DEVELOPING STUDENTS' UNDERSTANDING OF LINEAR EQUATIONS WITH ONE VARIABLE THROUGH BALANCING ACTIVITIES

A THESIS

Submitted in Partial Fulfillment of the Requirements
for a Degree of Master of Science (M.Sc.)
in
International Master Program on Mathematics Education (IMPoME)
Master of Mathematics Education Study Program
Faculty of Teacher Training and Education
Sriwijaya University
(In collaboration with Utrecht University)

By:

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FACULTY OF TEACHER TRAINING AND EDUCATION

SRIWIJAYA UNIVERSITY

JULY 2015

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JULY 2015

STATEMENT PAGE

I hereby:

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state that:

1. All the data, information, analyses, and the statements in analyses and conclusions that presented in this thesis, except from reference sources are the results of my observations, researchers, analyses, and views with the guidance of my supervisors.
2. The thesis that I had made is original of my mind and has never been presented and proposed to get any other degree from Sriwijaya or other universities.

This statement was truly made and if in other time that found any fouls in my statement above, I am ready to get any academic sanctions such as, cancelation of my degree that I have got through this thesis.

Palembang, July 2015

The one with statement

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ABSTRACT

Difficulties in learning algebra have remained to be problems that current curricula fail to solve. This study promotes a Realistic Mathematics Education (RME) based learning involving balancing activities to help students develop their notions of linear equations. Employing design research as an approach, we first develop hypothetical learning trajectories (HLT) of the learning we designed. The HLT was then tried out to 31 seventh graders in Indonesia in two cycles. Data gathered during the try out included video-records of classroom events, students' written works, and observation notes. The data were analyzed by reflecting actual findings against the HLT. The results suggest that balancing activities help students to develop their senses of algebraic representations from seeing them as objects, values within objects, into quantitative relationships. Data also showed that the activities have helped students to be more flexible in performing strategies to solve for equations. Problems in bridging students' understanding built through this study to be applied in wider contexts are suggested to investigate in further studies.

Keywords: Balancing activities, HLT, Linear equations, RME

ABSTRAK

Kesulitan siswa dalam pembelajaran aljabar masih menjadi masalah yang belum teratasi di kurikulum kita saat ini. Penelitian ini bertujuan menghadirkan pembelajaran matematika berbasis pembelajaran matematika realistik (RME) dengan melibatkan kegiatan-kegiatan menyeimbangkan untuk membantu siswa memahami konsep persamaan linear. Penelitian ini bertajuk penelitian desain yang diawali dengan mengembangkan dugaan lintasan belajar (HLT) siswa pada desain yang telah dibuat. HLT tersebut diujicobakan pada 31 siswa kelas VII di Indonesia dalam dua siklus. Data yang dihimpun melalui uji coba tersebut mencakup video pembelajaran di kelas, jawaban tertulis siswa, dan catatan lapangan. Data tersebut dianalisa dengan membandingkan fakta lapangan dengan HLT. Hasil analisis menunjukkan bahwa kegiatan menyeimbangkan dapat membantu siswa memaknai bentuk aljabar yang mereka kembangkan, mulai dari menganggap bentuk tersebut sebagai benda, nilai yang termuat pada benda, hingga hubungan antar nilai. Data lain juga menunjukkan bahwa kegiatan tersebut membantu siswa untuk tidak kaku dalam menggunakan strategi-strategi untuk menyelesaikan suatu persamaan. Hasil ini juga menghimpun penelitian berikutnya untuk mempelajarai bagaimana menjembatani pemahaman siswa yang telah dikembangkan melalui kegiatan ini untuk diaplikasikan di konteks yang lebih luas.

Kata Kunci: HLT, Kegiatan menyeimbangkan, Persamaan linear, RME

SUMMARY

Obstacles in students' learning of algebra have been challenges that many teachers are difficult to handle. Reasons due to the difficulties encountered by the students address two issues, that is, the content of the algebra itself which is different from (arithmetic) mathematics that students usually dealt with, and inability of teachers to present good algebra learning due to the absence of a guidance they could adapt in their teaching. These problems background the implementation of this study aiming at providing a local instruction theory for learning initial algebra, i.e. in the topic of linear equations.

Literature studies have been performed to some fields to well-address the design. The first is about the content of the school algebra and the changing roles of variables in conceptions of algebra. Specific look to Indonesian algebra curricula and classroom practices revealed a bad teaching behavior as that is predicted to be a problem in conveying algebra topics to students. The second area of our literature review concerns on linear equations with one variable. The review revealed sophistication hierarchy of strategies to solve linear equations with one variable, and four important subtopics needed to learn the concept. Studies about the potential uses of balancing activities were also discussed, focusing on both advantages and disadvantages of the activity. The next, three principles of Realistic Mathematics Education were presented including how to implement those concepts in mathematics classroom. The next issues are about teacher role, social and socio-mathematical norms, which was central to teaching reform. To end the literature study, a brief overview of how those theories supported the design is presented.

Design research consisting of three phases, that is, preparation and design, teaching experiment, and retrospective analysis is chosen as an approach to conduct this study. In the first phase, Hypothetical Learning Trajectories (HLT) is designed along with conjectures of students' performances. During the teaching experiment, the HLT was used as guidelines to conduct the lessons. To gather data, a number of techniques were employed, such as, video-recording classroom events, collecting students' written works, interviewing teacher and students, and making field notes. The data were reflected against the HLT to see whether they confirm the conjectures. Results of the analyses were used to revise the learning lines and produce the local instruction theory. These processes were done in two cycles, involving 4 students in the cycle 1, 27 students from different class and a teacher in cycle 2.

Learning conjectures for five main activities, distributed into six lessons, were made, that is, secret number, bartering marbles, formative evaluation, combining masses, and solving problems across contexts. The core activity, which is called balancing activities, were mainly contained in the bartering marbles. This part is divided into three, such as finding balance, maintaining balance, and finding weights, each aims to introduce balancing rules on a balance scale, equivalent equations, and solving for unknowns, respectively. Some other goals were also mentioned in relation to the students' understandings of algebraic representations and the strategies that students are expected to be able to perform.

Data of the cycle 1 showed positive confirmation of several aspects in the HLT, especially the balancing activities themselves. Minor revisions were made to ensure the applicability of the learning in wider subjects. The activity of combining masses

was removed from the learning line, since it was considered too easy, not showing the intended algebraic goals, and promoted strategies that have been introduced in the other meeting. Findings in cycle 2 suggest keeping the goals in learning lines. The last two activities, however, needed quite a major revision especially in the problems provided to be solved.

At last, it is concluded that balancing activities offered for algebra learning have helped students develop the students' views of algebraic representations, and make the students more flexible in performing algebraic strategies. A local instruction theory (LIT) for learning linear equations with one variable is then developed. Activities in the LIT promoted those applied in this study, but for the last two activities. These remained to be issues to further investigate in future studies.

RINGKASAN

Kesulitan siswa dalam pembelajaran aljabar menjadi tantangan tersendiri yang banyak guru gagal dalam mengatasinya. Ada dua hal yang dapat dijadikan alasan terkait kesulitan siswa dalam belajar aljabar, yakni isi dari aljabar itu sendiri yang notabene berbeda dengan matematika (aritmetika) yang sering dijumpai siswa, serta ketidakmampuan guru untuk menghadirkan pembelajaran yang bermakna karena tidak adanya sumber yang dapat dijadikan acuan dalam mengajar. Berangkat dari alasan-alasan yang telah dikemukakan, penelitian ini dilaksanakan dengan tujuan mengahdirkan teori pembelajaran local (HLT) pada topik persamaan linear.

Kajian literatur dilakukan pada beberapa bidang terkait untuk memastikan ketepatan desain pembelajaran yang disusun. Hal pertama yang dikaji terkait dengan isi dari kurikulum aljabar di sekolah dan penggunaan variabel yang ternyata berbeda-beda pada beberapa bentuk aljabar. Kajian secara lebih khusus mengenai pembelajaran aljabar di Indonesia berdasarkan kurikulum dan fakta lapangan mengungkap bahwa buruknya pengajaran materi aljabar di sekolah dapat menjadi alasan tidak tersampaikannya materi tersebut ke siswa dengan baik. Hal kedua yang dikaji adalah materi persamaan linear satu variabel (PLSV) itu sendiri. Kajian teori menemukan adanya tingkatan strategi untuk mencari penyelesaian PLSV. Selain itu, empat hal yang harus dikuasai siswa untuk dapat menentukan solusi dari suatu PLSV juga terungkap melalui kajian ini. Penelitian-penelitian yang melibatkan aktivitas menyeimbangkan juga dikaji untuk mengetahui kelebihan dan keterbatasan dari kegiatan tersebut. Kajian terhadap tiga prinsip dasar pada pembelajaran berbasis pendidikan matematika realistik dikemukakan dengan disertai penjelasan bagaimana menerapkan prinsip-prinsip tersebut dalam pembelajaran matematika. Isu peran guru, norma social dan sosio-matematika juga dibahas terkait pentingnya hal tersebut untuk mereformasi pembelajaran. Terakhir, aktivitas-aktivitas yang didesain disajikan berdasarkan teori-teori yang mendukungnya.

Penelitian ini menerapkan penelitian desain yang terdiri dari tiga tahapan, yakni persiapan dan desain, uji coba pembelajaran, dan analisis. Pada tahap pertama, dugaan lintasan belajar (HLT) didesain beserta praduga proses berpikir siswa. Dalam proses uji coba, HLT digunakan sebagai pedoman untuk melaksanakan kegiatan pembelajaran. Beberapa teknik diterapkan untuk mengumpulkan data selama proses ini, seperti merekam kegiatan pembelajaran, mengumpulkan jawaban tertulis siswa, mewawancarai guru dan beberapa siswa, serta membuat catatan lapangan. Data-data tersebut kemudian dibandingkan dengan HLT untuk melihat apakah pelaksanaan di lapangan sesuai dengan prediksi yang telah dibuat. Hasil analisis tersebut digunakan untuk memperbaiki lintasan belajar dan membuat teori pembelajaran local. Proses-proses ini dilaksanakan sebanyak dua siklus dengan melibatkan empat siswa pada siklus 1, dan 27 siswa dari kelas berbeda dan satu orang guru pada siklus 2.

Lintasan belajar yang dibuat terdiri dari lima kegiatan utama dan dibagi ke dalam 6 pertemuan. Kelima kegiatan tersebut adalah bilangan rahasia, tukar kelereng, evaluasi formatif, mengkombinasikan anak timbangan, dan menyelesaikan permasalahan pada berbagai konteks. Kegiatan inti, yang kami sebut kegiatan menyeimbangkan, termuat pada kegiatan tukar kelereng. Bagian ini dibagi ke dalam tiga kegiatan, yakni menemukan keseimbangan, mempertahankan keseimbangan, serta menentukan berat. Ketiga kegiatan tersebut masing-masing bertujuan untuk memperkenalkan aturan menyeimbangkan, konsep kesetaraan, dan menentukan nilai

yang tidak diketahui. Beberapa tujuan lain dari kegiatan tersebut juga dijelaskan terkait dengan upaya meningkatkan pemahaman siswa terhadap bentuk aljabar, dan strategi yang dapat mereka gunakan untuk menyelesaikan permasalahan aljabar.

Data dari siklus 1 menunjukkan kesesuaian praduga di HLT pada beberapa aspek, utamanya pada kegiatan menyeimbangkan itu sendiri. Meski demikian, perbaikan kecil tetap dilakukan untuk memastikan keterlaksanaan rencana dan lintasan belajar pada subjek yang lebih luas. Pada siklus ini, kegiatan mengkombinasikan anak timbangan dihapus dari lintasan belajar dengan beberapa pertimbangan, seperti: permasalahan yang diajukan dianggap terlalu mudah, tidak ada indikasi munculnya tujuan pembelajaran pada proses berpikir siswa, serta strategi yang dimunculkan pada dasarnya telah dikembangkan pada beberapa kegiatan sebelumnya. Temuan di siklus 2 menunjukkan hasil positif terhadap lintasan tujuan belajar yang telah disusun. Namun demikian, dua kegiatan terakhir masih memerlukan perbaikan cukup besar, utamanya pada permasalahan-permasalahan yang disajikan.

Terakhir, kami menyimpulkan bahwa kegiatan menyeimbangkan yang diajukan untuk membantu pembelajaran materi aljabar dapat membantu siswa mengembangkan pandangan mereka terhadap bentuk aljabar. Kegiatan ini juga membantu siswa untuk lebih luwes dalam menerapkan strategi pemecahan masalah aljabar. Teori pembelajaran lokal (LIT) untuk materi PLSV pun dikembangkan. Kegiatan-kegiatan yang tercakup dalam LIT ini menyadur kegiatan-kegiatan yang diterapkan dalam penelitian ini, terkecuali untuk dua kegiatan terakhir. Hal tersebut menjadi isu yang kami sarankan untuk dapat dikaji lebih lanjut pada penelitian-penelitian selanjutnya.

PREFACE

Praises be to the Allah SWT for His mercies and blessings this study can be completed. Peace be upon prophet Muhammad SAW, the good example and leader for the human beings.

This thesis contains report of a study on initial algebra learning implemented to seventh graders in a school in Palembang. The study is a culmination of knowledge learned during our master study in the International Master Program on Mathematics Education (IMPoME) conducted in Sriwijaya University (Unsri) and the Fruedenthal Institute for Science and Mathematics Education (FIsme), Utrecht University. In addition, this study is also conducted as a partial fulfillment of the requirements for earning a master degree in the program.

What motivated us to conduct this study was our concern on the teaching practice of algebra in Indonesia, which remains traditional and avoid students from good conceptual gains. That is why in this study, we try to promote a series of activities that perhaps can be used by mathematics teachers in their algebra teaching. A part of this study has been presented in The Third South East Asia Design/ Development Research (SEA-DR) International Conference, and will soon be accessible in its electronic proceeding. Another part is proposed to publish in the International Electronic Journal on Mathematics Education, Turkey.

We fully realized that this work is far from perfect, and thus we are open for any criticisms and suggestions for improving this study. At last, apart from its limitations, we do hope that this study can contribute something for a better practice of teaching mathematics.

Palembang, Juni 2015

Muhammad Husnul Khuluq

ACKNOWLEDGEMENT

This thesis is not purely an individual work. I would especially thank my Indonesian supervisors, Prof. Dr. Zulkardi, M.I.Komp., M.Sc. and Dr. Darmawijoyo, M.Sc., M.Si., for their supervisions during the experimental and analyses sessions; and my Dutch supervisor Mieke Abels for her great contributions during the designing process. Other parties also took parts for the completion of this thesis. Therefore, in this part, I am pleased to express my big appreciations to:

1. Prof. Dr. Badia Perizade, MBA, the Rector of Sriwijaya University (Unsri) for her motivations and financial supports since we were registered as a student in Unsri until the completion of this study.
2. Prof. Sofendi, M.A., Ph.D., and Dr. Hartono, M.A., the Dean and the Vice Dean for Academic Affairs of the Faculty of Teacher Training and Education (FKIP), for their sharing experiences and supports for the completion of this study.
3. Prof. Dr. Ratu Ilma Indra Putri, M.Si., the Head of Magister of Mathematics Education study program, for her motivations, guidances, concerns, and controls during our study both in Unsri and in Utrecht.
4. Dr. Maarten Dolk, the Coordinator of IMPoME program in the Freudenthal Institute for Science and Mathematics Education (FIsme), Utrecht University for his trust in selecting us to study in the Netherlands.
5. Prof. Robert K. Sembiring, the Chief of IP-PMRI and other PMRI boards, for awarding us the IMPoME scholarship; and for supports, sharing dreams, and inputs for improving the design we have made.
6. Prof. Dr. Supriadi Rustad, M.Si., the Director of Indonesian Directorate General for Higher Education (DIKTI) for sponsoring the IMPoME program.
7. Indy Hardono, MBA., the Team Coordinator Scholarships of Nuffic Neso Indonesia, for awarding us a Studeren in Nederland (StuNed) and facilitating our study in the Netherlands.
8. Prof. Dr. Hilda Zulkifli, M.Si., DEA, the Director of Postgraduate program of Unsri, and staffs for facilitating us the financial aids during our master study.
9. Prof. Dr. H. Arismunandar, M.Pd., the Rector of State University of Makassar (UNM); Prof. Dr. H. Hamzah Upu, M.Ed., former Dean of the Faculty of

Mathematics and Science (FMIPA) UNM; Prof. Dr. Abdul Rahman, M.Pd., current Dean of the FMIPA UNM; Sabri, S.Pd., M.Sc., a lecturer in mathematics department FMIPA UNM, for their recommendations, motivations, guidances, and trusts to enroll in the IMPoME program.

10. Dr. Maarten Dolk, Dr. H.A.A. (Dolly) van Eerde and Frans van Galen, lecturers in Fisme, who have assisted us for our data analyses.
11. Dr. Yusuf Hartono and Dr. Somakim, M.Pd., examiners and lecturers in Unsri, for inputs, suggestions, discussions, and criticisms for the work we have done.
12. Dra. Trisna Sundari, the Head of SMP Pusri Palembang; Ogi Meita Utami, S.Pd., model teacher; and participant-students from SMP Pusri Palembang for their kind helps facilitating the implementation of the study.
13. Lecturers in Sriwijaya University Language Institute (SULI) for their kind assistances, greets and smiles who have helped us accelerate our English.
14. Super IMPoME Unsri Batch V teams who have been a new family for the joys and sorrows we share. Also to other IMPoME friends in Surabaya for the unforgettable friendships we have built.
15. Seniors of IMPoME, particularly the IMPoME Batch IV for their warm welcomes, assistances, supports, and sharing information. You really have been a good model of older brothers and sisters.
16. My best friends, Abdul Ahkam and Muh. Akbar Ilyas, who have shared their dreams and together strived for making them come true.
17. Other parties we met during our studies in Unsri and Utrecht who have helped us much either for academic and for non-academic affairs.

CURRICULUM VITAE

Muhammad Husnul Khuluq (Husnul) was born in Limbung, South Sulawesi, on the 4th of January 1991. His parents, Irwan and Najmah, are mathematics educators in a university and a secondary school in South Sulawesi, respectively. He showed his interest in mathematics since he was a fifth grader after his first participation in a mathematics competition. In a moment, he represented his province for a National Science Olympiad (OSN) in the subject of mathematics. During his school ages, he participated and pioneered some learning communities in the fields of English, mathematics, and scientific writing. He also friend- tutored his friends and juniors in the community. His interest in mathematics and English made him decide to enroll in the International Class Program of Mathematics Education in the State University of Makassar in 2008. In 2012, he graduated the best in the program with a mini-thesis entitled Description of Students' Ability in Solving Programme for International Student Assessment (PISA) based Mathematical Problems. During his four-year study in the university, he has assisted several lecturers to conduct tutorials, remedial course, and handle presentations in several courses, like Everyday Mathematics, English for Mathematics, Introduction to the Fundamental of Mathematics and Introduction to Probability Theory. In 2013, he was awarded an International Master Program on Mathematics Education (IMPoME) scholarship that gave him chance to study Realistic Mathematics Education (RME) in Sriwijaya University and Utrecht University. Husnul has a great eagerness to be involved and contribute to the development of mathematics education in his country. Therefore, he tries to be updated of any educational policies and issues happening in the country. Currently, his field of interest is lower secondary mathematics due to his view of the period as a crucial moment for students' learning of mathematics.

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DEDICATION

I dedicate this thesis to:

My beloved parents, Irwan and Najmah,

My beloved brothers and sisters,

Khairunnisa
Muhammad Ikramurrasyid
Husnul Khatimah
Muhammad Alimul Hakim

the loves and spirits we share are beyond times and distances

IP-PMRI Teams

for the sincere works, dreams, and dedications to improve mathematics education in Indonesia

CHAPTER 1
INTRODUCTION

It is generally believed that algebra is very important, especially because it is a gateway to a higher level or a more applied mathematics. However, its notoriety as a difficult topic in mathematics has been a general issue among students in the world. The difficulty of algebra merely makes many students lazy, and thus tend to avoid mathematics once they have started to learn this topic (Cai, et al., 2005). The National Academy of Education (in Cai, et al., 2005) reported that many children indeed enjoyed studying mathematics and performed well in it when they were young. However, when they reached grade 4 or 5, they found it difficult and did not like the subject anymore. This issue should be taken into account, particularly because in many curricula, it is the period when algebra is first introduced to students.

In the case of Indonesian school mathematics, algebra is also a big issue. In addition to Indonesian students' difficulties in linear equations, Jupri, Drijvers, and Van den Heuvel-Panhuizen (2014a) and Mullis, Martin, Foy, and Arora (2012) reported how countries have performed in TIMSS mathematics 2011. The results show a very low performance of Indonesian students in algebra, particularly in questions that involve their reasoning.

Studies have been performed either to reveal what students find difficult when they learn algebra or to analyze causes of the students' difficulties. The study by Jupri et al. (2014a), for example, define five aspects in algebra in which many Indonesian students found struggles, such as, mathematization, algebraic expressions, applying arithmetic operations in algebra, dealing with equal signs, and understanding variables. These difficulties are identified by studying the algebra learning of grade 7 students of an Indonesian lower secondary school in the topic of linear equation. Of those five aspects, the mathematization was reported to be the most difficult part for students (Jupri, et al., 2014a). This aspect covers the students' ability to translate back and forth between the world of problem situation and the world of mathematics, and to reorganize within the mathematics itself.

Other studies, by Rosnick & Clements (1980) and Kaput (2000), also reported that the difficulties of the students are caused by several factors, but mostly by the way algebra is taught which remains very traditional. Moreover, some teachers (as reported in Kieran, Battista, & Clements, 1991) argued that algebra which involves using letters along with formal rules for operating the letters is really abstract for children. Therefore, it would not be wise to let them reinvent the ideas themselves. In other words, those teachers believed that algebra should not be taught in a constructive manner.

A reason behind the teachers' neglect to reform their algebra teaching, according to Kieran (1992), is the absence of a readily accessible form of communication that tells teacher how to implement algebra in class. In other words, they have neither guidance nor examples of a good algebra teaching from which they can learn to organize algebra learning.

Although a number of studies in algebra teaching have been conducted, most of them end up with only presumptions on the causes, what it impacts, and general predictive suggested solutions to the issues (see Pillay, Wilss, & Boulton Lewis,1998; Booth, 1988). Few studies (if any) focus on the learning and teaching algebra itself or provide practical suggestions that could be useful for teachers to improve qualities of their teaching.

Therefore, this study aims to build up a local instructional theory on algebra, particularly on the topic of linear equations with one variable. To address the aims, we will design learning materials and learning activities to promote the students' understanding and reasoning on algebra. The design makes the most of algebraic notions within balancing activities to facilitate students' learning on linear equations with one variable. So, the output of this study will provide practical solutions that the teachers and designers can use to reform their algebra teaching, as well as theoretical insights for designers or researchers to conduct deeper investigation.

Thus, this study will be addressed to answer the question; "*How can balancing activities support students' understanding of linear equations with one variable?*"

CHAPTER 2
LITERATURE REVIEW

2.1. School Algebra

2.1.1. Algebra and School Algebra

Forcing people to a fixed definition of algebra would hardly lead to a consensus. Some people might simply say that it is only mathematics involving symbols or letter; but of course, it should be a lot more. Freudenthal (1976) had explained the notions of algebra from a number of different perspectives. From geometrical views, for instance, Freudenthal defined algebra as a concept that relates between symbolic and extensive magnitude; or also interpreted as written numerals and real numbers. Meanwhile, from the way it is taught, algebra can mean knowledge of finding unknowns through systematic procedures.

Although for some people the usefulness of algebra is not explicitly visible, the concept has been indeed crucial in many applied disciplines, such as, physics, economy, geometry, and computer science (Cox, 2005). Its function as a language of mathematics would make it really required, especially in building mathematical models of life's phenomena. Historically speaking, Viete (in Usiskin, 1988) revealed that the invention of algebraic notions has had immediate effects on the development of higher level mathematics, like calculus and analytic geometry.

Algebra in school is a different case; it is often called school algebra. School algebra is seen as a step to 'real' algebra as well as the continuation of arithmetic learning (Usiskin, 1988). Mainly, school algebra contains two aspects called procedural and structural algebra (Kieran, 1992). The procedural part covers computational-related topics; often this leads to the operational aspects which closely relate to arithmetical skills for students. Sfard (1991) expressed that this part usually becomes an entrance for most people in their acquisition of algebraic knowledge. The second part, the structural, closely relates to the core concept of algebra itself. It focuses on the relationships among objects or quantities, rather than finding solutions of algebraic expressions.

Looking deeper into the content, Usiskin (1988) mentioned four different conceptions that build school algebra. Each conception implies different roles and uses of variables, expressions, and tasks. These conceptions also determine how a sub-concept should be taught. Those conceptions are:

Algebra as a generalized arithmetic

This conception covers ways to state the relationships among numbers. In this case, variables are treated as pattern generalizers. The concept often becomes a basis for numeric formulas. The main tasks to approach this concept is translating numerical patterns and then making a generalization.

Algebra as a study of procedures for solving certain kinds of problems

This conception starts with a generalized formula. Symbols given in the formula become the focus of attention, i.e. the students are asked to determine the values represented in the symbols. Only certain numbers would satisfy the conditions of the formulas. In this case, the symbol, which is indeed the variable, usually represents a constant.

Algebra as a study of relationship among quantities

This conception discusses how an algebraic expression states interrelated components. This gives insights on how changes of certain values (quantities) affect the balanced situations. Unlike the previous conception, the variables in this conception are not constants. Instead, they have various values. Simply, the variables are either arguments or parameters.

Algebra as a study of structure

The last conception contains a high level skill in algebra working, which is, theorems and manipulations. It discusses how an expression could be stated without changing its values. The variables are treated purely as objects. They are not to be solved, neither to find nor to relate each other.

These four conceptions are interrelated and are often used simultaneously in solving algebraic problems. Unfortunately, in many curricula, these conceptions are often not treated proportionally, with a tendency to the procedures (Brown, Cooney & Jones, 1990). As a consequence, the students tend to understand algebra as a set of rules and procedures that they have to memorize to be able to solve the problems.

Summary of these four conceptions are given in table 2.1. (also presented in Usiskin, 1998).

Table 2.1 Conceptions, uses of variables, and tasks building school algebra

Conceptions	Use of Variables	Tasks
Generalized arithmetic	Pattern generalizer	Translate, generalize
Study of procedures	Unknowns, constants	Solve, simplify
Relationship among quantities	Arguments, parameters	Relate
Study of structure	Arbitrary symbol	Manipulate, justify

2.1.2.School Algebra in Indonesia

In the Indonesian curriculum, algebra is given to students for the first time in the second semester of Grade VII. In that level, the students are directly confronted with new terms and concepts, with a few connections to what they have learned (see school textbooks by *Kementrian Pendidikan dan Kebudayaan,* 2013a, 2013b, 2014; Nuharini & Wahyuni, 2008). The mathematical contents given in the algebra chapter in the book includes (in order) 1) open and close sentences, 2) definitions of variables, equations, linear equations with one variable, and solution of linear equation, 3) application of linear equation, and 4) equality and inequality. Those mathematical ideas are presented very formally, involving mathematical symbols and expressions.

This condition becomes worse due to the teaching of mathematics in most Indonesian schools. A report from Organization for Economic and Cooperation Development (OECD, 2013) showed a high index of direct instruction in Indonesian classrooms. Such a traditional way of teaching could generate a view of algebra as a set of procedures disconnected from other mathematical knowledge and from students' real worlds (Herscovics & Linchevski, 1994; Kaput, 2000). As a consequence, many students found difficulties in working with algebra.

A study by Jupri, Drijvers, and Van den Heuvel-Panhuizen (2014a) found five categories that often become problems for many Indonesian students in their algebra studies. Those are usually found in 1) applying arithmetic operations, 2) understanding the notion of variables, 3) understanding algebraic expressions, 4) understanding the different meaning of equal signs, and 5) mathematization. The last category becomes the most common difficulty, i.e. in relating translating back and forth the world situations into algebraic words or within the mathematics contents itself.

2.2. Linear Equations with One Variable

One topic given to students in their early study of algebra in school is linear equations with one variable. The importance of this topic is viewed by Huntley and Terrel (2014) as a hallmark for students' algebraic proficiency in school. In many curricula, this topic mainly deals with solving two kinds of equations, namely, one-step and multi-step equations. This part will give an overview of how students usually deal with solving these kinds of equations and what knowledge they have to own to be able to solve problems on linear equations with one variable.

2.2.1.Strategies to solve linear equations with one variable

In investigations of students' learning on solving linear equations with one variable, researchers (Kieran, 1992; Linsell, 2007, 2008) found some strategies that students usually use to solve problems on linear equations with one variable, such as 1) guess and check, 2) counting techniques (known basic facts), 3) inverse operations, 4) working backwards then guess-and-check, 5) working backwards, then known facts, 6) working backwards, and 7) transformations.

