DESIGN RESEARCH ON MATHEMATICS EDUCATION: SUPPORTING 5th GRADE STUDENTS LEARNING THE INVERSE RELATION BETWEEN MULTIPLICATION AND DIVISION OF FRACTIONS

A THESIS

Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science (M.Sc.)
in
International Master Program on Mathematics Education (IMPoME)
Graduate School of Sriwijaya University
(In Collaboration between Sriwijaya University and Utrecht University)

By:
Septy Sari Yukans
NIM 20102812009

GRADUATE SCHOOL
SRIWIJAYA UNIVERSITY
MAY 2012

APPROVAL PAGE

Research Title	:	Design Research on Mathematics Education: Supporting 5th Grade Students Learning the Inverse Relation between Multiplication and Division of Fractions
Student Name	:	Septy Sari Yukans
Student Number	:	20102812009
Study Program	:	Mathematics Education

Approved by:

Supervisor I,	Supervisor II,
<u>Prof. Dr. Zulkardi, M.I.Kom., M.Sc.</u>	<u>Dr. Yusuf Hartono</u>

Head of Mathematics Education Department,	Director of Graduate School of Sriwijaya University,
<u>Prof. Dr. Zulkardi, M.I.Kom., M.Sc.</u> NIP 19610420 198603 1 002	<u>Prof. Dr. dr. H.M.T. Kamaluddin, M.Sc., SpFK</u> NIP 19520930 198201 1 001

Date of Approval: May 2012

ABSTRACT

This study aimed at supporting students learning the inverse relation between multiplication and division operations and the relation between two division operations involving fractions. This is a design study because it aims to five a local instructional theory for teaching and learning the inverse relation between multiplication and division and two division operations involving fractions. A Hypothetical Learning Trajectory (HLT) is plays an important role as a design and research instrument. It was designed in the phase of preliminary design and tested to four students joining the pilot teaching. The HLT is then revised in the retrospective analysis of the first cycle before it is used in the real teaching experiment with 26 students in the 5th grade of MIN 2 Palembang which had been involved in Pendidikan Matematika Realistik Indonesia (PMRI), or Indonesian Realistic Mathematics Education (RME) project since 2008.

The division problems designed involves the measurement and the partitive division cases which contain measuring activities using ribbons. From solving the measurement division problems, there are only approximately 30% of the students who can relate it with multiplication operations. Therefore, some more activities are added. After doing an activity about generating the multiplication and division equations, students can finally recognize the relation between the two equations. From solving the measurement and the partitive division problems, students recognize the relation between two division problems. In the end, they know that for every quotient number a, b, and c, the is a relation that if $a \div b = c$, then $b \times c = a$, or $a \div c = b$.

Key words: PMRI, RME, division of fractions, inverse relation between multiplication and division equations, inverse relation between two division equations.

Chapter I
Introduction

Many 5th graders of primary schools experiences great difficulties to learn divisions of fractions. Some studies relating to this topic found that the division of fractions was considered as one of the most difficult topics in arithmetic (Greg & Greg, 2007; Zaleta, 2008; Coughlin, 2010). This topic is also considered as the most mechanical, because it involves an algorithm to remember and to solve the divisions (Fendel and Payne, in Tirosh, 2000), like the invert-and-multiply algorithm which is used to solve division problems by multiplying the dividend with the inverse form of the divisor. Although the division of fractions is taught after the multiplication of fractions, some students hardly understood that the division has a relation with the multiplication. The topic of multiplication of fractions was also accompanied by an algorithm which is used by multiplying the two numerators and dividing it with the product of the two denominators.

A previous study conducted by Tirosh (2000) found that there are three main categories of students' mistakes when dividing fractions. The one which often occurs is the algorithmically based mistake. This mistake happens when the algorithm is viewed as a meaningless series of steps, so students may forget some of these steps or change them in ways that lead to errors. For example, instead of inverting the divisor before multiplying it with the divided, some students invert the dividend, or both the dividend and the divisor, or directly divide the numerator of the dividend with the numerator of the divisor over the result of dividing the denominator of the dividend by the denominator of the divisor. To avoid this mistake, it's much better to give students opportunities to really understand what a division of fractions is, instead of giving students with a set of rules.

Students in Indonesia have been learning fractions since they were in the 3rd grade of primary school. According to Indonesian curriculum, students in the 4th grade learn fractions' operations relating to addition and subtraction, and the 5th grade students learn about the multiplication and division of fractions. In the general classroom practices, the four algebraic operations relating to fractions tended to be introduced directly and students generally solved the operations in a more mechanical way. The classroom practices were more emphasize on remembering and using the algorithm, instead of giving students more activities to really understand the division problem, explore it, and find their own strategy to solve it.

The emphasis of using algorithms to solve algebraic operations relating to fractions is also found in some mathematics textbooks used by students in Indonesia. In some mathematics textbooks for the 5th grade of primary school, the topic of division of fractions is initiated by the introduction of the inverse form of a fraction, continued by showing one or two examples of problems and how to solve them using the division algorithm. Some other books provide a story problem relating to division of fractions which is solved by using the help of pictures/models or sometimes by using repeated additions or subtractions. However, in the classroom practices, the use of models or the repeated additions/subtractions to solve the division problems involving fractions rarely occurs.

Generally, the four algebraic operations, the additions, subtractions, multiplication, and divisions, are also considered as four different operations which have no connections each other. When learning about each algebraic operation, the other three operations are neglected. For example, the division operation of fractions was usually taught separately from the multiplication operations, whereas the two operations have a very strong relation.

There were some studies which aim to build students' understanding about the division of fractions. Zaleta (2008) used contextual situations and some concrete objects for students who learn the topic for the first time. This study mainly focused on the measurement division problems and only shows the very informal strategy on how to solve the division problems. In their study, Greg & Greg (2007) separated the division problems into two main cases: the measurement and the partitive divisions. They found that the measurement division could lead students to reinvent the common-denominator algorithm, and the partitive division could lead to the invert-and-multiply algorithm. However, Greg & Greg studied those two flows of vertical mathematization separately. Although there were some studies relating to the two cases of division of fractions, the studies didn't focus on how students could understand the relation between the division of fractions and the multiplication of fractions.

In this present study, 24 students from the 5th grade of MIN 2 Palembang, one of primary schools in Palembang, Indonesia, were participating in the 6 lessons which were designed for the division of fractions. The main goal of this study is to learn how students develop their understanding about the relation between multiplication and division, and the relation between two division problems involving fractions by exploring the measurement division and the partitive division problems. The two cases of division were given in some problems involving measuring activities using ribbons.

Ribbons were chosen as a model for the measuring activities in this study. It's based on the previous study conducted by Shanty (2011), a set of measuring activities with yarns could provoke students to the idea of multiplication involving a fraction and a whole number. Also, in a study conducted by Bulgar, the idea of using ribbons succeeded to promote students understanding of measurement divisions. She named the problems as Holiday Bows. To solve the problems, students found the number of shorter ribbons with a certain length that could be made from a longer ribbon. By exploring some measuring problems relating to measurement and partitive divisions, hopefully in the end of the present study students will know that if there is an equation $a \div b = c$, they can make the relations that $b \times c = a$ and $a \div c = b$.

To achieve that goal, I formulated two general research questions in this present study as follow.

1. *How can students learn the inverse relation between multiplication and division involving fractions by exploring measurement division problems?*
2. *How can students learn to recognize the relation between two division by exploring measurement and partitive division problems involving the same fractions?*

In the process of learning the inverse relation between multiplication and division and two divisions involving fractions, students did some measuring activities to solve some measurement and partitive division problems. A bar model was introduced to help them solve the problems. In order to know how students solve the division problems and how the model can give them help, I formulated two sub research questions as follow.

1. *How do students solve the measurement and partitive division problems?*
2. *How can the bar model help students solve the measurement and partitive division problems?*

Chapter II
Theoretical Framework

2.1 The Division of Fractions

In primary school, the division involving fractions isn't only represented as a division involving two fractions, both the dividend and the divisor are fractions, but it also can be represented as a division problem in which one of the dividend or the divisor is a whole number. Some studies show that all different cases of division involving fractions are difficult for students. The division of fractions is one of the most difficult topics for primary school students (Greg & Greg, 2007; Zaleta, 2008; Coughlin, 2010).

According to Tirosh (2000), there are three errors that primary school students often made when dealing with division of fractions problems. Those are: (1) algorithmically based errors; (2) intuitively based errors; and (3) errors based on formal knowledge. Algorithmically based errors occur as a result of rote memorization of the algorithm. This happens when an algorithm is viewed as a meaningless series of steps, so students may forget some of these steps or change them in ways that lead to errors. Relating to the intuitively based errors, Tirosh (2000) generalizes that there are three misconceptions that primary students often have relating to the division of fractions: (1) the divisor must be a whole number; (b) the divisor must be less than the dividend; and (3) the quotient must be less than the dividend. However, some division problems involving fractions usually contain some problems that may be confusing for students. The divisor of the division problems is not usually a whole number, the divisor isn't always less than the dividend, and the quotient isn't always less than the dividend. Beside those two errors, some students often make mistake when they are doing the more formal ways relating to division of fractions. This category includes incorrect performance due to both limited conception of the notion of fraction and inadequate knowledge related to the properties of the operation.

2.1.1 Two Cases of Division Problems

As well as division problems in whole numbers, division problems involving fractions for primary school students are also divided into two cases.

There are the measurement division and the partitive division cases. Some studies relating to division of fractions differentiate these two cases of division of fractions. These two cases of division have different meanings, quite different informal or preformal strategies to solve, and also different flows of vertical mathematization regarding to come to the more formal step of the topic. By the definition and the strategies students use to solve the problems, according to Zaleta (2006), these two cases of division are distinguished as follows.

a. The Measurement Division of Fractions

Measurement division is also called repeated subtractions. In this case of division of fractions, the total number of group which is going to be shared and the size of each group are known. The unknown is the number of groups which is going to share. Some examples of this kind of division of fractions are as follows.

(1) You have 3 oranges. If each student serving consists of ¾ oranges, how many student servings (or part thereof) do you have?

(2) Alberto is making posters, but his posters only use 2/3 of a sheet of paper. How many of Alberto's posters will those 3 ½ sheets of paper make?

To solve division problems involving measurement division, students have a tendency to solve them by using repeated subtractions or repeated additions. Students make a group of the known size and subtract the total value with the size of the group until the remainder of the total is not able to be subtracted again. The result is the number of groups with a known size which can be made from the total given.

For example, to solve the first example of this case of division of fractions, students are going to make some groups which size is ¾. By using repeated subtractions, they'll find how many times they subtract the total, 3 oranges, by ¾ until the total become zero. $3 - \frac{3}{4} - \frac{3}{4} - \frac{3}{4} - \frac{3}{4} = 0$. The number of groups which size is ¾ is four. Therefore, the result of the division problem is four. In the other hand, by using repeated additions, students can find how many times adding with the ¾ to get exactly 3.

b. The Partitive Division of Fractions

Partitive division is also known as the fair sharing. This case of division of fractions usually provides the total number which is going to be shared and the number of groups which are going to share, whereas, the size of each group is unknown. Some examples of the case of partitive division are as follows.

(1) You have 1 ½ oranges. If this is enough to make 3/5 of an adult serving, how many oranges constitute 1 adult serving?

(2) A group of 4 pupils share 3 loaves of bread. If they are going to share the bread equally, how big is the bread that each pupil gets?

One of some invented strategies that students use to solve this case of division of fractions is by distributing items from the total to each group, one or a few at a time.

For example, to solve the second example of partitive division, some students might have a tendency to distribute the 3 loaves of bread to the 4 pupils equally. Firstly, they are going to share the two loaves for four people equally, so each pupil will get a half of the bread. The remaining loaf is divided into four parts equally and then distributed to the four pupils, such that each pupil will get a quarter more bread. After finishing the distribution of the three loaves of the bread, students are going to count how much bread that each pupil gets. It's a half and a quarter of the bread, or ¾ loaves of bread.

2.1.2 Two Flows of Vertical Mathematizations

When looking at the numbers chosen for both the division cases, there are some differences between the measurement and the partitive division problems, especially when the problems are given to students who learn about this topic for the first time. In the measurement division problems, the dividend and the divisor are chosen in a way that the quotient will be a whole number. In the other hand, the divisor of partitive division problems tends to be a whole number, because it sounds strange to share some number of things for some numbers of groups if the number of groups isn't a whole number. In the case of division of fractions, especially for the partitive division, the dividend should be a fraction, if it isn't, then the problem will be a division problem involving whole numbers. Therefore, the quotient will be another fraction. Looking at these characteristics of the numbers chosen for both division involving fractions cases, it seems difficult to find a division problem written symbolically which can be formulated as story problems for both measurement and partitive divisions.

Because of some differences between the measurement and the partitive division problems, some earlier studies conducted relating to the division of fractions also differentiate the two cases of divisions. Greg & Greg (2007) study about the two cases of the division involving fractions. From their observation with primary school students relating to the measurement and the partitive division, they make two flows of vertical mathematizations as follows.

a. From the Measurement Division to the Common-Denominator Algorithm

   Starting from exploring problems relating to measurement division, students can come to the more formal strategy of using the common-denominator algorithm to solve the division problems. In the common-denominator algorithm, to solve division problems needs to make the denominators of both the dividend and the divisor the same. Then the result of the division problem is just dividing the numerator of the dividend by the numerator of the divisor.

   Here is an example to find the result of $\frac{2}{3} \div \frac{3}{4}$ by using the common-denominator algorithm. $\frac{2}{3} \div \frac{3}{4} = \frac{8}{12} \div \frac{9}{12} = \frac{8}{9}$.

b. From the Partitive Division to the Invert-and-Multiply Algorithm

   In the other hand, starting from exploring problems relating to the partitive division, students can come to use the invert-and-multiply algorithm, which is also more formal strategy to solve the division problems. In this algorithm, the division problem is converted into a multiplication problem with the inverse form of the divisor.

   Here is an example to find the result of $\frac{2}{3} \div \frac{3}{4}$ by using the invert-and-multiply algorithm. $\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$.

These two algorithms aren't easily achieved by primary school students in a short period of time. After exploring some different problems relating to each case of division problems in a long time, students will use some different approaches to solve the problems, and in the end, students will be guided to reinvent the two algorithms.

2.1.3 The Relation Between Multiplication and Division of Fractions

One of the main goals of this current study is to help students make a relation between multiplication and division of fractions. The relation here means that students know that for every division problem involving fractions, $a \div b = c$, means $b \times c = a$.

Previous study relating to multiplication of a fraction with a whole number was conducted by Shanty (2011). She uses the concept of distance and provides students with some rich measuring activities. The end goal of the study was that students would understand the concept of multiplication of a fraction with a whole number. The result of the study was that the fifth grade students who were learning the topic for the first time understand the concept of the multiplication by doing a set of instructional activities for 6 lessons.

Still relating to measuring activities, Bulgar (2003) used measuring activities with ribbons for fourth grade students to learn about the division of fractions, relating to measurement division. The problems given are a set of measurement division problems namely “Holiday Bows”. In the set of division problems, students are asked how many small ribbons (the length of each small ribbon is known) that can be made from a certain length of ribbon (the total length of ribbon which is divided). From the study, there are three means of students’ justification and reasoning to solve the problem. Firstly, students can convert the length of the ribbon (given in meters) into centimeters and do dividing with natural numbers. Secondly, students can really use a measurement unit and do measuring activities to see how many times it fits. The last is by using fractions.

Considering that the measuring activities can be used for the multiplication and the division of fractions, this present study will try to use the measuring activities for the case of the measurement division of fractions in which can also trigger students to solve the problems by using multiplication approach.

2.2 Realistic Mathematics Education

Realistic Mathematics Education (RME) is a theory of mathematics education which has been developed in the Netherlands since 1970s. This theory is strongly influenced by Hans Freudenthal's point of view of mathematics, that 'mathematics as a human activity' (Freudenthal, 1991). According to this point of view, students should not be treated as passive recipients of a ready-made mathematics, but rather than education should guide the students opportunities to discover and reinvent mathematics by doing it themselves.

Relating to the implementation of RME approach into the classroom, Treffers (1991) describes the five tenets of RME: (1) the use of contextual problems; (2) the use of models; (3) the use of students' own creations and contributions; (4) the interactivity; and (5) the intertwinement of various mathematics strands.

Below is the description of the use of five tenets of RME in this present study.

a. The use of contextual problems.

   Contextual problems are used in each activity designed in this present study about the division of fractions. The contextual problems designed are relating to measuring activities using ribbons. The mathematical activity is not started from a formal level, but from a situation that is experientially real for students. Therefore, even for students which are lower achiever students can understand the problems and can use hands-on activities with the real materials provided to solve the problems.

b. The use of models

   The models are used as bridges for mathematization. The models used can be models of the real situation, in which students will be able to solve some real life problems by doing some hands-on activities with the model, explore it, and get the mathematical idea behind it. Some models can also be models for thinking, in which the models can be used as a tool to solve any situational problems. The measuring activities with ribbons can prompt students to make some models, like rectangular bars and number lines.

c. The use of students' own creations and contributions

   The learning processes should give more spaces for students' own creations. In this case of lessons about division of fractions, the lessons should provide students some opportunities to solve the division problems by using their own way. They can create some models to illustrate the division problems or they can use some different ways which they create to solve the division problems.

d. The interactivity

   The lessons designed should provide some opportunities for students to have discussions among students, or students and the teacher. Discussions, cooperation, and evaluations among students and teachers are essential elements in a constructive learning process in which the students' informal strategies are used to attain the formal ones.

e. The intertwinement of various mathematics strands

   The lessons designed should intertwine to various mathematics strands. For example, the case of division of fractions can be intertwined with multiplication of fractions, addition, or subtractions of fractions. The use of some geometrical representation to solve some division problems also can develop students' skills and ability in the domain of geometry and measurement.

2.3 The Emergent Perspective

2.3.1 The Emergent Modeling

Gravemeijer suggested that instead of trying to help students to make connections with ready-made mathematics, students should be given opportunities to construe mathematics in a more bottom-up manner (Gravemeijer, 1999, 2004). This recommendation fits with the idea of emergent modeling.

Gravemeijer elaborated the model *of* and model *for* by identifying four general types of activity as follows (Gravemeijer, 1994).

(1) Situational activity, in which interpretations and solutions depend on the understanding of how to act and to reason in the context.

(2) Referential activity, in which model of the situation involved in the activity is used.

(3) General activity, in which model for more mathematical reasoning appears.
(4) Formal mathematical reasoning, which no longer depends on the use of models of and model for mathematical activity.

In this present study in which some activities and some problems relating to measuring will be given to students, some models to solve the problems may emerge. The measuring problems can be accompanied with some real materials that students can use to act and to reason the contexts by themselves. They can do some hands-on activities like measuring the length of the real materials with length measurement. The problems also can be modeled by making a drawing of the real situations. Relating to measuring problems by using the context involving ribbons, students can draw rectangular bars referring the ribbons, and then do measuring activity in their models with smaller measurement scale. When students can use a model for different cases of mathematical problems involving division of fractions, they are already in the general activity. Relating to this study, students can use a number line as a model for reasoning. In the end, students can use a more formal mathematical reasoning to solve the given problems, like using addition, subtraction, or multiplication of fractions as means to solve a division problem.

2.3.2 The Socio Norms and the Socio-Mathematical Norms

The socio and the socio-mathematical norms are important to be considered before doing the real classroom experience. The socio norms include some norms which are agreed by the teacher and the students relating to how to socially interact between the students and the teacher or among student in the classroom. As an example, a classroom may have a socio norm stating that students need to raise their hands before talking in a whole classroom discussion, student need to ask permissions from the teacher before leaving the class to go to a rest room, etc.

Besides the socio norms, a researcher should also consider the socio-mathematical norms relating to the topic designed. Regarding to the topic of division of fractions, the socio-mathematical norms that should be noticed can be about knowing whether students are allowed to solve some division problems by doing hands-on activities or not, etc.

The socio and the socio-mathematical norms in the classroom can be analyzed from conducting some classroom observations before the real teaching experiments are conducted. Therefore, a researcher can make a preparation regarding to the design for the students.

2.4 The Division of Fractions in Indonesian Curriculum

Fractions have been introduced to Indonesian students since in the second semester of 3rd grade of primary school. Students in this grade start to learn what fraction is, and how to sort fractions from the smallest to biggest or the other way around. In the 4th grade, students start to learn about simple operation relating to addition and subtraction of fractions. In the 5th grade, they start to learn about addition and subtraction within fractions which have different denominator, also how to multiply and to divide fractions.

The following table describes how fractions division fits into the Indonesian curriculum in the second semester of the 5th grade of primary school.

Standard Competence	Basic Competence
Number 5. Using fractions in solving mathematics problems.	5.1 Changing fractions to percentages and decimals and vice versa 5.2 Adding and subtracting fractions **5.3 Multiplying and dividing fractions** 5.4 Using fractions to solve problems involving ratio and scale

Chapter III
Methods

3.1 Research Approach

The present study is a design study, or a design research, which aims to provide or support theories relating to the topic of division of fractions and to design instructional materials relating to the topic for the 5th grade students in one primary school in Indonesia and to use the design in the classroom to support the students and to see the development of understanding of the students in the current topic. A design research is also known as a developmental research because instructional materials are developed. According to Freudenthal (1991) and Gravemeijer (1994), developmental research means to experience the cyclic process of development and the research so consciously, and then to report on it candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience.

There are three phases in the design research (Gravemeijer, 2004; Bakker, 2004). The three phases are respectively a preliminary design, a teaching experiment, and a retrospective analysis. Before conducting the design research, we need to design a Hypothetical Learning Trajectory (HLT). The HLT consists of three components, which are the learning goals, the mathematical activities, and the hypothetical learning processes which are the conjectures of how students' thinking and understanding will evolve in the learning activities. During the three phases of the design research, the HLT has some functions. In the preliminary design, the HLT guides the design of instructional materials which are going to be designed. The HLT is used as a guideline for the teacher and the researcher to do the teaching experiments in the second stage, and in the last stage, it is used to determine what the focus in the analysis is (Bakker, 2004).

