2015	Observed the mathematics lesson in the grade VA, and interview the mathematics teacher about general characteristic of the students.
Discussion	March-April 2015	A discussion with the teacher about the learning materials, reflection and improvement plan for each lesson before and after conducting it.
Pretest	March 6, 2015	Followed by 32 students of VA. It is aimed to check the students' prior knowledge related to the designed materials.
Interview	March 10, 2015	Interviewed 8 students based on the pretest result and in the end 4 students was chosen as the focus group based on the interview and further discussion with mathematics teacher.
First Meeting	March 16, 2015	Lesson 1: Color Costume
Second Meeting	March 17, 2015	Lesson 2: V and W Formation
Third Meeting	March 20, 2015	Lesson 3: Square Formation
Fourth Meeting	March 23, 2015	Lesson 4: Rectangular Formation
Fifth Meeting	March 27, 2015	Lesson 5: Triangular Formation
Posttest	March, 30 2015	Followed by 32 students of VA. It is aimed to confirm the students' progress and development of algebraic thinking.
Interview	March 31, 2015	Interviewed some students to gain more understanding toward their written works.

CHAPTER 4
HYPOTHETICAL LEARNING TRAJECTORY

Hypothetical Learning Trajectory (HLT) is the hypothesis used in a design research. Simon (1995) defined HLT as the prediction made by the teacher while he/she envision the students' way of thinking if they participate in the learning activity. It has three different roles for three different phases of the study. In the preliminary stage, the HLT is constructed and become the base of the designed materials. Later in the implementation, HLT is used as a guideline to conduct the lessons. In the retrospective analysis phase, HLT is employed to check the actual learning process happened in the classroom and will be revised or reformulated based on the findings.

Different with the hypothesis in experiment study which had to be done before the study conducted and cannot be revised during the implementation, HLT is flexibly adjusted based on the findings during the study. Here, we emphasize the word "hypothetical" in HLT as the foundation of our hypothesis. The teacher never sure whether his/her conjectures will really occurred during the lessons until the students really working on the problems. Therefore, even though the HLT is constructed as the preparation of the possible learning process of the students, there is no guarantee that it will exactly occurred as is predicted. Hence, it need to be revised and adjusted after reflect to the actual learning process.

In line with the role of the HLT, in this section we will describe some conjectures that we predict will be the learning route of the students during the learning activities. The conjectures will be completed by the learning goals, activity and the prediction of students' responses and how the teacher can support them if particular events are occurred.

Overall there are five lessons which covered the whole activities in this study which aim is to provide a meaningful algebraic activities for the young students. The activities will be about investigation of pattern which embodied in the visual representation. There are three kind of pattern used in this study: repeating pattern, growing pattern with constant difference and growing pattern with growing difference.

4.1 Lesson 1: Color Costume

4.1.1 Prior Knowledge

The students are have a prerequisite knowledge to:

- Order the whole numbers.
- Apply basic number operation to the whole numbers.
- Classify odd and even numbers.
- Find the factors and the multiples of certain numbers.

4.1.2 Learning Goals

- Identify the pattern of objects' arrangement.
- Predict 'the next' term of a regular pattern.
- Generalize the strategy to predict any term in a pattern.
- Evaluate the relation between the numbers.

4.1.3 Description of the Activities

The teacher starts the lesson by telling a plan of conducting a cultural event called Palembang Expo. The idea is the students will involve in a preparation for the dance performance. To catch the students' intention, the teacher may also discuss briefly about what kind of stuff they know as a part of dance, or whether they can do several traditional or modern dances.

The context used in the first lesson is about the plan to performing the *Saman* Dance, one of famous dance from Aceh. The idea is all of the students will participate. The school want to have different color of *Saman* costume. The first group will have the costumes in red and white, while the second group will use the pink, blue and yellow costumes. The arrangement of the dancer will be in alternate color, as can be seen in the Figure 4.1.

Figure 4.1. Saman dance costume

Phase 1: Introduction to the Problem

Using this context, the teacher will start the lesson by discuss about *Saman* Dance. After the students engaged with the context, the teacher narrow the discussion to make an order to the students using a number strip (see Appendix) which will represents the number of students in the formation and the color of costume they should wear. Ask them to thoroughly observe the alternate color, red and white. The aim of this activity are to (1) check the students' understanding about the context and (2) develop the students' awareness to the pattern.

Phase 2: Find the Next I

The teacher ask the students to work in pair and distribute the students' worksheet. There, the students can find a picture of a number strip which consists of 8 alternately red and white papers (see Figure 4.2). For the first step, the students are ask to draw the next two strips and predict the color of costume of the 12th and 25th students. The aim of this task is as a starting point to develop the students' initial strategy to find the 'next' sequence.

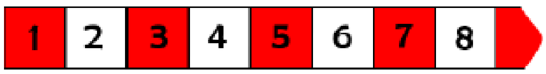


Figure 4.2. Number strips I illustration

Phase 3: See the Pattern

In a group of four, the students will continue to discuss about the pattern. To push the students to develop more general strategy, the third question on the worksheet is about finding the color of costume of the 100th student.

Phase 4: Find the Next II

The teacher ask the students to work with their pair to solve the second worksheet which discuss about the color of costume in the second group of *Saman* Dance (see Figure 4.3). The tasks asked the students to: (a) give the color sign for the next number strips and (b) predict the color of costume for the 11th and the 27th students.

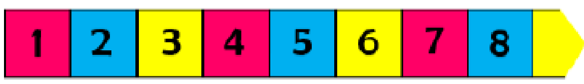


Figure 4.3. Number strips II illustration

Phase 5: Make it General

Still with their pair, the students are asked to solve the next problem which questioning about students' strategy to find the color of costume that will be used by the 100th student. After pair discussion, the teacher ask the students to compare their discussion with the other pair in their group. The teacher encourages the students to create a scientific environment in which they need to appreciate the others' opinions but in the same time try to convince the other to agree with their conjectures.

Phase 6: Structuring the Relation between Numbers

In a group of four the students will discuss the next problems. First is to determine whether a group of *Saman* dancers will have the same number of costume for each color. Second is about prediction of the costume color if the color of certain students' costume is given. The aim of this activity is to support the students' ability in evaluating the relation between numbers.

Phase 7: Class Discussion

The teacher will arrange a class discussion and ask the representation of some groups to present their works. The other students should react to the presenter. The teacher can also attract the students to discuss which strategy is counted as more sophisticated, more efficient or more effective to solve the similar problem.

Alternative Question

This problem merely given to the students who finish their work far before the other. The problem is: *Suppose that Mr. Rudi wants to create the Saman Dance with 100 students. He wants to modify the arrangement the arrangement of the dancers as you can see in Figure 4.4. His design is ORANGE-ORANGE-PINK-PINK-BLUE-BLUE repeatedly. What is the color costume of the last student? Explain your strategy!*

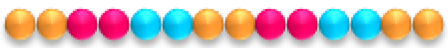


Figure 4.4. Saman dance costume colors illustration

4.1.4 Conjecture of Students' Responses and Guide for Teacher

In the following Table 4.1 and Table 4.2, we formulized a conjecture of how the students will be thinking about the given tasks in Worksheet 1 and 2, and how the teacher should react based on the students' responses.

Table 4.1

Conjecture and Guidance for Lesson 1 Worksheet 1

Problems	Conjecture of Students' Response	Guidance for Teacher
1. Continue the strips for the next two numbers!	The students continue to write the numbers and give the color sign on them.	• If the students continue drawing to check the costume of larger number of student, encourages them to think about general strategy by challenge them to predict the color of even larger number.
2. Can you predict the color of costume for the (a) 12th student? (b) 25th student?	• The students continue drawing. • The students consider the relationship between types of number (odd or even) with the colors.	
3. Explain how you predict the color of costume for the 100th student!	• The students make a string of 10 and multiply it ten times. • The students make a generalization that since 100 is even number and every even numbers is come in white then the costume color of the 100th student is white.	• If the students come up with odd-even idea, the teacher can challenge them to explain whether their conjecture will always true.
4. Is it possible to the 97th student to have the same costume color with the 43th student?	The students explain that since 97 and 43 are odd numbers than they will have the same color, which is red.	The teacher challenges the students to check their conjecture with another number, for instance: if the color of costume for the 93th student is red, what is the color of the 2nd one?
5. Suppose the 70th student get white for his/her costume, what is the costume color of the 1st one?	The students recognize the odd-even pattern, since 70 is even while 1 is odd, if 70 is red than 1 is white.	

Table 4.2

Conjecture and Guidance for Lesson 1 Worksheet 2

Problems	Conjecture of Students' Response	Guidance for Teacher
1. Continue the drawing for the next strip, what	The student will continue to draw the strips in the worksheet, by imitating or just write the symbol	• If the students use keep counting or drawing

Problems	Conjecture of Students' Response	Guidance for Teacher
is the color of the two-next strips?	of the color (for instance yellow-pink, or solely Y-P).	method, the teacher can encourages them to think a more general strategy by asking them to solve the larger number of lamp.
2. Can you predict the color of costume for the (a) 11th and (b) 27th students?	• The students are continue drawing. • The students think that every odd number is pink and even is blue, but then confuse with the yellow. • They may also think that every multiplicative of 3 will be yellow and 2 will be blue. But then confuse with the color of the costume for the students which numbers are multiplicative of 2 and 3. • The students see one pink, one blue and one yellow as a group. For instance if there are 11 dancers, there will be 3 groups and 2 left. Hence the 11th student will use blue costume. The similar strategy for the 27th student and the answer is yellow.	• If the students get stuck, the teacher can encourages them to observe the characteristics of students' numbers which come with pink and then blue and finally yellow. • If in the end no one of the students come with the general appropriate idea, the teacher can support them by asking about what happened to every students which number is the multiplicative of 3 and the student after it and the next after.
3. Explain your strategy to find the color of costume for the 100th student!	Divide 100 by 3 and observe that the remainder is 1. Hence, the color of costume is pink.	
4. If there are 53 students, would the formation has three equal color of costumes? Why or Why not?	The students realize that to have equal number of color, the number of students should be divisible by 3. Hence, the answer for this problem is no, because 53 is not divisible by 3.	If the students struggle with this idea, ask them to investigate whether 3 students will have equal color, how about 9 students and 10 students?
5. Is it possible to the 76th student to have the same costume color with the 121st?	The students use the structure of the number to check the color. For instance: if you divide 76 with 3 you have one as remainder, and so does 121:3. Hence, they will come with the same color which is pink.	If the students get struggle to solve this problem, the teacher can encourages them to find out the color of each student by using their previous conjecture.
6. If the 60th student use yellow costume. What will be the costume color of the 1st student?	The students observe the structure of number, in which 60 is divisible by 3 and it comes with yellow. Hence, the number which remain 1 if divide by 3 will have pink as its color.	

4.2 Lesson 2: Dance Formation

4.2.1 Prior Knowledge

The students are have a prerequisite knowledge to:

- Apply basic number operation to the whole numbers.
- Predict 'the next' of the pattern which involve the multiplicative numbers (*an*).
- Evaluate the relation between the numbers.

4.2.2 Learning Goals

- Predict 'the next' of the growing pattern with constant difference.
- Use words variable.
- Assess the conjecture for generalization.

4.2.3 Description of the Activities

Phase 1: Introduction

The teacher open the lesson by asking the students' experience in practicing a dance. The discussion will be narrowed to the formation of the dance. At the same time, the teacher shows several pictures of dance which formations are resemble to the geometrical patterns, such as square, circle, *V* shape, diagonal, vertical and horizontal. Here, the teacher can develop a short-friendly conversation with the students, such as ask about the name and the origin of the dance.

Figure 4.5. Example of *V* formation in *Panyembrama* dance

Phase 2: Find the Next

The teacher distribute the first students' worksheet and give them a chance to discuss it with their partner. The role of the teacher here is to make sure that the discussion is going well.

When the students stuck in particular problem, the teacher can give appropriate scaffolding. In the worksheet the students will find the following so-called *V* formation as can be seen in Figure 4.6.

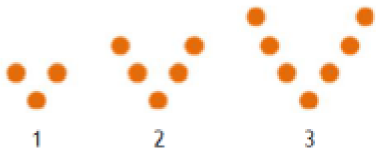


Figure 4.6. V dance formation

In pairs, let the students to work with the first four questions. They will investigate the number of dancers in small *V* formation and the relation between the number of pairs in *V* dance and the number of formation built.

Phase 3: Looking for Generalization

After a pair discussion, the students compare their work with another pair and continue to discuss the problem which ask about the number of dancers in 100th formation. After that, they continue with the last two problems in a group of four. The first problem is to give a reason whether two *V* formations can make one *V* formation. It is aimed to encourage the students to use the visual representation to illustrate their ideas. The second problem is to make board sign for the *V* formation. During the discussion, the teacher will check whether the students can makes an appropriate algebraic formula. If most of students can do that, the teacher can merely call for one group representative to share their idea in front of the class. However, if most students are struggle, a more deep discussion can be conducted.

Phase 4: Working with *W* Formation

In pair, the students will discuss the first and second problem in the second worksheet about *W* formation (see Figure 4.7).

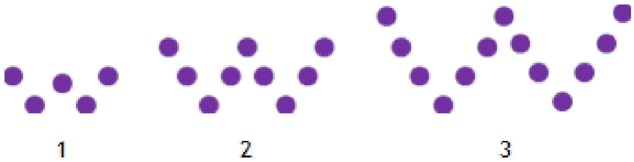


Figure 4.7. W dance formation

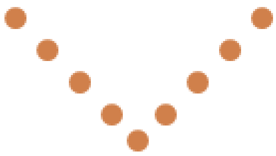
The question is aimed to help the students get the structure of *W* formation, by ask them to draw the formation of the fourth *V* formation, if the first until third formations are given. The next question is to evaluate whether 17 dancers can built this formation. After that, in a group of four they will compare their solution and continue to discuss the third problem. Here, the students should be able to evaluate and reason the given statement in the worksheet that two *V* formations can perform a *W* formation if one dancer is removed. The last, in their group the students are asked to formulate a words explanation to describe the relation between the numbers of dancer needed to create certain *W* formation.

4.2.4 Conjecture of Students' Responses and Guide for Teacher

In the following Table 4.3 and Table 4.4, we formulized a conjecture of how the students will be thinking about the given tasks in Worksheet 1 and 2, and how the teacher should react based on the students' responses.

Table 4.3

Conjecture and Guidance for Lesson 2 Worksheet 1

Problems	Conjecture of Students' Response	Guidance for Teacher
1. Draw the arrangement for the 4th formation!		The teacher encourage the students to check the shape of the dot formation given and see the change in each figure.
2. How many dancers will be in the 6th formation?	The students may keep counting, but since they are involved in similar activities in three lessons before, they may already look for the more general strategy, i.e.: by observe that the number of pair is equal to the number of formation. Hence, it 6th formation, there will be 6 pairs. And there will	

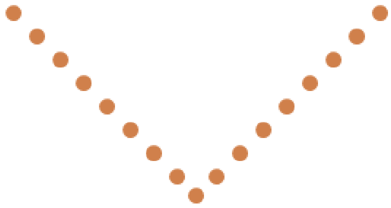
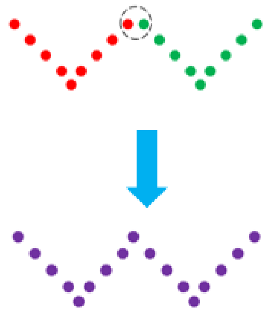
Problems	Conjecture of Students' Response	Guidance for Teacher
	be one in the middle, hence the total is 13 dancers.	
3. Draw the formation which consist of 17th dancers!		
4. How many pairs of dancers in the 45th formation?	45 pairs.	
5. Explain your strategy to find the dancers in the 100th formation!	There will be 100 pairs of dancers and one in the middle, so overall there will be 201 dancers.	
6. Can this *V* formation have 92 dancers? Why or why not?	The students may argue that *V* formation will never have 92 dancers because it always result in odd numbers.	The teacher encourage the students to do their own investigation by illustrate the condition given in the problem and ask them what will be happened.
7. The committee plan to combine two groups with *V* formation in one *V* formation, is it possible? Why or why not?	The combination of two *V* formations will not produce a *V* formation, but three of them will be. The reason is because the *V* pattern is always come with odd numbers. If two odd numbers are add together it will be even number. The students may also illustrate by using picture.	

Table 4.4

Conjecture and Guidance for Lesson 2 Worksheet 2

Problems	Conjecture of Students' Response	Guidance for Teacher
1. Draw the arrangement for the 4th formation!		To guide the students to formulate the words that explain their strategy, the teacher can ask the general rule to create the *V* formation (which is easier than the *W*) and then ask them to see the relation between *V* and *W* formation.
2. Tata compares the number of dancers in *V* and *W* formations. She claim that *W* patterns can be made from two *V*	Yes, because if you combine two *V* formations, you need to remove one in the meeting point to get *W* formation.	

Problems	Conjecture of Students' Response	Guidance for Teacher
formations which is subtract by one. Do you agree with her?		
3. Explain in words how you can work out the number of dancers when you are going to make certain *W* formation.	The number of dancers in *W* formation is equal to four times the number of formation add with one.	

4.3 Lesson 3: Dance Formation

4.3.1 Prior Knowledge

The students are have a prerequisite knowledge to:

- Use a word variable to represent a range of value or unknown.
- Formulate a strategy to find any term in a pattern.
- Seeing the relation among the numbers.

4.3.2 Learning Goals

- Predict 'the next' of the growing pattern with growing difference
- Create word formula to generalize a pattern.
- Find the number of unknown.

4.3.3 Description of the Activities

Phase 1: Get into the Problem

The teacher open the lesson by remind the students the topic they discussed in the previous meeting which is about the dance formation. The teacher told them that for the today's lesson they will investigate the other type of formation called square and triangular formation. The teacher can give the picture of Javanese Dance called *Serimpi* which use this position (see Figure 4.8).

Figure 4.8. Example of square formation in *Serimpi* dance

Phase 2: Find the Next

The teacher distribute the first students' worksheet and give them a chance to discuss it with their partner. The role of the teacher here is to make sure that the discussion is going well. When the students stuck in particular problem, the teacher can give appropriate scaffolding. In the worksheet the students will find a square formation. It is the dots pattern which illustrated as in the Figure 4.9.

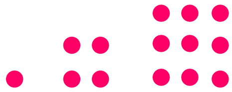


Figure 4.9. Dots pattern for the square dance formation

There are two main task in this steps, the first to draw the formation in the fourth figure and the second is to find the number of dancers in the tenth formation. The aim of those questions is as the warming up to encourage students to observe the regularity in the pattern.

Phase 3: Generalized the Square Pattern

Still with their pair, the students will discuss the third problem. The aim of this problem is encourage the students to think one step further. Here the students are asked to find the number of dancers in the 100th formation. After that, the teacher guide them to discuss their strategy with another pair. Then, in a group of four the students will discuss what words formula that can represent the relation between the number of dancers and the number of formation.

Phase 4: Can You Do the Reverse?

In a group of four, the students will discuss the last question which aim is to develop flexible thinking in algebra. Here, the students are asked to find the number of formation if there are certain number of dancers.

Phase 5: Try another Pattern

In the second worksheet, the students will also discuss about the similar problem which has triangular pattern as in the following figure.

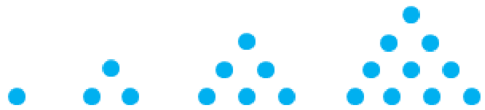


Figure 4.10. Dots Pattern for the Triangular Dance Formation

In pairs, they will works for the first three problems. After they discuss the strategy to figure out the number of dancers in the 100th formation, the teacher manage a mini discussion in a group of four. On the discussion the students will discuss which strategy that can help them to solve the problem efficiently. Next, they established a word formulas which represent the number of dancers in certain formation.

Phase 7: Class Discussion

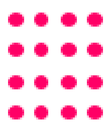
After all stages are done, the teacher conduct a class discussion on which the students will share their idea. A deep discussion should be given into the written formulation of the general strategy used by the students to determine the relation between the number of the dancers and the formation constructed.

4.3.4 Conjecture of Students' Responses and Guide for Teacher

In the following Table 4.5 and Table 4.6, we formulized a conjecture of how the students will be thinking about the given tasks in Worksheet 1 and 2, and how the teacher should react based on the students' responses.

Table 4.5

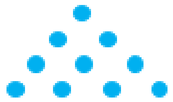
Conjecture and Guidance for Lesson 3 Worksheet 1

Problems	Conjecture of Students' Response	Guidance for Teacher
1. Draw the arrangement for the 4th formation!		• If the students generalization still depend on the number in previous formation (as conjecture b), the teacher may ask them to think

Problems	Conjecture of Students' Response	Guidance for Teacher
2. How many dancers will be in the 10th formation? 3. Explain your strategy to find the number of dancers in 100th formation! 4. Explain in words how you can work out the number of dancers when you are going to make certain formation.	To generalize, the students may come with different formula, i.e.: a. Since they compare the number of formations and the number of dots, they generalize that: the number of dancers will be equal to the square of the number of formation. Note: they may use "multiply a number by itself" instead of square. b. Look at the increase of the dots in each formations.  And generalize that in each formation, the number of dancers is increase as much as 2 times of the number of formation and then minus it one by one. Hence, the number of dancers in certain formation is the number of the dancer in previous formation added by two times the number of formation, minus one. Or, the number of dancers in a formation equal to the number of dancers in previous formation add with two times the number of previous formation, plus one. To calculate the number of dancers in previous formation itself, the students may come up with the idea of squaring the number as well.	whether it is efficient to use that method to find the number of dancers in 1000th formation. Support them to recognize that their strategy is correct, but they have to find out how to calculate the "number of dancer in previous formation" in efficient way. • If the students get the idea of squaring the number of dancers in the previous formation, the teacher can encourage them to realize the relation between their conjectures in (b) with the (a).
5. If there are 64 people, can you make the square formation? How many dancers you need more to make the next one?	The students will answer yes because 64 is a square number. They may conclude that 64 is a square of 8, hence the next square number is 81. Therefore, 64 should be added by 17 to get 81.	If the students can't decide whether 64 is square number or not, ask it with a simpler question, i.e.: "you said before that the number of dancers is equal to the number of formation times itself. Hence, can you find a number which will produce 64 if it multiply by itself?"

Table 4.6

Conjecture and Guidance for Lesson 3 Worksheet 2

Problems	Conjecture of Students' Response	Guidance for Teacher
1. Draw the arrangement for the 4th formation!		• If the students struggle, the teacher encourages them to observe the increased numbers in each figure.
2. How many people will be needed to make the: a. 6th formation b. 10th formation	a. Calculate $6+5+4+3+2+1 = 21$ b. Calculate $10+9+8+7+6+5+4+3+2+1 = 55$ They may calculate it one by one in order, but they also can use associative properties in this case.	• If the students calculate the sum of the numbers in problem 2 one by one, the teacher may encourage them to see the associative properties by observing relation between the pairs of number in end sides, ($\text{i. e.}: 6+1 = 5+2 = 4+3$). This strategy will be helpful later when they work with 100th formation.
3. Explain your strategy to find the number of dancers in 100th formation! 4. Write a sentence to describe the relation between the dancers needed and the number of triangle formation formed.	The students realize that they need to add the number of formation (let say n) with n-1, n-2 etcetera until n-(n-1). Hence, to determine the number of dancer in 100th formation, they will said add $100+99+98+\dots+1$. Since it involve a large number, they may find a way to calculate in less exhausted way. Here, if they realize that associative property is useful, they will structuring the number, for instance: $\overbrace{100+\underbrace{99+98+97+\dots+3+2}_{101}+1}^{101}$ They will have 50 × 101 = 5050.	

4.4 Lesson 4: Board Sign

4.4.1 Prior Knowledge

The students are have a prerequisite knowledge to:

- Create word formula to generalize a pattern.
- Find the number of unknown.
- Apply distributive, commutative and associative in whole numbers.

4.4.2 Mathematical Goals

- Find the member of the set which is represent by the given words formula.
- Classify the equal algebraic expressions.
- Determine the algebraic expression of particular pattern.
- Evaluate the relation between the numbers.

4.4.3 Description of the Activities

Phase 1: Introduction

As the continuous story of the previous meeting, the teacher will tell a story that the committee like their work for the square and triangular formation. Hence, the committee challenge them to create a new formation which followed certain rule as is given in the short letter below.

> Dear Mrs. Is
> This is the clue for your class's dance project!
> You will get the number of dancers (D) in certain formation if:
> **you take the number of formation add one to it and then triple it!**

Figure 4.11. Challenge letter

Phase 2: Create the Formation

In pair, the students will discuss three problems in the first worksheet. The task is to illustrate the first four formations and find the number of dancers in certain formation.

Phase 3: Find the Number

After that in group of four, the pairs compare their work and then back to work with their pairs again and discuss the last question in Problem A. The task is deal with finding the number of formation if there are certain number of dancers. This task is aimed to support the students' preliminary construction of solving linear equation. By the plot of the learning trajectory designed in this study, the students will likely to use "cover-up method" to solve this kind of problem. The students will compare their strategy in a group of four.

Phase 4: Which Signs are Correct?

In group of four, the students will discuss which equations are represent the words formula given in challenge letter. In the worksheet, the notion of algebraic formula is given. After this step, a whole class discussion can be conducted, not only to discuss which formulas are correct, but also to discuss the equality found in some algebraic expressions given.

Phase 5: Give the Sign

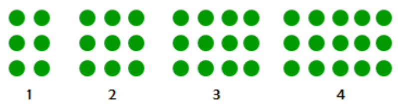
In a group of four the students will discuss the second worksheet. Here they will find four dance formations which they already discuss previously. In the previous meetings, they already formulate general formula in words to determine the relation between the number of dancers and the number of formation. The words formula will be given in the worksheet, depend on how the students were formulate it previously.

4.4.4 Conjecture of Students' Responses and Guide for Teacher

In the following Table 4.7 and Table 4.8, we formulized a conjecture of how the students will be thinking about the given tasks in Worksheet 1 and 2, and how the teacher should react based on the students' responses.

Table 4.7

Conjecture and Guidance for Lesson 4 Worksheet 1

Problems	Conjecture of Students' Response	Guidance for Teacher
Problem A 1. Draw the four first formations!		The teacher can encourage the students to give a reason behind the choice of the number and how it will relate to the formation constructed.
2. How many people will be needed to complete the 10th formation?	$(10 + 1) \times 3 = 33$	
3. Explain your strategy to find the number of people in the 100th formation!	In general, add your starting number with 99 and then add it with 1 and finally times it by three.	
4. Which number should be chosen	The students will work in reverse way, first divide 48 with 3 which	The teacher can challenge the students

Problems	Conjecture of Students' Response	Guidance for Teacher
to have 48 dancers in a formation?	result to 16 and then subtract 1 from it. The final result is 15.	to find a strategy to check their answer, by asking "how can you be sure that you answer it correctly?"
Problem B Which of those formulas represent the aforementioned formation correctly? A. $\boldsymbol{D = n + 1 \times 3}$ B. $\boldsymbol{D = 3 \times n + 1}$ C. $\boldsymbol{D = 3 \times (n + 1)}$ D. $\boldsymbol{D = 3 \times n + 3}$ E. $\boldsymbol{D = (n + 1) \times 3}$ F. $\boldsymbol{D = 3 \times 1 + n}$	The correct answer is the C, D and E options. The students is likely to have discussion about what is the difference with option A and E, also why A is incorrect while the other is correct. The similar discussion applied as well for option C and F. Here, the students can use their basic knowledge of properties in whole numbers which they usually use in the arithmetical operations. They may argue that A is different with E because A has no bracket. Hence they should multiply 1 by 3 first before it is added by any number *n* which makes it different with the original rules given by the committee.	The teacher will emphasize that this is another way to represent a general formula. The teacher will manage a discussion about the differences or the similarities between the given choices. The teacher can encourage the students to apply their basic knowledge in arithmetic (i.e.: associative properties) to check which formulas are consider as correct and which are doesn't. Also, the teacher can ask for confirmation: does some properties in arithmetic also work if you have changeable number like *n* in this formation?

Table 4.8

Conjecture and Guidance for Lesson 4 Worksheet 2

Problems	Conjecture of Students' Response	Guidance for Teacher
1. Write the algebraic formula for *V* Dance Formation. 2. Write the algebraic formula for *W* Dance Formation. 3. Write the algebraic formula for Square Dance Formation.	The students will write the correct algebraic expression for each pattern, i.e.: 1. $D = 2n + 1$ 2. $D = 4n - 1$ 3. $D = n^2$ 4. $D = 1 + 2 + \cdots + n$	The teacher should make sure that the students use the symbolization meaningfully, for instance when they write *n* ask them to explain what is the *n* stand for. Also, the students may confuse to find the relation between the numbers of formation in *W* formation with the number of

Problems	Conjecture of Students' Response	Guidance for Teacher
4. Write the algebraic formula for Triangular Dance Formation.	with D represents the number of dancers and n represents the number of formation. They may also use different variables for each problem.	dancers. Hence, if they merely come with the idea that W formation can be produce by two V formations minus 1 $((2 \times V) - 1)$, the teacher can bring up the discussion of how they can find direct relation between the number of dancers in V and W.

4.5 Lesson 5: Crown Maker

4.5.1 Prior Knowledge

The students are have a prerequisite knowledge to:

- Determine the algebraic expression of given pattern.
- Identify the relation between the numbers.

4.5.2 Mathematical Goals

- Illustrate the algebraic expression into words formula and the visual representation of it (and the reverse of it).
- Identify the range of an equation.
- Evaluate the role of chosen number *n*.
- Solve linear equation with one variable.
- Compare two linear equations with one variable.

4.5.3 Description of the Activities

Phase 1: Introduction to the Context

The teacher can start a lesson by bring up the topic of cultural event again. However, today the problem is not about the formation, but the head accessories called crown (*gelungan*) needed to complete the performance of traditional dance in Palembang, called *Gending Sriwijaya* and *Tanggai* Dance.

Figure 4.12. Illustration of Gending Sriwijaya and Tanggai dance

There are two worksheets used in this meeting. Both of them consist of similar and intertwined activities and the steps are more less the same.

Phase 2: Understanding Different Kind of Representations

In the first worksheet, the students and their pairs will discuss Aan's note which record the work of his father, called Mr. Husnul. Mr. Husnul is an artisan, sometimes he produce the accessories for the certain dance, based on the order. For this time, he works for the crown of *Gending Sriwijaya* Dance. There are 4 crowns produced for each day and from last production, Mr. Husnul already has 2. The following **Error! Reference source not found.** is the illustration of the crowns hich are keep in the table in Mr. Husnul's house.

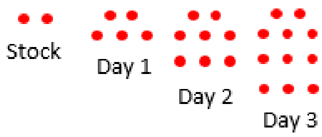


Figure 4.13. Dots pattern illustration of crown production in Mr. Husnul's house

In other hand, Aan made this notes to record his father's works:

2	5	8	11				

Figure 4.14. Aan's Recorder

The first problem is asked the students to explain the record above. In the second worksheet, the students will be given Mitha's notes which record the work of her mother called Mrs. Tyas, another artisan who works with crwon for the other dance called *Tanggai*. Here, again the students should explain the meaning of the algebraic expression: $S = 1 + 4n$.

Phase 3: How Many?

In a group of four, the students will discuss the rest of the problems. The problems is mostly about finding the number of crowns after a number of working days, vice versa. The aim of this problem is to emerge the students' strategy when they have to solve an equation in algebraic form.

Phase 4: Compare the Equations

In pair, the students will discuss about the number of crowns will be produced based on the given formula. The aim of this problem is to enrich the students' awareness toward the important of number *n* since here the students will observe that even Mr. Husnul has more stock from the last production, Mrs. Tyas will have more crowns after two days working because she can produce one crown more than Mr. Husnul in each day. Besides that, the students will also discuss that even Mrs. Tyas will have more crowns after the second day, in the first day they will have the same number of crowns. After the pair discussion, the students will discuss their work with another pair.

Phase 5: Discussion

The discussion will be focused on the strategy used by the students when they are going to find the proper number of *n* if the *S* is given, or vice versa.

4.5.4 Conjecture of Students' Responses and Guide for Teacher

In the following Table 4.7 and Table 4.8, we formulized a conjecture of how the students will be thinking about the given tasks in Worksheet 1 and 2, and how the teacher should react based on the students' responses.

Table 4.9

Conjecture and Guidance for Lesson 5 Worksheet 1

Problems	Conjecture of Students' Response	Guidance for Teacher
1. Explain your understanding towards Aan's notes.	The students may see Aan's notes as in the beginning his father already has two stocks from last production	• If the students can't understand the notes from Aan, encourage

Problems	Conjecture of Students' Response	Guidance for Teacher
	(given in the problem) and every day he produce 3 crowns.	them to see the relation between the numbers and the dots pattern which represent the number of crowns produced.
2. Can you fill the next three empty boxes?	2 5 8 11 14 17 20	
3. In 30 days, how many crowns will be available?	Each day there are 3 crowns produced, then in 30 days Mr. Husnul will has 90 crowns. Since there are 2 crowns from last year, the number of the crowns in their house will be 92.	
4. Write a general formula to represent the number of the produced crowns!	To generalize Aan's recorder, the students may write: the number of crowns equal to $2 + (3 \times \textit{the number of the working day})$ or write it in shorter way by symbolize the words variable into letter, i.e.: $S = 2 + 3 \times n$.	• If the students write the general formula in algebraic expression, the teacher may check their understanding toward the letter they used. Emphasize to the meaning of S and the meaning of n (for instance if the students formula is $S = 2 + 3 \times n$). Would S change if n change?
5. Suppose that there are 185 crowns in Mr. Husnul's house. Can you check how many days does Mr. Husnul already work?	The general formula for Aan's note is $2 + 3n$. Since that formula in particular n is produce 185 crowns and the students know that the constant 2 will always be there, the students can subtract 2 from 185, which is equal to 183. Each day Aan produces 3 crowns, hence the number of the day should be $\frac{183}{3} = 61$.	If the students get stuck, there are two steps of guidance that can be helpful. • The teacher encourage them to rewrite the algebraic expression which represent the number of crowns and the number of day working. • The teacher may encourage them to use cover up method. For instance, the students know that: $185 = 2 + 3n$ If they see $3n$ as one quantity first, let say the students cover $3n$, then they will realize that it equal to 183.

Table 4.10

Conjecture and Guidance for Lesson 5 Worksheet 2

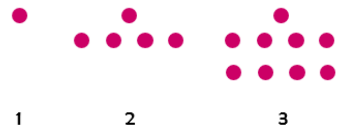
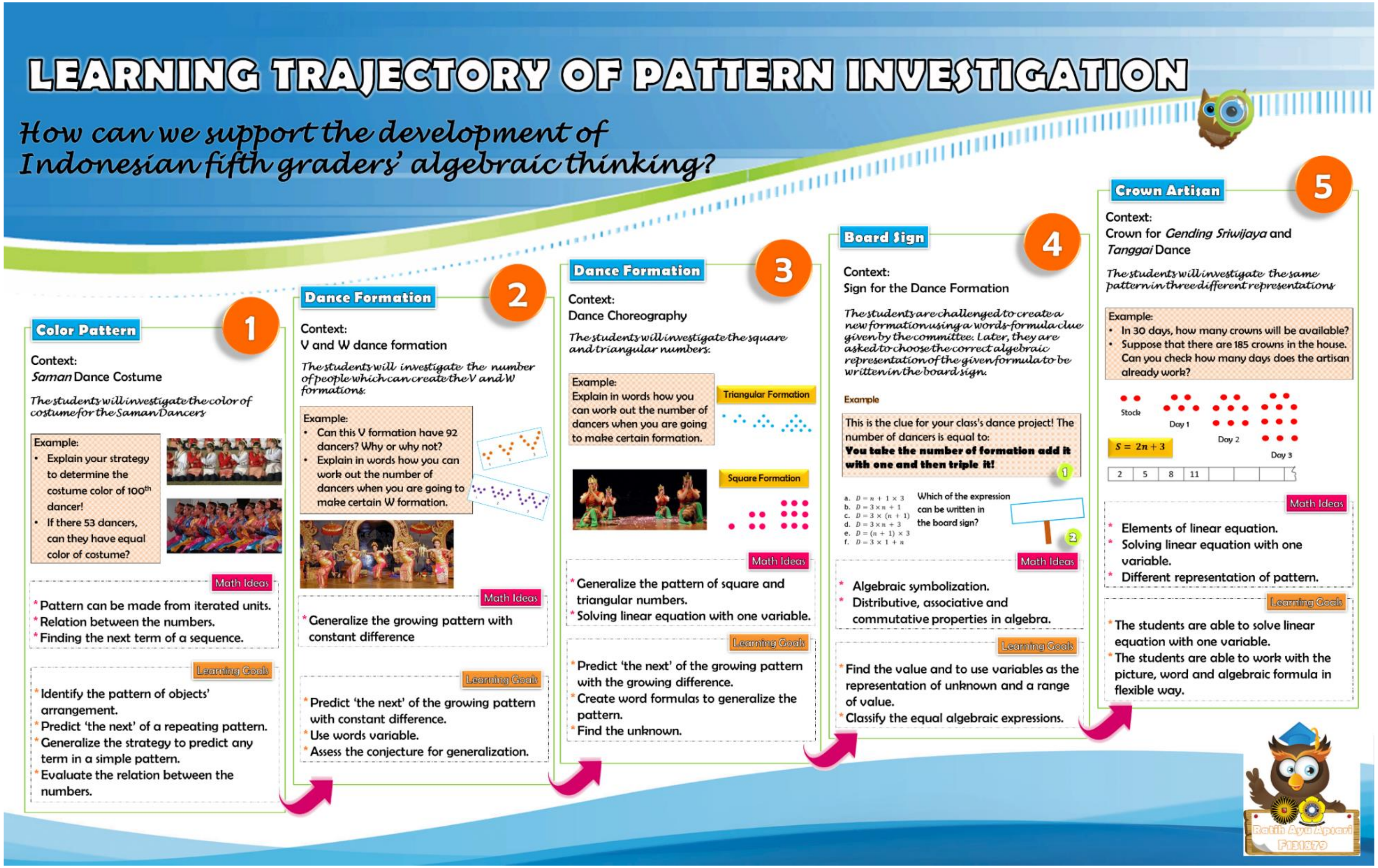
Problems	Conjecture of Students' Response	Guidance for Teacher
1. What can you explain about Mitha's note?	The students can explain that Mitha's mother merely has one stock from last year, but she can produce 4 crowns each day.	If the students get difficulties, the teacher can encourage them to see the relation between Mitha's notes and Aan's notes in the previous worksheet.
2. Illustrate the number of the available crowns in her house from the very beginning until the end of the second working day.		The teacher can give a support by ask them this following question, "if Mitha makes a notes like Aan, what number of crowns will be her first record?"
3. What value of *n* you take as the starting point in question 2? Explain why you choose that number!	The students may confuse what to choose: 0 or 1. The correct answer is 0.	
4. How many crowns will be at the Mitha's house if her mother just start working for the 10th day?	Mrs. Tyas already works for 9 days, hence the number of crowns in their house will be $1 + 4 \times 9 = 37$.	The teacher encourage the students to work with the algebraic formulas in flexible way. If Mrs. Tyas already works for 10 days, how the formula will looks like? If Mrs. Tyas already produce 73 crowns, how many days she already works?
5. Mrs. Tyas's target is to have 145 crowns at the end. Now, she already has 73 crowns. How many days she has to continue her work?	Mrs. Tyas has 73 crowns, hence she already works for 18 days. To produce 145 crowns, she needs to work for 36 days, hence she has 18 days more to work.	