Kieran (2006) named the first strategy 'trial-and-error substitution'. This strategy simply requires students' recognition of 'letters' as the representation of certain numbers in an algebraic expression and some sort of basic arithmetic skills. Here, the students should substitute any numbers and check whether the numbers fulfill the equation. Although this strategy is applicable to solve all kinds of equations, students should not really rely on it all the time, as they will have problems with relatively difficult questions, for example, ones that give fractional solutions.

The next two strategies, counting techniques and inverse operations, are often used to solve a one-step equation. Students who only rely on these strategies would not be able to solve the multi-step equation problems. The difference between these two strategies is observable when the students deal with problems involving a large number (Linsell, 2008). Often, the students who used counting techniques struggle in it.

Students' understanding of inverse operations would lead them apply the working backwards strategy to solve multi-step linear equations. Often, they combine this strategy with the other strategies after simplifying the expression into a one-step equation. The limitation of this strategy is when it deals with equations that involve variables in both sides.

The last strategy, transformation, is often called the formal strategy. This strategy requires students to treat equation as objects. Thus, they can manipulate things in the equations, reformulate it, and then find the solution. Relying on this strategy will likely help students solve problems in any kind of equation.

In her study, Linsell (2008) found strong evidence that these strategies are indeed hierarchical. Thus, the development of students' strategies indicates their level of understanding of algebra. Given this range of strategies, some teachers strictly limit the students to a single approach to solve equations: the formal one. This decision has been proven to be ineffective to build up students' understanding and visions toward algebra concept (Whitman in Kieran, 1992).

In an effort to introduce students to transformation strategy, many students found it difficult to understand 'equation' as a structure. This failure, according to Kieran (2007), can be recognized in three conditions, such as: 1) students' unsystematic and strategic errors when simplifying algebraic expressions, 2) students' neglecting to treat variables as objects, and 3) their misunderstanding of the equal sign. To anticipate this failure, basic knowledge should be given to students during their early study of algebra.

2.2.2.Basic knowledge to solve linear equations with one variable

A study by Linsell (2007) mentioned four basic concepts that the students have to master to be able to solve any linear equation problem systematically; those are, 1) knowledge of arithmetic structure, 2) knowledge of algebraic notation and convention, 3) acceptance of lack of closure, and 4) understanding of equality. Further, she explained that this basic knowledge has a strong relationship with strategies that students can use to solve linear equation problems.

Knowledge of arithmetic structure

The closed relationship between arithmetic and algebra leads to a view of algebra as a generalized arithmetic. This view requires students to have a good understanding of arithmetical notions before doing algebra. Moreover, Linsell (2008) emphasized that students' understanding of arithmetical structures has a dramatic effect on the most sophisticated strategy they are able to use when solving a linear equation problem.

The need for arithmetic in algebra is strengthened by other experts, like Van Amerom (2002), Booth (1988), and Usiskin (1997), who believed that arithmetic should become an entrance for algebra learning. In addition, Kieran (1992) stressed that intuitive precursors in arithmetic are absolutely needed to make students able to interpret $a + b$ as an addition of two objects; which is a higher level view of algebra. Therefore, she suggested involving some arithmetical identities with some hidden numbers in introducing the concept of equation to students. This hidden number could be changed progressively from finger (cover up), box drawing, and then finally letters. Such a way of introducing algebra would make the most of students' mathematical understanding of arithmetic.

Knowledge of algebraic notation and convention

This part is related to the use of symbols and the role of algebra as a language in mathematics. In school algebra, the use of symbols is crucial for students. Van Amerom (2002) explained that the symbols and notations that students produce during their algebra learning would be the basis for their reasoning ability.

A study by MacGregor and Stacey (1997) suggested that to help students' understanding of algebraic representations, algebra learning should be started on a concrete level, in which the students can produce and reflect on their own symbols against the true situation. This will make the symbols meaningful to them.

Acceptance of lack of closure

The notion of 'acceptance of lack of closure' was first developed by Collis in 1974. This idea focuses on students' ability to hold themselves back from finishing an operation; it simply tells about manipulating algebraic expressions. This issue is often encountered in discussions on equivalence or on solving a high level algebra problem. Students' difficulties with this notion are revealed by Wagner, Rachlin, & Jensen (as stated in Kieran, 2007), recognizing the struggles of many students to judge an equivalence without finding the unknown.

Kieran (2007) argued that teachers can help students build this knowledge by providing activities that develop the structure sense of students. The activity might give an image of the structures of two equivalent structures of expressions along with their decompositions and recompositions.

Understanding equality

Discussions about equality in school algebra usually involved the ‘equal sign’ and its meaning for students. Many students understand the equal sign as a signal for an answer, as they probably understand it in arithmetic. This becomes a problem, especially when students work with equations involving variables on both sides. Unfortunately, teachers often do not really pay attention to this problem (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). Further, Knuth, et al. (2005) presumed that this lack of attention could be the main cause of students’ bad performance in algebra in general.

In algebra, the ‘equal sign’ must be seen as a relational rather than operational symbol; this becomes a pivotal aspect of understanding algebra (Knuth, Stephens, McNeil, & Alibali, 2006).

2.3. Balancing Activity

The use of balancing activities is actually not new in learning; some of its applications can be seen in studies in physics or in mathematics itself (see Siegler & Chen (2002); Dawson, Goodheart, Draney, Wilson, & Commons (2010); and Surber & Gzesh (1984)). An explicit benefit of employing balance activities is that it involves the use of physical material, where students can directly observe and experience how the balance really works.

In algebra, the use of a balance scale could be a powerful tool to understand the idea of equations. Studies found many advantages of this tool, especially in bridging students’ procedural and informal knowledge to more structural understanding. Other advantages offered by the balance scale are given below:

Promoting the view of equations as objects

Treating expressions in algebra as objects is crucial in studying algebra. In teaching, teachers usually give a direct suggestion that an equation is indeed like a balance pivoted about the equal sign without any visualization (Sfard, 1991). This would hardly encourage students to imagine how it could happen and why they need to maintain a balance. Such questions need an indirect answer with a visual proof that can be done by presenting and experimenting with a real balance scale in the classroom. This idea is important because transferring the idea of transformation in algebra would require seeing an equation as an object to be acted on (Sfard, 1991).

Facilitating understanding of eliminations in algebraic operations

The second advantage of balancing activity is related with the first one, that is, to promote transformation strategy to students. Vlassis (2002) found that the balance model is an effective tool for conveying the principles of transformation, because the principles applied to create a balance situation on a balance scale really suit the process of solving equations. In this case, the balance scale gives meaningful insights on eliminating the same terms from both sides of an equation to obtain the value of the unknown.

Increasing representational fluency

Suth and Moyer-Pockenham (2007) explained that providing students with the balance model gives them opportunity to come up with multiple representations. Having experience with real balancing stimulates students to show their understanding in drawings, verbal explanations, or even formal representation freely. Their experiences become the sources of reflections when they are going to represent their ideas.

Beside the advantages, several studies also reported limitations of the balance activity. The first limitation deals with the activity's inapplicability to represent unknowns or expressions that involve negative numbers (Van Amerom, 2002). Another limitation is revealed by Surber and Gzesh (1984), i.e. that the balance activity seems incapable, and even confuses students, to work with reversible operations. Those limitations imply the need to present other supporting activities to cover what the balance scale cannot.

2.4. Realistic Mathematics Education (RME)

As explained in introductory part, innovations in learning and teaching algebra are really required. Therefore, in this study, a series of lessons is designed. The idea of Realistic Mathematics Education that makes the most of the applications of mathematics concepts in human's daily activities is chosen as a design heuristic that underlies activities in the design. This idea of RME based teaching has been adapted by countries including Indonesia with the so-called *Pendidikan Matematika Realistik Indonesia (PMRI).* A number of efforts have been performed to introduce the implementation to education components and authorities in the countries, like involving several schools to be pilots of PMRI classes, conducting researches on PMRI teaching, teacher trainings, and studying contexts that might be applicable in Indonesian classrooms (for further readings, see Putri, Dolk, & Zulkardi, 2015; Sembiring, 2010; Sembiring, Hoogland, & Dolk, 2010; Zulkardi, 2002).

The choice of RME as underlying of the proposed designs is mainly due to the three key principles of RME (mentioned in Gravemeijer, 1994), such as 1) guided reinvention, 2) didactical phenomenology, and 3) self-developed models.

Guided Reinvention

The idea of reinvention in RME is based on the view of mathematics as a process as well as a product of learning. The idea believes that students would learn better if they could discover the concept for themselves. Thus, students must be given opportunities to experience the process of how certain mathematical concepts were invented. This principle implies two components that should exist in mathematics teaching, such as, activities that lead into a concept in mathematics, and the mathematics concept itself.

To stimulate the process of reinvention, the teacher's role is crucial. Here, teachers should mainly act as a facilitator of learning. They scaffold their students' thinking process with questions. On the other side, the students also play a very important role. They are the main actors during the learning process. They do activities, explore the concepts, and try to reveal the mathematical ideas within the activities. In this phase, the students are required to be more self-reliant in doing their tasks.

Taking this principle into account, in facilitating students' learning on equations, balancing activities might be provided for students. During the process, the students can be given opportunities to explore algebraic concepts within the balance scale with minimal guidance. They will do experiments with the balance themselves, and represent what they find in their own representation.

Didactical Phenomenology

This concept is based on Freudenthal's (1999) belief that mathematics concepts, structures, and ideas were invented to organize and explain phenomena in the physical, social, or mental world. Connecting those applications with the learning process of young students is what we call didactical phenomenology. With this principle, students can recognize where the concept they are learning could be applied.

This principle recalls the need to present a context that allows students to show their range of mathematical ideas. It allows students to express their mathematical understanding by relating it to the contexts they are familiar with. Thus, it builds the students' common senses in the process of learning.

Based on the above explanations, there are two characteristics that situations should have to be considered a good context. First, the situation should contain an application of a mathematical concept, and second, it should support the process of mathematization. In other words, the context should contain mathematical concept, and be doable, and allow for reflection.

Self-Developed Models

Given the reinvention principle, the existence of a model is needed to facilitate a bottom-up learning process. The model would help students to bridge the contexts with mathematical concepts to achieve the learning goals. The model can be a scheme, description, ways of noting, or simply the students' understandings toward and uses of certain concepts to explain the more complex one.

In RME, it is very important that the students construct their own models. The students' initial model can be derived from their informal knowledge or strategies. During the lessons, the students are expected to formalize their initial understanding and strategies to work with a certain concept. This is why enhancing students' prior knowledge is required. The development of students' modeling becomes a concern in an RME teaching. Gravemeijer (1994) explained four levels of emergent modeling used in RME teaching such as, situational, referential, general, and formal.

Situational

This level is where a general context is first introduced. Thus, the model developed is still context-specific. The students should rely on their informal knowledge or experience to understand the situation.

Referential

This stage is where the promoted models, concepts, procedures, and strategies are explored. Those mathematical ideas are still context-bound. Hence, the students are working with problems within the context.

General

This phase starts when the discussion about procedures, strategies, concepts, or models become the focus. The movement from the referential to general model happens due to generalizations and reflections toward activities in the referential phase.

Formal

This highest level is shown in the students' uses of formal strategies or procedures in solving any related mathematical problems. Hence, the mathematical knowledge they have gained can be used to solve any mathematical problems across contexts.

These levels of modeling could be illustrated in figure 2.1:

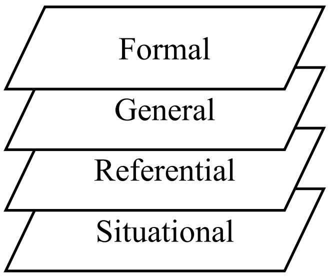


Figure 2.1 Scheme of modeling in RME

2.5. Teacher Role, Social Norms, and Socio-mathematical Norms

Efforts to reform teaching and learning in the mathematics classroom often deal with many aspects such as the teacher and classroom culture. Issues around the teacher's role have been central due to the learning shift from teacher-centered into students-centered. Issues about teachers' views on beliefs should have been a concern in the design of algebra learnings. Castro-Gordillo and Godino (2014) revealed that most teachers still hold traditional beliefs in algebra, which is central to results, rules, and procedures, as they were usually taught in their previous schools. This view should of course be repaired before they conduct the classroom.

Reforming teaching cultures will also have an effect on the classroom's atmosphere because students might not be accustomed to the new situation. However, Yackel and Cobb (2006) argued that as long as teachers can facilitate a good social interaction between and among students, the students can easily adapt to the new norms.

In mathematics, a shift in norms is also covered in a discussion called socio-mathematical norms. This notion focuses on the development of students' mathematical beliefs and values to become autonomous in mathematics (Yackel & Cobb, 2006). Socio-mathematical norm will specifically discuss the mathematical issues that the students encounter during the class, for example, students' perception of the most sophisticated, efficient, or elegant strategy. Such a notion is viewed to still have lack attention by current Indonesian teachers (Putri, Dolk, & Zulkardi, 2015).

In early algebra learning, investigating students' socio-mathematical norms is important, especially because the students are in transitions from arithmetical to algebraic thinking. A number of issues may be observed, such as students' views of a number of different strategies and different representations, completing a problem without finding the solutions, equal sign does not always separate operation and answers, and solving a complex problem by splitting it into simpler partitions. These ideas are probably new and different from what students encountered in arithmetic classrooms.

To be able to help children understand this shift of values, a teacher must be prepared for several things, such as, students' theory building, students' misconceptions, the role of representations, and how to move from misconceptions to knowledge (Even & Tirosh, 2002). It is important to highlight that in the reformed mathematics classroom; the teacher should identify the students' limited knowledge and use it as a basis for their learning.

2.6. Present Study

Based on the theoretical supports, this study tries to propose a series of algebra learning which is applicable in the Indonesian context. The series consists of six lessons that perhaps can help the students to have a good understanding of algebra, particularly of linear equations with one variable. Students' understandings of algebra observed in this study are limited to strategies they performed and their views of algebraic components within an algebraic expression (equations).

The first meeting

The first lesson tries to relate students' arithmetic knowledge with some basic ideas in algebra, under the view of algebra as a generalized arithmetic. Here, the students will play a 'guessing my number' game and then try to trace the key of the game. This activity will encourage students to show their initial algebraic representation. The teacher can also use this activity to correct students' misuses of the equal sign (if any) in arithmetic contexts. This step is important to ensure students' arithmetical fluency for the upcoming activities. This activity does not involve a balancing context, but it would be needed to cover the limitation of the balance (see the end of subchapter 2.3.).

The second meeting

The second meeting will be divided into two main parts. In the first part, the students will do the real balancing activity to find the relationship among three different objects. This activity will give the students insights about how the balance really works, which is essential to have them think about objects (see subchapter 2.3.1.). Afterward, the students are asked to record the real balancing activity, where they will need to make an expression of equality.

In the second part, the students will be asked to predict more balanced situations (based on the situations they encountered in the first part). This activity will help students to think of equivalent equations. Word representations of some students (if any) would not really be efficient here, and therefore, they would feel a need to change their way of representing a balance situation.

The third meeting

In the third meeting, the students will deal with solving a linear equation problem for the first time. This problem is given in the context of weight balancing. The activity will use the set of balance conditions they have collected from the previous meeting. A additional information about the weight of one object will be the basis for investigating the weight of the other two objects. This will lead the students to find the unknown by dealing with objects. At the same time, the additional information facilitates the students' understanding of letters (variables or other symbols) as the representations of quantities. This will lead the students into conception of relationship among quantities (see point c on subchapter 2.1.1.).

Also in this meeting, the students will see an animation of people that are struggling with a weight scale. Afterward, the teacher tells a set of instructions that the students can do to make the balance. Here, the students need to translate each step into an expression. This practice intends to show the students the process of finding unknowns using the idea of the balance, and also helps them relate the process to their way of solving on paper.

The fourth meeting

This meeting starts with a formative assessment session, where students have to solve problems related to what they have learned in the first three meetings. This test is needed to really make sure that the students have enough understanding to continue to the next discussion, which is a bit more formal. The assessment is followed by a classroom discussion and an investigation of a balanced condition, that require students to work with expressions rather than pictorial or other representations.

The fifth meeting

The fifth lesson is crucial to bridge students' movement from an informal to a formal concept of balance and solving equations. In this part, some problems that build students' flexibility in manipulating elements in equations are given. Instead of comparing weight, in this meeting other contexts will be involved to help students apply the idea of balancing in a wider context.

The sixth meeting

In the sixth meeting, balancing is not explicitly used in the context. Here, the students' uses of the balancing idea are observed. This activity is to ensure the knowledge transfer from the balance into formal algebra.

During the implementation of the lesson series, the researcher will try to investigate students' understanding of linear equations with one variable given the balancing context. This is to answer the general research question, *how can balancing activities support students' understanding of linear equations with one variable?*

To well address that question, critical looks on the students' strategy and representation will be performed. Therefore, we propose three sub-questions, such as:

1. *What strategies do the students use to solve problems of linear equations with one variable?*
2. *How do the students' algebraic representations develop during a learning sequence on using a balance to solve linear equations with one variable?*

CHAPTER 3
METHODOLOGY

3.1. Research Approach

This study generally aims to contribute an innovation in the early algebra teaching in Indonesia, particularly in the topic of linear equations with one variable. Thus, it provides a design proposal that would be field tested in an Indonesian classroom. The design covers learning materials as well as hypothetical learning trajectories (HLT) that altogether will suggest practical as well as theoretical inputs in the teaching of domain specific subjects.

To achieve the goals, design research has chosen as an approach to conduct this study. This is due to the suitability of the present study with the purposes of design research (as mentioned in Van den Akker, Gravemeijer, McKenney, & Nieveen , 2006), such as: 1) to increase the relevance of research for educational policy and practice, 2) to develop an empiric-grounded theory, and 3) to increase the robustness of a design practice.

This approach is more qualitative rather than quantitative. Therefore, the research questions addressed in this study would be answered based on verbal descriptions of how the design promotes students' learning, teacher's behaviors, and classroom conditions. This approach has a cyclic character with three main phases in each cycle, namely, 1) preparation and design, 2) teaching experiment, and 3) retrospective analysis. Analyses toward the cycles result in a new design or a follow-up cycle (Van Eerde, 2013).

3.1.1.Phase 1: Preparation and design

Gravemeijer and Cobb (2006) mentioned two main purposes of performing this phase; each is viewed from design and research perspective. The first aim, from the design perspective, is to formulate a local instructional theory. Meanwhile, from the research perspective, this phase aims to clarify the theoretical intents, including the learning goals (or end points) and the starting points, and a conjectured local instructional theory. Practically, Van Eerde (2013) mentioned steps that a researcher should do in this phase, that is, literature review, formulating research aims and questions, and developing an HLT.

Following the steps, in this study, the researcher first reviewed some studies reporting problems that usually happened in algebra classes, particularly in Indonesian classrooms. This was followed by studying Indonesian curriculum and its components, such as, syllabus, list of basic competences, mathematics textbooks, and teacher guides released by the ministry of national education of the country. Combining these information, the researcher came up with critical thoughts of the causes of students' difficulties in algebra and conjectured students' starting points based on what the students should have learned before learning the unit of algebra.

Afterward, the researcher reviewed other studies that focus on how to teach algebra in classrooms, to be able to figure out what the students should have during their algebra studies. The gap between students' conditions and the ideal algebra class reported in the studies enhanced the formulation of the goals and the general research question of the present study. In addition, reflections on the previous studies together with critical thought experiments of the researcher resulted in an HLT, which consists of a series of lessons, completed with conjectures and anticipations to possible responses of the students.

Learning materials were then developed. This included lesson plans, teacher guides, manipulative, and students' worksheets. Before they are used, those learning materials were first consulted with experts (in this case, a supervisor, lecturers, and education practitioners) and fellow researchers. Thus, by the end of this phase, a complete theory-based lesson series are ready to be field-tested.

3.1.2.Phase 2: Teaching experiment

The purpose of conducting teaching experiments in a design research is to evaluate and improve the local instruction theory developed in the first phase. Besides, it also gives illustrations of how activities in the designs work. Due to the cyclic characteristic of the design research, the teaching experiment should have at least two macro cycles; each contains a full series of learning. The macro cycles consist of micro cycles; each micro cycle determines activities in every single meeting. During the teaching experiment, the micro cycle may have a small change to adjust the learning based on the students' performances in the previous micro cycle. After the implementation of one series, the macro cycle is analyzed. The analysis results in a redesigning of the HLT, which then becomes the start of a new macro cycle. To have a clearer description about the macro and micro cycles in a design research, an illustration of the cyclic process of the design research is given by Gravemeijer (2004), as shown in figure 3.1:

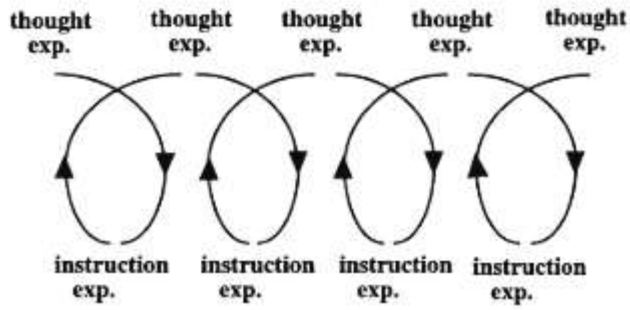


Figure 3.1 A macro cycle and micro cycles of design research

In this study, two macro cycles have been conducted. The first macro cycle is also called preliminary teaching experiment or pilot study. In this cycle, the local instruction theory developed in the preparation phase was experimented to a small group of students (four students). The class was handled by the researcher. The learning process was then analyzed in each micro cycle, to adjust the continuity of the learning sequence, and overall in one macro cycle, to produce a revised version of the local instruction theory. Reflections on the pilot study were mainly focused on how to develop the quality of the learning designs.

The second cycle was implemented in a larger group; in a normal classroom. In this phase, the class was conducted by the regular mathematics teacher. The teacher delivered the lesson based on the revised version of the local instruction theory. As in the pilot study, this learning was also analyzed during the micro cycle and later after the macro cycle. The analysis resulted in a local instruction theory of algebra teaching.

3.1.3.Phase 3: Retrospective Analysis

The retrospective analysis covers critical review, observation, and interpretation of what was happening in the classroom during the teaching experiment. In design research, this activity aims to develop the local instruction theory and improve the interpretative framework. Besides, Edelson (2002) also mentioned that this phase is important in terms of generalization of the theory developed in a study. Practically, this phase is done by comparing the hypothesized students' learning with the actual conditions during the teaching experiments. In this case, the HLT and research questions play important roles as guidelines for the analysis.

To conduct the retrospective analysis in this study, the researcher follows a task oriented approach (as described in Van Eerde, 2013). In this approach, the researcher will first have a look at the video of the lesson with the HLT and research questions as guidelines. During this process, the researcher will give remarks on interesting fragments. The role of the teacher and classroom dynamics are used to explain inconsistencies between the hypothesized and the observed learning. To help the process of analysis, Van Eerde (2013) suggested making a data analysis matrix comparing the HLT and the actual learning trajectory (ALT), like in table 3.1:

Table 3.1 Data analysis matrix for comparing HLT and ALT

Activity	Hypothetical Learning Trajectory (Conjectures of students' thoughts)	Actual Learning Trajectory (Observed class events)

3.2. Data Collection

In this subchapter, we structure our description of data collection techniques following the structure of the implementation of this study. This part aims to give an overview about what and how the data we are going to collect during the study.

3.2.1.Preparation Phase

Collecting data at this phase mainly aims to provide sufficient information about the subjects being studied, in this case, the teacher, students, and classroom environment. This is important to reveal the starting points of the students and the applicability of the design in the classroom, and to make an adjustment (if needed) with the initial step in our HLT. There are several techniques that we employed in this phase, such as, interview with the teacher, classroom observation, and conducting a pre-test.

Classroom observation and interview with teacher

The classroom observation generally aims to reveal the classroom norms, teachers' ways of teaching, and general overview of students' ability. Observation notes were made to record important information during the observation. To help the data gathering, an observation sheet was employed (see appendix).

Meanwhile, in the interview, the researcher tried to collect data on teacher's beliefs on algebra teaching, *Pendidikan Matematika Realistik Indonesia* (PMRI), and some technical matters related to the implementation of the study. This interview was semi-structured with some guiding questions provided (see appendix). The interview was audio recorded. An illustration of the data collected during the classroom observation and teacher's interview can be observed in table 3.2:

Table 3.2 Summary of data collection in the preparation phase

Topics	Data Sources	
	Classroom observation	Interview with teacher
Students' prior knowledge of the prerequisite topics	√	√
Classroom norms	√	
Pedagogical matters	√	√
Familiarity with PMRI (Indonesian RME)	√	√
Teacher's personality (teaching philosophy and beliefs, experiences in teaching the current level, educational background)		√
Classroom conditions (including the availability of learning materials)	√	√
Flexibility (how strict the school and the teacher with the following the curriculum, textbook, and innovation in teaching)		√

Pre-test

The pre-test mainly aims to reveal two things, that is, the student's understandings of the prerequisite knowledge, and the students' understandings toward the topic of the lesson. This test was organized in a form of written test (see appendix C). After the test, we would have the students' written works to analyze. In addition, an interview was also done to some students to clarify their answers and to get a deeper knowledge about the students' understandings toward the given problems. Interviews to students were video recorded.

Information on the students' mastery of the prerequisite knowledge is reflected on the starting point of our HLT. Meanwhile, the students' initial understandings toward the topic of the lesson were compared with their knowledge after the learning implementation, to see whether the learning really occurred.

3.2.2.Preliminary Teaching Experiment (Cycle 1)

In this phase, the design and the instruments were field tested to a small group (4 students). This field test perhaps provides suggestions to improve the design, the conjectures, and the instruments before they were used in a larger classroom. Data gathered during this phase included the students' written works, video recordings of classroom observation, and field notes. The learning session was handled by the researcher. A static camera was used to record the lessons. An observer also joined the class, made notes about the lessons, and watched over the camera. The result of the analysis of this phase would contribute to improvement of the HLT.

3.2.3.Teaching Experiment (Cycle 2)

The second teaching experiment was conducted in a larger group, which is different from the class of the students involved in the cycle 1. The class was handled by the regular teacher of the class. The revised HLT was used as a guideline for the teacher to conduct the lessons. The data gathered during this phase were expected to describe how the students learned during the 6 series of lessons, how the teacher facilitated students' learning, and how the designs helped students to learn.

To collect the data, the same techniques as in the first cycle was applied, that is, by utilizing video recordings, field notes on teacher's and students' performances, and the students' written works. Due to the large number of students involved in this phase, two cameras were employed to record the classroom activities; a static and a dynamic camera. The static camera was used to record the overall classroom activities, while the dynamic camera was used to zoom in a specific event during the class, for instance, the students' presentations or in their group's discussion. The classroom situation, including social norms and socio-mathematical norms were also observed during the class, particularly during the classroom's discussion and in the interaction between the teacher and her students and among the students themselves.

To collect more focused data, four students were chosen to be the focus group. The choice was made in a discussion with the teacher by considering the students' performances during the classroom observations, the pretest, and teachers' suggestions. The members of the groups were moderate (not too high and not too low in mathematics), but having a relatively good communication skill. During the class, the teacher gave no special interventions to the group.

3.2.4.Posttest

The posttest was given to students after the sixth lesson to ensure whether the students experienced a progress after attending the overall lessons. Therefore, some items in the posttest were made the same as in the pretest; some others are different, particularly those asking for students' higher level thinking. These different items perhaps showed if the students achieved higher than the expected learning goals. It is important to highlight that the students' performances in the posttest would not be the major data to reflect; it will just give insights about the progress of the students.

3.2.5.Validity and Reliability of Data Collection

To guarantee the quality of the data collected, issues related to validity and reliability will also be explained. The validity of the data collection tells its accuracy (internal) and its interventionist nature (ecological). Meanwhile, the reliability tells the independency of the researcher (internal) and the replicability (external) of the data collection (Bakker & Van Eerde, 2015).

To guarantee that the data we collected are valid internally, a number of data collections were employed, such as, interview, observation, field notes, and students' assignments. These different data collections perhaps strengthened each other in informing about the observed characteristics of the subjects. Thus, the explanations could be more accurate. Another effort to increase the internal validity was to increase the quality of the instruments. This was done through consultation with experts, in this case, the supervisors and some fellow researchers. A good-quality instrument merely impacted the quality of the data collected.

Meanwhile, the ecological validity was maintained by conducting the study in a normal classroom. In this case, the class was handled by the regular teacher, and the students involved are in one regular class. The general classroom norms will also be maintained, for example, greeting to teacher to start and end the lesson.

To ensure the internal reliability of the data collection, we used cameras instead of relying only on the observer’s notes. This use of technology perhaps minimized the intervention of the researcher in the data gathering. The video recordings indeed also maintain the external reliability of the data collection, since it provides virtual data that are replicable (Bakker & Van Eerde, 2015).

3.3. Data Analysis

To answer the research question, data we have collected were analyzed and interpreted. In general, the HLT was used as a guideline to analyze the data.