3.2 Data Collection

3.2.1 Preparation Phase

In this phase, we do some preparations before conducting the first cycle of the research. We prepare an HLT for six meetings in the 5th grade of MIN 2

Palembang, an Islamic primary school in Palembang, Indonesia. The school has been involved in the *Pendidikan Matematika Realistik Indonesia* (PMRI), the Indonesian Realistic Mathematics Education (RME) since 2009. We will choose a class which consists of 24 heterogeneous students, students' skills and students' level of understandings are different. The age of the students are about 10 to 11 years old and they have learned some topics relating to fractions since they are in the 3rd grade. However, the students haven't ever learned about the division of fractions before. In the present study, the 5th grade students will learn about the division of fractions for the first time.

Besides preparing the HLT, we will also collect some data before conducting the preliminary teaching experiment (first cycle). We will do a classroom observation (in a class in which the research will be conducted), an interview with the mathematics teacher, and a pretest (for students in the class where the research will be conducted).

a. Classroom Observation

The classroom observation will be done in two classes. We will do a classroom observation in the class where the research will be conducted and in the class where we will choose only 6 heterogeneous students who will join the piloting. During the classroom observation, we will investigate the socio or socio-mathematical norms in the classrooms, the nature of the classroom discussions, the strategy that students use to solve some problems relating to the topic learned (that will be about addition, subtraction, or multiplication operations involving fractions), and how the classroom management is (see Appendix 3 for the full classroom observation scheme).

To collect data from the classroom observation, we would like to use a video recorder and field notes. The video recorder will be put in the corner of the classroom to record the general things happening in the classroom. The field notes are used to mark some important things happening in the classroom which are based on the classroom observation scheme.

b. Interview with the Teacher

We will conduct an interview with the teacher. The interview will be about the teacher's experience relating to teaching the topic of division of fractions, some difficulties that she had, how's students attitude towards the topic, the variety of the students, and how she organized the classroom. Besides conducting a semi-structured interview with the teacher (see Appendix 2 for the interview scheme with the teacher), we will also have a discussion with her. The discussion will be about the hypotheses of the suitability of the HLT for the students. It aims to see from the teacher's point of view, whether the students will be able to solve some problems which will be given in the six lessons.

To interview the teacher, we would like to use a video recorder. From the recorder, we would like to make a transcription, and then analyzing the transcript.

c. Pre-Tests

The pretest will be given to all of the 26 students in the 5th grade where the study will be conducted and to the 6 students who will participate to the piloting. The pretest is given after students learn about the multiplication operations involving fractions and before they learn about the division of fractions. We would like to know students' prior knowledge relating to some prerequisites topics before they learn about the division of fractions, and whether they already have their own strategies to solve some simple division problems involving fractions.

The pretest will contain 5 problems which are solved in a paper. Students will be given a set of papers including the problems given and blank spaces to scratch and to write down their solutions (the pretest problems are in Appendix 4).

3.2.2 Preliminary Teaching Experiment (First Cycle)

The preliminary teaching experiment will involve the participation of 6 students from a 5th grade class which is not the class where the real teaching experiment will be conducted. This is the first cycle of the design research. The initial HLT will be tried out here. We will see whether the activities prepared are appropriate to students with different levels (the six students joining the first cycle have different achievements in mathematics).

In this first cycle, I will be the teacher and the researcher at once. There will be a colleague responsible for the video recorder. During the preliminary teaching experiment, data will be collected by using a video recorder, a field note, and the written documents. The video recorder will be used to record all things happening in the small group discussion. Besides teaching, I will also make notes of some remarkable things happening in the small group. The written documents include students' group worksheet to solve the problems given in each meeting to be solved in small groups (2-3 students in each group), individual worksheets which will be given in some meetings after the whole group discussion.

After conducting the preliminary teaching experiment, we will analyze the results, and revise the initial HLT. The revised HLT will be used to the second cycle, in the real teaching experiments involving 24 students in one classroom.

3.2.3 Teaching Experiment (Second Cycle)

The teaching experiment will be conducted in a 5^{th} grade class with 26 heterogeneous students. During the teaching experiment, the mathematics teacher will teach the classroom, implementing the revised HLT. Data in the teaching experiment will be collected in the same ways as the preliminary teaching experiment (first cycle). There'll be some written tests (written documents), video recorders, and field notes.

Two video recorders are used in each lesson conducted. One video recorder is put in the corner of the classroom to see the general things happening in the whole classroom, the discussion in the whole classroom, like when the teacher is speaking and the students are listening to the teacher, or when there's a whole classroom discussion, it is used to see how the students explain to the teacher and the other students in the classroom. Besides, it is also used to remember the sequences of learning in the classroom.

The other one video recorder will focus on what's happening in one group of students. It's quite difficult to record all of the discussion in all groups, when students are discussing the given problems. Therefore, one video recorder will be put focusing on one group of students, so I will still can follow what's happening in the discussion among students (from students to students) in the group when they are discussing the given problems.

3.2.4 Post-test

The posttests will be given after completing the five lessons about the fractions division. In the posttests, there'll be 6 questions relating to the measurement and partitive divisions involving fractions. The post test is used to diagnose thoroughly students' final development after joining the six lessons about fractions division.

3.2.5 Validity and Reliability

The validity concerns the quality of the data collection and the conclusions that are drawn based on the data. According to Bakker (2004), validity is divided into two definitions, namely internal validity and external validity. In this present study, we will only do the internal data validation. Internal validity refers to the quality of the data collections and the soundness of reasoning that led to the conclusion. To improve the internal validity in this present study, during the retrospective analysis we will test the conjectures that have been generated in each activity. The retrospective analysis involves some data collected from some different ways; data from the video recording, field notes, written documents, and interviews. Having these data allows us to do the data triangulation so that we can control the quality of the conclusion. Then, data registration will ensure the reliability of the different data collected from different methods. More about the reliability of the data, in the data analysis we will also do trackability and inter-subjectivity. We will give a clear description on how we analyze the data in this study so people will easily understand the trackability. In addition, inter-subjectivity will be done to avoid the researcher's own viewpoint towards the data analysis. Therefore, some colleagues will participate in the discussion relating to analyzing some data.

3.3 Data Analysis

3.3.1 Pre-test

From the students' worksheets for the pre-test, we aim to see students' current knowledge relating to some prerequisites topics needed before they can come up to the division of fractions and relating to their natural strategies to solve some simple division of fractions problems. There are four main points which are focused on from the pre-test as follows.

(1) To know whether students know some representations of fractions.
(2) To know whether students are able to solve the addition, subtraction, and multiplication operations involving fractions.
(3) To know whether students are able to use a number line in mathematics operations (addition or multiplication)
(4) To know whether students have their own strategies to solve some simple division problems involving fractions if they haven't learned about this topic before.

Data collected from the pretest will be used as starting points to redesign the initial HLT.

3.3.2 Preliminary Teaching Experiments (First Cycle)

Data from the video recorder used to record the six meetings with the six students in the preliminary teaching experiment will be registered in a video registration. There'll be some transcriptions of some important and interesting discussions happening during the meetings. The video transcripts, the field notes, and the students' written documents will be analyzed thoroughly. In the analysis, we will see whether the hypothesis that has been made from the initial HLT really occurs in the small group discussion.

The result of analyzing the first cycle will be used to revise the initial HLT before it is going to be given to the students in the real teaching experiments.

3.3.3 Teaching Experiment (Second Cycle)

Data in the second cycle (the real teaching experiments) are collected by using the similar ways as data collected in the first cycle. There'll be data from the video recorders, the field notes, and the students' written documents. We will see the development of students with different levels in understanding the topic of the division of fractions. Whereas the levels of the students will be differentiated by looking at the strategies they use to solve the given problems. From the iceberg of the division of fractions, we can see the levels of the students, whether they are still in the informal level, the pre-formal level, or already in the formal level. We will also see whether the models proposed in the lessons can promote students to understand the topic of division of fractions.

3.3.4 Post-test

Data from the post-test will be used to see the endpoints of students' understanding relating to the topic learned. From comparing the result of the pretest and the posttest, we can see what new things that students learn until the end of the lesson, or if students didn't know about a certain knowledge or don't have a certain skills in the beginning, will they improve and become knowledgeable and skillful in the end of the lesson.

Chapter IV
Hypothetical Learning Trajectory

Before starting the cycles of the design research, the Hypothetical Learning Trajectory (HLT) is formulated. This HLT about the division of fractions consists of students' starting points, which is the current existing knowledge of the students just before learning the topic of the division of fractions, the main learning goals which are going to achieve, some learning goals for each lesson conducted, some mathematical tasks for each lesson, and the conjectures of students' thinking. In this HLT, I will design some activities for six lessons about some parts of division of fractions for the 5th grade students.

All the six lessons designed will fit to the five tenets of RME as mentioned in the chapter of theoretical frameworks. For the mathematical tasks given to the students, I will use some story problems relating to measuring activities involving the division of fractions. Some measuring activities which are used are about measuring and making some partitions of ribbons, and measuring the length of a distance. The task in each activity will be given as measurement division problems or partitive division problems. All problems will provide students opportunities to solve in some different ways. Some problems can be solved by doing or exploring with the real materials, and all problems give opportunities for students to do modeling.

There are two main goals that we want to achieve in this present study. The first goal is to support students understanding and making relations between division and multiplication involving fractions. It means that students will understand that a division problem can also be represented as a multiplication problem and in advance they can make a relation, that if they know $a \div b = c$, then they will also realize that $b \times c = a$. The second goal of this present study is to support students understanding the relations between two division problems. The relations here is that if they know $a \div b = c$, then they know that $a \div c = b$.

4.1 Activity 1: Measuring Activities

4.1.1 Description

The first activity relating to the division of fractions is started from doing measuring. The mathematical task given to students is done in some small groups. The problem is a measurement division problem which asks to find how many souvenirs made of ribbon that can be made from a given length of ribbon. Some students may see the problem as a multiplication problem with fractions and some of them may solve the problem by using multiplication, since they have learned about this topic before.

4.1.2 Students' Starting Points

a. Students know how to measure length by using length measurement (in meter or in centimeter)
b. Students can convert the length in meter into centimeter or the other way around
c. Students can do additions, subtractions, and multiplications involving fractions
d. Students haven't learned about division of fractions before
e. Students have experiences with whole numbers, that dividing is more difficult than multiplying.

4.1.3 Mathematical Goals

a. Students can make some partitions of a ribbon by measuring
b. Students can use multiplication as a mean to solve a division problem

4.1.4 Mathematical Task

Materials: ribbons, length measurement in meter or in centimeter

To prepare the celebration of Kartini's Day which will be held in the April 21st, some people together with the householders of the Pakjo sub district are preparing to make some small souvenirs made of colorful ribbons for the visitors who will come to the celebration in the hall. They decided to make some big flowers, key chains, and some small flowers. To make one big flower, they need one meter of ribbons, and to make one key chain and one small flower, they need respectively a half meter and three quarters meters of ribbons. At the current time, the committee of the celebration only has 9 meters of ribbons. The committee decides to create some souvenirs from the 9 meters of ribbon, and creates the other souvenirs later after buying more supply of ribbons. Can you help the committee to estimate how many big flowers, key chains, and small flowers that can be made from **all** of the 9 meter of ribbon?

4.1.5 Conjectures of Students' Work

There are some possibilities that students may do to solve the given problem.

a. Using whole numbers to solve the problem

   Because students can convert length in meters into centimeters or the other way around, some of them may convert the length in centimeters, so they will solve the problem with whole numbers, not fractions. After getting the answer, they can convert the length back to meters.

b. Using measurement scale to really measure the length

   Some of them may really need models to solve the problem. They will measure with a length measurement to measure the total ribbons, and then finding how many small parts that they can make from all the total length of ribbons by measuring the length of each souvenir.

c. Using fractions

   Some of them may use the numbers to solve the problem. They will find the length to make some souvenirs by using the multiplication involving fractions (they have learned about this topic before learning the division of fractions). For example, to make one key chain needs $\frac{1}{2}$ meter of ribbons. Therefore, they need 2 meters of ribbon to make 4 key chains, $\left(4 \times \frac{1}{2}\right)$. One small flower needs ¾ meter of ribbon, so there are 3 meters of ribbon used to make 4 small flowers. The remaining length of the ribbon is 4 meters, which can be used to make four big flowers.

d. Drawing models

There are some models that students may draw to solve the problem. To represent the situation, the ribbon, the possible models that may appear are the bar model or the number line model.

Figure 1. A bar model for measurement division problems

4.2 Activity 2: Making Relations between Multiplication and Division

4.2.1 Description

In this measuring activity, students will do a task in some small groups to find how many small ribbons with a given length that can be made from a given length of ribbon (as the total). Some different approaches to do the task may appear. Some students may use a multiplication approach, in which they see the problem as a multiplication problem, so they are going to find how many times they have to multiply the length of the small ribbon to fit or to exceed the total length. Some others may see the problem as a division problem. Although they may measure the total length by making groups of the smaller length, in the end they reason that the number of small ribbons that can be made is got by dividing the total length with the length of each ribbon.

Two approaches to solve the problem using a multiplication or a division and how to write the numbers using mathematical symbols as a multiplication or as a division will be the main focus in the whole classroom discussion.

4.2.2 Students' Starting Points

a. Students know how to measure length by using length measurement (in meter or in centimeter)
b. Students can convert the length in meter into centimeter or the other way around

c. Students can do additions, subtractions, and multiplications involving fractions
d. Students can do measuring (with the real objects or by making models) to solve a measurement division problem

4.2.3 Mathematical Goals

a. Students can find some strategies to solve measurement division problems
b. Students can rewrite the problems in mathematics equations as a multiplication problem or a division problem
c. Students know the relation between a multiplication and a division involving fractions

4.2.4 Mathematical Task

To make some decorations for the hall to prepare the celebration of Kartini's Day, your job is to find out how many small ribbons that can be made from the packaged lengths for each color ribbon. Complete the table below.

White Ribbon	Length of each small part	Number of parts
1 meter	½ meter	
1 meter	1/3 meter	
1 meter	¼ meter	
1 meter	1/5 meter	
Blue Ribbon	Length of each small part	Number of parts
2 meters	½ meter	
2 meters	1/3 meter	
2 meters	¼ meter	
2 meters	1/5 meter	
2 meters	2/3 meter	
Gold Ribbon	Length of each small part	Number of parts
3 meters	½ meter	
3 meters	1/3 meter	
3 meters	¼ meter	
3 meters	1/5 meter	
3 meters	2/3 meter	
3 meters	¾ meter	

Write down in a mathematical sentence

a. From 1 meter white ribbon, you can make 2 ribbons which the length is ½ meter.
b. From 1 meter white ribbon, you can make 4 ribbons which the length is ¼ meter.
c. From 2 meters blue ribbon, you can make 3 ribbons which the length is 2/3 meter.
d. From 3 meters gold ribbon, you can make 4 ribbons which the length is ¾ meter.

4.2.5 Conjectures of Students'Work

There are some possibilities that students may do to solve the problem.

a. Using whole numbers, by converting the length (in meter) into centimeters, and then solving the division in whole numbers.
b. Using measurement scale

   Some students may still need measurement scale and really measure the exact length of the ribbon to find the answer.
c. Looking at the pattern of some problems

   Some students may notice that they can make 2 parts having length ½ meter from 1 meter ribbon, 4 parts having length ¼ meter from 1 meter ribbon, etc. So for each unit fraction, the number of small ribbons that they can make is the denominator of the unit fraction times the length of the total ribbon.
d. Writing down the problems as multiplications or as divisions

   For the additional questions below the table, some students may write down the problem by using multiplication sign or division sign. As an example, from 1 meter of ribbon, we can make 2 small ribbons having length ½ meter can be written as follows.

   $2 \times \frac{1}{2} = 1$ or $1 \div \frac{1}{2} = 2$.

4.3 Activity 3: Making Partitions

4.3.1 Description

In this activity students will solve problems relating to partitive division. The given problems will also relate to making partitions of a certain length of ribbons. However, the length of each part is not given. From a certain length of ribbons, students are asked to find the length of each part if the number of partitions (all part has the same length) is known. All of the problems relating to the partitive division will involve division problems which the divisor is a whole number.

In this activity, students will be allowed to solve the partitive division problems using real materials. The length of the ribbon will be given in meters. If they are not able to solve the problem in meters, they are also allowed to change the length into centimeter first. Therefore they will do a division in whole numbers. However, in the end students need to reconvert the length into meters again. There may be some modeling activities in this problem. I will also allow students to use models to solve the partitive division problems.

4.3.2 Students' Starting Points

a. Students know how to measure length by using length measurement
b. Students can fold a ribbon into two, three, four, six, eight, or some other simple parts
c. Students know repeated subtractions or additions in the division or multiplication involving whole numbers
d. Students know how to multiply two fractions, a fraction with a whole number, or two whole numbers
e. Students can convert meter into centimeter or the other way around
f. Students can do a division in whole numbers

4.3.3 Mathematical Goals

a. Students know how to make partitions from a given length of ribbons
b. Students can use models to solve partitive division problems

4.3.4 Mathematical Task

Materials: Ribbon, length measurements (in centimeters or in meters)

Aji follows his father who becomes one of the committee of the Kartini's Day to come to the city hall to prepare the celebration. Because there are only a few people coming, Aji is asked to help the committee to cut some ribbons. One of the committee members gives him 2 meters of red ribbon, one meter of yellow ribbon, and three quarter meter of green ribbon. He is told to cut the red ribbon into four equal parts, also to cut the yellow and the green ribbon, each of the ribbons is cut into three equal parts. Aji is struggling to divide the ribbon into the parts asked. Can you show him how to divide it? How long does the length of each ribbon after being cut?

4.3.5 Conjectures

There are some possibilities of students' answers to solve the problem.

a. Some students may be able to just imagine the situation of the problem, like they imagine if they have a 2-meter ribbon, and if they're going to divide it into 4 equal parts, it means that each of the 1-meter should be divided into two. They'll get two equal parts for each meter, so they'll get exactly four equal parts from the two meters, in which each part has ½ meter in length.

b. Some students may use the multiplication operations to find the answer, as follows.

   4 x … = 2

   3 x … = 1

   3 x … = ¾

c. Students may convert the length into centimeters. From dividing the 1 m, or 100 cm, ribbon into three equal parts, the length of each part is about $33\frac{1}{3}$ cm. From dividing the ¾ m, which is 75 cm, into three equal parts, the length of each part is 25 cm.

d. Some students may find incorrect fractions. In the ¾ ribbon, they divide it into three equal parts. They may think that each part should be 1/3 because they divide it into three equal parts.

e. Some students may use models to solve the problem. Some models that may appear from the story using ribbons are bar models or the number line model.

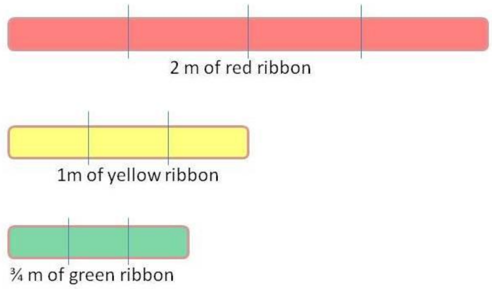


Figure 2. Using models to solve partitive division problems

4.4 Activity 4: Making Relations between Two Division Problems

4.4.1 Description

In this activity, students will be given some measurement division and partitive division problems. From the given measurement and partitive division problems, students need to make a relation between the two problems. The relation here is that if there is given an equation $a \div b = c$, they will consider that $a \div c = b$. A pair of problems consists of a measurement division problem and a partitive division problem which have the same numbers involved, so the relation can be seen by making the two equations.

4.4.2 Students' Starting Points

a. Students know some strategies to solve measurement division problems or partitive division problems
b. Students can make an equation involving division of fractions from a given situation
c. Students know that for a, b, and c are whole numbers, if $a \div b = c$, then $a \div c = b$

4.4.3 Mathematical Goals

Students know the property in divisions involving fractions that for a, b, c are rational numbers, if $a \div b = c$, then $a \div c = b$.

4.4.4 Mathematical Task

a. (*Measurement division problem*) From a 3-meter ribbon, Indah is going to make some flowers. To make one big flower from ribbon needs $\frac{3}{4}$ meter. How many flowers that she can make? Make a mathematics equation from this problem!

b. (*Partitive division problem*) Sinta is going to divide 3-meter ribbon into 4 equal parts. How long is the length of each part? Make a mathematical equation from this problem!