Table 4.11

Conjecture and Guidance for Lesson 5 Worksheet 3

Problems	Conjecture of Students' Response	Guidance for Teacher
1. Who will produce the more crowns? Mrs. Tyas or Mr. Husnul Why?	The students can explain that Mitha's mother will produce more because each day she has 4, while Mr. Husnul only 3. But in the first day they will have the same number of crown.	Ask the students to consider the number of each day's production.
2. Is it possible for them to have the same amount of crowns? Explain!		



LEARNING TRAJECTORY OF PATTERN INVESTIGATION
How can we support the development of Indonesian fifth graders' algebraic thinking?
Color Pattern
1
Context:
Saman Dance Costume
The students will investigate the color of costume for the Saman Dancers
Example:
• Explain your strategy to determine the costume color of 100th dancer!
• If there 53 dancers, can they have equal color of costume?
Math Ideas
* Pattern can be made from iterated units.
* Relation between the numbers.
* Finding the next term of a sequence.
Learning Goals
* Identify the pattern of objects' arrangement.
* Predict 'the next' of a repeating pattern.
* Generalize the strategy to predict any term in a simple pattern.
* Evaluate the relation between the numbers.
Dance Formation
2
Context:
V and W dance formation
The students will investigate the number of people which can create the V and W formations.
Example:
• Can this V formation have 92 dancers? Why or why not?
• Explain in words how you can work out the number of dancers when you are going to make certain W formation.
Math Ideas
* Generalize the growing pattern with constant difference
Learning Goals
* Predict 'the next' of the growing pattern with constant difference.
* Use words variable.
* Assess the conjecture for generalization.
Dance Formation
3
Context:
Dance Choreography
The students will investigate the square and triangular numbers.
Example:
Explain in words how you can work out the number of dancers when you are going to make certain formation.
Triangular Formation
Square Formation
Math Ideas
* Generalize the pattern of square and triangular numbers.
* Solving linear equation with one variable.
Learning Goals
* Predict 'the next' of the growing pattern with the growing difference.
* Create word formulas to generalize the pattern.
* Find the unknown.
Board Sign
4
Context:
Sign for the Dance Formation
The students are challenged to create a new formation using a words-formula clue given by the committee. Later, they are asked to choose the correct algebraic representation of the given formula to be written in the board sign.
Example
This is the clue for your class's dance project! The number of dancers is equal to:
You take the number of formation add it with one and then triple it!
1
a. D = n + 1 × 3
b. D = 3 × n + 1
c. D = 3 × (n + 1)
d. D = 3 × n + 3
e. D = (n + 1) × 3
f. D = 3 × 1 + n
Which of the expression can be written in the board sign?
2
Math Ideas
* Algebraic symbolization.
* Distributive, associative and commutative properties in algebra.
Learning Goals
* Find the value and to use variables as the representation of unknown and a range of value.
* Classify the equal algebraic expressions.
Crown Artisan
5
Context:
Crown for Gending Sriwijaya and Tanggai Dance
The students will investigate the same pattern in three different representations
Example:
• In 30 days, how many crowns will be available?
• Suppose that there are 185 crowns in the house. Can you check how many days does the artisan already work?
Stock
Day 1
Day 2
Day 3
S = 2n + 3
2 | 5 | 8 | 11
Math Ideas
* Elements of linear equation.
* Solving linear equation with one variable.
* Different representation of pattern.
Learning Goals
* The students are able to solve linear equation with one variable.
* The students are able to work with the picture, word and algebraic formula in flexible way.
Ratih Ayu Apsari
F131879

CHAPTER 5
RETROSPECTIVE ANALYSIS OF THE HYPOTHETICAL LEARNING TRAJECTORY

Previously in Chapter IV, the Hypothetical Learning Trajectory (HLT) designed for this study were described. As explained in the research method, we employed the HLT as a guideline to conduct the teaching experiment and generate the analysis for this study. In this present chapter, the results and findings during the study will be elaborated. The story will be classified based on two cycles of this research, the first and the second cycle. Each cycle consists of the same phases, which are pretest, five lessons about patterns investigation and posttest.

The first cycle was conducted in a small group, consists of four fifth grade students of VB MIN 2 Palembang. In this stage, the researcher become the teacher because the main aim of this cycle is to check whether the designed activities will working properly in the “real” students’ learning activities. Any related information were used to improve the HLT. The revised version of the HLT will be used as a guideline to the second cycle. The second cycle involved 32 students of VA in the same school. In the second cycle, the mathematics teacher in the particular class was involved to be the teacher. She was asked to follow the trajectory based on the revised version of the HLT which is embodied in the learning materials given to her.

During the analysis, both of the cycle will be specified into three major aspects: teaching experiment, retrospective analysis and improvement of the HLT based on the analysis we made. The implementation section will give the reader general idea of how the activities were conducted. The retrospective parts were formulized after doing analysis using constant comparative method, including the steps of watching all videos, select interesting segment and finding the contradiction or supporting fragment in different lessons (for detail see Chapter 3, page 21). The researcher also compare the findings from video registration and students’ written works with the field notes during the implementation and the observation sheets which was fulfilled by an observer (see Appendix 3).

The results of the analysis of the first cycle will contribute to the adjustment of the HLT in the second cycle, while the result of the second cycle will be used to establish the new version of the HLT which is answering the question of this study: "*how can patterns support the development of students' algebraic thinking?"*

5.1 Analysis of the First Cycle

5.1.1 Pretest

The pretest was conducted on February 18, 2015 to the 30 students of VB class MIN 2 Palembang which aimed is to get an insight of students' prior knowledge related to the topic. Besides that, it is also used to give a consideration in order to select the focus students which will be involved in the pilot experiment.

The students work individually to solve six problems in approximately 45 minutes. There are two main topics tested on the pretest: arithmetic and algebra. The arithmetic field focus on students' comprehension in doing basic operations with whole numbers. The algebra sector focus on the students' preconceive idea about the existence of pattern (see Teacher Guide) for the complete problems.

Based on the analysis of the students' written work, we chose six students. Those students were not the highest achiever due to the test result, but they have unique reason during the pretest. We discussed it with the teacher and she gave some remarks, especially in term of the students' willingness to speak and their participation in daily teaching and learning in their classroom. Together with the teacher, we finally chose four from them to be selected as the focus students for the first cycle who followed the fifth designed lessons. Those students are Adi, Kayla, Dela and Rafi. After selected the focus students, we interviewed them to support the findings from worksheet and to get more insight about their way of thinking.

In the following discussion, we will describe the findings and the reflections for each pretest items.

Problem 1

The first problem were about the number series. There are five series as can be observed on the following Figure 5.1. The students were asked to continue the next three numbers of the series based on their knowledge.

a. 52, 53, 54, ..., ..., ...
b. 97, 98, 99, ..., ..., ...
c. 1, 3, 5, ..., ..., ...
d. 92, 94, 96, ..., ..., ...
e. 1, 1, 2, 3, 5, 8, ..., ..., ...

Figure 5.1. Number series problem

Most students were able to continue the series, except for the last question. Only one student who realized the relation between the adjacent numbers.

Problem 2

In the second problem the students had to fulfill the following number pyramid (see Figure 5.2).

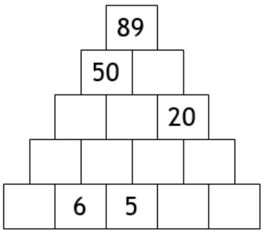


Figure 5.2. Pyramid number

The students hardly understood the problem, until the researcher explain it in front of the class and give one example of it. After that, the students expressed their understanding by saying "Oh!" and started to work on it. However, when we analyzed the students' written works, most of them are only correctly answer the first empty box. The highest score achieved is three correct empty boxes. See the example of students' answer in the following Figure 5.3.

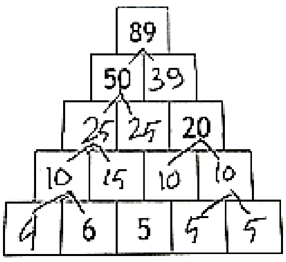


Figure 5.3. The student's answer for pyramid number problem

Based on the Figure 5.3, we observed that most of the students did not consider the given numbers in several boxes.

Problem 3

The third problem is doable for the students. Figure 5.4 shows four questions given in this problem.

a. $10 + 8 \times 4 =$
b. $39 \times 11 =$
c. $185 : 5 - 4 =$
d. $6^2 =$

Figure 5.4. Basic arithmetical problems

From those problems, the most common error is found in the order of the operation. For instance in question (a), some students got confuse whether to do multiplication or addition first.

Problem 4

The aim of this problem is to observe students' conception toward a pattern and its structure. In question (a), they were asked to determine whether the beads come in odd or even numbers, without counting it by ones. Figure 5.5 shows the problem 4(a).

Figure 5.5. Bead string problem

The words "without counting" confusing the students because as they knew they have to count to decide it. They researcher finally let them do it with any method they want to use, even with counting. When we analyzed the students, we found that most of the students were using counting by ones method and counting it separately between the black and the orange beads and add the result.

Question (b) is similar with (a), the students had to determine whether the number of the blocks given in the Figure 5.6 are odd or even.

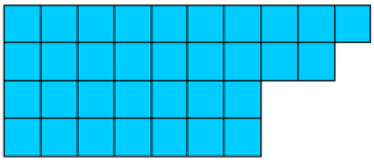


Figure 5.6. The blocks

The strategy used by the students is still counting by ones. Figure 5.7 shows quite different method. Even she also count it by one, she consider the structure of the picture as well.

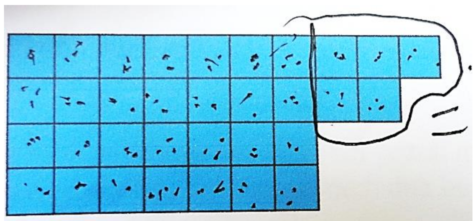


Figure 5.7. Student's strategy to determine the odd/even blocks

Figure 5.7 shows that the students consider that there is "a strange" part of the picture and she circled it to emphasize. The result of this problem gave an insight that the students were not accustom to think out of the box. They value mathematics as a learning branch on which have to be counted precisely.

Problem 5

The fifth problem is to determine which picture has the most dots (see Figure 5.8).

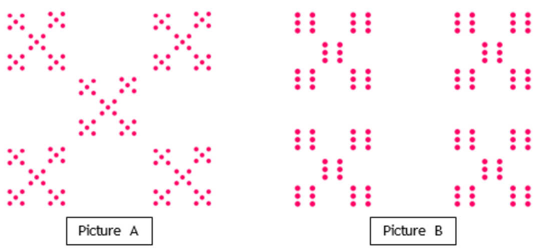


Figure 5.8. Which one has the most dots?

To solve this problem all of the students tried to count the dots. Most of them used multiplication, for instance the example of Rina's works in the following Figure 5.9. Even though almost all of them focus on finding the exact numbers before comparing the number of dots, some are incorrectly answer it due to error in calculation.

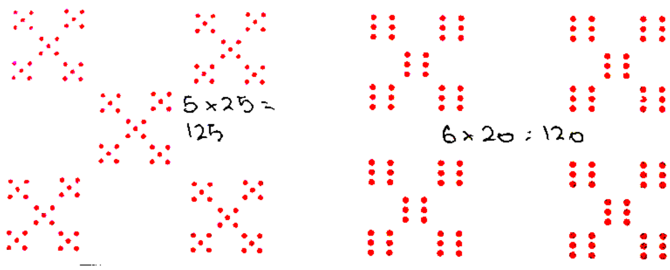


Figure 5.9. Using multiplication to count the dots

Problem 6

This problem is aimed to check the students' awareness toward a growing pattern. In the students' question sheet, the students were given a stamp which can produce two flowers each time it is stamped into a cloth. There also an illustration of the number of produced flowers after certain number of stamping (see Figure 5.10).

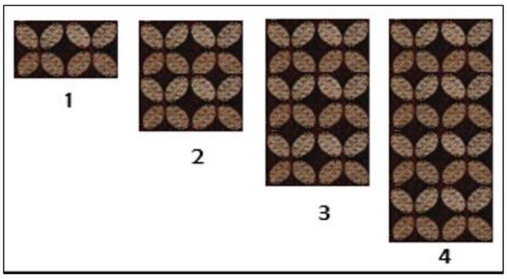
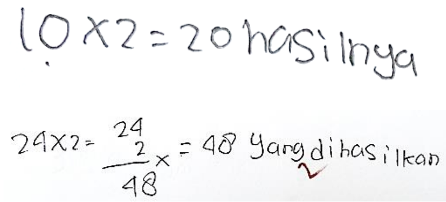


Figure 5.10. Flower stamping

First, the students were asked to continue draw the result of fifth stamping. Most of them were able to do that. After that, the students had to figure out the number of flower after the 10th and 24th stamping. Most of the students can do it without listing all number of produced flowers from the first until the given number of stamping. They used multiplication method, as they realized each time they stamp they will have two flowers. The following Figure 5.11 shows one example of students' written works.



Translation:
10 × 2 = 20 the result

Translation:
24 × 2 = 48 the result

Figure 5.11. Student' solution for stamping problem

Based on the result of the pretest we concluded that most of students have sufficient general prior knowledge to follow the designed learning trajectory. More specific to the selected focus students, their characteristics are summarized in the following Table 5.1.

Table 5.1

The Characteristics of the Participant in the First Cycle

No.	Students' Name	Characteristics
1.	Dela	Representation of low achievement students, not really good in multiplication.
2.	Kayla	Representation of middle-low achievement students, good in basic arithmetic.
3.	Adi	Representation of relatively high achievement students, good in basic arithmetic, creative.
4.	Rafi	Representation of middle-high achievement students, frequently doing error in calculation.

5.1.2 Lesson 1

After analyzing students' works toward the pretest, six students were chosen. The researcher discuss the result with the teacher and by the help of the teacher's consideration, four students were officially selected to be focus students of the research.

The first lesson conducted on February 23, 2015 with four students of VB MIN 2 Palembang who had been selected to be the focus students of the first cycle. The students had to work in pairs to solve the given problems. The general aim of the lesson 1 is to enhance the students' awareness toward pattern which is embodied in the particular arrangement of certain object. There are two related activities given during the lesson as follows.

The first activity which is also the opening of the overall lesson during the study used the context of cultural exhibition in Palembang where the school is asked to be the participant. The students was enthusiast hearing that all of the fifth grader will perform *Saman* Dance, a traditional dance from Aceh. Hence, it was quite easy to grab students' attention at the time. Further, the researcher explained that the dancers will use the red and white costume and to make it fair the headmaster will create a number string. Each students will get a number which is indicate their position in the dance and the color of their costume as well. The following Figure 5.12 illustrated how the headmaster plans to order the dancers based on their numbers.

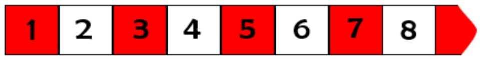
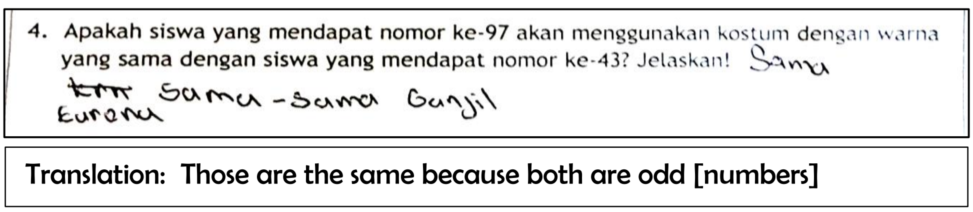


Figure 5.12. Number string of the dancers' position in the first activity

As a warming up question, the researcher asked the students about the color of costume used by the ninth and the tenth dancers. Both of pairs answer it correctly by keep counting from the strings. After that, they were asked to solve the second until the fifth problems with their pairs.

Problem 2 and 3 basically the same, it asked the students to develop their generalization ability by seeing the regularities in a pattern. The second problem was about the color of costume used by the 12th and the 25th dancers while the third is the 100th dancers. While working with this problem, as predict in the HLT the students were able to see the relation between the alternate colors with the odd-even number's type.

Problem 4 asked the students to see the relation between the numbers, by asking them evaluate whether the 97th and 43rd dancers will use the same color of costume. Both of pair were able to solve it by directly pointing to the odd-even number's type as is also predicted in the HLT. Figure 5.13 shows the example of argument given by the second group.


4. Apakah siswa yang mendapat nomor ke-97 akan menggunakan kostum dengan warna yang sama dengan siswa yang mendapat nomor ke-43? Jelaskan! Sama ~~krn~~ Karena Sama-sama Ganjil

Translation: Those are the same because both are odd [numbers]

Figure 5.13. Second group's solution toward the fourth problem

To finish this activity, the students were asked to evaluate the pattern in different order. The aim of this activity is to fostering the students to be aware with the order, since different order may leads to different conclusion. The question was describe that the first costume's color is unknown but the 70^{th} students use red, the students were looking for the color of the 31^{st} dancer. The students were able to solve this question easily as well. They need to relate the odd-even numbers with the colors help them to think in flexible way.

The second activity is about a plan to create a *Saman* Dance Group with three different costumes: pink[1], blue and yellow. The first part of the worksheet consists of three leading questions: (1) continue the strip until the next two numbers, (2) find out the color of costume used by the 11^{th} and 27^{th} dancers and (3) the 100^{th} dancer. To solve the first two questions both of the pairs used keep counting method. They recorded their works as in the following Figure 5.14.

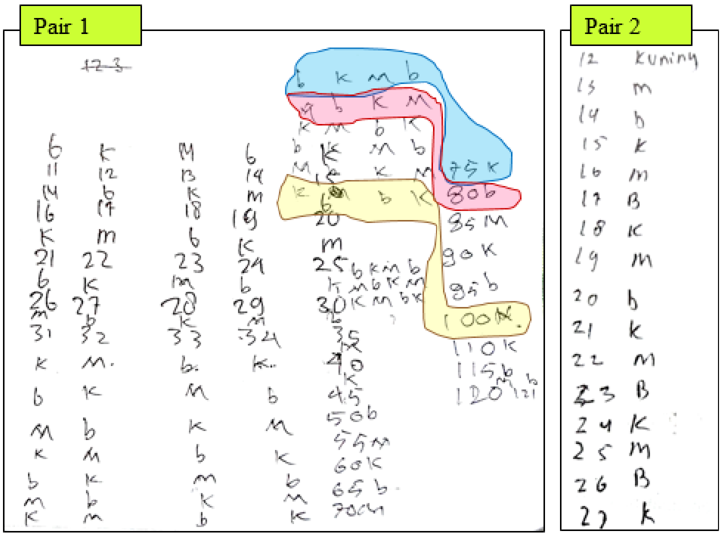


Figure 5.14. Students' strategy to solve the first part of problem

Figure 5.14 shows that the first pair organized their counting in every 5 while the second group merely continue the calculation. The symbol *b, k, m* come from "*biru*", "*kuning*" and "*merah*" which stand for blue, yellow and red respectively.

[1] It should be pink, but since the printer color was looks like red, the students keep saying the color is red instead of pink.

The first pair listed all numbers and its colors (see the colored marks given in the Figure 5.14(Pair 1)). Differently, the second pair employed three different approaches. On their worksheet they explained that they use 4 as their reference. Since 100 is the multiplication of 4, the color of the 100th dancer will use the same color of the 4th. During the discussion they used another method. Fragment 1 illustrates the students' argument toward the problem.

First, according to Rafi they should use 3 as the reference and multiply it with 33. It comes with 99 is equal to yellow color, then the 100th will be red.

Fragment 1: Use 3 as the Reference

[1] Rafi	:	Because, red-blue-yellow [pointing to the given picture].
[2]		The third color is yellow.
[3] Researcher	:	Yes
[4] Rafi	:	Nah, multiply 3 with 33. It will be 99.
[5]		If you add 1 more it will be 100. Nah, that is red.

When Rafi gave argument, Adi tried to interrupt. According to him, they should use the 10th. Since the 10th dancer will use red, the 100th will also use red.

Fragment 2: Use 10 as the Reference

[1] Adi	:	The easiest strategy …
[2] Researcher	:	What is that?
[3] Adi	:	8 is blue, 9 is yellow, [and] 10 is red …
[4] Researcher	:	Pink …
[5] Adi	:	Just the same, Yuk …
[6]		It should be multiplied by 10.
[7]		The color is still pink …

To deal with the difference answer given by the first and the second pairs, the group discussion conducted in the last session. The discussion was intended to encourage the students to aware what is the "good reference" to be used in this case. However, the cross questions given during discussion was not support them to realize the contradiction in their argument. Hence, the students still not be able to generalize what kind of reference which can help them to do generalization.

The fourth problem is to determine whether a formation with 53 dancers can have the equal number of dancers in each costume color. The students were not able to meaningfully investigate the term of "equal colors" if there are three kind of colors. Instead of using grouping or division methods, the students come with arbitrary answer as: "using blue color" and "the same because you have three colors".

The problem for the fifth task is to determine whether the 76th dancer will use the same costume color with the 121st. The students solved it by finding the color costume for each of them and matched it. Since those numbers used different costume colors, they concluded that the answer is no.

In the last problem, the students encountered different pattern. The information stated that if the 60th student use yellow, what about the 19th? This problem was not clear for the students, they answer it by relate the previous pattern that the 1st student uses red, the 2nd uses blue and the 3rd uses yellow. They did not consider that it might have different pattern since the 60th student uses yellow.

5.1.3 Lesson 2

The second lesson conducted on February 24, 2015. The general aim of the lesson 2 is to enhance the students' awareness toward growing pattern with constant difference. There are two worksheets given, the first is about *V* pattern and the second is the *W* pattern. The context is about a formation of Balinese Dance called *Pahnyembrahma* which will be performed in Palembang Expo as well as the *Saman* Dance which have been discussed in the previous meeting.

There are seven problems given in the *V* pattern task. In the introduction, the researcher showed the illustration of three consecutive *V* formations. The similar picture also given in the worksheet and the students were asked to draw the next formation. The task is continued by finding the number of dancers in the 6th formation.

The first pair used a so-called recursive formula by adding two dots from the previous formation. Their strategy can be observed in the following Figure 5.15 and Fragment 3.

Figure 5.15. Adding two strategy

Fragment 3: Adding Two Strategy

[1] Researcher : What are you doing here?
[2] Did you add by two?
[3] Do you agree, Dela?
[4] (*Dela nodded*)
[5]Researcher : Doesn't it by one?
[6]Kayla : No.
[7]Researcher : Why?
[8]Kayla : If you add by one you get even number.

The fragment of Kayla's and Dela's discussion showed that they started to think how to preserve the general structure of *V* formation. Hence, they added two dancers on top of the previous formation to get the current formation. They use this strategy to find the number of dancers in the sixth formation as well.

The second pair did it differently. They continued to draw until the fifth formation before they construct the general formula. They separated the part of the formation which "has pair" and the "single" one, as is can be seen in the Figure 5.16.

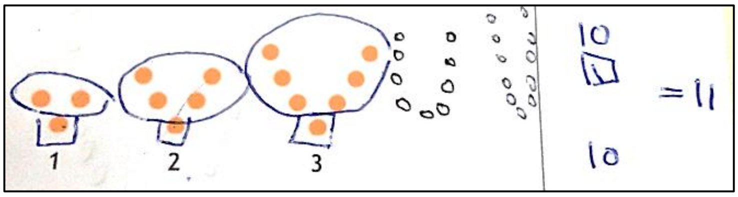


Figure 5.16. The second pair's strategy to the first task

Based on the Figure 5.16 we can observe that the students moved from drawing to symbol representation by using numbers. They saw the relation between the number of dancers which has pairs and the number of formation. Hence they concluded that in the fifth formation they will have ten dancers in the "paired" section and one dancer in the middle.

The second task is to bridge the students to see the relation between the number of formation, the number of dancers which have the pairs and the total number of dancers. The questions are about to draw the *V* formation which has 17 dancers and to find how many pairs will be in the 45th formation.

To solve the third question, both pairs started to draw with the one dot which represent the dancer in the middle and continue to add two until they got 17 dancers.

The second pair got conflict when solving the fourth question. Rafi argue that there will be 22 pairs, because if there are 45 dancers and one in the middle the 44 dancers will be in 22 pairs. He got confused with the “number of pairs”, “number of dancers” and “number of formation”. Adi said that in the first formation there is one pair, in the second formation there are two pairs, etcetera. Hence in the 45th formation there will be 45 pairs. After a while, the researcher jump into discussion and brought a new conflict as can be seen in Fragment 4.

Fragment 4: Pairs in the Formation I

[1] Researcher	:	How many pairs will be in the 45 formation?
[2] Rafi	:	22 pairs
[3] Researcher	:	How many pairs will be in the 22 formation?
[4] Rafi	:	11 pairs
[5] Researcher	:	How many pairs will be in the 11 formation?
[6] Rafi	:	5 pairs
[7] Researcher	:	How many pairs will be in the 10 formation?
[8] Rafi	:	5 pairs
[9]		Eh, how come?

After that, Adi re-explained his idea about the number of pairs in each formation. Now, he added his explanation by pointing to the figure as is showed in the Fragment 5.

Fragment 5: Pairs in the Formation II

[1] Adi	:	1 pair (*pointed to the figure of the first formation*),
[2]		2 pairs (*pointed to the figure of the second formation*),
[3]		3 pairs (*pointed to the figure of the third formation*).
[4] Rafi	:	Oh, yes! Right!

The fifth problem is to explain the strategy find the number of dancers in the 100th formation. The first pair got lost here. They combined adding and doubling method, using the fifth formation as the reference to find the other number of dancers in other formation. Hence, they wrote that in the 10th formation there will be 22 dancers, the 15th formation will have 33 dancers and so on (see Figure 5.17).

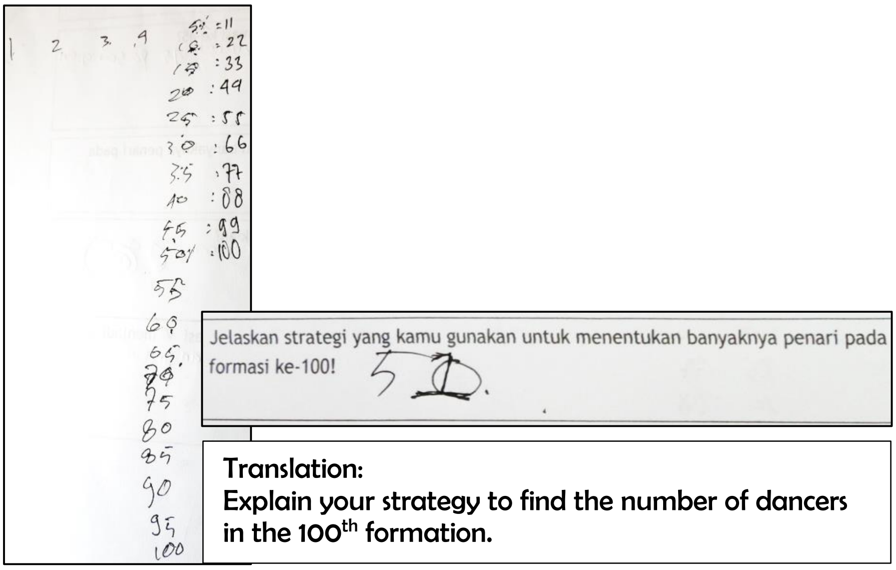


Figure 5.17. The first pair's strategy to solve the fifth problem

When it come into the 50th formation, they wrote 100. All of sudden they got confused which was the row for the number of dancers and which for the number of formation. They mixed everything up and they concluded that in the 100th formation there will be 50 dancers (see the conclusion made in the right side of Figure 5.17). But then they remembered that there will be one person in the middle, and then they changed 50 into 51.

The second pair used the same strategy they used to solve the first problem. They found that there will be 200 dancers in the "paired" section and one in the middle. Their answer can be observed in the following Figure 5.18.

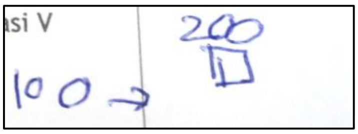
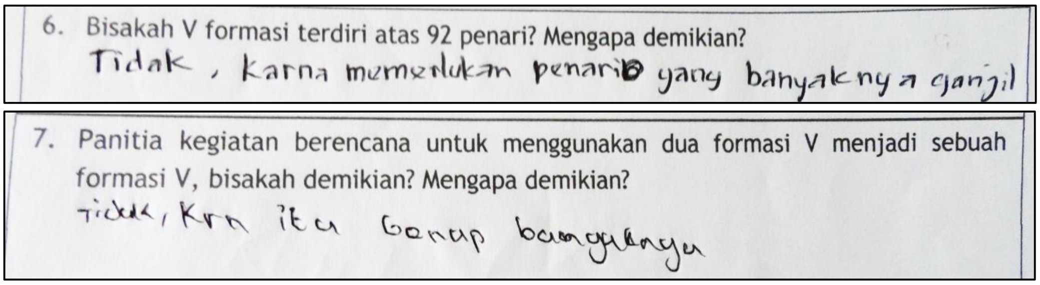


Figure 5.18. The second pair's strategy to solve the fifth problem

After both of pairs were done to discuss this problem, we conducted a group discussion. During group discussion, the students discussed the method they used to solve the problems. The second group explain their way of seeing the structure of *V* formation which help them to find the number of dancers in any formation, without the need of exhausted listing.

After the discussion, the students continued the task. They were asked to determine whether 92 dancers can perform a *V* formation. After that, they should explain whether the combination of any two *V* formations can make a *V* formation.

The second pair who have an insight about the structure of *V* formation from the very beginning, stated that the answer is “no” for both problems. They conclude that any *V* formation will need an odd numbers of dancers. Their answer can be observed in the Figure 5.19.


6. Bisakah V formasi terdiri atas 92 penari? Mengapa demikian?
Tidak, karna memerlukan penari yang banyaknya ganjil

7. Panitia kegiatan berencana untuk menggunakan dua formasi V menjadi sebuah formasi V, bisakah demikian? Mengapa demikian?
Tidak, Krn itu Genap banyaknya

Translation of students’ answer:
6. No, because you need odd number of dancers.
7. No, because it will produce even numbers.

Figure 5.19. The Second Pair's Strategy to Solve the Last Tasks of *V* formation

The first pair who had a new understanding of how the structure of *V* formation use a picture to explain. Now, they concerned with the shape of *V* formation before they jump to number. They answered by pointed out the important of one person in the middle to perform a *V* formation. Furthermore they argue that with 92 dancers or if two *V* formations are combined, there will be no one dancer in the middle. Hence, it will not construct a *V* formation.

Fragment 6: Could it be a *V*?

[1] Kayla	:	No, you can't.
[2]		Because no one becomes alone.
[3] Researcher	:	What do you mean by that?
[4] Kayla	:	This all have their pairs, no one stands alone behind.
[5] Researcher	:	Oh, you mean no one has no pair?
[6] Kayla	:	Yes, it is *V* formation.

Based on the Fragment 6, we can see how Kayla considered the structure of *V* formation. Previously she and Dela always busy only with numbers (see Figure

5.17), but after got a new insight of how they can use the help of picture to work with mathematical world, she started to think about generalization: why she always need the odd numbers.

In the discussion, both pairs use the picture to explain their arguments. As observe from their answers, they try to relate the odd-even number properties with the shape of *V* formation. They conclude that *V* formation will never have even number as its number of dancers, because to create a *V* shape, there always one person in the end of the formation.

The lesson continued to the second activity about *W* pattern. In the worksheet, the three-first-*W* formation was given and the students had to draw the fourth. Both of pairs were able to do this, however they have a difficulty in make the parallel dots in the same line because there was no grids or supporting lines given in the worksheet.

Next, the students need to evaluate the relation between the number of dancers in the *V* and *W* formation. The question is, "is it true that a *W* formation can be constructed by combining two *V* formations which is subtract by one". Both of them express their agreement, their reason can be seen in the Figure 5.20.

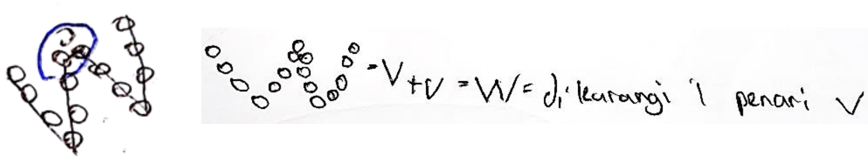


Figure 5.20. Remove one

The last task is to creating a word generalization which represent the students' strategy to find the number of dancers. Here, the students got difficulty to understand the problem itself, especially to meaningfully understand the word "unknown", when they were asked to generalize it into "unknown" number formation.

5.1.4 Lesson 3

The third lesson conducted on February 27, 2015. The general aim of the third lesson is to support the students' awareness toward growing pattern with growing difference. There are two designed activities for this lesson, square and triangular formations related problem. The context is about creating a square formation for a Javanese Dance.

The students were given a picture of the three first square formation and for the initial task, the students were asked to draw the fourth formation and then determine the number of dancers in the 10^{th} and 100^{th} formations.

The first pair used the addition strategy. They recognize that they had to add dancers above and in one side of the previous formation. At first they need to draw each formation by one, as is can be seen in Figure 5.21.

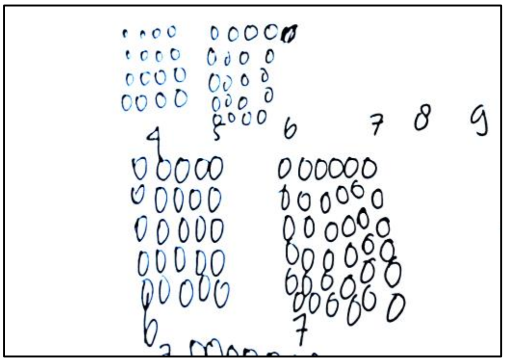


Figure 5.21. Drawing one by one

After a while, they felt tired to do that and continue drawing in other strategy. They developed a more efficient drawing technique as can be seen in Figure 5.22.

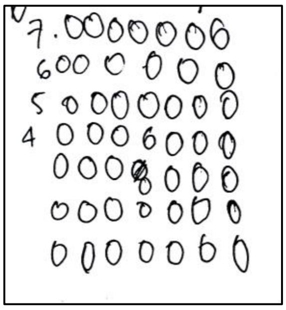


Figure 5.22. Drawing the additional dancers

Even though the first pair tried to keep continue method, they were not able to solve the second and the third task. They found the answer after discussed it with the second pair in the group discussion session.

The second pair was using different strategy. Instead of focused on the number of added dancers in each formation, they investigated the relation between the number of dancers in the sides of the square. Their discussion recorded in the Fragment 7.

Fragment 7: Rows and Number in Rows

[1] Researcher : How you get 100 dancers?
[2] Adi : Look at this. In the 3^{rd} formation you have 3 dancers.
[3] (*Pointed to the number of dancers in each row of the third*
[4] *formation*).
[5] Here, 4.
[6] (*Pointed to the number of dancers in each row of the fourth*
[7] *formation*) and so on.
[8] So [in the] 9^{th} [formation] there will be 9 [dancers].
[9] if it is [in the] 10^{th} [formation] there will be 10 [dancers].
[10] And then you go down, you have 10 [rows].
[11] Then, you find it, 10 times 10 is 100.

To check the students' generalization, the researcher asked further question related to their strategy.

Fragment 8: Generalize for the 100^{th} Formation

[1] Researcher : How about this problem? (*Pointed to the third question*).
[2] How about the 100^{th} formation?
[3] Adi : Doddi will answer [Doddi is Rafi's other name].
[4] Rafi : In the 100th formation, 100-100-100.
[5] [refer to the 100 rows and each row has 100 dancers].

However, in their worksheet they did not write the final answer of 100 times by 100. When the researcher asked them about it, they answered it correctly.

As previously mentioned, the first pair was not able to finish the second and the third problem. After seeing the first pair stuck too long, we conducted a group discussion. During the discussion, Adi and Rafi shared their idea about how they find the number of dancers in certain formation. At first they explain that they used multiplication strategy. However, Kayla and Dela did not understand why they did multiplication. Fragment 9, Fragment 10 and Fragment 11 illustrated their discussion.