3.3.1.Preparation Phase

Overall, analysis toward the data gained in the preparation phase aims is needed to adjust the starting points in the HLT. Data sources in this phase included classroom observation and interview with the teacher, pretest and interview with some students.

Classroom observation and interview with the teacher

Data gathered in this phase were observation notes and audio recordings. The observation notes were re-read and analyzed to mainly see the classroom norms applied in the class, and the classroom interactions. Some remarks were noted down to later be consulted with the teacher during the interview session. Important explanations that were audio-recorded were transcribed and matched with the field notes to see the consistency of the information.

Pretest and interview with some students

Data obtained from the pretest included students’ written works and video recordings of the interviewed-students’ explanations. The result of the pretest was also used to select the focus group, in such a way that students in the focus group should vary in their level of knowledge observed from the result of this pretest. Meanwhile, to analyze the video recordings of the students' explanations during the interview, we first replayed the video and transcribed interesting segments. Qualitative data on students' prior knowledge of the prerequisite materials and the topic of the lesson were critically gathered. This was reflected on the starting points of the HLT.

3.3.2.Preliminary Teaching Experiment (Cycle 1)

Generally, the preliminary teaching experiment aims to improve the design before it was used in the second cycle. Analysis to the data gathered was done by observing how a small group of students react to it. In this phase, data collected were video recordings of the learning process, observation notes, and students' written works. The data were analyzed with task oriented approach, that is, by analyzing in the level of activities (Van Eerde, 2013). HLT was used as a guideline to analyze the data. The analysis was done in each micro cycle and macro cycle. The data analysis performed as in the following steps:

Analysis of the micro cycle

Video recordings

The video of one lesson was watched along with the HLT to make a comparison between the students' actual learning and the proposed HLT. Here, we focused on analyzing the conjectures in the HLT, whether it covered all the possibilities of the students' ideas, and whether the suggestions we gave for the conjectures really helped the students in learning.

Observation notes

Observation notes were made by an observer during the class session. These notes were used to help the analysis of the video recording. The observation notes told the researcher crucial learning moments that should be reviewed in the recordings.

Students' written work

The students' written works were gathered by collecting worksheets and exercises given to students during the class. Analysis of these works was done by interpreting the students' answers; observing whether their works and explanations in the video recording confirmed each other.

The video recordings, observation notes, and the students' written works in one lesson were compared each other to see how and what the students have learned during the lesson. This gave us ideas about the students' end points after one lesson, which will be used as a starting point for the next lesson. Adjustments to the HLT of the following meeting were done according to this result.

Analysis of the macro cycle

The analysis of the macro cycle of the preliminary teaching was performed after the posttest. Here, reflections on every micro cycle were taken into account to re-build the new version of the HLT. Thus, modifications toward the design can be performed.

3.3.3.Teaching Experiment (Cycle 2)

Data analyses of the teaching experiment in the second cycle were mainly the same as those applied in the first cycle. Special concerns were given to the performance of the focus group. However, the overall classroom performances were also looked to gain information about the classroom condition. The result of the analysis of this cycle was used to answer the research questions and to generate a local instruction theory of algebra learning. Therefore, in analyzing the macro cycle of this phase, we looked carefully not only to the HLT but also to the research questions.

3.3.4.Posttest

Analysis of the posttest was similar to that of the pretest. Qualitative analysis was done by interpreting the students' written works. This interpretation was confirmed to the students through an interview after the test. The result perhaps enriched the conclusion and answers to the research questions.

3.3.5.Validity and Reliability

The reliability of data analysis and the validity of the conclusions drawn in a study is a key to a good quality study. Therefore, these issues become essential. Bakker and Van Eerde (2015) mentioned that in data analysis, the researcher must pay attention to the soundness of the conclusion (internal validity), generability and transferability of the results (external validity), the independency of the researcher in data analysis (internal reliability), and the transparency of the analysis (external reliability).

Internal validity

Data triangulations were performed by comparing data from different sources to ensure the credibility of the result. Reflections toward conjectures of a specific episode were also tested over the other episodes. This ensured whether the interpretations toward the data were accurate. Comparison between the HLT and the actual learning will also improve the internal validity, as it provided evidence to draw a conclusion.

External validity

Efforts to ensure the transferability of the analysis were done by presenting transparent explanations about the classroom situations, students' characteristics, and conclusion drawings in every stage of analysis. This should be enough for readers to justify the suitability of the study in their environment, and how the HLT could be adjusted in their situation.

Internal reliability

Internal reliability in data analysis is kept by minimizing subjectivities in interpreting data. This was done by discussing important episodes of the lesson with the supervisors, and fellow researchers. Thus, the final conclusions covered the combination of multi-interpreters, instead of the researcher's own interpretation.

External reliability

To improve the traceability of the data analysis, transparent reports on how the researchers come up with the interpretations or conclusions were provided. Here, we provided the readers with evidences and steps on how the data were treated.

3.4. Research Subject

This study was implemented to the seventh grade students of SMP Pusri Palembang. The school is a project school of the PMRI. Four students were involved in the pilot study. A regular class, different from the class of the pilot students, consisting of 27 students were taking part in the teaching experiment two. The mathematics teacher of the class was also involved. In the real teaching experiment (cycle 2), we also chose a group to be the focus. This was done to gain specific data of students' progresses. There were no special treatments given to the focus group, except in video-recording the class events (the group sat in the focus of the camera).

CHAPTER 4
HYPOTHETICAL LEARNING TRAJECTORY (HLT)

As elaborated in the methodological chapter, HLT plays a very important role in every phase of design research. It was first developed in the preparation phase, and then used as a guideline for teacher during the teaching experiment. Meanwhile, in data analysis, the researcher reflected on the HLT against the students’ actual learning to produce an empiric based local instruction theory of mathematics. The construction of an HLT is first postulated by Simon (1995) as a thought process about the reflexive relationship between a design of activities and consideration of students’ thinking if they participate in the activities. Further, he mentioned three components that should exist in an HLT, that is, learning goal, learning activities, and hypothetical learning process (HLP).

The first thing we have to develop to make a good HLT of learning is a clear conceptual goal of the learning; also called teacher’s goal (Simon, 2014). The formulation of the goal should sound mathematical, since the lesson is indeed addressed to help students learn a particular mathematical concept. A learning goal should be practical; it must be sufficient to illustrate the role of mechanism, and must be based on students’ prior knowledge as a starting point (Simon & Tzur, 2009).

After setting the learning goal, a set of learning tasks are developed. Simon (1995) mentioned that these tasks will be the key part of the instructional process. The activities will facilitate the students’ learning to be able to reach the intended mathematical goals. In arranging the sequences of the tasks, a researcher must have thought about the learning lines that connects the students’ initial understanding and the final goal. This activity will produce what is called hypothetical learning process.

The next, we have to think in more details about the learning procedures. In this phase, the researcher shall do thought experiment about students’ reaction toward the proposed activities. The hypothetical learning process is set in a way that gives activity-effect relationship. Thus, for each activity, the researcher develops conjectures of how students-participants will respond to it. The conjectures might vary due to differences among the students. Therefore, suggestions of how teachers should respond to ensure the students back to the learning path should also be clear in the HLT.

For the present study, we have developed a hypothetical learning trajectory (HLT) for learning linear equations with one variable for grade VII of Indonesian lower secondary school. This HLT involves 6 series of lessons, with a pretest and posttest before and after the 6 lessons. The pretest was also included in our HLT; and we named it as lesson 0, since the result of the pretest was considered the starting point of the first meeting.

4.1. Lesson 0: Pretest

As explained in the methodological chapter, the pretest has two main purposes, namely, 1) to identify the students' understandings of the prerequisite knowledge, and 2) to check the students' understandings of the topic of the lesson. In the present study, students' prerequisite knowledge that is observed in the pretest covers students' arithmetical understanding needed to solve problems on linear equations with one variable. Meanwhile, the students' understanding of the topic of the lesson assessed in the pretest relates to the meaning of variables and basic steps to solve a linear equation problem.

To achieve these two goals, four items have been developed (see appendix). The first two items were addressed to reveal students' understanding of the prerequisite knowledge, while the other two were used to check their understandings of solving linear equation problems. Overall, during the pretest, the teacher should not give any intervention to students' answers. However, the students' answers in question 1 and 2 would be discussed in the beginning of the first meeting.

Conjectures of students' thoughts and suggestions for teacher

In solving the first question, the students might have problems to answer part b and c (see figure 4.1). The question 1b asked the students to find a number that satisfies an arithmetic expression. The question 1c actually asked a quite similar question, but this item allowed more than one solution.

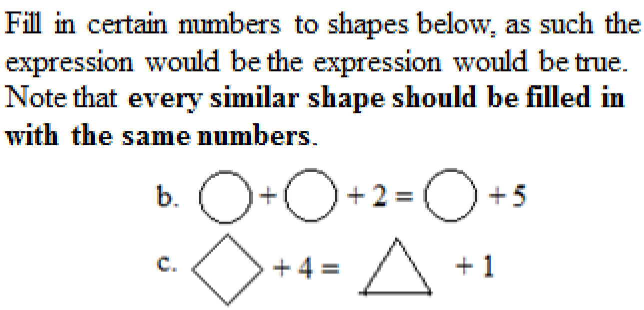


Figure 4.1 Question 1b and 1c in the pretest

To answer the question 1b, we predicted that many students would try to substitute some integers (starting from 1), and then check the answer. This student will keep doing the same with other numbers until they find out that '3' satisfies the expression (as illustrated in figure 4.2.)

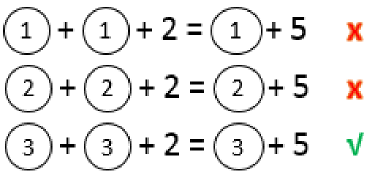


Figure 4.2 Prediction of students' answer to problem 1b in the pretest

This way of answering is considered enough in this phase. Thus, the teacher should not force the students to perform other strategies to solve the problem. However, the teacher still needed to check if there would be other students solving the problems using another strategy.

Meanwhile, to solve the question 1c, we predicted that some students might try to directly guess the numbers to fill in the two shapes. This arbitrary guess-and-check would hardly lead the students into the right answer (see left in figure 4.3). Therefore, later in the follow up meeting (in lesson 1), the teacher should rise this issue and ask the students to think of a more systematic way of guess-and-check, that might be performed by some other students (see right in figure 4.3.).

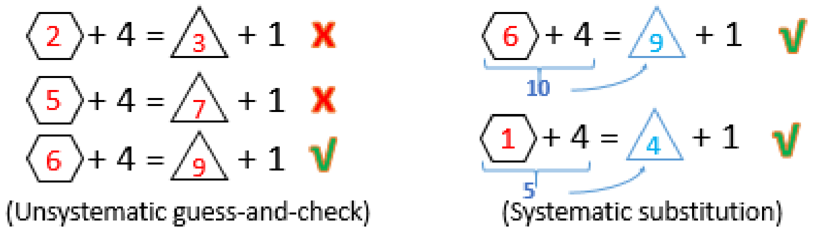


Figure 4.3 Prediction of students' answers to problem 1c in the pretest

The next question in the pretest asked the students to trace a series of arithmetical operations to find a certain number. The question gives the students a clue: *"if you multiply the number by 3, and subtract the result by 5, the result will be 7"*. This clue might lead some students to again guess-and-check the answer. Such a strategy could lead the students into a correct solution, but probably after a number of error trials.

Other students might have performed reversed operations (or working backwards). In their answers, they might (or may not) use certain notations to represent the unknown number. An example of students' possible answer with reversed operations is illustrated in figure 4.4.

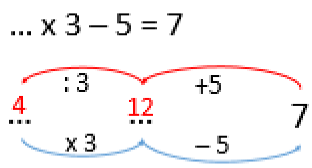


Figure 4.4 A possible answer for problem 2 in the pretest

This representation shows a good arithmetical understanding of the students, which would also be the goal of the first meeting. Thus, if all the students have come up with this way of answering (which we thought, some students would not), then the teacher could just go through the first meeting to just recheck their way of representing unknowns, and not focusing on the arithmetical matters anymore.

Question 3 and 4 in the pretest should not be discussed with the students during the class since they would be used again in the posttest to check the students' progress.

However, students' answers in both questions would show how far the students have recognized the use of variables and performed procedures to solve for a linear equation problem.

4.2. Lesson 1: Secret numbers

Starting points and learning goals

Before attending this lesson, students must have been able to perform basic arithmetic operations. It is also important that the students knew about the inverse operations and have experiences on it. Although the students might have not used variables, they might have found it in a form of placeholder (or label) in arithmetic.

Therefore, in this lesson, we intended to build the students' understanding of:

1. Making sense of equal sign as indicators of equal relations
2. Variables as the representation of a (generalized) number.
3. Using algebraic expression to record (generalized) arithmetical processes.

Activities, conjectures of students' thoughts, and suggestions for teachers

To achieve the goals, two main were activities presented to students, namely, secret number (part 1), and secret number (part 2). These two activities made the most of students' arithmetical knowledge to understand basic ideas of the use of symbols in algebra. Therefore, brief discussions on the first two problems of the pretest would be required in the beginning of this lesson (see subchapter 4.1).

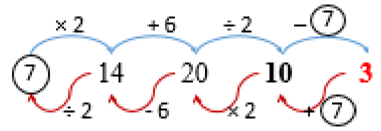
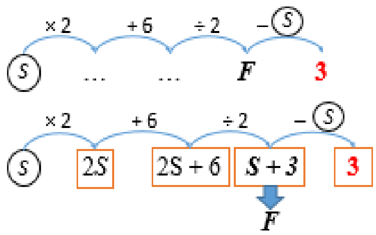
Secret number (part 1)

The activity adopted the idea of guessing-my-number game. Here, the teacher would ask the students to think of a number, and then give some instructions that the students have to follow. After a series of instructions, the teacher would guess the students' numbers. The task for students would be to figure out how the teacher guesses the number. This activity would require students to trace the arithmetic operations they have performed, and to make a generalization of the operations. This might give chances for teachers to introduce a function of variable as a tool for generalization. In this phase, the students might use any symbols to represent their numbers in a more general form. It can be a drawing, or an abbreviated letter.

Table 4.1 gives an overview of the possibility of students' thinking and some suggestions that the teacher might do if the students come up with that idea.

Table 4.1 Overview of hypothetical learning process in Secret Number (Part 1)

Prediction of students' thinking	Suggestions for teachers
While following the instructions, the students might make the following representations (suppose the students' secret number is 7): Style 1 $7 \times 2 = 14 + 6 = 20 \div 2 = \mathbf{10} - 7 = \mathbf{3}$ Style 2 7 14 20 $\mathbf{10}$ $\mathbf{3}$ Style 3 $7 \times 2 = 14$ $14 + 6 = 20$ $20 \div 2 = \mathbf{10}$ $\mathbf{10} - 7 = \mathbf{3}$	Teacher should ask the students to show these different representations (if appeared) on blackboard. He/she might invite students to discuss why style 1 is not an appropriate representation.
When asked to figure out the trick, at first, the students might relate their secret number (*S*), the first final result (*F*), and 3 (the number appeared the same for every students at the end).	The teacher highlights the students' first conclusion, and discusses with the students how they may notate them. Here, the teacher might wait for the students' proposal or make a convention with the students to use abbreviated letters (*S* and *F*).
Here, the students might recognize that: $F - 3 = S$ (or with any representations of *F* and *S*, the students might make)	(*it is possible that the class discussion convents on other representations, if this happens, the teacher has to follow the convention*). Afterward, the teacher might rise another issue about the generality. He/ she might post question like, "why is that so?" or "can it be applied to any number trick?" or "why must the result be 3?".
The student might continue to trace the arithmetic operation they have performed to find out the reason behind the trick. Here, the students can come up with the following representations:	The teacher discusses the power of arrow representation OR help students with style 3 relate each step. Again, the teacher should ask to think of generality. Here, the question can be: "can you figure out how we could get the same result at the end?" OR quite explicitly asking "can you show with your representation, why for any secret

Prediction of students’ thinking	Suggestions for teachers
Style 2  Style 3 $7 \times 2 = 14$; $14 + 6 = 20$; $20 \div 2 = \mathbf{10}$; $\mathbf{10} - 7 = \mathbf{3}$ $14 \div 2 = 7$; $20 - 6 = 14$; $\mathbf{10} \times 2 = 20$; $\mathbf{3} + 7 = \mathbf{10}$ Students with style 2 will probably make an arrow representation, and thus they will come up with the idea of working backwards and inverse operations. The students with style 3 will probably do the same, but they might fail to relate each step.	number we choose, the result goes to 3?”.
To answer the question, the students might do repeated trials with several numbers. Some of the might propose to do tracing back by utilizing symbols that they have agreed in advance. The students’ answers could be: 	Teacher conducts a discussion of how to represent the arithmetical process into an equation. In the discussion, the teacher can name this generalized arithmetical process as “*trick formula*”

It is important to highlight (as we also noted in our teacher guide) that the teacher should encourage any representations of the students. In this phase, the teacher should not interfere students’ ways of writing, unless the students themselves decide to change it.

Secret number (part 2)

The second main activity is also a secret number. However, in this part, the students would be challenged to make their own secret number, including a set of instructions that others should perform to play the game. In this game, the students in group should first write their ‘*trick formula*’ as the key to the secret number game they developed. Afterward, they demonstrate their trick, and let other students participate. During this activity, the students may come up with ideas, as given in table 4.2.

Table 4.2 Overview of hypothetical learning process in Secret Number (part 2)

Prediction of students' thinking	Suggestions for teachers
Students might make an arrow representation to formulate their trick. Here, they may use symbols to control over the sequences.	Teacher asks the students to try out their secret numbers. Afterward, he/ she might challenge the other groups to find out the trick. Teacher might ask the students to write their trick formula on the blackboard and let other groups comment on their trick. Teacher might rise question like, "are you sure that your trick would be applicable for any numbers people might choose?" "Can you explain, why?"
The students might come up with answer that represents the general applicability of the symbol, like, "*because this actually represents whatever number you want*".	Teacher highlights the keyword, and asks students to make conclusion of what they have learned.

With the two main activity presented in lesson 1, we expect that the students can build up their understanding of the uses of symbols, and basic algebraic expression. It might be that the students have successfully produced an equation, here. But, the teacher should not force the students for it. In this phase, some students might have used letters to represent the generalized number given in their trick formula, however, it is possible that other students still neglect to use it, and are more convenient to make a drawing. For this phase, the teacher may let the students use their own representations.

4.3. Lesson 2: Finding balance

Starting points and learning goals

Students' understanding of variables becomes the starting points of the students to participate in this activity. Their insights about equations may have been built also in the previous activity. However, there is still no guarantee if they could make an equation. In this phase, the students would do balancing activities with a real balance scale. Thus, the students' experiences with the balance scale contribute to students' prior knowledge for this lesson. The activities given in the second lesson aim to:

1. Introduce the rules of balancing to students
2. Introduce the use of letters (or symbols) to represent objects
3. Build the students' understanding of equivalent equations

Activities, conjectures of students' thoughts, and suggestions for teachers

Activities in lesson 2 were all related to balancing activities. There were two main activities that the students should do in this meeting, namely, bartering marbles (part 1), and bartering marbles (part 2). In those activities, the students were first introduced to a problem in context, where two children with their collections of three-different-sized marbles. The problem would lead the students to the idea of bartering by considering the weight of the marbles.

Bartering marbles (part 1)

In the first part, the students would compare the weight of 3-different-sized marbles in a real balance scale. The task would be to make as many balanced conditions as possible on a balance scale, and then record it in their worksheets. The task was set to perform in group. Some conjectures have made, as can be observed in table 4.3. These cover the students' strategies to make the balanced conditions and the way they represented the balanced conditions.

Table 4.3 Overview of hypothetical learning process in the bartering marbles (part 1)

Conjectures of students' thoughts	Suggestions for teachers
The strategy the students might use to find the balance conditions:	
➢ Some students might do arbitrary trials. They might combine two kinds of marbles first, and then estimate how those might relate. They would do combination of others by considering the result of the previous trial.	➢ Teacher suggests the students to record every trial they have made, to prevent them repeat measuring. After a certain time, the teacher might ask the progress of the group, and encourage them to be more flexible in combining.

Conjectures of students' thoughts	Suggestions for teachers
➢ Other students might combine the same objects on one arm, and combine the other two objects on the other arm.	➢ Teacher might ask the students to explain their strategy to make the balance, like, "what did you do to get into the balance" OR "what will you do if the scale tends to the right?

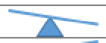
The way the students represent the balanced conditions might vary, such as:

➢ Ordinary listing

LEFT	RIGHT	RESULT
1 small + 1 big	2 small + 1 medium	(not balanced)
3 medium	1 small + 1 big	(not balanced)
1 big	2 small + 1 medium	(balanced)

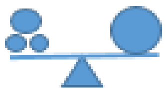
➢ Teacher should let the students with these different representations show and explain their representations.

➢ Making tables

Left			Result	Right		
S	M	B		S	M	B
1	1	-		-	-	1
1	2	-		-	-	1
2	1	-		-	-	1

➢ The teacher can offer the students to comments on the benefits of each other's representations, and then ask the students' preferences. The teacher might address question like: "if you are asked to do the same tasks, which representation do you think will you use? Why?".

➢ Draw the balance and the marbles



➢ Write an equation of the balanced conditions

$$2S + 1M = 1B$$

(Some students may not use the plus sign '+'. Instead, they use the word 'and' to state the combination of objects)

(It is also possible that the students do not use the equal sign yet '=', but use word 'is balance', or 'has equal weight', or 'equals')

➢ This moment should also be used by the teacher to discuss whether the plus and the equal signs are appropriate to use in this case.

By the end of this activity, the teacher would provide the students with a poster hang on the blackboard, where students can write any balanced conditions they have made. This poster will be used in the next activity.

Bartering marbles (part 2)

In this part, the students would be asked to predict more balances, given the same conditions as in part 1. Here, the students would not be allowed to use balance scales anymore, and would not be restricted by the number of marbles they have. Thus, they could only rely on the list of balanced conditions they have made in the previous activity. Table 4.4 gives predictions of what the students might come up with when performing this task.

Table 4.4 Overview of hypothetical learning process in the bartering marbles (part 2)

Conjectures of students' thoughts	Suggestions for teachers
It might not be really easy for students to think of any new relations. They may keep trying to combine the existing marbles, and can probably reveal a new relation.	➢ Teacher encourages students to keep imagining the balance. (If necessary) he might give clue to combine the existing balanced conditions. He can ask, "what do you think make them balanced?" or "what does it mean if $2S + 1M = 1B$ (or taking another example from the known balance) ➢ The keywords are 'imagine', 'relate', and 'think of the balance'.
The students might look at relationship between the balance conditions they have found, like $2M = 1B$ and $4M = 2B$, where they can see that the latter is double the former one, and try to generalize this into "finding new balanced combinations by multiplying or dividing the quantities in each balance with the same amount"	➢ This reveals the equivalence under the multiplication and division. Teacher must note this notion and highlight it in the discussion and conclusion part.
The students might think to add the same amounts to both sides from a balanced condition, like: $2S + 1M = 1B$ into $2S + 2M = 1B + 1M$	➢ This will lead the students to the equivalence under the addition and subtraction.
The students might think to exchange certain marbles with other combinations as they realized that they have the same weight. For instance, from $2M = 1B$ and $2S + 1M = 1B$ into $2M = 2S + 1M$	➢ Teacher must note and emphasize the point of understanding the consequence of the balance. The idea will lead the students to substituting with the same quantity

It is important that the teacher highlighted each idea, and invited students to formulate/ summarize any actions that maintained the balanced situation on a balance scale. The teacher may also ask students to compare the use of letters in the current meeting (representation of objects) with that in the previous meeting (representation of numbers).

4.4. Lesson 3: Finding unknowns

Starting points and learning goals

Based on what the students would have done in the first two lessons, we assumed that the students would have been able to make equations with symbols. The symbols could be understood by students as generalizations of numbers or representations of certain objects. In addition, from activities in lesson 2, the students have understood basic rules to work with a balance scale, particularly on how to combine objects on the scale while maintaining its balance conditions. However, during the two activities, the students would only work with direct objects; they have not built relationships of the concepts with quantities. That is why activities in the third lesson will be addressed to:

1. Introduce the use of symbols to represent quantities
2. Use the rules of balancing to find unknowns

Activities, conjectures of students' thoughts, and suggestions for teachers

To help the students achieve the learning goals, two main activities would be delivered, namely, bartering marbles (part 3), and weighing beans. Both would not involve physical activities anymore.

Bartering marbles (part 3)

This activity is the continuation of the two activities involving marbles in the lesson 2. Here, students would be presented with the list of balanced situations that they have created during the bartering marbles part 1 and part 2. In this part, the weight of the small marbles (3 gram) would be informed. The task for students would be to find the weight of the other two marbles. This task aims to build students' understanding of variables as representation of quantities, and algebraic expressions as tools to represent the relationship among quantities. Some conjectures of how the students will deal with the problems are provided in table 4.5.

Table 4.5 Overview of hypothetical learning process in the bartering marbles (part 3)

Conjectures of students' thoughts	**Suggestions for teachers**
The students might first have in mind that the weight of the other two medium and the big marbles must be more than 3 gram.	➢ Teacher encourages students to write that information, so that, they can reflect their answers with that information. ➢ The teacher might encourage students to use the list of balanced conditions (equations) they have made in the previous meeting
Students, who neglect to use algebraic expressions in the previous meeting, might have difficulty. Other students will use the lists of equations of the balances (from the previous meeting), they may select certain equations and change (substitute) S (symbols for the small marbles) with '3'.	➢ Encourage the students to try to use work with algebraic expressions (or symbols). ➢ Teacher encourages the students to make a new equation, after they substituted the S. This will ease them think to find the solution, since the unknown things in the expression became less. (*This step is important to promote the use of variables to represent quantities*).
Some selections may be helpful, and some others may not.	➢ Discussion about the students' selection of equations should also be conducted. So, the students can recognize which representations are more helpful to find the solutions.
Students who fail to make a good selection probably will do guess-and-check. Here, they will substitute 'S' with '3', and then guess arbitrary values for 'M' and 'B'. Afterward, they check the result. $2S + 1M = 1B$ combinations of balance that the students select $S = 3$, Suppose $M = 5$ and $B = 7$ Then, we have: $2 \times 3 + 5 = 7$ ✗	➢ Ask the students to rethink of their selections, like: "why do you choose this relation?", or "why do not you choose another relation?", or "what do you think make you hard to find the solution?".
Big probability that the students will start to find the 'M' instead of the 'B' due to direct relations that 'M' and 'S' have. Here, they may do the following: $1M = 2S \rightarrow 1M = 2 \times 3 \rightarrow 1M = 6$	➢ In the discussion, the teacher should encourage students to show (write) these steps completely which show how changes happen.
Afterward, they may do the same thing to find the B from M, or from M and S.	➢ The words "we do the same to both sides" or "to maintain the balance, once

Conjectures of students' thoughts	Suggestions for teachers
	we add or remove something to certain arm, then we have to do the same with the other arm" is important to constantly say during the process. In the discussion, the teacher might also raise issue of treating '$2S$' as '5×3'.

This activity would help students to find the unknown by identifying the relationship among quantities. Systematic substitutions would become the modest strategy promoted in this activity.

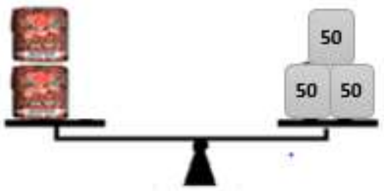
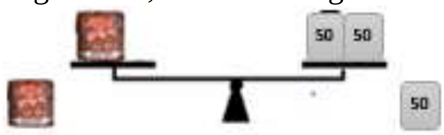
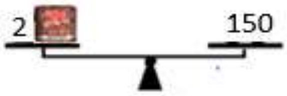
Weighing beans

This activity would start with a story about efforts to find the weight of two bags of beans with a two-armed balance scale and combination of 50g masses. Here, the students would not do the real balancing; they would just see how people in the story struggle to find the balance. The final balance condition is given in figure 4.5. The task for students was to find out the weight of one bag of beans.

Figure 4.5 Balance conditions in the weighing beans problems

Some conjectures about the students' thinking when solving this problem are presented in table 4.6. Here, we focus on how the students might represent a balanced condition. There was no explicit clue about how the students should make their representations, which was intended to see the students' development in representing situations on a balance scale.