4.4.5 Conjectures of Students' Work

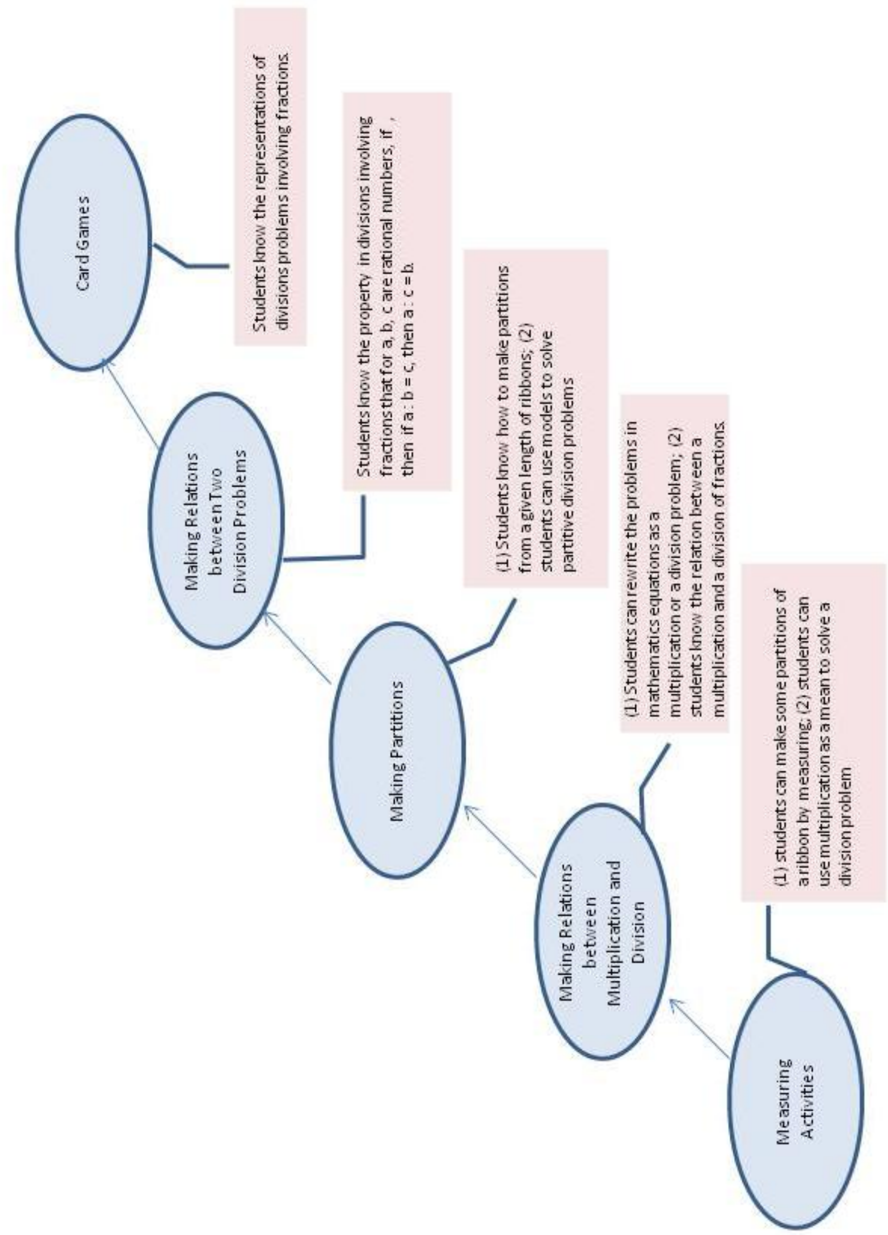
From the first problem, students may make an equation involving division of fractions: $3 \div \frac{3}{4} = 4$. In the other hand, for the second problem they may make an equation $3 \div 4 = \frac{3}{4}$. After getting these two different equations, students will discuss how this relation can happen in the whole classroom discussion.

4.5 Activity 5: Card Games

4.5.1 Description

In this activity, students will play a card game, in which they need to make some groups of the cards, in which each group consists of some representations of a division of fractions problems. For each case of division problems, there'll be three different representations. The first is represented by long sentences, in which the situation is clearly described by words. The second is a representation with a number line or a rectangular model, and the third is a representation with mathematical symbols, like using repeated addition, or making the denominators the same.

4.5 The Visualization of the Learning Trajectory



Chapter 5

Retrospective Analysis

This chapter gives descriptions and analysis of data collected from the pilot and the teaching experiment done in this present study. This design study was conducted in two cycles of design study; the pilot teaching and the real teaching experiment. The Hypothetical Learning Trajectory (HLT) which had been designed in the chapter 4 was used in the pilot teaching. After doing the retrospective analysis of the pilot teaching, the HLT was revised before it was used in the second cycle of the design research as a guideline to conduct the real teaching experiment.

There were four students participating in the first cycle of this present study. Those were from a class which is not the class where the real teaching experiment would be held.

5.1 Pilot Teaching

The pilot teaching was conducted with a group of four students, excluded from the real teaching experiment class. There was a student with high achievement in mathematics, two average students, and one quite slow student. The four students were chosen by the mathematics teacher.

The initial HLT was evaluated in this cycle. There were five activities with some conjectures which had been made, predicting what might occur in the classroom. During the retrospective analysis, the evaluation of the initial HLT would include whether the conjectures made would occur or not, analysis of some interesting parts of each activity, and the analysis of some causes which caused the conjectures didn’t occur. In addition, input was also obtained from students’ strategies and struggles when doing the sequence of activities in the HLT.

5.1.1 Pre-Assessment

The four students were examined in the pre-assessment with a set of four types of problems. There were two problems measuring students’ understanding about some fractions and their representations in a drawing, a problem relating to addition and multiplication operations in fractions, a problem relating to simple measurement division and a problem about simple partitive division. The last two problems were included to see whether students have their own nature way to solve the two kinds of division problems although they had’t learned about the topic.

From the result of the pre-assessment, some students had difficulties to represent a fraction into a drawing in a bar model. Students were still struggling to solve the simple division problems involving measurement and partitive divisions. However, students were able to use a ruler to measure the length of a given ribbon and could do additions and subtractions involving fractions. In addition, students hadn’t learned about the multiplication operation involving fractions and whole numbers in the formal classroom.

5.1.2 Activity 1: Measuring Activities

The first activity in the initial HLT was aimed to know whether students could make some partitions by measuring activities and whether students could use a multiplication involving fractions to solve the problem. The problem was to find the number of three souvenirs that could be made from a given length of ribbon. Each of the three souvenirs needed a certain length of ribbon. Students were challenged to make the best use of the given length of ribbon, 4 meters, to make the three souvenirs, respectively have 1 meter, ½ meter, and ¾ meter. This problem gave students opportunities to find as many combinations as possible of the number of each souvenir such that all of the given length of ribbon was used.

To find the combination of each souvenir which could be made from the ribbon given, all of the four students in the first piloting used a trial-and-error strategy, by converting the length given into centimeter and using the real ribbon to really see whether they had done correctly or not.

Because the length measurement given to the student was in centimeter, students need to convert the length from meter into centimeter. At first, they had difficulties to convert the length. It was unpredicted before, because the topic of length unit conversion had been learned by the students since they were in the fourth grade. Therefore, the students were reminded about the length unit conversion.

They were traditionally taught the topic of length conversion by remembering something called 'length stairs', in which they had 7 staircases, the kilometer is on the top of the staircase and the millimeter is in the lowest staircase. By making a drawing of the length stairs, students realized that they went two steps downstairs from meter to centimeter, so they should multiply the length with 100. Therefore, they were convinced that 1 meter ribbon is equal to 100 cm.

Students were able to reason how many centimeter they should measure for the ½ meter and the ¾ meter. They knew that ½ meter is a half of the 1 meter, so it is a half of 100 cm, which is 50 centimeter. They also knew that a quarter meter is a half of a half meter, which means that a quarter meter is a half of the 50 centimeter, which is 25 cm. So, the ¾ meter is the result of adding the ½ meter with the ¼ meter, so it is 75 cm.

Figure 3. Using the real object

After converting all the length into centimeter, students used the real object, the ribbon and the length measurement to solve the problem. To find the combination of the souvenirs made, students used the trial-and-error strategies. They measured a 1-meter, which was the length of the big flower souvenir, and then tried the number of souvenirs for the rest two souvenirs which could be made from the remaining ribbon.

In the initial HLT, it was predicted that students might use repeated addition or the multiplication involving fractions to solve the problem. However, those two strategies didn't occur. That was because students hadn't learned about the multiplication operation involving fractions before. The length of the ribbon which was only 4 meter might also influence students not to use the multiplication. The addition was very much easier to solve the problem.

After doing the activity, students could make partitions from a given length of a ribbon into some parts in which the length of each part had already been known. The lengths which were given in meter were converted into centimeter, and then students used the real object to clearly experience the situation in the problem. They were only able to find a combination of the number of the three souvenirs, and had difficulties to find other combinations. That might because students weren’t used to solve open problems when they were studying in their classroom before.

In the revised HLT, the HLT2, this activity would be integrated with the second activity relating to measuring activities with the measurement division problems. In the second activity, students would try to find some strategies to solve the measurement division problems. One of the strategies would be using multiplication operations involving fractions to support students solving division problems. This first activity would be removed because this problem would be very difficult for students who only could solve the problem by using length measurement if the total length of the ribbon would be extended. To find as many combinations as possible and to promote students using the multiplication operation involving a whole number (the number of souvenir) and a fraction (the length of the ribbon needed to make a souvenir) needs a longer length of ribbon. As a result, students would have difficulties to really measure the length by using the real object if the length of the ribbon would be extended.

5.1.3 Activity 2: Making Relations between Multiplication and Divison

It's a big activity in which before making relations between multiplication and division, students would do some measuring activity to find the number of short ribbons which could be made from a given length of ribbon. After getting the number of short ribbons, they would be given some statements regarding to the findings they had made during the measuring activity. Then, they would be guided to generalize some mathematical expressions involving the multiplication and division operations.

There were three mathematical goals formulated in the initial HLT. Students were expected to know some strategies to solve measurement division problems, could write mathematical equations based on statements relating to measuring with ribbon, and in the end students could learn the inverse relationship between the multiplication and the division involving fractions.

a. Using Real Object to Solve Measurement Division Problem

In the first lesson, students were working to find the number of partitions which could be made from a given length of ribbon. Students were still using the real object to solve the problem. Again, they converted the length from meter into centimeter. Some students were able to do division in whole numbers to find the number of partitions, after they converted the length into centimeter.

Some students were only using the length measurement without really measuring the ribbon to find the number of partitions.

Figure 4: Using length measurement in centimeter to measure

The four students didn't have difficulties to solve the problems when the fractional part of the length could easily be converted into centimeter. However, they really had difficulties to convert some fractions into centimeter, like a third and two-third. Therefore, the researcher who was also the teacher during the pilot teaching added one small activity relating to find how many centimeter a third meter is.

In the initial HLT, it was expected that students might use the multiplication operation involving fractions to solve the problem. To find the number of partitions which could be made, students would find a whole number in which if they multiplied it with the size of the partition, the product would be the total length of the ribbon. However, there was no student who used the multiplication operation. Like the first activity, this conjecture didn't occur probably because students hadn't learned about the multiplication operation involving fractions formally in the classroom.

Students were also expected to use models to solve the problem. However, there was no student who used the model, like a bar model which was seen as a ribbon. To promote students making drawings to solve the problem, in the HLT2 the students' worksheet there would be a space for students to give reasons how they got the answer. The teacher would also guide the students in using the bar model to solve the problem.

b. Converting a Third Meter into Centimeter

In order to find the number of partitions which could be made from a given length of ribbon if the size of each part was a third meter, the four students in the pilot teaching were struggling to find how many centimeter a third meter is.

A student tried to find a number in which if she added it with itself three times, she would get 100. Then, she realized that she could do by multiplying a number with three to get 100. She picked a number and she multiplied it with 3 until she got a number resulting 100 if it is multiplied with 3.

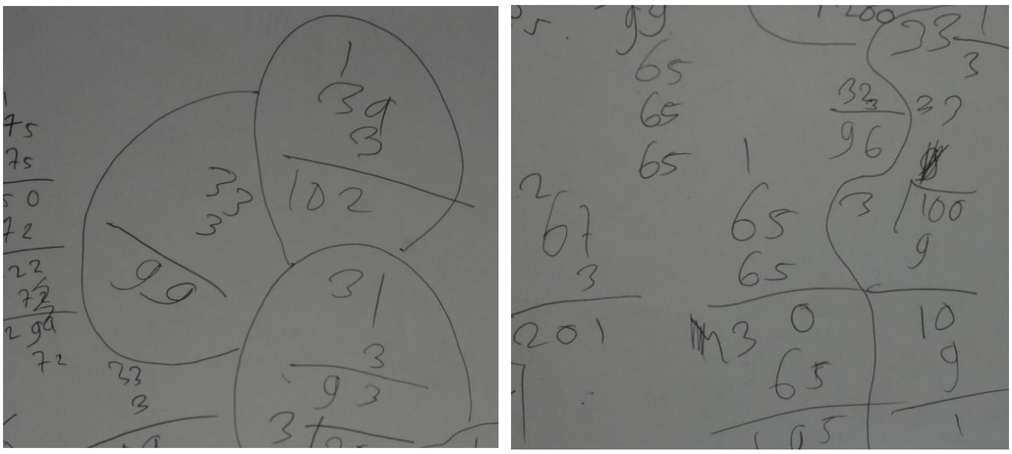


Figure 5. Converting a third meter into centimeter

She tried to multiply 3 times 39, 3 times 31, and 3 times 33. She got 99 and she said that it was a bit more than 33 (in the left side of the figure). Then, she did a long division algorithm to divide 100 by 3. She got $33\frac{1}{3}$ and she finally said that a third meter of ribbon equals to $33\frac{1}{3}$ cm (see the right side of the figure, where she divided a 100 with 3 in the long division algorithm).

Other students were still struggling to convert the length. The researcher guided the students to use the real object. The students were asked to find how they should fold a 1-meter ribbon to get a third meter. They knew that the ribbon should be folded into three equal parts, so they folded the ribbon into three equal parts.

Figure 6. Folding a 1-meter ribbon into three equal parts

In the left figure, a student was folding the ribbon into three equal parts. Then, they measured the length by using a length measurement in centimeter. They found that the length of each part was 33 cm.

The students got confused because when they added three 33 cm, they only got 99 cm, not 100 cm. Then, the student who used the long division algorithm to divide 100 by 3 showed the other students how she got the $33\frac{1}{3}$ cm. Finally, all students agreed that it's true that $\frac{1}{3}$ m equals to $33\frac{1}{3}$ cm.

There are some strategies which could be done to convert a third meter into centimeter. The four students in the piloting group were using the real object, ribbon and length measurement in centimeter. They knew that a third is got from dividing a 1-meter ribbon into three equal parts, so they folded the 1-meter ribbon into three. Then, they measured the length by using length measurement. They measured 33 cm in the length measurement. After getting the hypothesis of the length in centimeter, they did a justification to check whether they would get 100 cm if they added three 33 cm. Students could also use trial and error strategy to find how many cm a third meter is. They could try a number in which if they multiplied it with 3, they would get 100 cm.

Measuring with the real object and estimating the length by using trial-and-error strategy in fact still couldn't give students a solution of the problem. They were not sure that the number they had got was really the conversion in centimeter from a third meter. They knew that the number would be between 33 and 34 cm. To really get the number, a long division algorithm involving whole number was used. They divided 100 by 3, so in the end they got $33\frac{1}{3}$ cm. After making a justification, they were sure that $33\frac{1}{3}$ cm equals to a third meter.

c. Generating Mathematical Equations

After exploring measuring activities with ribbon, students were given some statements. They were asked to translate the words statements into some mathematical equations. The goal was to guide students generating mathematical equations involving multiplication and division operations with fractions. This sub-activity was done in the third day of pilot teaching. At that time, students had learned about the multiplication operation involving fractions in their classroom.

Students already knew how to multiply a fraction with a whole number and to multiply two fractions.

There were given some statements from the measuring activity which had been done before. For each statement, students were asked to determine as many mathematical statements as possible. Students could use a multiplication, a division, or a repeated addition equations to express the words statement.

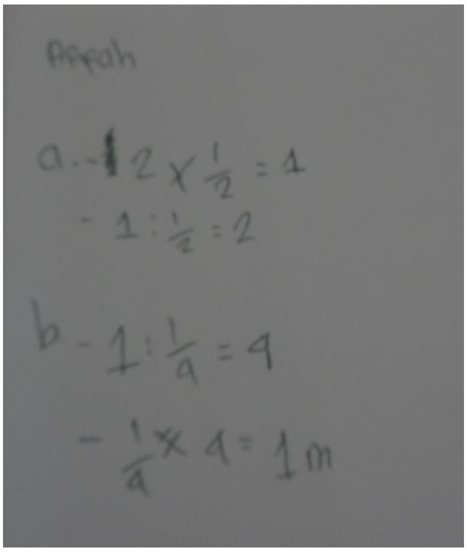


Figure 7. Converting words statements into mathematical equations

From the figure above, a student was making two mathematical equations, one is expressed by using multiplication operation involving fraction, and the other is using division operation. The first statement was stated as follows.

"From a 1-meter white ribbon, we can make 2 partitions each having length ½ meter."

The student could make two mathematical equations from the statement above. She didn't write the algorithm, both the multiplication and the division algorithm in her equations. For the multiplication equation, $2 \times \frac{1}{2} = 1$, it was meant that if there are two ribbons each having length ½ meter, the total length of ribbon needed to make the partitions is 1 meter. Whereas the division equation, $1 \div \frac{1}{2} = 2$, was meant that if there is a 1-meter ribbon which is divided into some equal parts, each having length ½ meter, then the number of partitions which could be made is 2.

There was a student who wrote another division equation from the statement. She wrote $1 \div 2 = \frac{1}{2}$, which might mean that if there is a 1-meter ribbon which is divided into two equal parts, then the length of each partition is ½ meter.

From the activity of generating mathematical equations, it was found that students could make a relationship between the given words statement relating to the measuring activity and the mathematical expressions involving multiplication and division operations of fractions. The measuring activity can be explained in some words statements which can easily be converted into two mathematical equations involving multiplication and division.

In the next activity, based on the two kinds of mathematical equations which were generated from the words statement, students would identify the relation between the two.

d. Making Relations between Multiplication and Division

In the discussion, students were guided to see the relationship between the two mathematical equations, one was expressed as a multiplication equation and the other is expressed as a division equation. Students recognized that for every pair of multiplication and division equation, the three numbers used were all the same. Only the position of the number was different.

When students were given a division equation, then they could find the multiplication equation using the three numbers used in the division. For example, if there was given an equation $1 \div \frac{1}{4} = 4$, then they knew that if they multiplied 4 times ¼, they would get 1. The researcher which was also the teacher guided the students to name each number involved. 1 in the division equation is the dividend, ¼ is the divisor, and 4 is the quotient. Therefore they could make a relationship that if the quotient of the division is multiplied with the divisor, then they'll get the dividend of the division equation.

In this activity, students were only concluding the inverse relation between the multiplication and the division equation orally. Therefore, for the next activity, the next students would be given a sheet of paper, which would be called the reflection page, so they could write down what they knew about the inverse relation between the multiplication and the division.

5.1.4 Activity 3: Partitive Division Problems

In this activity, students would solve some partitive division problems. The goals were that students would be able to find some strategies to solve the partitive division problems and to use models to solve the partitive division problems. The problems given were still using the story of measuring with ribbons.

The task of the student was to find the length of each partition of ribbons which was made by dividing a given length of total ribbon into some equal parts. Looking at students' strategies, students tended to solve the problem by converting the length from meter into centimeter. They also could use repeated addition and multiplication involving a whole number and a fraction to solve the problem.

In the worksheet, the task of the students was to find the length of each partition if a 2 meter ribbon is divided into four equal parts, a ¾ meter is divided into three equal parts, and a 1 meter ribbon is divided into three equal parts.

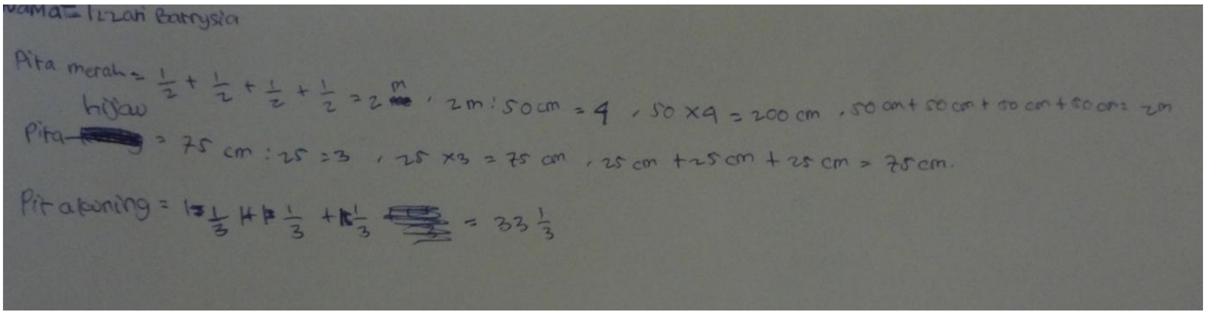


Figure 8. Solving partitive division problems

In the figure above, students wrote some mathematical equations, using the repeated addition involving fractions and also repeated additions involving whole numbers, multiplication involving whole numbers, and a division involving whole numbers.

In the first line, the student added four halves and she got 2 meter. Then she divided a 2 meter by a 50 cm and she got 4. She multiplied a 50 with 5 and she got 200 cm. At last, she added four 50 cm and she got 2 m. It seems like she sometimes used fractions to solve the problem, but sometimes changed the meter into centimeter to avoid working with fractions, especially in the multiplication and the division operations. It was probably because dividing or multiplying with whole numbers was much easier to do than dividing or multiplying with fractions.

The other three students in the piloting group were also used more formal operations using multiplication or division by converting the length from meter into centimeter. Sometimes they used repeated additions involving fractions. However, there was none of them who made a drawing of the situation to solve the problem.

5.1.5 Activity 4: Making Relationship between Two Division Equations

The goal of this activity was to support students to learn the relation between two division equations involving fractions, that if there is a division equation $a \div b = c$, then $a \div c = b$. In this activity, students would be given problems which were given as mathematical equations. Students could make their own stories based on the mathematical equations and then they could solve the problem by using their own strategies.

The first problem was to find the quotient of $2\frac{1}{2} \div \frac{1}{4}$. All of the four students made a division story which was the case of measurement division problem. One of the four students made a story about dividing a 2 ½ meter ribbon into some equal parts in which the length of each part was ¼ meter. The question was to find the number of partitions which could be made.

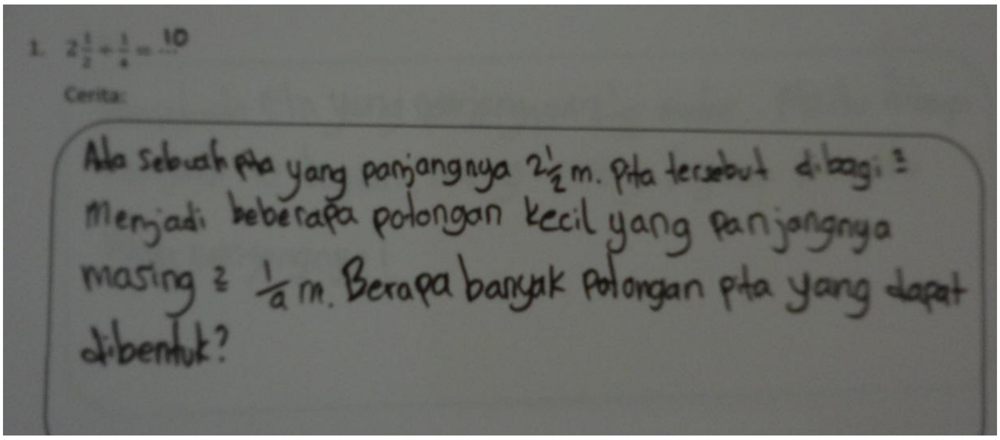


Figure 9. Making stories from mathematical equations

The answer of one of the students below shows how they used a numbered line to solve the division problem. She also used the multiplication as a mean to find the answer of the division problem.