Fragment 9: Why You Multiplied It?

[1] Rafi : Multiplication
[2] Researcher : What you multiply?

[3] Rafi : 10 times 10.
[4] Researcher : What is the 10?
[5] Adi : It is the number of dancers in each formation.
[6] Kayla : Hah?
[7] Researcher : Can you explain it to them? (*Talked to Adi and Rafi*)
[8] Adi : Hah?
[9] Researcher : Please explain why you multiply 10 and 10.

Fragment 10: The Formation

[1] Adi : Look at this
[2] Researcher : Yes, continue it.
[3] Adi : Why I got 10 times 10, because here 3
[4] (*Pointed to the picture of the 3rd formation*),
[5] Nah 3 times 3 is 9.
[6] So, that's mean in the 10th formation there will be 10 times 10.
[7] Researcher : Do you understand it (*Talked to* Dela *and Kayla*)
[8] Kayla : No
[9] Researcher : He said in the third formation it should be 3 times 3.
[10] Which one do you mean 3?
[11]Adi : If [in the] 4th [formation], 4 times 4.
[12]Researcher : Which one do you mean by 3 (repeated the previous question).
[13] … (*The students talked out of topic*)
[14]Adi : 1, 2, 3 – 1, 2, 3, 3 times by 3
[15] (*Pointed to the picture, counted each dancer in each row*)

After a while, Kayla got an insight of how to determine the number of dancers.

Fragment 11: "Aha!" Moment

[1 Dela : Ah, that is 3 and 3
[2] (*gave an invisible marked in the third formation,*
[3] *see the illustration of it in the* Figure 5.23).
[4] Researcher : How about the 4th [formation]?
[5] Dela : 4 times 4.

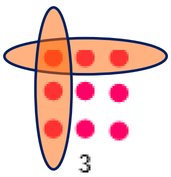


Figure 5.23. The illustration of Dela's invisible marks

After that, the students continued their worksheet. The task is to write their strategy to find the number of dancers needed if they want to construct certain formation. The formulation of the problem was not very clear for the students and the students were confuse to explain their strategy in words as well.

As their responses, the first pair merely drew the second, third and fourth formations in order to answer this problem. The second pair formulated their answer as “*by determine from each row*”.

The next problem is to evaluate whether 64 dancers can perform a square formation. To deal with this problem, the first pair used keep counting method by drew the dots continuously until reach 64 dancers.

The second pair using squaring approach. Since they realized that each formation has “a number of formation times itself” formula to determine the number of dancers on it, they tried to find a number that fulfill it. To figure out the number, they used guess and check method. They started by squaring 4 and then 6, 7 and finally 8. Fragment 12 shows Adi’s mental calculation.

Fragment 12: Can We Have 64 By Squaring A Number?

Adi	:	So, 64. 4 x 4 = 16. The 6th formation … So, the seventh formation … [The] 8th [formation] ... 8 times 8 …

The last problem in this activity is to determine the number of additional dancers if the committee want to create the “next” formation from the formation with 64 dancers. The students were able to solve this problem by finding the number of dancers in the 9^{th} formation and then subtract it with 64.

The second activity is about triangular formation, which is representation of a triangular numbers. In the students’ worksheet, the students found the picture of three-first-triangular formation and they were told to draw the fourth.

The second pair, Rafi and Adi, focused on the notion of “triangle” instead of seeing the structure of the given picture. Hence, at first they didn’t draw the correct next pattern. Figure 5.24 is the picture from Rafi and Adi.

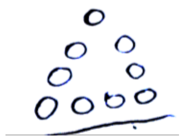


Figure 5.24. Incorrect Representation of Triangle Formation

The first pair was arguing among them. Kayla thought as Adi and Rafi's answer while Dela able to recognize the shape of triangular pattern. Dela argued that in the next pattern there will be the additional dots as in number of the formation. Hence, in the fifth formation there will be five dots more from the fourth formation. Her answer can be seen in the Figure 5.25.

Figure 5.25. The fifth triangular formation

Due to the time limitation, the activity continued in the next activity. Reflect on students' struggle in seeing the shape of triangular formation, the researcher revised the worksheet by added the fourth formation there and asked the students to draw the fifth formation. Finally, four of them come with the same conclusion of the shape of the triangular formation.

The second problem is asking about the number of dancers in 7th and 10th formations. Both of the pairs used addition method to solve this problem. They saw the structure vertically. To solve the problem, the first pair needed the help of drawing to do it (Figure 5.26).

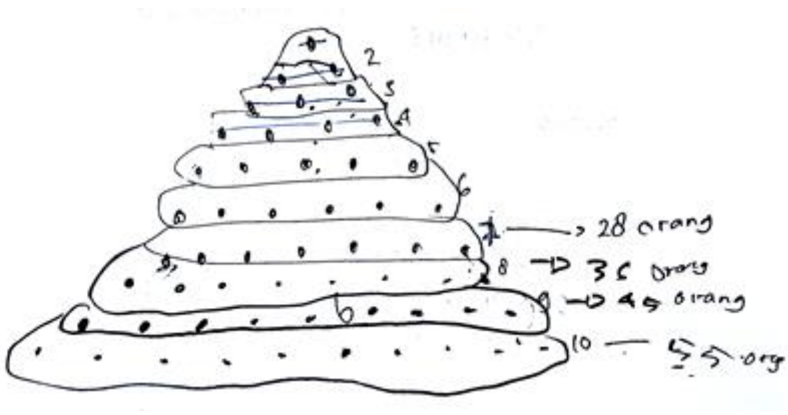


Figure 5.26. Drawing the triangular formation

The second pair saw the addition from right to left as they marked their worksheet like in Figure X.

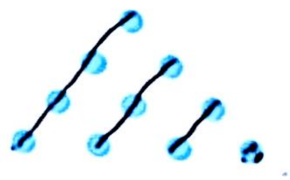


Figure 5.27. Second pair's point of view toward the structure of the triangular pattern

Without drawing, they found that the number of dancers will be the same as they add from one until the number of formation. Their strategy is transcribed in the following Fragment 13.

Fragment 13: Adding One to Seven

[1] Rafi : Four plus three plus two plus one
[2] Adi : A minute, Rafi.
[3] Rafi : Seven plus three …
[4] Adi : Wait me …
[5] (*Adi got lost*)
[6] Researcher : Rafi, can you shows your method to Adi?
[7] Why it can be 1 + 2 + 3 + 4?
[8] Rafi : Count from this (*pointed to the picture*)
[9] Adi : Oh, I got it (*marked the picture as can be seen in the*
[10] *previous* Figure 5.27).
[11]Researcher : If you want to find the 7^{th} formation, what will you do?
[12]Rafi : The 7^{th} formation? [Adding] One until seven.
[13]Adi : How many you get?
[14]Rafi : 1 + 2 + 3 = 7, 7 + 4 = 11
[15]Adi : 1 + 2 = 3, 3 + 3 = 6…
[16]Researcher : Write it down [the calculation], dear. So you will not confuse.
[17](*The students keep adding until they get the result*).

In the end they concluded that to find the number of dancers in a triangular formation, they need to add from 1 until the unknown number. Even though they were able to generalize the relation between the number of dancers and the number of formation, when adding the numbers, both of pairs did not try to find more efficient strategy. They just keep adding the first number until the last in the right order without, for instance, calculate it by grouping each 10.

5.1.5 Lesson 4

The fourth lesson conducted on March 2, 2015. There are two main activities in this lesson. First, the students were engaged in a problem in which they had the general formula of a dance formation and they were asked to draw the fourth first formations. The formula is "you take a number of formation, add it with one and then multiply the result with 3". After drawing the formations, they were asked to determine the number of dancers in six arbitrary formations using the given formula and then generalize its strategy to find the number of dancer in the 10th and 100th formation. The last question is to determine the number of formation which has 48 dancers.

The students were not able to solve the drawing tasks. However, they successfully find the number of dancers in certain formation. The first pair's written work is showed in Figure 5.28. They were correct in using calculation, but they did not relate it with their drawing.

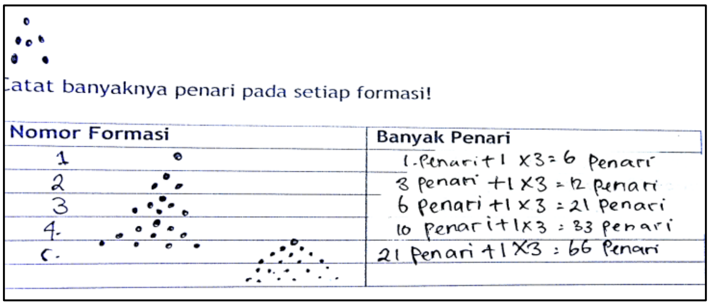


Figure 5.28. First Pair's Solution for Lesson 4 Task 1

The second pair doesn't have difficulties to solve the activity except for drawing task. For the last question, they record their addition, start from the 10th until they got 48 dancers (see Figure 5.29).

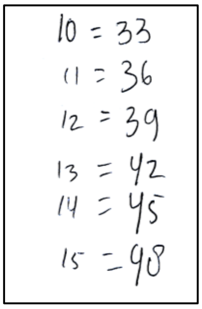


Figure 5.29. Second Pair's Solution for Lesson 4 Task 1

In the second activity, the students were work with a formal algebraic expressions. The researcher told a story that the committee wants them to make a board sign which indicates the formula they used to build the formation in Activity 1. There are six options and the students had to judge which of those are correct and explain the reason. The options are: (1) $D = n + 1 \times 3$, (2) $D = 3 \times n + 1$, (3) $D = 3 \times (n + 1)$, (4) $D = 3 \times n + 3$, (5) $D = (n + 1) \times 3$ and (6) $D = 3 \times 1 + n$.

The students solved this problem by using two approaches. First, they tried to replace n with numbers and check whether it gives the same answer as the given word formula. Second, the students observed the structure of algebraic expression, including the existence of brackets and the order of operations (multiplication first and then addition). Figure 5.30 shows the example of students' reasoning.

c. yang c walaupun (n+1) di blakang tetapi ada tanda kurung jadi yang ada tanda dihitung pertama

Translation:
C, even though (n+1) is placed behind, it has a bracket. Hence it will be calculated first.

Figure 5.30. One example of students' translating the formal algebraic expression

Even though the students were able to work with the given problems in the fourth lesson, the activity was not really meaningful. The students seemed to merely focus on numbers without seeing the role of variable. The notion n come meaningless. Therefore, a lot of revision was applied for the fourth lesson (see more in the 5.2: Improvement of the HLT 1 section).

5.1.6 Lesson 5

The fifth lesson conducted on March 3, 2015. In this occasion, the students were working in three activities which have four different representations of a pattern, including words formula, visual representation, number string and formal algebraic expression.

The first activity is about an artisan called Mr. Husnul who create a crown for Gending Sriwijaya Dance, a typical dance of Palembang. In the worksheet there are a dots representation of the produced crowns in Mr. Husnul's house. It is said that he has two crowns stock from last year and nowadays he can produces three crowns for each day. There also a number string with the number of crowns from the starting until the third day production. After giving the context, the researcher asked the students' understanding toward the notes which embodied in the number string form. The students were able to recognize the relation between the words formula and the visual representations of it in dots and in the number string. They also able to continue the next numbers of the pattern. See Figure 5.31 to observe the students' argument.

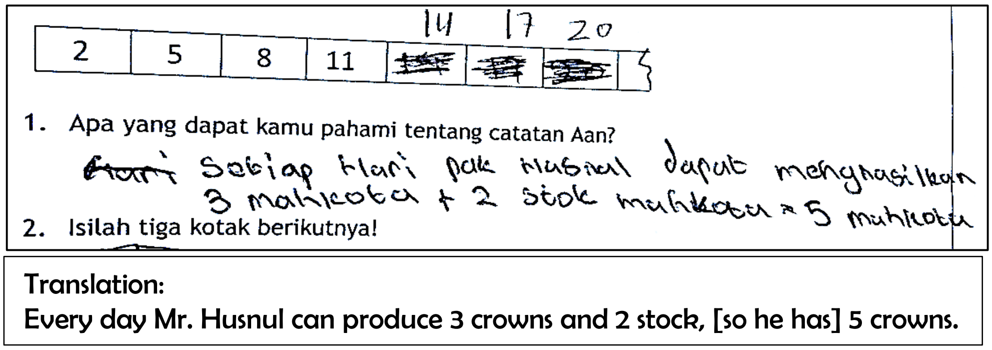


Figure 5.31. Understanding the number string

The next task is to determine the number of dancers if Mr. Husnul already works for 30 days. The first pair used keep listing method by fulfill the string until the 14th day working. After that, they used the result on the 10th days as their new starting point and incorrectly multiplied it with 30. Figure 5.31 shows their answer.

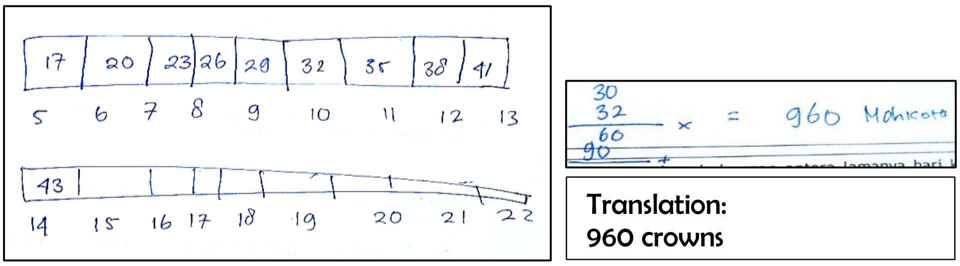


Figure 5.32. Listing and multiplying strategy

The second pair multiply the working days, which is 30, with 3 and add 2 to its result. They also able to re-stated the words formula as the answer for the fourth question which asks about the general formula of the relation between the length of day works and the number of produced crowns. The expectation was they able to write the algebraic formula, but in the end it didn't come as it.

The fifth question is to determine the number of Mr. Husnul's working days if there are 185 produced crowns. The second pair used division method, they divide 185 with 3 and got 61. The remainder is 2 and they considered it as the stock from the previous year.

The second activity is about another artisan called Ibu Tyas, she create the crown for Tanggai Dance, another typical dance from Palembang. Here, the students should have been given the algebra formula of the crown production. However, since the algebraic formula was not conceptually built in the students' cognitive structure, the researcher decided to skip this activity.

We moved to the third activity in which the students were asked to compare the number of crowns produced by Mr. Husnul and Ibu Tyas. The researcher told them in words formula that Mr. Husnul can produces three crowns each day and has two stocks from last year, while Ibu Tyas has one stock and able to creates four crowns each day. Fragment 14 illustrated the discussion of the students to answer the aforementioned problem.

Fragment 14: Who Has the Most?

[1] Researcher	:	Which one your answer, Kayla and Dela? Who has the most?
[2] Kayla	:	Both.
[3] Dela	:	The same.
[4] Researcher	:	Rafi and Adi?
[5] Rafi & Adi	:	That one. (*Pointed to Bu Tyas' works*)
[6] Researcher	:	Oh, who has 1 and 4[each days], right?
[7]		Dela said both of them will have the same.
[8] Dela	:	Eh, no! [She seemed to lose her confidence]
[9] Rafi	:	Yes, it is.
[10]Adi	:	It will be the same if you add it, if it doesn't it will be different.
[11] Dela	:	2 + 3, 1 + 4
[12]Researcher	:	2 + 3, 1 + 4, that's for what day's work?
[13] Dela	:	For the first day.
[14]Researcher	:	The first day, in the second day who will has the most?
[15]All students	:	This (*pointed to Bu Tyas' works*)
[16]Researcher	:	Why?
[17] Dela	:	This merely add by 3 while the other add by 4.
[18]Researcher	:	How about in the 10^{th} day?
[19]Rafi & Adi	:	This [pointed to Bu Tyas' work.
[20]Researcher	:	Is it possible if Bu Tyas has less crown [from Mr. Husnul]?
[21]Rafi & Adi	:	No.

5.1.7 Posttest

After conducting the fifth lessons of pattern investigation, we conducted a posttest on March 6, 2015. The posttest, which consisted of four problems, was given to the four focus students (see complete problem in Teacher Guide). After doing the test, the students were interviewed to recheck the understanding of researcher toward the students' written works. The following discussion will discuss each of the posttest's item.

Problem 1

In the first problem the students encountered a number string with five repeating colors. The students were asked to figure out the color of the 27th, 550th and 159.638th from the following pattern given in the Figure 5.33.

Figure 5.33. Number string problem of posttest

To solve this problem, the focus students employed different strategy. Dela for instance, stated a non-mathematical argument. She said since 2 comes in blue and so does 7, the 27th should be blue. During the interview the researcher asked what will happen with the color of number 23, since 2 is blue and 3 is purple. She cannot answer and realized she had to find another method.

In one hand, Kayla used keep counting method until she got 27 and then gave up to answer the rest. On the other hand, Adi and Rafi observed the pattern and find the regularities of it. One example of Adi's argument recorded in the following Fragment 15. The fragment illustrates how Adi shared his idea in finding the color of the 550th strip. At first he answered it as green, but during his explanation he changed it.

Fragment 15: The Color of the 550th Number

[1] Adi	:	27 is blue
[2]		550…
[3]		... [Thinking]
[4]		It should not be green
[5] Researcher	:	What it should be, then?
[6] Adi	:	It should be red.

[7] Researcher : How could you get red?
[8] Adi : Red?
[9] Researcher : Yes?
[10]Adi : 500 is red
[11]Researcher : How could you know that?
[12]Adi : Because it is the multiplication of 20
[13]Researcher : 20 can be multiplied to be 500?
[14]Adi : Yes
[15] So, this 500 is red.
[16] And then this 50, the 40 is red.
[17]Researcher : Why?
[18]Adi : Because 20 is red, hence 40 is red
[19]Researcher : So if 20 red, 40 is also red...
[20] Then?
[21]Adi : This is should be red.
[22]Researcher : How about the 50?
[23]Adi : 50 is also red.

Based on the students answer Adi and Rafi are developing a good reasoning, but Kayla and Dela still have a lack in determining the unit pattern of a pattern. The unit pattern refer to the smallest pattern of a repeating pattern which will be repeated. This sense is important to develop the very basic generalization ability, to know when a special feature of a pattern will occur in the future.

The second question asked the students to observe the strip with yellow, green and red colors. They have to investigate what is the color of the strip if they add the numbers in the yellow with green strips. After that in the third question they need to fill the addition table as in the following Figure 5.34.

+	Merah	Kuning	Biru	Ungu	Hijau
Merah					
Kuning					Merah
Biru					
Ungu					
Hijau	Hijau				

[2]

Figure 5.34. Addition pattern table

[2] Written in Bahasa, *merah* = red, *kuning* = yellow, *biru* = blue, *ungu* = purpe, *hijau* = green

To solve this problem, Dela, Kayla and Adi randomly assigned the color in the table. Differently, Rafi checked the color given, for instance red and red is red because the result of it is always red. The following Figure 5.35 shows his solution for this problem.

+	Merah	Kuning	Biru	Ungu	Hijau
Merah	merah ~~kuning~~	kuning	Biru	ungu	hijau
Kuning	Kuning	Biru	ungu	hijau	Merah
Biru	Biru	ungu	hijau ~~kuning~~	merah ~~kuning~~	kuning
Ungu	ungu	biru	kuning	merah	hijau
Hijau	Hijau	ungu	biru	kuning	merah

Figure 5.35. Rafi's solution for addition pattern

Problem 2

The second problem is aimed to check the students' awareness toward the structure of a visual representation of a pattern. In the worksheet, three types of pattern are given including square, triangular and pentagonal numbers. The students need to continue those patterns. The problem is given in the following Figure 5.36.

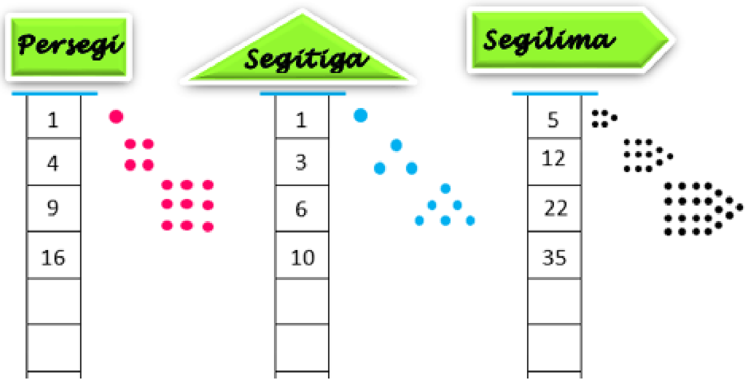
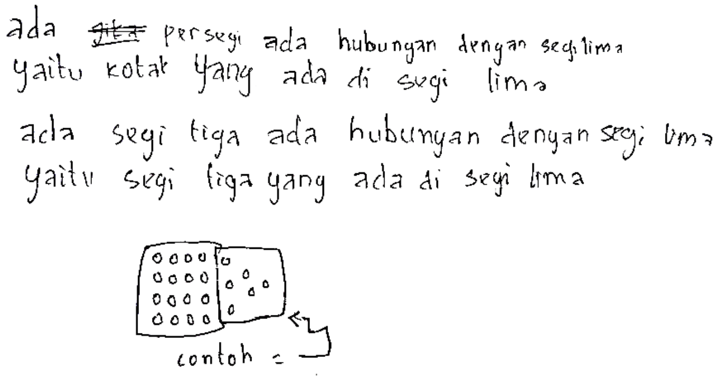


Figure 5.36. Square, Triangular and Pentagonal Pattern

Adi and Rafi were able to fill the box and gave the appropriate illustration. In addition, when it is asked to investigate the relation between those three patterns, Adi got an insight that the pentagonal numbers can be constructed by combine the square and triangular numbers. The following Figure 5.37 shows his explanation.



Translation:
There is. The square has its relation with the pentagon, which is the square part in the pentagon.
The triangular has a relation with the pentagon, which is the triangle part in the pentagon.

Figure 5.37. A square and a triangular performed a pentagonal pattern

Problem 3

The third problem is similar with the sixth problem of the pretest, which is about the stamping problem. Given a stamp, which can produce two flowers in each stamping. The students had to: (a) draw the number of flowers after the fifth stamping, (b) determine the number of flowers in the 10th stamping, (c) 100th stamping, (d) 199th stamping and (e) n^{th} stamping.

All students have no difficulty to draw the flowers of the fifth stamping. They also able to determine the number of flowers in certain number of stamping, except Rafi. Somehow he mixed the way he saw the structure of pattern. From the given picture as in Figure 5.38, he drew different representative of the 10th stamping and got 200 flowers.

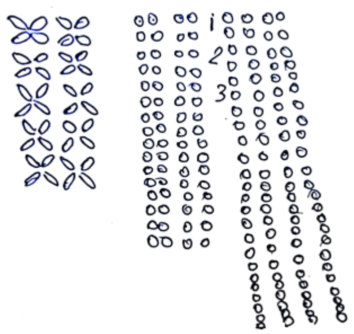


Figure 5.38. Rafi's illustration for 10th stamping

For the generalization until *n* stamping, only Adi who were able to state that *n* is any number of stamping. Hence, he concluded that the flowers will be as much as two time the number of stamping.

5.2 Improvement of the HLT 1

Reflect on the observation of the preliminary lessons in the first cycle, we found out that the pattern activities is helpful to enhance the students' awareness toward generalization of an algebraic pattern. The generalization is supported by the ability of the students in seeing the structure of a pattern which is visualized in geometrical form.

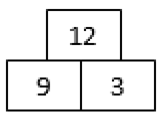
However, we realized that the students encountered several difficulties in doing some activities, especially in order to understand the word of "any number of *n*" which refer to a variable and to use "algebraic expression". The forced learning toward formal representation of a variable, especially in the fourth lesson, creates a cognitive gap in the students' way of thinking. Instead of using their understanding of pattern and generalization as they did in the previous meetings. Here, even though the students used the algebraic notation explicitly, they did a less algebra. Hence, we concluded that a formal algebraic notation is not suitable yet for the students. Therefore, we made some minor and major revisions toward the learning trajectory as can be seen in the following description.

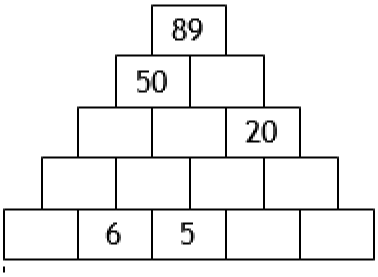
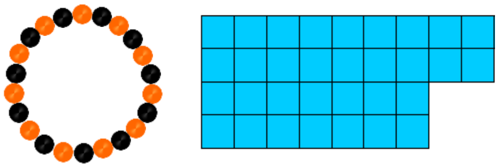
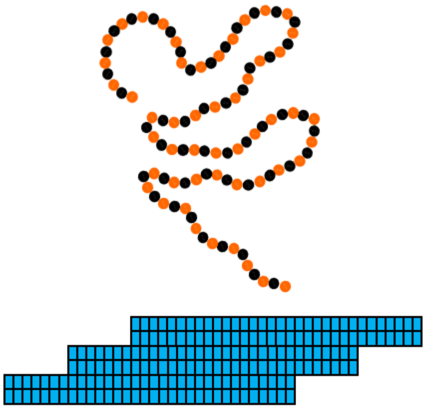
5.2.1 Pretest

For the pretest, we made a minor revisions about the clarity of the questions as can be seen in the following Table 5.2.

Table 5.2

Finding and Revision Plan for Pretest

No.	Original Problem and Finding	Revision
2.	The second problem is hardly understood by the students. The instruction is clear, but not enough to help the students who never work with this kind of problem have an idea of what to do. Original question: *The value in a square is the sum of the two numbers directly below it.*	Add an example of how to fulfill it. Can be done just for a pyramid with two levels, i.e.: 

No.	Original Problem and Finding	Revision
		
4.	The fourth problem is aimed to encourage the students to "not counting by one" when determine whether the specific objects (in this case the string of beads and a group of blocks) come in even or odd numbers. However the instruction is confusing, the students keep asking what it is mean by "without counting" and in the end they merely counting it by one. Also, the beads and the blocks were not too tiring to be counted by one, that's might be one reason why the students' did not trying to find other efficient strategy. 	Delete the words "without counting" and use longer beads and blocks in the given pictures, so the students will get difficulty to count it by one and develop more sophisticated strategy to handle it. Add more beads and blocks. 
5.	The term "without counting" in the fifth problem were also confusing.	Leave out the word "without counting". The problem merely asked the students to explain their strategy in order to decide which picture has the most beads.

5.2.2 Lesson 1

There are minor revisions we made for the first lesson. This is about the formulation of question in Bahasa Indonesia which was confusing the students and an unclear information given in the problems. The revision applied for the second worksheet which is about three colors costume for *Saman* Dance.

Table 5.3

Finding and Revision Plan for Lesson 1

No.	**Original Problem and Finding**	**Revision**
1.	The color of costume was stated as pink-blue-yellow, but the printing result was not clear and the pink color is more looks like red than pink.	We tried to re-print but still the pink is looks like red. Hence, we changed the statement of the color costume as: red-blue-yellow.
3.	The problem is to find the color of costume used by the 100th dancer from the following pattern.  During the first cycle, the students used multiplication of 10 strategy. For instance if the color of the 10th student is pink, then the number of 30th is also pink. This strategy is correct when it is used to deal with two alternately colors, but not with 3 colors.	Add this possibility in the HLT and teacher guide, so the teacher will be prepare when the students use this strategy (see Table 5.4).
4.	My Indonesian translation for the following problem wasn't clear. *If there are 53 students, would the formation has three equal color of costumes? Why or Why not?*	Revise the Indonesian translation of it.
6.	Original question: *If the 60th student use yellow costume. What will be the costume color of the 19th student?* When working with this problem, the students thought the pattern is continued from the given strip for the previous questions. In fact, they encounter a new pattern arrangement here.	Add more information that the pattern is different and add the information of the color of costume of the three consecutive number before ask them to find the color of the first student's costume. Revision: *Suppose that we don't know the color of costume used by the first dancer, but the 60th dancer uses red, the 61st uses blue and the 62nd uses yellow. Find out the color of costume used by the first dancer!*

In line with the revision plan for the teacher guide which will be used to guide the students when solving the costume color of certain number of dancers (problem 2 and 3), the following Table 5.4 will describe in detail.

Table 5.4

New Conjecture of Students' Way of Thinking for Worksheet II Question 2 and 3

<table>
<tr><th>Problems</th><th>Conjecture of Students' Response</th><th>Guidance for Teacher</th></tr>
<tr>
<td>2. Determine the color of costume used by: (a) the 11th dancer and (b) the 27th dancer.</td>
<td>

The students continue to draw it one by one.
The students use the previous method as they used in the Worksheet I, for instance stated that the odd number dancers will use red costume, while the even number dancers will use yellow costume. But then they get confuse with the yellow costume.
They may conclude that every multiplication of 3 is yellow, every multiplication of 2 is blue. But then they confuse what will happened with the multiplication of 2 and 3 (for instance the multiplication of 6, 12, etc.).
Uses the reference of 10th, but then confuse why the 11th strip does not come with the same color as the 1st one (the following picture can be seen in the of the Teacher Guide).

If the students did not recognize this contradiction while using this strategy, the
</td>
<td>

If the students use counting by one method, keep asking them bigger and bigger number of dancers until they find it is tiring to count by one and develop more advance strategy.
If the students use reference of 10, bring that idea to the class discussion and ask the students to participate in detecting the error of that strategy.

For instance:
"Let's consider the strategy from the X Group (mentioned the name of the group and ask one representation of them to present t in the front of the class".
If the students' explanation not really clear, the teacher can help by showing a table given in the Teacher Guide which illustrated the general idea of it.
<table>
<tr><td>P</td><td>B</td><td>K</td><td>P</td><td>B</td><td>K</td><td>P</td><td>B</td><td>K</td><td>P</td></tr>
<tr><td>B</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>
"Okay, do you have any opinion regarding this strategy? Do you think we can use it?" (wait until the students rise their hand).
If all of the students agree to use this method, the teacher can shows the contradiction based on the order of the colors. "Let's have a look on it. What will be the color of the 11th dancer?" (The students will ask it as blue).
"Well, if we double the first strip, the 11th dancer will have to have
</td>
</tr>
</table>

Problems	Conjecture of Students' Response	Guidance for Teacher
3. Explain how you determine the color of costume used by the 100th student!	teacher can bring this in the discussion. • Using the multiplication of 3 as their reference. Each dancers whose number is the multiplication of 3 will use yellow. If one more than the multiplication of 3, the dancer will use red. If their numbers are two more than the multiplication of 3, the dancers will use blue costume. • Use the reference of 10 as explain the second problem. • Use the reference of 3 or others which are the multiplication of 3 (i.e.: 6, 9, 12, etc.). • Divide100 with 3 and check the remainder. Since the remainder is 1, then the answer is red.	the same color as who?" (The students will answer it as 1, who use the red costume). The students will realize their error. The teacher can give more question as, "So, do you think it is wise to use 10 as our reference?" (The students will answer as 'no'. If there still the students who say 'yes', asks their reason. Let the discussion continue and asks the students who say 'no' to give their arguments as well). Finally, the teacher asks for concluding remark. "Now, what do you think will work the best to be the reference? So if it be multiplied the order of the color arrangement will not change." (The students will give several answers. The correct answer is the multiplication of 3, is it okay to use 3, 6, 9, 12 …). • If the students use the strategy of dividing 100 with 3 and check the remainder (if it is 0 then the answer is yellow, if it is 1 then the answer is red, while if the remainder is 2 the answer is blue), ask them to test their prediction to the other bigger numbers. • If the students encounter difficulties during the discussion, ask them to investigate the characteristics of the dancers' numbers which use the red/yellow/blue costumes. Start with yellow, because it is the multiplication of 3 and therefore the easiest to be investigated. For

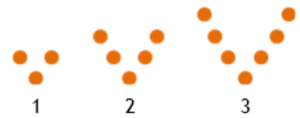
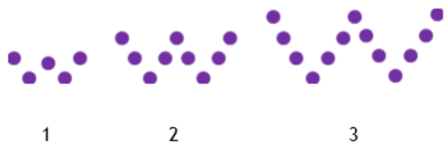
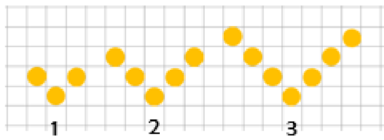
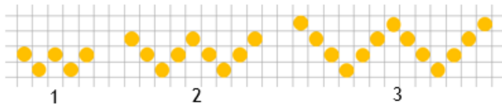
Problems	Conjecture of Students' Response	Guidance for Teacher
		instance: "Look, what numbers will come with yellow costume?" • If the students cannot achieve the conclusion, the teacher can support them by asking, "What is the color for the numbers which is the multiplication of 3? How about if you have 1 remaining?"

5.2.3 Lesson 2

We made a small revision for the second lessons. During the first implementation, we found that the students hardly see the structure of *W* formations because they got lost when seeing the parallel dots.

Table 5.5

Finding and Revision Plan for Lesson 2

No.	Original Problem and Finding	Revision
1.	The dot positions for *V* and especially *W* formations was not very clear, the students sometime get difficulties to draw.  	Make a grid, so the parallel dots will be obvious.  
2.	The formulation problem of combining 2 *V* into a *W* was not challenge the students. "*Tata compares the number of dancers in V and W formations. She claim that W patterns can be made from two V formations which is subtract by one. Do you agree with her?"*	Edit the main of the problem into the following revision: *"She claim that W patterns can be made from two V formations. Do you agree with her?"*

5.2.4 Lesson 3

We made some revisions here, in term of the context and the work load for the students. We realize that the time allocation is not that much to be able to work with 2 activities (square and triangular number pattern) in a day. We also notice that the triangular formation is quite hard for the students to be learned in the third meeting.

Table 5.6

Finding and Revision Plan for Lesson 3

No.	Original Problem and Finding	Revision
1.	We use context of girl's typical dancing for the second and the third meeting and the male students were not pleased and feel unmotivated.	Change the context for the third meeting into formation of martial art flash mob, which is involving both of girls and boys. 
2	Time allocation was not enough to do both of square and triangular formation.	Only focus with the square formation for the third meeting and move the triangular number pattern for another meeting.

5.2.5 Lesson 4

The fourth lesson in the first cycle was done properly but it is not meet the important goals of this study. Hence, we made a major revision here. Instead of working with variables and all kind of abstract representation of algebra, we will focus on exploration and elaboration of the structure of a pattern. We will optimizing the use of visual support to do it. The following HLT is a new trajectory for the fourth lesson. In this meeting, the students will working with the rectangular formation. The context is still continue from the previous meetings, which is about a cultural exhibition at Palembang Expo.

Topic: Creating a Rectangular Formation

a. Prior Knowledge

The students are expected to:

- Create a general formula to a particular pattern.

- Predict the “next terms” of a pattern.
- Evaluate the relation between the numbers.

b. Learning Focus

Through the learning process, the students are expected to:

- Evaluate the structure of a pattern.
- Create a general formula of a pattern.

c. Description of the Activities

Phase 1: Transformation

The teacher open the lesson by bringing up the context of cultural exhibition in Palembang Expo. The setting is continuing the previous meeting, which talk about a formation for Martial Art Flash Mob. Previously, they designed a square formation in order to fulfill the request of the committee. But, due to some conditions, the committee plan to change it into different formation. The teacher will explain this to the students. The teacher can use the following sentence, “Children, we have a problem with the square formation we made yesterday. The committee discuss it with the cameraman from PAL TV, the local television which will documenting all event, and that person refuse the committee plan. He said, the square formation will not give a good view on TV. He suggest the committee to use a rectangular form. The committee wants our help again. What do you think about it? Can you do that? Now, please discuss it with your pairs, how to transform a square into rectangular formation by adding a minimum dancers in each formation”.

The teacher distribute the worksheet and ask the students to work with their pair. The aim of this activity is to enhance the students awareness toward the structure of a pattern by observe its visual support.

Phase 2: Find the Rectangular Formula

After the students familiar with the shape of the rectangular formation, in group of four the students will discuss the number of dancers in certain rectangular formation. The given questions will lead them to generalize a general formula of a rectangular formation.

Phase 3: The Introduction of Triangular Formation

The activity will be continued by asking the students to equally divide the dancers in the rectangular formation into two groups: the dancers with black costumes and white costumes. This problem will guide them to the triangular formation. Again, this activity is aimed to: (1) enhance the students' awareness toward a structure of a pattern from its visual representation, (2) encourage the students to develop a reasonable argument of the properties of the even numbers (even numbers always can be divided by 2 and the multiplication of an even number with a whole number is always even).

Phase 4: Class Discussion

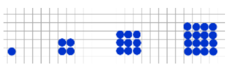
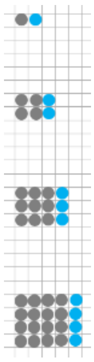
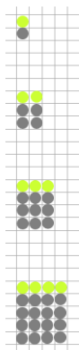
The teacher will arrange a class discussion and ask the representation of some groups to present their works. The other students should react to the presenter. The teacher can also attract the students to discuss which strategy is counted as more sophisticated, more efficient or more effective to solve the similar problem.

d. Conjecture of the Students' Responses and Guide for Teacher

The following Table 5.7 shows the conjecture of students' responses toward the given problem and a guidance for the teacher to manage the discussion and to support the students in constructing their understanding.

