SUPPORTING STUDENTS' REASONING IN ADDING FRACTIONS THROUGH MEASUREMENT ACTIVITIES IN GRADE FOUR OF PRIMARY SCHOOL

MASTER THESIS

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STATE UNIVERSITY OF SURABAYA
POSTGRADUATE PROGRAM
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A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Master Program of Mathematics Education, State University of Surabaya

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ABSTRACT

Anwar, Lathiful. 2011. *Supporting Students' Reasoning in Adding Fractions through Measurement Activities in Grade Four of Primary School*. Thesis, Mathematics Educations Study Program, Postgraduate Program of Surabaya State University. Supervised by: (I) Prof. Drs. Ketut Budayasa, Ph.D., and (I) Prof. Dr. Siti M. Amin, M.Pd..

Keyword: addition of fractions, students' reasoning, design research, measurement activities, measure, operator, emergent modeling

One of reasons why fractions are a topic which many students find difficult to learn is that there exist many rules calculating with fractions. Some previous researcher confirmed that the problem which students encounter in learning fraction operations is not firmly connected to concrete experiences.

For this reason, a set of measurement length activities was designed to provide concrete experiences in supporting students' reasoning in addition of fractions, because the concept of fractional number was derived from measuring. This design research aims to investigate how measurement activities could support students' reasoning and reach the mathematical goals of addition of fractions. Consequently, design research is chosen as an appropriate means to achieve this research goal and a sequence of instructional activities is designed and developed based on the investigation of students' learning processes. Students and a teacher of grade 4 in elementary school in Indonesia (i.e. SD Islam At Taqwa Surabaya) are involved in this research.

The result of the teaching experiments showed that *measuring activities* could stimulate students to acquire the idea of a addition of fractions. Furthermore, the strategies and drawing as visualization of situation used by students in solving problem could gradually be developed, through *emergent modeling*, into a *bar model* as a *model of* situation and finally become *model for* mathematical reasoning. In solving measuring context related to addition of fractions problem, emergent modeling played an important role in the shift of students' reasoning from concrete experiences in the situational level towards formal mathematical concepts of addition of fractions.

Chapter I
Introduction

A. Background

There are many researches in mathematics education that paid attention in the area of "understanding of fraction". The reason is that because fraction is a topic in which many teachers find difficult to understand and teach (Ma, 1999), and many students find difficult to learn (Clarke, Roche, Mitchell & Sukenik, 2006; Gould, 2005; Streefland, 1991). Among the factors that fractions in particular difficult to understand are their many representations and interpretations (Kjlpatrick, Swafford, & Findell, 2001).

Hasenmann (1981) and Keijzer (2003) found that one of the reasons why fractions is difficult for children is that there exist many rules in calculating fractions, which are more complicated than those for natural number. In addition, memorizing rules, concepts and lack of knowledge of basic concepts brings the difficulties in using the knowledge. Consequently, these difficulties cause students to do operations instead of understanding the mathematical concepts and making sense the operations of fractions.

In Indonesia, operation of fractions is taught from grade 4 in the second semester to grade 6 of elementary school. Soejadi (2000) stated that most of mathematics teachers in Indonesia base their teaching on teacher-centered instead of student-centered learning. Consequently, teachers use most of the contact time for explaining and solving mathematics problems, while students remain passive and simply copy what their teacher writes on the black board. In addition, mathematics problems used in assessment activities focus merely on algorithms and procedures and they lack elements of practical applications (Suryanto, 1996). Consequently, students have been trained for the skills and should have mastered such procedures even they do not 'understand'.

However, mastering the procedure is also important, but mastering the procedure without understanding it is worthless. This is the reason why; there is a need to emphasize a shift-thinking from procedure to understanding. Hasemann (1991), Kamii & Klark (1995), and Streefland (1991) confirmed that the problems that students encounter in learning fractions, especially when operations on fractions are not firmly connected to concrete experiences or significant situations. Consequently, in exploring the question of how to facilitate the transition process from concrete experiences via modeling fractions to formal reasoning and understanding several fraction, Explorative activities could be mentioned. It is known that the concept of natural numbers was derived from counting and the concept of fractional number was derived from measuring (Freudenthal, 1983; Streefland, 1991).