Table 4.6 Overview of hypothetical learning process in the weighing beans activity

Conjectures of students' thoughts	Suggestions for teachers
The students will recognize the efforts that Iman (subject in the story) used to make a balance.	The teacher might show pictures of each step and asks some students to interpret the pictures.
The students might look at the balance conditions, and start to move from certain things to find a new balance.	Removing similar things is also an idea of equivalence that the teacher should emphasize in the discussion.
Unfortunately, they will find unfriendly situation that does not allow them perform so:  	Regarding some students' decisions to keep removing things, the teacher might bring it into classroom discussion. The teacher might ask, "do you think it is true? can you explain?" or "why do you remove one bag of beans and one 50g mass?" or "do you think the result will still be balance?"
Some might still do removing, but find a wrong answer, like following:  	Another option is that the teacher can ask to students to reflect on their conclusion. The teacher can ask, "so how weigh do you think is one bag of beans? Is it 50g, as you removed, or 100g as it stays on the scale?". Further, the teacher might suggest to change the object with numbers or work with equations, by asking "how many gram are the masses in total?" or can explicitly offer "I think we have to find another way".
The students replace the masses with numbers. They may also do the same for the beans; they might change it into variables or just leave them as drawings.  OR $2B = 150$	The teacher encourages both ways; drawings and equations. In the discussion, he should relate those two representations.
Some students might be able to find the answer with this representation, but some others might have difficulty, especially if they do not really think of	To help the students who find difficulty to solve the expression, the teacher might ask them to interpret the picture or the representation. The teacher can

Conjectures of students' thoughts	Suggestions for teachers
the interpretation of the representation	ask, "can you say it in words what did you understand from the picture?"
The students might say "the weight of 2 bags of beans is 150 gram".	The teacher can ask, "so, what is the weight of only 1 bag of bean?"
This will help them think to find the weight of 1 bag of beans.	

Activities in the third meeting might become the students' first experiences in solving for unknowns in a linear equation problem with one variable. The rules to maintain the balance will be really employed here, either to find the answer or to reflect the strategies that the students take.

4.5. Lesson 4: Formative Evaluation

Starting points and learning goals

Basic ideas to solve for unknowns using balancing strategies were given to students in the first three lessons. Thus, we assumed that the students would have been able to solve simple problems involving real balancing scale. Therefore, the students could move to a more mathematics situation, especially in their strategy to find the unknown in a given linear equation problem. To ensure the movement to the next phases, lesson 4 is given to check the students' readiness. Therefore, we formulate the learning goals for the lesson 4, such as:

1. Evaluating students' understanding of the use of variables, algebraic expressions, and solving for unknowns in a balance scale.
2. Facilitating the students' movements from balance drawings into equations.
3. Using the idea of balancing to solve an algebraic representation (an equation) of a situation on a balance scale.

Activities, conjectures of students' thoughts, and suggestions for teachers

To reach the goals, two activities would be conducted. The first one is an individual formative assessment, and then the second one would be solving for unknowns that they should perform in pairs.

Formative Assessment

In this activity, the students would be given three problems to solve individually. During the test, the teacher should not interrupt students' answers. Afterward, cross-checking would be managed to conduct. This moment will also be used to conduct the classroom discussion about the problems. The first problem in this formative task would ask students to relate objects on a balance scale, drawings, and a balance formula. Reflection on this problem might help students interpret a balance formula in their own minds. A table will be provided in the question as shown in figure 4.6.

The following table illustrates three balanced conditions. Complete the blank cell!

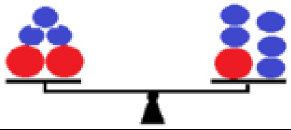
Left	Right	Balance drawing	Balance formula
2 small marble 1 big marble	...	...	$2s + 1b = 5s$
...	...		...
...	...	...	...

Figure 4.6 Question 1 on the formative assessment

We predict that the students might have no difficulty to complete the table. However, it is important during the classroom discussion that the teacher should give emphasis on the relationship between the balance drawings and the balance formula. We also predict that some students would only use letter (abbreviation of a word) to state the objects on the left and the right arm of the balance.

In the second question, the students should compare four different combinations of objects on a balance scale to identify one in the figures which is wrong. This activity demands students' understanding to manipulate and maintain the balanced situation in a balance scale, which is indeed a concept of equivalent equations. The four different combinations are given in figure 4.7.

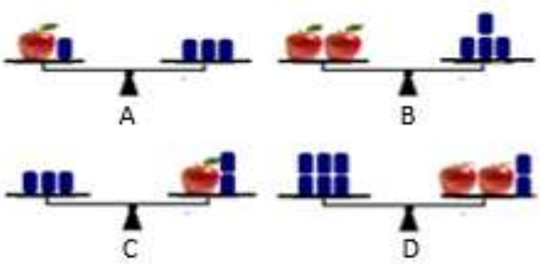
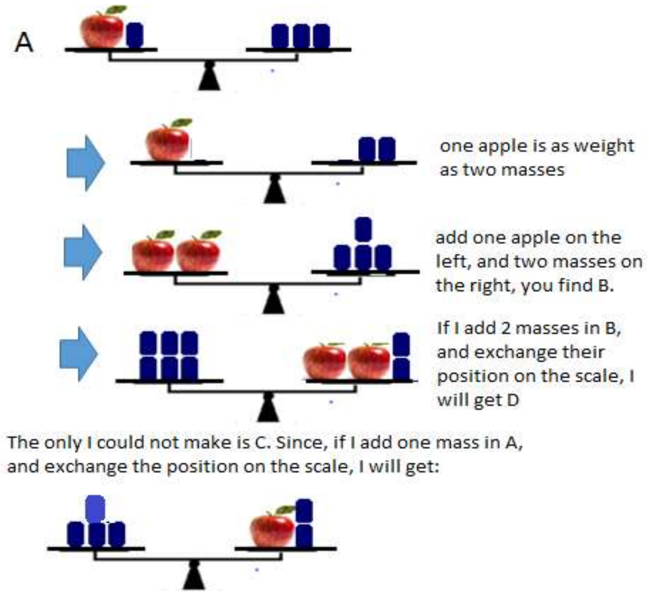
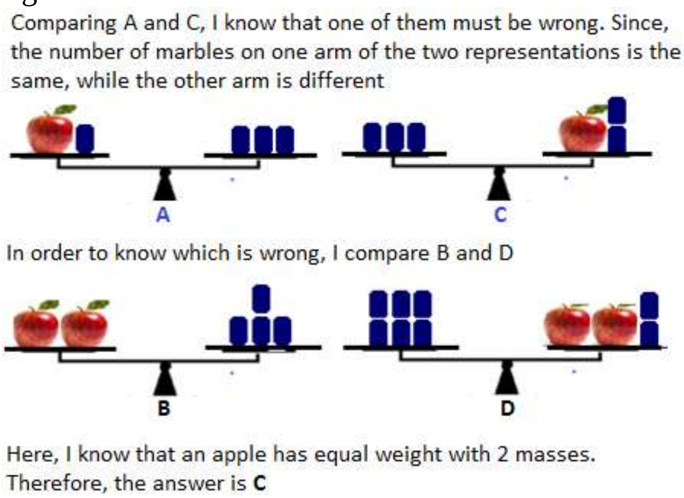


Figure 4.7 Combinations of balance to compare in formative assessment question 2

This problem would evaluate students' understanding of the rules of balancing and equivalent equations, which would be required for modifying equations in more advanced problems. Various strategies might be performed by students to solve this problem, as summarized in table 4.7:

Table 4.7 Overview of hypothetical learning process in question 2 of the formative assessment

Conjectures of students' thoughts	Suggestions for teachers
Students might find the weight of an apple in each representation. Thus, they will find that there is one, which is C, that gives a different result. This means, that the answer is C.	It is possible that all the students come up with this idea. Thus, in classroom discussion, the teacher might ask, "can anyone identify the mistake without solving the problem?" or probably the teacher can give a quite direct leading question like, "can you determine which is wrong by only comparing the figure?".
Some students might choose one of the four combinations, and try to construct the other three combinations, while maintaining the balance. One that they could not construct would be the answer.	The teacher should encourage these two strategies.
For example, if the students choose to start from A, then they will do the following:	The only thing that the teacher should do is to ensure whether

Conjectures of students' thoughts	Suggestions for teachers
 They probably can start with another option. Some students might also compare options A-and-C, and B-and-D (since both give the same number of apples) by moving the masses. Afterward they can compare the result with the result of comparing the other two. This strategy can be illustrated in the following figure: 	the students really understand what they do. In classroom discussion, the teacher should encourage those different ways of solving. He must convince students that they are indeed the same ideas.

The last problem in the formative test, question 3, presents solving for unknowns. A figure is provided, however, the students are free to choose if they want to work with the figure or using algebraic expression, as shown in figure 4.8.

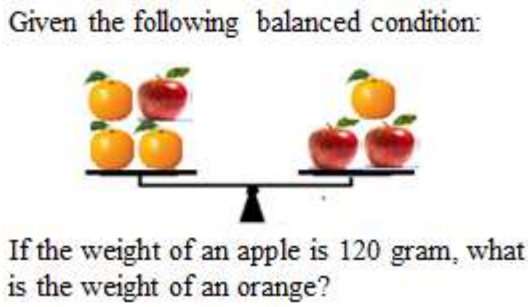
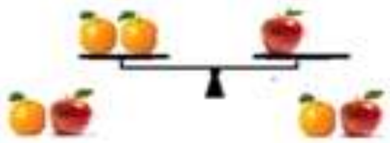
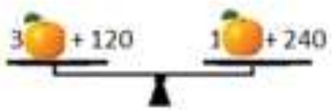


Figure 4.8 Question 3 of the formative assessment

What would be interesting to observe from students with this question is the way the students represent the problems before they solved it. This is covered in the overview of hypothetical learning process given in table 4.8.

Table 4.8 Overview of hypothetical learning process in question 3 in the formative assessment

Conjectures of students' thoughts	Suggestions for teachers
Some students might first remove similar things from both arms of the balance. So, they would find:  So, they know that the weight of an apple equals the weight of two oranges. Thus, they can find the answer.	Encourage this way of working, and compare it with the work of the other students.
Some other students might change the apples into numbers and count the number of oranges in both sides. Thus, the students will have:  Afterward, they simplify the expression to find the answer (this conjecture is possible as they have done	This way of working may confuse students, especially if they do not really understand their representation. Thus, if the students with this representation get stuck, the teacher might ask the students to first interpret or say what they represented.

Conjectures of students' thoughts	Suggestions for teachers
the similar thing in lesson 3).	
Some other students might directly translate the problem into algebraic equations, and then solve it. These students can do: $3J + 1A = 1J + 2A$ Afterward, the substitute the weight of the apple into the equation. Here, they will get: $3J + 120 = 1J + 2 \times 120$ And then, solve the problems: $2J = 120$ Thus, $1J = 60$.	The teacher should not force the other students to do the problem this way. However, he might ask the students who come up with this idea (if any) to explain their way of answering. The other students might give comments or ask about it. At the end, the teacher should encourage students to perform strategies that they really have understood.

Discussions about the assessment might take longer time than is provided. The teacher, however, should not be influenced by time. It would be suggested for the teachers not to move to the next lesson, unless the students have shown a good understanding of the idea behind the questions in the formative assessment.

If there would be more time, the teacher could give another problem to students, which is, finding mass. This question would also ask the students to solve for an unknown. However, the worksheet would require students to state the problem into an equation. The situations presented in this activity are shown in figure 4.9. In this problem, the balance scale could not be iterated. Therefore, they could not remove (real) thing from both sides to simplify the problem.

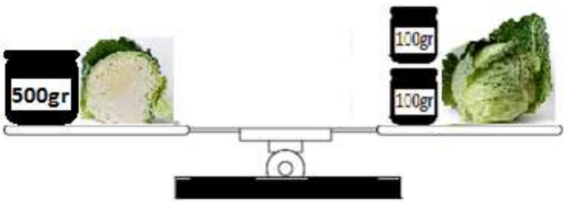
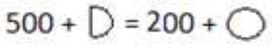


Figure 4.9 Situation given in 'finding mass' problem

The table 4.9 provides conjectures of students' thoughts and suggestions for teachers, if those conjectures appeared in classroom.

Table 4.9 Overview of hypothetical learning process in ‘finding mass’

Conjectures of students’ thoughts	Suggestions for teachers
The students might translate the picture into the following expressions: $500\ +\ ½\, cabbage\ =\ 200 + 1\ cabbage$ OR  OR $500\ +\ ½\, C\ =\ 200 + 1\ C$	The teacher might let the students continue with those representations. But, in the discussion, the shows his preference to use letters by constantly using that kind of representations.
Realizing the need to keep the balance, the students might think to just remove same amounts of the masses. Thus, the remaining work would be: $300 + \frac{1}{2}C = 1C$ Big possibility that the students will get confused to decide their next step.	The teacher might emphasize this removal by writing something like “minus 200 from both sides”. The teacher should stimulate the students’ thinking by asking, like “what do you think you can do?” or “if it is not a ½ do you think you can solve the problem? Why do not you do the same?” or “what do you think you can relate from the equation?”
Some students might think to remove the half part of C from both sides. Thus, they will arrive into the result: “$300 = \frac{1}{2}C$”. So, “$1C = 600$” Other students might see the relation between a half and a full unit. So, they can come up with: $300 + \frac{1}{2}C = \frac{1}{2}C + \frac{1}{2}C$ which means that $\frac{1}{2}C = 300$ and $1C = 600$	The teacher can ask the students to clarify their steps. He should emphasize the steps that maintain the equality of the expression, for example, the change from $\frac{1}{2}C = 300$ to $1C = 600$.

4.6. Lesson 5: Manipulating Balance

Starting points and learning goals

It is assumed that the students have been able to solve linear equation problems in the context of balance before attending this meeting. However, the students might not be really flexible yet in working with it; they might still rely on trying to find the unknown once they found an equation. In this case, students’ ability to manipulate the equations would still need to be developed. In addition, during the first four meetings, we still have no idea whether the students can recognize a linear equation problem given in a non-weighing context. Therefore, the purposes of the present lesson are to:

1. Build students' flexibility in manipulating quantitative relationships
2. Introduce the students to linear equation problems in a non-weighing context

Activities, conjectures of students' thoughts, and suggestions for teachers

During the fifth meeting, the students will be given two problems, namely, combining balances and buying fruits. Classroom discussions in this meeting would be focused more on the mathematical aspects rather than the context. This would guarantee the applicability of the students' knowledge in a more general situation.

Combining masses

The first problem that the students would solve is 'combining balance'. In this problem, the students would be asked to arrange different masses on a balance scale to weigh mung-beans (700 gram). The provided masses weigh: 1 gram, 2 gram (2 pieces), 10 gram, 20 gram, 50 gram, 100 gram, 200 gram (2 pieces), and 1 kg (normally, a 500-gram mass also exists, but in the story we said that it was gone).

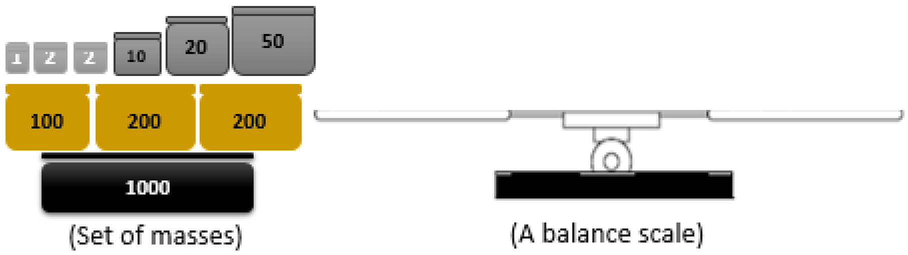
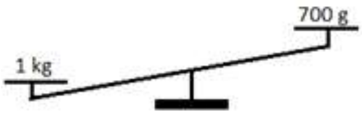
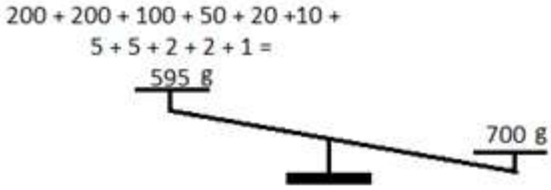
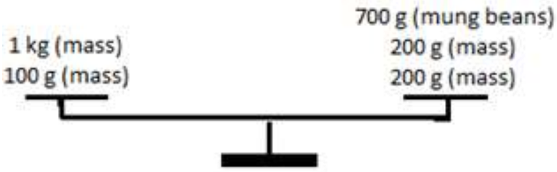
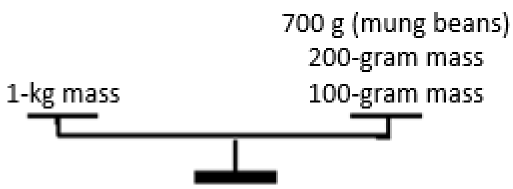


Figure 4.10 Masses and a balance scale given in 'combining masses'

This activity perhaps builds students' flexibility in relating quantities. This also might give the students insights of how to find unknowns without solving it; or also called acceptance of lack of closure. Students' possible thoughts when working with this activity are given in table 4.10.

Table 4.10 Overview of hypothetical learning process in combining masses

Conjectures of students' thoughts	Suggestions for teachers
Some students might put the mung beans (700 gram) on one side alone, and then try to combine the existing masses to get the 700.  (They may think that they should not involve the 1kg mass, because it weighs more than 700 gram). These students will end up with combining all the other masses, but the 1kg, on one left of the balance scale, and found the following:  This might lead them into thinking that, 'even if we combine all masses, but 1 kg, the weight does not reach 700 gram'. So, it might be impossible to find to make 700 gram.	➢ The teacher opens a discussion by asking whether it is really impossible to weigh 700 gram at a time, using the provided masses. ➢ Student's doubt answer should be utilized to bridge them think. Teacher might ask question like "why do we only manipulate one arm?" or "can't we put another mass together with the 700?" ➢ If none of them appears with the idea of manipulating both arms, the teacher can ask simpler problem to think of, like, "how do you think we can make 30gram from 20 and 50 gram masses?"
The students will find the idea of manipulating both sides and come up with the result:  OR  	Teacher asks the students to explain their strategy.

It is also possible that the students still draw the masses and the mung beans in their answer. However, it is suggested to focus discussion more on the mathematical aspect rather than the representation in this phase.

Buying Fruits

The context raised in this problem would relate weights of certain fruits with their prices. This task aims to facilitate the students' movement from weighing contexts. Although, they can still find the word 'weight' in the question, the meaning of variables appeared in the representations would tell 'the price' instead of 'the weight' of each kg fruits. The meaning of variables should be emphasized in classroom discussion. Students' possible performances when dealing with this activity are summarized in the table 4.11.

Table 4.11 Overview of hypothetical learning process in 'buying fruits'

<table>
<tr><th>Conjectures of students' thoughts</th><th>Suggestions for teachers</th></tr>
<tr><td>The students might have no problem in translating "the price of 1 kg of grapes, and 5 kg of duku is Rp.64.000,- into mathematical words".

They probably translate into:
$$1G + 5D = 64000 \qquad (1)$$However, some students might have difficulty in understanding, "the price of 1 kg grape is three times as much as the price of 1 kg duku".</td><td>The teacher might give a simpler illustration by asking, "if the price of 1 kg duku is Rp. 5.000,-, how much do you think the price of 1 kg grape is?", or "how may you interpret the question", or "what information have you got from the story in the question?".</td></tr>
<tr><td>The students might get the idea of the relation between the price of the duku and the grape, that is:
$$1G = 3D \qquad (2)$$For the next step, some students might do guess-and-check.

Here, they first guess the price of the duku, and then find the price of the grape using relation (2).
Afterward, they may check the result by substituting the prices into relation (1).
<table>
<tr><td>Price of 1 kg duku (guess)</td><td>5000</td><td>6000</td><td>7000</td><td>8000</td></tr>
<tr><td>Price of 1 kg grape</td><td>15.000</td><td>18.000</td><td>21.000</td><td>24.000</td></tr>
<tr><td>1 kg grape + 5 kg duku</td><td>40.000</td><td>48.000</td><td>56.000</td><td>64.000</td></tr>
</table>
</td><td>The teacher might let these students go with their ideas. However, later in the discussion, they might discuss and compare this strategy with other strategies.</td></tr>
<tr><td>Other students might substitute the equation</td><td>Discussion about the most efficient</td></tr>
</table>

Conjectures of students’ thoughts	Suggestions for teachers
(2) into equation (1), like what they have done in lesson 2 and 3. Here, the result would be: $3D + 5D = 64.000$ $\rightarrow \quad 8D = 64.000$ $\rightarrow \quad D = 8.000$ And the price of 1 kg G would be Rp.24.000,-.	and effective strategy should be conducted. “efficiency implies the fastest way to the solution, while the effectiveness implies the correctness of the answer”.

4.7. Lesson 6: Balancing across contexts

Starting points and learning goals

Students’ understandings of algebra built in the first five lessons become the basis for their learning in the 6th meeting. Therefore, in general, there will be no new topic given to students during this meeting. However, the researcher thinks that the students’ experiences in the non-weighing contexts built in lesson 5 are not enough yet to ensure their understanding. This becomes the reason why we choose to add this lesson. The proposed learning goal for this meeting is for students to apply the idea of the balance to solve linear equation problem across contexts.

Activities, conjectures of students’ thoughts, and suggestions for teachers

In this meeting, the students will be given two problems in general contexts. These are to ensure the transfer of knowledge from the contextual based into the concept. The two problems are about parking rates and units of temperature

Parking rates

In the ‘parking rates’ problem would present a story of two parking lots with different charging systems. The balanced condition would be given at a time when the two parking areas charge the same amount to their customers. The task for students was to figure out how this situation happens. Here, they would need to compare the two payment systems. To have a clearer view of the problem, see figure 4.11.

Ulil and Ruslan were riding their own motorcycle to a mall. The mall has two parking lots; outer and underground.

The rate of the outer parking is Rp.2000,- in the first hour plus Rp.1000,- in the next hours. Meanwhile, the fee for the underground parking is Rp.4.000,- in the first hour plus Rp.500,- in the next hours.

As they arrived, Ulil suggested to park in the outer parking since he thought it would be cheaper. Ruslan disagreed, he parked his motorcycle underground. He also thought that it would be cheaper.

After hours in the mall, they left the mall and found out that they had to pay the same amount.

Figure 4.11 Situation given in 'parking rate' problem

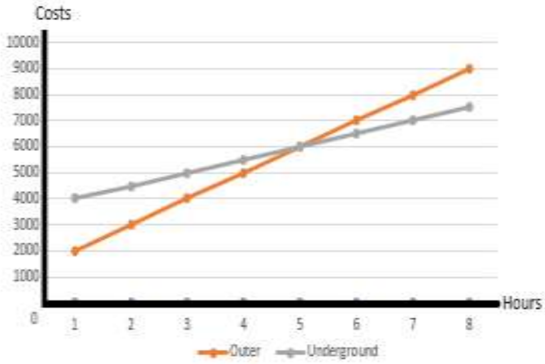
It is important to ensure the students' understanding of solving the parking rate problem, since it becomes the main goal of the overall lesson. Therefore, the classroom discussion must be engaging to all of the students. If necessary, the lesson might end in this activity to ensure the students' understanding. There are some possibilities that the student might come up with when dealing with this problem. These are summarized in the hypothetical learning process in table 4.13.

Table 4.12 Overview of hypothetical learning process in 'parking rates'

<table>
<tr><th>Conjectures of students' thoughts</th><th>Suggestions for teachers</th></tr>
<tr><td>Some students might list the possibilities of parking costs in the outer and underground park per hour.

These students will find:
<table><tr><th>Hours</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr><tr><td>Outer</td><td>2000</td><td>3000</td><td>4000</td><td>5000</td><td>6000</td></tr><tr><td>Underground</td><td>4000</td><td>4500</td><td>5000</td><td>5500</td><td>6000</td></tr></table>So, they would notice that actually the costs differ, except if they park 5 hours.</td><td>The teacher might let these students work, and later compare their strategy with other strategies (if any).

If there is no other students who come up with other strategies, the teacher might ask students to think of the parking costs of each place, like, "how much do you think the parking cost if they stay 3 hours?" or "can you think of a more general way of representing the parking cost of each alternative?".</td></tr>
<tr><td>Some students might be able to generalize the cost, and found the following:</td><td>The teacher might discuss with the students why the answer differs from what they have found when they list.</td></tr>
</table>

Conjectures of students' thoughts	**Suggestions for teachers**
$O(outer) = 2000 + 1000 \times h\ (hour)$ $U(Underground) = 4000 + 500 \times h\ (hour)$ Afterward, they continue working to find the h. Students might find the h and conclude that the number of hours they spent in mall is h; which is a wrong answer.	The teacher might also address discussion on what the 'h' in the equations represents. He might ask students to check their formulas per hour. And then, they will get into another formula or redefining the 'h', and then find the correct answer.
Some students might get stuck with the expressions and have no idea of what to do to determine h.	The teacher might help by asking question like, "in the question, it says that Ulil and Ruslan paid the same amount. How do you think we could relate this with the formula?"
Students might think of relating those two equations, and then find the answer.	Teacher might continue to the next question, that is, "what will happen if both Ulil and Ruslan stayed longer? Who will pay the least amount?"
Big possibility that the students know that the outer park would cost more. Their reasons might vary, such as: ➢ Comparing the parking costs of both areas for an hour extra. Here, they will find that in the outer part, they need to pay Rp.7000,-, while the in the underground part, they would only pay Rp. 6500,-. ➢ Observing the growth of the parking costs. Here, the students will argue that the outer park would be more expensive, since it adds by Rp.1000,- each hour, while the cost for the underground park only adds by Rp.500,- ➢ Observing the difference between the outer and underground, where they can see, that the longer they park, the smallest the gap between the outer and the underground park costs. After a certain time, it would be the same, and the underground park turned cheaper.	The teacher should encourage this idea. It can be also that the teacher leads the students to plot the results into a graph, and show all the students arguments in the graph. The graph would be: 

Unit conversion

This problem would be presented in a formal context that explicitly shows mathematical formula to solve. The formula tells relationships between two units of temperature; Fahrenheit (F) and Celcius (C), such as: $F = \frac{9}{5}C + 32$. Through a story in the context, we tell the students the value of F and ask them to determine the value of C. In this activity, the students would work fully with algebraic representation. Thus, it would ensure the students' ability to solve any representations of linear equation problems. Predictions of students' learning process through this problem are presented in table 4.13.

Table 4.13 Overview of hypothetical learning process in 'unit conversion'

Conjectures of students' thoughts	Suggestions for teachers
At first, the students might substitute the 'F' in the equation with the 70. Thus, the equation would be: $70 = \frac{9}{5}C + 32$ Some of the students might feel strange since they usually find the constant on the right side.	(If the students feel strange) the teacher might illustrate with the balancing. He might give two conditions to students the first is putting the masses on the right, and the other one is putting them on the left.
The students will continue working with the equation, as follows: $70 = \frac{9}{5}C + 32 \rightarrow 70 - 32 = \frac{9}{5}C$ $\rightarrow 38 = \frac{9}{5}C$	The teacher might ask what $\frac{9}{5}C$ means. He might ask questions like, "suppose we know the degree in celcius, or we know that C is 15, what F will be?"
Many of the students will probably get stuck here.	This will give illustration to students on how to operate $\frac{9}{5}C$ means, and will think of reversed operations to solve the problem.
Some students might still get hard to figure out what to do. Some students might try to guess the number; they do trial-and-error. This might also lead into the correct answer.	If this happens, the teacher can simplify the form, using a decimal representation. The teacher might also keep this way to show to students that there is not only one fixed way to solve math problems. However, the teacher would promote working with algebraic representation, since the trial-and-error may not be efficient in time.

CHAPTER 5
RETROSPECTIVE ANALYSIS

Testing toward the hypothesized trajectory of learning of linear equations with one variable proposed in chapter 4 was done during the teaching experiments. In this chapter, the applicability of those hypotheses has been analyzed in comparison with the actual learning conditions observed during the teaching experiments. The analysis would be based on the data collected in the form of video recordings, students' written works, and field notes. This chapter would be structured following the order of activities during the implementation, that is, preliminary observation, preliminary teaching experiment (cycle 1), teaching experiment (cycle 2), and posttest. In each part, the discussion will focus on development of students' strategies, algebraic representations, and how those two related to the balancing activities given during the learning process. In addition, issues regarding classroom norms and socio-mathematical norms observed in this study are presented inclusive in the analysis of the cycle 2.

5.1. Preliminary Activities

This part includes classroom observation, an interview with teacher, a pretest, and interviews with students. Those activities were meant to gain insights of students' understandings of the topic of linear equations with one variable, classroom conditions, and teacher's belief in teaching innovations before the learning implementation. The significance of this part is for readers to reflect the generality of findings in this study.

During the preliminary activities, it was revealed that subject students have learned about the linear equations weeks before the learning implementation. Moreover, when the researcher attended the target class for an observation, the students have been discussing problems on inequalities with one variable. An interview with the teacher confirmed this situation. However, further looks at students' performances during this session and on the other preliminary activities revealed some struggles by students, in general, to understand this topic. This shows the needs for students to re-discuss the topic; and therefore, the researcher and teachers decided to continue working with this group subjects.

In order to give an overview of what was happening during the classroom observations, the fragment 5.1 shows a conversation between the teacher and students during the preliminary session. In the fragment, the class were discussing about how to find the solution for the inequality $\mathbf{1 - 3y < 42}$.