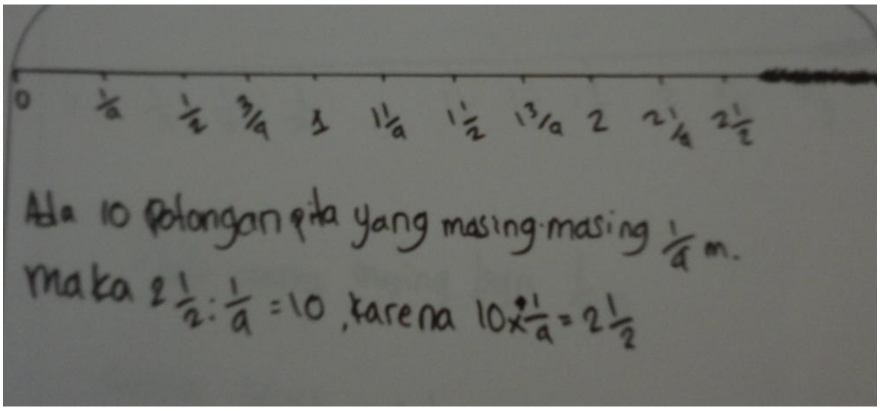


Figure 10. Using a number line to solve a division problem

In the figure above is the answer made by one of the four students. She draw a number line starting from 0 to 2 ½ which was the length of the ribbon divided. She divided each 1 meter into four equal parts and she named in each mark with fractions which were the multiplication of ¼. She stopped at 2 ½.

It looks like she knew that the length of each part was ¼ meter, so the fraction corresponding to the marks she had made was the multiplication of ¼. Below the bar she wrote that there are 10 partitions of the ribbon each length is ¼ m. Therefore, in the last sentence she concluded that $2\frac{1}{2} \div \frac{1}{4} = 10$, because $10 \times \frac{1}{4} = 2\frac{1}{2}$.

In the last sentence, she wrote that the answer of the division equation was 10 because if she had 10 times the length of each partition, then she would get the total length of the ribbon. It seems like she already related the division problem with the multiplication to solve the problem. For the division equation, she wrote that the quotient was 10 because if she multiplied the 10 with the ¼ then the product would be the length of the ribbon. She had already thought in terms of the division and the multiplication.

The next problem was to find the quotient of $2\frac{1}{2} \div 10$. The four students in the piloting were making stories about the partitive division case. In their story, they had a 2 ½ meter ribbon which was divided into 10 equal parts. The task was to find the length of each part. The student realized that in the previous problem they had made a drawing in a number line about a ribbon in which the total length was 2 ½ meter. They made partitions for every ¼ meter, then they got 10 equal parts of the drawing. Therefore, for the second problem, they realized that the length of each partition was ¼ meter.

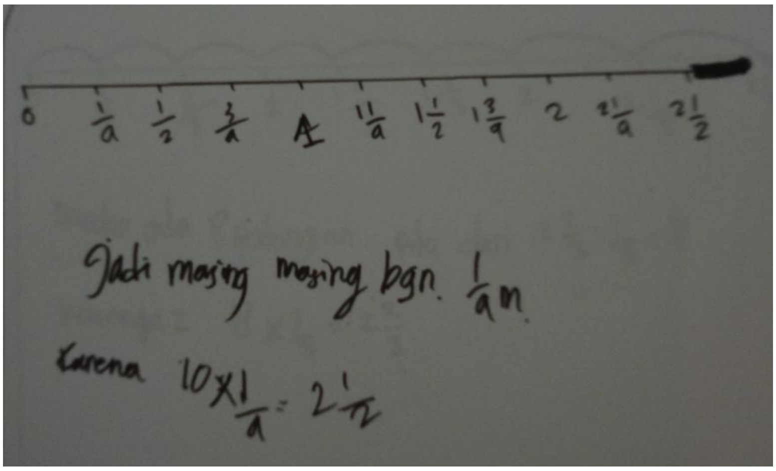


Figure 11. Solving partitive division problem

From the drawing she draw the same number line as she draw to solve the first problem. below the drawing, she wrote that the length of each partition was ¼ meter, because $10 \times \frac{1}{4} = 2\frac{1}{2}$. The student was able to also make use the multiplication involving fractions to find the quotient of a partitive division case. She was already able to make relations between the division and the multiplication involving fractions.

For the next pair of problems, students recognized that there was a relationship between every two division problem. After getting the quotient of the first division problem, they checked the next problem and see that the division involved the same two numbers. They made a conjecture that the answer might be the other number used in the first division. Then, they checked by using a drawing or the multiplication to see whether the answer was right.

After solving some pair of problems, the four students were discussing the relationship between the division problems they had solved before. They were asked whether they saw something intriguing from the pattern of the number.

They recognized that the numbers used in the two division equations were the same. Then they could generate another division equation if they were given a division equation involving fractions. As an example, if they were given a division equation $2 \div \frac{1}{4} = 8$, then they could generate another division equation $2 \div 8 = \frac{1}{4}$. Finally, with the guidance from the researcher, they could orally say that if the dividend of a division equation was divided by the quotient, then the answer would be the divisor of the first division equation. In the next cycle of the design research, students would be given a reflection sheet to write down the relation between the two division equations.

5.2 Real Teaching Experiment

The real teaching experiment was conducted after the first cycle of pilot teaching, starting from 16th March 2012. There were 26 students from the 5th grade of MIN 2 Palembang, one of Islamic primary school in Palembang, Indonesia, participating in the real teaching experiment. The mathematics teacher, Mrs. Risnaini, taught the class herself by following the teacher guide given.

During the real teaching experiment, the researcher with a colleague became the observer in the class. There were two video cameras used during the real teaching experiment, one was used to capture the general discussions happening in the whole classroom, and the other video camera was used to see carefully some discussions happening in the focus group observed. There were four students in the focus group, one good student, two average students, and one low achiever student.

5.2.1 Pre Assessment

The pre-assessment was given three weeks before the real teaching experiment would be conducted. When the assessment was given, students in the class had studied about an introduction of fractions and their representations, additions and subtractions of fractions having the same denominator or not. They hadn't learned about multiplication of a fraction with a whole number, multiplication of two fractions, and the division of fractions.

a. A Fraction and Its Representation

From the 26 students doing the pretest, there were only 8 students who answered the first two problems relating to make a representation of a fraction in a drawing correctly. There were two drawings of bars, representing two ribbons which length is 1 meter and 2 meters. The students were asked to shade or to give marks on the drawing, showing how long the ¾ of the drawings is.

To find the ¾ from the drawing of ribbons in the first two problems, most students were dividing the drawing into four equal parts. However, there were only a few students who found that the ¾ is the three out of the four parts. Some others were only dividing the bar into four equal parts but didn't give mark or shade the bar showing how long the ¾ is.

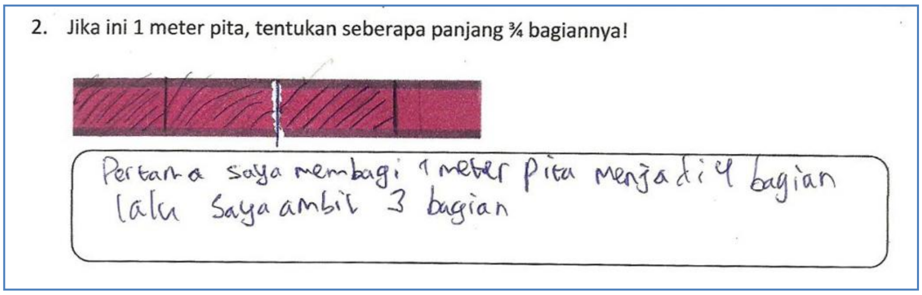


Figure 12. Shading 3 parts out of 4 to represent the ¾

The figure is the work of a student, namely Rafly, to estimate the length of the ¾ of a 1 meter ribbon. In the box he wrote how he estimated the position of the ¾. He wrote that he firstly divided the 1-meter ribbon into four equal parts and he took the three parts. From the drawing, there are three marks dividing the drawing of 1-meter ribbon into four parts and there are three parts shaded. He knew that the ¾ is three out of four, in this case three parts out of the four parts. He could represent a fraction into its pictorial representation in a bar model and he could express a fraction into how many out of something.

Some students could estimate the position of ¾, but they couldn't give clear reasoning why they answered so. Below is the work of Alhikmah.

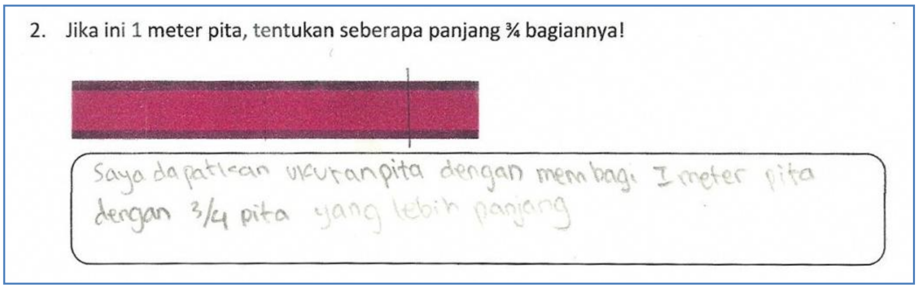


Figure 13. Estimating the position of ¾ of 1 meter ribbon

Alhikmah gave a mark on the drawing, close to the end of the drawing. In the box she wrote how she estimated the position of the ¾ of a drawing of 1-meter ribbon, but her reason was not clear enough. She wrote that she got the size of the ribbon (asked) by dividing the 1-meter ribbon by ¾.

The position of the ¾ on the drawing can be accepted, that students already put the mark in the position where it's the real ¾ part of the drawing. However, they didn't give explanation how they could give mark on that position.

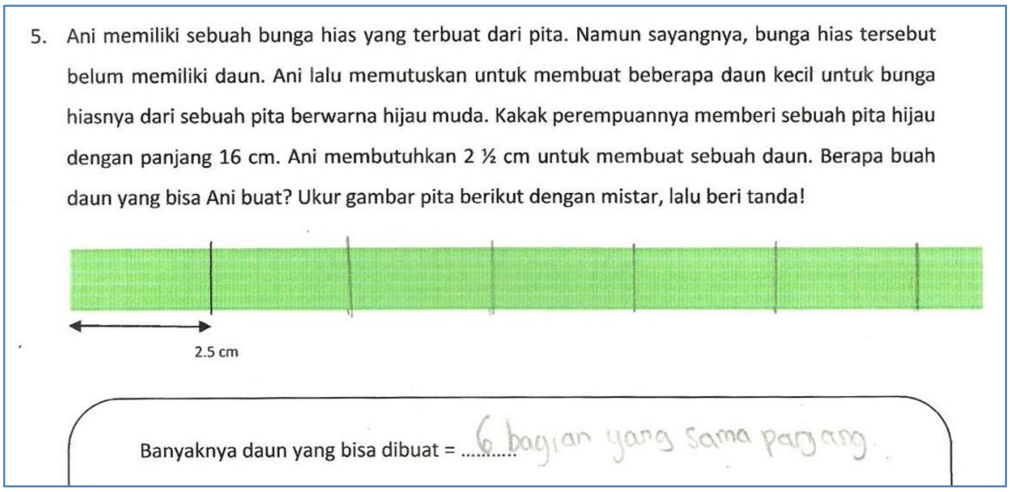
Other students' mistakes to determine the position of ¾ from two drawings of 1-meter and 2-meter ribbon are incorrect to find how many parts they had to divide the drawing. Some students divided the drawing into five equal parts. They shaded the three out of the five parts. Some other students were able to make 3 marks so they had divided the drawing into four. Then, they had shaded the three out of the four. However, they didn't divide the bar into four equal parts. There are three equal parts having almost equal size, but the rest one part is so much different in size compared to the other three.

From the result of the students' pre-assessment for the first two problems, although the topic of fractions had been introduced since students were in the third grade of primary school, there were still some 5^{th} grade students who were struggling to represent a fraction in a bar model.

b. Repeated Additions or Multiplication of Fractions with Drawings

Mostly all students were able to find the number of short ribbons having length 2 ½ cm from a given drawing of ribbon having length 16 cm. The drawing of ribbon exactly had length 16 cm. Students were told to use a ruler to measure the length. The goal of solving this problem are to know whether students can use a ruler as a length measurement to measure a given length of ribbon and to know whether students can add two or more fractions.

To solve the problem, students started measuring a 2 ½ cm from an edge of the drawing and continued to measure until they found that there was no longer short ribbon which could be made.


5. Ani memiliki sebuah bunga hias yang terbuat dari pita. Namun sayangnya, bunga hias tersebut belum memiliki daun. Ani lalu memutuskan untuk membuat beberapa daun kecil untuk bunga hiasnya dari sebuah pita berwarna hijau muda. Kakak perempuannya memberi sebuah pita hijau dengan panjang 16 cm. Ani membutuhkan 2 ½ cm untuk membuat sebuah daun. Berapa buah daun yang bisa Ani buat? Ukur gambar pita berikut dengan mistar, lalu beri tanda!

2.5 cm

Banyaknya daun yang bisa dibuat = ...6 bagian yang sama panjang.

Figure 14. Using a ruler for measuring

The figure was the work of Alhikmah to solve the problem. In the problem, there was a story relating to making some small leaves for making some flowers made by ribbons. The leaves were made from a green ribbon, in which the length to make a small leaf is 2 ½ cm. There was a 16 cm green ribbon left, and students were asked to estimate how many small leaves which could be made.

Students used a ruler to measure a 2 ½ cm of ribbon. Some students started measuring from the left side of the drawing and fitting the edge of the drawing with the mark 0 on the ruler. Some of them could give marks on the drawing, starting from the middle of 2 and 3 in the ruler, which is the 2 ½ cm, and then continue to 5 cm, 7 ½ cm, up to 15 cm. Some other students measured a 2 ½ cm one by one. They fit every end of 2 ½ cm with 0 and started measuring another 2 ½ cm using a ruler.

Students finally found that there were 6 parts of short ribbons having length 2 ½ cm which could be made. They neglected the 1 cm ribbon and only focused on the number of short ribbons which they could make.

From the result of answering the second problem, the 5th grade students could already use a length measurement in centimeter to measure the length of an object, and they could do additions involving some benchmark fractions when they were measuring with the length measurement.

c. Solving Measurement and Partitive Division Problems

Most students were not able to solve the two problems. They left the answer sheet blank, some others were making drawings in the answer sheet, but they were not able to solve the problem. While most students left the answer of the problem blank, Alhikmah gave an answer for the last problem about partitive division as follows.

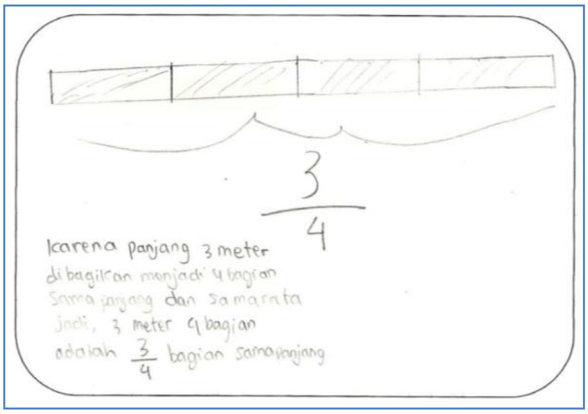


Figure 15. Solving the partitive division problem in the pretest

The problem was to find the length of a ribbon which was got from dividing a 3 meter ribbon into four equal parts. She made a drawing of a bar and divided it into four equal parts. She gave a mark, ¾, below her drawing. Then, in the left bottom of her worksheet, she wrote, "because the 3 meter length is divided into four equal parts, the 3 meter 4 parts is ¾ equal parts."

It seems like she could find a fraction from a given length which is divided into some equal parts. The fraction representing it was the total length divided by the number of partitions made. She could solve a partitive division problem relating to measuring activity to find the length of a partition made by dividing a total length of ribbon into some equal parts.

From the result of the pretest, it was found that there were only a few number of students who were able to make a correct representation of a given benchmark fraction. Students were able to use a ruler as a length measurement to measure the length of ribbon. They could add length of ribbons involving fractions. However, they hadn't been able to use multiplication of fractions and they were not able to solve problems relating to measurement and partitive division problems.

5.2.2 Activity 2: The Measurement Division

The first activity of the real teaching experiment was started from the teacher's explanation, recalling students' understanding relating to length conversion, from meter to centimeter and the other way round, the use of ribbons for making handicraft, and giving a small problem to students. The problem posed by the teacher was about length conversion from meter to centimeter or the vice versa. She asked how to measure a ribbon having length 1 meter by using a ruler which shows only centimeters.

All students didn't have any idea how to solve the problem. They didn't know how to use ruler (in centimeter) to measure 1 meter ribbon. Therefore, the teacher guided students to remember the stair of length unit, and asked how many staircases are needed in order to arrive at cm if they start from m. Most students remembered that there are 2 staircases, which means they need to multiply 1 meter by 100. They finally concluded that 1 meter equals to 100 centimeters.

After recalling students' knowledge relating to length conversion, the teacher gave the first problem relating to measurement division to determine the number of small ribbons having length 1 ½ meters which can be made from a 3-meter ribbon. The teacher told the problem orally. There were only a small number of students in the classroom who raised their hands, showing that they had an answer of the problem. There was no student who wanted to use the real materials (ribbons and length measurement) to solve the problem although the teacher had offered students to use them.

One student answered that there would be six small ribbons having length 1 ½ meter which could be made from a 3-meter ribbon. When he was asked to explain their reasoning, he got confused and became unsure with his answer. Another student said that there would be two small ribbons which length is 1 ½ meter which could be made from a 3-meter ribbon. He made a drawing of bar which represented the ribbon in the whiteboard.

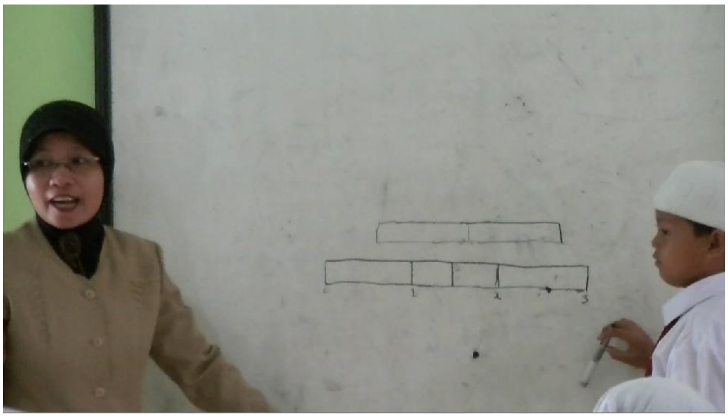


Figure 16. Guiding students to solve a measurement division problem

The first drawing he made was a bar which he said the length was 3 meter and it was divided into two exactly in the middle. He said that the sign in the middle divided the 3 meter ribbon into two parts in which the length of each part was 1 ½ meter. The teacher asked him to make a drawing in which he could show a more convincing way to divide the ribbon. The teacher asked the student to give mark in the drawing which showed where the position of 1 meter, 2 meter, 3 meter, and 1 ½ meter were. The teacher and students in the classroom agreed that in the left edge of the bar was marked as 0.

The student said that 1 ½ meter means that the length of each part is one meter and a half meter more. Therefore, in the drawing, he made the mark marking the 1 ½ in the middle of the mark 1 and 2. He said that the length starting from the mark 1 into the middle of the mark 1 and 2 was a half meter. In the left side of the mark between the 1 and 2, the length of the ribbon was 1 ½ meter and so was the length in the right side of the mark. Therefore, from the drawing he concluded that the number of parts having length 1 ½ meter which could be made from a 3-meter ribbon is two.

After having a short explanation, remembering students about some prerequisites topics, and solving one problem relating to measurement division, the 26 students in the classroom were grouped, in which each group consisted of 4-5 students. All students in group discussed the given problems relating to measurement division. Students could solve the problem by using any strategies they wanted. However, some real materials (ribbons, scissors, markers, and length measurement in centimeter) were given in case students needed them.

Using Real Objects to Solve the Division Problems

The focus group observed in the real teaching experiment took the real materials and started working cooperatively to solve the problem. The first part of the given problem was to find the number of small ribbons which could be made from a 1-meter ribbon. The four focus students started to measure a 1-meter ribbon, and then tried to measure the smaller ribbons (½ meter) by using the length measurement which was in centimeter. They converted the length, all into centimeter. From one of the edge of the ribbon, students in the focus group started measuring as long as 50-centimeter, which equals to ½ meter. They found that from the 1-meter ribbon, they could make two 50-centimeter ribbons, or two halves. Therefore, the number of small ribbons having length ½ meter which could be made is two.

The focus group still converted the length given in the problem in meter into centimeter when they solved the other problems. For example, to find the number of small ribbons each having length ¼ meter which could be made from a 1-meter ribbon, they tried to find a whole number in which if they multiplied it with 4, the result will be 100 (100 cm), or 1 meter. They got 25, because 4 times 25 is 100. Therefore, they concluded that ¼ meter equals to 25 cm.

To find how many small ribbons having length $\frac{1}{5}$ meter which can be made from 1-meter ribbon, the four students in the focus group were still trying to find a whole number in which if they multiplied it by 5, the result would be 100. So they found 20 cm, because 5 times 20 equals to 100. Therefore, they concluded that $\frac{1}{5}$ meter equals to 20 cm. Next, they used the length measurement to predict how many times they should fold the 1-meter ribbon if they had some small parts having length 20 cm. Finally they found that it was 5 partitions of 20 cm which they could make from a 1-meter ribbon.