According to this situation, the researcher conducted design research that has purpose to develop theories about both the process of learning and means designed to support that learning (Cobb, Paul & Gravemeijer, 2006). The design research presented in this research is design research which particularly focuses on the relation among fractions as theme and use Realistic Mathematics Educations (RME) approach with measurement length as the context of the activities.

B. Research questions

Based on the explanation at the background the researcher formulated two research questions as follow:

1. *How can measurements activities support students' reasoning in adding fractions?*
2. *What kind of models used by students to support their reasoning in adding fractions?*

C. Research Aim

Based on research question, the aims of the research are:

1. Describing how measurement activities can support students' reasoning in adding fractions.
2. Describing what kind of model(s) used by students to support their reasoning in adding fractions and describe the role of the model(s)

D. Definition of Key Terms

1. Measurement is the process or the result of determining the magnitude of a quantity. Measurement length is the process of determining the magnitude of length. Measurement activities are activities in determining the magnitude of quantity, such as length.

2. Support means to help. Reasoning is the process of drawing conclusions based on evidence or stated assumptions. Support students' reasoning means to help student in processing of drawing conclusions based on evidence or stated assumptions.

3. Fraction is a number that indicates the quotient of two quantities, especially the quotient of two whole numbers written in the form a/b, whereas b is not equal to zero.

4. Adding fractions or addition of fractions in this research is addition of positive fractions with the same denominators and different denominators. For instance, 1/5 + 3/5 and 1/5 + 2/3.

5. Model is representation of situation/problem. Model describes the process of solving a contextual problem with the help of formal mathematical knowledge. Models are used as mediating tools to bridge the gap between situated knowledge and formal mathematics.

E. Assumptions

This research has a assumption related to the research subject, it is described as follows: the researcher assumes that the students' works represent the students' thinking process.

F. Significances of Research

There are some various significances of this research that are expected to be useful for the development of science, especially mathematics education. The significance of this research is presented as follows:

1. Significances for researcher:
   a. Providing an overview of how the processes of students' thinking and reasoning in constructing their understanding about addition of fractions.
   b. Providing an overview of how to design instructional activities through a Realistic Mathematics Education approach in addition of fractions that can support students' thinking and reasoning in adding fractions.
   c. Completing final task (master thesis) in mathematics education study program at the postgraduate program.
2. Significance for stakeholders: As a reference for stakeholders (i.e. teachers, curriculum developers) to design an instructional activities about a certain topic particularly the topic about addition of fractions.

Chapter II

Theoretical Framework

In order to construct groundwork of our research, the researcher developed the theoretical framework. Some literatures are studied to identify the basic concepts that required to do addition of fractions. Furthermore, this literature reviews are useful in designing instructional activities in which students will gain more insight in the addition of fractions

In this research, measurement activities are explored as experience-based activities and contextual situation to build upon students' reasoning and reach the mathematical goals of addition of fractions. Furthermore literatures about realistic mathematics education is needed in explaining and investigating how can measuring contexts can support students' thinking from informal to more formal mathematics in adding fractions.