Table 5.7

Conjecture of Students' Responses for the 4th Lesson (Revised)

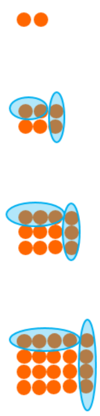
Problems	Conjecture of Students' Response	Guidance for Teacher
1. With the minimum number of added people, could you transform this square formation into rectangular formation? 	We gave a space in the worksheet that can be used by the students to draw the additional dancers. This is the possible correct answers:  Or 	If the students add the dancers not in the minimum number, the teacher can ask them to re-think about it, by asking: "If you add less people, can you still have a rectangle?"

<table>
<tr><th>Problems</th><th>Conjecture of Students' Response</th><th>Guidance for Teacher</th></tr>
<tr><td>2. Investigate how to find the general formula of rectangular formation!</td><td colspan="2">The final goal of the task is to have the students formulize the general formula of a rectangular pattern. To support them, some preliminary questions were asked to help them in exploring and elaborating their ideas.</td></tr>
<tr><td>a. The number of dancer in the 5th formation.

b. The number of dancer in the 10th formation.

c. The number of dancer in the 99th formation</td><td>The number of dancers in the 5th formation is 30 people.
The number of dancers in the 10th formation is 110 people.
The number of dancers in the 99th formation is 9900 people.

We predict the students will use these following strategies:
• Continue from the previous drawing.
• Count the addition in each formation.



Or record it on the table or not in the table

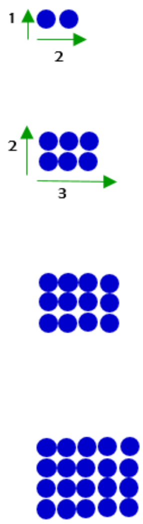
<table>
<tr><th>No. Formasi</th><th>Banyak Penari</th><th></th></tr>
<tr><td>1</td><td>2</td><td></td></tr>
<tr><td>2</td><td>6</td><td>+4</td></tr>
<tr><td>3</td><td>12</td><td>+6</td></tr>
<tr><td>4</td><td>20</td><td>+8</td></tr>
<tr><td>5</td><td>30</td><td>+10</td></tr>
<tr><td>6</td><td>42</td><td>+12</td></tr>
<tr><td>7</td><td></td><td>+14</td></tr>
<tr><td>8</td><td></td><td></td></tr>
<tr><td>9</td><td></td><td></td></tr>
<tr><td>10</td><td></td><td></td></tr>
</table>

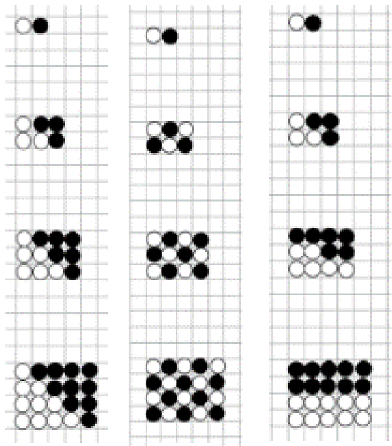
• Start from the square formation Add the number of dancers in a square formation with the number of the formation.</td><td>If the students keep using drawing method, challenge them to find the number of dancers in bigger number of formations.
If the student merely able to generalize by using square formula rule, ask them to think further without finding the square first. The teacher may asks: "Can you find another strategy? The strategy by which you can directly find the number of dancers in the rectangular formation without finding the number of dancers in the square formation first?"
During the class discussion the aforementioned type of answers should be discussed. Hence, the students will think about the most efficient answers.</td></tr>
</table>

Problems	Conjecture of Students' Response	Guidance for Teacher

Previously, we know that the number of dancers in a square formation is equal to the square of the number of its formation. Hence, for instance, in the 10th formation there will be $10^2 = 100$ dancers.

- Consider the shape of the rectangle and find the relation between the length, the width and the number of formation. They may conclude that the length of a rectangle is equal to the number of formation, while the width is equal to the number of formation + 1.
- They may also see the relation between the length and the width of the rectangular and the number of dancers and conclude that the number of dancers is equal to length × width.



Problems	Conjecture of Students' Response	Guidance for Teacher
So, in general how could you determine the number of dancers in the certain formation?	The students will make a conclusion based on their strategies.	
3. Reconsider the following rectangular formation. Divide the dancers on the formation such that half of them will use black costumes and the other half use white costumes. Investigate why you can always do that in each formation!	 (a) (b) (c) A square formation always have an even number of dancers. This happen due to one of the number of formation (or the number of formation+1) is an even number. Hence, even × odd = even. Since the number of dancers is an even number, it can be divided by 2. The first aspect is important to encourage the students to elaborate that the multiplication of a number with an even number will always an even number. The second point is to encourage the students to reason about the properties of even numbers.	There are two important aspects to be considered by the teacher when giving a guidance questions for the students' discussion: (1) Does the number of dancers in the rectangular formation always even? (2) Why we can always get an equal number of dancers in black and white costumes in each formation? The teacher also needs to ask the students to carefully look the structure of the picture they made when solving the problem. For instance, the picture (a) will give a better illustration than the picture (c). If no one divide the rectangular formation as in the picture (a), the students can bring it up in the discussion, for instance: "I have an idea to divide it as this picture [shows the picture]." "What do you think about it? Can it give an easy insight to the audience that we have two equal groups with black and white costume?"

Problems	Conjecture of Students' Response	Guidance for Teacher
		"Have a look in this configuration, do you have an idea what kind of shape which is resemble to the arrangement of this black and white costume?"

5.2.6 Lesson 5

We made a major revision for the fifth lesson as well. We decided to focus on the geometrical proof of an algebraic number pattern through this lesson. The students will working with triangular formation. Here, the students should be able to work flexibly in three different formations: square, rectangular and triangular formations. We prepared a few problems only for this lesson to make sure the discussion going in depth and rich.

Topic: Triangular Formation

a. Prior Knowledge

The students are expected to:

- Create a general formula to a particular pattern.
- Predict the "next terms" of a pattern.
- Evaluate the relation between the numbers.

b. Learning Focus

Through the learning process, the students are expected to:

- Developing a strategy to find the value of variable as an unknown number.
- Formulizing a reasonable argument by observing the structure of pattern supported by its visual representation.

c. Description of the Activities

Phase 1: The Half

The teacher will start the lesson by discussing the pattern of white and black costume arrangement as is agreed in the previous meeting. "Observe this figure [showing the illustration of the black-white costume arrangement in the rectangular formation, see Teacher Guide]. Do you notice what is the shape constructed by the dancers in white costume?". "Can you draw the next formation of dancers in white costume?" The students distribute the worksheets and ask the students to work in pairs.

Phase 2: The Fastest Method

After the students are able to recognize the shape of triangular formation, the teacher distribute the next worksheet which is about finding the number of dancers in the certain triangular formations. Specifically, they have to find the number of dancers in the 5th, 10th and 100th rectangular formation which use the white costumes. It is the same as if we ask them to find the number of dancers in the 5th, 10th and 100th formations of a triangular formation. The tasks should be done in group of four. The aim of this activity is to enhance students' structure sense and developing an efficient strategy to deal with a growing pattern with growing difference.

Phase 3: Unknown Formation

The task is continued by a challenge to find the number of formation while the number of dancers with white costume are given. The aim of this problem is to encourage the students to develop a strategy to find the value of unknown. In the problem, it is given that there are 210 dancers with white costume. The students should find the number of its rectangular formation. A class discussion will be conducted after the students finish in discussing this problem.

Phase 4: Proof Using Visual Support

After the class discussion, the students will investigate a problem related to triangular and square formations. In the worksheet, the students will see a prediction given by a student called Laura who claims that two consecutive triangular formation can perform a square formation. The students have to evaluate whether the statement is always work. The aim of this problem is develop the students' ability in doing geometrical proof which is supported by a visual representation support.

Phase 5: Class Discussion

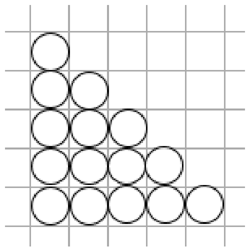
The teacher will arrange a class discussion and ask the representation of some groups to present their works. The other students should react to the presenter. The teacher can also attract the students to discuss which strategy is counted as more sophisticated, more efficient or more effective to solve the similar problem.

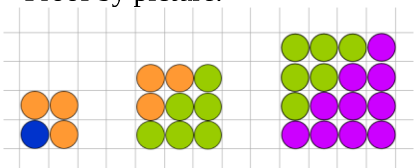
d. Conjecture of the Students' Responses and Guide for Teacher

The following Table 5.8 shows the conjecture of students' responses toward the given problem and a guidance for the teacher to manage the discussion and to support the students in constructing their understanding.

Table 5.8

Conjecture of Students' Responses for the 5th Lesson (Revised)

Problems	Conjecture of Students' Response	Guidance for Teacher
1. Draw the dancers with white costume in the 5th rectangular formation!		If the students use counting from 1 until the number of formation (see the second strategy of the second problem), ask them to find a faster strategy to do the addition. For example:
2. In the 10th formation, how many dancers who use the white costume?	• Continue to draw the 10th formation of triangular formation. • Observe the growing pattern in each formation and then add: 1 + 2 + 3 + 4 + … + 10 = 55 people. • Count from the number of dancers in the 10th formation of the rectangular formation and divide it by 2 The number of dancer in the 10th formation of the rectangular formation is $10 \times 11 = 110$ people. Hence, in the 10th formation of the triangular formation there will be 110 divided by 2 = 55 dancers.	$\overbrace{1+\underbrace{2+3+4+5+6+7+8+9}_{11}+10}^{11}$ $5 \times 11 = 55.$
3. Write your fastest strategy to find the number of dancers in the 100th formation who use the white costumes.	• The students may explain that to add from 1 until 100 is the fastest startegy. • Other students may argue that count from the numebr of dancers in the rectangular formation which is divided by 2 is the fastest.	The teacher asks the students to discuss which strategy give the fastest, the most understandable and the most accurate based on their opinion. Based on the efficiency and accuracy, they may choose the "rectangular

<table>
<tr><th>Problems</th><th>Conjecture of Students’ Response</th><th>Guidance for Teacher</th></tr>
<tr><td></td><td></td><td>formation divided by 2” strategy, because addition from 1 until 100 has a high level of error possibility, rather than multiplying 100 and 101 and divide its result by 2.</td></tr>
<tr><td></td><td>The students may do an error by multiplying the number of dancers in the 10th formation with 10.</td><td>The teacher can shows the illustration of the dancers who use white costume in the 10th formation and re-draw it 10 times (the illustration is given in Teacher Guide).</td></tr>
<tr><td>4. What is the number of rectangular formation which has 210 dancers using white costumes?</td><td>Find out the number of total dancers in the rectangular formation, which is $210 \times 2 = 420$.
And then find two consecutive numbers which multiplication result is 420.
Probably, the students will start by checking that $20 \times 20 = 400$ and try 20×21.
And then, they conclude that 420 orang terletak pada formasi ke-20.</td><td>The teacher can asks the students to observe the relation between triangle-rectangular- and square formations. For instance:
“If you have 210 dancers in white costume, what is the total number of dancers?”</td></tr>
<tr><td>5. Laura predict that if she adds the number of dancers in two consecutive triangular formations, she will get a square formation!
Do you agree with her? Help Laura to explain her prediction!
</td><td>Proof by observing the relation between the number of dancers between square and triangular formations.
<table>
<tr><th>Triangular Formation</th><th>Square Formation</th></tr>
<tr><td>$1 + 3 = 4$</td><td>4</td></tr>
<tr><td>$3 + 6 = 9$</td><td>9</td></tr>
<tr><td>$6 + 10 = 16$</td><td>16</td></tr>
<tr><td>$10 + 15 = 25$</td><td>25</td></tr>
<tr><td>$15 + 21 = 36$</td><td>36</td></tr>
</table>
Proof by picture.
</td><td></td></tr>
</table>

5.2.7 Posttest

We made a number of revisions for the posttest items. We modified the formulation of the problems and focus on the observing structure related problem. After conducting the lessons in the second cycle, we also added some problems which was prepared as the discussion topic, but due to the time limitation it was not given to the students. The test items consist of the following topic: (1) repeating pattern which is represented by the number strings consist of five repeating colors, (2) testing the students' reference of unit pattern which is represented by string of beads with three repeating colors, (3) structure and the relation between the pattern which is represented by a V and W problems, (4) geometrical proof of triangular and square related problems, and (5) an additional problem of Fibonacci series. The complete problem can be seen in the Teacher Guide.

5.3 Analysis of the Second Cycle

5.3.1 Pretest

After reflected to the implementation of pretest and analyze the responses of the students in the first cycle toward the designed problem, we revised some format of the problems and tested the new designed test to the subject of the second cycle. We conducted the pretest of second cycle on Friday March 6th, 2015 for 32 students of VA MIN 2 Palembang. The aim of the pretest in the second cycle is to check the students' prior knowledge regarding to the topic. Due to the limitation of the researcher, not all students in the classroom will be able to be deeply observed in each meetings. Hence, we need to select a focus students. After checking the result of students' written works we selected 8 students to be interviewed and reflect from the both result and discussion with the teacher, we selected four students as the focus group.

Problem 1

The first problem contains five questions about continuing pattern: (a) a number series less than 100, (b) a number series from before to more than 100, (c) a series of odd numbers, (d) a series of even numbers, and (e) Fibonacci series. Most students were able to do the four first questions. None of them were able to solve the pattern of Fibonacci.

The problem is about number pyramid, same as is used in the first cycle. We added an example of how to fulfill the pyramid. Therefore, the students in the second cycle have better understanding toward the instruction of the problem compared with the students in the first cycle. In the following Figure 5.39, Arkam shows a good consideration of the relation between numbers in the upper and lower sides of the pyramid. Unlike the majority of the students who got lost in the last and one from last level of the pyramid, he properly managed the number he chose.

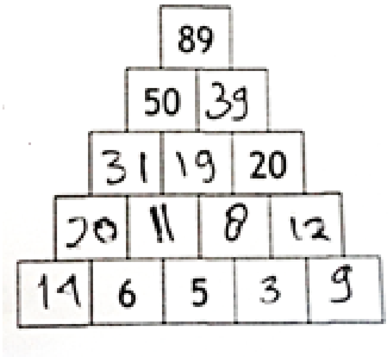


Figure 5.39. Number pyramid solution

Problem 3

The third problem is about basic arithmetic rules and operations, including adding, subtracting, multiplying, dividing and squaring. Similar with the result in the first cycle, most students were able to solve for questions in this problem section. However, some students have trouble in determine the rule of operations when more than one operation is included, i.e. addition and multiplication with has no brackets in the operated numbers. Some errors occurred in the aforementioned cases.

Problem 4

The fourth problem is about determining whether the given pictures have an odd or even numbers of objects on it. The problem was modified from the first cycle by adding more objects in the picture to encourage the students using keep counting method. The pictures for question (a) and (b) is as in the following Figure 5.40.

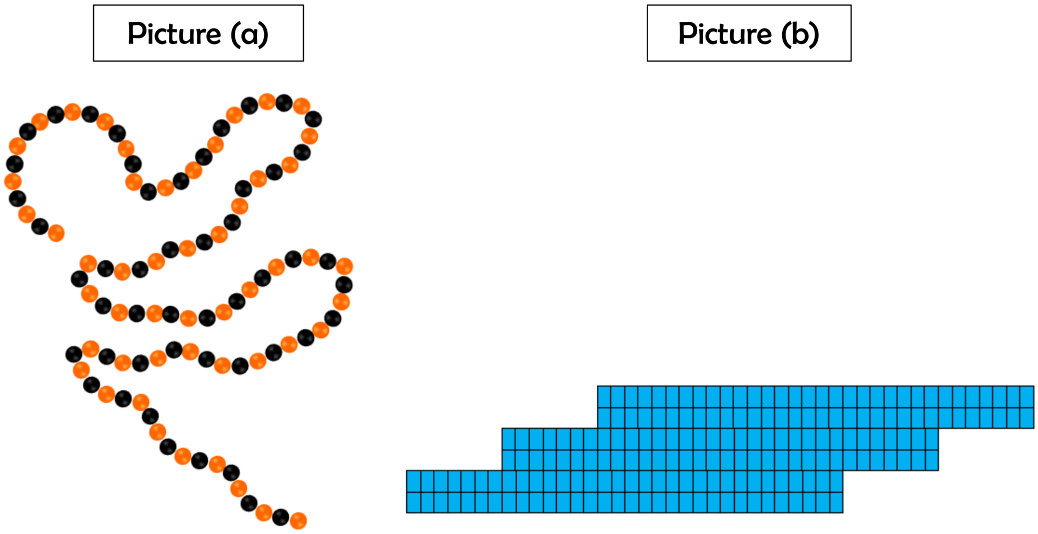
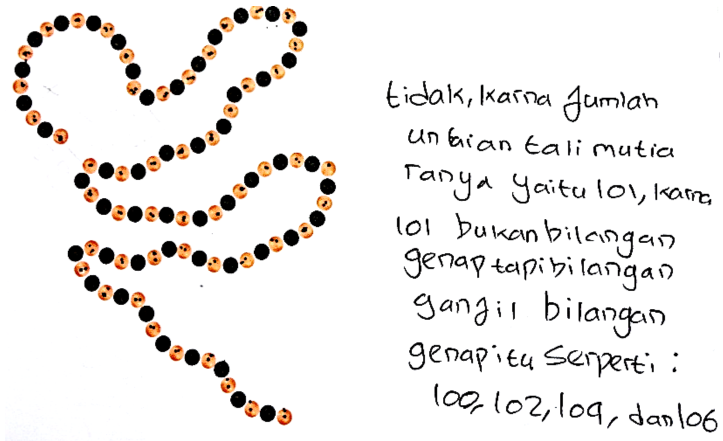


Figure 5.40. Does the Picture A has even beads?
Does the Picture B has even beads? Why?

The expectation is the students will find it is tiring to count the beads or the blocks one by one, because it is a lot of work. However in fact, counting by one is still become the students' favorite method. See one example of students' work in Figure 5.41.

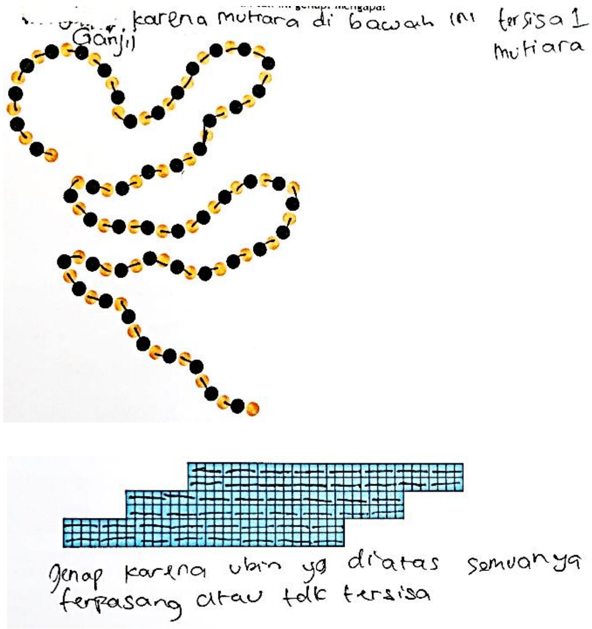


Translation:
No, because the number of beads is 101, and 101 is not an even number, but odd number. The example of even number are 100, 102, 104 and 106.

Figure 5.41. Counting by one strategy

Some other counted it separately between the orange and the black beads for Picture A. There are also the students who counted in group of ten.

Only a few students who thought about more efficient strategy by pairing the beads and grouping the blocks in four instead of counting by one. The following Figure 5.42 gives the picture of students' works.



Translation:
Odd, because the beads below has one left beads.

Translation:
Even because all of the floor above has a pair, no one left.

Figure 5.42. Odd or Even? Conclude without Counting

When the researcher interviewed one of the students who employed this method, called Meilia, she came with a reasoning as can be seen in the Fragment 16.

Fragment 16: How Could You Know The Picture Has Even Number of Beads?

[1] Researcher	:	Why you answer that the beads have an odd number?
[2] Meilia	:	This have 1 left.
[3]		So, at first I make it even, 1-2
[4]		(*pointing to the beads and counting*) and then
[5]		I make a line (*draw imaginary line above of her answer sheet*)
[6] Researcher	:	Can you show it?
[7] Meilia	:	Like you make a line, for every 2 [beads]
[8]		So, they will be in pairs.
[9]		Orange and black is become a pair.

Problem 5

The fifth problem is asked about which picture has the most beads (see Figure 5.43).

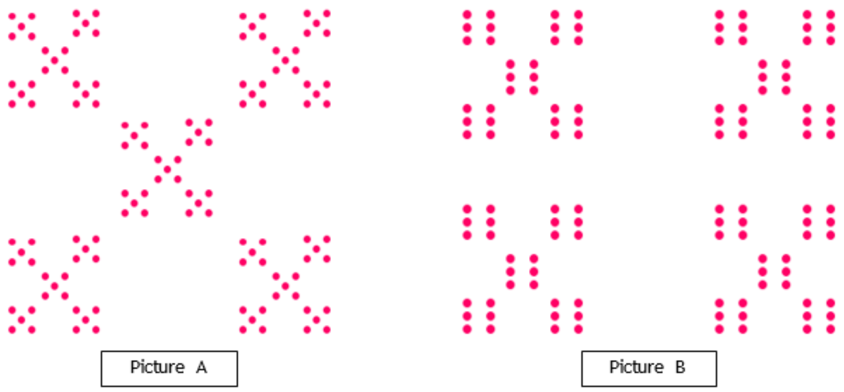
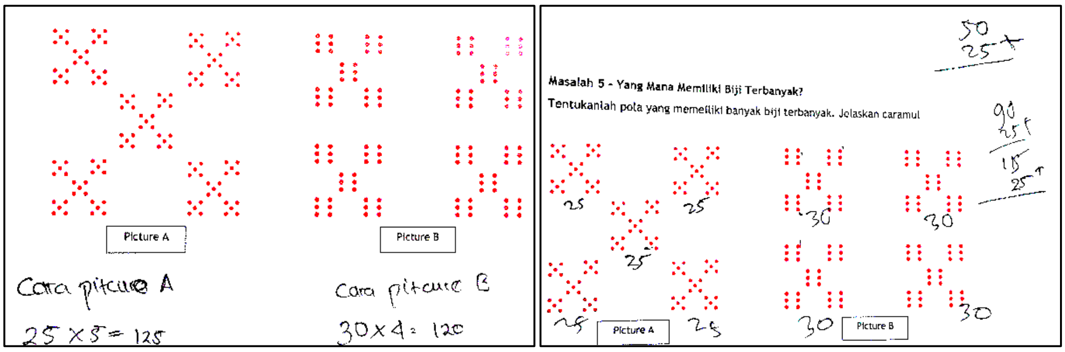


Figure 5.43. Which picture has the most beads?

Most students were able to solve this by doing multiplication (see Figure 5.44 (a)) or addition (see Figure 5.44 (b)).



(a) (b)

Figure 5.44. (a) Applying multiplication, (b) Applying addition

Problem 6

In this problem, the students were given a picture of a stamp which can produce two flowers in each stamping. To support them, the illustration of the number of flowers produced in the first, second and the third stampings are showed in the students' sheets. The first task is to draw the flowers produced in the fifth stamping. Most students were able to do it. Since the empty space given was limited, the students were able to change the direction of the pattern. In the given illustration of the four-first stamping's results, the flowers were order vertically, two for each lines as in the Figure 5.45.

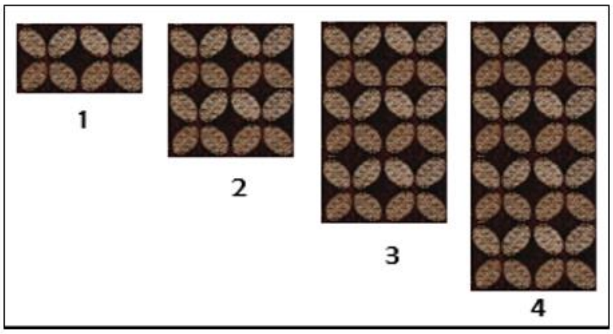


Figure 5.45. Illustration given in the problem

For the next questions which is asked to determine the number of flowers in the 10th and 24th stampings, most students were able to use the information given in the problem, that the rule is they have two flowers in each stamping. Hence, they were allowed to multiply the number of stamping with two, to get the number of produced flowers. The following Figure 5.46 shows students' solution and strategy toward the aforementioned problem.

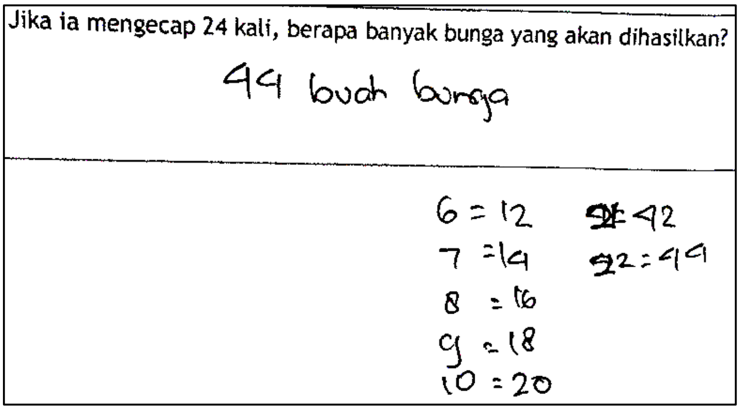


Figure 5.46. Students' record the number of stamped flowers

Reflect from the result of the students' written works, interview and discussion with the teacher, we chose four students: Ferdi, Meilia, Naurah and Arkam to be the focus group. They were selected because they have sufficient prior knowledge in basic arithmetic and based on their willingness to cooperate which is showed during the interview. The focus group is a group of students who will be observed in detail during the research. They were selected because the researcher cannot thoroughly observe all students one by one during the lesson.

5.3.2 Lesson 1

The first lesson of the second cycle was conducted on March 16, 2015. It was started by giving the context of Palembang Expo, the same context as is used in the first cycle. After that, they were told to work with their pair to find out the color of costume used by the certain number of dancers. The problem is the same as in the first cycle: it was started by questioning the color costume used by the 9th, 10th, 12th, 25th and then the 100th dancers.

In general, unlike the focus student in the first cycle, the students in the second cycle were not directly see the relation between colors and numbers. At first, the focus group use keep counting method, until they found the color of costume which is used by the 25th dancers. However, to find the 100th dancer color's costume, they develop another idea by using halving method. The students' argument can be seen in the following Fragment 17.

Fragment 17: Halving

[1] Researcher	:	If you continue to count it, how will you find the costume color
[2]		of the 100th dancer?
[3] Arkam	:	Divide.
[4] Researcher	:	Divide by what number?
[5] Naurah	:	Divide by 2.
[6] Researcher	:	Divide by two? And then?
[7] Arkam	:	Halving 50.
[8] Naurah	:	Halving again.
[9] Arkam	:	25 is red, and 25 + 25 add again until you get 100, that also red.

But then, Arkam and Naurah realize that something going wrong with their calculation by using halving method. Arkam started questioning whether 100 is even or odd number. To support Arkam, Naurah started to write down all numbers on her book, she back to keep counting method. In the end, they didn't come yet to the conclusion, because the teacher open a discussion to answer the first part of the task.

As predict in HLT, another pairs in the class directly pointed out the relation between even and odd and the color of costume. Some other pairs also used ten as their reference. Since the 10th dancer used white costume, they argued that the 100th dancer will use white costume.

After the class discussion, the students were continue to solve the question number 4 and 5. The fourth problem asks whether the 43rd and the 97th dancer will use the same color costume. To solve the problem, most of the students were use the conclusion they got from previous number: red is for odd numbers and white is for even numbers. Hence, they said the 43rd and the 97th dancers will use the same color, which is red.

The fifth question is about finding the costume color of the 31st dancer if the 70th dancer use red and the first dancer's costume color is unknown. The aim of this problem is to encourage the students to think about the order of a pattern. The red-white based on odd-even numbers can be completely different if the order changed. Therefore, in this problem they should think first if the 70th dancer use red, what will be the odd/even – red/white rules. After discussing what will be the color of costume used by the first dancer, they find out that the 31st student will use white costume. Their discussion can be seen in the following Fragment 18.

Fragment 18: Different Order

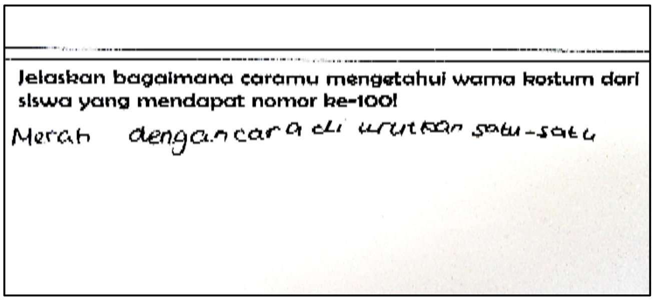
[1] Arkam	:	This is weird!
[2] Researcher	:	Why do you think it is weird?
[3] Arkam	:	Why the 70th dancer uses red costume?
[4] Researcher	:	Previously, what is the costume color of the 70th dancer?
[5] Naurah	:	White.
[6] Researcher	:	White. Now (s)he uses red.
[7]		So, do you think the order is the same as this?
[8]		(*Pointed to do given picture for number 1-3*)?
[9] Arkam	:	No (*Naurah shook her head as her agreement toward Arkam's*
[10]		*responses*).
[11]Researcher	:	So do you think the arrangement is different?
[12]Naurah	:	That's mean number 1 is white.
[13]Researcher	:	The first dancer uses white …
[14]Naurah	:	Number 2 is red.
[15]Researcher	:	Number 2 is red, so what is the color of the 70th dancer?
[16]Naurah	:	Red.
[17]Researcher	:	How about the 31st?
[18]Naurah	:	White. If the first dancer uses red, the 31st is also red.
[19]		If the first uses white the 31st uses white.

From the Fragment 18 we can see that the students aware that different arrangement can provide different rules for the pattern.

After conducting the class discussion, due to the technical issue in the classroom, we had to stop the teaching and learning session, even though we still have one activity to go. Hence, the second activity was given as the homework. The students were asked to do it with their pairs and collected it the day after.

The first part of the task is consist of three questions which is illustrated by a number string from 1 until 8 which come with different alternate colors: red, blue and yellow. The first problem is to determine the costume color used by the 9th and 10th dancers. To solve this, all of the students use keep counting method.

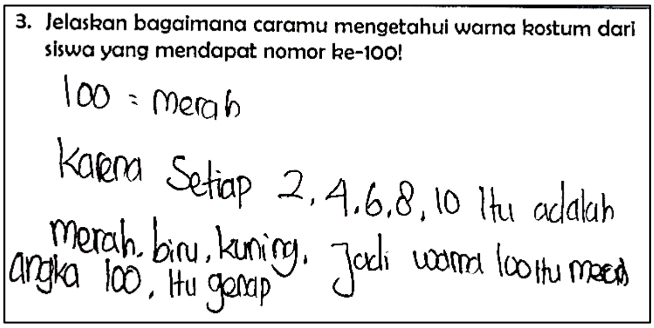
The next questions are to determine the color of costume used by the 11th, 27th and finally the 100th dancers. Some students, including the focus groups, kept use keep counting method or incorrectly employed odd-even rules (one example given in Figure 5.47).



Jelaskan bagaimana caramu mengetahui warna kostum dari siswa yang mendapat nomor ke-100!

Merah dengan cara di urutkan satu-satu

Translation:
Red, by counting one by one.



3. Jelaskan bagaimana caramu mengetahui warna kostum dari siswa yang mendapat nomor ke-100!

100 = merah

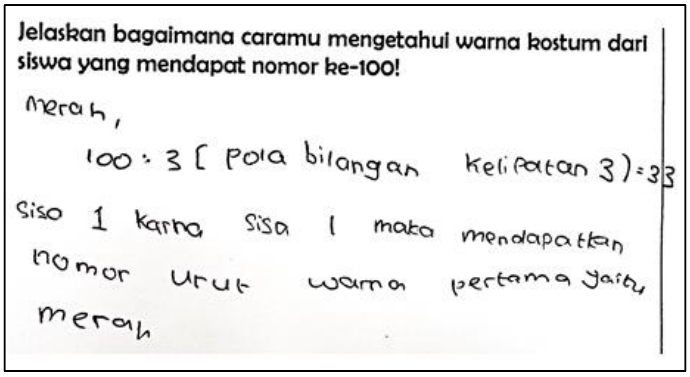
Karena setiap 2, 4, 6, 8, 10 itu adalah merah, biru, kuning. Jadi warna 100 itu merah angka 100, itu genap

Translation:
100 = red
Because every 2, 4, 6, 8, 10 are red, blue, yellow. So the 100th is red because 100 is even [number].

Figure 5.47. Students' counting by one (above); incorrect reference (below)

Figure 5.47 shows that the students didn't recognize the important of finding the unit pattern in a repeating pattern. They still employed the previous method as in the first activity and merely check for odd and even numbers without realizing there is other color involves there and the rule are no longer working.

By contrary, there are also the pairs who figure out the "unit pattern" of the given pattern and used it as their reference (see Figure 5.48).



Translation:
Red,
100:3 (the multiplication of 3) = 33.
The remainder is 1, so it get the same as the first number which is red.

Figure 5.48. Use 3 as a reference

The example of students' works in the Figure 5.48 shows the students recognize that there is a pattern which is continuously repeating. They figured out that the unit pattern is three colors: red-blue-yellow, which means 3 is important reference for them. Hence, they work with 3 and find the closest number with 100 which they can have by finding the multiplication of 3. They conclude that 99 is the number and it will have the same color as number 3. Hence the 100th dancer will use the same color as the next of the 3rd dancer used.

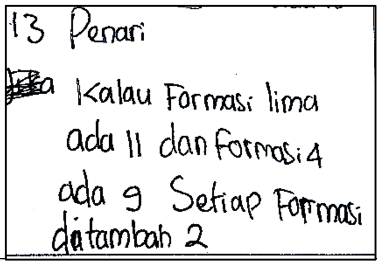
Actually to support the students develop their idea of unit pattern, a media had been prepared (see Teacher Guide). The media is used to emerge the contradiction from the students' responses when they used incorrect reference number. Later, the visualization given by the media can be helpful to show the important of reference number and to figure out which one is the unit pattern of the repeating pattern.

5.3.3 Lesson 2

The second lesson conducted on March 17, 2015. The context is still use the story from the previous meeting which about Palembang Expo. The teacher grabbed the students' attention by showing them a series of dance formation from different culture in Indonesia. After that, the topic zoomed into a traditional dance from Bali called Pahnyembrahma Dance which formed in a *V* formation. The students were asked to investigate the number of dancers which can perform a *V* shape.

After giving the context, the teacher spread out the worksheet and asked the students to discuss the first two problems together with their pairs. The first question is to draw the forth formation. The students got no difficulty in continue the formation. The provided grids help them a lot in drawing the *V* shape neatly.

The second question is to find out the number of dancer in the 6th formation. To solve this problem, most students use "adding by two from previous formation" method as predicted in HLT. One example of students' written work can be seen in Figure 5.49.



Translation:
13 dancers
The 5th formation has 11 dancers and the 4th formation has 9 dancers.
Each formation added by two.

Figure 5.49. Add by two

Figure 5.49 above is come from Meilia and Ferdy, one of the pair in our focus group. They observe the change in each number of formation as adding two dancers in the top of formation. Therefore, they develop recursive formula which connecting the number of dancers in previous formation with the addition of two dancers in every upcoming formation.

Another pair in focus group, Arkam and Naurah, used different way of seeing the structure of *V* pattern. Instead of seeing the addition of two dancers in each next formation, they observed the number of dancers in each sides and in the middle. The following Fragment 19 will give an insight of Arkam's and Naurah's way of thinking.

Fragment 19: The Dancers in the Sides and in the Middle

[1] Arkam & Naurah	:	Six in the right side and 6 in the left side and one in the
[2]		middle.
[3] Researcher	:	How about in the seventh formation?
[4]		How many dancer will be in each sides?
[5] Arkam & Naurah	:	Seven.
[6] Researcher	:	How many in the middle?
[7] Arkam & Naurah	:	One.

After the students finish with their works, the teacher manage a class discussion to discuss those problems. After that, the teacher asked the students to work again with the next part of worksheet which consists of four questions.

Problem number 3 is to draw a *V* formation which consist of 17 dancers. To solve this problem, Arkam and Naurah used the similar method which they applied to solve the previous problem. During their discussion, Naurah got confuse with the relation between one dancers in the middle and the total number of dancers. The following Fragment 20 shows their thinking.

Fragment 20: Add or Subtract?

[1] Researcher	:	How do you know how many dancers should be in each sides?
[2] Naurah	:	Divide by two and subtract by one.
[3]		Eh, divide by two and add by one.
[4]		Eh (*look at Arkam, asked for help*)
[5] Researcher	:	Divide by two and then add or subtract by one?
[6] Arkam	:	Divide by two and you will have one more left.
[7] Researcher	:	One more left? Where will you put that one?
[8] Arkam	:	In the middle.

As we predicted in HLT, some pairs got difficulty to solve this problem because they cannot understand that the given information is the total number of dancers, not the number of formation. In the following Fragment 21, one example of the students' confusedness is given.

Fragment 21: The Number of Dancers or Formation?