Fragment 5.1 Students' struggles to solve $1 - 3y < 42$

Teacher	What is the sign of 1?
Students	Positive
Teacher	To remove 1, what should we do? To remove it, we should subtract it with?
Students	**(Long hesitations). (Some students tried to guess: 3; 42 minus 3).**
Teacher	42 minus 1. This is just the same as? Here, the 1 we had, we should subtract it with?
Students	$\mathbf{3y}$
Teacher	This one, 42 minus?
Students	1
Teacher	One. So, the 1 here should also be subtracted with?
Students	42
Teacher	$1 - 1 - 3y$. That's the same, right? But because this $1 - 1$ results in 0, so we may not write it.

Continue working, the teacher seemed to drive students to the correct way of solving the problems. However, students seemed not to understand the guidance. Finally, the process went on as if the teacher solved the problem on her own. This is recorded in fragment 5.2.

Fragment 5.2 Teacher directs students to solve the problem

Teacher	Now we have $-3y < 41$. To make it only y, **what do we usually do**?
Students	Divide.. Reverse..
Teacher	Reverse? Reversing from smaller '<' into larger '>'; and from 41 into?
Students	(together) **41 over 3**
Teacher	3 or $\mathbf{-3}$?

8	Students	(together) **3**
9	Teacher	Why $-\mathbf{3}$?
10		(long hesitations. students seemed confused)

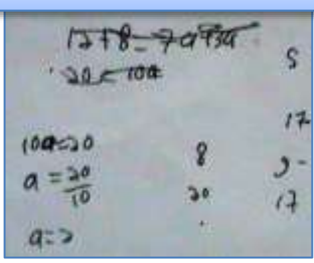
Analyzing the fragment 5.1 and 5.2 resulted in some important information about the conditions of the subjects prior to the learning with balancing activities, such as, the students' difficulties, classroom situations, and the teacher-students interactions. The analyses are combined with data gathered from the pretest, interviews, and field notes to gain more elaborated explanations about those aspects.

Students' struggles in their initial algebra learning

The first data revealed concern on the difficulties of the students when working with linear equation problems. Despite the 'inequality part', the fragment 5.1 and 5.2 clearly show some struggles, especially when performing the transformation strategy (fragment 5.1 line 3-5; fragment 5.2 line 1-2 and 6-9). In the conversation, the teacher has tried to provoke students with incomplete-sentence clues. However, the students just remained silent or started to guess inarguably.

Such a failure to perform the so-called formal strategy is also observable in students' written works during the pretest (an example is shown in figure 1). Here, the student seemed to try to separate between terms with variables and the constants. His strategy was to move some terms across the equal sign. The problem was in the changes of value signs. Here, the student seemed to only memorize a rule like: 'if moving across the equal sign, change value'. As a result, a mistake like shown in figure 5.1 became common, as they often missed or forgot to perform certain step.

Question: Find the value of 'a' that satisfies: $\mathbf{3a + 12 = 7a - 8}$



Student's answer

$12 + 8 = 7a + 3a$

$20 = 10a$

$10a = 20$

$a = \frac{20}{10}$

$a = 2$

Figure 5.1 Example of a student's struggle (Bagus) to solve a linear equation problem

Another strategy used by most students to solve problems of linear equations is guess-and-check. To majority of the participant-students, this strategy seemed to be their modest. In the interview, students often explained that they “first guessed any numbers”, “used their own logics”, or “try out numbers” and then checked whether the numbers matched with the expected results. This strategy works in any equations. However, it might not be efficient to use in a non-simple form of equations. For that, Linsell (2009) categorized it the lowest in the hierarchy (see in chapter 2).

Another struggle is in the algebraic representations. The simplest problem was that students hardly identified variables, constants, and coefficients in an equation; especially in a complex form. In addition, many students could not identify variables represented in other symbols than letters. Students’ lack of understandings of variables can be seen in an interview with a student prior to the learning implementation, like what is shown in fragment 5.3.

Fragment 5.3 Researcher asked students to define variable

Researcher	Have you ever heard the term variable? What is variable?
Amel	Variable is… (thinking loudly) **a letter behind a number**
Researcher	Letter? Behind number? So, you mean.. for example.. how does it usually look like?
Amel	Letter before number.. letter before number
Researcher	For instance, $y + 7 = 10$ (writing while pronouncing). Is there any variable here?
Amel	(thinking) yes.. y
Researcher	Letter behind number, right? Where is the number?
Amel	(confused).. hmm… isn’t it **a letter in… in an arithmetic operation**?
Researcher	Hmm.. so what if $\square + 7 = 10$. Is there variable?
Amel	Hmm.. maybe.. (looking confused)

Misunderstanding about locations of each component in an equation also happened to some students. Some students strictly limit themselves that variables must be on the left side of the equal sign, while the constants are on the other side. They even proclaimed that putting variables on the right of equal sign was an error and would not lead to a correct answer. Those problems (according to Kieran, 2007) implied failure to understand an equation as a structure. Further, she explained that it might cause difficulties in performing the transformation strategy to solve for equations.

Classroom situations and teacher-student roles

The preliminary activity, particularly the classroom visit also tried to study the conditions of the targeted classroom in its daily basis mathematics teaching. Overall, data, as recorded in the video and field notes, show a typical traditional mathematics teaching in the class. In this occasion, teachers seemed to take over most of the class works. Although communications between teacher and students occurred in some sessions, interactions among the students were likely unobservable.

As can be observed from the fragment 5.2, the teacher seemed to dominate the problem solving activity (line 1-2, 4-5, 7, and 9). The way the teacher posed questions also seemed leading to telling rather than thinking. Thus, the students only needed to continue the incomplete sentence of the teacher to answer. Extra remark could be addressed to the line 16-17 when the teacher asked, “What do we usually do?” Such a question seemed to tell students the strategy they need to solve the given problem. Further, it might also signal that the teacher encouraged her students to only memorize procedures to solve for equations. Unluckily, the students did not really show a good understanding, and that they frequently missed the teacher’s expectations.

The power of the teacher and students’ over-trusts to teacher are also recorded in the field note. In many occasions, students would keep asking the teachers whether their answer is right or wrong. As a consequence, there was lack of interactions among the students. In addition, the teacher would be very busy to answer the students’ never-ending questions.

The results of this preliminary session suggested minor adjustments in our teaching plan, such as to minimize times allocated for introducing the topic of the learning. The lesson plan was also improved by adding more details providing some guiding questions that teachers could employ during the teaching (the version used for the cycle 2 can be seen in appendix). This was meant to help the teacher organize the discussion and scaffolding sessions.

5.2. Preliminary Teaching Experiment (Cycle 1)

The first cycle of the teaching experiment aims to pilot-test the designed learning materials and the proposed learning lines. This phase was conducted in 6 meetings with various numbers of activities in each meeting. Unlike the chapter 4, the organization of this subchapter will follow the activities instead of the meeting itself, since in the implementation, activities given in certain meetings are sometimes split or merged. There are four main activities conducted, such as: 1) secret number, 2) bartering marbles, 3) formative evaluation, 4) manipulating balance, and 5) linear equations across contexts. A pretest is also a part of this cycle, but in this report, we count it as a preliminary activity reported in the previous part. Discussions in this part might not be in deep. The focus would be on how the design will be used in the next cycle. More elaborated discussions of each part would be presented in the analysis of the second cycle.

5.2.1.Activity 1: Secret Number

This activity focused on the arithmetic prerequisites to learn linear equations and its algebraic representations. In this activity, the students were playing guess-my-number games, making their own tricks, and trying to record a series of arithmetic operations, as instructed in the game. The mathematical goals of this activity are to build students' conceptions of equal signs; promote the use of symbols and algebraic representations to represent generalized arithmetic procedures.

Strategies the students employed during the meeting

As shortly mentioned in the result of the preliminary activity, many students did rely on a guess-and-check strategy to solve for unknown. Although discussion about strategy is not the main aim of this lesson, a new strategy (for students) was revealed, that is, working backwards. The strategy appeared when the students were assigned to make their own secret number trick and then guess other students' secret numbers. Fragment 5.4 shows how Anggi has applied the working backward strategy in her trick.

Fragment 5.4 Student’s uses of working backwards strategy

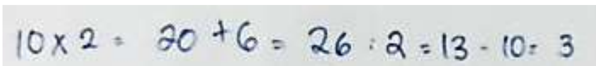
Anggi	(*Think of a number*, *add* 5, *multiply* 2, *divide by* 2, *subtract by* 2, *and then add* 2) {initial instructions}
Researcher	How to find the result then? {the secret number} (miswording, but understandable by the students)
Anggi	The final result is subtracted by 2 $\{-2\}\ldots$ add 2 $\{+2\}\ldots$ divide by 2.. ehh.. multiply it by 2 $\{\times 2\}\ldots$ divide by 2 $\{\div 2\}\ldots$ minus 5 $\{-5\}$

This strategy may not be applicable for all kinds of equations, yet it might help students to later gain insights of the transformation strategy. At least, this might add student’s references of strategies they could use to solve for equations. That the students now knew another approach, which might be more effective than the guess-and-check for certain problems, indeed indicated a progress in the students’ sense of problem solving. Perhaps, this would contribute to the students’ flexibility in employing strategies to solve for linear equation problems.

Students’ algebraic representations

The use of equal signs

As aimed, the secret number activity during the first cycle successfully invited the students’ common mistakes in misusing equal signs to appear. Figure 5.2 shows how the students have recorded arithmetic operations they performing following the teacher’s instruction in the first stage of the secret number activity.

(a) 

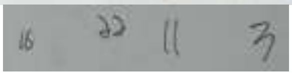
(b) 

Figure 5.2 Students’ ways to record a series of arithmetical operations

The two ways of presenting the series of arithmetic operation in figure 2 have precisely been predicted in the HLT. In the figure 5.2a, which is the most common in this study, the students simply used equal signs to state the results of operation just before it. In this occasion, the students treated the equal sign only as a symbol that leads to a result, not as relational function. To react, as planned, the researcher asked the students to read their answer aloud in a slow tempo. Such an action has helped students to realize their mistakes, as recorded in fragment 5.5.

Fragment 5.5 Students realize their mistake in employing equal signs

Researcher	Anggi, would you please read
Anggi	10 times 2 equals 20 plus 6 equals 26 over 2 equals 13 minus 10 equals 3
Students	(long hesitations) (re-read the sentence slower and slower){seemed that they realized something incorrect}
Researcher	Try to think of it. Try to think of the sentence.
	(…)
Gaby	Ahh.. different.. $10 \times 2 = 20; 20 + 6 = 26.$
Students	Aaahh…

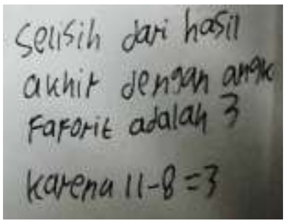
After the conversation, the class agreed to the misuses of the equal sign. They concluded that what are separated by equal signs must have the same value. The class, then, continued to discuss how to best represent those arithmetic operations. This ended up with combining the figure 5.2b with arrow language.

The use of symbols to represent unknowns

The HLT predicted that the students might write the secret number differently from other numbers in the series of arithmetic operations. Such writing was planned to be an initial way of noting unknown. However, as shown in the figure 5.2, this hypothesis seems to be not confirmed in the very first part of the activity. As the lesson continued to making tricks and guessing other’s numbers, an indication of students’ uses of symbols was detected in a student’s scratched work. There is no further confirmation about this answer from the student since he did not express this during the math congress. However, his constantly using such kind of noting (circling the secret number, when he tried to trace his trick) perhaps can indicate his awareness of the distinct of the secret number to the other numbers. This way of writing is given in figure 5.3.

Figure 5.3 A student constantly circled the secret number

During the lesson, the researcher also found a chance that teachers should have made the most of to lead the students into the use of symbols that represent arithmetic numbers, as can be observed in figure 5.4. The figure shows the students' notices of the key to the trick of the secret number game by the teacher.



(Translation)
The difference between the final result and the favorite number is 3, since $11 - 8 = 3$

Figure 5.4 Student's prediction of the key to the trick

From the answer, it seemed that the students have noticed some important numbers in the series of operations, that is, the final result, the secret number, and the difference. This might open chance for researchers to ask students to symbolize and state the relation between those three numbers, as a formula. This idea is then included in the list of scaffoldings for in the second cycle.

Concluding remarks

Video registration and field notes (observation sheet) report that the classroom went well. The students were really enthusiastic to play and try to find the key of the trick. Discussions occur along the sessions, between the teacher and students and among the students themselves. The proposed goals are feasible. Thus, the researcher decides to keep this activity in the next cycle with minor adjustments. The modification made, such as, adding the notions of important numbers aims to help students to think about the trick. In addition, it might open more chances for students to use and understand symbols to represent unknown numbers in a series of arithmetic operation.

5.2.2.Activity 2: Bartering Marbles

This part is eventually the core of the so-called balancing activities in our study. There were three main activities performed by the students in this part, such as 1) finding balance, 2) maintaining balance, and 3) finding weights. The activity was organized in two days. A problem in a social context about bartering 3-different sized marbles was given in the beginning of the session (see attachment). In this case, the students were led to think that weighing (or equalizing weights) could be a solution to the proposed problem.

Part 1 (Finding balance)

In this part, subject students were trying to combine three different sizes of marbles on a balance scale. The task was to find and list as many balance combinations as they could. The combinations are proposed as possible barter combinations (see chapter 4). This activity aims to let the students experience and observe equalities as well as rules of balancing in a real situation. In addition, the way the students represented the balance combinations would be used as a starting point to promote their algebraic representations. In this case, students would be directed to use letters instead of words or pictures to represent the marbles. So, the students might get impressions that variables (letters) might represent certain objects. An illustration of what was happening in the classroom during this activity is given in figure 5.5.

Figure 5.5 Students were trying to find a balance

Students' strategies

This activity is the first step to introduce the balancing strategy to students. Rules applying to find balances were easily recognized by the students. In a session, the students explained that what they needed to do to make it balance is to simply add or remove something from an arm. If a balance scale is more to the right, then they need to add some on the left or remove certain amount in the right. This idea would be reflected when the students later do equality that did not involve any balance.

Furthermore, this activity was also meant to stimulate the students' awareness of relationship among quantities, when they were started to become lazy to use the real balance. A conversation between two students in group 1 (given in fragment 5.6) indicates the starts of the students' awareness of quantitative relations.

Fragment 5.6 Students started to think of quantitative relations

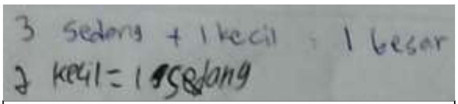
(*After finding a combination of 9 small=1 big, Bagus and Sabil tried to find another combination. Sabil first put 2 big marbles on an arm and some small marbles on the other arms of the balance scale*)

Bagus	Can we only use the small ones? 2 big and how many of the small?
Sabil	[add small marbles until he thought it was balance]
Bagus	[change his hand with a hook to hold the balance scale] [put the balance scale on his table, and then raised it slowly. He observed it] [put the balance again, and again raised it] [he observed and found it balance]
Bagus	[counting silently] 11, 12, 13, 14, 15, 16, 17, 18, 19.

Observing the fragment 5.6, Bagus had started to think about moving from a certain combination to another combination. However, when he tried to test his conjectures, he did not find confirmation about his idea, which made him confused. Despite the correctness of the results, this student seemed to have thought of the quantitative relations among objects, which made it possible to relate the combinations one another. This indicator confirmed our HLT on how the activity builds students' sense of quantities in balancing activities. Further, this understanding could grow to be a 'model of' for students once they found problems of equal balances

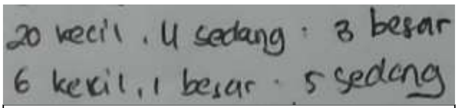
Students' representations

Another concern of this activity is on the students' representations, in this case, the way the students listed the combination of balances they have discovered during the exploration with a balance scale. As obtained in this study, the two groups showed different way of recording (shown in figure 5.6).



3 medium + 1 small = 1 big
2 small = 1 medium

(a) Group 1



20 small, 4 medium = 3 big
6 small, 1 big = 5 medium

(b) Group 2

$1B = 3S + 1K$

B= *Besar* (Big); S=*Sedang* (medium); K=*Kecil* (Small)

(c) Classroom Discussion

Figure 5.6 Different ways of representing balance combinations

The different ways of write-reporting the balance combinations led into discussions of the uses of plus sign '+'. It was not so hard that students finally agreed about the uses of the plus sign. Discussion about the use of equal sign was also conducted, in which, the students all argued about the equal weight of the combinations of objects in the two arms of the balance scale.

The initiation of the use of letters in this study was also done in this session. Although it is proposed by teachers, it is likely that the students can easily understand the meaning, as they confirmed such a representation. The letters used in this phase might not function as a variable yet. They are simply symbols that represent objects or an alphabet that has extended form. However, such a view is important for students to bring up when they do a more symbolic algebra. Moreover, treating variables as objects in the equation is seen as a requirement to build a structural view of algebra representations and to perform the transformation strategy to solve linear equation problems (Jupri, Drijvers, & Van den Heuvel-Panhuizen, 2014b).

Concluding Remarks

Overall, the class went as it planned. The inaccuracy of the tools employed in this study on one side has affected students' performance especially when they tried to move from real weighing (with balance) into mental weighing (predicting balance). However, it can also be treated as a starting point to propose the mental weighing, rather than the real weighing.

This activity can be a good starting point for students to investigate and experience equalities in a balance. The use of more accurate tools (if possible) is suggested, or otherwise, discussion about the less accuracy of the tool could be informed just before the investigation. In our case, the latter one was applied due to resources limitation. As the consequence, an adjustment was done for the upcoming activity, that is, by providing a new list of balance combinations.

Part 2 (Maintaining balance)

The second part of bartering marbles gave the same problem as that in the first part. The difference is that, in this activity, the students were no longer provided with balance scales. So they should do a mental instead of real weighing. In other words, they predict the balances. For this occasion, the students might use the combination of balances they had from the previous activity as a starting point to think of the other combinations (for this occasion, we provided them with the same lists). Thus, it would force students to think of relationships among the combinations of balance. The provided balanced combination is given in figure 5.7. The mathematical goal of this activity is related to equivalent equations.

DAFTAR KOMBINASI SEIMBANG

1. $1\boldsymbol{B} = 3\boldsymbol{S} + 1\boldsymbol{K}$
2. $2\boldsymbol{S} = 5\boldsymbol{K}$
3. $3\boldsymbol{B} = 7\boldsymbol{S} + 13\boldsymbol{K}$
4. $4\boldsymbol{K} + 1\boldsymbol{B} = 5\boldsymbol{S}$
5. $3\boldsymbol{S} + 3\boldsymbol{K} = 1\boldsymbol{S} + 8\boldsymbol{K}$
6. $5\boldsymbol{K} + 1\boldsymbol{B} = 1\boldsymbol{B} + 2\boldsymbol{S}$

Symbol explanation:

B = *Besar* (Big)
S = *Sedang* (Medium)
K = *Kecil* (Small)

Figure 5.7 List of the provided balance combinations

Students' strategies

For Bagus and Sabil who were indicated to have realized the existence of the relationships among the balance combinations, there seemed no struggles to find their first new combination. The other group of Anggi and Gaby, however, did. But finally, after being asked to imagine the combinations of balance on a balance scale, they could have their first proposal. The fragment 5.7 shows Anggi's proposed answer.

Fragment 5.7 Student's strategy to maintain balance

Researcher	How?
Anggi	The number 2 here is 2 medium equals 5 small; which means, if we double it; 4 medium is equal to 10 small ones.. Isn't it?
Researcher	Hmm… are you sure it is right?
Anggi	[mumbling]
Researcher	Why do you think if it is doubled, then it is still balance?
Anggi	Hmm.. **Isn't it balance from here? So, if we add with another balanced combinations**, it is still balance..

It is clearly observable in the fragment 5.6 how Anggi has figured out that doubling a balance combination will still maintain its balance. Here, she could explain that combining two balance situations will give a new balance combination. Besides doubling, a number of strategies to modify while maintaining balance were revealed during this study, such as, multiplying with constant, adding another balance, adding and subtracting with similar objects, and exchanging positions. Moreover, a quite advanced strategy "substitution" was also revealed. This finding is recorded in a conversation between the researcher and Sabil in fragment 5.8.

Fragment 5.8 Students use substitutions to find a new balance combination

Researcher	How about this? [pointing to $2B = 17K$]
Sabil	That one.. Hmm.. That's from.. number 1 also.. In number 1, it says $1B = 3S + 1K$ So, if $2B$.. the $2B$ is $\boldsymbol{6S}$ and $2K$ So, $2B$.. Hmm..
Bagus	1 medium (1S) is equal to **2-and-a-half small ($\boldsymbol{2.5\ K}$)**.. Right? {referring to equation 2.. $2S = 5K$}

Sabil	**If we multiply it by 6**, then it is 15.. plus the 2 small, it comes to 17 small ones ($17K$)..

Although Sabil and Bagus have not used the terminology ‘substitution’, it is clear from the fragment 5.8 from line 6-to-8 that they have substituted $1S = 2.5K$ into $2B = 6S + 2K$ to find $2B = 15K + 2K = 17K$. The appearance of substitution strategy was not anticipated in our initial HLT. It was expected that this idea might occur in the bartering part 3 (just after this session). The success of Sabil (and Bagus; not in the transcript) to use the idea of substitution showed a rather their advanced understandings of quantitative relationships. This shows a high level of flexibility and the acceptance of lack of closure, which is a big step to understanding of transformation strategy.

Student’s Representations

Issues on the students’ representations in this part are focused into two, that is, the students’ reactions to the uses of letters (fully) and the stage to which the students have treated the letter in their representation. As what have learned in the first part of the bartering marbles, the students might just simply understand the letters as a short-term for objects it represented. Hence, they just mixed up with words and letters. As they were suggested to work with only letters (stimulated with the list of balance), the students seemed to have no struggle. Thus, the researcher was convinced that using only letters for the upcoming representations would not be a problem for students.

The fact that students concluded some steps that can maintain the balance, such as, multiplying, adding, removing, exchanging, and etc. indeed shows an indication of progress views in the letters the students used. Thus, in this step, the students actually did not only treat the letter as a pure object, but having something else to operate. In this stage, the students might have developed a sense of variable.

Concluding remarks

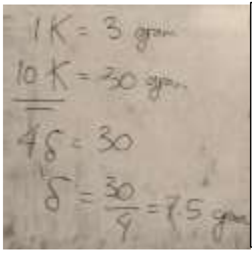
Investigations to find more balance combinations seemed to be challenging for students. Everyone is actively involved in thinking and arguing process during the class. However, there was a tendency that students often employed the same steps to generate more balances, after they were convinced with their initial findings. This brings suggestions to be applied to the second cycle to focus more on the way the students found the balance, rather than finding as many balance combinations as possible.

Part 3 (Finding weights)

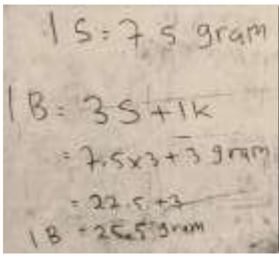
The last part of the balancing activities employed the list of balance combinations that students have found in the first two parts. In this occasion, the students were challenged to find the real weight (in gram) of each size of marbles. Here, the weight of the small marble was informed (3 gram) while the weight of the other 2-sizes remained to be the students' task to find. This activity might become the first in the series of lessons where students employ balancing strategy to find the unknown. In addition, the students' representation will reform into equation in this stage.

Students' strategies

To give you an overview of how the students work in this part, we present you the figure 5.8. Part (a) of the figure shows the work of Anggi and Gaby to find the weight of the medium-size marble, while the (b) is Bagus' and Sabil's work to find the weight of the big-size marble.



$1K = 3\ gram$
$10K = 30gram$
$4S = 30gram$
$S = \frac{30}{4} = 7.5gram$



$1S = 7.5\ gram$

$1B = 3S + 1K$
$\quad = 7.5 \times 3 + 3\ gram$
$\quad = 22.5 + 3$
$1B = 25.5\ gram$

(a) (b)

Figure 5.8 Students' strategy to find the weight of the medium and big marbles

The first group (show in figure 5.8a) decided to employ a balance combination they have found in the second part (which is $4S = 10K$, derived from $2S = 5K$). Later, they substituted the weight of the small size marble into K in the combination of balance. Hence, they found that $4S = 30$. The next, they employed inverse operation to find the S. This idea was likely similar to working backwards they have learned in the first part of the lesson (the secret number). As can be observed, the students' tendency to guess-and-check is completely diminished. They, with no doubt, substituted the value of an unknown into the equation to find the value of the other unknown.

Students' representations

As can be observed from the figure 5.8, in order to find the weight of the medium (*S*) and the big (*B*) size marbles, the students decided to change the value of the small marble (*K*) in their balance combination. Such an action shows that students understood the letter not only as an ordinary object, but one that has value. This understanding of letters both as objects and as values has helped the students to solve the problem. On the one hand, the students can easily manipulate the combination (as they thought it is an object) as well as perform some arithmetic operations (since there is a value in it).

Classroom situations

The problems raised in this activity are open in a way that a single answer can be approached from different ways. This perhaps grew the students' interests in the class, especially during the math congress. The task was really durable for students. Hence, this part would be continued in the next cycle without any modifications.

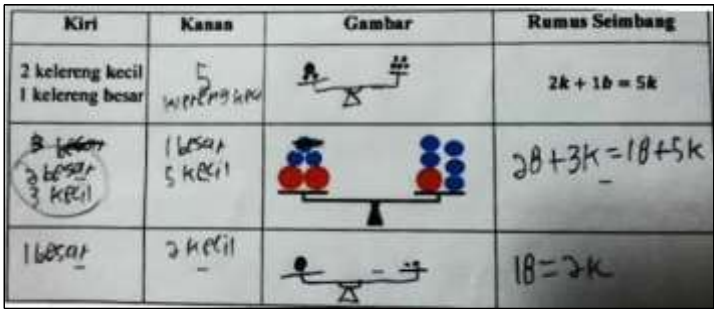
5.2.3.Formative Evaluation

The formative evaluation, in this case is the mid test, was meant to ensure the students' readiness to work with a considerably more formal algebra. The focus of this assessment was on the students' understanding of balance situations, its representations, and its uses to solve for unknowns. To conduct this part, the researcher first gave time for students to solve the problems individually. Afterward, cross-checking and discussions were conducted. In addition to this evaluation test, a problem involving weighing context was also given by the end of this part.

For the purpose of clear reporting, this part is organized following the order of the questions in the test. The focus was whether the items are reasonable to keep, modify, or need to be deleted. The indicator was on how far the students reflected the idea they have learned to solve this item. Students' performances and representations during the test are also presented in brief.

Problem 1

This problem aims to strengthen students' mind about the relationship between objects on a balance scale, pictorial representation, and the so-called balance formula (introduced in the balancing part 1). The task is presented in an incomplete table that students have to fill. An example of student's answers can be observed in figure 5.9.



Note:
Besar = Big (B)
Kecil = Small (S)
Kiri = Left
Kanan = Right
Rumus seimbang = Balance formula

Figure 5.9 An example of student's answer in problem 1

There was almost no problem encountered by the students to solve this problem, except when completing the last row (all empty cells). What the researcher intended was actually only to look at the relations horizontally (the objects, the picture, and the balance formula). However, some of the students also concerned about the vertical relationship. Thus, they tried to make the third combination which is equivalent to the first two combinations. The good thing of the students' views is that the idea of equivalent equations is also involved in this task. This inspired the researcher to modify the question for the cycle 2, as such; the third combination must be equivalent to the first two. Adjustment in the instruction was made to perform this modification.

Problem 2

The second task asked students to compare four different results of weighing (see figure 5.10). The task is to determine one that does not fit the result of the other three.

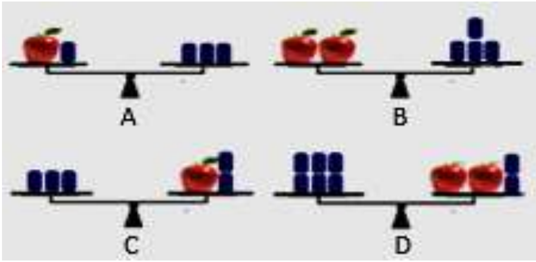


Figure 5.10 Comparing balance in problem 2 of the formative evaluation

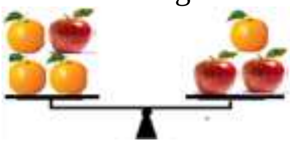
To solve the problem, most of the students tried to solve each option first (determine the number of masses fit with one apple), and then compared the result. One student tried to just move from one picture to another picture. She tried option A and C first, and then found that they were different. Here, she concluded that the wrong result must be C. The idea of either solving or maintaining balance situations is well observed through this question. However, our observer suggested exchanging the option A with C, since some students seemed to start with an assumption that option A is true, and found it correct when they substitute the result to options B and D. The suggestion made sense and thus it was taken to modify the question for the next cycle.

Problem 3

This task provides the students with a picture of weighing result on a balance scale. As observed in the figure 5.11, there has been considerable progress of the students from what they did in the pretest. The figure shows a question and an example of a student's answer (Bagus) when solving the problem.