One of strategies students used to solve the measurement division problem was by converting the length from meter into centimeter. It was because they would have whole numbers instead of fractions which were easier to calculate.

Figure 17. Using length measurement to find the number of partitions

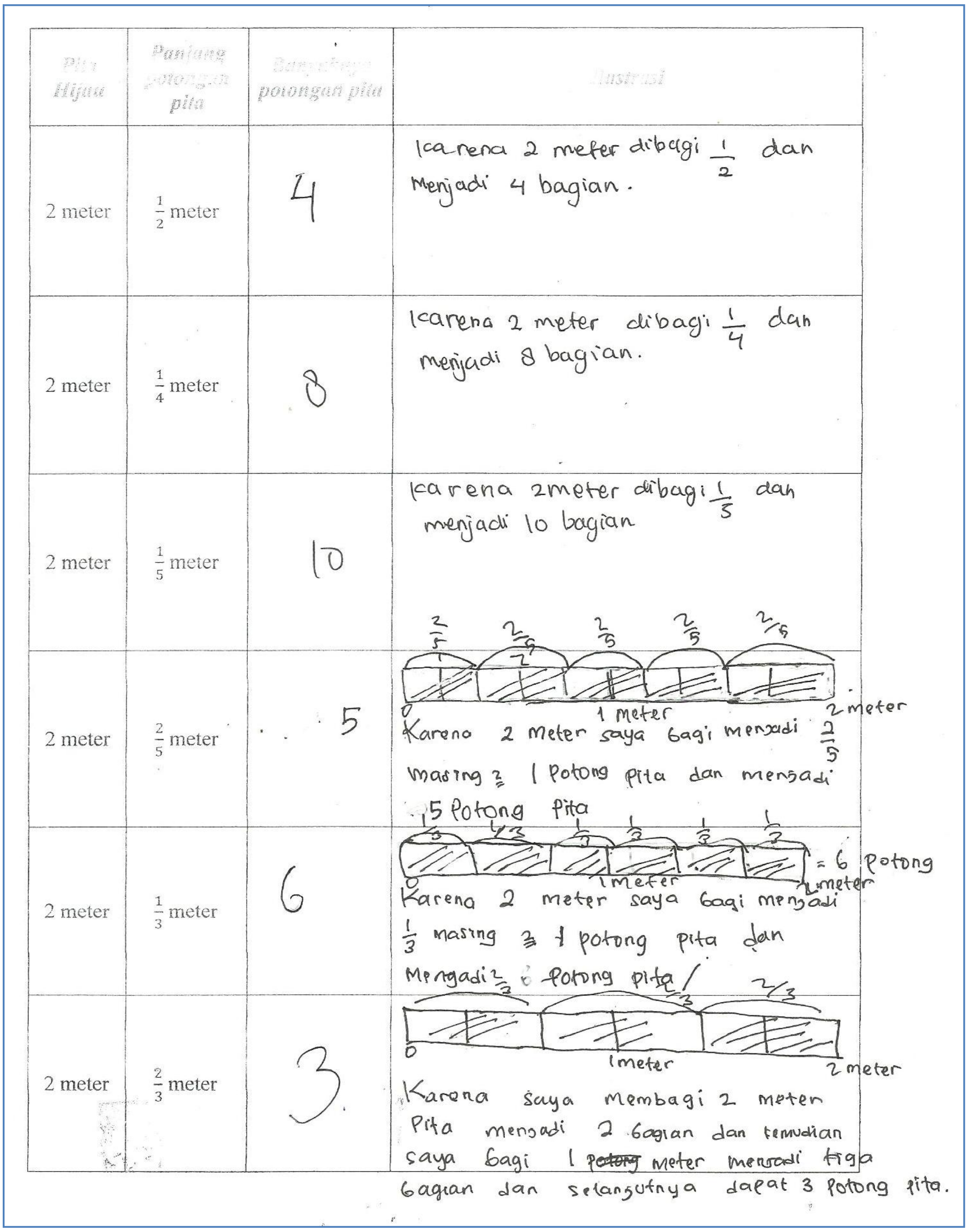
The focus group students who always converted the length from meter into centimeter didn't find difficulties to convert length which are fractions which result to whole number if they are multiplied by 100. For instance, $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}$. However, they were struggling to find the number of small ribbons each having length $\frac{1}{3}$ meter because they couldn't find a number in which if they multiplied it by 3 would result 100. They had difficulties to convert how many centimeter a $\frac{1}{3}$ meter is. So they left the problem which the length of each small ribbon is $\frac{1}{3}$ meter.

Using Models to Solve the Measurement Division

The four students in the focus group were using real materials to solve problems relating to measurement division. In the first set of the problems given, which consisted of four problems, the length of the total ribbon divided was 1 meter. However, in the next sets of problems, the length of the total ribbon was 2 and 3 meter. It's true that they would need a lot of work to do if they measured the real ribbon which length is exactly 2 and 3 meters. They found it very cumbersome and they tried to use different strategies to solve the next problems.

After the four problems relating to measurement division where the length of the ribbon divided is one meter, there are some other problems which are discussed by the students. In the next 6 problems, the length of the ribbon divided is two meter. The length of small ribbons made from dividing the 2-meter ribbon are $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5}, \frac{1}{3}, and\ \frac{2}{3}$. The four students in the focus group easily solved the first three problems, which are to find the number of partitions which could be made from 2-meter ribbons in which the length of each partition respectively $\frac{1}{2}, \frac{1}{4}$, and $\frac{1}{5}$

meter. However, to find the number of partitions in which the length of each part are $\frac{2}{5}$, $\frac{1}{3}$, and $\frac{2}{3}$ meters was difficult for them.



Pita Hijau	Panjang potongan pita	Banyaknya potongan pita	Ilustrasi
2 meter	$\frac{1}{2}$ meter	4	karena 2 meter dibagi $\frac{1}{2}$ dan menjadi 4 bagian.
2 meter	$\frac{1}{4}$ meter	8	karena 2 meter dibagi $\frac{1}{4}$ dan menjadi 8 bagian.
2 meter	$\frac{1}{5}$ meter	10	karena 2meter dibagi $\frac{1}{5}$ dan menjadi 10 bagian
2 meter	$\frac{2}{5}$ meter	5	$\frac{2}{5}$ $\frac{2}{5}$ $\frac{2}{5}$ $\frac{2}{5}$ $\frac{2}{5}$ 0 1 meter 2 meter Karena 2 meter saya bagi menjadi $\frac{2}{5}$ masing 2 1 potong pita dan menjadi 5 potong pita
2 meter	$\frac{1}{3}$ meter	6	$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$ = 6 potong 0 1meter 2 meter Karena 2 meter saya bagi menjadi $\frac{1}{3}$ masing 2 1 potong pita dan mengadi $\frac{2}{3}$ 6 potong pita /
2 meter	$\frac{2}{3}$ meter	3	$\frac{2}{3}$ $\frac{2}{3}$ $\frac{2}{3}$ 0 1meter 2 meter Karena saya membagi 2 meter pita menjadi 2 bagian dan kemudian saya bagi 1 ~~potong~~ meter menjadi ~~tiga~~ bagian dan selanjutnya dapat 3 potong pita.

Figure 18. The focus group's worksheet

The second set of problems relating to measurement division was to find the number of partitions which the length of each part was given from a 2-meter ribbon. The first problem was to find the number of parts in which the length of each part is ½ meter. In the column illustration, students gave reasons why they answered that the number of partitions is four. They wrote, "*because 2 divided by ½ becomes 4*".

From the classroom observation, field notes, and video recorder, the four students didn't use real materials to solve the first three problems in this second set of problems. From the interview, they said that they already knew that if the length of the ribbon divided was 1 meter, the number of parts having length ½ meter which could be made would be 2 (from the previous set of problems). Therefore, for the two meter ribbon, the number of parts would be four.

To solve the second and third problems in this second set of problems, students in the focus group were still using information they knew from the first set of problems. They didn't need real materials to measure in order to find the answer. They knew that in a 1-meter ribbon, they could make 4 small ribbons having length ¼ meter, so they concluded that there would be 8 small ribbons having length ¼ meter which could be made from a 2-meter ribbon.

The four students in the focus group seemed struggling to solve problem numbered 4, 5, and 6 in this second set of problems. To find the last three problems, the teacher prompted the four students in the focus group to make a drawing of the situation. The fourth problem was to find the number of small ribbon having length $\frac{2}{5}$ meter which could be made from a 2-meter ribbon. One of the students, Vanya, had made some drawings in her book. The teacher came to the group and suggested all of the students in the group to solve the problem by making drawings of ribbons like Vanya did. Vanya looked confused to use her drawing to solve the problem. Therefore, the teacher gave some guidance for the group to solve the problem. Here is what the teacher said during giving the guidance.

Teacher : Can you make a drawing of this problem? How long is the ribbon? 2 meters? Can you find…can you make a drawing of 2-meter ribbon? Let's draw first the 1-meter. How long is the part? $\frac{2}{5}$ meter? From the 1 meter ribbon, can you find where the $\frac{2}{5}$ meter is? How many times you have to divide? So, how can you determine the $\frac{2}{5}$ meter?

Figure 19. Determining a $\frac{2}{5}$ meter from a drawing of 2-meter ribbon

Students in the focus group were still struggling to solve the problem by using drawings. One of the students from focus group, Ajib, saw how the teacher guided another group in the class. The group was also trying to determine the number of ribbons having length $\frac{2}{5}$ meter from a 2-meter ribbon. The teacher gave guidance for the group. The students in the group had already made a drawing of 2 meter ribbon, in which each meter of it had already been divided into five equal parts, so they had 10 equal parts from the 2 meter ribbon. The teacher gave guidance for the group relating to how to determine the number of small parts having length $\frac{2}{5}$ meter which could be made from the drawing of 2-meter ribbon.

Teacher : Now, can you determine how long the $\frac{2}{5}$ meter starting from this drawing (pointing at one of the edge of the rectangular model)? Where's the $\frac{2}{5}$? Make a mark on it! (the student make a mark). Yes, that's $\frac{2}{5}$. And then, can you make another $\frac{2}{5}$ again? Don't do it irregularly! Continue from here (pointing at the end of the first $\frac{2}{5}$)! So the ribbon has been cut until here, right? Can you give mark on where should we cut another $\frac{2}{5}$ again? (students are marking). Ok, give sign "$\frac{2}{5}$" here, so we know that it is $\frac{2}{5}$. And then, where's another $\frac{2}{5}$? Yes, and then? Ok, and then? Ok. Now, count how many $\frac{2}{5}$ you have made? (student: five!) So, how many parts you have made?

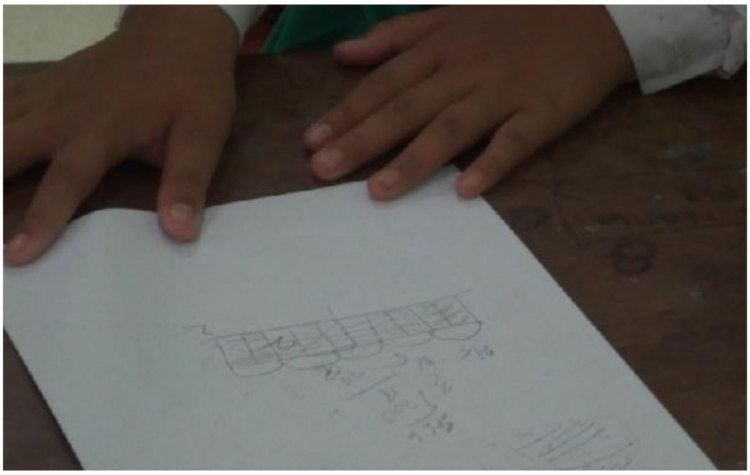


Figure 20. Making a drawing to solve a division problem

After looking at the teacher explanation, Ajib and his friends in the focus group start solving the problem. With the guide of the teacher, they could find the number of small parts having length $\frac{2}{5}$ meter from a 2-meter ribbon.

Ajib and Vonny discussed the problem, and the other two students in the focus group listened to the discussion. They knew that to get the $\frac{2}{5}$ meter of ribbon, they should divide a 1-meter ribbon into five equal parts. Therefore, they had 10 equal parts of $\frac{1}{5}$ meter in all 2 meters ribbon. Then, they shaded every two $\frac{1}{5}$-meter part, so they got 5 parts of $\frac{2}{5}$ meter from their drawing. They concluded that there are 5 parts of small ribbons having length $\frac{2}{5}$ meter which could be made from a 2-meter ribbon.

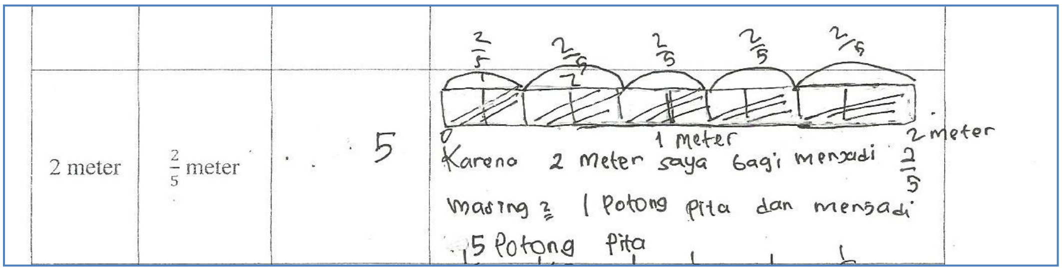


Figure 21. The focus group's answer to find the number of partitions made from 2-m ribbon

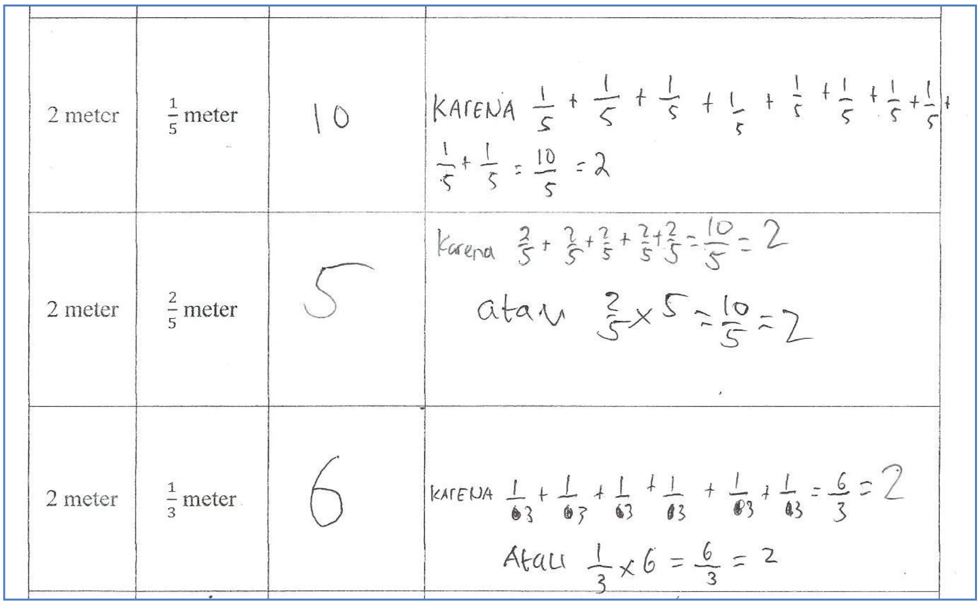
The focus group was good at using models to solve the problem. They were still using the models to solve the next problem relating to dividing a 3-meter ribbon into some parts in which the length of each part was given. They didn't have difficulties to find the number of partitions although the length of each part known were not fractions which result to whole number if they are multiplied by 100. They could find the number of parts if the length of each part were $\frac{1}{3}$ and $\frac{2}{3}$ meters.

The last problem relating to the measurement division was still about finding the number of part from a total length of ribbon in which the length of each part is given, but the total length of the ribbon couldn't evenly been divided. There was a remainder from the division. Although in the HLT it was predicted that students wouldn't be able to express the result as a fraction, at least there would be some students know that there is a remainder and they couldn't make a new part of small ribbon from it.

The focus group also realized the situation in the last problem. They were trying to find the number of small ribbon having length $\frac{2}{3}$ meter which was made from a 3-meter ribbon. They started to solve the problem by dividing each 1-meter ribbon by 3, so they got 9 small parts of ribbon having length $\frac{1}{3}$ meter. To get the $\frac{2}{3}$ meter, they shaded every two $\frac{1}{3}$-meter parts, so in the end they got 4 groups of $\frac{2}{3}$ meter and there was a small length of $\frac{1}{3}$ meter left. They knew that the leftover is $\frac{1}{3}$ meter and they couldn't make any new small ribbon of $\frac{2}{3}$ meter from the leftover.

Conjectures which were made in the HLT predicted that there would be at least four different strategies to solve the problem relating to measurement division. However, there were only two strategies which were used by students in the focus group. They used the real materials (ribbons and length measurement) and change the length from meter into centimeters, and they also used models referring the situations in the problems. The other two conjectures made were using more formal strategies to solve the problem. The first strategy is to use repeated addition, and the second is to use the relation between multiplication involving fractions to solve the problem.

A more formal approach to solve the problem was used by a group of students in the classroom during the real teaching experiment. They used repeated additions, adding the length of the part (the shorter ribbon) many times until they got the total length of ribbon, and also used multiplication involving fractions to solve the problems. They used multiplication involving a fraction and a whole number, multiplying the length of each part with a whole number in which the result would be the total length of the ribbon. Then, the whole number is the number of parts made.



2 meter	$\frac{1}{5}$ meter	10	KARENA $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{10}{5}=2$
2 meter	$\frac{2}{5}$ meter	5	Karena $\frac{2}{5}+\frac{2}{5}+\frac{2}{5}+\frac{2}{5}+\frac{2}{5}=\frac{10}{5}=2$ atau $\frac{2}{5}\times 5=\frac{10}{5}=2$
2 meter	$\frac{1}{3}$ meter	6	KARENA $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{6}{3}=2$ Atau $\frac{1}{3}\times 6=\frac{6}{3}=2$

Figure 22. Using more formal approaches to solve the division problems

The group of students who used more formal strategies to solve the problem didn't need real materials to solve the problem, nor did make a drawing of a bar representing the ribbon.

Classroom Discussion

In the end of the activity of the measurement division which was held for two meetings, each meeting was approximately 70 minutes, there was a classroom discussion, in which all of the groups were discussing their findings. The teacher made a table of the number of parts which could be made from a given length of ribbon in the whiteboard and asked the students to write down their answer. There was also a column of illustration in which students could write down their reason of their answer. Some students could write down the mathematical expressions by using repeated addition or multiplication of a fraction with a whole number. In the HLT, the lesson relating to making the mathematical sentences would be learned in the third meeting.

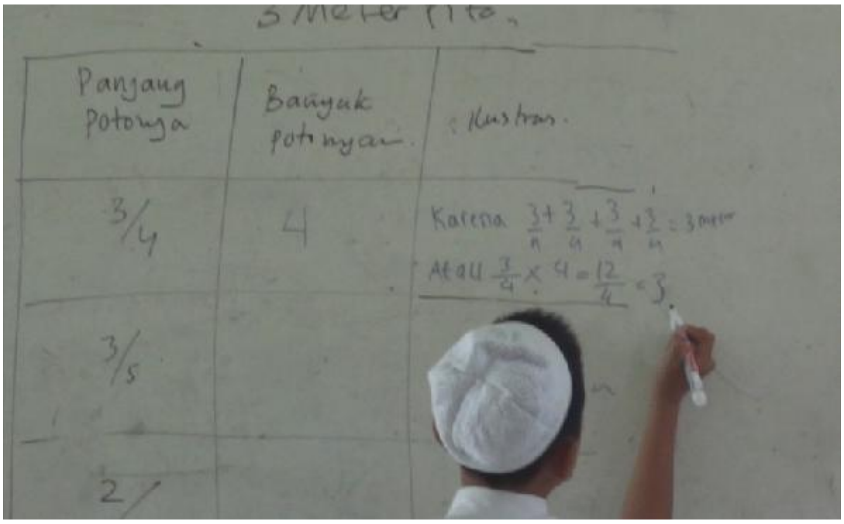


Figure 23. Discussing some strategies to solve division problems

There were two mathematics equations written by students in the classroom during the discussion. One was written as repeated addition. Students were adding the length of each part by itself many times until they got the total length. They found that there are 4 parts of small ribbon having length ¾ meter which could be made from a 3-meter ribbon because they needed to add four three-quarters together in order to get 3 meters. Another expression was written as a multiplication involving a fraction and a whole number. Students found that there are 4 parts of small ribbon with length ¾ meter from a 3-meter ribbon, because if they multiply the ¾ by 4, they got the total length 3 meters.

Students in the focus group hadn't written down the mathematics expressions. They solved the problem by exploring the real materials or making a drawing of the ribbon and making some partitions of it. After there were some students writing down some mathematical expressions, in the third meeting they are expected to have some idea to write down some mathematical expressions of the given word expressions.

5.2.3 Activity 2: Finding the Multiplication and Division Equations

The goal of this activity is that students can find mathematical equations involving multiplication and division of fractions from a given statement relating to measuring activity done in the first and second meetings. In the two meetings before, students had found the number of small ribbons in which the length is given from a longer ribbon. To find the number of parts, some students had reasoned by using repeated additions or using multiplication operation involving a fraction and a whole number. Some of them had known how to express a words statement by using those two operations, but some were still struggling to express it. From the previous two meetings, there was no student who wrote an equation using division operation.

The third lesson was started from a question given by the teacher which aimed to remind students about the previous meetings relating to measuring activities. After getting the answer of the problem, the teacher guide students to make a statement in words relating to the given problem, and then wrote down some mathematical equations which could be made from the statement.

The problem posed by the teacher was to find the number of small ribbons having length ¾ meter which could be made from a 3-meter ribbon. There was immediately a student who raised his hand, and voluntarily wrote his answer in the whiteboard. The student who raised his hand, Rafly, is the best student in the class. He immediately shouted that there are 4 small ribbons of ¾ meter which could be made from a 3-meter ribbon. Other students in the classroom agreed that there are 4 parts.