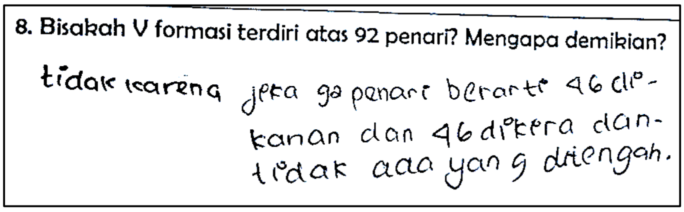
[1] Zahrah	:	Should I draw in each square?
[2]		(*Pointed to the given grids in the worksheet*).
[3] Researcher	:	No, what is asked? The total number of dancers is 17.
[4] Jihan	:	35.
[5] Researcher	:	Which information given in the worksheet?
[6]		17 dancers or 17^{th} formation?
[7] Jihan	:	17 dancers.
[8] Davina	:	The answer is 35 dancers.
[9] Researcher	:	Suppose we have a formation with 3 dancers.
[10]		How the picture of it will be?
[11]Rachel	:	That is the first formation.
[12]All students	:	(*Drawing a V formation with 3 dancers*).
[13]Davina	:	5.
[14]Jihan & Zahrah	:	One in the right, one in the left and one in the middle.
[15]Researcher	:	One in the right, one in the left and one in the middle.
[16]		Okay, so you have 3 dancers. How if you have 17 dancers?

[17]Jihan & Davina : 35 for each.
[18]Researcher : Are you sure?
[19]Zahrah : Eh.

After that the researcher asked them again to draw a formation with 3 dancers, with 5 dancers, etcetera. Then, asked them to re-read the question. Finally, they realize that 17 is not the number of formation but the number of dancers that should be in the formation.

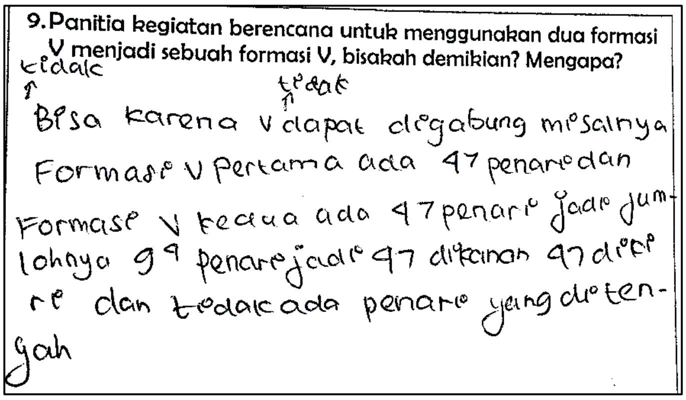
Due to the time limitation, today's lesson were not able to discuss all designed materials. Hence, the last part of *V* formations related problems were given as homework and the second activity about *W* pattern, was skipped.

The homework is consists of two related questions about the possibility of constructing a *V* formation (a) with 92 dancers and (b) from combining two *V* formations. To solve this problem, the focus students used the method of middle-right-left rule. Their reasoning can be seen in the Figure 5.50.


8. Bisakah V formasi terdiri atas 92 penari? Mengapa demikian?

tidak karena jika 92 penari berarti 46 di-kanan dan 46 dikira dan-tidak ada yang g ditengah.

Translation:
Cannot, because of there is 92 dancers there will be 46 dancers in the right and 46 dancers in the left side and no one will be in the middle.


9. Panitia kegiatan berencana untuk menggunakan dua formasi V menjadi sebuah formasi V, bisakah demikian? Mengapa?

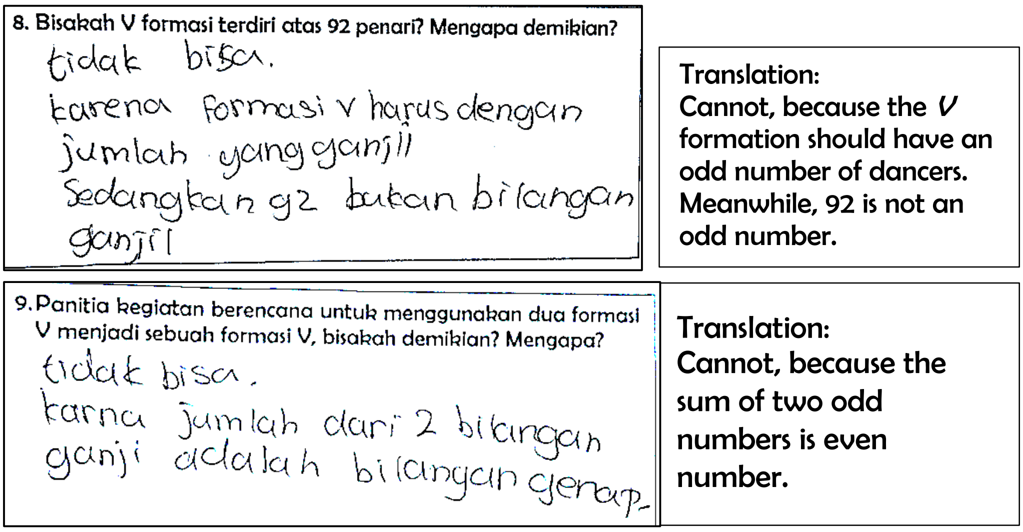
tidak tidak
Bisa karena V dapat digabung misalnya Formasi V pertama ada 47 penari dan Formasi V kedua ada 47 penari jadi jumlahnya 94 penari jadi 47 dikanan 47 dikiri dan tidak ada penari yang di tengah

Translation:
Cannot, because the *V* formations cannot be combined.
For instance the first *V* formation consists of 47 dancers, the total will be 94 dancers. That's means 47 in the right and 47 in the left and no one will be in the middle.

Figure 5.50. Middle-right-left

From their worksheets, we can conclude that the students cannot explain yet in general the reason behind no one will be there to be the "middle dancer".

Other students, for instance Nur and Fauzan, started to write it in general form by considering the odd-even number of dancers and its relation with the characteristics of total number of dancers which can perform a *V* formation. Their answer can be observed in the Figure 5.51.


8. Bisakah V formasi terdiri atas 92 penari? Mengapa demikian?

tidak bisa.
karena formasi v harus dengan jumlah yang ganjil
Sedangkan 92 bukan bilangan ganjil

Translation:
Cannot, because the *V* formation should have an odd number of dancers. Meanwhile, 92 is not an odd number.

9. Panitia kegiatan berencana untuk menggunakan dua formasi V menjadi sebuah formasi V, bisakah demikian? Mengapa?

tidak bisa.
karna jumlah dari 2 bilangan ganji adalah bilangan genap.

Translation:
Cannot, because the sum of two odd numbers is even number.

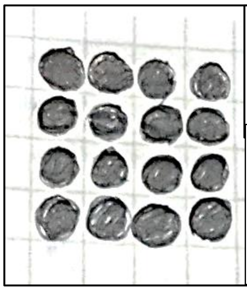
Figure 5.51. Odd or even?

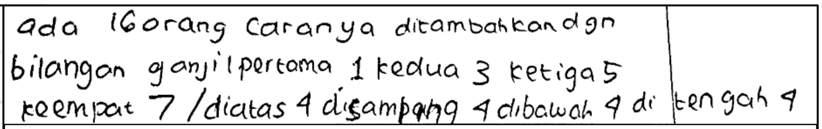
From their reasoning, Nur and Fauzan developed a more sophisticated mathematical idea that a *V* formation is always come with an odd number of dancers. They might have conclude that from the shape of *V* and their previous investigation about the number of pairs that will be in a *V* formation. Nur and Fauzan might realize there always one dancer who has no pair. Hence, the number of dancers is always odd.

5.3.4 Lesson 3

The third lesson conducted on March 20, 2015. The context is about creating a flash mob formation for Indonesian Traditional Martial Art called Pencak Silat. The formation should always be in square shape. The teacher showed the picture of the first three formation and asked the students to draw the fourth formation with their pairs.

There are different methods used by the students. These methods are similar with the conjectures in HLT. It represent the students' way of thinking in seeing the structure of the square formation. Figure 5.52 for example, shows that the students were using addition strategy.

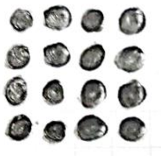


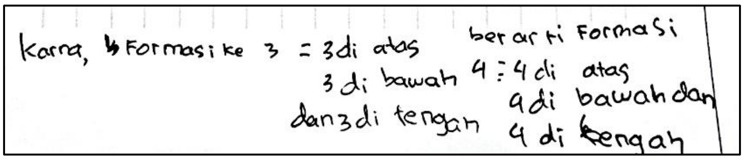


Translation
"There are 16 people, the strategy is by adding the odd number: first with 3; the second with 3, the third with 5 the fourth with 7 OR 4 above, 4 in sides, 4 below and 4 in the middle."

Figure 5.52. Arkam and Naurah's addition strategy

Arkam and Naurah were initially using addition strategy by checking how many dancers should be added in each upcoming formation. They added another answer (the answer after the word "OR") after hearing other students' strategy during the class discussion. The strategy which is pointing the number of dancers in each side was given by Zahrah and Jihan. The following Figure 5.53 illustrates their point of view.

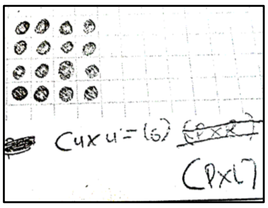




Translation:
Since on the third formation = 3 above, 3 below and 3 in the middle;
That's mean in the fourth formation =
4 above, 4 below and 4 in the middle.

Figure 5.53. Seeing the dancers on each rows

More advance strategy is given by Hafidz and Diana who seeing the multiplication of the sides of the square as a good approach to start with. See Figure 5.54 to observe their works.



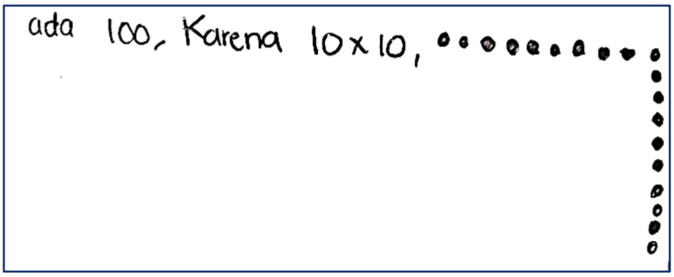
Translation:
The strategy is
(4× 4 = 16)
(Length × Width)

Figure 5.54. See the sides only

Even though the answer given in the Figure 5.54 is the most sophisticated answer we figured out that it is not easy for the students to come to that point. In the next discussion we will show the students' struggle to move from addition to the multiplication strategy.

The next three questions are to determine the number of dancers in the 10th, 15th and 100th formations. To solve this problem the students used quite similar strategy, which can be classified as addition and multiplication. When the students used addition, they need to know the previous number of dancers before they find the dancers in current formation. This strategy will lead the students to construct a recursive formula.

Another strategy is by using multiplication. Here, the students used properties of square. They observed that the total number of dancers in a square formation is equal to the result of squaring the number of dancers in a side of the square. This strategy will lead the students to generate a general formula, which means they can find the number of dancers in any formation without the need of finding the number of dancers in the previous formation. The following Figure 5.55 shows one answer from Reno and Felis who employed the multiplication strategy by checking the dots in the sides only. It is similar with the strategy given in the Figure 5.54, but it has more clear representation.



Translation:
There are 100,
because
10 × 10

Figure 5.55. Reno and Felis' multiplication strategy

Shifting from addition to multiplication or in other words from recursive to general formula is a crucial movement in development of algebraic thinking. The focus group encountered this struggle as well. Naurah, Ferdi and Meilia using addition strategy and refuse to think another method. Differently, Arkam noticed the pattern. He proposed an idea, similar with Figure 5.54, which is considering the number of dancer in the sides of the square, but the others, who did not ready yet to come to that conclusion, refused it. The researcher was trying to bring up Arkam's idea into group discussion. The following Fragment 22, Fragment 23 and Fragment 24 will give an illustration of the students' shift from recursive to general idea.

Fragment 22: How if We See the Sides of the Square?

[1] Researcher : Let's observe the picture given on the previous worksheet.
[2] Previously, Arkam said that in the third formation,
[3] there will be 3 people on the side and 3 people above
[4] Naurah : You are wrong if you say 3 on the side and 3 above!
[5] Meilia : Yes! Because in the middle is also 3!

Naurah and Meilia was not convinced at that time. After that they continue to work by themselves, which use addition strategy.

Fragment 23: Keep Adding

[1] Researcher : What are you doing, Ferdi?
[2] Ferdi : I am adding.
[3] Researcher : Adding. If I ask the 100th formation, will you add it until then?
[4] Ferdi : Yes
[5] Researcher : Sure? Don't you have other strategy?
[6] Ferdi : No...
[7] Researcher : Are you really going to add it until the 100th formation?
[8] Researcher : Let's think about it! Naurah, Arkam, do you have other strategy?
[9] (*Arkam, Naurah, Ferdi and Meilia continued to counting*)
[10]Researcher : Lets we observe the shape of the formation.
[11] (*The teacher came*).
[12]Teacher : Ferdi, now let's try the fifth formation.
[13]Teacher : How many people are there in a row?
[14]Ferdi : 5 people.
[15]Teacher : 5 people, try to draw it.
[16](*Arkam, Naurah, Ferdi and Meilia drew the fifth formation*)
[17]Teacher : 5 people, how many row?
[18]Ferdi : 5
[19]Teacher : So, also what is the total?
[20]Ferdi : The total is ... 25 people.

After a while, the teacher were left the group. The focus students continued their discussion. Finally they got a new insight.

Fragment 24: From Addition to Multiplication

[1] Ferdi : Oh this is multiplication! 10 times 10!
[2] Naurah : Yes!
[3] Arkam : Squaring...
[4] Ferdi : What squaring are you talking about!

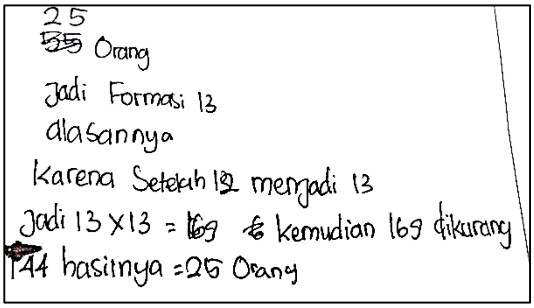
[5] (*Said something in local language*)
[6] Researcher : Why? Why?
[7] Researcher : 10 times 10 is 100, why?
[8] Meilia : We find another strategy!
[9] Naurah : Since there are 10 rows
[10]Meilia : Suppose a row is fulfilled by 10 people
[11] and then multiply it with the number of people in that row.
[12]Researcher : There is a row with 10 people and then multiply it?
[13]Meilia : There are 10 rows, in each row there are 10 people
[14] So, 10 times 10
[15]Researcher : So, 10 times 10.
[16] How it will be in the 15th formation?
[17]Meilia : 15 times 15

From the Fragment 22 we can observe how the students who not ready yet with more advance strategy refused to agree with the other's student idea (see line 3 of Fragment 22). Here the role of teacher is very important to bridge the gap between students' strategies. Instead of directly express her/his agreement toward particular strategy, the teacher should give the students chance to explore more and to construct more sophisticated idea. It can be done by ask questions which challenge the students to think more and emphasize some important ideas which students' missed from their friend's explanation.

Similar strategy was used to find the 100th formation. After that, class discussion was conducted and some students were presenting their works in the front of the class.

The last part of this activity is ask the students to think about the number of dancers if certain number of dancer is given. In the worksheet, the students were informed that there are 144 dancers. To solve this problem, the students used their prior knowledge about multiplication. Actually the school is demand the students to memorize multiplication until 15. Hence, most of them easily solve this problem because they now 12 multiply by 12 is equal to 144.

Next, the students were ask to find how many dancers will be needed if the students want to create the next formation. This problem was solved by using similar strategy as the previous question. The students argued that the next formation will be the 13th formation which consist of 169 dancers. Hence, the committee will be 25 dancers more. Figure 5.56 is one example of students' works given by Arkam and Naurah.

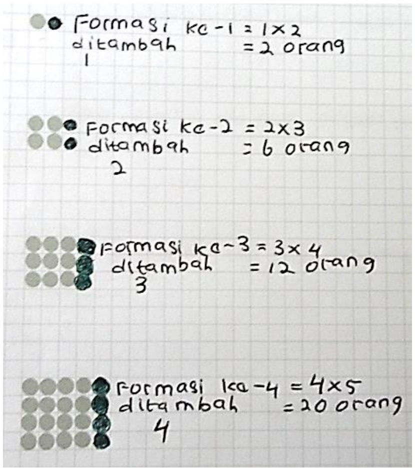


Translation:
25 people
So in the 13th formation
The reason
Because after 12 is 13
So, 13 × 13 = 169
After that subtract 144 from 169, the result is 25 people.

Figure 5.56. Arkam's and Naurah's answer for the square formation

5.3.5 Lesson 4

The fourth lesson conducted on March 23, 2015. The context was still continuing from the third lesson, which is about Pencak Silat flash mob. The teacher explained that after discussing with the representative of Palembang Television (PalTV), the committee decided to modify the formation into rectangle. Hence, the first task is to change the first-four-square-formations they had from previous meeting into rectangular shape by adding the minimum number of dancers. The aim of the task is to encourage the students to see the structure of the visual representation of a pattern. Figure 5.57 shows one of example of students' work.



Translation:
- **1st formation, add by 1**
  1 × 2 = 2 dancers
- **2nd formation, add by 2**
  2 × 3 = 6 dancers
- **3rd formation, add by 3**
  3 × 4 = 12 dancers
- **4th formation, add by 4**
  4 × 5 = 20 dancers

Figure 5.57. The example of students' solution of square modification

This problem was not too easy for the students. Some errors were occur, including (1) the students added the dancers randomly as long as the square can be formed a rectangle without considering the instruction of “add as minimum as they can” (see Figure 5.58, the students’ corrected their first answer which was not considering the “minimum” instruction) and (2) the students were not focus to the growth in the “next formation”. Hence, some students were having the same number of dancers for the first until fourth rectangular formations.

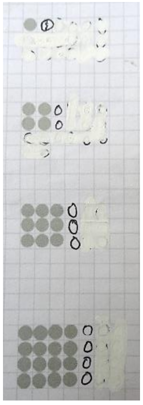


Figure 5.58. Forgot the minimum

The activity continued by finding the number of dancers in the5^{th}, 10^{th}, 99^{th} formations and asked the students to create a word generalization of what strategy they used to create a rectangular formation.

To solve this problem, there are two strategies employed by the students. First, they likely to go with the context, which promote the use of square formation before creating the rectangle. The following Fragment 25 illustrate the students’ thought using the aforementioned strategy.

Fragment 25: From Square to Rectangle

[1] Meilia : Yuk, Yuk (*Older sister/she called the researcher*)!
[2] Does the fifth formation means as the fifth row?
[3] Researcher : I don't know, why dont you ask Ferdy, ask Arkam, ask Naurah?
[4] What do you think Ferdi? Arkam? Discuss it with Meilia.
[5] Just now, she asked what does it mean by the fifth formation?
[6] Arkam : Find the square of 5, it is 25 ... and then add with 5.

Second, the students investigated the shape of rectangular formation and then constructed its own general formula as is showed in the Fragment 26.

Fragment 26: Rectangle Formula

[1] Researcher	:	Let's think other strategy.
[2]		Do you think there will be other way to solve it without finding
[3]		the number of dancers in the square formation first?
[4] Ferdi	:	Yes, multiplication.
[5] Researcher	:	Multiplication of?
[6] Ferdi	:	Which formation?
[7] Researcher	:	The same formation, without …
[8] Arkam	:	Multiply by 6.
[9] Researcher	:	Multiply by 6, what will be multiply by 6?
[10]Naurah	:	What is that what is that?
[11]Arkam	:	6 times 5.
[12]Researcher	:	Why 6 times 5?
[13]Arkam	:	Hah?
[14]Researcher	:	Why 6 times 5? Can you show with your drawing?
[15]Ferdi	:	6 times 5 is 30.
[16]Researcher	:	Yes, but why (it can be used)?
[17]Ferdi	:	[*because*] the result is 30
[18]Arkam	:	Rectangle.
[19]Ferdi	:	5 times 5 is 25, add with 5 the result is 30.
[20]Researcher	:	And where you get 5 times six 6?
[21]Ferdi	:	Because 6 times 5 is 30.
[22]Arkam	:	Each rows have 6.
[23]		So, 6 times 5 …

Due to time limitation, the second part of the activity was not discuss and become a homework, which should been done in pairs. The task is to divide the dancers in rectangular formation fairly into two groups, white and black costumes. The students' got a worksheet, which consists a picture as can be observed in the following Figure 5.59.

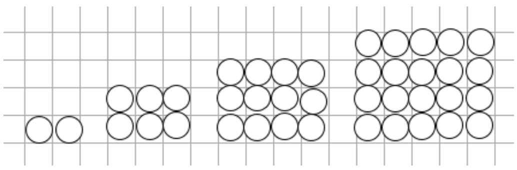


Figure 5.59 Illustration Given in the Students' Homework

To do this task, the students can do anything they want, based on their own creativities as long as the dancers who use white costume as many as the students who use black. The example of students' works can be observe in the discussion of Lesson 5.

5.3.6 Lesson 5

The fifth lesson conducted on March 27, 2015. Before the teacher came to the current materials, she remind the students about how to find square and rectangular formation. The refreshment process took around a half of the lesson time. As the implication, not all prepared materials for the new topic were able to discuss today.

The lesson is relate to the previous meeting, which was about the formation for Pencak Silat flash mob. As is mentioned in the previous meeting, the committee decided to use rectangular formation. For homework, the students were asked to divide the dancers in the formation fairly into two groups, one use black while the other use white.

From the collected homework, we found that some students tried not to just consider the "fair-sharing" rule but also the esthetic value of a stage performance. The following Figure 5.60 gives the example of students' works.

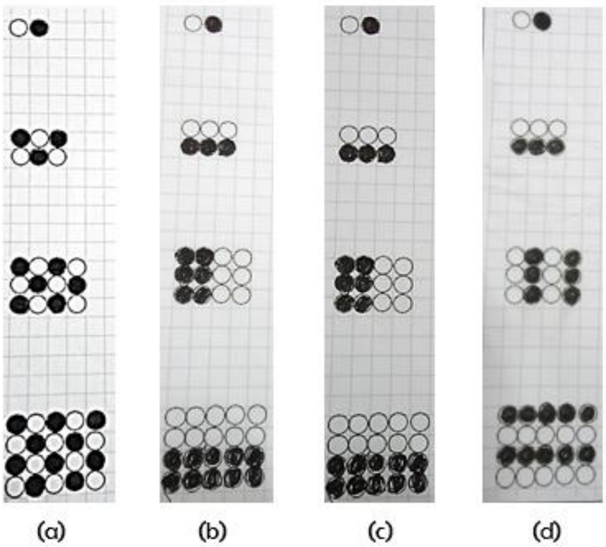


Figure 5.60. Example of students' works in dividing the rectangular formation

After briefly discussed some of students works, the teacher announced that the committee chose a unique pattern embodied in triangle shape. Specifically, each rectangular formation can be divided into two triangular formations. The teacher used the Appendix of the Teacher Guide to show the students the illustration of two triangular formations in a rectangular formation as following Figure 5.61.

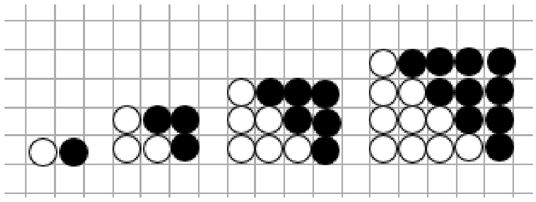
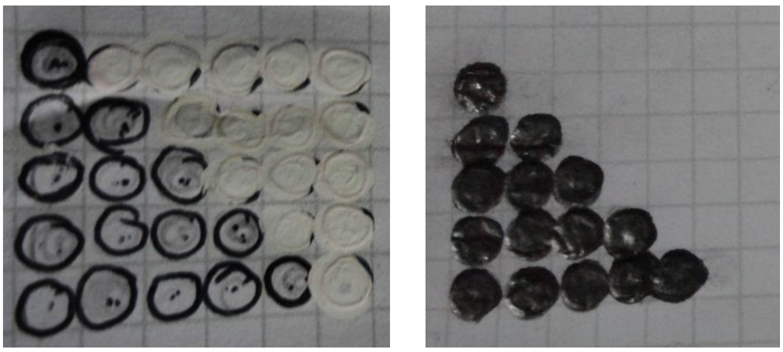


Figure 5.61. Dividing rectangular into triangular formation

The first task is to draw the dancers with white costume in the fifth formation. The focus students solved it by firstly drew the dancers in the fifth rectangular formation, divided them by two and erased half of them. Other students used similar method but they saw the triangular shape differently at the beginning. Instead of directly used the construction of rectangular formation, they checked the addition of dots in each formation and then concluded that the next formation will have one more dot in its row. Hence, the fifth formation will be constructed by one dot in the first row, two dots in the second row etcetera until five dots in the fifth row. Figure 5.62 will show the difference between those students' worksheets.



(a) (b)

Figure 5.62. Different methods to draw a triangular formation

The Figure 5.62(a) has a pen correction in the half of their picture for the fifth formation. This method is likely to happen because the students use the information given in the context, which said that the rectangular formation will be divided into two rectangular formations.

In one hand, the second pair (Figure 5.62(b)) drew the fifth formation by observed the addition of dots in each formation. After drawing the fifth formation, they tried to find other strategy without keep adding one dot in each lines. Their discussion is recorded in the following Fragment 27.

Fragment 27: Checking Strategy

[1] Jihan : Divide it fairly.
[2] Zarah : If there are 30
[3] (*refer to 30 dancers in the 4th rectangular formation*)
[4] Zarah : It will be 15.
[5] Researcher : You have to divide it fairly?
[6] Researcher : What do you mean by that?
[7] Jihan : Because the instruction is white only.
[8] (*Refer to the task which merely ask about the dancers in the*
[9] *white costume*)
[10]Zahrah : 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15
[11] (*counting by one to check*)
[12]Zahrah : Yes, correct! 15, 15.

From lines 8 and 9, we can observe that Zahrah find a need to proof it manually. Even though previously Jihan already mentioned that they can use halving (or based on their words it should be fairly divided), she ensured their calculation by checking the number of dots one by one.

The second task is about finding the number of dancers in the 10th formation whose costume are white. All students were able to solve this problem by using halving the number of dancers in the 10th rectangular formation strategy.

The third task is to find the number of white-costume-dancers in the 100th formation. In the previous problem Naurah and Arkam, already pointed out that the number of dancers in a triangular formation will be a half of the number of dancers in a particular rectangular formation. Related to that, in the previous meeting they were easily solving the problem of finding out the number of dancers in the 100th rectangular formation. However, in this activity Naurah and Arkam got confuse with the number of dancers in a row. At first, they argued that the 100th rectangular formation will has 110 rows and each row consists of 100 dancers.

Fragment 28: How Many Dancers Will Be in The Rectangular Formation?

[1] Researcher : I want to ask this first.
[2] How did you know it should be 100 by 110?
[3] Naurah : Ehm ... This is a rectangular formation.
[4] Researcher : So, a rectangular formation? Do you mean in the 100 formation?
[5] Naurah : Yes
[6] Researcher : What number multiplied by what number?
[7] Arkam : 100 times 110
[8] Researcher : Are you sure 110?
[9] Arkam : Yes, I am
[10] Naurah : Eh?

[11]Researcher : How many rows?
[12]Naurah : 110 rows
[13]Researcher : And how many dancers in each row?
[14]Arkam : 100
[15]Arkam : 110
[16]Naurah : 100 just add by one
[17]Researcher : So how many it will be?
[18]Naurah : 200
[19]Naurah : Ehh ... (*She took her pen and tried to drawing*)

Seeing the students were struggling, the researcher tried to give scaffolding by remind them about square formation before jump into rectangular formation. Based on the Fragment 28 (especially lines 13, 14, 15 and 17), Arkam and Naurah cannot decide how many dancer will be added into each rows if they modified a square into rectangular form.

Fragment 29: How Many Dancers should be Add?

[1] Researcher : How about the 100th formation of a square?
[2] Arkam : 100 by 110
[3] Naurah : 100 by 100
[4] Arkam : How come?
[5] Naurah : Square! (*Refer to square formation*)
[6] Arkam : Oh, right!
[7] Researcher : Then, how about the rectangular?
[8] Naurah : 110 … eh 101 … eh …
[9] Researcher : Well ... 100, 101, 110, 120 or 200?

They kept struggling until the researcher asked Naurah to draw her thinking about modifying a square into rectangular. Naurah made the following Figure 5.63. In addition, her explanation is noted the Fragment 30.

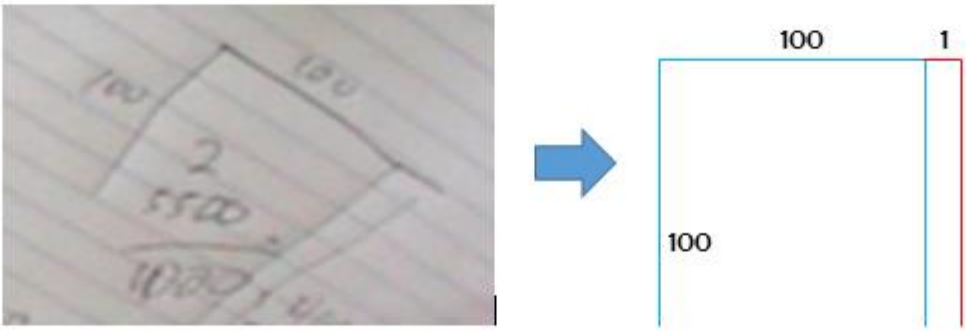


Figure 5.63. Where will you add the dancers?

Fragment 30: How Many Dancers should be in a Row?

[1] Naurah : (*Drawing*)
[2] Researcher : Okay, do you want to draw a square here? 100 and 100?
[3] Naurah : (*Nodded*)
[4] Researcher : So how many dancers should be added?
[5] Where will you put them?
[6] (*Pointed to an invisible column in the left side of her picture*)
[7] Naurah : Over here (*drawing an additional line*)
[8] Researcher : Okay, how many row you add here?
[9] Naurah : One row (*but instead a row, she drew a column which cover the*
[10] *left side of a* 100×100 *illustration of a square formation*
[11] [see Figure 5.63]).
[12] Researcher : A row? So how many dancers will be in a row?
[13] Naurah : 101

The fourth question related to the inverse problem. In this stage, the students were asked to determine what number of rectangular formation with 210 dancers in white costume.

Fragment 31: From Triangle to Rectangle Formation

[1] Researcher : Okay Ferdi, how do you get 20?
[2] Ferdi : Nah, 210 + 210 = 420.
[3] Researcher : He eh …
[4] Ferdi : Nah, then you find it.
[5] 20 time 20 equal to 400.
[6] Since you want to make a rectangular formation you should add
[7] 20 more. It makes 420.
[8] Hence, it should be the 20th formation.

Based on the Ferdi's explanation, we can observe that he used several steps to find the unknown number. First, he used the structure of the triangular formation. Since there are 210 dancers in white costume, means there is a triangular formation with 210 dancers. He doubled the numbers which indicates doubling two triangular formations. The combination of those formations construct a rectangular formation. Then, he used trial error strategy, but not totally guessing, to find out what number of rectangular formation it is. He tried to predict the closest number to 420. He realized that squaring method is works to find the square formation, which will be modified into rectangular formation (see Fragment 31 lines 5, 6 and 7). In the end, he successfully concluded that the number of formation is 20.

Due to time limitation, the last problem of this activity was not discussed and be used as one of the posttest item.

5.3.7 Posttest

After conducting the five series of lesson, we conducted a posttest to check the students' transferability in answering the pre-algebra related problem, especially in pattern investigation problems. The test was conducted on March, 30 2015 for the 32 students of VA MIN 2 Palembang. Some of the posttest problems are different with the items used in the first cycle (read the reflection and the reason of revision in 5.2: Improvement Plan of HLT 1). We will discuss each problem in detail. The complete test can be seen in Teacher Guide.

Problem 1

The first problem is related to a number string with five repeating colors (see Figure 5.64). The questions are: (a) finding the color of the 27th strip, (b) evaluating whether the 550th and the 11th strips will have the same color, and (c) explaining how to find the color of 159.638th strip.

Figure 5.64. Number string with five colors illustration

To solve this problem, there are some students who still used keep counting method. Some others employed different strategies, for instance dividing by five or checking the last number only. Rina is one of the students who applied divide by five strategy, since there are five repeating colors in the strip. She found that if she divide 159.638 with 5, the remainder is 3. Hence, its color should be 3. Figure 5.65 shows her answer.

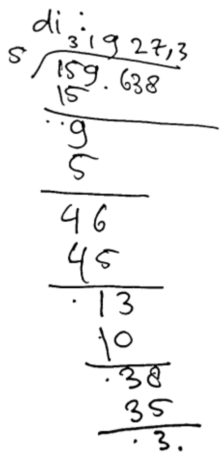
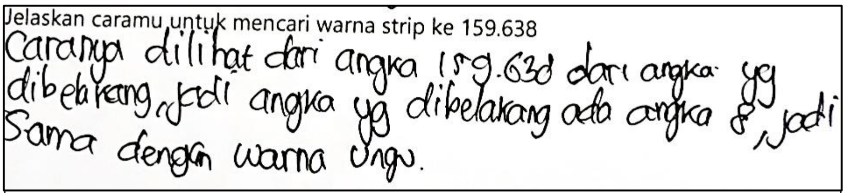


Figure 5.65. Divide by 5

Fariz used checking the last digit strategy. He might be used 10 as his reference, skip 0, and realize that 11 will come with the same color as 1; 12 as 2; 13 as 3; 44 as 4; 100 as 10 etcetera. His reason can be seen in the Figure 5.66.



Jelaskan caramu untuk mencari warna strip ke 159.638

Caranya dilihat dari angka 159.638 dari angka yg dibelakang, jadi angka yg dibelakang ada angka 8, jadi Sama dengan warna Ungu.

Translation:
The strategy is by checking the number of 159.638. From the last digit of that number, which is 8, is equal to purple.

Figure 5.66. Check the last digit only

Problem 2

We curious to find a way to push the students' development of "finding more efficient strategy by seeing the structure of pattern" issue. Hence, we included the similar problem of odd/even beads as we used in the pretest, with more number and color of beads. For the posttest, we used three color: orange, black and blue. The task is to determine whether the beads can be divided in to three equal groups. The given beads is as in the following Figure 5.67.

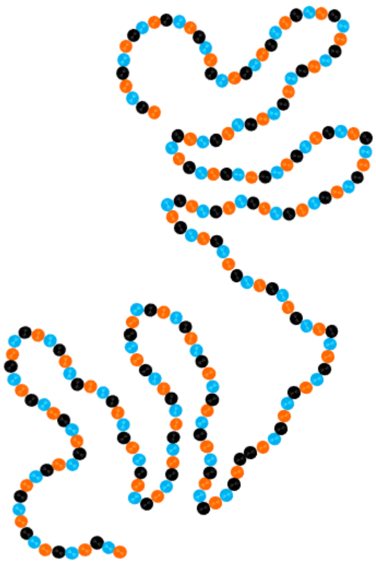


Figure 5.67. Three colors of beads

As can be seen, the order of the beads' colors are orange-black-and blue. The last bead have orange as its color. In the middle of the string, there are two consecutive black beads and continued by orange, which means one blue is missing. This is happen due to human error in the researcher side. Our first intention is merely use orang-black-blue repeating pattern which will be ended with an orange bead.

However, the error in the middle provides another discussion. For the students who directly check the colors in general, the string seem to have orang-black-blue repeatedly. For others who did keep counting or observe the problem in detail, it might be different case.

The majority of the students used counting by one strategy. It is surprising because the number of the beads are not small, but they didn't find it is tiring to count it one by one. See one of the students' counting method in the Figure 5.68.

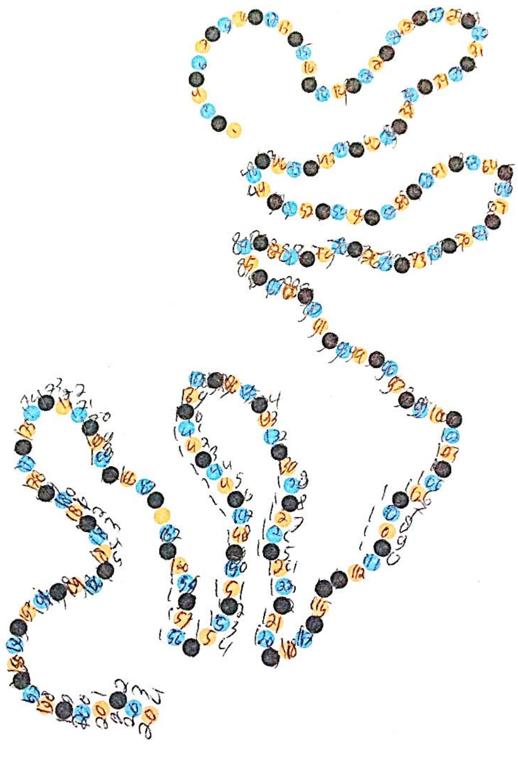
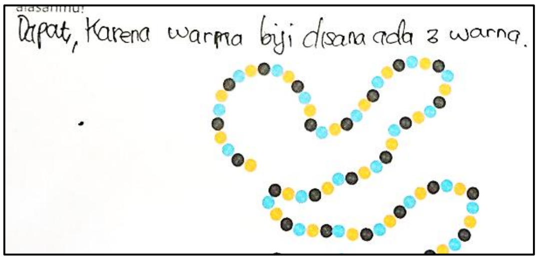


Figure 5.68. Counting by one strategy

Differently, Fariz merely checked the global view of the pattern and concluded as "Yes, it can be. Because there are 3 colors of beads". In this point, he started to develop a structure sense in his mind, he can conclude whether the string has equal number of beads by checking their colors only. However, he forgot to consider that the string ended by orange, instead of blue. His answer can be seen in Figure 5.69.



Translation:

It can, because the string has three colors.

Figure 5.69. One example of students' reasoning of three color beads

The same logic, but more advance structure sense, used by Meilia, who concluded that "It can't be divide by 3 because it has one bead left" (see Figure 5.70). As Faris, Meilia considered the structure of the pattern in general and seemed not to noticing that there are two blacks in the middle and one blue is missing.