A. Addition of fractions

Bezuk and Cramer (1989) explained that operations with fractions should be delayed until the concepts and ideas of the comparing and equivalent fractions are firmly established. On the other hand, Kieren (1976) proposed that the concept of fractions consists of some interpretation such as operator and measure. He also proposed that understanding of fractions depends on gaining an understanding of each of these different meanings. The researcher described some ideas and concepts supported to addition of fractions as follows:

1. Interpretation of Fractions

There are some interpretations of fractions such as ratio, operator, quotient, and measure. The operator and measure interpretations are considered necessary for developing proficiency in additive operations on fractions (Fosnot & Dolk, 2002; Charlambos, et al , 2005). The both interpretations, operator and measure, are described as follows:

In *the measure* aspect, fraction can represent a measure of a quantity relative to one unit of that quantity. Lamon (1999) explained that the measure interpretation is different from the other constructs in which the number of equal parts in a unit can vary depending on how many times someone partition. This successive partitioning allows to "measure" with precision. The researcher spoke of these measurements as "points" and the number line served as a model to demonstrate this. More specifically, a unit fraction is defined (i.e., 1/a) and used repeatedly to determine a distance from a preset starting point (Lamon, 2001). For example, 3/4 corresponds to the distance of 3 (1/4-units) from a given point. No wonder why this latter personality of fractions has systematically been associated with using number lines or other measuring devices (e.g., rulers, hand spam) to determine the distance from one point to another in terms of 1/a-units.

In *the operator* aspect, Clark (2007) explained that a fraction can be used as an operator to shrink and stretch a number such as 3/4 x 12 = 9 and 5/4 x 8 = 10. It could also be suggested that student lack of experience with using fractions as operators may also contribute to the common misconception that multiplication always makes the result is bigger and division always makes the result is smaller.

2. Comparing and equivalent Fractions

The usual approaches in comparing fractions are finding common denominators and using cross-multiplication. These rules can be effective in getting the correct answers but these rules do not require understanding about the size of the fractions. If children are taught these rules before they have the opportunity to think about the relative sizes of various fractions, there is little chance that they will develop any familiarity with number sense about fractions size. Comparison activities (which fraction is more?) can play a significant role in helping children develop concepts of relative fraction sizes (De Walle, 2008). He also recommended that the use of a region or number line model may help students who are struggling to reason mentally.

The general approach to help students to have an understanding of equivalent fractions is by giving them opportunities to use contexts and models to find different names for a fraction. In this stage students are expected that this is the first experience that a fixed quantity can have multiple names (De Walle, 2008). In finding the equivalent fractions, the procedures should never be taught or used until the students understand what the results means. As a concept, two fractions are equivalent if they represent the same amount or quantity, for instance the same length. Through the activities, students are expected to realize that to get an equivalent fractions, doubling the numerator and denominator by the same nonzero number.

3. Addition Fractions through Measurement Length

It is known that the concept of natural numbers was derived from counting and the concept of fractional number was derived from measuring (Freudenthal, 1983; Streefland, 1991). Consequently, in order to teach addition of fraction, the researcher used measurement of length as the context.

There are two reasons why measurement activities are used as the context. The first, measurement comprises an aspect of practical skill that is important in daily life. The second, measuring numbers represents a specific aspect, because they refer to an "environment" in which the number exists, for instance measuring distance of 3 meters (or ½ of the rope measuring 6 meters).

A foundational idea in the teaching of measurement is the concept of the unit, the unit must be compatible with the property being measured (Charlambos, 2005). When measuring objects, a unit is chosen first and iterated to measure a property and there may be part of a unit left over at the end. When dividing a whole into some parts, the parts (e g quarters) are adjusted until the set number of equal parts leave no remainder (without gaps and without overlapping).

However, the notion of fractions as numbers appears to be simplistic. Lamon (1999) also refers to a qualitative leap that students need to undertake when moving from whole to fractional numbers. Linking the measure aspect of fractions to the partitioning process, Lamon (1999) considers performing partitions other than halving as a necessary skill for the development of the measure personality of fractions. To develop the measure personality of fractions students should also be able to use a given unit interval to measure any distance from the origin (e.g., zero). This means that students should be capable of locating a number on a number line and, conversely, be able to identify a number represented by a certain point on the number line (Hannula, 2003).