The mathematics teacher emphasized a statement, "if we have a 3-meter ribbon and we are going to divide it into some equal parts, each having length ¾ meter, there will be 4 equal parts made". From the statement, she asked students to write down the mathematical expressions expressing the situation.

In front of the class, Rafly wrote three mathematical equations he knew. He had a repeated addition, a multiplication involving a fraction and a whole number, and a division equation. It was surprising because Rafly used the invert-and-multiply algorithm, an algorithm to solve a division of fraction. The conjectures made in the HLT don't include this kind of situation. The HLT was made with the assumption that students hadn't known about the invert-and-multiply algorithm because they hadn't learned about division of fractions before in the classroom.

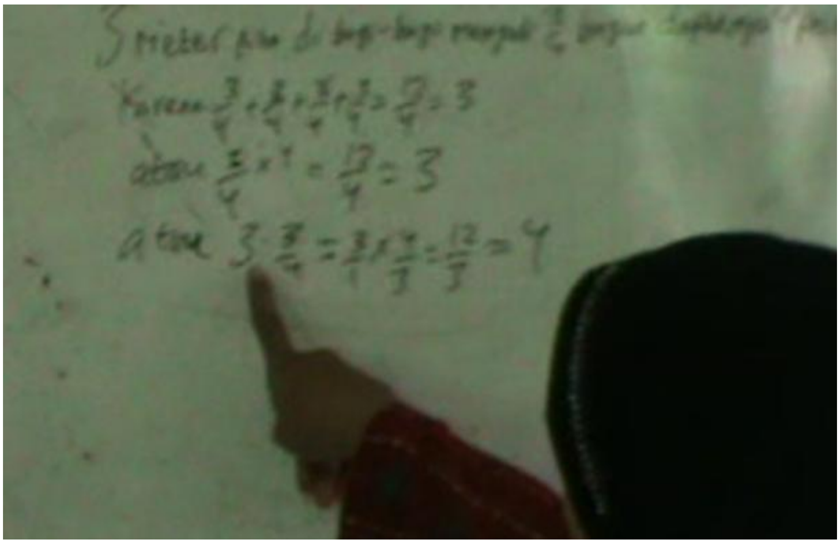


Figure 24. The teacher looked at the inversion algorithm for the division of fractions

A student from the focus group, Ajib, asked the three mathematics equations made by Rafly critically. He saw that in the first two equations, equations in repeated additions and multiplication, there was a 3 in the right side of the equal sign. Different from those two equations, the last equation expressed by using division has a 4 in the right side of the equal sign.

Answering Ajib's confusion, the mathematics teacher gave meaning of every number written in the division equation. She prompted all students with questions like, "what is 4 here? What does 3 here mean?" so the equation would be more meaningful. She emphasized that the 4 in the division equation stands for the number of parts made, the 3 means the total length of the ribbon divided, and the ¾ is the length of each part of the ribbon. So, if there is a total number of ribbon divided equally in which the length of each part is given, the result is the number of part. In the other hand, for the multiplication operation, if there is given the length of each part of the ribbon and the number of parts made, the product is the total length of the ribbon.

The teacher then gave students a set of four problems relating to making mathematical equations using multiplication and division from a given statement in words. Students were working in groups to do the answer sheet.

In the previous activity (activity 2), students had been working to find the mathematical equations using multiplication and division operation from a given statement relating to measurement division. A statement was given and students could write down as many mathematical expressions as possible in their answer sheet regarding to the given statement. The following is an example of the statement and students' mathematical expressions.

Statement: "*From a 1-meter ribbon, we can make 2 shorter ribbons each having length ½ meter*"

Students' answers:

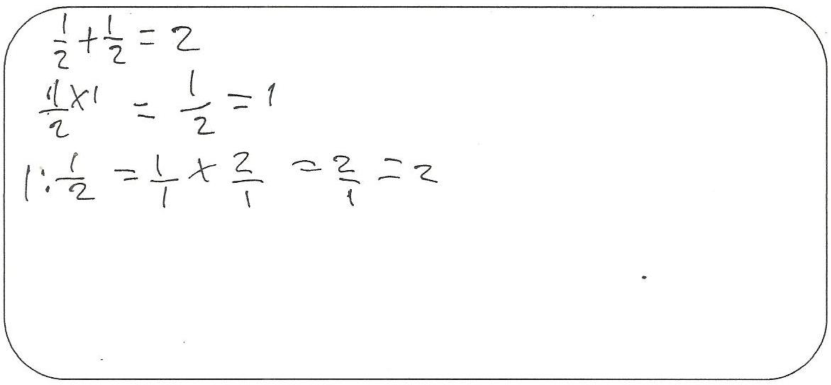


Figure 25. Generating mathematical equations from a given statement

The above picture is the answer of the students from the focus group. The first mathematical sentence is a repeated addition. They add the length of two shorter ribbons in which the length of each is ½ meter, and they answered 2 meters, whereas the total length of the ribbon should be 1 meter. It was probably because students were thinking informally but writing down the equation as they thought it formally. When students wrote $\frac{1}{2}+\frac{1}{2}=2$, students might informally think that they had one half and another one half, so they had two halves. Therefore they wrote down 2 in the right side of the equation.

The second answer is using multiplication operation. The answer is incorrect. Instead of multiplying ½ with 1, the sentence should multiply the ½ with the number of parts made, which is 2. The third equation is a division equation. They divide the total length of ribbon with the length of each part and the result is the number of the part. However, they used the invert-and-multiply algorithm to find the result of the division. It seems that they had seen the algorithm used by Rafly before.

After some minutes, they were discussing their findings together in a small classroom discussion. The result showed that all groups could make the mathematics equations using multiplication and division operations. However, students were using the invert-and-multiply algorithm again to solve the division, although in the statement, all the three variables, the number of parts, the length of each part, and the total length, were given.

5.2.4 Activity 3: Discussing the Relation of Multiplication and Division of Fractions

After making mathematical equations involving fractions, the students were having a classroom discussion, led by the teacher, to see the inverse relation between a multiplication and a division of fractions. The aim of the activity is to support students finding the inverse relation between multiplication and division operations involving fractions. In other words, students are expected to know that if there is given a division equation, then the result of multiplying the result of the division with the divisor is the dividend. If we write it using three numbers, a, b, c, then the relation can be shown as $a \div b = c$ and $b \times c = a$.

After discussing the given problem in groups, the discussion was set so that all students could share their findings together with other students from other groups. The teacher had a table of multiplication and division equation stuck in the whiteboard. Then, she asked students to write down the mathematics equations involving multiplication and division in the table.

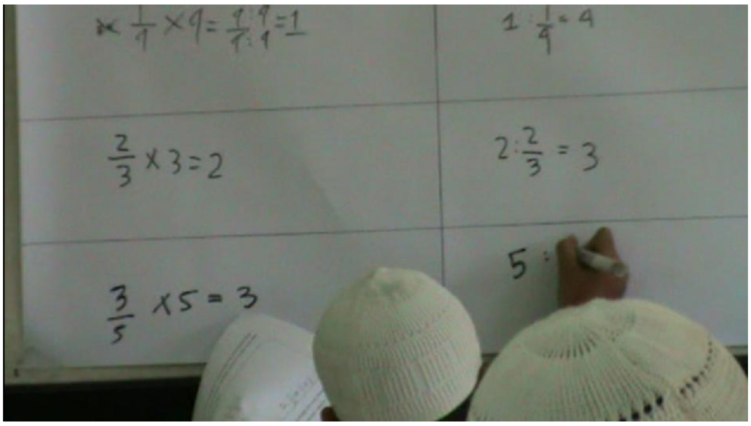


Figure 26. Students were writing down some mathematical equations

After the four statements in words had been translated into two mathematics equations involving multiplication and division operation, the teacher guided students to look carefully at the two equations. She asked students to see the pattern of the numbers. Every pair of the multiplication and division equation involved the same three numbers. Each number in the equations had the same meaning. For example, in the equation $\frac{2}{3} \times 3 = 2$, the $\frac{2}{3}$ was the length of each part, 3 was the number of partitions which could be made, and 2 was the total length of ribbon. The three numbers used in the division equation also stood for the same meaning. $2 \div \frac{2}{3} = 3$ had a meaning that if a 2-meter ribbon is divided into some parts in which the length of each part is $\frac{2}{3}$ meter, there will be 3 parts of ribbons made.

From those four pairs of equations using multiplication and division operations involving fractions, the teacher prompted students to say something about the relationship. Then, they were given a new worksheet to write down what they knew about the two related equations.

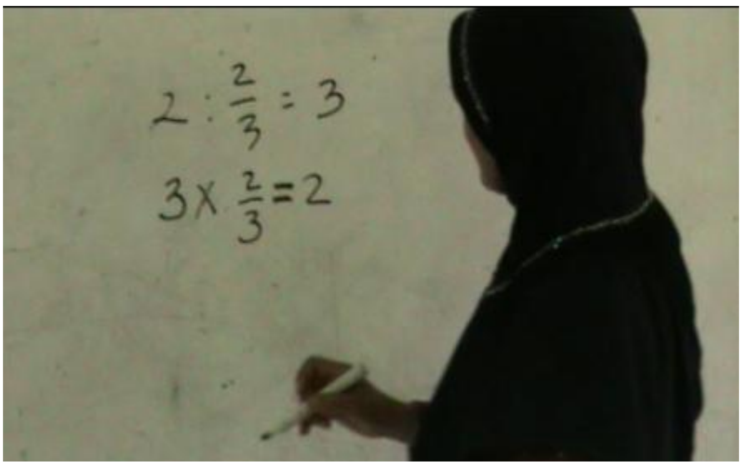


Figure 27. Teacher guided students to find the inverse relation

In the end of the third lesson, there was no student who was able to really say the relationship. They knew that those two equations involved the same number and each number had their own meaning. However, they were still struggling to see the relationship.

5.2.5 Activity 4: Playing Card Games

In the fourth meeting, the activity of making relations between the multiplication and division of fractions was continued. Because in the third meeting, students didn't seem very enthusiastic to follow the lesson, the card game was given in the beginning of this fourth meeting.

Students in group were competing to match the given card having four different colors. The quickest group which was able to match is the winner. The card game has four different colors with four different representations. There are four blue cards which contained four different statements in words. The statements were all about measurement division involving fractions and whole numbers. Four orange cards represented the drawings of the four different situations, four red cards for mathematics equations using multiplication operation, and four green cards for mathematics equations using division operations. A group of card consisted of a blue, orange, red, and green cards, each card is one.

During the card game, all students were enthusiastically participating in the game. They worked cooperatively in their group.

Figure 28. Students were playing with card games

Students were reading the cards one by one. They highlighted at the numbers used in a card and tried to make a group of cards having different colors which had the same number. In less than 5 minutes, there were some groups of students who had matched the card correctly. The focus group was the runner-up of the game.

After playing with the game, the teacher guided students to look at the two equations using multiplication and division which were on the red and blue cards. She made another table for multiplication and division operation on the whiteboard and asked students to mention the equations so she could write them down in the table.

Figure 29. Discussing the relation between multiplication and division

Like what she did in some last minutes of the third meeting, she prompted students to look carefully at the two equations in multiplication and division. She asked the pattern of the numbers used in those two equations and guided students to give meaning for the three numbers. In those two equations, the three numbers were for the total length of ribbon, the number of parts, and the length of each part made.

After having discussion with the whole students, there were some students, not from the focus group, trying to give comments on the equations. They looked at the first pair of multiplication and division equations written in the table,

$$5 \times \frac{1}{2} = 2\frac{1}{2}\ and\ 2\frac{1}{2} \div \frac{1}{2} = 5.$$

With the guide of the teacher, they gave meanings of each number in the two equations. They said that the 5 in the equation is the number of small ribbons, ½ is the length of each ribbon, and the 2 ½ is the total length of ribbon. From the division equation, they mentioned the dividend, the divisor, and the result. Finally, they concluded that if the result of the division is multiplied with the divisor, the result is the dividend.

During the discussion, most students were only listening at the discussion. Some of them gave response to some small questions asked by the teacher, for example, “is there any similarity of the multiplication and the division equations?”, “are there any new numbers which are not used in the multiplication equation which are used in the division equation?” Most students responded that there were something “similar” from the two equations and they knew that the two equations had three same numbers. They knew that the order of the numbers used in the two equations was different but they couldn't say out loud their thinking.

When leading the discussion, the teacher gave an illustration how the multiplication and the division work in the whole numbers. She gave a division equation and she put meanings in each numbers used in the equation.

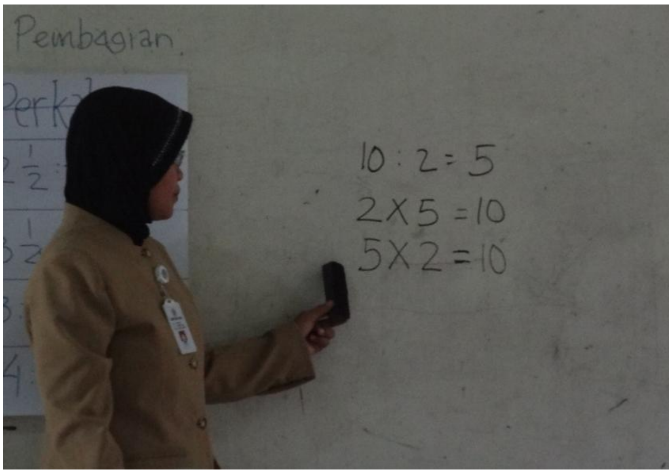


Figure 30. Looking how the division and multiplication in whole numbers

In the whiteboard, the teacher wrote a division equation, $10 \div 2 = 5$. She gave meaning that if 10 is the total length of a ribbon (in meter) and 2 is the length of each part of ribbon which is made from making some partitions of the ribbon, then there will be 5 partitions of the ribbon.

There was a piece of paper given to students where they could write down what they knew relating to the relation of the multiplication and division equations. They solved it individually. We call the paper as a reflection sheet, where they could reflect or recall anything they know relating to the topic which had been learned. They could give an example of problems and the strategy to solve it, and also the place to write down the relation between multiplication and division equations written in the table in the whiteboard.

The four students in the focus group had similar answer. In their reflection sheet, they only gave some examples of multiplication and division statements which involve the same number. For instance:

i. For multiplication: I multiply 5 number of ribbons with ½ and the result is 2 ½

   For division: I divide 2 ½ by ½ and the result is 5 number of ribbons

ii. For multiplication: I multiply 13 number of ribbons with ¼ and the result is 3 ¼

   For division: I divide 3 ¼ by ¼ and the result is 13 number of ribbons

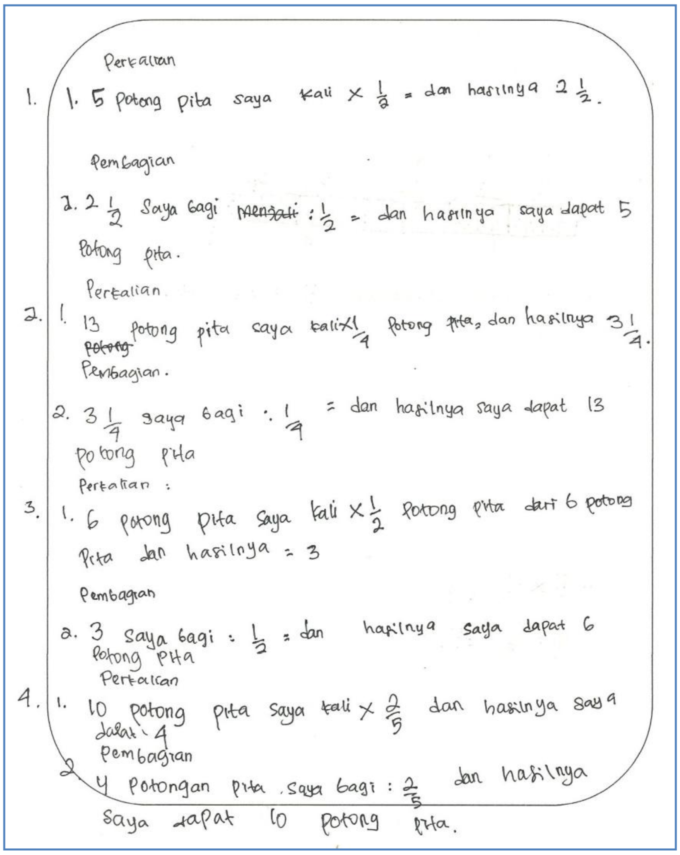
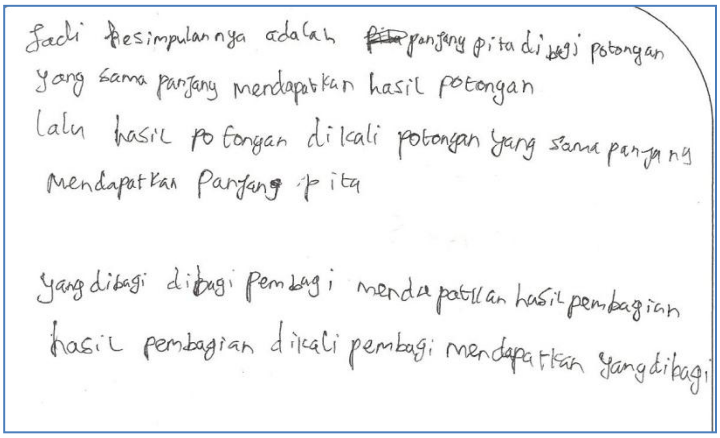

Perkalian
1. 1. 5 potong pita saya kali $\times \frac{1}{2}$ = dan hasilnya $2\frac{1}{2}$.
Pembagian
2. $2\frac{1}{2}$ saya bagi ~~mengali~~ $: \frac{1}{2}$ = dan hasilnya saya dapat 5 potong pita.
Perkalian
2. 1. 13 ~~potong~~ potong pita saya kali $\times \frac{1}{4}$ potong pita, dan hasilnya $3\frac{1}{4}$.
Pembagian.
2. $3\frac{1}{4}$ saya bagi $: \frac{1}{4}$ = dan hasilnya saya dapat 13 potong pita
Perkalian :
3. 1. 6 potong pita saya kali $\times \frac{1}{2}$ potong pita dari 6 potong pita dan hasilnya = 3
Pembagian
2. 3 saya bagi $: \frac{1}{2}$ = dan hasilnya saya dapat 6 potong pita
Perkalian
4. 1. 10 potong pita saya kali $\times \frac{2}{5}$ dan hasilnya saya dapat 4
Pembagian
2. 4 potongan pita saya bagi $: \frac{2}{5}$ dan hasilnya saya dapat 10 potong pita.

Figure 31. The conclusion made by the students in the focus group

Students in the focus group only gave examples of statements relating to multiplication and division involving fractions and whole numbers. They knew that both the multiplication and the division used the same numbers and their own functions. However, they couldn't make the real mathematical equation for the statement.

There are some interesting answers from other students from other groups. Rafly could make a general conclusion in his reflection sheet.


Jadi kesimpulannya adalah panjang pita dibagi potongan
yang sama panjang mendapatkan hasil potongan
lalu hasil potongan dikali potongan yang sama panjang
mendapatkan panjang pita

yangdibagi dibagi pembagi mendapatkan hasil pembagian
hasil pembagian dikali pembagi mendapatkan yangdibagi

Figure 32. Rafly's conclusion about the inverse relation

The translation and some interpretation of the answer sheet which Rafly wrote:

"*The conclusion is that if the length of a ribbon is divided by the length of parts having the same length, the result is the result of partitioning. Then, the result of partitioning is multiplied with the length of parts having the same length, the result is the length of a ribbon.*

Dividing a dividend with a divisor gets the quotient. Multiplying the quotient with the divisor gets the dividend."

Looking at the conclusion which Rafly had made, he didn't use some small examples to show the relationship. He was able to see the inverse relation between the multiplication and the division and he was also able to make a good generalization of the inverse relation.

5.2.6 Activity 5: The Partitive Division

The aim of having this activity was to give opportunities for students to find some strategies to solve partitive division problems. The problems given were still problems relating to measuring ribbons. In the problems, the length of the total ribbon divided and the number of partitions made were given. The task of the student was to find the length of each smaller ribbon made from the division.

There were some conjectures made in the revised HLT. To solve the partitive division problems, some students might use the real ribbon with the exact length and make some partitions of it. Some others might convert the length given in meter into centimeter and did a division in whole numbers. Some others might make a drawing of the situation, like making bar models like they had done before to solve the measurement division problem. Another conjecture was some students might use the relation of multiplication involving a fraction and a whole number to solve the problem. As an example, in order to find the length one small ribbon made from dividing a two meter ribbon by four, they would find a fraction in which if they multiplied it by 4 would result 2.

After a set of partitive division problem consisting four smaller problems was given to students, the four students in the focus group were discussing the problem together. They read the problem and tried to understand the problem. They knew that the problem asked to find the length of a small ribbon made from dividing a given ribbon into some equal parts.

The first problem was to find the length of smaller ribbon if there was a 2-meter ribbon which was divided into 4 equal parts. To solve the problem, they immediately made a drawing, a bar model, and gave mark that the length of it was 2 meter. They also put the mark, marking 1 meter.

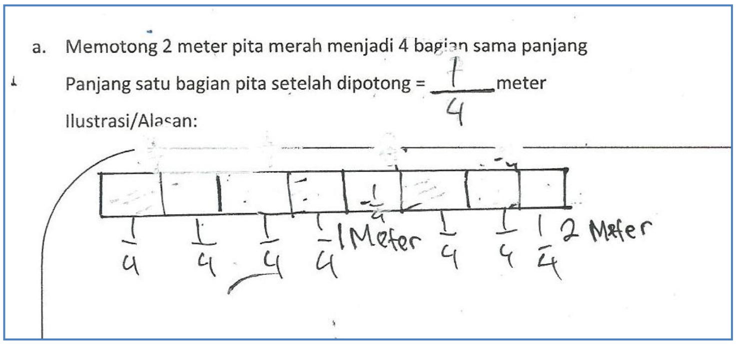


Figure 33. Dividing a 2-meter ribbon into four equal parts

In the picture above, the students in the focus group were dividing each meter ribbon into four equal parts, but actually the case was to divide the 2-meter ribbon into four equal parts. They got eight equal parts for all the 2-meter ribbon. Their final answer was ¼ meter.