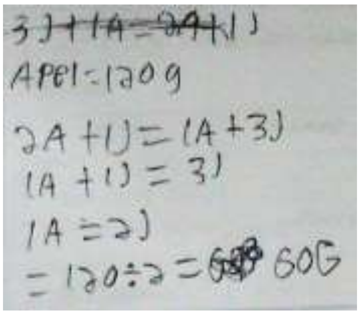
Question:

The picture below shows a result of balancing fruits:



If the weight of an apple is 120 gram, what is the weight of an orange?

Student's answer:



Apple = 120g
$2A + 1J = 1A + 3J$
$1A + 1J = 3J$
$1A = 2J$
$= 120 \div 2 = 60G$

Note: A = *Apel* (APPLE); J=*Jeruk* (ORANGE)

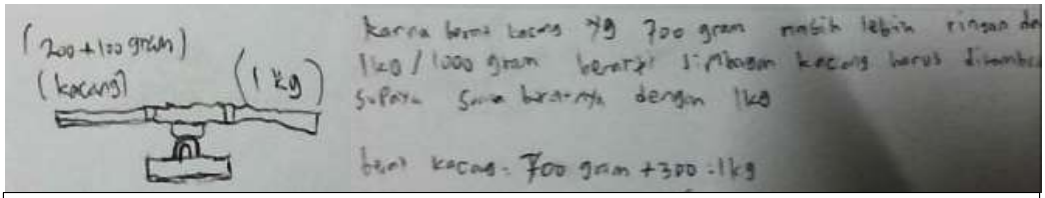
Figure 5.11 Problem 3 and an example of student's strategy to solve the problem

What is interesting from the strategy shown in the figure 5.11 was the movement from each step, for example, from $2A + 1J = 1A + 3J$ into $1A + 1J = 3J$. Here, the student removed $1A$ from each side. When asked to explain his answer, he said that we could throw that away, and it is still balance. This might give a clue where his idea came from. Here, the student presumed that he needed to simplify the situations on the balance scale by reducing the forms while maintaining the balance. With this understanding, this students had successfully been safe from mistakes, that he often made prior to the learning implementation (see figure 5.1). This might give an evident of how the balancing activities have helped students enhance their strategy to solve problems of linear equations.

Zooming out to the class, the students performed different strategies, such as, guessing-and-checking, substituting, and eliminating when solving this strategy. Hence, the researcher believed that this question is good to maintain in the second cycle.

5.2.4.Combining masses

This part was intended to help students to be flexible in combining quantitative relationships. Here, the students were provided with a number of possible masses to combine to weigh a plastic of mung beans. In response to this problem, the students seemed to just easily found the right combination of masses and drew it on a balance scale representation. An example of a student's answer can be observed in figure 5.12.



Since the weight of the beans 700 gram is still lighter than 1 kg/ 1000 gram, we need to add more masses on the arm of the beans, as such; the weight would be the same as 1 kg.
The weight of the bean = 700 gram + 300 = 1kg

Figure 5.12 A student's answer and argument for the combining masses problem

As can be observed from the figure 5.12, the students seemed to only employ basic rules of finding balance to solve this problem. It is likely not observable that this problem requires them to really manipulate the quantities. Furthermore, reflecting to the overall structure of the design, the researcher presumed that this step might not be too essential or it is redundant with what they have learnt in some meetings preceded it. Taking these reasons into account and considering additional activities for another part (will be explained later), the researcher decided to exclude this part from the learning line for the second cycle.

5.2.5. Solving linear equation problems across contexts

This session was split into two parts. The first part, the buying fruits, related weighing context to another aspect, while the second part, the parking rates, involved no weight at all.

Part 1 (Buying Fruits)

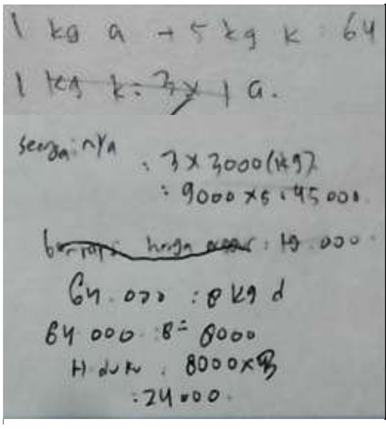
A problem chosen to deliver in this part was intended to bridge the students into a more general use of the concept of balance. In this case, we tried to help the students to generalize the balancing weights into another component balance. Weight was still involved as a stimulus to the balance concept. However, it was not the main thing to work on. Instead, they have to concern about the price of the fruit. The context of the problem is given in figure 5.13, and the task is to determine the price of fruits per unit of weight.

> Umi has just bought 1 kg of grapes and 5 kg of duku with a total price Rp.64.000,-. She said that the price of the duku is very cheap, that is why she bought a lot.
>
> The price of 1 kg grape is three times as much as the price of 1 kg duku. But, she forgot the exact price of each.

Figure 5.13 Buying fruit problem

Students' performance to solve the 'buying fruit' problem

As an overview of a student's performance in this activity, the figure 5.14 is presented. To focus on are the strategy and the representation the student employed.



1kg a + 5 kg k = 64
1kg k = 3× 1a

If
= 3×3000 (1kg)
= 9000×5= 45.000
So, the price of grape = 19.000

64.000 = 8kg d
64.000 : 8 = 8000
Price of Duku = 8000×3 =24.000

Figure 5.14 Student's answer when solving 'buying fruits' problem

It is observable from the student's work that the student tried to perform two strategies. In the first chance, he tried to make guesses-and-checks by first assuming the price of one of the fruits (he did it more than once, but the other trials are not presented in this report). With the strategy, he came into a conclusion (we have no idea how he got it), but then he was unsure about the answer. We presumed that in this phase, the student has experienced a disadvantage of the guess-and-check strategy. Continued working, he tried to perform a seemingly formal way of solving, and finally he found an answer (although, indeed the answer is wrong. The price of the duku is supposed to be the price of the grape).

Looking into the way he represented his answer, our first concern was about the use of letters 'a' and 'k' to represent the price of '***a**nggur* = grape' and 'du**k**u' respectively. In his second-last-row answer, he seemed to just be flexible in translating the symbol he used into the supposed meaning. A source of mistake he made was when interpreting the question. Here, he translated "*the price of 1 kg of grape is three times as much as the price of 1 kg duku*" into "1 kg k = 3×1a". This problem seemed to stimulate the students' algebraic representations and strategies (as shown by the students in figure 5.14). Hence, the researcher was contented to keep the problem for the second cycle.

Part 2 (Parking rates)

This part presents non-weighing context, neither does it relate with the weight. This was to ensure the generality of the balancing approach for students to solve the problems. The students dealt with problems of two parking lot tariffs, the outer (with Rp.2000 in the first hour plus Rp.1000,- for the next hours) and the underground (with Rp.4000 in the first hour plus Rp.500,- for the next hours). A case given in the problem where two people working in different places should pay the same amount. The student's task was to find out how long those two people were in the mall.

The strategies the students employed to solve this problem were nearly the same. They tended to list possibilities and end just when they found the similar tariff for both parking areas. None of the students tried to make equations to work with. This possibility has been anticipated in the HLT. Thus, the class continued to ask the students to think of a more algebraic approach. A problem occurred, especially, when the students need to generalize numbers. This problem is observable in fragment 5.9.

Fragment 5.9 Students' struggles to generalize numbers

Researcher	Suppose it is 10 hours. If it is 10 hours, how much should Ulil pay?
Sabil	2000 plus 9 times 1000 $\{2000 + (9 \times 1000)\}$
Researcher	What about Ruslan?
Sabil	4000 plus 9 times 500 $\{4000 + (4 \times 500)\}$
Researcher	Well. Right? What if 20 hours?
Sabil	**2000 plus 19 times 1000** $\{2000 + (19 \times 500)\}$
Researcher	Here?
All students	**4000 plus 19 times 500** $\{4000 + (19 \times 500)\}$
Researcher	Hmm… If it is n hours?
	(very long hesitations)
Researcher	Why?
Sabil	Minus 1 hour. $n - 1$. $\{2000 + ((n - 1) \times 1000)\}$
Researcher	So, here, how much?
Sabil	4000 plus n … (**hesitations; thinking**) minus 1 (**doubt**)

As can be observed from the fragment 5.9, the teacher actually has tried to help the students to think of generalizing arithmetic numbers with algebra by presenting some examples involving the number patterns. The patterns have been readable by the students (line 2, 4, 7, 9, and 15), but the idea of generalizing has not been conveyed yet (line 11 and 15). This failure indicated the need to a more emphasis on a function of variable to generalize arithmetic patterns. As a reflection, more emphasis on this idea is suggested in the secret number activity (as it should aim). Another possible reason why this problem might occur was the choice of number which is easily traceable for students. Thus, it seemed that they are allowed to go back to their old way of solving, which is easier to conduct for this case. Therefore, we decide to add another similar problem that gives less probability for students to trace manually.

5.2.6. Improving the Learning Line for the Second Cycle

Based on the analyses toward the activities and tasks given in the first cycle, adjustments were made. Some modifications are major, and some others are minor. The minor changes had no impact to the changes in the learning line. They just affected the teacher guide, especially the scaffoldings proposed to teachers to use. Additional trajectories of students' thought are also categorized as minor revisions. Meanwhile, the major change suggested changing in the learning line.

In the first activity, minor changes were made. The first one was on the emphasis on important numbers. This stress was meant to ease students to work and to let more chances of using symbols. The second was to highlight the use of symbols to state generalized numbers in the concluding remark of the activity. The second activity, bartering marbles, would be kept overall, except for a little adjustment in the beginning of the part 2. It was planned to use the students' lists of balance combinations from the part 1 to go on in this activity. However, due to the less accuracy of the tools, the list would be provided. An investigation about the students' understandings of coefficient would be of the concerns from the second part of this activity.

Items tested in the formative evaluation would also remain nearly the same. The proposal change was for the 2nd question to change the order of the option (see figure 5.15). This was due to the observer's analysis that some students tended to make a presumption that the option A (in the context of the question) must be right.

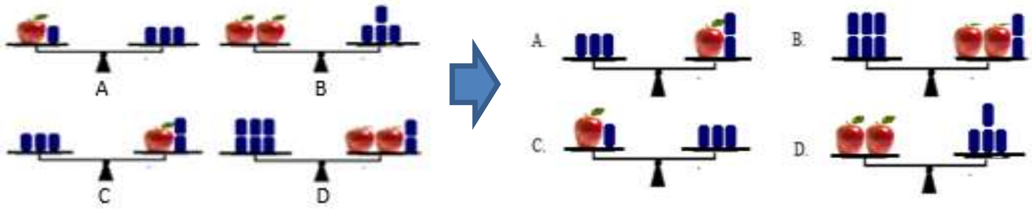


Figure 5.15 A change of the options in question 2 of the formative evaluation

The next activity, combining masses or also called manipulating balance, would not be used in the second cycle. This was due to observations to students' performances during the meeting that showed redundancy in the students' strategies to solve the problem. It was also argued that this activity was a bit out of the learning line. As a replacement, more problems on linear algebra across contexts would be given. This added time allocations for performing the last activity. Thus, it was planned to give an additional problem to students resembling the 'parking rate' problem that will not allow them to list manually.

The overall decisions of either keeping, revising, removing, or replacing activities from the first to the second cycle is presented in table of comparison between the HLT and ALT for the pilot study.

Table 5.1 Comparison between HLT and ALT for cycle 1

Activity	Hypothesized learning events	Observed learning events	Decision for cycle 2
Secret number	**Strategies** - Students use working backwards to trace back the number trick **Representations** - Students misuse equal signs when writing series of guessing-my-number instructions - Students realize the misuse	 - A student employed working backwards, other students only observed relations between their secret numbers and final result - Three students misused the equal sign; another student did not employ any equal sign - Students recognized the	Overall activities were kept with modifications in scafflodings. Using terms important numbers were suggested to encourage students see numbers that can be

Activity	Hypothesized learning events	Observed learning events	Decision for cycle 2
	of equal sign through linguistic approach	misuses after repeated-reading and make sentence what they have pronounced	generalized and to stimulate the use of letters or other symbols in their trick.
	- Students employ letters to state secret numbers in their records	- The use of letters were not observable. However, an indication of using symbols was found. Two students employed words to explain the trick, which could be developed into letter or symbols	
	- Students recognize generalized arithmetics presented in letters in their keytrick	- Students were able to make their own secret number trick and convinced that it would work for any numbers. Letters, however, were not involved.	
Find balance	**Strategies** - Students see concepts of equalities represented in a balance scale, and make sense efforts to find balance	- All students could explain strategies to make balanced combinations of marbles on a balance scale. - Combinations that students found vary due to the less accuracy of the tool	This part is fully used in the cycle 2. However, list of balance formulas should be provided for the upcoming meeting to ensure that the combinations used by students are equivalent. This change has directly been adapted in this pilot study.
	- Students start to think of relationships among combinations they found	- Two students were indicated to try mental weighing showing that they have had a sense of relations among the balanced combinations.	
	Representations - Students show various ways of representing combinations of balance	- Two pairs of students made different representations, words-and-punctuations and words-and-operation signs.	
	- Discussions about the different representations result in a convention of using letters to represent objects in the balance, and then named balance formula	- All students agreed to use letter-and-operation signs combinations to state the balance combinations due to its efficiency of writings. They also agreed to name them balance formulas.	
Maintain balance	**Strategies** - Students realize the existing relationship	- Students found out that the balanced situations could	There seems no essential obstacles found

Activity	Hypothesized learning events	Observed learning events	Decision for cycle 2
	among balance formulas that they can make new balance formulas by combining the existing ones, adding or subtracting each sides with equal things, and exchanging objects on both arms of the balance scale. - Some students may have thought to substitute a certain formula into another formula. **Representations** - Students have no problems to use balance formulas during the meeting since they saw it as objects - Students start to see letters in the balance formulas not only as objects, but ones that have values	be maintained by adding, subtracting, doubling, multiplying, and exchanging positions of objects represented in the balance formulas. - A pair of students have substituted a formula into another formula to find a new balance formula. - Students clearly recognized the letters and symbols represented in balance formulas as objects on balance scale. - Asked to find more balance formulas without physical weighing, students thought of the weights represented in each object represented in the letter.	in the implementation of the activity, the goals are shown; thus, we decide to keep this activity without no changes. For cycle 2, a goal to observe was added, that is, to see students' understandings of coefficient.
Finding weights	**Strategies** - Students select certain combinations from balance formulas, substitute the known values, and perform balance strategy to find the value of the unknowns **Representations** - Students see changes from balance formulas into equations after substituting certain values to formula	- Both pairs of students selected a helpful balance formula, substituted the known value to it, and find the weight of the medium marble. Only a pair of them can find the weight of the big marble; another pair found obstacle to select a balance formula to find the weight of the big marble (no direct formula leading to it). - Students' choice of formula did not really show the regular form of linear equations with one variable. Thus, students might not recognize them	This activity is kept without revision. It is just expected that in the cycle 2, certain groups will choose equations that would better show the changes from balance formulas to equations.

Activity	Hypothesized learning events	Observed learning events	Decision for cycle 2
	- Students recognize equations as both as objects and quantity relationships	as equations - Students used equations to find the weight of two-size-marbles from the weight of of one size marble. This indicates quantity relationships.	
Formative evaluation	**Problem 1** (see figure 5.9) Students recognize relations among objects, figures, and balance formulas given in the table	- All students filled in appropriate expressions given in the first two rows - Three students thought of finding another combination which is equivalent to the other two, which they could not find since the first two rows were not equivalent	Immediate response: adjust to the first 2 rows. Cycle 2: took into account finding another equivalent combination for the last row
	Problem 2 (see figure 5.10) - Some students decide the wrong figure by first solving each problem - Some students directly compare between choices and find the strange result	- Three students solved each option first; another student tried to move from a combination to another combination. - Some students simply presumed the option A is true (which is indeed true) and started working with the option.	Exchange options A and C, as such, the students (if they started to presume that the (new) option A is true), will not find correct answer.
	Problem 3 (see figure 5.11) - Students translate situations in the figure into balance formulas, subsitute the known element, and perform transformation strategy to solve the problem	- Various strategies were performed like guess-and-check, substiutions, and removing equal amounts from both sides. - Two students worked with balance formulas; the other two worked with the figure	The problem was completely used in the second cycle
Combine masses	- Students struggle to find the combination since they may not think of manipulating both arms.	- Students easily solve the problem using their arithmetic senses.	The activity was not applied in the 2nd cycle since it did not show the intended goal.

Activity	Hypothesized learning events	Observed learning events	Decision for cycle 2
Problems accross contexts	**Buying fruit problem**		
	- Students translate the situation in the problem into equations	- Students used letters (abbreviated words) and write relations between two objects (duku and grape) discussed in the story.	This problem was again used in the second cycle
	- To solve the problem, the students can either list possibilities given in the relation or substitute a relation into another relation	- Students first tried to guess-and-check the answer, but found it helpless. Thus, he tried to perform a rather formal way of solving and found an answer.	
	Parking rates		
	- Students list the possible answers and found the problem - When asked to perform a rather algebraic way, students may see patterns of the two payment tarriffs, generalize the pattern into algebra expression, and then solve the problem.	- All students listed possibilities of answers and find the expected answer. - Students found difficulties to generalize the arithmetic patterns within the charging methods. Although, with helps, they finally arrived into generalized arithmetic forms, students seemed to not understand the generalizing process yet.	We decided to add a special problem for introducing ways to generalizing arithmetical operations, replacing the combining masses problem.

5.3. Teaching Experiment (Cycle 2)

The learning implementation in the cycle 2 followed the revised learning line reflected from the happenings in the field test in the cycle 1. The class was ran by a regular mathematics teacher, and participated by students in one complete class. The experiment lasted in 5 days, excluding a posttest afterward. Small technical problems occurred during this cycle was caused by some school scheduled agendas that cut the implementation of this lesson twice for nearly a week each. These considerably long breaks required some recalls for activities that were mostly related.

The students in the class were divided into groups consisting of 4-to-5 students each. One group (group 1) was chosen as a focus group, the progress of which will dominate the report in this part. Classroom data and performances of other students would also be explained supporting the data from the focus group. The focus group consisted of four students whom have been interviewed just before the teaching experiment was conducted. The ability of the students in the focus group varied (observed through the test and interview session results), and was believed to represent the whole class characteristics (based on the teacher interview and classroom observation results). Initially, of the four students, one (Ria) was really rigid to formal mathematics and denied the use of informal way. Another student (Amel) put much relies on informal strategies, like a guess-and-check. Meanwhile, the other two (Badri and Aldi) were considered average students.

5.3.1. Secret Number

Analysis of the cycle 1 suggested little changes from what have been conducted in the cycle. There were two parts of this activity; the first one was playing and exploring a like guess-and-check game, and the second one is making tricks for the game. This activity concerned about the students' initial uses of algebraic elements, such as, symbols and equal signs for arithmetic purposes. Apart from it, a brief look at students' strategy to solve the problem perhaps also gives insights to strategies that students might have been able to perform.

In general, this strategy went differently from what was planned due to a slip up by the teacher when trying to explore the instructions provided for the game. In this case, she missed an important step (stopping and guessing students' secret numbers after the 4^{th} instruction; this was meant to emphasize the first important number), and swapped the last two steps; as such the final result (F) was the secret number itself, which unintentionally lost the stress for the 2^{nd} important number (check the lesson plan meeting 1 in the attachment; and the last paragraph of secret number cycle 1). As consequences, the task became less challenging and the emphasis of important numbers became weird to ask. As a solution, additional activity entitled learning to make secret number game was shortly conducted in the beginning of the upcoming meeting (see additional activity in meeting 2 in the attachment).

Students' strategies

A strategy shown by the students during this activity was an inverse operation. This strategy is observable in the students' typical way to make a secret number game trick (as shown in figure 5.16). In the figure, the students realized that to turn the numbers back to its origin (after some operations), they need to inverse the operation they have performed. The students' arithmetic understandings probably have a great influence in the proposal of this strategy.

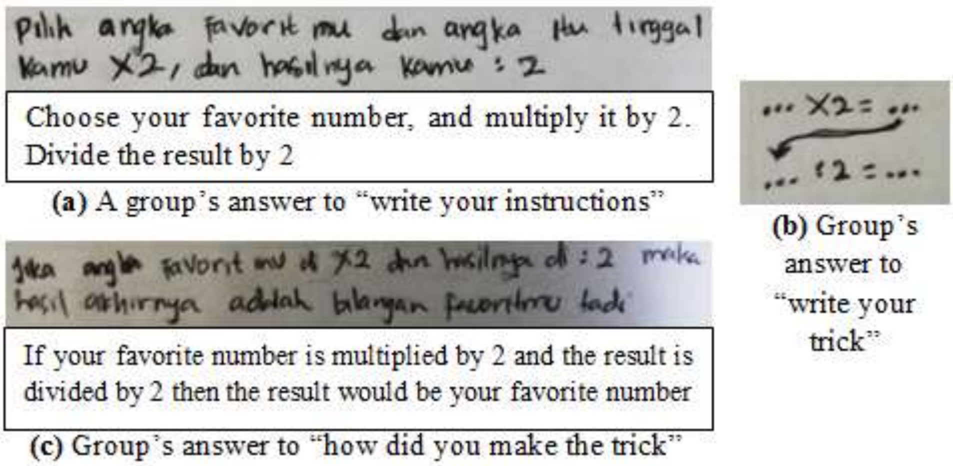


Figure 5.16 Students' typical strategy to create their own 'secret number' trick

Working backwards, as appeared in the first cycle, was not shown during the meeting. This might be caused by the similar numbers appeared in the origin (secret number) and the final answer, which stimulated the students to think of the inverse operation.

Students' representations

Despite the students' strategy, the researcher also puts concerns about the students' representations, which might support their understanding of algebraic representations, such as the use of equal signs and symbols.

The use of equal sign

Like what was found in the first cycle, all students, except one (in figure 5.17b), seemed to misuse the equal sign when recording the series of operations (see figure 5.17a). In the beginning of the math congress, they did not even realize that what they have written was indeed incorrect. A student argued that the equal sign should have led to the results of operations. In reaction to it, the teacher invited the students to pronounce the first two operations separated by the first equal sign carefully and loudly, for instance, $18 + 9 = 27 - 3$ (from figure 5.17a). After a number of repetitions, some students began to feel strange and realized the mistake they have made. They then agreed that a correct way of stating it was, for example, a downward writing as shown in the figure 5.17b.

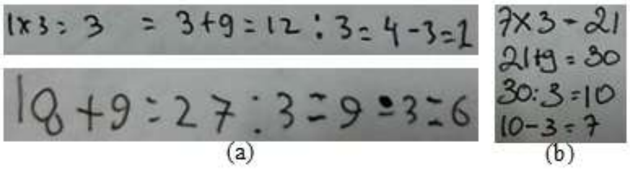
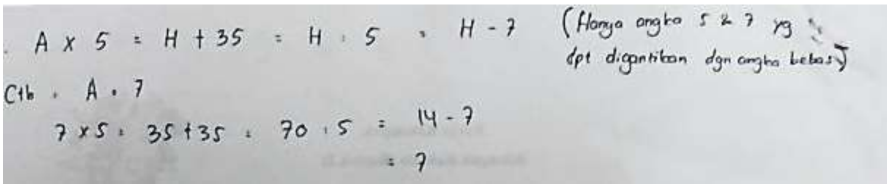


Figure 5.17 Students' records of arithmetic instructions

The fact that students understood equal sign as a sign of results of certain operations might come from their experience in working with arithmetic. This difficulty (according to Kieran, 2007) is common and tends to avoid students from performing the formal strategy in their initial algebra learning. The secret number activity helps students to treat the equal sign more as a relation rather than as an indication of results. Such an understanding might be helpful for students to later understand the existence of non-numbers element in an equation.

The use of symbols

Another task given in this session asked the students to create and explain their trick of secret number games. From this task, the students' ways of writing their secret number in their trick formula became a focus. To ease the analysis, examples of a group's answer is provided in the figure 5.18.



$A \times 5 = H + 35 = H \div 5 = H - 7$ For example: $A = 7$ $\mathbf{7} \times 5 = 35 + 35 = 70 \div 5 = 14 - 7 = \mathbf{7}$	(Only 5 & 7 can be substituted with any numbers)

Figure 5.18 Students' uses of symbols in 'secret number' activity

Apart from the mistake in their uses of equal sign (in figure 5.18), it is observable that the students have decided to use letters to state their secret number. The '*A*' in their representation might stand for '*Angka*' which means 'number', and the '*H*' stands for '*Hasil*' which means 'result'. In the upright corner, the students added explanation that perhaps indicates their awareness of generalized number. However, there was no further explanation about this representation during the class. Otherwise, the students probably just treat the number simply as a label.

Another evident that is probably more powerful to confirm the students' awareness of the existing of generalized numbers is shown in fragment 5.10. The conversation occurred in the focus group. Although they did not involve any algebraic symbols (or letters), they seemed to realize that the trick they have made is applicable for any numbers (line 84 and 90).

Fragment 5.10 Students realized the generalized numbers

1	Group 1	(They tried their trick and found that it worked)
2	Amel	**Would you try other numbers? Up to you. Any numbers**.
3		(look convinced; she challenged her friends)
4	Aldi	How many instructions are there?
5	Badri	Four. 1, 2, 3, 4.
6	Badri	(Recheck with another number)
7	Teacher	Would it be right if… if the secret number is not two?
8	Amel	**Yes. Correct**. Let's try.
9	Badri	(Try with 3)
10	Amel	Try zero. The easy way usually is 0.

Concluding remarks

As briefly explained in the beginning of this session, there was incident in the beginning of the session. This incident confused not only the students but also the researcher and the teacher of what to do. However, after some discussions, the class can be brought back normal.

However, in general, the students were still enthusiastic with the game and felt challenged to find out how the trick worked out. The goals of the activity were all observable, though with some differences with what we had in the cycle 1. Overall, the researcher is contented with the value of this activity as a good way to discuss the students' misunderstandings of equal signs or to stimulate the students' uses of symbols for arithmetic purposes.

5.3.2.Bartering Marbles

As it was on the first cycle, the bartering marble activity was divided into three parts. In the first and the second part, the students explored equalities in a balance scale through physical and mental weighing of marbles, respectively. In the third part, they were dealing with equations to solve.

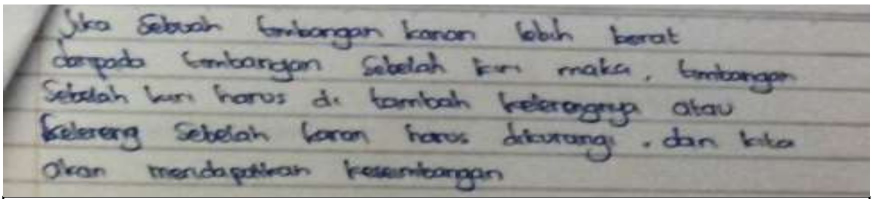
Part 1 (Finding balance)

As explained in chapter 4, the first activity of the bartering marbles provides the students with experiences in exploring concepts of equalities within a balance scale. The class started with a discussion about a problem of bartering three different size marbles. In the discussion, the teacher proposed weights as a standard to do the barter, as such, the combination of marbles might only be allowed to barter with another combination that has equal weights. A balance scale and a number of marbles were provided for every group. They were working with the tools to find as many barter combinations as possible (see figure 5.19). The purpose of this activity was to let the students experience efforts to reach a balance condition as well as to stimulate them to recognize quantitative relationships among the combinations.

Figure 5.19 Illustration of the students' working with balance scales

Students' strategy

Similar to what was found in the first cycle, the students could easily recognize basic rules to balance a balance scale (see figure 20). This rule was also discussed in a simple question-and-answer from teacher to students. The step perhaps would be useful for them to reflect on when working with equalities in mathematics. Essentially, the rule has implied relationships among quantities (in this case, weight) of objects on the right and the left arms of a balance scale. However, it did not seem that the students were aware of it during this stage.



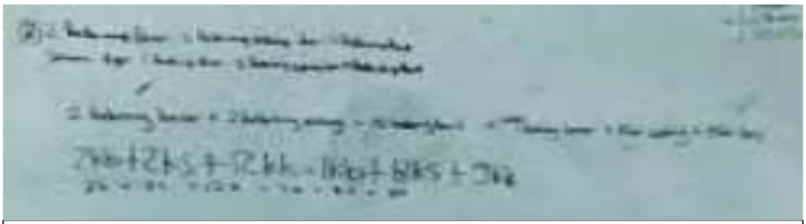
If a balance scale **is heavier on its right arm** than on its left, **then** we have to **add more marbles to its left**, **or** we must **take away some** marbles **from its right arm**. Next, we will find it balance.

Figure 5.20 Student’s easily recognize basic rules in balancing

Unlike the cycle 1, none of the students seemed to conjecture about the quantitative relationships among the combinations of balance. As they were asked to find more and more balance scale, they just continued playing with their balance scale, measuring, observing, and adding their balance combination lists. The students’ non-awareness of the relationships among combinations of balance was probably due to the limited time for them to explore the balance. This might not be a big problem since they would still work with the problems in next session, where they would be forced to think of such relationships.