The number line has been acknowledged as a suitable representational tool for assessing the extent to which students have developed the measure interpretation of fractions and for teaching *the additive operations* of fractions (Keijzer, 2003). Even though, previous research suggests that students face certain difficulties in placing numbers on the number line. In particular, it has been found that students count the partition marks on the number line rather than the intervals of the number line, use a wrong unit, especially when the number line has a length of two units, and fail to locate a fraction on the number line when the line is divided into parts equal to a multiple or a sub-multiple of the denominator of the given fraction (Baturo, 2004). Therefore, Smith (2002) suggests that to fully develop the measure personality of fractions students also need to master the order and equivalence of fractions.

There are five cluster that precede operation with fraction, namely producing fractions and their operational relations, Generating equivalencies, Operating through a mediating quantity, Doing one's own productions, and On the way to rules for the operations with fractions (Streefland, 1993).

Streefland (1993) describes the sequence of addition of fractions as follows:

a. *Producing fractions*

The activities here are concentrated on providing rich contexts at the concrete level. In solving the contextual problem, fractions is produced by means of partitioning and measuring context (Keijzer, 2003; Streefland, 1991). Attaching a length to a given unit also measures. The fraction that at first described the part-whole relationship now becomes a fractions in a measure. Through this activity, students will realize about the interpretation of fractions such as measure and operator.

b. *Generating equivalencies*

Partitioning as activity for producing fractions has its sequel in the treatment of situations in which division is better concealed. This also holds for increasing precision in the comparing and equivalent of fractions (Streefland, 1991). This means that the mathematical ideas under consideration will be applied more broadly. This also takes place in problem involving distance (length) relate to addition of fractions problem.

c. *Operating through a mediating quantity*

The point of this, it is to determine the length of all sort of combinations in which fractions appear. This is indirect method of determining the addition of fractions (Streefland, 1991; Fosnot & Dolk, 2002). The idea of common whole or common denominator can be of service in mediating quantity.

d. *Doing one's own productions*

In this stage, attention is paid to take fractions apart and put them together in order to acquire skill in producing equivalent fractions and to sharpen students' own concept of the operations. It means that students are able to solve problems in a more and more refined manner at the symbolizing level. This take place through using a variety of 'model of situations' and through applying productions methods which become more formal. The visual models here can be of service in illustrating length. A number line and bar can also be applied for this purpose.

e. On the way to rules for the operations with fractions

Free productions at a symbolizing level focuses the attention on taking fractions apart and putting them together, keeping in mind production of equivalent of fractions and developing ideas for the operations (Streefland, 1991).

Phrasing of formal rules as an activity is not considered up to this stage. On the other hand, as many activities as possible are directed towards stimulating the students to contribute their own informal ways of working.

B. Realistic Mathematics Educations

Realistic Mathematics Education (RME) is a theory of mathematics education that offers a pedagogical and didactical philosophy on mathematical learning and teaching as well as on designing instructional materials for mathematics education. The central principle of RME is that mathematics should always be meaningful to students.

The term ‘realistic’ in RME means that problem situations should be ‘experientially real’ for students. This does not necessarily mean that the problem situations are always encountered in daily life. Students can experience an abstract mathematical problem as real when the mathematics of that problem is meaningful to them. Mathematical learning should be an enhancement of common sense (Freudenthal, 1991).

Students should be allowed and encouraged to invent their own strategies and ideas, and they should learn mathematics on their own authority. At the same time, this process should lead to particular end goals. This raises the question that underlies much of the RME-based research, namely that of how to support this process of engaging students in meaningful mathematical and fractions problem solving, and using students' contributions to reach certain end goals.

The theory of RME is adjusted to mathematics education, because it includes specific characteristics on and design principles for mathematics education. Characteristics and principles of RME are described in the following sections.

1. Five characteristics of RME

There are five characteristics of the Realistic Mathematics Education (Treffers, 1987):

1. *Phenomenological exploration.* A rich and meaningful context (concrete or abstract) should be explored to develop intuitive notions that can be the basis for concept formation. In our sequence activities, we use some contextual problems as starting point in learning activity such as designing hanger, cutting scout rope, track of bike racing, et cetera. For instance, designing hanger context will be explored by students to develop their intuitive notation or symbol of fractions.