The second problem was to cut a half meter ribbon into two equal parts and it asked the length of each part after the division. Vonny, who wrote on the answer sheet for the focus group draw a bar again. She said that it was a half meter. Then, Ajib said that they should make a line, dividing the bar exactly into two equal parts in the middle of the bar. They knew how to divide it in the drawing, but they had difficulties to determine the fraction representing the length. They said that it was a half meter, but they suddenly realized that the total length of ribbon divided was a half meter. So the length should be less than a half meter.

The teacher prompted the students to make a grid line which was made by extending the bar. Students in the focus group had made a bar of ½ meter, and they made a new bar having length 1 meter.

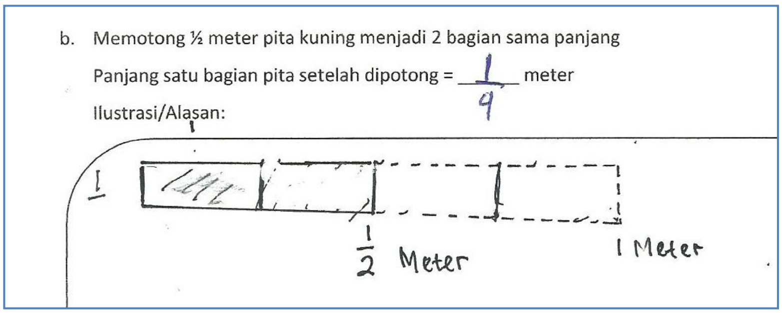


Figure 34. Dividing a half meter into two equal parts

From the picture, students had a 1-meter ribbon. They agreed that to determine the fraction of one part ribbon shaded, they should divide the other ½ meter (drawn in grid lines) into two equal parts, so they had four equal parts of ribbon. After having a drawing of 1 meter ribbon which is divided into four equal parts, students could easily recognize that the fraction representing one of the four parts is a quarter. Therefore, their answer of the length of each part of the ribbon made by dividing a half meter ribbon into two equal parts was ¼.

The third problem was to divide a three-quarter ribbon into three equal parts and students are asked to determine the length of each part. They solved the problem similarly as they solved the second problem. Students in the focus group made an extension of the bar into 1 meter, and they divide the ¾ into three equal parts. They found that there were four equal parts and the length of each ribbon asked in the problem is one part. Hence, they concluded that the length of each part was a quarter meter.

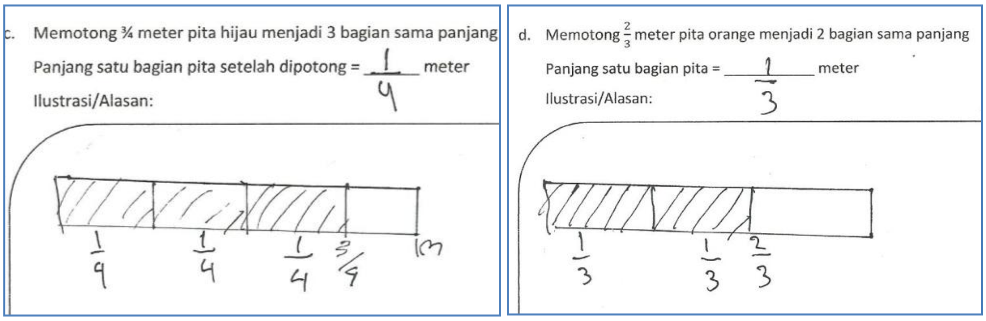


Figure 35. Extending the total length of the ribbon into 1 meter

Again, in the last problem given, which was to determine the length of small ribbon got by cutting a $\frac{2}{3}$ meter ribbon into two equal part, students in the focus group made a drawing of 1 meter ribbon, divided it by two, and gave marks on the $\frac{2}{3}$. After dividing the $\frac{2}{3}$ meter into two equal parts, they got three equal parts of ribbon

The four students in the focus group didn't convert the length from meter into centimeter and then making some partitions using real ribbon as they did when they solved the measurement division problems in the first activity. They had used the bar model to solve the measurement division problem since the second set of problems given in the first activity. They found it difficult to convert some length, like $\frac{1}{3}$ meter, into centimeter, and they found that the bar model could be used far easier. They didn't continue to use the real materials because they found it very tedious to solve problems involving a longer length of ribbon.

5.2.7 Activity 6: Making Stories

After solving partitive division problems, students were given a set of problems written in mathematical equations. To solve the problems, students gave meaning first to the numbers used in the equation. For example, the problem is written as $2 \div \frac{1}{2} = \cdots$. In order to solve the problem, students could give meaning for the 2 and the ½ in the equation. They could say that 2 is the total length of ribbon and ½ is the length of each smaller ribbon made. In a given box, students wrote their own story referring to the equation. For instance, find the number of small ribbon made from dividing a 2-meter ribbon into some smaller parts in which the length of each part is ½ meter. Then, there was a given space for students to write their strategies to solve the problem.

In the example above, the mathematical sentence is translated into words statement which is the case of measurement division. The problems given were arranged in pair. A pair of problem, which was two mathematical equations both involved division operations, was set to be translated into a measurement division problem and a partitive division problem. For an example written above, the pair is an equation $2 \div 8 = \cdots$. The 8, the divisor, written in the second equation is the result of the first equation, and the result of the second equation is the divisor of the first equation. The illustration relating to measuring activity for the second equation can be made using the case of partitive division, like "if a 2-meter ribbon is divided into 8 equal parts, how long is the length of each part?"

5.2.8 Activity 7: Discussing the Relation between Two Divisions of Fractions

After making stories from some mathematical expressions and answering the problems, a classroom discussion was held. In the classroom discussion, students discussed their findings together in the classroom, led by the mathematics teacher. The complete mathematical equations involving the two cases of division were written in a table in the whiteboard. Then, a discussion was to find the relation between the two division equations was held. The aim of the discussion was to support students finding the relation of the two division equations, that if there is given a division equation $a \div b = c$, then there will be another division equation which could be made from the three numbers used, $a \div c = b$. In other words, if there is a division equation, then the result of dividing the dividend with the answer of the first division is the divisor of the first division.

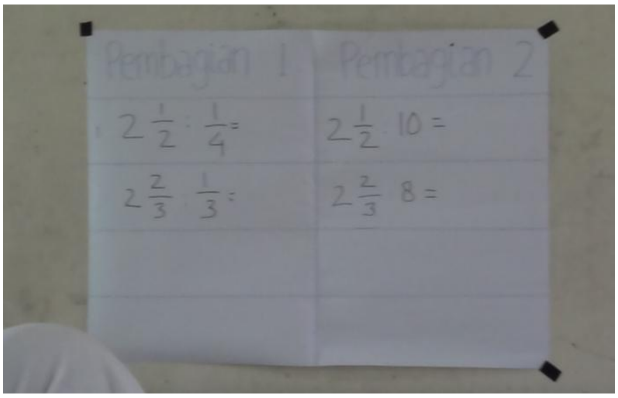


Figure 36. Discussing the relationship between two division equations

The mathematics teacher guided students to find the relationship. She made a table in the whiteboard and asked students to fill the answer based on the problems from the previous activity that they had done. After the table was completed, students then discussed some new things that they recognized. The teacher asked them to see how the position of the numbers in the first and the second division.

Students recognized that the three numbers used in the two equations were the same. With the guide of the teacher, they named the three numbers as the total length of ribbon, length of each partition, and the number of partitions. They finally could conclude that if the total length of ribbon is divided by the number of partitions, the quotient would be the length of each partition, and if the total length of ribbon is divided by the length of each partition, the quotient would be the number of partitions.

For more general conclusion, the teacher guided students to see those three numbers as the dividend, the divisor, and the quotient of a division equation. Therefore they could make a generalization that if the divisor is divided by the quotient of the division equations, they would get the divisor of the division equation.

Starting from a measuring activity, a statement involving the total length of ribbon, the number of partition, and the size of each partition could be made. From this statement, three mathematical equations using the three variables could be made. Then, after some discussion, a generalization of the relation between those three equations could be revealed.

Answering The First Sub Research Question

There were two research questions posed in this present study. The first research question was to see what strategies students used to solve the measurement and partitive division problems.

There were some students' strategies which were used to solve the measurement and partitive division problems.

a. Using real objects to solve the problem and converting the length into centimeter. Students converted the length which in the problem was given in meter into centimeter, and then measured using the length measurement to find the answer of the problem. For the partitive division problem, they converted again the length into meter to say the size of each partition made.
b. Using drawings to solve the measurement and partitive division problems. In the measurement division, students divided each meter ribbon in their drawing into some equal parts correspond to the fractional part of the size of the partitions. They saw the denominator of the fraction which was the size of the partition and divide each meter of their drawings into the number of the denominator of the fraction. In the partitive division, to divide a fractional part of the size of the total which was less than 1 meter, they made an extension of the drawing to compare the fractional part and the whole 1 meter. After having the 1 meter, they knew where the position of the fractional part of the partitions, and they could determine the number of partitions made from the fractional part of the total.

c. Using repeated addition to solve the measurement division problems. To solve the measurement division problems, students added repeatedly the size of the partitions until they got the total length of the ribbon divided.
d. Using multiplication of a fraction with a whole number. In order to find the number of partitions which could be made in the measurement division problem, students were trying to find a whole number in which if they multiplied it with the length of each partition, the answer would be the total length of the ribbon. In the partitive division, students were trying to find a fraction which is the length of each partition in which if they multiplied it with the number of parts they were asked to make, the answer would be the total length of the ribbon.
e. Using the invert-and-multiply algorithm in the division of fractions. Although the topic of the division hadn't been taught formally in the class, there were already some students who knew about the inversion algorithm and they could use it to solve the problem.

Answering The Second Sub Research Question

Another sub research question in this study is to find how the model can support students solving the division problem. In order to answer this sub research question, in the end of the lesson relating to measurement division, students were given a problem to be solved individually. The problem was to find the number of partitions having length $\frac{4}{5}$ meter which could be made from a 4 meter ribbon.

Students answered the problem by using models or using a more formal approach, like using repeated addition and multiplication involving fractions. Some others also used the inversion algorithm to solve the problem. There was none who still use real materials to solve the problem.

Zooming in into some of students' answers who answered the problem by making a drawing, or using the bar model, there were some findings showing how the students made use the bar model to solve the division problem relating to measurement division. Some students were able to solve a measurement division problem by making a good illustration in the bar model they'd made, but some students were still struggling to use the bar model.

Regarding to using the bar model as a tool to solve the problem, here are students' answers responding the measurement division problem given.

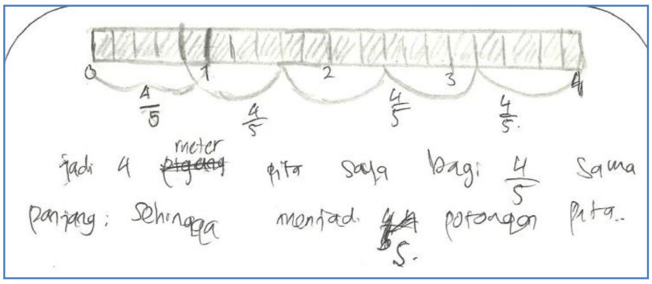


Figure 37. Aldy used a model to solve a measurement division problem

Aldy made a drawing of a 4-meter ribbon which he put marks in every 1-meter ribbon in the drawing. He divided each meter into five equal parts, and then he made some jumps of four parts. He had 5 jumps of 4 parts from the left side of the bar model to the right. From 0 to 4. Below the drawing he wrote, "*So, I* *divided the 4 meter ribbon with $\frac{4}{5}$ in the same length, so I got 5 partitions of the ribbon.*"

Aldy and some students in the class were using the same way to solve the division problem using the bar model. Some of them were able to determine the number of partitions made in order to get the length of the fractional part. They were also able to determine the number of steps they had to take from the partitions they had made. Students could use the bar model to illustrate the situation in the problem and to find the number of partitions they could make.

Different from Aldy, Isma split the 4-meter bar model into four 1-meter bar models. In her four bars, which each bar was supposed to be a 1-meter ribbon, she made partitions of each bar into our parts. Then, to find the number of partitions which could be made, she made a jump of four parts from the first bar to the fourth bar.

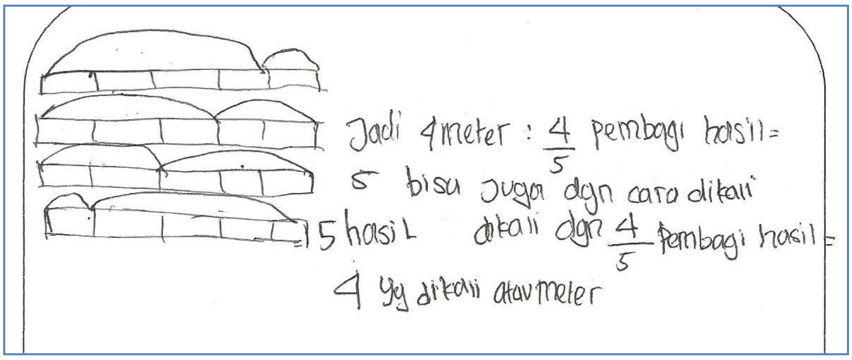


Figure 38. Splitting the bar model

In the drawing, Isma represented the four meter ribbon into four bars. To determine the length of each part, she divided each bar into five parts, and she made a jump of every four parts. The one part leftover from the first bar was combined with the three parts from the second bar. She did the same until she did the last jump, dividing all the four bars.

In the right side of her drawing, she wrote "*So, the 4 meter* ÷ $\frac{4}{5}$ *of the divisor = 5. We also can do with the multiplication. The answer 5 is multiplied with the* $\frac{4}{5}$ *of the divisor = 4 which is multiplied or meter.*"

Although some students were able to use the model correctly to support them solving measurement division problems, there were still some students who couldn't use the bar model correctly to solve the problem.

Sayyid draw a bar model representing the 4 meter ribbon. He made partitions as many as 5 in each meter of his drawing. He knew that he could get the length of the partition, $\frac{4}{5}$ m, by dividing making a jump of four parts from the partitions he had made. However, he skipped one part in his drawing.

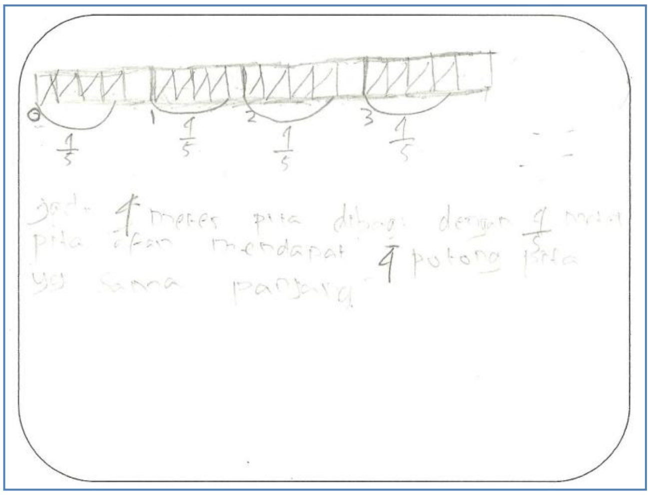


Figure 39. Skip jumping

He started his jump of four parts from the sign marking every meter of ribbon in his drawing. Therefore, he had $\frac{1}{5}$ meter left in each meter of ribbon. In his conclusion he wrote that from the 4 meter ribbon which was divided by $\frac{4}{5}$ meter ribbon, there would be 4 partitions of ribbon which had the same length.

This student could determine the number of partitions for each meter of ribbon and the number of jumps to get the length of the fractional part of the ribbon. However, he neglected the $\frac{1}{5}$ meter from every 1-meter ribbon. It was probably because in the beginning the teacher didn't give emphasis that students should maximize all of the length of the ribbon when making partitions.

Other students were able to determine the number of partitions they should make for each meter ribbon, but unable to determine the number of jumps.

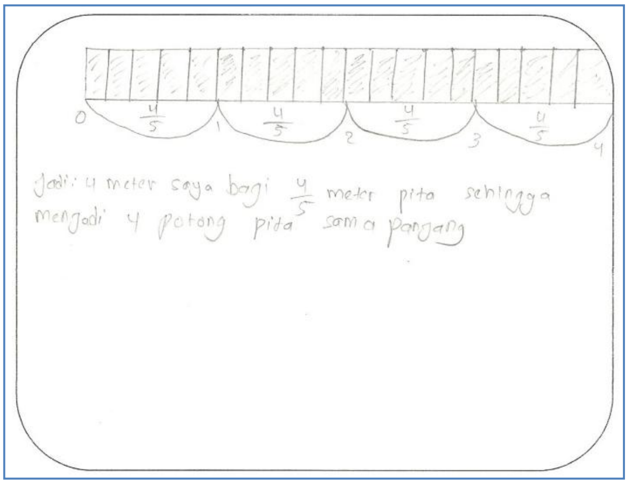


Figure 39. Incorrectly making a jump of $\frac{4}{5}$

There were still some students who weren't able to determine the number of jumps from the partitions they'd made in their drawings. Instead of making a jump of 4 parts in his drawing, Agus incorrectly made a jump of 5. He got 4 partitions, instead of 5 partitions.

Some other students prefer to use more formal approaches, like repeated additions or multiplication operations involving fractions. Some of those who used more formal approaches were able to use the bar model correctly, but some of them were not really use the bar model to solve the problem. They already got the solution of the problem by using the more formal approaches and draw the bar model fit to the solution they've got. So, the bar model was not used as a tool for them to solve the problem.

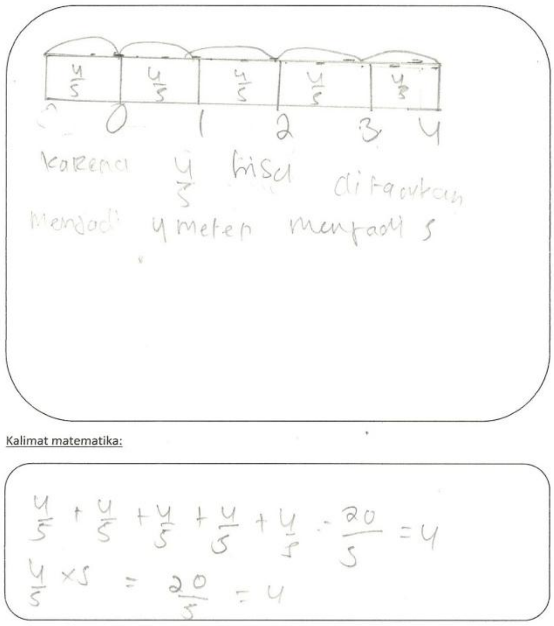


Figure 40. Incorrectly use the bar model

In the below box in the drawing, the student could use a repeated additions, adding five four-fifths and getting the total length 4 meter. She also could use multiplication involving fractions, multiplying 5 times $\frac{4}{5}$ to get the total length. In the upper box, the student wrote below the bar, "*because* $\frac{4}{5}$ *could be added to become 4 meter, become 5.*"

She probably meant that the $\frac{4}{5}$ could be added many times with itself until it would get 4, as many as 5 times adding. From her solution she probably meant that the number of partitions which could be made was 5, she got it from the multiplication and the repeated additions. Then, she made a drawing in the bar model, showing a bar model which was divided into five equal parts, and she wrote $\frac{4}{5}$ in each part she made. Looking at the position where she give mark in her $\frac{4}{5}$ meter, she might not use the bar model to find the solution of the problem. She only draws a bar which was meant to illustrate the problem, but she did it incorrectly.

Looking at how students used the bar model to solve division problems, approximately 55% of the students could use the bar model correctly to help them solving the division problems. Others were still struggling to use it. The bar model could be introduced for students who were only able to solve the problem by using real materials. When the length of the ribbon divided became longer and made them tedious to use real materials, the bar models could be used as a help to solve the problem.

The bar model could be used to find the number of partitions in a measurement division problem. In order to use the bar model to solve measurement division problems, students should be able to determine the number of partitions for each meter and the number of jumps they should take.

Chapter VI

Conclusion and Recommendation

6.1 Conclusion

This present study has shown how 5^{th} grade students learn the inverse relation between multiplication and division operation, and the relation between two division operations involving fractions from solving measurement and partitive division problems. The measurement and partitive division problems are presented as measurement problems with ribbons which can give students opportunities to explore the problems and use many strategies to solve them. A bar model is introduce as a help for students who are struggling to solve division problems.

Division problems involving stories about measuring ribbons can be expressed as a measurement division problem or a partitive division problem. In order to give help for the students to learn the inverse relation between multiplication and division operations of fractions, measurement division problems with ribbons are given. There are some strategies students used to solve the measurement division problems. Approximately, 70% of the students are still using real objects, making a drawing, using repeated additions, or converting the length given in meter into centimeter and doing multiplication or division in whole numbers. Others are able to use multiplication operations involving a whole number and a fraction, and even to use the invert-and-multiply algorithm to solve the problem, although the topic of division of fraction hasn't been learned in the formal classroom. However, students are still not able to see the relationship between the multiplication and division operations.

In order to support the discussion discussing the inverse relation between multiplication and division operations, there are two statements which are posed. From the two statements, students are guided to find mathematical equations involving multiplication and division operations of fractions. After generating the two mathematical equations, the discussion is held, and students can find the inverse relation between the multiplication and the division operations.

Measuring problems can also be represented as a measurement division problem or a partitive division problem. In the measurement division, the number of partitions is asked, whereas in the partitive division, the size of the partition is asked. From solving some pairs of measurement and partitive division problems, students can recognize the relation of the two. After getting the solution for the measurement division problem, students can make a hypothesis of the solution of the partitive division problem. With the guide of the teacher, students can see the inverse relationship between the two division problems.