Translation:

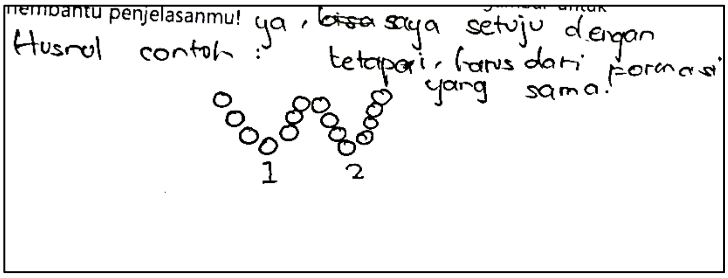
No, it cannot. Because one it has one bead left.

Figure 5.70. Meilia's reason for the beads problem

Problem 3

The third problem is about *V* and *W* formations. It is asked the students' opinion whether two *V* formations can construct a *W* pattern. In the students' sheets, the picture of three-first formations of *V* and *W* patterns are given.

Most of the students answer it as yes, because they said two *V* is a *W*. For instance in the Rama's answer as the following Figure 5.71.



Translation:
Yes, I agree with Husnul. For instance, (giving the picture). But it should have the same number formations.

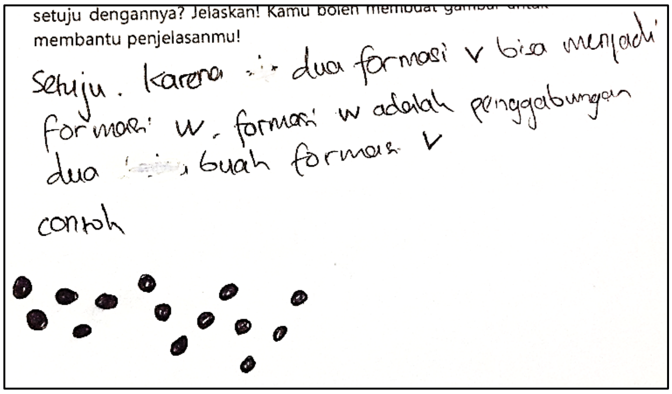
Figure 5.71. 2 *V* is a *W*.

Fewer students realized that they need to remove one dancer if they want to combine 2 *V* in order to get a *W* formation. Figure 5.72 shows the students' way of thinking.

Translation:
Disagree, because there is one excessive [dancer].

Figure 5.72. You have an excessive dancer

Some students expressed their agreement toward the statement, but they drew it differently. One of the example is Arkam's answer in the Figure 5.73.



Translation:

Agree, because two *V* formations can be a *W* formation. *W* formation is the combination of 2 *V* formations.

For example: [giving an illustration]

Figure 5.73. One example of students' agreement but showing disagreement

When the researcher interviewed him, we found that Arkam didn't really think that the combination of 2 *V* formations is really a *W*. He said it can be, but one dancer should be removed. His argument recorded on the Fragment 32.

Fragment 32: Remove One

[1] Arkam	:	*W* is the combination of 2 *V*.
[2] Researcher	:	Which *V*? Can you show me?
[3] (*Arkam marked his answer sheet* (see Figure 5.73))		
[4] Researcher	:	Is there any dancers which is appear in both of *V* formations?
[5] Arkam	:	Yes, the middle on (pointed to the picture)
[6] Researcher	:	So, what do you think about that?
[7]		Is there any condition when you combine 2*V* and create a *W*?
[8] Arkam	:	Yes, you should subtract one dancer.

The consistency of his idea is supported by his answer to the next question, which asked to determine the number of dancer in the 100th formation of *V* and *W* formation. For the *V* formation, he used “pairs and middle” structure. He employed the same thing during the second lesson. Figure 5.74 shows Arkam’s answer.

b. Berapa banyak lingkaran yang diperlukan untuk membentuk formasi V ke-100?

201 karena 100 berpasangan dan 1 ditengah

Translation:
201 [dancers], because 100 pairs and 1 in the middle

Figure 5.74. Dancers in the 100th *V* formation

To solve the number of dancer in the 100th W formation, Arkam related the structure of *V* and *W* formation. He used his idea of combination of 2 *V* formations is a *W* formation if one dancer is removed. His argument can be seen in the following Figure 5.75.

c. Berapa banyak lingkaran yang diperlukan untuk membentuk formasi W ke-100?

401 karena kalo dibelah menjadi formasi V harusnya 402 karena dikurangi I. I ditengah bawah dan I ditengah bagian atas. Semua berpasangan

Translation:
401 [dancers], because if you separate it into V formations you will have 402 [dancers], but you need to subtract 1.

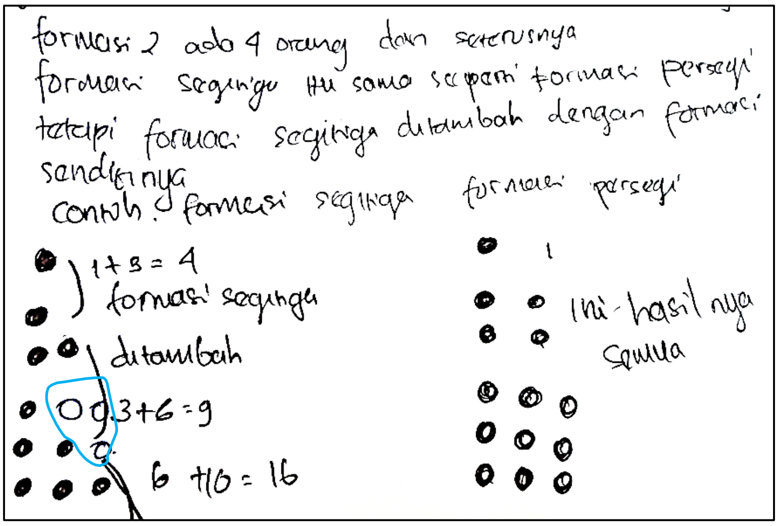
Figure 5.75. Dancers on the 100th *W* Formation

Problem 4

In the fourth problem, the students deal with the triangular and square number pattern. Given a table of numbers in triangular and square pattern, and a statement that a girl called Laura predicts that if she combine two consecutive triangular numbers, she will gets a square numbers. She tries some numbers, i.e. $1 + 3 = 4$; $3 + 6 = 9$. The students need to argue whether Laura prediction is correct for other triangular numbers.

Since in the problem triangular and square numbers are given in the "number" form without a visualization support, the students hardly drew the illustration of the problem. When the researcher asked, they got confused how to use the picture of triangular and square number pattern to proof Laura prediction. Most of students continue to add some other consecutive triangular numbers and find that the result is a square number.

Some students used visual support. One of them is Arkam. His answer can be observed in the Figure 5.76.



Translation:
The 2nd formation has 4 people and etcetera. The triangular formation is the same with the square formation, but the triangular formation should be added by itself. [3]

Figure 5.76. Arkam visualization for triangular and square numbers

Problem 5

The fifth problem basically prepared only for the students who did the posttest faster than the others. But during the implementation, all students asked to do the same problem. The problem is to continue the Fibonacci series. In the students' sheets, given the number of: $1, 1, 3, 5, 8$ ... and the students were asked to continue until the next three numbers.

[3] The dots in the blue field is made by Arkam during the interview section, when the researcher asked him to show where he combine two consecutive triangular numbers

It was also given in the pretest and all of the students failed to solve the pattern. However, in posttest some students were able to solve it because they got insight of the relation between the adjacent numbers. In the following Fragment 33, Arkam shows his understanding toward the pattern.

Fragment 33: Fibonacci Series

[1] Arkam	:	$1 + 1 = 2$
[2]		$1 + 2 = 3$
[3]		$2 + 3 = 5$
[4]		$3 + 5 = 8$
[5]		$5 + 8 = 13$
[6]		$8 + 13 = 21$
[7] Researcher	:	What is the next?
[8]		$13 + 21 = 34$

5.4 Improvement of the HLT 2

Overall the analysis of the second cycle shows that the students learn algebra when they doing pattern activities. Learning algebra here refer to the development of algebraic thinking which is emphasize on the ability of thinking about general rule that lying on the structure of a pattern.

Reflect to the findings in the second cycle we conclude that the word “generalization” is still abstract for them. Hence, the question which ask directly to formulize a “words formula which represent the relation between, for instance, the number of dancer and the number of formation,” is need to be changed. To remove the word “generalization” but still ask the students to construct a generalization in their mind, we need to make the number larger. As we observe, the 100th number is not that tiring for the students, since some of them are willing to count or add one by one [see lines 3, 4, 5, 6 and 7 of Fragment 23). By adding bigger numbers in the problem, the students will be pushed to find more advance strategy because they cannot count by one forever.

The second remark is about the important of the “unit pattern” as the reference that the students should be looking at when they work with a repeating pattern. When we observe the students’ works with repeating patterns related problem (see Lesson 1, pretest problem number 3 and posttest problem number 2), we found that the students were not established the sense of unit pattern yet. Hence, they confused with what kind of reference number that can be helpful for them to work with. For instance, if the pattern is repeating in every three items, they should use the multiplication of 3 as their reference point instead of using 10.

The third remark is to optimizing the support of visualization to help the students get insight of the structure of pattern, especially for the second lesson which is about *V* pattern. The preliminary questions on the third lesson was to encourage the students to find the different between “the total number dancers” and “the number of pairs” in certain number formation; and in the same time also finding the relations between those three aspects. We found that the students struggle to understand that number of pair is not the number of total dancers and vice versa. One alternative that might be helpful is to add the problem related to the pairing. For instance by provide a situation in which a *V* formation is given, but somehow the picture was not clear and only a half part of it is showed. See Figure 5.77 for the illustration.

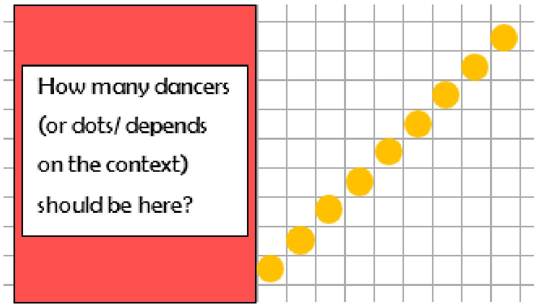


Figure 5.77. Finding the pair number

By working with the aforementioned problem, the students will likely to understand that “pairing” has to do with equal number of object in both sides of the V formation and they will have one more in the middle.

5.5 Summary of Comparison between HLT and ALT

Previously, we explain in detail how the actual learning trajectory (ALT) was took place during the lesson. In this section, the summary of comparison between what we predict as students’ responses as is written in HLT and the actual condition in the classroom.

Table 5.9

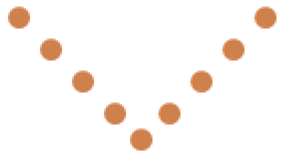
Comparison between HLT and ALT Lesson 1 Cycle 1

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
	Worksheet 1	
1. Continue the strips for the next two numbers!	The students continue to write the numbers and give the color sign on them.	The students were able to determine the next color of the number string.
2. Can you predict the color of costume for the (a) 12th student? (b) 25th student?	• The students continue drawing. • The students consider the relationship between types of number (odd or even) with the colors.	• One pair made a list of the numbers and the related colors, not by drawing but the idea is the same. • Another pair saw the relation between odd and even numbers with the colors.
3. Explain how you predict the color of costume for the 100th student!	• The students make a string of 10 and multiply it ten times. • The students make a generalization that since 100 is even number and every even numbers is come in white then the costume color of the 100th student is white.	
4. Is it possible to the 97th student to have the same costume color with the 43th student?	The students explain that since 97 and 43 are odd numbers than they will have the same color, which is red.	• The students saw the relation between odd and even numbers with the colors.
5. Suppose the 70th student get white for his/her costume, what is the costume color of the 1st one?	The students recognize the odd-even pattern, since 70 is even while 1 is odd, if 70 is red than 1 is white.	• The students understood that the arrangement of the dance's costume is different with the previous number problems. However they were able to use the characteristics of odd and even number with the new color arrangement.
	Worksheet 2	
1. Continue the drawing for the next strip, what is the color of the two-next strips?	The student will continue to draw the strips in the worksheet, by imitating or just write the symbol of the color (for instance yellow-pink, or solely Y-P).	The students wrote the initial colors of the dancers' costume.
2. Can you predict the color of costume for the	• The students are continue drawing. • The students think that every odd number is pink	• Both of the pairs applied listing method.

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
(a) 11th and (b) 27th students?	and even is blue, but then confuse with the yellow. • They may also think that every multiplicative of 3 will be yellow and 2 will be blue. But then confuse with the color of the costume for the students which numbers are multiplicative of 2 and 3. • The students see one pink, one blue and one yellow as a group. For instance if there are 11 dancers, there will be 3 groups and 2 left. Hence the 11th student will use blue costume. The similar strategy for the 27th student and the answer is yellow.	
3. Explain your strategy to find the color of costume for the 100th student!	Divide 100 by 3 and observe that the remainder is 1. Hence, the color of costume is pink.	• The first pair kept listing. • The second pair used a reference. They discuss whether using 3 or 4 as the reference number.
4. If there are 53 students, would the formation has three equal color of costumes? Why or Why not?	The students realize that to have equal number of color, the number of students should be divisible by 3. Hence, the answer for this problem is no, because 53 is not divisible by 3.	The students didn't understand the meaning of the problem.
5. Is it possible to the 76th student to have the same costume color with the 121st?	The students use the structure of the number to check the color. For instance: if you divide 76 with 3 you have one as remainder, and so does 121:3. Hence, they will come with the same color which is pink.	The students checked the color of the 76th and the 121st dancers.
6. If the 60th student use yellow costume. What will be the costume color of the 1st student?	The students observe the structure of number, in which 60 is divisible by 3 and it comes with yellow. Hence, the number which remain 1 if divide by 3 will have pink as its color.	The students didn't realize that the arrangement of the costume's color was changed.

Table 5.10

Comparison between HLT and ALT Lesson 2 Cycle 1

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
	Worksheet 1	
1. Draw the arrangement for the 4th formation!		The students drew the V formation. But they seemed to confuse where to put the parallel dots, because there was no gridlines in the worksheet.
2. How many dancers will be in the 6th formation?	The students may keep counting, but since they are involved in similar activities in three lessons before, they may already look for the more general strategy, i.e.: by observe that the number of pair is equal to the number of formation. Hence, it 6th formation, there will be 6 pairs. And there will be one in the middle, hence the total is 13 dancers.	• The first pair used adding two method to determine the number of dancer for each next formations. • The second pair used the structure of "pairs in the sides and one in the middle" rule as is predicted in the HLT.
3. Draw the formation which consist of 17th dancers!		The students drew from the one dot in the bottom and add two until they got 17.
4. How many pairs of dancers in the 45th formation?	45 pairs.	Before they found the correct answer, the students were confuse whether they will have 45 pairs or 45 dancers in total.
5. Explain your strategy to find the dancers in the 100th formation!	There will be 100 pairs of dancers and one in the middle, so overall there will be 201 dancers.	The students were using the middle-pairs rule as is predicted in the HLT.
6. Can this *V* formation have 92 dancers? Why or why not?	The students may argue that *V* formation will never have 92 dancers because it always result in odd numbers.	The students realized that the number of dancer in the *V* formation should be odd, hence 92 dancers will not be able to form it.
7. The committee plan to combine two	The combination of two *V* formations will not produce a *V* formation, but three of them will be. The reason is because the *V*	The students used the same logic as for the problem 6. Two *V* formations will have even

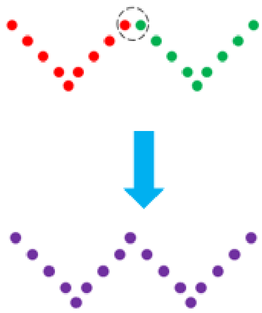
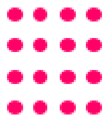
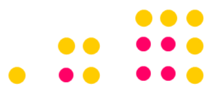
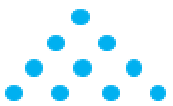
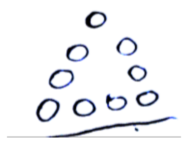
Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
groups with *V* formation in one *V* formation, is it possible? Why or why not?	pattern is always come with odd numbers. If two odd numbers are add together it will be even number. The students may also illustrate by using picture.	number of dancers. Hence, it cannot be another *V* formation.
	Worksheet 2	
1. Draw the arrangement for the 4th formation!		The students confused to draw because there was gridlines to help them checked the parrallel lines.
2. Tata compares the number of dancers in *V* and *W* formations. She claim that *W* patterns can be made from two *V* formations which is subtract by one. Do you agree with her?	Yes, because if you combine two *V* formations, you need to remove one in the meeting point to get *W* formation. 	The students used the picture to explain their answer, as is predicted in the HLT.
3. Explain in words how you can work out the number of dancers when you are going to make certain *W* formation.	The number of dancers in *W* formation is equal to four times the number of formation add with one.	The students didn't get the meaning of word "unknown". They argued that they will never know because the number of formation itself is unknown.

Table 5.11

Comparison between HLT and ALT Lesson 3 Cycle 1

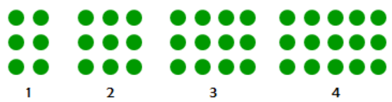
Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
	Worksheet 1	
1. Draw the arrangement for the 4th formation!		The students were able to continue drawing by observe the addition row and column in each formation.
2. How many dancers will be in the 10th formation? 3. Explain your strategy to find the number of dancers in 100th formation! 4. Explain in words how you can work out the number of dancers when you are going to make certain formation.	To generalize, the students may come with different formula, i.e.: c. Since they compare the number of formations and the number of dots, they generalize that: the number of dancers will be equal to the square of the number of formation. Note: they may use "multiply a number by itself" instead of square. d. Look at the increase of the dots in each formations.  And generalize that in each formation, the number of dancers is increase as much as 2 times of the number of formation and then minus it one by one. Hence, the number of dancers in certain formation is the number of the dancer in previous formation added by two times the number of formation, minus one. Or, the number of dancers in a formation equal to the number of dancers in previous formation add with two times the number of previous formation, plus one.	• The first pair used the addition strategy by the help of make a drawing of it. • The second pair used the squaring method as is predicted in HLT (answer a)

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
	To calculate the number of dancers in previous formation itself, the students may come up with the idea of squaring the number as well.	
5. If there are 64 people, can you make the square formation? How many dancers you need more to make the next one?	The students will answer yes because 64 is a square number. They may conclude that 64 is a square of 8, hence the next square number is 81. Therefore, 64 should be added by 17 to get 81.	The first pair continued drawing until they get 64. The second pair used guess and check strategy. They started by squaring 4, 6, 7 and finally 8.
	Worksheet 2	
1. Draw the arrangement for the 4^{th} formation!		In the beginning three of the students got lost about the triangular formation. Instead of drawing like the example of the given first three formation, they draw like this: 
2. How many people will be needed to make the: a. 6^{th} formation b. 10^{th} formation	c. Calculate $6+5+4+3+2+1 = 21$ d. Calculate $10+9+8+7+6+5+4+3+2+1 = 55$ They may calculate it one by one in order, but they also can use associative properties in this case.	Both of the students do the addition from 1 until the number of formation, which represents the number of dots in the last row. They didn't develop any advance strategy in doing addition.
3. Explain your strategy to find the number of dancers in 100^{th} formation!	The students realize that they need to add the number of formation (let say n) with $n-1$, $n-2$ etcetera until $n-(n-1)$. Hence, to determine the number of dancer in 100^{th} formation, they will said add $100+99+98+\ ...\ +1$. Since it involve a large number, they may find a way to	The students concluded that they need to add 1 until 100.

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
	calculate in less exhausted way. Here, if they realize that associative property is useful, they will structuring the number, for instance: 101 100+99+98+97+ … +3+2 +1 101 They will have 50 × 101 = 5050.	
4. Write a sentence to describe the relation between the dancers needed and the number of triangle formation formed.	The students will write their strategy as is used to solve the previous problem. For instance: by adding 1 until the number formation.	The students generalized that they need to add 1 until certain number of formation. They were able to generalize but didn't try to find more efficient strategy to do the calculation.

Table 5.12

Comparison between HLT and ALT Lesson 4 Cycle 1

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
Problem A 1. Draw the four first formations!		The first pair focused on the shape of formation without considering the number of dancers on it. The second pair skipped the drawing task and merely focused on the number of dancers. The students were able to solve the rest of the problem by following the instruction: multiply the number with 3 and add it with 1.
2. How many people will be needed to complete the 10th formation?	$(10 + 1) \times 3 = 33$	
3. Explain your strategy to find the number of people in the 100th formation!	In general, add your starting number with 99 and then add it with 1 and finally times it by three.	

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
4. Which number should be chosen to have 48 dancers in a formation?	The students will work in reverse way, first divide 48 with 3 which result to 16 and then subtract 1 from it. The final result is 15.	The students used listing method, started from the number of dancers in the 10th formation until they get 48 dancers.
Problem B Which of those formulas represent the aforementioned formation correctly? A. $\boldsymbol{D = n + 1 \times 3}$ B. $\boldsymbol{D = 3 \times n + 1}$ C. $\boldsymbol{D = 3 \times (n + 1)}$ D. $\boldsymbol{D = 3 \times n + 3}$ E. $\boldsymbol{D = (n + 1) \times 3}$ F. $\boldsymbol{D = 3 \times 1 + n}$	The correct answer is the C, D and E options. The students is likely to have discussion about what is the difference with option A and E, also why A is incorrect while the other is correct. The similar discussion applied as well for option C and F. Here, the students can use their basic knowledge of properties in whole numbers which they usually use in the arithmetical operations. They may argue that A is different with E because A has no bracket. Hence they should multiply 1 by 3 first before it is added by any number n which makes it different with the original rules given by the committee.	The students were able to do the problem by checking the organization of the expressions. They pay attention with the brackets and the order of the operations.
	Worksheet 2	
1. Write the algebraic formula for V Formation. 2. Write the algebraic formula for W Formation. 3. Write the algebraic formula for square Formation. 4. Write the algebraic formula for triangular Formation.	The students will write the correct algebraic expression for each pattern, i.e.: 1. $D = 2n + 1$ 2. $D = 4n - 1$ 3. $D = n^2$ 4. $D = 1 + 2 + \cdots + n$ with D represents the number of dancers and n represents the number of formation. They may also use different variables for each problem.	The second worksheet was not discussed due to the time limitation and after consider the students' response of the first worksheet on which they stop to think algebraically but merely focus on the notation and symbol meaninglessly.

Table 5.13

Comparison between HLT and ALT Lesson 5 Cycle 1

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
	Worksheet 1	
1. Explain your understanding towards Aan's notes.	The students may see Aan's notes as in the beginning his father already has two stocks from last production (given in the problem) and every day he produce 3 crowns.	The students were able to recognize the notes of Aan.
2. Can you fill the next three empty boxes?	2 | 5 | 8 | 11 | 14 | 17 | 20	The students were able to see the addition of number in each formation.
3. In 30 days, how many crowns will be available?	Each day there are 3 crowns produced, then in 30 days Mr. Husnul will has 90 crowns. Since there are 2 crowns from last year, the number of the crowns in their house will be 92.	• The first pair used listing method until the 14th day working and then they incorrectly used the result of the 10th day production to be multiplied with 30. • The second pair multiplied 30 and 3 and then add the result with 2.
4. Write a general formula to represent the number of the produced crowns!	To generalize Aan's recorder, the students may write: the number of crowns equal to $2 + (3 \times \textit{the number of the working da}$ or write it in shorter way by symbolize the words variable into letter, i.e.: $S = 2 + 3 \times n$.	The students were not really engaged with the symbolization.
5. Suppose that there are 185 crowns in Mr. Husnul's house. Can you check how many days does Mr. Husnul already work?	The general formula for Aan's note is $2 + 3n$. Since that formula in particular n is produce 185 crowns and the students know that the constant 2 will always be there, the students can subtract 2 from 185, which is equal to 183. Each day Aan produces 3 crowns, hence the number of the day should be $\frac{183}{3} = 61$.	The second pair used division method, they divide 185 with 3 and got 61. The remainder is 2 and they considered it as the stock from the previous year
	Worksheet 2	
1. What can you explain about Mitha's note?	The students can explain that Mitha's mother merely has one stock from last year, but she can produce 4 crowns each day.	This activity was skipped because the students were not ready to use algebraic

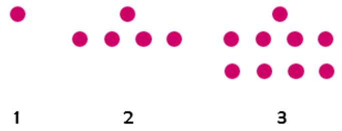
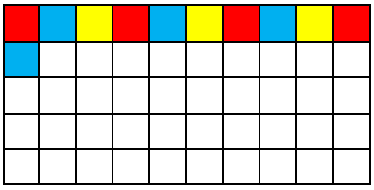
Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
2. Illustrate the number of the available crowns in her house from the very beginning until the end of the second working day.		symbolization in meaningful way.
3. What value of *n* you take as the starting point in question 2? Explain why you choose that number!	The students may confuse what to choose: 0 or 1. The correct answer is 0.	
4. How many crowns will be at the Mitha's house if her mother just start working for the 10th day?	Mrs. Tyas already works for 9 days, hence the number of crowns in their house will be $1 + 4 \times 9 = 37$.	
5. Mrs. Tyas's target is to have 145 crowns at the end. Now, she already has 73 crowns. How many days she has to continue her work?	Mrs. Tyas has 73 crowns, hence she already works for 18 days. To produce 145 crowns, she needs to work for 36 days, hence she has 18 days more to work.	
	Worksheet 3	
1. Who will produce the more crowns? Mrs. Tyas or Mr. Husnul Why? 2. Is it possible for them to have the same amount of crowns? Explain!	The students can explain that Mitha's mother will produce more because each day she has 4, while Mr. Husnul only 3. But in the first day they will have the same number of crown.	The students were able to evaluate that Mrs. Tyas has more crown after their first production by considering that she has 4 crowns each day.

Table 5.14

Comparison between HLT and ALT Lesson 1 Cycle 2

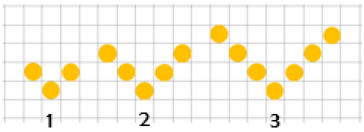
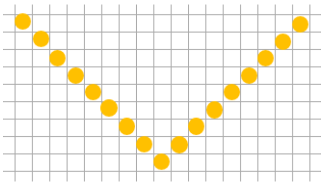
Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
	Worksheet 1	
1. Continue the strips for the next two numbers!	The students continue to write the numbers and give the color sign on them.	The students were able to determine the next color of the number string.
2. Can you predict the color of costume for the (a) 12th student? (b) 25th student?	• The students continue drawing. • The students consider the relationship between types of number (odd or even) with the colors.	• Some students made a list of the numbers and the related colors, not by drawing but the idea is the same. • Some students used the multiplication of 10. • Some students saw the relation between odd and even numbers with the colors.
3. Explain how you predict the color of costume for the 100th student!	• The students make a string of 10 and multiply it ten times. • The students make a generalization that since 100 is even number and every even numbers is come in white then the costume color of the 100th student is white.	
4. Is it possible to the 97th student to have the same costume color with the 43th student?	The students explain that since 97 and 43 are odd numbers than they will have the same color, which is red.	• The students saw the relation between odd and even numbers with the colors.
5. Suppose the 70th student get white for his/her costume, what is the costume color of the 1st one?	The students recognize the odd-even pattern, since 70 is even while 1 is odd, if 70 is red than 1 is white.	• The students understood that the arrangement of the dance's costume is different with the previous number problems. However they were able to use the characteristics of odd and even number with the new color arrangement.
	Worksheet 2	
1. Continue the drawing for the next strip, what is the color of the two-next strips?	The student will continue to draw the strips in the worksheet, by imitating or just write the symbol of the color (for instance yellow-red, or solely Y-R).	The students wrote the initial colors of the dancers' costume.
2. Can you predict the color of	• The students continue to draw it one by one.	• Some students applied listing method.

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
costume for the (a) 11^{th} and (b) 27^{th} students?	• The students use the previous method as they used in the Worksheet I, for instance stated that the odd number dancers will use red costume, while the even number dancers will use yellow costume. But then they get confuse with the yellow costume. • They may conclude that every multiplication of 3 is yellow, every multiplication of 2 is blue. But then they confuse what will happened with the multiplication of 2 and 3 (for instance the multiplication of 6, 12, etc.). • Uses the reference of 10^{th}, but then confuse why the 11^{th} strip does not come with the same color as the 1^{st} one (the following picture can be seen in the Appendix of the Teacher Guide).  If the students did not recognize this contradiction while using this strategy, the teacher can bring this in the discussion. • Using the multiplication of 3 as their reference. Each dancers whose number is the multiplication of 3 will use yellow. If one more than the multiplication of 3, the dancer will use red. If their numbers are two more than the multiplication of 3, the dancers will use blue costume.	• Some students incorrectly used the odd-even rules. • Some students used the multiplication of 3 reference to determine that every dancers whose number is the multiplication of 3 will use yellow, etc.
3. Explain your strategy to find the color of costume for	• Use the reference of 10 as explain the second problem.	

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
the 100th student!	• Use the reference of 3 or others which are the multiplication of 3 (i.e.: 6, 9, 12, etc.). • Divide100 with 3 and check the remainder. Since the remainder is 1, then the answer is red.	
4. If there are 53 students, would the formation has three equal color of costumes? Why or Why not?	The students realize that to have equal number of color, the number of students should be divisible by 3. Hence, the answer for this problem is no, because 53 is not divisible by 3.	The students were able to solve this by dividing 53 with 3. Since it has a remaining, they concluded that the colors will not equal.
5. Is it possible to the 76th student to have the same costume color with the 121st?	The students use the structure of the number to check the color. For instance: if you divide 76 with 3 you have one as remainder, and so does 121:3. Hence, they will come with the same color which is pink.	The students checked the color of costume used by the 76th and the 121st dancers.
6. Suppose that we don't know the color of costume used by the first dancer, but the 60th dancer uses red, the 61st uses blue and the 62nd uses yellow. Find out the color of costume used by the first dancer!	The students observe the structure of number, in which 60 is divisible by 3 and it comes with red. The 61st is blue, so the 1st is also blue.	• Some students stated that it should be blue without gave their reason. • Most of them merely stated that it should be red without argument as well. But we analyzed that the students who said "red" seemed to not consider that the arrangement was changed.

Table 5.15

Comparison between HLT and ALT Lesson 2 Cycle 2

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
	Worksheet 1	
1. Draw the arrangement for the 4th formation!		The students were able to draw the V formation.
2. How many dancers will be in the 6th formation?	The students may keep counting, but since they are involved in similar activities in three lessons before, they may already look for the more general strategy, i.e.: by observe that the number of pair is equal to the number of formation. Hence, it 6th formation, there will be 6 pairs. And there will be one in the middle, hence the total is 13 dancers.	• Some students used adding two method to determine the number of dancer for each next formations. • Some students used the structure of "pairs in the sides and one in the middle" rule as is predicted in the HLT.
3. Draw the formation which consist of 17th dancers!		• Some students drew from the one dot in the bottom and add two until they got 17. • Some other evaluate that there will be 8 pairs in 17 dancers and 1 in the middle before they draw it.
4. How many pairs of dancers in the 45th formation?	45 pairs.	Most students were struggle whether they will have 45 pairs or 45 dancers in total.
5. Explain your strategy to find the dancers in the 100th formation!	There will be 100 pairs of dancers and one in the middle, so overall there will be 201 dancers.	The students were using the middle-pairs rule as is predicted in the HLT.
6. Can this *V* formation have 92 dancers? Why or why not?	The students may argue that *V* formation will never have 92 dancers because it always result in odd numbers.	The students realized that the number of dancer in the *V* formation should be odd, hence 92 dancers will not be able to form it.
7. The committee plan to combine two groups with	The combination of two *V* formations will not produce a *V* formation, but three of them will be. The reason is because the *V* pattern is always come with odd	The students used the same logic as for the problem 6. Two *V* formations will have even number of dancers. Hence, it cannot be another *V* formation.

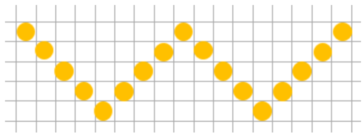
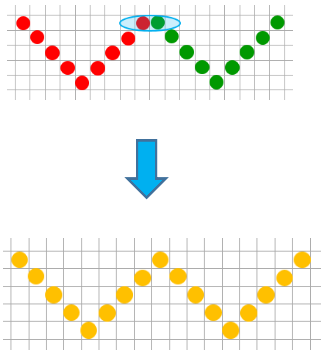
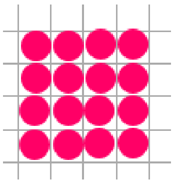
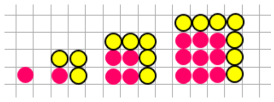
Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
V formation in one *V* formation, is it possible? Why or why not?	numbers. If two odd numbers are add together it will be even number. The students may also illustrate by using picture.	
	Worksheet 2	
1. Draw the arrangement for the 4th formation!		Due to the time limitation, the second worksheet was not discussed.
2. Tata compares the number of dancers in *V* and *W* formations. She claim that *W* patterns can be made from two *V* formations. Do you agree with her?	No, because if you combine two *V* formations, you need to remove one in the meeting point to get *W* formation. 	
3. Explain in words how you can work out the number of dancers when you are going to make certain *W* formation.	The number of dancers in *W* formation is equal to four times the number of formation add with one.	The students didn't get the meaning of word "unknown". They argued that they will never know because the number of formation itself is unknown.

Table 5.16

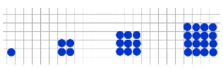
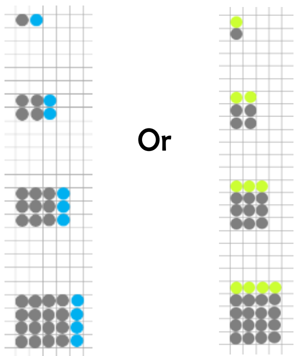
Comparison between HLT and ALT Lesson 3 Cycle 2

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
1. Draw the arrangement for the 4th formation!		There are different point of view in seeing the structure of square pattern and it is influenced the students' generalization. • Some students used the addition strategy by checking the additional dancers they need to add in each next formation. • Some students used the squaring method.
2. How many dancers will be in the 10th formation? 3. Explain your strategy to find the number of dancers in 100th formation! 4. Explain in words how you can work out the number of dancers when you are going to make certain formation.	To generalize, the students may come with different formula, i.e.: a. Since they compare the number of formations and the number of dots, they generalize that: the number of dancers will be equal to the square of the number of formation. Note: they may use "multiply a number by itself" instead of square. b. Look at the increase of the dots in each formations.  And generalize that in each formation, the number of dancers is increase as much as 2 times of the number of formation and then minus it one by one. Hence, the number of dancers in certain formation is the number of the dancer in previous formation added by two times the number of formation, minus one. Or, the number of dancers in a formation equal to the number of dancers in previous formation add with two times the number	

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
	of previous formation, plus one. To calculate the number of dancers in previous formation itself, the students may come up with the idea of squaring the number as well.	
5. If there are 144 people, can you make the square formation? How many dancers you need more to make the next one?	The students will answer yes because 144 is a square number. They may conclude that 144 is a square of 12, hence the next square number is 169. Therefore, 144 should be added by 25 to get 169.	Most of the students used intelligence trial and error. They started by knowing the square of 10 is 100 and move forward.

Table 5.17

Comparison between HLT and ALT Lesson 4 Cycle 2

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
1. With the minimum number of added people, could you transform this square formation into rectangular formation? 	We gave a space in the worksheet that can be used by the students to draw the additional dancers. This is the possible correct answers:  	• Some students added the dancers without considering the word "minimum" • Some students consider the minimum dancers that should be added.

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)

2. Investigate how to find the general formula of rectangular formation!

   a. The number of dancer in the 5th formation.

   b. The number of dancer in the 10th formation.

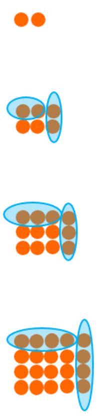
   c. The number of dancer in the 99th formation

Conjecture of Students' Responses (HLT)

The number of dancers in the 5th formation is 30 people.
The number of dancers in the 10th formation is 110 people.
The number of dancers in the 99th formation is 9900 people.

We predict the students will use these following strategies:

- Continue from the previous drawing.
- Count the addition in each formation.



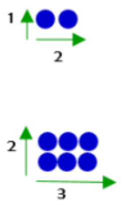
Or record it on the table or not in the table

No. Formasi	Banyak Penari	
1	2	
2	6	+4
3	12	+6
4	20	+8
5	30	+10
6	42	+12
7		+14
8		
9		
10		

- Start from the square formation
  Add the number of dancers in a square formation with the number of the formation.

Students' Actual Responses (ALT)

- Most students used the square formula which is added by the number of formation, to get the rectangular formation's dancers. For instance, in the 10th rectangular formation there will be $(10 \times 10) +$ 10 dancers, which is equal to 110 dancers.
- Some students were able to see the relation between the dancers in "length" and "width" sides with the total number of dancers by using multiplication relationship. For instance, in the 10th formation, there will be 10×11 dancers, which is equal to 110 dancers.

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
	 Previously, we know that the number of dancers in a square formation is equal to the square of the number of its formation. Hence, for instance, in the • Consider the shape of the rectangle and find the relation between the length, the width and the number of formation. They may conclude that the length of a rectangle is equal to the number of formation, while the width is equal to the number of formation + 1. • They may also see the relation between the length and the width of the rectangular and the number of dancers and conclude that the number of dancers is equal to length × width. 	
So, in general how could you determine the number of dancers in the certain formation?	The students will make a conclusion based on their strategies.	The students tend to use verbal generalization easier than write it.