2. *Using models and symbols for progressive mathematization.* The development of students' intuitive or informal notions towards more formal mathematical concepts is a gradual process of progressive mathematization, the process by which a student extends his understanding from the concrete to the formal. From the contextual problem, students might use a variety of models, schemes, diagrams, and symbols to support their mathematization, provided these instruments are meaningful for the students and have the potential for generalization and abstraction. The double number line is used, for example, to solve the contextual problem of addition of fractions. See section 3 for instructional activities in the hypothetical learning trajectory.
3. *Using students' own constructions and productions*. It is assumed that what students make on their own is meaningful for them. Hence, using students' constructions and productions is promoted as an essential part of instruction.
4. *Interactivity.* Students' own contributions can then be used to compare and reflect on the merits of the different models or symbols. In our instructional activity, we provide small group discussion and mathematical congress, whole-class discussion. It is expected that students will learn from each other in small groups or in whole-class discussions.
5. *Intertwinement.* The important point of this characteristic is to consider an instructional sequence in its relation to other domains. This means, for instance, that theory and applications are not taught separately, but that

theory is developed from solving problems. For instance, other domains related to our instructional sequence are multiplication of fraction and whole number, ¼ x 12, and also measurement of length.

2. Three principles of RME

In addition to those characteristics, there are three principles of RME for *design* in mathematics education, such as guided reinvention, didactical phenomenology, and emergent models (Gravemeijer, 1994).

The first principle: Guided reinvention. Reinvent mathematics occurs when students progressively mathematize their own mathematical activity under the guidance of teacher and the instructional design (Treffer, 1987). The first principle of RME, guided reinvention, which states that students should experience the learning of mathematics as a process similar to the process by which mathematics was invented (Gravemeijer, 1994).

There are three methods in guide reinvention, the first method is a 'thought experiment'. In this method, we should think of how students could have reinvented the mathematics at issue themselves. The second method is to study the history of the topic at issue. The third method, elaborated by, is to use students' informal solution strategies as a source (Streefland, 1991). In our design research, we use this method. We support students' solutions in getting closer to the end goal.

The second principle: Didactical Phenomenology. Didactical phenomenology is the study of concepts in relation to phenomena with a didactical interest. Our challenge is to find phenomena by the concepts that are to be taught (Freudenthal, 1983). For example, in sections 4.5 and 4.6, our conjecture about students' thinking is that students add fractions through addition of whole number and then move back to fraction by using the idea of fractions as measure and operator and double number line as model in computation strategy.

The third principle: Emergent Models. In second principle of RME about progressive mathematization, we try to find models that can help students make progress from informal to more formal mathematical activity. In case of addition of fractions, line (track) of racing in combination with whole number in bottom and fractions in above, next we call a double number line, that represent the track of bike racing will be envisioned to become model of situation to solve contextual problem and later a model for more formal reasoning (Fosnot & Dolk, 2003; Streefland, 1991). Gravemeijer (1994) described how models-of a certain situation can become model-for more formal reasoning.

The levels of emergent modeling from situational to formal reasoning are shown in the following figure:

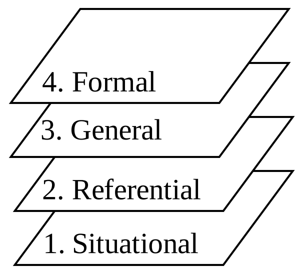


Figure 2. l. Levels of emergent modeling from situational to formal reasoning

The implementation of the four levels of emergent modeling in this research is described as follows:

1. *Situational level*

   Situational level is the basic level of emergent modeling where domain-specific, situational knowledge and strategies are used within the context of the situation. Measurement activity provides informal knowledge of addition of fractions to students when students have to determine the total part of track. There are some addition of fractions concepts that are elicited by this activity, such as interpretation of fractions, comparing fractions, equivalent of fractions, common denominator. In this level, students still use their own production of symbolizing and model of thinking related to the situation.