Students’ representation

The task of this part asked students to write down the combination of balances they have found from the activity of balancing. This task aims to see their initial representations that later would be developed into algebraic equations. Various ways of representations appeared during the activity (see figure 5.21). Therefore, the teacher managed to conduct a classroom discussion on how to best write the representation.



2 big marbles, 2 medium marbles, and 12 small marbles is equal to 1 big marble, 8 medium marbles, and 9 small marbles

2 big marbles + 2 medium marbles + 12 small marbles = 1 big marble +8 medium marbles + 9 small marbles

$2\,kb\ +\ 2\,ks +\ 12\,kk\ =\ 1kb\ +\ 8\,ks\ +\ 9kk$

$2b + 2s + 12k = 1b + 8s + 9k$

Note:

kb = **k**elereng **b**esar (big marble)	**b** = **b**esar (big)
ks = **k**elereng **s**edang (medium marble)	**s** = **s**edang (medium)
kk = **k**elereng **k**ecil (small marble)	**k** = **k**ecil (small)

Figure 5.21 Students' initial representations (from sentences to letters)

The figure 5.21 shows how the students' ways of listing progressed during the math congress. Starting the discussion, the teacher first asked a group who wrote the longest (as she observed during the exploration session) to write their answer on whiteboard. Afterward, she invited other students to comment and suggest a better way of writing the combinations. Words like simpler, shorter, and more efficient way of writing were employed by the teacher during this session. In the end of the discussion, the class agreed to employ letters (as symbols for certain objects) instead of the long sentence to record the combination of balance.

Although involving letters instead of words or sentences, such a way of representing objects is easily accepted by the students. In other words, the students simply recognized the *b*, *s,* and *k*, for example, as the representation of big, medium, and small marbles consecutively. This understanding, later when we have developed the representations into equations, would help the students to associate letters in an equation to certain objects. Thus, they would be helped to treat the equations as real objects rather than only as mathematical objects.

One thing that the teacher might have missed to discuss in this session is the why of the uses of symbols, for example, the plus and the equal signs, in the representation.

However, there was no question, neither complaint about the uses of those symbols. This might show their understanding to those symbols' functions and meanings for the purpose of the listing.

Part 2 (Maintaining balance)

In the second part, the students continued to find more balance combinations, but without utilizing physical tools anymore. The purpose of this activity was to encourage the students to think of quantitative relationships among the objects. A list of balance combinations was provided as a starting point to explore more combinations (see figure 5.22). The main mathematical goal intended in this activity was the concept of equivalence.

DAFTAR KOMBINASI SEIMBANG

1. $1\boldsymbol{B} = 3\boldsymbol{S} + 1\boldsymbol{K}$
2. $2\boldsymbol{S} = 5\boldsymbol{K}$
3. $3\boldsymbol{B} = 7\boldsymbol{S} + 13\boldsymbol{K}$
4. $4\boldsymbol{K} + 1\boldsymbol{B} = 5\boldsymbol{S}$
5. $3\boldsymbol{S} + 3\boldsymbol{K} = 1\boldsymbol{S} + 8\boldsymbol{K}$
6. $5\boldsymbol{K} + 1\boldsymbol{B} = 1\boldsymbol{B} + 2\boldsymbol{S}$

Figure 5.22 List of balance combinations and an example of a group's answer
(B = *Besar* 'Big', S = *Sedang* 'Medium', K = *Kecil* 'Small')

Generally, the class went well, although the students looked confused of what to do in the beginning of their group's work, which might due to the teacher's explanation which was rather unclear. Thus, to help the students understand the lesson, the teacher together with the observer and the researcher went around the class to ensure that the task's instructions were understandable.

Students' strategies

In order to find more balances, some students found struggles, either to think of the way to find the new balance combinations or to give arguments for their answer. Such a difficulty was also experienced by students in the focus group. Fragment 5.11 provides an illustration of how the focus students discuss their new balance combination to justify whether it is indeed correct (balance). Prior to the conversation in the fragment, the group found out a new combination $9k + 2b = 1b + 7s$ by adding objects in the 4th and the 5th combination from the provided list of balance.

Fragment 5.11 Students' struggles to justify a new balance combination

Teacher	Why did you add it?
Teacher	Why do you think it is balance?
Teacher	Try to imagine.. try to think of the marbles and the balance scale
Amel	**I cannot imagine, I don't know the weight**.
Aldi	(pointed at certain combinations)
Teacher	How? How Aldi?
Aldi	This one is too many, this one is less
Amel	What?
Aldi	This one is too many. It should not be $9k$ and $7s$
Amel	**But this one is $\mathbf{2b}$**, (while here is only) $1b$.. Aaaaah...
Aldi	Let's assume this $1b$ is 5, 1b is 5
Amel	No, it couldn't
Aldi	**$\mathbf{5k}$?**
Amel	Couldn't. **Here $\mathbf{1b}$ is $\mathbf{3s}$** $(+1k)$ [pointing to combination 1 in the provided list]

If we read a student's statement in the fragment 5.11 line 5, the student seemed to think that she would not be able to make a new relation, unless they knew the real weight. She even presumed that mental weighing (by imagining the activity of balancing) was not doable as they did not have any idea about the weight. However, her statement (in the line 5) perhaps might not be enough to give a judgment that she did not think of the quantitative relation yet. As we look further to her argument (in line 11 and 15) for the correctness of the new combination, she has considered the value that each form (in this case $9k + 2b = 1b + 7s$; $1b = 3k$ and $1b = 3s + 1k$) represents. This indicated that the student has thought about the quantitative relationships among objects in the balance combination. The discussion continued until the teacher invited the class to have a math congress.

In the math congress, the teacher asked students classically to mention any strategies they have found to maintain the balance. This session is illustrated in the fragment 5.12. As clearly mentioned, at least, the students in the class have gained the notions of equivalence under multiplication, subtraction, addition, and division.

Fragment 5.12 Students' ways to find more balances

Teacher	Now, I would like to ask. The question is easy. Please the others, be silent.
Teacher	How can you find more balance combinations?
Darma	**Multiplying**.. multiply them by 2.
Teacher	What else Daffa?
Daffa	**Subtracting** (removing). But only with **the same thing**, like, a medium with another medium, subtracts.
Bimo	**Dividing**..
Amel	Combining.. we can make new combinations using the previous combinations.
Teacher	What should we do?
Amel	Combining them [moving her hands like gathering things].. combining them.. **adding them**.

Students' understanding of those strategies to maintain balances was confirmed by a number of new combinations they have found by the end of the lesson (see figure 5.23). Discussions with students during their group works also supported this claim. Reasons like "because I know that these 2 combinations have been balance, so when I combined them, the result will still be balance" are common.

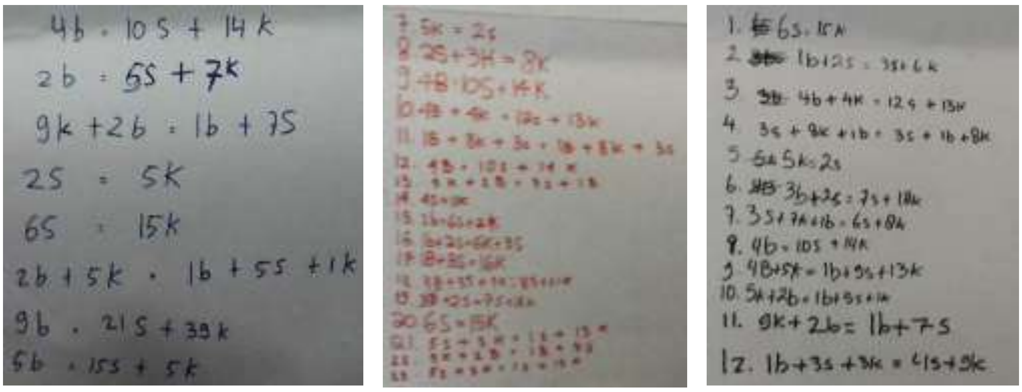


Figure 5.23 New combinations of balance obtained by the students
(B = *Besar* 'Big', S = *Sedang* 'Medium', K = *Kecil* 'Small')

Students' representations

The first remark in related to the students' understandings of the representations is that there seemed to be no difficulty at all for students to recognize what are represented in the lists of combinations. As what they have discussed together in the previous part of the bartering marbles, they easily identified and treated the letters as objects, i.e. different kinds of marbles.

The second remark is, when the students began to work, they would be forced to make relations between representations. Here, they were required to think of value or other things in the representations which are comparable. In this case, the students will consider the weight of the objects (due to the context of balancing). In that manner, the students needed to treat the representation not simply as an ordinary object, but as one that has values (weights). In this sense, the idea of letters as a representation of quantities started to emerge.

Although the students presumably have begun to assume the letters as quantities, they might have no idea of coefficient yet. Thus, at this stage, the students probably just saw, for instance $2b$, as a symbol for two big marbles (objects) or the weight of two big marbles. There is no sufficient evident yet (and probably because it did not happen yet) that the students understand the $2b$ as $2 \times b$ or 2 times the weight of a big marble. Later in the upcoming meeting, this idea was perhaps to develop.

Part 3 (Finding weights)

Combinations of balance that the students have found during the first two parts of the bartering marble activity were employed in this last part. Given the information about the weight of the small marble, the students were allowed to choose any of the balance combinations that might help them to find the weight of the other two kinds of marbles. Here, the symbol (letters) within the list of balance combinations is explicitly related to a certain quantity. Thus, the letters will not only stand as representations of objects, but also values, quantities, or in this case the weights of the marbles. The work was organized in pairs.

The implementation of this part in the second cycle seemed to be very condensed due to time constraints. So, there was almost no time allocated for a classroom discussion. As a solution, the teacher and the researcher decided to only invite a pair of students to come to the whiteboard and explain their answer to the class. The other students were asked to give comments on it (indeed none of them asked, until the class was over). Thus, in this analysis, we would only focus to the students' written works to solve the problem.

Students' strategies

There were at least three steps that the students should do to complete the task (finding the weight of the medium and the large marbles), such as, 1) selecting appropriate combinations (from the list of balance combinations) to work on, 2) substituting the known value to the combination, and 3) finding the value of the unknowns. Overall, the students performed quite well in those three steps (as can be observed in figure 5.24). The existing errors obtained in the students' works might just due to their less thoroughness when working (figure 5.24 right; missed-substituting).

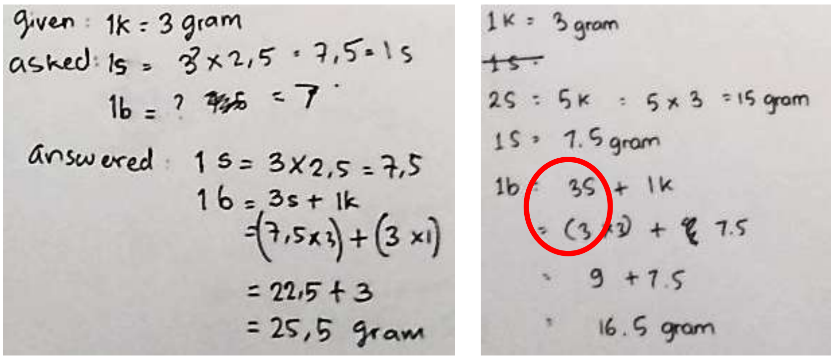


Figure 5.24 An example of student's answer
k = ***kecil*** (small), s = ***sedang*** (medium), b = ***besar*** (big)

The process of selecting combinations to work with in the first step required students to identify which of the combinations would be helpful for them to solve the problems. In this case, they, at least began to work with a combination that related only two objects; one of them must be the small marbles (the weight is known). This process involved the students' ability in terms of acceptance of lack of closure. Based on an observation to the class' works, it was found that the students in general could find the helpful relation. Most of them chose to start with the $2s = 5k$ to first find the s, and then continued with $1b = 3s + 1k$ to find the value of b.

Students' representations

In order to perform the 2^{nd} and the 3^{rd} step, the students' understandings of algebraic representations played its role. When the students have to substitute the weight into the representation, they must have known that the letter involved in the combination has a value, which they indeed work with instead of with the object itself. In addition, these steps also strengthen the students' insights on the coefficients that probably have not developed in the previous stage. For instance, when they substituted the value of s (7.5 gram) and k (3 gram) into the combination $1b = 3s + 1k$ to find the weight of the big marble, they need to be able to recognize the $3s$ as $3 \times s$ and $1k$ as $1 \times k$. Furthermore, they may made sense of the role of coefficient in the representation by relating it to their previous view of the symbols as the objects and the values within them.

It was actually expected that this last part of the balancing activity could show to students a clear form of linear equations with one variable. However, if we observed from their answers and performances here and the following parts, it seemed that the students did not really see that they indeed have been working with the linear equations with one variable. This might be due to the combination of balance the students chose to work that led them to one-step linear equation, which is considered the simplest form of equations. From the figure 5.24, for instance, the first linear equation with one variable the students found is $2s = 15$, while afterward, they just directly substituted value and find the unknown.

Therefore, as a recommendation for the future implementation of this part, an additional similar task might be given, but the combinations of balance to choose might be limited. In essence, the combinations should lead students to work with, for example, $2k + 3b = 4b + 1k$, as such, when the students substitute the weight of k (the small marble, 3 gram), they will find $6 + 3b = 4b + 3$ which is a more advanced form of linear equations. In this manner, the students perhaps can fully realize that they work on linear equations, which might strengthen the position of this understanding to reflect on when doing more formal linear equations with one variable.

Concluding remarks

Generally speaking, these three sessions of the bartering marble activities were worth implementing. The tasks really encouraged students to show their understanding (in the context of balance) and their representations. The mathematical goals proposed in the activities were overall observable and well-directing students to the learning lines. The last part, however, might need some improvements to ensure the bridge of the considerably informal level of mathematizations and the more formal ones. Observations during the classroom session (shown in the video and observation sheets) also showed the students' enthusiasm with the tasks. They seemed to enjoy weighing on a balance scale, exploring more and more balance combinations, and then find out the weight of each marble.

However, the researcher and also observer noted some (generally) practical problems during the class sessions. The first remark is on the teacher-students' interactions. In most of the sessions, the teacher seemed to really dominate the classroom discussions. Although the students in some occasions also expressed their opinions, it was usually the teacher posed the questions. As a result, the interaction mostly happened from the teacher (asking) to her students (answering). In other cases, interactions among students were barely observable. The students were rarely asking to their friends, even in the presentation session. Thus, if they have problems, they would just ask the teacher and wait for the response.

The second remark was due to the openness of the tasks, particularly in the first two parts of the bartering marbles activity. As the students were asked to find as many balance combinations as they could, some students probably assumed this task as a competition among groups. Thus, they would hardly be asked to stop working. As a result, some presentations were not really effective since other students still thought of more answers to their task.

The last one which was the teacher's inconsistency in using the terms 'strategy', 'combination', and 'trick'. This was again found in the math congress during the first two parts, as the students were asked to combinations and explain their strategy to make or maintain balances. As a result, some questions probably sounded confusing for children. Thus, they were sometimes seemingly lost in the discussion.

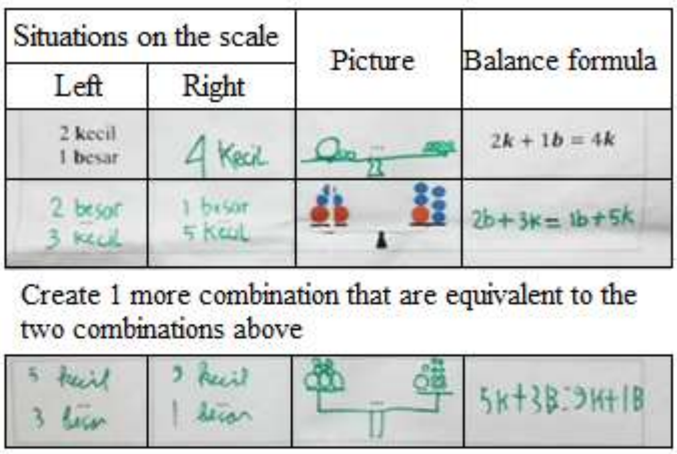
5.3.3.Formative Task

After participating the secret number and the balancing marbles activities, we assumed the students have learned enough prerequisites topics in informal ways. A bridge to the formal was also given in the last part of the balancing activities. However, it was felt important to make sure that the students have really understood those informal materials before they really moved onto the formal ones. Therefore, this activity was performed. The focus of this evaluation would be on the students' recognitions of algebraic representations used in the class and the basic strategies to solve for unknowns in a linear equation with one variable involving balance. There were five items given during this activity, the structure of which would be used as that of the discussion of this session.

This meeting was divided into two parts; a test and a discussion session. The test was organized as an individual test. During this session, both the teacher and the researcher were not allowed to interfere the students' works, unless if the students asked for clarifications for the questions on the test. This session lasted in 25 minutes. Just after the session, the students were discussing the answer. In this occasion, the teacher organized cross-checking (the students swapped their works with their neighbor). The discussion was handled by the teacher.

Problem 1

The first problem aimed to see the students' understanding of the balance formula, since we expected them to only employ and work with such a way of stating equalities for the next activities. In this task, we asked the sudents to see the relation between objects on the balance, pictures, and the representations.



Situations on the scale		Picture	Balance formula
Left	Right		
2 kecil 1 besar	4 Kecil		$2k + 1b = 4k$
2 besar 3 kecil	1 besar 5 Kecil		2b+3k= 1b+5k

Create 1 more combination that are equivalent to the two combinations above

5 kecil 3 besar	9 kecil 1 besar		5k+3B=9k+1B

Figure 5.25 Student's answer in the question 1 of the formative test

The figure 5.25 presented an answer appeared in the classroom discussion. This answe is considered representative to overall students' works, as can be observed from their written works. The works either in their paper or in the math congress indicated a good understanding of the balance formula. This implied that the students perhaps can translate back-and-forth the situations given on the real balance or on the picture from/ into a balance formula. This understanding is hoped to help the students in reflecting equations (later on) with the marbles even when working in a formal stage. Nevertheless, this finding might not be enough yet to justify whether the students have been able to generalize their understandings in other situations but the balance scale. This suggested, for future implementation, adding follow up questions asking them to translate other situations (than marbles) also into a balance formula.

The change from the cycle 1 (question for the last row) did not distruct the students' performance in general. Their considerably good understanding of equivalence in a balance scale helped them to find a new combination of balance easily.

Problem 2

The second item of the formative evaluation (as shown in the figure 5.26a) tried to show to students a disadvantage of relying on the guess-and-check strategy while at the same time promoting the balancing activity. A problem of filling in numbers as they have worked on during the preliminary session was given in a harder-to-guess number. During the working session, only two of total students finally decided to try another strategy than the guess-and-check. Here, they employed the formal strategy they have learned from their previous class to solve the problem. One succeeded, but the other one missed in a step (see figure 5.26b). However, it was clearly shown from their worksheet that all the students have tried to use the guess-and-check, and found it unhelpful. This satisfied our first goal of presenting this activity.

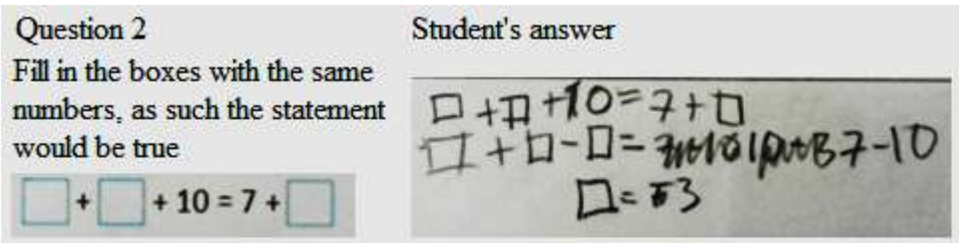


Figure 5.26 Problem 2 in the formative evaluation

The students' failures to solve this problem suggested the students' difficulty to move from solving by observing objects. In other words, they seemed to have seen the object (on balance) first to then consider the value within the things. This might also indicate that the students' understanding, in this phase, was still restricted in the informal level of solving linear equations. In order to promote the balancing strategy to solve this problem, the teacher invited the students to conduct a math congress. Unfortunately, the discussion was seemingly result- and procedure- oriented instead of understanding and reflecting (as shortly transcribed in fragment 5.13).

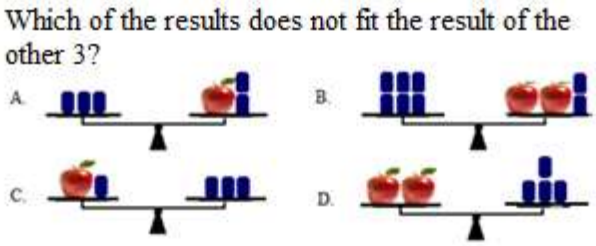
Fragment 5.13 Discussing the problem 2 of the formative evaluation

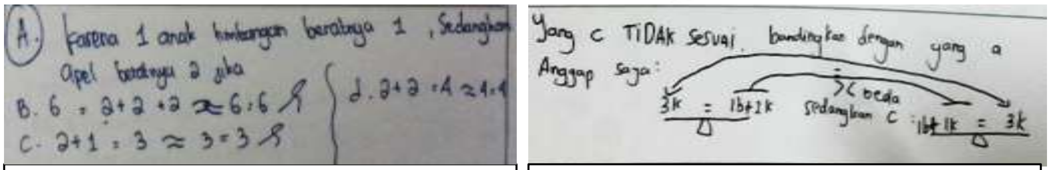
Teacher	Number 2, what makes it hard?
Students	Difficult. Negative (shouted each other)
Teacher	No, we should not think of the negatives. Look. Just now, I said, if on the left is 10,000 and on the right is 10,000, then we took from both 1,000 equally. How's the result?
Students	9000
Teacher	Well, 9000. So, is it still the same?
Students	Yes, the same.
Teacher	Thus, **anything equal things, from which we took equal amounts, the result must still be the same**.

The teacher's scaffoldings (recorded in the fragment 5.13 line 3-5, and 10-11), in our view, were less precise and too guiding. If we observed, the student's difficulty was indeed not in treating the objects in balance, but on reflecting the situations which was not in the context of balance into the context of balance. Thus, the supposed scaffolding should force students to relate the situation in the question to that on the balance. The line 10-11 from the fragment 5.13 also seemed very explicitly telling the students to remove equal things from both sides. In our sense, they should have been able to perform this if they have situated the problem into the context of balancing.

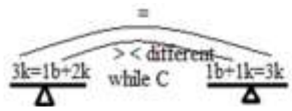
Problem 3

The purpose of the third question in the formative test was to see the students' uses of the idea of equivalences to justify a wrong result of weighing. The strategy that students employ in this process is of the concerns. To begin the analysis, the figure 5.27 that showed the question and some students' answers are presented.





(The answer is) A, because 1 mass weighs 1, while an apple weighs 2. If: B. $6 = 2 + 2 + 2 \approx 6 = 6$ C. $2 + 1 = 3 \approx 3 = 3$ D. $2 + 2 = 4 \approx 4 = 4$	The C IS NOT FIT. Compare with A. Assume:



a b

Figure 5.27 Question 3 of the formative test and strategies of students to solve it

Solving this problem, there were two most common strategies observable in the students' worksheets. The first one was simplifying each of the four options, and then comparing the results. With this strategy, the students could see that in each option, but option A, gave the same result. The second modest strategy was by substituting certain quantities that might belong to the both the apple and the masses (see figure 5.27a). Another student with this strategy gave an explanation of his strategy in a presentation session. The fragment 5.14 and 5.15 present the conversation between the teacher and the student during the session.

Fragment 5.14 A student explained his way of solving the question 3

1	Teacher	Which one did you look first? Yesterday, which one did
2		Aldi look first?
3	Aldi	Looking at? Ones that do not suit
4	Teacher	Yes, the not suit. What did not suit?
5	Aldi	C and D

Aldi	B and C… and D
Teacher	C and D? How's next?
Aldi	They are balance
Teacher	Balance, and then what?
Aldi	In turn, when we see the A, **we know one apple equals 2 masses…** On the left, on the left side of the balance are 3 masses… On the right, there are 2 masses. And one apple is equal to 2 masses. **So, on the right, there are 4 masses indeed**.
Aldi	So, this scale is not balance.

The fragment 5.14, particularly from line 10 to 14, shows overall steps that students have performed. In brief, the student first tried to simplify the balance, and then substituted the result into the four options. From the fragment 5.14 and the figure 5.27a, the student seemed to make a pre-assumption of the weight of the apple and the masses. This possibility was neglected in the fragment 5.15 that shows the continuation of the conversation between the teacher and the student. Here, the student explained how he could find the idea of 1 apple is equal to 2 masses (in the fragment 5.14) or how he made an assumption of the weight of both the apple and the mass.

Fragment 5.15 Student’s strategy to solve question 3 of the formative test

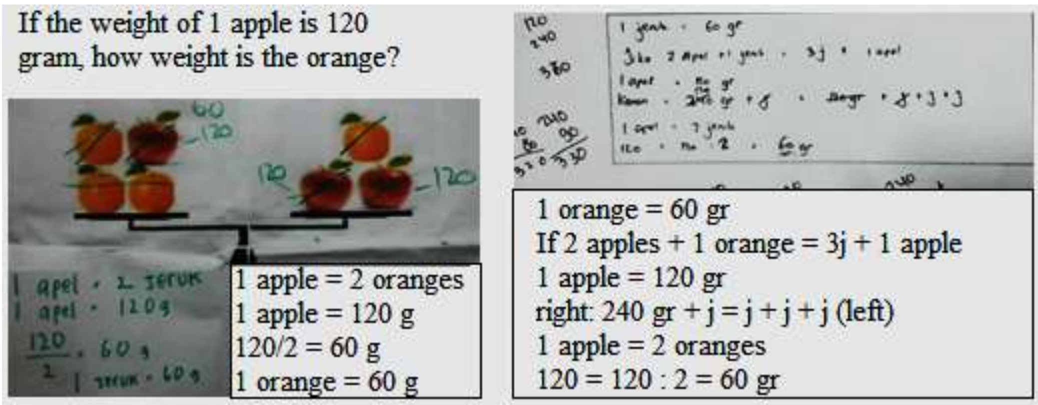
Teacher	Now, we ask Aldi, where did you find that one apple equals two masses? In which part Aldi can find the part A?
Aldi	In option B, C, and D
Teacher	How? How? How can you find it? How?
Aldi	From the total masses. Do you want to write on it?
Teacher	No, no need. How? How?
Aldi	We can see from the total masses
Teacher	Total masses. For example?
Aldi	**We equalize the result. So, these are two masses, and here are 6** [pointing option B]
Aldi	So, **here we have 2 apples**, and these, **the rests are 4 masses. There are 2, so, one apple is equal to 2 masses**. **And we add another apple which is 2 masses**. So, the total masses are 6.
Aldi	So, in B, one apple equals 2 masses
Teacher	Yes. Also with C and D?
Aldi	Yes

It is clear from the fragment 5.15 line 9-10 that the student started to work by simplifying the objects on balance in the option B (see figure 5.27). To solve this problem, the student compared the number of masses in both sides of the balance scale, and then removed the same amount from them. Thus, he found the rest which was 2 apples on one side and 4 masses on the other side; this implied that the weight of 1 apple equals that of 2 masses. Next, he checked his finding by substituting 1 apple with two masses, and found out that both side weighs 6 masses. Using this result, he continued to check the other options, and found that C and D gave the same result.

Another strategy (as shown in figure 5.27b) was a bit different. As can be observed from the figure, the student directly tried to compare the objects on the balance scale of options A and C. Here, he noticed similar objects on the right of the balance A, and the left of the balance C, which implied equal weights. However, objects on the other arm of each balance indicated a different weight. Thus, the student concluded that the wrong answer must be either and not both A or C. Unfortunately; the student directly concluded that the answer is C without considering the options B and D; as what was worried by the observer of in the first cycle. Despite the correctness of the answer, the student's way of thinking employed the so called acceptance of lack of closure, which is, an ability to hold not to solve problem directly. Performing this strategy might become a basic for manipulating algebra forms.

Problem 4

The problem 4 was the first question in the formative test that explicitly asked the students to solve a problem on a balance scale. The aim of this task was to see the student's understandings of balance strategy in the context of balance. Some strategies were observable from both the math congress and the students' written works (see figure 5.28). The figure 5.28a was by a student (Vio) in the math congress, and the figure 5.28b was the work by Amel in her worksheet.



a b

Figure 5.28 Student's answer to problem 4 in the formative test
(Note: **j** = ***Jeruk*** = Orange)

The first student, Vio, solved the problem by first realizing the existing equal weights of objects on the left and right arms of the balance scale; and thus she removed it (she crossed 1 orange and 1 apple from two sides). Afterward, she wrote the remaining objects by words and equal signs (1 apple = 2oranges). Substituting the known information, she could easily solve the problem. Crossing or removing things seemed very simple and helpful here, but it should have started with recognizing balancing strategy that allows denial of equal amounts. Understanding of this strategy perhaps might give sense to a formal strategy that the students have performed (memorized and usually ruined) prior to their participation in this class. In addition to Vio's strategy, during the math congress, the teacher asked students if they had different ways to solve the problem. The fragment 5.16 showed two students that claimed their answer different from the first strategy.