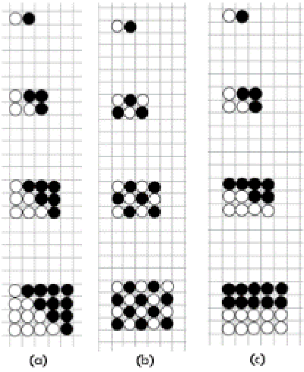
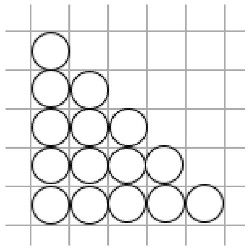
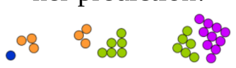
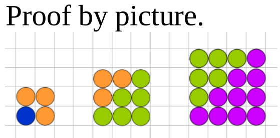
Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
3. Reconsider the following rectangular formation. Divide the dancers on the formation such that half of them will use black costumes and the other half use white costumes. Investigate why you can always do that in each formation!	 A square formation always have an even number of dancers. This happen due to one of the number of formation (or the number of formation+1) is an even number. Hence, even × odd = even. Since the number of dancers is an even number, it can be divided by 2.	The students divided it variously by considering not only the task instruction which state "fair colors of white and black", but also the artictic value of it. None of them divided as in prediction (a). Some of them drew as (b) and (c), the other used the similar but with some more modification in between. Even though the division was different, the students kept the "fair" rules in their works.

Table 5.18

Comparison between HLT and ALT Lesson 5 Cycle 2

Tasks	Conjecture of Students' Responses (HLT)	Students' Actual Responses (ALT)
1. Draw the dancers with white costume in the 5^{th} rectangular formation!		The students did it using two general methods: • By drawing the complete formation first (of the rectangular) and then colored the black and remove them. • By drawing one dancer and go down until reach the number of dancers (i.e. in the 5^{th} formation: 1-2-3-4-5 dots in each lines).
2. In the 10^{th} formation, how many dancers who use the white costume?	• Continue to draw the 10^{th} formation of triangular formation. • Observe the growing pattern in each formation and then add: 1 + 2 + 3 + 4 + … + 10 = 55 people. • Count from the number of dancers in the 10^{th} formation of the rectangular formation and divide it by 2	Most students used the square-rectangular-triangular formation method (as is predicted in the third prediction for the problem number 2, second prediction of number 3, and in the prediction for number 4).

<table>
<tr><th>Tasks</th><th>Conjecture of Students' Responses (HLT)</th><th>Students' Actual Responses (ALT)</th></tr>
<tr><td></td><td>The number of dancer in the 10th formation of the rectangular formation is 10 × 11 = 110 people. Hence, in the 10th formation of the triangular formation there will be 110 divided by 2 = 55 dancers.</td><td>For instance, the 10th triangular formation, is constructed by a half of the 10th rectangular formation, which is constructed by 10th square formation which is added by 10.</td></tr>
<tr><td>3. Write your fastest strategy to find the number of dancers in the 100th formation who use the white costumes.</td><td>The students may explain that to add from 1 until 100 is the fastest startegy.
Other students may argue that count from the numebr of dancers in the rectangular formation which is divided by 2 is the fastest.
The students may do an error by multiplying the number of dancers in the 10th formation with 10.</td><td></td></tr>
<tr><td>4. What is the number of rectangular formation which has 210 dancers using white costumes?</td><td>Find out the number of total dancers in the rectangular formation, which is 210 × 2 = 420.
And then find two consecutive numbers which multiplication result is 420.
Probably, the students will start by checking that 20 × 20 = 400 and try 20 × 21.
And then, they conclude that 420 orang terletak pada formasi ke-20.</td><td></td></tr>
<tr><td>5. Laura predict that if she adds the number of dancers in two consecutive triangular formations, she will get a square formation!
Do you agree with her? Help Laura to explain her prediction!
</td><td>Proof by observing the relation between the number of dancers between square and triangular formations.
<table>
<tr><th>Triangular Formation</th><th>Square Formation</th></tr>
<tr><td>1 + 3 = 4</td><td>4</td></tr>
<tr><td>3 + 6 = 9</td><td>9</td></tr>
<tr><td>6 + 10 = 16</td><td>16</td></tr>
<tr><td>10 + 15 = 25</td><td>25</td></tr>
<tr><td>15 + 21 = 36</td><td>36</td></tr>
</table>
Proof by picture.
</td><td>Due to the time limitation this problem was not discussed.</td></tr>
</table>

5.6 General Remarks on Students' Learning Early Algebra

Reflect to the teaching experiment phase, we figured out that early algebra activity can be helpful to prepare the students to develop an algebraic thinking which will be the important aspect of learning algebra in higher education. To start with, a pattern investigation related problem is one of the powerful activity which can provide a rich learning environment for the students, in which they can observe the series of objects and elaborate its characteristics to identify other objects which also belong to the same pattern. The use of visual support which represents the number series, provides an essential part of learning. It helps the students to catch the whole structure of a pattern which useful to create a generalization and in the same time to evaluate the member of pattern. In this following discussion we would like to emphasize the conclusion for each important aspects of the research.

About the Pattern

The first lesson of our design is about repeating pattern. When the students working on it, they need to pick a correct unit pattern and later on develop which reference can be helpful for them to work more flexible with numbers. The illustration for this point is given by the learning trajectory of the second worksheet in the first meeting, which is about a dance arrangement with three different color costumes: red-blue-yellow.

The big idea of this lesson is when the students realize it has something to do with the unit pattern: the smallest pattern arrangement which is going to repeat in the series. Once they realize that the unit pattern is three colors which is red, blue and yellow, they can use it to choose the "smartest reference number". For instance, if the problem is about to find the color of costume used by the 25th dancer, of course they can merely divide it by 3 and check for the remainder. However, they may also choose another number as their reference, i.e. 12 and doubling it to get 24 and check the remainder. They will still in the right path. Differently, if the students are not connecting the idea of pattern, unit patter and reference number, they may randomly choose 10 as their reference. Hence, they may state that the 10th dancer will use red costume, they double it to 20 and incorrectly conclude that the 25th dancer will use yellow.

The visual support of the repeating pattern is useful to use when the students get stuck of the chosen reference number. It will be easier for the students to see the aforementioned error when they are asked to present their idea by coloring the table as is illustrated in the following Figure 5.78.

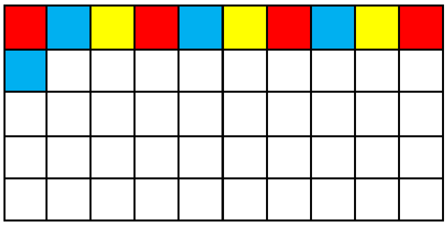


Figure 5.78. Example of incorrect reference number

The second lesson is dealing with the growing pattern with constant difference and it will be continued to the growing pattern with growing difference in the third until fifth activities. When the students working with these activities, it is important for them to see the basic structure of the given pattern. For instance, the triangular pattern. The first step to recognize this pattern is by checking the shape of it. We can observe the difference of how the students in our first cycle got lost of the "template" of the triangular pattern. In contrary, the students in the second cycle were quite good in seeing the structure of triangular pattern and even able to connect it with two other patterns: square and rectangular.

About the Context

Overall, the chosen context which is about dance formation was support the learning trajectory. We highlighted that in our first teaching experiment the context were not too powerful to keep the students talk about formation, number of dancers and the shape represented by its' formation. This is happened due to sudden change of the formalization level of mathematics which create a cognitive shock in students' mind. The gap was created by the tasks given in the fourth meeting of the first cycle which require the students to work with formal representation of variables and equations. At that time, the context were just standing as a meaningless daily-world problems, its existence doesn't contribute to the students' knowledge construction.

Different condition showed by the reflection of the second cycle. The context is used as a starting point which played important role of the students' point of view on the structure of the pattern. For instance, when the context was saying about the plan to modify the square into rectangular pattern. The way students' generalize the rectangular pattern was mostly influenced by square general formula, before they were able to find another independent generalization by considering the shape of rectangle.

About the Organization of the Task

For each meeting, the arrangement of the task were similar. It started by the introduction of the visual representation of the pattern, asked the students to draw the next member of that pattern, and then finding certain other members. The "next pattern" asked were getting bigger until reach the 100^{th} number. It is aimed to encourage the students to develop a general idea of the pattern, because once the number get high the students mostly will get tired to keep drawing or counting by one. The finding showed that it is useful to achieve the generalization goal.

The Notion of Variable

The algebraic equation was not explicitly introduced during the second cycle, but the students did more algebraic skill than in the first cycle. In the first cycle, especially in lesson 4 in which the symbol of *n* was introduced, the students were able to use it, but they didn't meaningfully understand it. The *n* appear as abstract object and the students used it in abstract way, they stop to talk about the changeable value of *n* or the hidden value of it. Differently in second cycle, the students were not directly used the variable in formal notation, but they were working with the role of variable as changeable value which can be dependent factor or independent factor. For instance, in the square formation related problem. When the students looking for the number of dancer in the 10^{th} formation, they valued the number of formation as independent variable which is influenced the total number of dancers as the dependent variable. Also, the students also seeing the variable as unknown number which has certain exact value.

Besides that, the most important point of this study is to introduce the role of variable as the representation of the range of value, which has important role to develop generalization ability. It is related to the role of variable as a changeable value, but focus on the relation between variables and the representation of the whole pattern and what kind of values that can be a member of the variables. For instance, in the square formation, the relation between the number of formation and the total number of dancers in each formation is given by squaring relation: the students need to multiply the number of formation with itself. The students can also evaluate what kind of number which can be the number of dancers, as they know it always come in square number form.

The Development of Strategy to do Generalization

To do generalization, the students were guided by the organization of the task which is started by picture, ask them to draw, find the "close-next" number and finally to the "far-next" number. The drawing method is useful to help the students realize the visual shape of the pattern. When drawing, some of them mostly also think about what is the difference between the previous, the current and the next member of the pattern. This critical thinking lead the students to the construction of recursive formula. From recursive formula the students start to develop other point of view in seeing the structure of a pattern. As the n^{th} number getting bigger, it will be tiring if they continue to use recursive formula because they need to count the previous number before able to find the next. The shift from recursive to general formula in this study is supported by the given visual representation. The geometrical form representation of the number pattern plays two important role in the present study: (1) as a visualization of the context and (2) as a mathematical models.

The Development of Strategy on Finding the Unknown

The most common strategy used by the students to find the unknown was trial and error or guess and check, but they used their prior knowledge to do intelligent guessing. Here, the students was not randomly guessing, they find the closest number that they already know. For instance to find what is the number of formation which has 420 dancers in the rectangular formation, the students started by find the square number closest to 420.

We realized that the task given in the designed materials was not much focused on the development of the strategy to solving linear equation. Hence, the students' strategy was limited to the basic method.

Algebra as a Language

The development of the algebraic language is shows as the students formulated their own general formula for a pattern. We found that, when the problems were directly asked to "create a general formula", the students got confuse. But they did generalization when the problems ask them to work with the numbers. Hence, we keep the number bigger to push the students find a way to explain how they solve the problem. Some students also used a symbol to represent their thinking. For instance in Figure 5.54, the student wrote $p \times l$ which is stand for length times width and that represent the number of dots in each rows and the number of rows given in the square formation.

5.7 Classroom Norms and Teacher Role

Before we conducted a lesson, we interviewed with the math teacher of VA MIN 2 Palembang, named Istiarti Sri Sa'diah, and observed one of her meeting on the class. We gained several important background of her (the result of the interview with her can be seen in Appendix 2), and highlighted that she is an experienced teacher who already teach for about 30 years but never applied RME approach in her classroom. She keep her lesson in a direct teaching and learning, on which she transfer the materials, including formula and everything related to it, write some problems and asks the students to work on it. For her, RME seems to take longer time than what she can afford in each semester and she afraid that a group/pair discussion will create an unconducive classroom situation.

We also observed the students of VA and found that they did a very little interaction related to mathematics during the lesson. When the teacher asked their opinion, they will rise their hand and answer it. They didn't ask a question. The interaction is only from teacher to students and the students will reply it in return.

Reflect to the preliminary observation of teacher beliefs on RME and classroom norms, the researcher tried to manage a discussion step by step. For the first meeting, we emphasized the class setting which will allow the students to discuss with pair and group of four. During the discussion the students should be encouraged to talk with their pairs, have the same amount of work (no child ignore the lesson by doing another activity) and the teacher should not transfer the materials.

During the discussion, the researcher showed the teacher guide and gave an example of how the opening of the lesson is expected based on the design. In the implementation, the teacher imitated the steps given in the teacher guide and let the students to work with their pairs.

The first lesson was the very first time for the students to work in group during their mathematics class in grade V. Hence, the students had to adjust themselves. Not all students were directly able to work with their partner. Some students too dominance while others are too shy to contribute in the discussion. We prepared for this condition, because we understood that the students are not accustomed to learn in this way. Hence, during the lesson the teacher kept moving around the class and reminded the students to talk with their partners or groups.

One important remark on the first meeting is the class discussion was not properly conducted. As the teacher let the students to work with their pairs, they lost their attention when it comes to the whole class discussion. They seemed to keep working when the representative of some groups were present their works in front of the class or answer the teacher's questions. Reflect to the first implementation, in the next discussion, the teacher and the researcher agreed to ask the students to stop working with their worksheet while the class discussion conducted. The teacher also need to explicitly drag the class's attention to the discussion.

In the second implementation, the attention to the class discussion were better than previous meeting. However, we found that it still teacher center instead of the students. The teacher become the one who stand in the front of the class, ask the students to answer from their seats and then repeat the students' words to the whole class. After the class finished, the teacher and the researcher discussed about it and agreed to give more chance to the students to perform in front of the class.

In the third lesson, the group work getting better, even though some students still need some remainders to work with their partners. During the class discussion, the teacher gave a chance to the students to present their works in front of the class. The students were not accustom to do the presentation, they looked very shy and tend to merely read their discussion in small voice. The teacher repeatedly asked them to louder their voice.

After the class discussion, the teacher and the researcher discussed about the aforementioned condition of the students. We discussed that it might be helpful if the students draw or write their works on the whiteboard. To support this plan, the teacher asked the researcher to prepare for larger worksheet which can be used as a display on which the students can write it and stick it on the whiteboard when they are going to present it. In addition, we also noticed that one thing still not elaborated yet, which is about socio-mathematical norms. Here, frequently the teacher merely asked the students with the best answer to present their works in front of the class. Hence, the other students with less sophisticated method usually erase their answer and copy what is written on the white board. Hence, we agreed that in the next meeting we will try to pick different type of answers and started with the basic idea and move to more sophisticated one.

We tried this plan in the fourth meeting. It is going relatively well and the “big sheet” help the students in present their works. However the socio-mathematical norms issue was still need to handle more. Hence, in the discussion after the implementation, we talked about it again. Now, not only that the correct answer which can be presented in the class discussion, the researcher proposed to give the equal chance to the students who do an error which can provides a rich discussion. In the fifth meeting, we tried to do that, but due to the time limitation it was not elaborated as is planned.

The following Figure 5.79 represents the improvement focus in each discussion before the implementation.

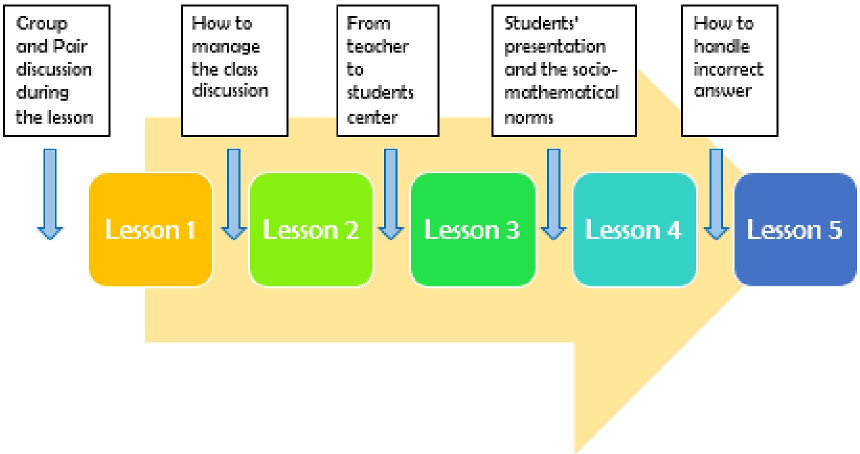


Figure 5.79. Discussion focus for classroom norms

In the end of the teaching experiment, even though some of the norms in the classroom were developed, especially in students' own construction, interaction between students and the teacher role as the organizer of the discussion, some important aspects were still need to be enhanced. We highlighted that the students interaction in term of questioning and responding to the others are still lack, also the students' confidence in present their works and the discussion about socio-mathematical norms.

One important issue during the implementation is the time management. We expected the students will be able to work in more various problem during the lesson which gave us 2×30 minutes for each meeting, but in reality some problems cannot be discussed. Some reasons behind it is because the teacher was likely to focus on not too important issues and she asked the students to discuss on it quite longer than it should be and sometime we got technical issue in the classroom which make the lessons disturbed. Also, because the pair and group's work setting is not common in the experiment class, the teacher encountered some difficulties to handle the class discussion, to give equal attention to each pair/group. She tried to manage it and it took longer time. However, we understand that five meetings will not easily changed everything. Hence, in general we satisfied with the cooperation of the teacher and the students as the participant of the current study.

CHAPTER 6
CONCLUSION AND RECOMMENDATION

The main goal of the present study is to contribute to an empirical grounded local instruction theory in learning algebra in preliminary class, especially using patterns activities. Gravemeijer (2004) underlined the definition of local instruction theory as the instructional sequences which is embracing the theories of how the students will learn about specific topic, what kind of classroom cultures and the powerful support that can be used by the teachers to enhance it.

In this final chapter, we would like to summary the main findings of the result in two major segments by explaining the reflective and prospective components. The reflective part will describe the conclusion of the study by answering the research question. The prospective part will give a recommendation based on the analysis of the implementation of this study can be improved for the next study of pre-algebraic lesson. The recommendation also important to enhance the power of the designed theory if it is going to be implemented in the classroom setting.

6.1 Conclusion

6.1.1 Answer for Research Question

In the following discussion, we would like to answer the research question of this study which is:

"How can patterns support students' algebraic thinking?"

In general, patterns activity is emphasize to the students' ability in seeing the continuous series which can be generalized. This study employed two functions of pattern. First, it is served as the visualization of problem. The patterns is the problem itself. From the study, we found that the students were easily engaged in the activity because it used of familiar context, got support through visual representation, and arranged progressively. In each lesson we designed a levelling problem of patterns on which the students will develop the strategies, from basic to advance, in order to find the "next term" of a pattern.

Mostly, in the beginning the students will start by imitating the shape of the basic pattern in order to understand the structure behind it. After that they may continue drawing to determine the next member of it, or if they not so good in drawing or find it is tiring to draw one by one, they will do listing method which leads them to the recursive formula. The use of recursive formula is characterized if the students need to find the n^{th} term before get the $(n+1)^{\text{th}}$ term. They may find it tiring to use recursive formula as the n getting bigger. They will observe the structure once again and try to figure out what is the general formula which will help them predict any member of the given pattern.

However, the aforementioned path of learning is not always followed by the students. Some students may see the structure of the general formula from the very beginning while the others need to do some more exploration. It is make a sense since the students' development of the knowledge construction is unique, it is depend on how the cognitive schemes built on them. The activities offered in this study is contribute to the students' habit in seeing the structure of the pattern itself. Structure sense is the important aspect of doing generalization. Different way of seeing the structure of a pattern may leads to different way or level of doing generalization. A very clear example was showed in the third activity of the second cycle, where the students work with the square formation and had develop different envision toward it (see Figure 6.1).

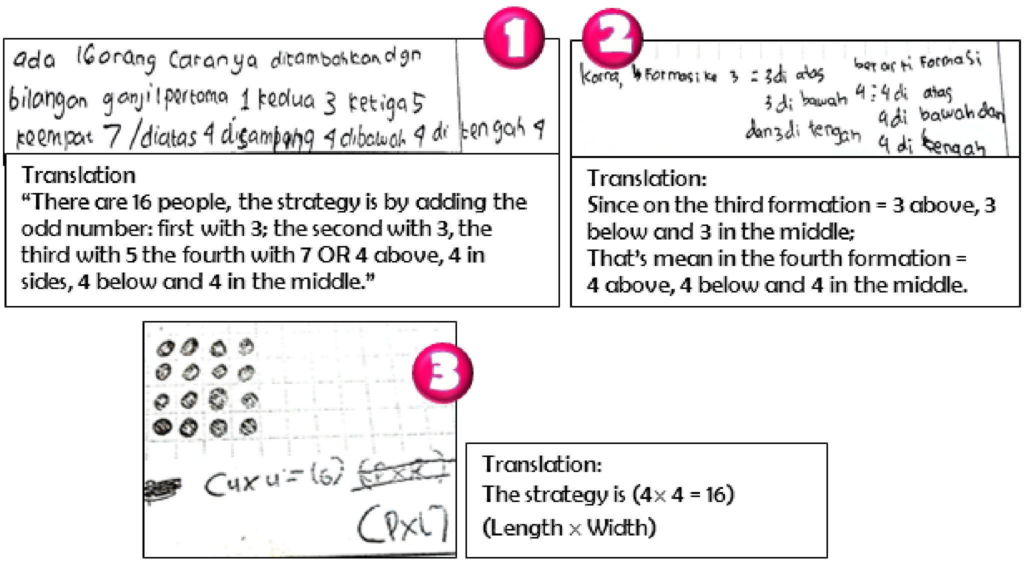
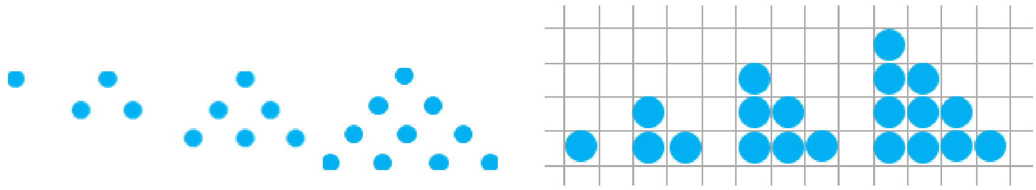


Figure 6.1. Example of different way of seeing the structure of square pattern

To optimize the use of pattern, we agreed that the chosen pattern should have a unique structure which can be explored and contribute to the development of the sense of structure. As the example, we point out the different ways to illustrating the triangular pattern in the first and the second cycle (see Figure 6.2).



(a) (b)

Figure 6.2. Comparing two representations of triangular pattern

Both of the pictures in Figure 6.2 are a triangular pattern. However, based on our findings, Figure 6.2 (b) which was used in the second cycle, gave a richer experience for the students to develop an algebraic thinking than Figure 6.2 (a) which were used in the first cycle. During the first cycle, the students were suddenly asked to observe the structure of triangular formation without any guidance of how they can see its structure Hence there were only a unique solution: you should add from 1 until forever. Case closed. The algebraic thinking will stop there. However, the pattern in the second lesson was started fro square, rectangular and finally triangular in the last lesson which give more support for them and also to challenge them to observe what it is behind the triangular shape. What makes it is growing one more in each formation? What can I use rectangular formation to get the triangular? How square formation related to it? The students also questioning themselves further, “Should I merely 1 until forever? Can I use different method? What will happen if I combine two triangular patterns?” Even if the students did not ask themselves as is expected, the task arrangement ask them to do so.

We also found that in this study in order to enhance the use of pattern activities is by creating a situation on which the students find the important of finding a more advance strategy. This can be done by set the number bigger and bigger until the students feel the urge of general formula because they cannot rely on the drawing, listing or recursive formula method.

Another important point from this study is about the abstract level of the language of algebra used in the designed problem. As we conducted the first cycle of this study, we found that introducing the formal algebraic expression or ask the students to create a generalization by using the unknown number called *n* are problematic.

First, the introduction of "unknown" term. The translation of the word "unknown" in Bahasa Indonesia was confusing the students. They argued that they will never know, for instance, how many dancers in the unknown number of formation of the certain shape formation, because it simply "unknown". Therefore, we modified the problems in the second cycle by asking for bigger number only or ask it verbally of how the students will work to solve a very big number. Here, even though the students didn't directly mention "this formula works for any number of *n*", the statement like "squaring (the number formation)" if they are going to find a number of dancers in a square formation is already an impressive generalization.

Second, when we tried to introduce the formal algebraic expression, the students started to work only with numbers and letters, and ignored the main role of variable. This condition different with the result of (Carraher et al., 2006) which successfully implemented a lesson in early algebra which is introduce the algebraic notation for even a second grader of elementary school. On the previous research the students were able to do generalization and to symbolize the notion of variable of their generalization. We analyzed that the students in the first cycle of the present study were not achieve the same things because: (1) the idea of the "unknown" term itself still abstract for them, (2) the variable concept is not fully constructed yet, and (3) the organization of the task itself was too jump, since the students had to move from the focus of doing observation of the structure to be able to do generalization, to the focus of the abstractness of the language usage. Therefore, we diminished the use of abstract notation in the second cycle of the study.

Overall, the general LIT of patterns for early algebraic lesson which is constructed from this study can be observed in the following subsection.

6.1.2 Local Instruction Theory on Learning Early Algebra Using Patterns

The aim of this study is to contribute to the theory of learning algebra for the fifth grader of elementary school. The specific activity chosen to be the main elaborated field for the students is patterns activity. The following Table 6.1 summarized the global view of instructional designed activities.

Table 6.1

Local Instruction Theory in Pattern Activities

Activity	Main Goals	Global Description of the Activity
Activity 1: Costume Color	• Identify the pattern of objects' arrangement. • Predict 'the next' term of a regular pattern. • Generalize the strategy to predict any term in a pattern. • Evaluate the relation between the numbers.	The students need to find the regularities in the two types of repeating pattern in the arrangement of dance's costume colors. The unit pattern of the first task is red-white colors. The unit pattern of the second task is red-blue-yellow colors.
Activity 2: *V* Formation	• Predict 'the next' of the growing pattern with constant difference. • Use words variable. • Assess the conjecture for generalization.	The students will observe a formation of dance which is resemble to the shape of *V* alphabet. They need to discuss what number of dancers which can construct a *V* formation. The *V* formation is the representative of a growing pattern
Activity 3: Square Formation	• Predict 'the next' of the growing pattern with growing difference • Create word formula to generalize a pattern. • Find the number of unknown.	The students will discuss about square formation which is one example of growing pattern with growing difference. The context is about flash mob which use the shape of square. The students need to figure out what is the relation between the number of dancers and the number of the n^{th} constructed formation.
Activity 4: Rectangular Formation	• Evaluate the structure of a pattern. • Create a general formula of a pattern.	The students will discuss how to modify the square into rectangular formation and generally explain the total number of dancers in certain number formation of rectangular formation.
Activity 5: Triangular Formation	• Developing a strategy to find the value of variable as an unknown number. • Formulizing a reasonable argument by observing the structure of pattern supported by its visual representation.	The context is about the arrangement of the costume colors used by the dancers in rectangular formation. Half of the dancers will use white costume, while another half use the black one. Each group of the same color costume dancers construct a triangular shape. In this activity, the students will discuss the relation between square-rectangular and triangular formation.

6.2 Recommendation

Overall we suggest a further research in field of early algebra which is bridging the arithmetic and algebraic aspects for young students to help them develop their algebraic thinking. After reflect to the findings and its' analysis, we remarked some recommendation that might be useful for further research as well for the implementation of the learning design in the classroom.

First, if the students' characteristics are relatively hard to be tired in using keep listing method or using recursive method even for the requirement of big number (up to 100), consider to use even bigger number. This is important to encourage the students think about more efficient strategy because they cannot do basic idea of listing or keep counting forever. If they keep in basic arithmetical way of thinking, they will not achieve the generalization skill.

Second, this study was not give a wider chance to the students to develop the strategy of finding unknown. The students were able to solve the finding unknown related problems mostly by using trial and error or intelligence guessing strategies. In the further study, the future researcher may consider to design a patterns activities which also encourage the students think about more advance strategy, like cover up method.

Third, the further research may also consider to focus not only toward the generalization of arithmetic but also in proof it geometrically using the help of visual supports. To support the chance of doing geometric proof, the design can involve the use of concrete object. Hence the students can, for instance, freely moving one object to another place, substituting several objects and observe what will happen if certain objects be combined with others.

We will end this remark by pointing to the time and task management issue. As is explained in the analysis part, some of the designed problems were not discussed during the study due to the time limitation (some are used in the posttest some are dropped). This condition can be a consideration when a teacher wants to implement the same learning trajectories. The teacher can prepared the problems as is used in this study, however if the students took longer to solve the problems or the time allocation in the school was limited, the teacher can merely discuss a part of it.

REFERENCES

Anderson, T., & Shattuck, J. (2012). Design-based research: A decade of progress in educational research? *Educational Research, 41*, 16-25. doi:10.3102/0013189X11428813

Bakker, A. (2004). *Design research in statistics education: On symbolizing and computer tools.* (Doctoral Dissertation). Utrecht: CD-beta press.

Bakker, A., & Van Eerde, H. A. (2015). An introduction to design-based research with an example from statistics education. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to Qualitative Research in Mathematics Education* (pp. 429-466). New York: Springer. doi:10.1007/978-94-017-9181-6_16

Ball, D. L. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education.* Arlington: RAND.

Barab, S., & Squire, K. (2004). Design-based research: Putting a stake in the ground. *Journal of the Learning Sciences, 13*, 1-14. doi:10.1207/s15327809jls1301_1

Brawner, B. (2012). Teaching and learning with technology: Reforming the algebra classroom. *Southwest Teaching and Learning Conference* (pp. 1-8). San Antonio: Texas A&M Univeristy.

Capraro, M. M., & Joffrion, H. (2006). Algebraic equations: Can middle-school students meaningfully translate from words to mathematical symbols? *Reading Psychology, 27*, 147-164.

Carraher, D. W., Schlieman, A. D., Brizuel, B. M., & Earnes, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education, 37*, 87-115.

Dekker, T., & Dolk, M. (2011). From Arithmetic to Algebra. In P. Drijvers (Ed.), *Secondary algebra education: Revisiting topic and themes and exploring the unknowns* (pp. 69-87). Rotterdam: Sense Publishers.

Denscombe, M. (2010). *The good research guide.* Berkshire: McGraw Hill.

Desforges, C., & Cockburn, A. (1987). *Understanding the mathematics teacher*. Sussex: Falmer Press.

Drijvers, P. (2003). *Learning algebra in a computer algebra environment.* (Doctoral Disertation). Utrecht: CD-beta press.

Drijvers, P., Dekker, T., & Wijers, M. (2011). Patterns and formulas. In P. Drijvers (Ed.), *Secondary Algebra Education: Revisiting Topics and Themes and Exploring the Unknown* (pp. 89-100). Rotterdam: Sense Publisher.

Drijvers, P., Goddijn, A., & Kindt, M. (2011). Algebra education: Exploring topics and themes. In P. Drijvers (Ed.), *Secondary algebra education: Revisiting topics and themes and exploring the unknown* (pp. 5-26). Rotterdam: Sense Publisher.

Egodawatte, G. (2011). *Secondary school students' misconceptions in algebra.* (Doctoral Dissertation). Retrieved from https://tspace.library.utoronto.ca/bitstream/1807/29712/1/EgodawatteArachchigeDon_Gunawardena_201106_PhD_thesis.pdf.pdf

Frambach, J. M., van der Vleuten, C. P., & Durning, S. J. (2013). AM last page: Quality criteria in qualitative and quantitative design research. *Academic Medicine, 88,* 552.

Freudenthal, H. (1974). Soviet research on teaching algebra at the lower grades of the elementary scholl. *Educational Studies in Mathematics, 5,* 391-412.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning, 6*, 105-128. doi:10.1207/s15327833mtl0602_3

Gravemeijer, K. (2011). How concrete is concrete? *IndoMS-JME, 2*, 1-14.

Gravemeijer, K., & Bakker, A. (2006). Design research and design heuristics in statistics education. In A. Rossman, & B. Chance (Ed.), *Proceedings of the Seventh International Conference on Teaching Statistics* (pp. 1-6). Voorburg: International Statistical Institute.

Herbert, K., & Brown, R. H. (2000). Patterns as tools for algebraic reasoning. In B. Moses (Ed.), *Algebraic Thinking, Grade K-12: Readings from NCTM's School-Based Journals and Other Publications* (pp. 123-128). Reston VA: National Council of Teachers of Mathematics.

Jacob, B., & Fosnot, C. T. (2007). *Young mathematicians at work: Constructing algebra.* Portsmouth, NH: Heinemann.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on childrens' algebraic reasoning in elementary school. *Journal for Research in Mathematics Education, 38,* 258-288.

Jupri, A., Drijvers, P., & van den Heuvel-Panhuizen, M. (2014). Difficulties in initial algebra learning in Indonesia. *Mathematics Education Research Group of Australasia,* 1-28. doi:10.1007/s13394-013-0097-0

Kementerian Pendidikan dan Kebudayaan. (2006). *Kurikulum 2006: Kompetensi dasar SD/MI (Curiiculum 2006: Basic competence for elementary school).* Jakarta: Author.

Kementerian Pendidikan dan Kebudayaan. (2013). *Kurikulum 2013: Kompetensi dasar SD/MI (Curiiculum 2013: Basic competence for elementary school).* Jakarta: Author.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator, 8*, 139-151.

Kvasz, L. (2006). The history of algebra and the development of the form of its language. *Philosophia Mathematica, 14,* 287-317. doi:10.1093/philmat/nkj017

Lannin, J. K. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning, 7,* 231-258.

Lee, L., & Wheeler, D. (1987). *Algebraic thinking in high school students: Their conceptions of generalisation and justification.* Montreal: Concordia University.

Ma, H. L. (2007). The Potential of patterning activities to generalization. In J. In Woo, & e. al. (Ed.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education. 3*, pp. 225-232. Seoul: PME.

National Council of Teachers of Mathematics. (1989). *Principles and standards for school mathematics.* Reston, VA: Author.

Plomp, T. (2007). Educational design research : An introduction. In T. Plomp, & N. Nieven (Ed.), *Proceedings of the Seminar Conducted at the East China Normal University* (pp. 9-35). Shanghai: SLO.

Quinlan, C. (2001). From geometric patterns to symbolic algebra is too hard for many. In B. J., P. Perry, & M. M. (Ed.), *Numeracy and Beyond, Proceedings of the Twenty-Fourth Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 426-433). Turramurra: MERGA Inc.

Renze, J., & Weisstein, E. W. (n.d.). *MathWorld*. Retrieved from Wolfram Web Resource: http://mathworld.wolfram.com/Algebra.html

Rivera, F. (2011). Toward a visually-oriented school mathematics curriculum. *Mathematics Education Library, 49*, 21-38.

Sampsel, A. (2013). Finding the effects of think-pair-share on student confidence and participation. *Honors Projects*, Paper 28.

Sharma, B. K. (2014). Interactional concerns in implementing group tasks: Addressing silence, dominance, and off-task talk in an academic writing class. *Innovation in Language Learning and Teaching*, 1-18. doi:10.1080/17501229.2014.914522

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114-145.

Tall, D., & Thomas, M. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics, 22*, 125-147.

Treffers, A. (1987). *Three dimensions: A model of goal and theory description in mathematics instruction - the Wiskobas project.* Dordrecht: D. Reidel Publishing Company.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford (Ed.), *The ideas of algebra: K-12* (pp. 8-19). Reston, VA: National Council of Teachers of Mathematics.

Van den Akker, J., Gravemeijer, K., McKenney, S., & Nieveen, N. (Eds.). (2006). *Educational design research.* London: Routledge.

van den Heuvel-Panhuizen, M. (2000). Mathematics education in the Netherlands: A guided tour. In *Freudenthal Institute CD-rom for ICME9* (pp. 1-32). Utrecht: Utrecht University.

van Eerde, H. (2013). Design research: Looking into the heart of mathematics education. *The First South East Asia Design/Development Research (SEA-DR) International Conference* (pp. 1-11). Palembang: Sriwijaya University.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*, 458-477.

Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, *49*, 379-402.

APPENDICES

LEARNING MATERIALS

Teacher Guide which is consists of lesson plans, students' worksheets, the overview of the activities and classroom materials can be downloaded at:
https://drive.google.com/open?id=0B3BCtmkBNL_dR2U2azhCTWdUaVE
(English Version), or
https://drive.google.com/open?id=0B3BCtmkBNL_dRTIybWVZNnF5NmM
(Indonesian Version)

APPENDIX 1

THE CLASSROOM OBSERVATION SCHEME

Classroom Norms

- Do the students ask for teacher's agreement for any steps they make when solving a problem?
- Is there any particular rule, like "the students should raise their hand before asking or saying or answering something" or "ask permission to the teacher before leave the class"?
- Do the students brave to share their ideas?
- Do the teacher remind the students to listen to the other?
- Do the students ask the teacher if there is something unclear for them?

Socio-Mathematical Norms

- How the teacher scaffolds the students when they get stuck?
- Can the students have different answer with other students?
- Can the students have different answer with the teacher?
- How the teacher manages different opinions in the class?
- Do the students are encouraged to think about other solutions or strategies?
- Does the teacher appreciate more efficient and effective strategies used by the students? Do they discuss it together?
- How the teacher react to the incorrect responses/answers?

Pedagogical Aspect

- How the teacher open the lesson?
- How the students are managed during the lesson, are they work individually, in pair or in group?
- Does the task's instructions are clear?
- Is there any class discussion?
- How the teacher conduct the lesson, by explaining? By giving a notes in the blackboard? Letting the students do some problems without clear guidance? Or by giving a chance to the students to solve some guided problems?
- Does the teacher gives a worksheet for the students?
- What the teacher do when the students work with their tasks?
- How does the interaction between teacher and students?
- How does the interaction between students?
- Do the teacher share equal chance to the students to participate in discussion?
- How the teacher deals with irrelevant behavior of the students?
- Does the teacher follow the book strictly?
- How the teacher end the lesson?