6.2 Recommendation

All of the measurement and partitive division problems in this present study involve measuring activity with ribbon. The problems can give students opportunities to explore deeper by using real objects. It also can be represented as a drawing in a bar model which can easily be understood by the students. Multi-strategies occur. Beside using real object and making some drawings, this problem can provoke students to use repeated additions involving fractions, multiplication operation between a whole number and a fraction, and also can be represented as a division problem, so students can solve a division problem by using the inverse multiplication operation. Therefore, the problems relating to measuring activities can also be used in the topic additions of fractions which have the same denominators and also in the topic multiplication of fractions with a whole number. If we change the numbers used in the problems into whole numbers, the measurement and partitive division problems involving measuring activity with ribbon can also be used to introduce the inverse relationship between multiplication and division operations in whole numbers.

All of the measurement and partitive division problems used in this present study are problems which involve fractions and a whole number. In other words, for the measuring activities, the number of partitions is always a whole number, or the total length of ribbon can evenly be divided by the size of each partitions. By modifying the problem, the problems can also be used for a division problem involving fractions which also result a fraction. Looking at the flow of vertical mathematization, we can add some more activities in the measuring problems to support students understand the idea of the two algorithms in the division of fractions, which are the invert-and-multiply and the common denominator algorithm.

For more recommendations, in order to build students' understanding about the relation between the multiplication and the division involving fractions, we have to make sure that students in the classroom haven't learned about the two algorithms for the division of fractions. Otherwise students will be more sure to use the algorithm to solve the division problem, instead of using the inverse multiplication to solve the division problem.

References

Bakker, A. 2004. *Design research in statistics education: on symbolizing and computer tools*. Utrecht: CD-B Press, pp. 38-39.

Bulgar, Sylvia. 2003. *Children's sense-making of division of fractions*.

Cobb, Paul & Gravemeijer Koeno. 2006. *Educational design research*. London & New York: Routledge (Taylor & Francis group)

Cramer, K. Monson, D. Whitney, S. Leavitt, S. Wyberg, T. 2010. *Fractions and problem solving*. Journal of Mathematics Teahing in the Middle School Vol. 15, No. 6.

Freudenthal, H. 1991. *Revisiting mathematics education*. Dordrecht: Kluwer Academic Publisher.

Gravemeijer, K. 1994. *Developing realistic mathematics education*. Research group on mathematics education. Utrecht: CD-B Press.

Gravemeijer, K & Cobb, P. 2006. *Educational design research: Design research from a learning design perspective*. UK: Routledge

Gregg, J. Gregg, D. 2007. *Measurement and fair-sharing models for dividing fractions*. Journal of Research in Mathematics Education Vol. 12: 490 – 496.

Roux, A. Olivier, A. Murray, H. 2004. *Children struggling to make sense of fractions: an analysis of their argumentation*. South African Journal of Educaion Vol. 24: 88 – 94.

Shanty, N. 2011. *Design research on mathematics education: investigating the progress of Indonesian fifth grade students' learning on multiplication of fractions with natural numbers*. A thesis of a master program, Indonesia: Universitas Sriwijaya.

Tirosh, Dina. 2000. *Enhancing prospective teachers' knowledge of children's conceptions: the case of division of fractions.* Journal of Research in Mathematics Education Vol. 31, No. 1, 5 – 25.

Zaleta, C. 2006. *Invented strategies for division of fractions*. Proceedings of the 28th annual meeting of the North American Chapter of the International

Group for the Psychology of Mathematics Education. Mexico: Universidad Pedagogica Nacional.

Zaleta, C. 2008. *Oranges, posters, ribbons, & lemonade*. Journal of Mathematics Teaching in the Middle School Vol. 13, No. 8.

Appendix 1: The Teacher Guide

A. Activity 1: Measuring Activities

Time Allocation : 70 minutes

Mathematical Goals:

a. Students can make some partitions of a ribbon by measuring
b. Students can use multiplication as a mean to solve a division problem

1. Introduction

In the beginning of the lesson, ask students if they know some souvenirs or handicraft which are made from ribbons, make open discussions about this topic, let students share their experience relating to creating something from ribbons. If students are not responding or have a little experience relating to creating some things from ribbon, show students a real thing or a picture of a flower made of ribbon, and then tell a brief story about making the flower. Emphasize on the length needed to make one flower from ribbon.

The teacher can give some questions like:

- What kind of handicraft that you ever make from ribbons?
- Do you still remember how big is the length of the ribbon needed to make one thing?
- Do you think how many things that you can make from 2 meters of ribbon?

2. Giving Problems

Telling an introduction story about the enthusiasm of people living in Pakjo subdistrict to prepare the celebration of Kartini's Day and some struggles that the children in that subdistrict have when they are going to help preparing the celebration. Ask students to sit in groups, each group consists of 4 to 5 students. Then, give the written problem to the students.

Problem:

Materials: ribbons, length measurement in meter or in centimeter

To prepare the celebration of Kartini's Day which will be held in the April 21^{st}, some people together with the householders of the Pakjo sub district are preparing to make some small souvenirs made of colorful ribbons for the visitors who will come to the celebration in the hall. They decided to make some big flowers, key chains, and some small flowers. To make one big flower, they need one meter of ribbons, and to make one key chain and one small flower, they need respectively a half meter and three quarters meters of ribbons. At the current time, the committee of the celebration only has 9 meters of ribbons. The committee decides to create some souvenirs from the 9 meters of ribbon, and creates the other souvenirs later after buying more supply of ribbons. Can you help the committee to estimate how many big flowers, key chains, and small flowers that can be made from **all** of the 9 meter of ribbon?

Ask students to read the problem carefully in the group, and give them opportunities to ask whether they miss some information from the problem. While students are working in groups, choose secretly some groups which have some different answers which are interesting to discuss in the whole classroom discussion.

Tell students that you already prepare the 9-meter ribbons and some length measurement scales and ask them if they need the real ribbon and length measurement to solve the given problem.

3. **Discussions**

In the discussion, let students share their different strategies to solve the problem. Make sure that some different strategies are shown and give time for each group to write down their strategies in the whiteboard, and then to explain how they solve the problem.

Conjectures of Students' Work	**Teacher's Responses**
Instructional activity: *To find the number of souvenirs for the Kartini's Day*	
There are some possibilities to divide the 9 meters of the ribbon. Some students may be able to divide all of the 9 meters of ribbon, or some of them may have some leftover ribbons.	The teacher should remind the students that the ribbon should be evenly divided, without any leftover.

<table>
<tr><td>Some students may change the meter into centimeters to find the answer of the problem.</td><td>The teacher lets students to change the scale, like changing the 1 m into 100 cm, and a half meter into 50 cm, and then dividing the whole number. However, don’t forget to ask students to reconvert again their answers into meter.</td></tr>
<tr><td>Some students may ask how long is the length the ribbon used to make each of the souvenirs. They may also ask whether they can make only one kind of souvenirs, for example, to make only the big flowers.</td><td>The teacher should say that the students can make any number of the souvenirs from all of the 9 meters of ribbons. However, they are not allowed to make only one or two kinds of the souvenirs. For each of the souvenirs, at least they have made one.</td></tr>
<tr><td>Some students may be struggling to divide. They cannot make any models to solve it, or they cannot use the repeated addition or subtraction, or use the multiplication operations among fractions.</td><td>The teacher can give the students the real 9 meter ribbon, so the students can measure it using the length measurement to determine how long is the ribbon used to make each souvenir.</td></tr>
<tr><td>Some students may use additions or subtractions, which are combined by multiplication. For example, they’ll make the three quarter into a nicer number, like making two of three quarter, so there’ll be 2 small flowers from one and a half meter of ribbon. One key chain which has length a half meter, so they already use 2 meters of ribbon. The rest 7 meters can be used to make 7 big flowers. And other combinations.</td><td>The teacher can make the following table in the white board, and let the groups write down their findings.
<table>
<tr><th>Group</th><th>Length</th><th>Big Flower (1 m)</th><th>Key Chain (½ m)</th><th>Small Flower (¼ m)</th></tr>
<tr><td rowspan="2">1</td><td>Total</td><td></td><td></td><td></td></tr>
<tr><td>Each</td><td></td><td></td><td></td></tr>
<tr><td rowspan="2">2</td><td>Total</td><td></td><td></td><td></td></tr>
<tr><td>Each</td><td></td><td></td><td></td></tr>
</table>
Then, the teacher asks one or two groups to explain and write down their strategies to solve the problem in the whiteboard.</td></tr>
<tr><td>Some students may make a drawing of the ribbon by rescaling the length. Instead of drawing the 9 meter of ribbon, which is impossible, they’ll make 9 cm of the ribbon. Then they’ll measure the length for each of the souvenirs with a ruler and predict how many souvenirs they can make to fit the 9 cm.</td><td>The teacher let the students to explain and to write down their strategy in the whiteboard. This will become an example to do the next activity, relating to measuring and partitioning with a measurement scale.</td></tr>
</table>

Give emphasize on the strategies of using repeated additions to solve the problem. If some students notice that they can solve the problem by using multiplication involving fractions, ask students how to express the problem a multiplication problem.

Talk about the expression (the equation that can be made relating to multiplication involving fractions).

4. **Making Conclusions**

In the end of the discussion, ask a simple division problem for students, like if there are 6 meters of ribbon, how many small flower (each length ¾ m) that can be made. Some students will probably say 8 small flowers. Ask the reason why. Try to guide students to come to the expression that the paper flower will be 8 because $8 \times \frac{3}{4} = 6$.

B. Activity 2: Making Relations between Multiplication and Division

Time allocation : 70 minutes

Mathematical Goals:

d. Students can rewrite the problems in mathematics equations as a multiplication problem or a division problem
e. Students know the relation between a multiplication and a division involving fractions

1. **Introduction**

In the beginning of the activity, the teacher can remind students about the previous activity about how to make partitions from a given length of ribbon by measuring. Tell students that in this activity, they are not going to measure with the real ribbon anymore. Ask them to make drawings/models to solve the given problems.

2. **Giving Problems**

Ask students to work cooperatively in a small group with 4 – 5 students in each group. Give the problem to the students and then give time to them to read and to understand the problem. Ask if there is some information which is not clear.

There are two problems given in this activity. Let students finish the first problem with the table first. Make the first classroom discussion to discuss the first problem. From the answers get, ask students to make mathematical equations from a word statement as given in the second problems. Students may make an equation involving multiplication operation or division operations. Compare how a word statement can be represented in these two ways in the second classroom discussion.

Problem 1:

To make some decorations for the hall to prepare the celebration of Kartini's Day, your job is to find out how many small ribbons that can be made from the packaged lengths for each color ribbon. Complete the table below.

White Ribbon	Length of each small part	Number of parts
1 meter	½ meter	
1 meter	1/3 meter	
1 meter	¼ meter	
1 meter	1/5 meter	
Blue Ribbon	Length of each small part	Number of parts
2 meters	½ meter	
2 meters	1/3 meter	
2 meters	¼ meter	
2 meters	1/5 meter	
2 meters	2/3 meter	
Gold Ribbon	Length of each small part	Number of parts
3 meters	½ meter	
3 meters	1/3 meter	
3 meters	¼ meter	
3 meters	1/5 meter	
3 meters	2/3 meter	
3 meters	¾ meter	

Problem 2:

Write down in a mathematical sentence

e. From 1 meter white ribbon, you can make 2 ribbons which the length is ½ meter.
f. From 1 meter white ribbon, you can make 4 ribbons which the length is ¼ meter.

g. From 2 meters blue ribbon, you can make 3 ribbons which the length is 2/3 meter.
h. From 3 meters gold ribbon, you can make 4 ribbons which the length is ¾ meter.

3. Discussions

a. First Discussion

   The first discussion is held after all students finish the first problem.

   Let students fill the result they've got in the similar table written in the whiteboard.

   Choose two problems and discuss the strategies to solve the problem together in the classroom discussion. Make a rectangular model or a number line to solve the problems.

b. Second Discussion

   The second discussion is held after all students finish the second problem. Students may not take much time to solve the second problems. Ask them to say how they represent the problem in a mathematics statement. Ask students until getting two different representations, one is represented as a multiplication equation, and another is represented as a division equation.

   Ask students how these two different representations can happen, why a sentence can be made as a multiplication or a division equation at the same time. Ask also whether the two mathematics equations are correct.

4. Making Conclusions

After the second discussion, guide students to realize that for every division involving numbers (whole numbers or fractions), the result of multiplying the quotient and the divisor is always the dividend. Give two or three small problems to be answered orally.

C. Activity 3: Making Partitions

Time Allocation : 70 minutes

Mathematical Goals:

c. Students know how to make partitions from a given length of ribbons
d. Students can use models to solve partitive division problems

1. Introduction

In the beginning of the lesson, challenge students to divide a ribbon into four equal parts. Ask one or two students to demonstrate in front of the classroom. The length of the ribbon divided hasn't been known yet. Then, ask a question to students, to find the length of each part, after the ribbon is divided, when the total length is given.

2. Giving Problems

Ask students to work in small groups, 4 – 5 students in each group. Give the problem to the students to be solved together in their own group.

Problems:

Materials: Ribbon, length measurements (in centimeters or in meters)

Aji follows his father who becomes one of the committee of the Kartini's Day to come to the city hall to prepare the celebration. Because there are only a few people coming, Aji is asked to help the committee to cut some ribbons. One of the committee members gives him 2 meters of red ribbon, one meter of yellow ribbon, and three quarter meter of green ribbon. He is told to cut the red ribbon into four equal parts, also to cut the yellow and the green ribbon, each of the ribbons is cut into three equal parts. Aji is struggling to divide the ribbon into the parts asked. Can you show him how to divide it? How long does the length of each ribbon after being cut?

3. Discussions

In the discussion, let students share their different strategies to solve the problem. Make sure that some different strategies are shown and give time for each group to write down their strategies in the whiteboard, and then to explain how they solve the problem.

Conjectures of Students' Work	Teacher's Responses
Instructional activity: *To find the length of each part of the three different colors ribbons.*	
Some students may be able to just imagine the situation of the problem, like they imagine if they have a 2-meter ribbon, and if they're going to divide it into 4 equal parts, it means that each of the 1-meter should be divided into two. They'll get two equal parts for each meter, so they'll get exactly four equal parts from the two meters, in which each part has ½ meter in length.	The teacher should let the students who have this kind of thinking to explain their idea. Then, ask them to make a visualization on how they divide the ribbon, by providing a drawing of a ribbon which is agreed that the length is 2 meter. See if the students are able to divide the model, as if they are dividing the real 2 meter ribbon.
Some students may be struggling to divide.	The teacher can draw a model of the ribbons, then ask the students to show how to divide them. If they are not able to divide or to estimate the length of each part of the ribbons, the teacher can give the students the real two-meter, one-meter, and three-quarter-meter of ribbons with different colors. Let them fold the ribbons, and then measure the length of each part.
Some students may use the multiplication operations to find the answer, as follows. 4 x … = 2 3 x … = 1 3 x … = ¾	The teacher can choose the students who work with this strategy to explain in the classroom. Ask them why they make the blank space, and why they multiply the first with four, the second and the third with three.
Students may convert the length into centimeters. From dividing the 1 m, or 100 cm, ribbon into three equal parts, the length of each part is about 33 $\frac{1}{3}$ cm. From dividing the ¾ m, which is 75 cm, into three equal parts, the length of each part is 25 cm.	The teacher should tell the students that they should convert the length back into meter. If the students are not able to do that, give an example of a 'benchmark' length, like how many meter is 50 cm. Maybe most of them will immediately know that 50 cm is equal to a half meter. Then ask them how to do that. If there's no student who explains why the 50 cm equals to a half meter, tell students to make a fraction. One meter is equal to 100 cm, so if there's 50 cm there'll be $\frac{50}{100}\ m = \frac{1}{2}m$. Let the students continue to find how many meters the 25 cm is by

	making fractions with denominator 100.
Some students may find incorrect fractions. In the ¾ ribbon, they divide it into three equal parts. They may think that each part should be 1/3 because they divide it into three equal parts.	If the students can find a correct fraction for the 1 m ribbon, which is 1/3, ask them to compare whether the 1/3 that they get by dividing the ¾ m ribbon equals to the 1/3 getting from dividing the 1 m ribbon into three parts. The other way is that the teacher can ask students to convert the ¾ meters into cm, which is 75 cm, and then divide it into three parts, so each part should be 25 cm. Ask them to find how many meter the 25 cm is.

4. **Making Conclusions**

In the end of the lesson, give a problem to students to be solved individually for some minutes, and then guide them to discuss the answer together in the classroom. The teacher can ask a question like, "Andi divides 4 ½ meters of ribbon into three equal parts. How long is the length of each part?" Let students solve the problem by converting the length into centimeter (but remind them to convert the length back into meter), by making a model, making a number line, or using multiplication involving fractions to solve the problem.

D. Activity 4: Making Relations between Two Division Problems

Time Allocation : 70 minutes

Mathematical Goals:

Students know the property in divisions involving fractions that for a, b, c are rational numbers, if $a \div b = c$, then $a \div c = b$.

1. **Introduction**

Remind students about some division problems involving fractions which are the measurement division or the partitive division. The teacher can give two simple questions for students. As examples, "Aji is going to divide a 3-m rope into 12 parts. How long is the length of each rope?" or

"How many ¼ m ribbons that can be made from a 3-m ribbons?" Try to use problems which involve the same numbers.

2. **Giving Problems**

Ask students to work in pairs. Let them discuss the two problems given.

Problems:

c. (*Measurement division problem*) From a 3-meter ribbon, Indah is going to make some flowers. To make one big flower from ribbon needs $\frac{3}{4}$ meter. How many flowers that she can make? Make a mathematics equation from this problem!

d. (*Partitive division problem*) Sinta is going to divide 3-meter ribbon into 4 equal parts. How long is the length of each part? Make a mathematical equation from this problem!

After solving the problems, ask some possible answers for each problem to all students. Some of them may have the same answer. If there are any different answers, ask students who have different answers to explain their argument, or ask one of the students having common answer to explain their reasoning.

Then ask two or four student to write down the mathematics equations from the two problems. Guide student if they are not able to write down the mathematics equations, or offer students who are able to do that first.

Then, give the following problem for students to be discussed in pairs again.

Problem:

Dian and Rani are creating handicraft from ribbons. Each of them has $5\frac{1}{3}$ meters of ribbon. Dian is going to make some big flowers from the ribbon. She predicts that each of the big flowers will need $\frac{2}{3}$ meter of ribbon. After dividing the ribbon into some pieces in which each piece has length $\frac{2}{3}$ meter, Dian gets 8 pieces of ribbon.

In the other hand, Rani is going to make 8 key chains from the ribbon, so she divides her $5\frac{1}{3}$ meters of ribbon into eight equal pieces. She hasn't divided her ribbon, but Dian says to her that she will get $\frac{2}{3}$ meter for each piece of her ribbon. Dian is very sure with her prediction. Do you believe what Dian has said? Give your opinion!

3. **Discussions**

   The main focus in the discussion is the relation between the two given division problems. Guide students to make relations between the first two division problems by firstly finding the answers, and then making mathematical sentences from the two problems.

   Ask students about the relation between the two division problems, until they can notice themselves that when they divide the dividend with the quotient of the division, then the result is the divisor of the division.

4. **Making Conclusions**

   In the end of the lesson, guide students to conclude the lessons learned from this activity. Guide them to make a general conclusion, that if $a \div b = c$, then $a \div c = b$.

E. Activity 5: Playing with Card Games

Time Allocation : 20 minutes

Mathematical Goals:

a. Students understand that a division problem can be represented as written story problems, pictorial representations, or mathematical symbols
b. Students know two division problems which are alike, that for every division $a \div b = c$, there is another division $a \div c = b$.

Description:

In this game, students will be grouped into some small groups, each group consists of 4-5 students. All of the groups will be competing in the game, in which they will try to match the cards as fast as possible. The fastest group finishing to match all the cards will be the winner. However, there are some cards which are empty. In those empty cards, students are asked to make by themselves the missing representation of the division problems.

Appendix 2: The Teacher Interview Scheme

1. Background of the teacher.
   - How long has the teacher been teaching?
   - Is the teacher graduated from mathematics education or not?
   - Has the teacher ever taught the topic of division of fractions?
2. The teacher's experience relating to teaching the topic of division of fractions.
   - How does the teacher usually teach the topic of division of fractions? The use of models? The use of some daily life problems? Directly going to the more formal approach?
   - What are the difficulties relating to the topic?
   - Where do the sources used for teaching come from?
3. How the teacher usually organizes the classroom.
   - Does the teacher already have a list of groups of students in the 5^{th} grade?
   - If yes, how does the teacher organize the students in each group, is it homogeneous students (students with the same or almost the same ability are in the same group)? Or is it heterogeneous?
4. The variety of the students.
   - How many students are there in the classroom? How many boys? How many girls?
   - Approximately how many high achiever students? Low achiever students? (or it can be seen from the list of students' rank from the previous semester)
5. Using realistic approach in teaching mathematics.
   - Does the teacher ever use realistic approach in teaching mathematics?
   - What does the teacher know about realistic approach?
   - What's the teacher point of view relating to teaching with realistic approach?

Appendix 3: The Classroom Observation Scheme

1. How's the nature of the classroom discussion?
    - Is it mostly teacher explanation?
    - Any discussion?
    - If there's any discussion, do students in each group work cooperatively? Is there a real discussion in each group or only dominated by the work of one or two pupils?
    - Do the students speak quite often?
    - Are the students shy while they are given a chance to speak?
    - Is it a horizontal or vertical interaction?
    - Is there something like a math congress in which each group has the opportunity to speak and to share their findings?
2. How's the strategy that students use to solve some problems?
    - The topic observed may be a topic relating to addition, subtraction, or multiplication operations involving fractions (the topic of division of fractions is given after those three operations). Therefore I want to know whether students use a more informal, preformal, or more formal strategies when dealing with the problems.
    - Do students accustom working with models?
3. How's the classroom management?
    - How does the teacher open/close the class?
    - How much time given for the explanation, for the group discussion, for classroom discussions, for individual work?
    - Does the teacher use a mathematics textbook, some other resources, like internet, etc?
    - Does the teacher use some multimedia?
    - Does the teacher use some problems from daily life situations?
4. How's the socio norm in the classroom?
    - Is there any hand-rising rule?
    - Are the students allowed to go outside the classroom during the lesson?

- Do the students listen to the teacher’s explanation when she’s explaining in the classroom?
- Do the students laugh when there are some students making some mistakes?