Fragment 5.16 Other strategies to solve problem 4

Teacher	Anyone has a different way? Wawan, how?
Wawan	**This one is 120, this is 120, and this is 120 [writing 120 beside each apple]**
Wawan	**This one is the same [pointing to an apple from both sides].This one [pointing to the other apple on the right] is distributed into these [pointing to the two oranges], because there is one more orange.**
Wawan	So, 1 orange is 60gram.
Teacher	Why did you think that way?

10	Wawan	Don't know. It just came.
11	Teacher	One more idea. Amel, try. A different way.
12	Amel	**I tried one by one**.
13	Teacher	Different from Vio (the first strategy)?
14	Amel	**I tried, 10 not balance. 20 not balance. Until 60.**
15	Teacher	Hmm.. so, you change the orange?
16	Amel	Yes.

If we observe the strategy by Wawan in the fragment 5.16 line 2-7, there might be an impression that the strategy was not mathematically different from that of Vio. However, the teacher probably did not recognize this similarity. He just asked the student to explain his answer without further exploring the mathematical differences of the two strategies. The last strategy (also shown in the fragment 5.16, line 12 and 14) was a guess-and-check by Amel. She explained that she has tried from 10, 20, up to 60 and later found that the 60 satisfied the answer. Again, this chance to promote the balancing strategy was missed by the teacher. Whereas, observing and discussing the different strategies of students are of the aspects of socio-mathematical norms. This might become a suggestion for future cases that the existing differences should be discussed to strengthen the so-called the elegancy or the sophistication of a strategy (Putri, Dolk, & Zulkardi, 2015; Yackel & Cobb, 2006).

Another suggestion for this part is on the student's ways of recording or writing their answer. Although, it was planned that the student would be allowed to freely use their own representations to express their ideas, in this task, the students might have been explicitly introduced or asked to record the answer using symbolic language. To conduct this, the teacher might let students to first show their answer (with their own style of writing), and then, the teacher can propose the more efficient of writing (as discussed in the balancing marble part 1). This might help students to be more accustomed to algebraic representations as well as to a better organized answer.

Problem 5

The last problem in this formative test was actually planned to be presented just after the third balancing marble activity. However, due to time constraint, the researcher decided to include this question in this session. The purpose of the question was to give initial non-marble context of comparing the weight of objects on a balance scale. This highly related with the basic idea of performing the balance strategy. The figure 5.29 shows the question and a strategy appeared in the math congress.

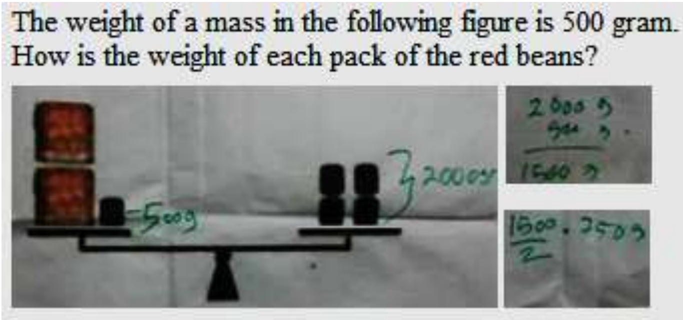


Figure 5.29 Student’s answer to problem 5 of the formative test

The answer shown in the figure 5.29 is actually very common in the student’s written works. Here, the goal of the question seemed partially hidden, especially in making the students aware of the balancing strategy. In their answer, the student seemed to employ no algebra skills. They just used their arithmetic knowledge and intuition to complete the problem. Hence, we might suggest diminishing this activity for future implementations of this series of learning.

Concluding remarks

The class was well organized during the written works. There were almost no interfering interactions between the teacher and her students and among the students. This gave original results of students which might really reflect their current understanding toward the lesson. Various strategies and representations were shown by the students.

Important note is made particularly in the math congress. During the session, the students were rather passive in terms of questioning; but more active in explaining answers. In some of the occasions, both the researcher and the observer noticed a chance for teacher to let the students ask, instead of asking herself. However, because probably also the students did not really show eagerness to ask, the teacher mostly dominate the questioning. Another concern observed during the session was about the missing to make the most of the students’ different ways of both solving problems or displaying answer. Such a difference would be really suggested to make the students think of the advantages of a certain strategy and representation. The last remark was on the time constraints. The two sessions of this activity was planned to organize in one meeting; 30 minutes of working and 40 minutes of discussion. However, the working time lasted longer for about 15 minutes, which made the discussion was less effective in this day. As a result, the researcher decided to allocate more times to re-discuss the formative tests in the upcoming meeting.

Overall, the researcher saw that this middle test has considerably essential role in the learning line, especially because it is in-between the informal and the formal knowledge. This activity ensures students' readiness for a more abstract notion of linear equations, which might be really different from the mathematics they usually deal with.

5.3.4.Solving problems across contexts

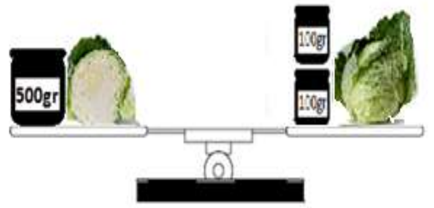
This activity was divided into two parts. In the first part, the students were working with problems that were still related to weights, while in the second part the students were dealing with non-weighing contexts at all. All the questions in the sessions were served in story contexts. As proposed in the cycle 1, an additional problem that is in line with the parking rate problem was provided. This problem gave sub-questions to stimulate the students' uses of algebra instead of listing to solve the parking rate problem.

In our plan, this activity should be conducted in two full meetings. However, since the previous activity took a half of the first allocated meeting for this session, the class was run by utilizing the remaining times. Even so, the class went quicker, and thus the time did not seem to be a problem during the implementation anymore.

Part 1 (Weight-related context)

There were two problems in different contexts presented in part 1, that is, calculating weight and buying fruits. In the first problem, students were explicitly asked to state a balance situation into a mathematical form before starting to solve the problem. This was meant to direct the students to employ mathematical symbols (which might be considered more algebraic way of working) in their problem solving process. Such an instruction was absent in the second problem since the researcher intended to see the students' preferences of the representation to work with by that time.

To give a brief overview before reading the analysis, we present you (in short version) the two problems of this part in figure 5.30. Later in our analysis, the first problem will be mostly related to the students' representation, while the second problem will be discussed more in the students' strategy.



Assuming that the weight of cabbages are the same:

1. Fill in the blanks to state the balance conditions you observed from the figure.
   . . . + . . . = . . . + . . .
2. Continue your work to find the weight of 1 cabbage.

Umi has just bought 2 kg of grapes and 5 kg of duku with a total price Rp.88.000,-. She said that the price of the duku is very cheap, that is why she bought a lot.

The price of 1 kg grape is three times as much as the price of 1 kg duku. But, she forgot the exact price of each.

1. Find out the weight of each 1kg duku and 1 kg grape.
2. If you buy 2 kg grapes and 2 kg dukus from the shop, how much should you pay?

Figure 5.30 Two problems with weight-related context

Student's strategy

There are some obstacles that students found out when they have to deal with the second problem. One of the obstacles is observable in the fragment 5.17. The fragment showed a conversation of two students of the focus group when they were trying to solve the problem.

Fragment 5.17 A student's strategy and struggle

1	Ria	Manual, Del. Let's do it manually.
2	Amel	**Let's just change all into duku**
3	Ria	**This is grape, not duku**
4	Amel	**The price, the price**
5	Amel	So, this one becomes duku
6	Amel	Naah.. So, the price of one duku is equal to 8000
7	Ria	How could it be 8000?
8	Amel	Because 88 over 11 is 8

As you can read in line 2-4 from the fragment 5.17, Ria seemed to have a difficulty to treat the grape as duku. This might imply that the student was still seeing the real objects instead of the value within it. As a result, she could not find any relations to compare. Another student seemed to have fully seen the values as she understood from the question. She knew that what they were talking about was essentially not about the objects, but the price (value) attached to them. Thus, she could eventually make a relation and state a fruit in terms of the other fruit. As the discussion continued (in the fragment 5.18), the reason behind Ria's misunderstanding seemed to reveal.

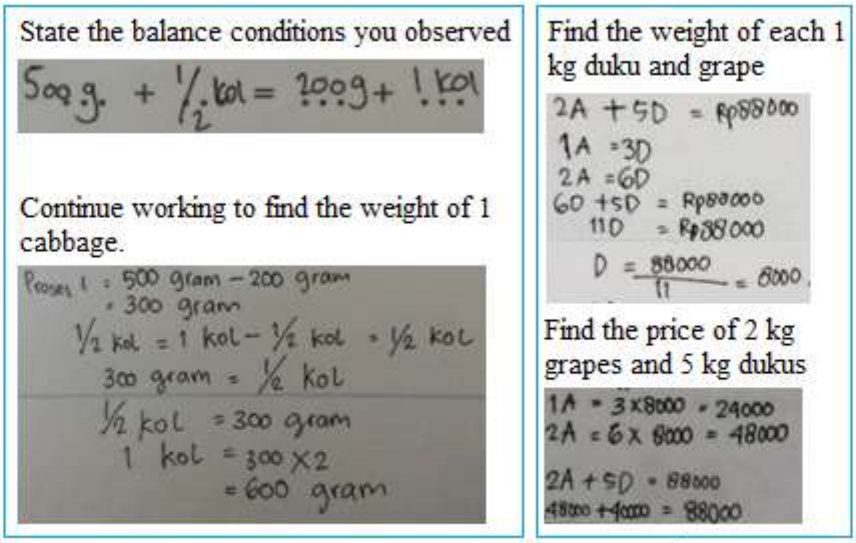
Fragment 5.18 Students misunderstood stories in the question

1	Teacher	**What is asked** is the price or the fruit itself?
2	Ria	**The fruit**
3	Teacher	The fruit or the price?
4	Amel	**The price**. Yes, it was the price
5	Ria	**Yeah, the price**
6	Teacher	So, the fruits, do we still get the grapes and the dukus?
7	Amel	Hmmm.. still.. God's willing..
8	Teacher	Still, because what we equalize was?
9	Ria	The price, right?? Right

Our first indication about the cause of Ria's difficulty is simply due to her less thoroughness to look at the question. As shown in fragment 5.18 (line 2), she first thought that the question asked about the fruit, and not the price. That is why she was hardly being able to think of the quantities. The second one was her ability to treat the fruit both as objects and as objects that have value was not well developed yet. If the second reason became the case, then further reflections toward the depth of the discussion of the bartering marbles (especially the 2^{nd} and the 3^{rd} part) should have been made.

Students' representations

To begin the analysis of the student's algebraic representation, figure 5.31 is first served. The figure shows two answers, each from the calculating weight and buying fruit problem. The answers belong to the same pair of the focus group students.



a b

Figure 5.31 Examples of students' answer
(Note: *Kol* = cabbage; D = **D**uku; A = ***A****nggur* = Grape)

Observing the two answers given in the figure 5.31, the first remark we could see was the way the students wrote the unknown parts. In the figure 5.31a, the students decided to state "*kol* (cabbage)" instead of using letters during their works. Meanwhile, in the figure 5.31b, the same students choose to use letters standing for the objects while they were working, whereas, the two problems were indeed given in the same worksheet. This might explain why many students chose to employ words instead of letters to solve story context algebraic problems. It does not mean that the students were not able to work with letters/ variables, yet they might not feel the need to use it. This might also suggest that this question was less successful to stimulate the students' feel of necessities to use algebra representation to work.

The second remark highlights the students' ability to create/ make algebra representations from a story context (as shown in figure 5.31b) as well as working with the problem. It is observable that the students could understand well the representation they have made and employ it to solve the problem. This indicated that the student's senses or meanings to algebra representations would be really helpful for them to work in a formal level of algebra. The evident we had, however, did not merely show that the students have been able to reflect and understand certain representation which was not from their own production. This might suggest additional problems to include in the future implementation of this work.

Part 2 (Non-weight related contexts)

To conduct this part, the researcher decided to give two almost similar problems; the first part with guidance to work with algebra representation, while the in the second part the students were allowed to decide their own strategy. In the first activity, the question explicitly asked the students to write formula for calculating total profits. Symbols are explicitly suggested like the total profits (*total* ***u****ntung*: ***U***) and the number of stuffs (*banyaknya* ***b****arang:* ***b***) in the question (see attachment).

Student's strategy

Like what was found in the previous part (of the weight-related contexts), the students treated the two similar problems differently. In the first problem, where they were forced to use symbols, they solved problems by neatly involving the symbols. They managed to make equations that told the situations of the question, and then solved it algebraically (more often by utilizing the inverse strategy). However, when moving to the next problem, they went back to manual strategy, in this case, listing possibilities (as shown in figure 5.32).

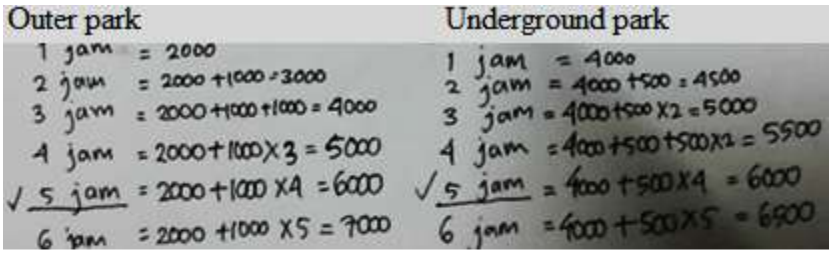


Figure 5.32 Students list possibilities to find the answer
(Note: *Jam* = Hour)

That students might come up with such a strategy shown in the figure 5.32 has been conjectured. A similar finding was also obtained in the cycle 1. At the time, the emphasis of the reflection was just on presenting another task (the first problem of this part) just before this problem that showed them the advantage of working with algebra representations, where they actually had performed quite well in it. However, the durability of listing in this activity probably made the students still convenient to work with it. Hence, we might suggest employing another problem which was in favor with the balancing strategy to substitute this problem in the future.

Student's representation

Unlike when working with the parking rate problem, when solving the problem of selling handicrafts, the students used symbols as they were forced to do. In this occasion, the students would find the algebraic way more beneficial, since the expected answer was not that easy to obtain with listing. As you can see from 5.33, the students first recognized the relation between the total profit and the number of stuffs sold, as they wrote as $U = b \times 700$. At last, he made an equation by first substituting the value of U (94.500; given in the question), before later solve the problem.

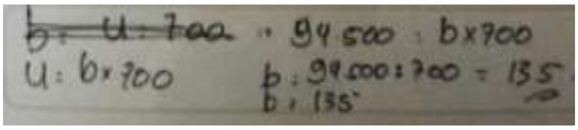


Figure 5.33 Students solve problem using symbols
(Note: U=*total **u**ntung* (total benefits); b=*banyak **b**arang* (the number of stuffs))

As what was concluded in the weighing-related context part, the students' representations when working seemed to not merely reflecting anything but preferences. Any representations the students produced, either in words or letters, were understandable by them and equally helping them to perform algebraic tasks.

Concluding remarks

Overall, the class involved all the students in the investigation. Although the teacher still dominated questionings in the math congress, the students were enthusiastic in explaining and arguing about the answer. As afore explained, certain problems might not really helpful to encourage students to show their algebraic performances. Substituting questions were perhaps proposed to replace those items.

5.4. Remarks After Learning Implementation

In order to complete and well conclude the explanation about the students' progresses in and after the implementation, a test was given post to the classroom sessions. Interviews were also done to all the focus group students, some other non-focus students, and the teacher. The result of those activities is described in this part, in comparison with the students' preliminary knowledge before the learning implementation.

Student's strategies

The most comparable thing related to students' strategy in the initial and last parts was on their flexibility in performing strategies to solve (direct and indirect) problems of linear equations. Various strategies were observable in the students' answer sheets in the posttest, such as, removing the same things from equations, structured guess-and-check, and substitutions. Formal procedures as some students usually performed incorrectly also appeared; but the common mistake is now barely observable. This situation has clearly improved from what the students did before the class, where they only relied on memorizing steps and unorganized guess-and-check.

However, it was observable during the interview after posttest that students' prior learning on algebra (before this implementation) still affected strategies they performed after the learning. In several occasions, some students still mentioned changing signs, inverse the operation, converse the number, and many others indicating their uses of their old understanding. It was rather unclear whether the students have made sense of those ways from what they just learned or just simply still memorize the rules they have learned.

Student's representations

Students' understanding of formal representation of linear equations also gets better and better. If some students prior to this implementation claimed that the variables (or letters) must be positioned in the left arm of the equal sign; they by now understood that positions would not matter. They have also built up insights of the variability of values represented in an algebra representation. In that case, they knew that the variable in an expression can have many values, but (in case of linear equation with one variable) only one that might make the equation true.

An item to see how the students have understood the algebraic representations, given in the posttest can be observed in figure 5.34

Have a look at the following form:

$$3 - 2n = n + 12$$

Circle statements that you thought are true about the form (The answer CAN BE MORE THAN ONE)

a. The form is neither true nor false
b. The form is wrong due to n appearing on the right of equal sign
c. The form is wrong, because n is not a number
d. We have no idea about n
e. n can be changed with any numbers
f. n should be changed with -3

If you have other remarks about the forms, write it here ….

Figure 5.34 An item in the posttest

Of six students we interviewed, all agreed to statement $5a$ arguing that the form can be identified true or false after determining the value of n. None of them agreed with $5b$ including the students who believed it to be true prior to the learning implementation. In a talk to a pilot student, the student compared the equation with objects on a balance scale, where changing positions would not matter to the balanced situations. Thus, the position of n in the equation would not matter. This recognition of the relation between the informal and formal form is an indicator of the students' structure sense (see Jupri, Drijvers, & van den Heuvel-Panhuizen, 2014b).

Regarding the statement 5e, five students agreed that n can actually be changed with any numbers; however, they needed to substitute it with -3 to make the form true. Another student argued that the n can only be changed with -3, otherwise it leads to a wrong result. In our view, both opinions actually have shown the students' notices of the variability of the n in the form. However, the latter one might see n more on the perspective of a solution for an equation.

CHAPTER 6
CONCLUSION AND SUGGESTION

6.1. Conclusions

This part contains essential parts of the results and analyses of the present study which respect to the research question about roles that the balancing activities play in improving the students' understandings of linear equations. There were two main concerns, each answers a subquestion raised in this study. The first part was on how the activities promote students' abilities in performing strategies to solve problems of linear equation with one variable, and the second one relate to the development of students' algebraic representations. As a closing remark, the final proposal of (temporary) local instruction theory developed during this study is also presented.

6.1.1. Students' strategies during the learning implementation

As explained in the analysis part of both cycle 1 and cycle 2 of this study, the students seemed to be more flexible in selecting various strategies when dealing with linear equation problems after participating in this study. It was noticed that prior to the implementation of these lesson series, many students only relied on guess-and-check strategy. Some others have been able to perform the so-called formal strategy, but with common mistakes indicating their less understanding to it.

The activities proposed in this study helped students to make sense of some possible ways of solving problems that make them not too strictly limiting themselves to one fixed strategy. The secret number activity, for instance, has promoted working backwards and inverse operations as a way to trace back series of operations to find an unknown origin number. This perhaps adds students' references and sophistication of strategies to solve linear equation problems (Linsell, 2009).

In the second activity, three sessions of learning with balances have been employed to promote balancing strategies, which is believed to be an approach to algebraic transformation strategy. The activities first tried to build up students' visions of equalities by direct-experiencing and observing situations on a balance scale. From the activity, the students made sense of ways to create and maintain balanced conditions in weighing activities. The results of this observations were later reflected when worked on problems of linear equations.

The first and the second activity of this lesson were conducted and intended to strengthen the students' understanding of linear equations in an informal level. As a bridge to formal, the third activity was first done by checking the students' acceptance of the first two lessons. In this part, reliances on guess-and-check strategies were still observable in some students, but they found it disbenefit to employ, as the problems proposes has managed to do. Thus, the students were encouraged to think of other strategies to deal with linear equations. Indications of balancing strategies have succesfully been performed by some other students. Here, strategies like removing equal things from both sides of balances or equations was performed.

More general situations were presented in the forth and the fifth activity to ensure that students could apply strategies they have learned from the balancing activities in wider contexts. Positive confirmations were identified both from students' written works and their performances in class. In this occasion, some students have reflected on their knowledge of balance to treat objects in equalities. This notion has helped them perform and make sense of the formal strategy. Although the students have been able to perform the formal strategy, some of them seemed to still rely on exhausted manuals, especially if the strategy is still durable. A reflection might be proposed in this case related to the power of context and selection of problems to give in the last two parts of the learning.

6.1.2. Students' understanding of algebraic representations

An important issue in algebra, especially in its early learning is the existence of symbols and ways the students see and treat them. This aspect has been an aspect to concern on during the designing, implementations, and analyses of this study. Aspects of algebraic representations that became central in our design was the use of letters or symbols which is in favor with variables, and other symbols employed in equations like the equal sign and operations signs.

The secret number activity puts more concerns on the students' uses of equal signs and symbols to represent unknowns in a series of arithmetic operations. The students' uses of those components in this meeting were considered their initial representation. The activity revealed misuses of equal signs by all participant students due to their views of the function of equal sign simply as a symbol that indicates result of an operation. Such an understanding led them to neglect the function of equal signs as an indication of equal relations. This misuse of equal signs is often paid lack attention by teachers and has caused bad performance in algebra in general (Castro-Gordillo & Godino, 2014; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). Further, Knuth, Stephens, McNeil, and Alibali (2006) explained that building insights of the relational function of the equal sign is considered a pivotal aspect of understanding algebra. During the math congress of this session, the teacher highlighted this error through linguistic approach of the meaning of the words "*sama dengan*" or "be equal to". Also in this activity, especially observed in the cycle 2, the students employed letters to label a missing number in an arithmetic operation of secret number.

Involvement of letters in arithmetic was also encouraged in this activity, in which, the students used abbreviated letters to label the secret number in the series of instructions of guessing-my-number game due to their insights of the generality of the arithmetic processes they have produced. This insight is viewed by Sfard (1991) as a bridge between arithmetic and algebra.

The key activity to understanding different faces of variables was the bartering marbles. This section consists of three parts with promotes progressive notions of variables from stage to stage. Part 1 of this activity asks students to list all combination of balances they found. Various ways of representing balanced combinations were shown here, starting from a very long sentence, shorter sentence, combinations of words and operation signs, shorter symbols, and finally left the shortest as a combination of letters and operation signs. Promoting the most efficient or in this case the shortest way of stating balance combinations has left students with understandings of the involvement of letters in the so-called balance formula as a representation of objects. This was the first view of students about letters in a formula. Short discussions were also conducted to ensure the students' fully awareness of changing of certain words into mathematical symbols, such as, "*dan* = and' into a plus sign '+' and "*setara dengan* = equivalent with" into an equal sign '='. This notion perhaps prevent students from what is called parsing obstacle in algebra.

The next part presented balancing formulas and asked students to work with instead of other representations. This made the students accustomed to working while relating symbols and objects they stood for. As the students were assigned to combine the balance formulas while maintaining their balance, the students started to shift their treating of letters as objects into values or quantities within the objects. Yet, the students have not used those ideas to solve problems.

As they came in to the last activity, where they needed to perform some operations in terms of finding the unknowns given in the balance formula, the students started to make relations between quantities involved. In this occasion, they might have realized that each letter involved in a representation can tell the value or the quantity of the other letters in the same representation. The students' insights to this relation presented a notion of quantitative relationships symbolized in their representations. This activity was also intended to show to students an initial form of linear equations with one variable that came from their balance formula. However, students seemed not to catch this idea well.

After the series of learning, we observed good understandings of algebra representations of students, particularly ones that they produced. This, however, does not merely indicate neither does it guarantee the students' fluency in treating representations produced by someone else. This case was barely observable during this study, and thus suggested to observe in future studies.

6.1.3. Local instruction theory for learning linear equations with one variable

Based on the analyses of both the first and the second cycle, we come up with a proposal of an instruction theory of learning linear equations with one variable, which can be used by teachers as a guide to set their lesson plans. As explained in the analysis of the cycle 2, in general, the activities involved in the instruction theory (presented in table 5.2) are those we conducted in this study. However, improvements of the instruments and conjectures may still be needed especially for the last two steps in the learning.

Table 6.1. Local instruction theory for learning linear equations with one variable

Activity	Goals	Descriptions
Secret number	- Build relational conceptions of equal signs	- Teacher gives gives instructions of playing guess-my-number game to students and later guess the students' number; the students are

Activity	Goals	Descriptions
		asked to record all arithmetic operations they have performed. - Students show their records of operations in the math congress; misuses of equal signs are highly expected to appear that students have to discuss during the math congress.
	- Promote the use of symbols to state hidden number	- Teacher challenges students to guess other students' secret number; students write the way they found the number. It is expected that certain symbol would be used by students to state the secret number before they found it.
	- Promote the use of symbols as a generalized number	- Students are challenged to make their own secret number instructions and also write the trick to guess the number in their tricks. - The students may try to learn teachers' secret number instructions for some numbers, and see the patterns. They may give marks on some important numbers, like the final result and the secret number itself. - The students make their own proposal of secret numbers following that performed by the teacher.
Balancing activities		
Finding balance	- Observe principle of equalities on a balance scale	- Teacher presents a story of bartering marbles that asks students to combine 3-different-size marbles on a balance scale and find as many balance as they could.
	- Promote students' representations of equalities	- Students are asked to report all the balanced combinations they have found. The way the students present the balanced combinations is seen to be their representation of equalities. - In the math congress, the teacher invites students to discuss the use of equal signs to state the balanced situation.
	- Promote the use of letters to represent objects in a formula	- Teacher organizes a math congress and asks a group with the longest representations to write their combination of balance first. - Teacher invites other students to propose a more efficient way of writing the balanced combination; combination of letters-numbers-operation signs is promoted and later named balance formulas.
Maintaining balance	- Build understanding of equivalences	- Students are asked to find more balance combinations using the list of balance formulas they have found in the previous activity (or provided by teacher). In this sense, the new balance formulas are

Activity	Goals	Descriptions
		equivalent to the provided ones. - Math congress is conducted to check the students' awareness of equivalent concepts (different forms, but still balanced).
	- Facilitate views of using letters to represent values within objects	- In order to be able to combine balance formulas, students should not see letters involved in the balance formula not as objects but as objects that have values (weights). Thus, they indeed combine the weights of the objects.
Finding weights	- Facilitate changes from balance formulas into equations	- Teacher informs the weight of a size of marbles, and then challenges the students to determine the weight of the other two sizes using the balance formulas they have. Here, students will substitute the known weight into their balance formulas; some formulas will change into equations with one variable.
	- Promote views of equations as relationship among object-quantities	- In this sense, the students may see the equation as relating two groups of objects that have equal weights.
	- Use balance strategy to solve for an equation	- Seeing the object-element in the equation, the students may propose an idea of removing or adding equal amounts to both sides of the equations. In this sense, balanced situations are still maintained, and the unknown in the equation can be easier to determine.
Mid evaluation	- Evaluate students' understandings of one-variable linear equations situated on balance scales	- Teacher provides questions related to what students have done up to the current meeting. The questions mainly involved situations on a balance scale. - Math congress is conducted while the students cross-check other friends answers one another.
From balance scale to algebra	- Bridge movement from equations situated on balance scale into more general application of equations	- Students work on problems that did not involve balance scale with balancing approach. - Teacher facilitates discussions by presenting the non-involving-balance-scale situations on a balance representation, and asks if students can work with it.
	- Make mathematical model of situations	- Teacher present several situations and ask students to translate them into mathematical models.
Solving problems accross contexts	- Solve problems any forms of linear equations with one variable	- Teacher gives problems of linear equation with one variable to students involving any applications of the concepts.

6.2. Suggestions

Although findings during the experimental sessions positively confirmed advantages of this study, several aspects remained to be explored, investigated, and revised. These remaining tasks gave our first suggestion to future researchers to study more about the powerfulness of contexts provided in this study. This issue is due to difficulties of some students to apply the idea they have learned in the context of balance into a larger context. Other tasks might also be focused on the last two sessions of this study, which is, the part of vertical mathematization. During this study, we felt the needs to improve the problems we had as such they could strongly be related to balancing activities.

Despite our optimistic views on the benefits of this study, we are fully aware that the involvement of researchers during the study is a bit too much, especially in some essential parts that teachers should have been able to handle themselves in real classrooms. The researchers realized fully realized that the consequences of such actions in terms of reducing the ecological validity of the study. However, the situations in the field seemed to force the researchers to take some intereference; otherwise, the lesson might not run well. We were contented that such occurence would not happen if the teachers fully understood the activities in the design. Thus, communications to teachers must have been built more intensively. Also, we highly suggest to teachers who are willing to conduct this study to carefully learn the provided lesson plans, particularly on the classroom organization matters and scaffoldings during the group work and math congress. It was important to stress that this learning is designed to have a nearly full of students-centered works with minimal guidance of teachers.

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