- Does the teacher checks the students understanding (for instance by ask them to summarizing the point of lesson)?
- Is there any formative assessment?
- How does the time management

Students' Characteristic

- How does the students' participation in the classroom? Are they active or they mostly shy and quite in the classroom?
- Do they listen to the teacher?
- Do they listen to the other students?
- Is there any dominated students?
- Do the students become overactive when the teacher let them to work in pair or group?
- What is students' general difficulties in learning mathematics?
- How is their motivation in learning?

Implementation of PMRI

- Does the teacher use realistic problem to start the learning activities?
- Do the students have a chance to construct their own understanding?
- Is there any intertwinement between the learning topic with other topic in mathematics or other subjects?
- Is there mathematical model used in the lesson?

THE TEACHER'S INTERVIEW SCHEME

Teacher's Background

- What is your educational background?
- How long have you been teaching in elementary school?
- How long have you been teaching the fifth grader students?
- Are you willing to cooperate to try new things in term of teaching and learning?

Teaching Experience Related to the Research's Topic

- Do you have experience in teaching arithmetical topic?
- If yes, how do you usually teach them, for instance to find the unknown number of the equation (as is written in the syllabus)? Do you want to change it in the future? What difficulties encountered by the students related to the topic?
- If no, how will you teach it?
- Do your students able to do basic arithmetical operation like addition, subtraction, multiplication and division?

Classroom Norms

- How you manage your class's discussion?
- Do you have agreement with your students related to what students' should and shouldn't do during your class?

Socio-Mathematical Norms

- What you usually do when your students get stuck?
- How you manage different opinions in your class?
- Do you usually encourage your students to think other solutions or strategies?
- Do you discuss more efficient or effective strategies with your students?
- How you react to the incorrect responses/answers?

Pedagogical Aspect

- How you usually manage your students in the class? Are they work individually all the time or in pair or in group?
- If you usually use group work, how you grouping your students?
- How you usually open your lesson?
- Do you mostly use textbook? What book do you use to teach mathematics in fifth grade?
- Do you use other resources?
- Do you prepare a worksheet?
- What do you do to encourage the passive students to involve in discussion?
- Do you usually give a homework?

Students' Characteristics

- What is the average age of the fifth grader students in your class?
- How many people are there in your class?
- How is the students' condition in your class? Are they high, moderate or low achievement's students?
- How about their motivation in learning?
- Do they actively participate in the class?
- Is there any dominating students?

Knowledge about PMRI

- Have you heard about PMRI?
- Have you participate in the workshop or seminar related to PMRI?
- Have you implement PMRI in your classroom?
- If no, why? Will you try it in the future?
- If yes, is it continue or just for certain topic? Will you do it also in the future?
- What do you think about PMRI? Do you see any difficulties in its implementation?
- Are you use daily live context or problems in your class?

APPENDIX 2

SUMMARY OF THE INTERVIEW WITH THE TEACHER

Teacher's Background	The name of the teacher is Mrs. Istiarti Sri Sa'diah, S.Pd.I. She has been teaching in primary school for 28 years after she graduated from teacher training in field of elementary education and Islamic studies.
Teaching Experience Related to the Research's Topic	As the primary school teacher, she is specialized in teaching in higher grades. This year is her first time to teach mathematics for fifth grader (she mostly teaching for the sixth grader). She is not familiar with the theme of "pre-algebraic" lesson, because for her algebra is a secondary school's topic. However, when the researcher gave an example of number series, she agreed that some simple series are learnt in primary school, when the students learn about jumping number, for instance. In addition, since she always teach higher grades in elementary school, she never teach such of basic arithmetical lesson which is similar with the present study.
Classroom Norms	She will open the class, deliver the materials, creates some routine problems and asks the students to solve it by themselves.
Socio-Mathematical Norms	Sometimes, she asks the students to find "simpler form of answer" for instance in the fraction topic.
Pedagogical Aspect	The teacher always had her class to work individually, no group discussion were allowed to prevent the loudness of the students' voice which can disturb other's classroom. The teacher and the students have the same textbook, but rarely use the problem there during the lesson. The teacher mostly creates her own problems, so she can adjust the level of the problems with the students' condition in the classroom.
Students' Characteristics	There are 32 students of VA MIN 2 Palembang. The students are relatively active and have a good motivation in learning mathematics. Some students may have less participation during the lesson, while some others are likely be dominated students.
Knowledge about PMRI	The teacher had heard about PMRI but she never apply it or follow a training on it previously. She argues that PMRI will take much time and she afraid that she will not be able to deliver all materials in one semester if she uses it.

APPENDIX 3

OBSERVATION SHEETS

Observer[4] :

Date :

Classroom Management[5]

No.	Aspect	Explanation
1.	Social Norms	
2.	Socio-mathematical Norms	
3.	Interaction between teacher and students	
4.	Interaction between students (pair and group discussions).	
5.	The appropriateness between teacher guide and the implementation	
6.	Time Management	

Accomplishment of Learning Goals Lesson 1

No.	Aspect	Explanation
1.	Students are able to identify the patterns (red-white and pink-yellow-blue).	
2.	Students are able to predict the 'next' term of a sequence.	
3.	Students are able to do generalization for particular pattern.	
4.	Students are able to evaluate the relation between numbers.	
5.	Strategies used by the students to generalize a pattern/predict the next term.	
6.	The clarity of the task's instruction.	
7.	Feasibility of the task.	
8.	Effectiveness of the scaffoldings.	

Accomplishment of Learning Goals Lesson 2

No.	Aspect	Explanation
1.	Students are able to identify the shape of V patterns.	
2.	Students are able to predict the 'next' term of a sequence.	

[4] The observer of this study is Rudi Hartono, a colleague from IMPoME Batch V

[5] To be fulfilled in each lesson

No.	Aspect	Explanation
3.	Students are able to do generalization for particular pattern using a daily words formula.	
4.	Students are able to evaluate the relation between pattern (combination of 2 V)	
5.	Strategies used by the students to generalize a pattern/predict the next term.	
6.	The clarity of the task's instruction.	
7.	Feasibility of the task.	
8.	Effectiveness of the scaffoldings.	

Accomplishment of Learning Goals Lesson 3

No.	Aspect	Explanation
1.	Students are able to identify the shape of pattern (square number's pattern)	
2.	Students are able to predict the 'next' term of a sequence.	
3.	Students are able to do generalization for particular pattern using a daily words formula.	
4.	Strategies used by the students to generalize a pattern/predict the next term.	
5.	Students are able to find the number of unknown.	
6.	Strategies used by the students to find the unknown number.	
7.	The use of variable.	
8.	The clarity of the task's instruction.	
9.	Feasibility of the task.	
10.	Effectiveness of the scaffoldings.	

Accomplishment of Learning Goals Lesson 4

No.	Aspect	Explanation
1.	Students are able to modify a square formation to the rectangular formation using the smallest additional number of dancers.	
2.	Students are able to predict the next of rectangular pattern.	
3.	Students are able to generalize the rectangular pattern using daily word's language.	
4.	Students are able to reason in general why a rectangular numbers can be divided by 2.	
5.	The clarity of the task's instruction.	

No.	Aspect	Explanation
6.	Feasibility of the task.	
7.	Effectiveness of the scaffoldings.	

Accomplishment of Learning Goals Lesson 5

No.	Aspect	Explanation
1.	Students are able to identify the triangular pattern.	
2.	Students are able to predict the next of triangular pattern.	
3.	Students are able to generalize the triangular pattern using daily word's language.	
4.	Students are able to find the unknown number.	
5.	Students are able to develop a reasonable argument to prove the relation between triangular and square pattern.	
6.	The clarity of the task's instruction.	
7.	Feasibility of the task.	
8.	Effectiveness of the scaffoldings.	

APPENDIX 4
PAPER PUBLICATION[6]

Understanding Students' Transition from Arithmetic to Algebraic Thinking in the Pre-Algebraic Lesson

ABSTRACT

Purpose — An Early algebra activities is important to prepare a young students to develop algebraic thinking before they enter algebra lesson in secondary school. Hence, this research was aimed to develop a local instructional theory in learning early algebra for elementary school students in Indonesia using Pendidikan Matematika Realistik Indonesia (PMRI) approach.

Method — Related to the aim of the study which is to design a learning materials, this article is a part of the design research study which is following three steps of designing: preparation, teaching experiment and retrospective analysis. The teaching experiment was conducted in fifth grader of MIN 2 Palembang, Indonesia, academic year 2014/2015. The data was gathered through students' written works, video registration and field notes. Those information was analyzed qualitatively using constant comparative method.

Findings — From the analysis we found that the students use the help of visual support of the given algebraic pattern to move from arithmetic-typical thinking

[6] When publishing this thesis, the paper was submitted in to Malaysian Journal for Learning and Instruction (MJLI).

which is showed by the existence of recursive formula to the algebra-typical thinking which is represented by the construction of the general formula.

Significance — Reflect to our findings, we recommend the authorities in Indonesia to consider the use of pre-algebraic lesson in form of algebraic pattern investigation which is visualized in geometrical representation, for the elementary school students.

Keywords: Algebra, pre-algebra, early algebra, elementary school, recursive, generalization, PMRI

INTRODUCTION

Most problems in algebra teaching and learning in secondary school have been blamed to the lack of connection between what the students learned in primary level and what they are going to learn in the higher education (Dekker and Dolk, 2011). The students spent a lot of time in arithmetic during their elementary years. It is understandable since reflect to the history, arithmetical competencies is important as the basic of algebra. However, to keep the students working only in field of arithmetic, leads to the limitation of students' mathematical conception. As a finding from Lee & Wheeler (1987), the students seemed to not believe that algebra is generalization of arithmetic, since for them arithmetic will never be generalized. The implication of this tendency is the students see these two branch of mathematics, arithmetic and algebra, as two separated knowledge. There is nothing to do with arithmetic when they are in algebra class and vice versa.

Moreover, the typical learning of algebra in Indonesia creates an image that algebra is merely concern with symbol and notation, with finding the solutions for *x* and *y*. Hence, the crucial aim of the learning algebra which is developing the students' algebraic thinking is mostly forgotten. Instead of understanding algebra as the relation between the numbers' structure and generalization, ironically, school algebra had been transformed into mechanical process to solve algebraic expressions.

A recent study by Jupri, Drijvers & van den Heuvel-Panhuizen in 2014, figured out that the current learning style of algebra in Indonesia, which is not only rigidly separate the relation between arithmetic and algebra, but also preserve the formal method of teaching it, has contribute to the students' low performances in algebra (as indicated in TIMSS 2007, PISA 2009 and PISA 2013). Therefore, they called to the reformation on teaching and learning algebra in Indonesia. In line with that, a pre-algebraic related activity is strongly recommended to be included in Indonesian preliminary classroom.

As the students in earlier grades in primary school are already engaged in a number of arithmetic related topics, the arithmetical thinking already builds on their cognitive schema. Hence, the activities which focused on the transition from the typical arithmetical problems to the algebraic problems are needed. For instance, before they are in arithmetic class, the students learn that 1 is an odd number and 2 is an even number. Later, in early algebraic class they will identify which numbers are characterized as odd and which are even.

In line with that, National Council of Teachers of Mathematics (1989) recommended the use of pattern activities as a solution to bridge the shift from arithmetic to algebra, and at the same time to start algebraic lessons in early grades. The pattern investigation could be designed in a different form, for instance pictorial or geometrical representations, number patterns, patterns in computational procedures, linear and quadratic patterns and repeating patterns (Zaskis & Liljedahi, 2003).

Even though a number of studies pointed out the use of pattern investigation activities as a connection to the preliminary algebraic lesson, the study of this field is not done yet. As in Indonesia which is not familiar with this kind of lesson, to start a pre-algebraic lesson, we need to understand the students' knowledge construction when they are given an algebraic related problem.

Reflected to the aforementioned background, the general aim of this study is to develop a local instruction theory in learning early algebra. In specific we would like to understand how patterns support the development of algebraic thinking. Therefore, the present study is going to answer the following research question: "*how the students' move from arithmetic to algebraic thinking?*"

METHOD

In line with the aim of the study which is to *design* a local instructional theory and the research question which is to *understand* how the designed materials will support the students' transition from arithmetic to algebra, design research was deliberately chosen as the appropriate approach.

The study followed three steps of design research, which are: preparation, teaching experiment and retrospective analysis (Plomp, 2007). In the preparation stage, we studied literatures, checked for the current condition of teaching and learning trends in Indonesia and designed a learning materials in form of Hypothetical Learning Trajectory (HLT).

HLT is the hypothesis used in the study. It is called *hypothetical* due to its characteristic which can be adjusted during the research, based on the findings of the study. It is formulated in three main aspects: the learning goals, the activities and the prediction of how the students will responses towards it (Simon, 1995).

The second step of the study is teaching experiment, in which we implemented the learning lines in the “real” classroom situation. We chose a partner school called MIN 2 Palembang, Indonesia. It is a state Islamic elementary school in Palembang. There are two teaching experiments in this study. The first is the pilot where only four students of VB class of MIN 2. The second teaching experiment involved the whole class of VA MIN 2 and their mathematics teacher. The aim of the teaching experiment in general is to check the appropriateness of the HLT when it be implemented in the real classroom environment. The last phase of this study is retrospective analysis. Here, the collected data was carefully analyzed using qualitative analysis called constant comparative method (Bakker, 2004). The result from analysis phase contribute to the adjustment of the HLT.

Previously, it is mentioned that this study is going to design a learning materials in pre-algebra related topic. Questioning the traditional algebraic lesson, which is formally given to secondary school students, we will use arithmetic as a first step in a big leap to formal algebra. Pattern investigation will be used as the main activity of the study and it will be visualized in geometrical form to give a less abstract elaboration field for the students.

The designed learning activities are based on the principles of Pendidikan Matematika Realistik Indonesia (PMRI) which is adapted from Realistic Mathematics Education (RME) approach. Based on PMRI point of view, which is influenced by Freudenthal (1974) thought, mathematics should be valued as a human activity. Therefore, instead of transfer the knowledge of mathematics to the learners, PMRI demand to a rich learning environment which gives the students chance to do mathematizing.

There are five tenets of PMRI which become the foundation of designing patterns activities in early algebraic lesson (van den Heuvel-Panhuizen, 2000; Zulkardi, 2002). First, the use of context. We used a local and familiar context of dance formation as the starter of the lesson. Second, the use of visual support to bridge the “real world” in students’ mind with “abstract world” in algebra. As previously explained we used pattern activity to bridge the students’ transition from arithmetic to algebra. To support it, a visual representation of the pattern will be given as the illustration of how does the number emerged in the dance formation problem. Third, we emphasize the students’ own production. Related to the third point, we also took care of how the interactivity between teacher and students or among the students themselves should be done. The social norms in the classroom was discussed earlier with the teacher to make sure that the teacher understand her role as a classroom manager who gives the chance for the students to learn and discuss with their pairs or groups. The last is the intertwinement principle which means the designed is not stand alone but connected to other learning strands. In this case, the designed pre-algebraic lesson is related to at least three branch of mathematics: arithmetic, algebra and geometry (as the pattern embodied in the geometrical shape representation). Also, since the context is about dance formation, it is also relate to the art, culture and social issues in Indonesia.

In this paper we will only focus on the result of the second teaching experiment and in detail we will only zoom in into the third designed activity.

RESULTS

Before we implemented the designed materials, we conducted a pretest which is aimed to gain an information about students' prerequisite knowledge in arithmetic which will be the base of the learning algebra. From the pretest we found that most of the subject of the study are capable to do basic arithmetical problems. Hence, the pre-algebraic lesson is possible to be implemented.

The lesson was started by a repeating pattern activities which is continued by growing pattern with constant difference activities in the next meeting. Then, the students were encounter a growing pattern with growing difference related problem in the third meeting. By thoroughly analyzed the learning moment of the students in this activity, we have some insight of how the students use their arithmetical base to move more advance in generalization which is the heart of algebra.

The third meeting on the second teaching experiment was about a square pattern. There are three learning goals in this lesson, which are: (1) predicting the "next pattern", (2) create a general formula for the square pattern and (3) evaluate the structure of pattern. The third lesson conducted on March 20, 2015. The context is about creating a flash mob formation for Indonesian Traditional Martial Art called Pencak Silat. The formation should always be in square shape. The teacher showed the picture of the first three formation (see Figure 1) and asked the students to draw the fourth formation with their pairs. The second question is to find out the number of dancers in the 10^{th}, 15^{th} and 100^{th} formations.

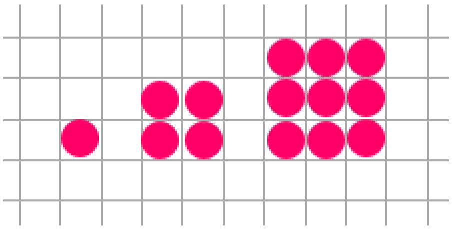


Figure 3. The first three formations of the square pattern

In our HLT we predict that the students will do drawing to find the number of the dancers in the “close” next formation. Some of them may also check the difference in every formations and develop a recursive formula as showed in Figure 2.

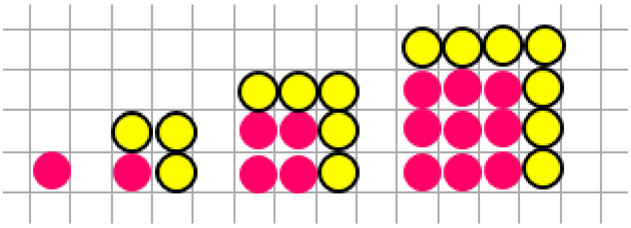


Figure 4. Addition strategy

The rest may observe the structure of the square pattern and construct a general formula of it as can be seen in the following Figure 3.

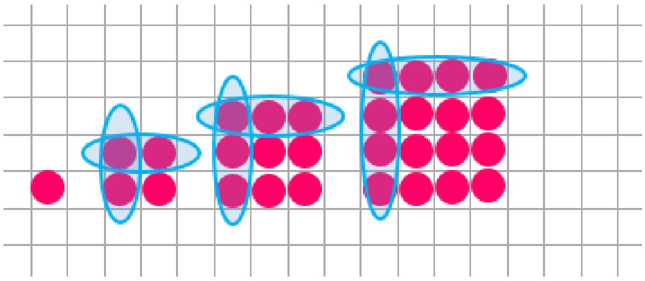


Figure 5. Multiplication strategy

During the implementation, our conjectures are fairly proved. There are different methods used by the students, including: (1) check for the addition in each formations, (2) check for the number in each rows and count for the number of rows in each formations and (3) check the number of dancers in a horizontal and a vertical sides.

These methods represent the students' way of thinking in seeing the structure of the square formation. Figure 4 for example, shows that the students were using addition strategy.

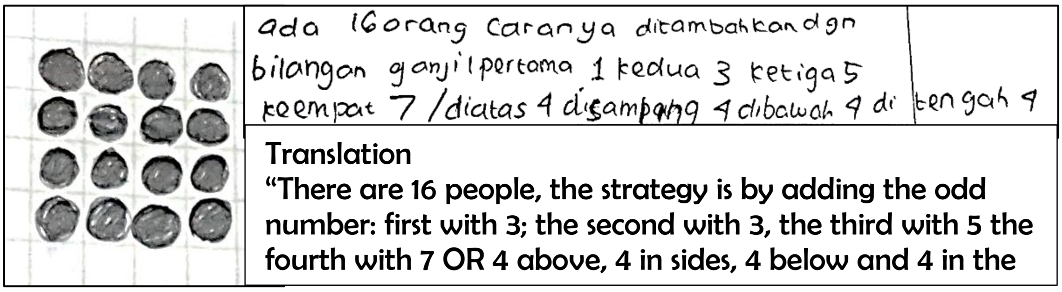
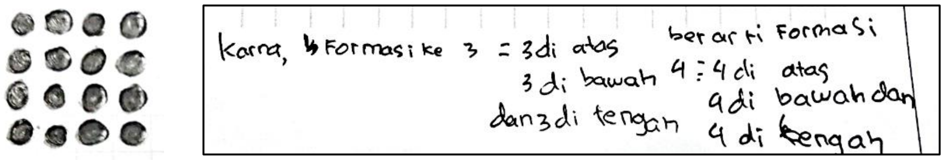


Figure 6. Arka's and Nora's addition strategy

Arka and Nora were initially using addition strategy by checking how many dancers should be added in each upcoming formation. They added another answer (the answer after the word "OR") after hearing other students' strategy during the class discussion. Based on their observation, they recognize that the addition of dancers is always come in odd numbers and it is always growing.

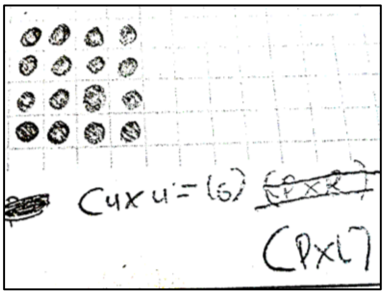
The strategy which is pointing the number of dancers in each sides (as copied by Arka and Nora in the end of their answer sheet) was originally given by Rachel and Davina who present their work in the class discussion. The following Figure 5 illustrates their point of view.



Translation:
Since on the third formation = 3 above, 3 below and 3 in the middle;
That's mean in the fourth formation =
4 above, 4 below and 4 in the middle.

Figure 7. Seeing the dancers on each rows

More advance strategy is given by Hafidz and Diana who saw the multiplication of the sides of the square as a good approach to start with. See Figure 6 to observe their works.



Translation:
The strategy is (4× 4 = 16)
(Length × Width)

Figure 8. See the sides only

After all students done with the first task, the teacher managed a class discussion and ask two pairs with different strategies, as explained in Figure 4 and

5, to present their works. We didn't showed the answer as is given in the Figure 6 because we afraid the students who didn't get about it will not thinking by themselves and directly follow it. We realize that even though the answer given in the Figure 6 is the most sophisticated answer, it will not be easy for the students to come to that point.

After the class discussion, the lesson was continued by the next problems. Now, the students can work in group of four (two closest pairs joined to become a group) had to determine the number of dancers in the 10^{th}, 15^{th} and 100^{th} formations. To solve this problem the students used quite similar strategy, which can be classified as addition and multiplication. When the students used addition, they need to know the previous number of dancers before they find the dancers in current formation. This strategy will lead the students to construct a recursive formula.

Another strategy is by using multiplication. Here, the students used properties of square. They observed that the total number of dancers in a square formation is equal to the result of squaring the number of dancers in a side of the square. This strategy will lead the students to generate a general formula, which means they can find the number of dancers in any formation without the need of finding the number of dancers in the previous formation. The following Figure 7 shows one answer from Reno and Felis who employed the multiplication strategy by checking the dots in the sides only. It is similar with the strategy given in the Figure 6, but it has more clear representation.

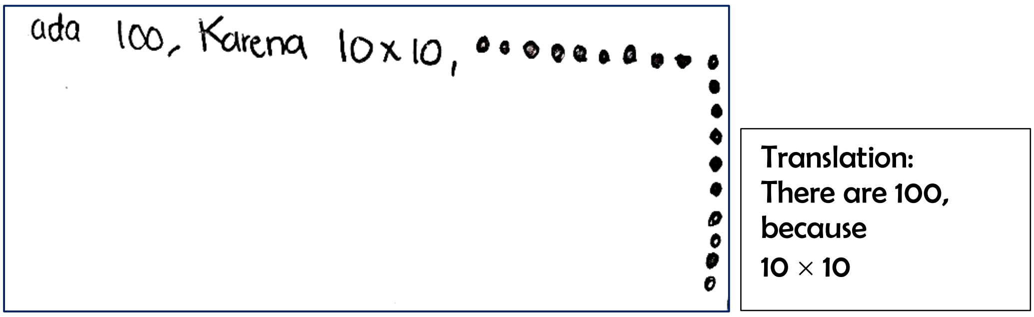


Figure 9. Reno and Felis' multiplication strategy

Shifting from addition to multiplication or in other words from recursive to general formula is a crucial movement in development of algebraic thinking. Most of students encountered the same struggle. The following discourses will give an insight of how the students tried to discuss, failed to agree and finally realized what the other proposed to them.

In a group consists of four students namely Arka, Nora, Feri and Meli, different way of seeing the structure of the square pattern happened. Arka realized that the dancers in the sides of the square contribute to the total number of dancers. He focused on an outer vertical column and an outer horizontal row of the square (see the general illustration in the Figure 3). Differently, Nora, Feri and Meli using addition strategy and refuse to think another method.

The researcher was trying to bring up Arka's idea into their group discussion. It is not easy since the three of the group member simply refused the idea without thinking further of its possibility. Later on, the teacher came up to the group and tried to give a scaffolding to encourage them to think about more efficient strategy. The following Fragment 1, Fragment 2 and Fragment 3 will give an illustration of the students' shift from recursive to general idea.

Fragment 34: How if We See the Sides of the Square?

[1] Researcher : Let's observe the picture given on the previous worksheet.
[2] Previously, Arka said that in the third formation,
[3] there will be 3 people on the side and 3 people above
[4] Nora : You are wrong if you say 3 on the side and 3 above!
[5] Meli : Yes! Because in the middle is also 3!

Nora and Meli was not convinced at that time. After that they continue to work by themselves, which use addition strategy.

Fragment 35: Keep Adding

[1] Researcher : What are you doing, Feri?
[2] Feri : I am adding.
[3] Researcher : Adding. If I ask the 100th formation, will you add it until then?
[4] Feri : Yes
[5] Researcher : Sure? Don't you have other strategy?
[6] Feri : No...
[7] Researcher : Are you really going to add it until the 100th formation?
[8] Researcher : Let's think about it! Nora, Arka, do you have other strategy?
[9] (*Arka, Nora, Feri and Meli continued to counting*)
[10]Researcher : Lets we observe the shape of the formation.
[11] (*The teacher came*).
[12]Teacher : Feri, now let's try the fifth formation.
[13]Teacher : How many people are there in a row?
[14]Feri : 5 people.
[15]Teacher : 5 people, try to draw it.
[16](*Arka, Nora, Feri and Meli drew the fifth formation*)
[17]Teacher : 5 people, how many row?
[18]Feri : 5
[19]Teacher : So, also what is the total?
[20]Feri : The total is ... 25 people.

After a while, the teacher were left the group. The focus students continued their discussion. Finally they got a new insight.

Fragment 36: From Addition to Multiplication

[1] Feri : Oh this is multiplication! 10 times 10!
[2] Nora : Yes!
[3] Arka : Squaring...
[4] Feri : What squaring are you talking about!

[5] (*Said something in local language*)
[6] Researcher : Why? Why?
[7] Researcher : 10 times 10 is 100, why?
[8] Meli : We find another strategy!
[9] Nora : Since there are 10 rows
[10]Meli : Suppose a row is fulfilled by 10 people
[11] and then multiply it with the number of people in that row.
[12]Researcher : There is a row with 10 people and then multiply it?
[13]Meli : There are 10 rows, in each row there are 10 people
[14] So, 10 times 10
[15]Researcher : So, 10 times 10.
[16] How it will be in the 15th formation?
[17]Meli : 15 times 15

From the Fragment 1 we can observe how the students who not ready yet with more advance strategy refused to agree with the other's student idea (see line 3 of Fragment 1). Here the role of teacher is very important to bridge the gap between students' strategies. Instead of directly express her/his agreement toward particular strategy, the teacher should give the students chance to explore more and to construct more sophisticated idea. It can be done by ask questions which challenge the students to think more and emphasize some important ideas which students' missed from their friend's explanation. Similar strategy was used to find the number of dancer in the 100th formation.

DISCUSSION

Algebraic thinking is defined by Kieran (2004) as the ability to focus on relations between the numbers. It consists of the "generalized arithmetic (using literal symbols), the development of mathematical models and the development of the language of algebra" (Dekker and Dolk, 2011, p.70).

Indonesian typical task is not so much allow the students to do their own construction of knowledge. Hence, since it is a new project in early grades, we need to change it bit by bit and make it as a habit. At the first cycle of the teaching experiment, we tried a sudden change in the type of the task, in which we directly ask the students to find a general formula of a pattern. The result was unpleasant, because the students didn't get the meaning of "general formula", yet we forgot to consider that the young children didn't know about algebra as a generalization of arithmetic.

Reflect to that, we changed the learning trajectory and the organization of the task. To do generalization, the students were guided by the organization of the task which is started by picture, ask them to draw, find the "close-next" number and finally to the "far-next" numbers. The drawing method is useful to help the students realize the visual shape of the pattern. When drawing, some of them mostly also think about what is the difference between the previous, the current and the next member of the pattern. This critical thinking lead the students to the construction of recursive formula. From recursive formula the students start to develop other point of view in seeing the structure of a pattern.

As the n^{th} number getting bigger, it will be tiring if they continue to use recursive formula because they need to count the previous number before able to find the next. From the analysis of the result we can observe that the students who use addition strategy is much more in arithmetical thinking and it is not easy for them to directly move to the general formula, by using the multiplication of the two sides of the square (relating it to the square formula). However, as is recorded in Fragment 2, Indonesian students are mostly hard worker. Even the 100^{th} term of series is not tiring enough for them to have the students think about more efficient strategy. Hence, to use even bigger numbers for the “next-asked numbers” and not just until 100, is highly recommended.

From their discussion in Fragment 3, we also can find that their shift happened when they realize that *there are 10 rows and each row has 10 dancers*. This thought is a crucial learning moment when the students stop to think about the added dancers in each formation and started to think in global form a square. They tested their conjecture for the other number of formation which they already found previously using the addition strategy and they convinced that it will always work. We understood that proof by example is never be valued as a proper way to proof. However, we valued the way of students’ thinking who consider the shape of pattern, as their idea is based on “the number of rows is equal to the number of dancers in the rows” which is guaranteed by the square basic fact which stated that all sides are equal.

CONCLUSION

This study is a part of design research study which aim is to produce a compatible learning trajectory in learning early algebra for Indonesian primary school students. As a part of it, this paper would like to emphasize what kind of activities that contribute to the development of learning algebra in early ages and what kind of support needed to optimize it. Based on the reflection of the research’s findings, we understand that the students progressively move from arithmetic to algebraic thinking when they figure out the “shortcut” to minimize the effort to find the next series of the algebraic pattern. Some students may did it faster than the others, but one good example of how the students struggle when they didn't get it as can be seen in Fragment 1, 2, and 3 give us a rich elaboration field.

Generally, the illustration of the arithmetical series which can be generalized, give a support for the students to start their thinking. The geometrical form representation of the number pattern plays two important role in the present study: (1) as a visualization of the context and (2) as a mathematical models.

As a visualization of context, the picture of the square pattern (see Figure 1) help the students to understand that the dancers will not only in the sides, but covering the area of the square. Hence, the number of dancers will be the same as the area of the square.

As a mathematical model, the visualization help them to recognize the structure of the series, a skill that tend to be valued as an abstract concept for them if it is directly started with numbers. As the series grows, the students will lay on the important aspect of the structure. Hence, we conclude that in the students' transition from arithmetic to algebraic thinking, is depend on how they see the structure of the pattern. This is called as structure sense (Zazkis & Liljedahl, 2002) which have a relation with personal reference while doing generalization. In line with that, we suggest that the implementation of pre-algebraic activity which is aimed to bridge the transition from arithmetic to algebraic thinking is using the pattern activities which is embodied in the visual representation. Also, the asking "next series" of the pattern is need to be considered to promote the students start to think in more efficient way and lead them to think in general.

ACKNOWLEDGEMENT

The authors are thankful to the teacher, Ms Istiarti Sri Sa'diah and the students if VA MIN 2 Palembang academic year 2014/2015 for their participations in this. This study is a part of a larger research which was conducted to fulfill the requirement of finishing master degree on mathematics education. The first author is an awardee of IMPoME scholarship program which is funded jointly by DIKTI, Indonesia and NUFFIC NESO, the Netherlands.

REFERENCES

Bakker, A. (2004). *Design research in statistics education: On symbolizing and computer tools.* (Doctoral Dissertation). Utrecht: CD-beta press.

Dekker, T., & Dolk, M. (2011). From arithmetic to algebra. In P. Drijvers (Ed.), *Secondary algebra education: Revisiting topic and themes and exploring the unknowns* (pp. 69-87). Rotterdam: Sense Publishers.

Freudenthal, H. (1974). Soviet research on teaching algebra at the lower grades of the elementary scholl. *Educational Studies in Mathematics*, *5*, 391-412.

Jupri, A., Drijvers, P., & van den Heuvel-Panhuizen, M. (2014). Difficulties in initial algebra learning in Indonesia. *Mathematics Education Research Group of Australasia*, 1-28. doi:10.1007/s13394-013-0097-0

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator*, *8*, 139-151.

Lee, L., & Wheeler, D. (1987). *Algebraic thinking in high school students: Their conceptions of generalisation and justification.* Montreal: Concordia University.

National Council of Teachers of Mathematics. (1989). *Principles and standards for school mathematics.* Reston, VA: Author.

Plomp, T. (2007). Educational design research : An introduction. In T. Plomp, & N. Nieven (Ed.), *Proceedings of the Seminar Conducted at the East China Normal University* (pp. 9-35). Shanghai: SLO.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26,* 114-145.

van den Heuvel-Panhuizen, M. (2000). Mathematics education in the Netherlands: A guided tour. In *Freudenthal Institute CD-rom for ICME9* (pp. 1-32). Utrecht: Utrecht University.

Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics, 49*, 379-402.

Zulkardi, Z. (2002). *Developing a learning environment on realistic mathematics education for Indonesian student teachers* (Unpublished doctoral dissertation). University of Twente: The Netherlands.

LETTER OF CONSENT

We hereby give consent:

Student Name	:	Ratih Ayu Apsari
Student Number	:	06022681318077
Study Program/Class	:	Mathematics Education/IMPoME
Article Title	:	Understanding Students' Transition from Arithmetic to Algebraic Thinking in the Pre-Algebraic Lesson

To publish the article to International Journal namely Malaysian Journal of Learning and Instruction (MJLI).

Supervisor I,	Supervisor II,
Prof. Dr. Ratu Ilma Indra Putri, M.Si	Dr. Darmawijoyo, M.Sc., M.Si
NIP 19690814 199302 2 001	NIP 19650828 199103 1 003

Paper Subimission Confirmation   Inbox x

Ratih Ayu Apsari 22 Jun (1 day ago)

Dear Editor MJLI, my name is Ratih, on June 17 I submitted a paper to MJLI bu...

Nor Arpizah Binti Atan 14:38 (23 hours ago)

to me

Dear Ratih

Thank you for your submission. We will sent the letter sooner.

Nor Arpizah Atan | Managing Editor|
UUM Press| Universiti Utara Malaysia | 06010 UUM Sintok, Kedah | Malaysia |
| Faxs :049284792 |

From: Ratih Ayu Apsari [mailto:ra.apsari@gmail.com]
Sent: Monday, June 22, 2015 1:50 PM
To: Nor Arpizah Binti Atan
Subject: Paper Subimission Confirmation

...

NOTULEN
THESIS DEFENSE

Student Name	:	Ratih Ayu Apsari
Student Number	:	06022681318077
Study Program	:	Magister PendidikanMatematika
Research Title	:	Understanding Students' Transition from Arithmetic to Algebraic Thinking in the Pre-Algebraic Lesson
Place of Defense	:	FKIP Kampus Bukit Besar, Sriwjaya University, Palembang, Indonesia
Date of Defense	:	30 June 2015/11.00-12.00 WIB

No	Examiner	Questions/Suggestions	Reaction
1	Prof. Dr. Zulkardi, M.I.Komp., M.Sc.	Consider to use the word "context" in your title because your thesis is using Indonesian local context, when you are going to publish your work.	Have been explained in the thesis defense.
2.	Dr. Somakim, M.Pd.	What is the contribution of your thesis with Kurikulum 2013?	Have been explained in the thesis defense.
3.	Dr. Yusuf Hartono	• What is the different between arithmetic and algebra? • What kind of difficulties found in learning algebra in relation with arithmetic? • How can your research contribute to make a closer connection between arithmetic and algebra?	Have been explained in the thesis defense.

Palembang, 30 June 2015

Supervisor I,

Prof. Dr. Ratu Ilma Indra Putri, M.Si
NIP 19690814 199302 2 001

Supervisor II,

Dr. Darmawijoyo, M.Sc., M.Si
NIP 19650828 1991103 1 003

Approved by,
Head of Mathematics Education Department

Prof. Dr. Ratu Ilma Indra Putri, M.Si.
NIP 19690814 199302 2 001

CURRICULUM VITAE

Ratih Ayu Apsari born on April 22, 1991 in Denpasar, the capital city of Bali. She started her formal education by attending a kindergarten named TK Bakti 2 Denpasar in 1995, followed by a primary school SD Negeri 5 Padang Sambian in 1996 – 2001. In 2002 – 2005, she registered as a student of SMP Negeri 2 Denpasar, a local-state junior high school in Bali. After that she continued her study at SMA Negeri 4 Denpasar until 2008. For four years after graduated from senior high school, she studied mathematics education at Universitas Pendidikan Ganesha and attained her first academic degree, S.Pd, in 2012. In 2013, she was accepted to be one of the awardees of the IMPoME scholarship, a collaboration program between Universitas Sriwijaya and Utrecht University which aimed is to produce a professional lecturer in mathematics education. She finished her master in 2015.

After finishing her master, she plan to teaching mathematics and continue doing research in mathematics education, especially in field of arithmetic, pre-algebra and algebra. She loves both of teaching and researching. If you interested in her works, she can be contacted through her email address: ra.apsari@gmail.com.