2. *Referential level*

   The use of models and strategies in this level refers to the situation described in the problem or, in other words, referential level is the level of models-of. A class discussion encourages students to shift from situational level to referential level when students need to make representations (drawings) as the models-of their strategies and measuring tools in the measuring activity. As an addition, the "draw number line" activity also served as referential activity in which students produced their own draw (line) to represent their way in measuring length. In this activity, student-made line became model-of the situation. See sections 3, chapter II.

3. *General level*

   In general level, models-for emerge in which the mathematical focus on strategies dominates over the reference to the contextual problem. Student—made line produced in "making our own number line" became model-for measurement when they turned to be "blank number line" as means for measuring. In this level, the blank line were independent from the students' strategies in the measuring activity.

4. *Formal level*

   In formal level, reasoning with conventional symbolizations is no longer dependent on the support of model-for mathematics activity. The focus of the discussion moves to more specific characteristics of models related to the concept of addition of fractions.

3. Addition of fractions in the Indonesian curriculum for elementary school

Fractions have been introduced to elementary school students since semester 2 grade III with learning that is focused on identifying and comparing fractions. Furthermore, in grade IV semester 2, this topics are repeated and improved, including adding fractions. So in class IV semester 2, this is the first time students learn to add fractions, which then repeated and improved in class V and VI. The following submitted details of Standard Competence (SC) and Basic Standard (BS) per grade level that gave rise to the sum of learning fractions, as well as examples of indicators that are translated from the those BS.

Table 2.1. Addition of fractions for elementary school in the Indonesian curriculum

Standard Competence (SC)	**Basic Competence (BC)**
The Second Semester of Grade IV	
6. Use fractions in problem solving	6.1 Explain the meaning of fractions and sequence of fractions (repetition and improvement the topic and comparing fractions) 6.2 Simplifying the various forms of fractions. To achieve this KD then students should have competence about the concept of equivalent fractions, fractions familiar mix, and decimal fractions. But students have not learned to change the denomination of one form into another form. 6.3 Addition of fractions. Competencies students should be able to add fractions with same denominator and are not the same. Example elaboration of indicators for KD 6.3 as follows: • Determine the result of the sum of 2 or 3 pieces of plain with same denominator. • Determine the result of the sum of 2 or 3 pieces of plain with different

	denominator. 6.4 Subtraction of fractions 6.5 Solving problems associated with Fractions

C. Hypothetical Learning Trajectory

The design research presented in this research is particularly concerned with the collaboration between addition fractions as a theme and use Realistic Mathematics Education (RME) approach. It means that the aim is to develop a *local instruction theory* on addition fractions under the guidance of the domain-specific instruction theory (RME theory), that serves as an empirically grounded theory on how a set of instructional activities can work, not merely to provide teachers with instructional activities that work in the classroom.

The *local instructional theory* consists of conjectures about a possible learning process, together with conjectures about possible means of supporting that learning process. Simon (1992) used a "*hypothetical learning trajectory"* as an instrument to bridge the gap between the instruction theory and teaching experiment.

A hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process–a prediction of how the students' thinking and understanding will evolve in the context of the learning activities. (Simon, 1995: 136).

A conjectured local instruction theory is made up of three components: (1) mathematical learning goals for students; (2) planned instructional activities and the tools that will be used; and (3) a conjectured learning process in which one anticipates how students' thinking and understanding could evolve while engaging in the proposed instructional activities (Gravemeijer, 2004).

1. Mathematical Learning Goals

There are two kind of the goal in our local instructional theory such as mathematical goals and learning goal for students. The both mathematical and learning goal we write as following table:

Table 2.2. The mathematical learning goals

Goals	
Mathematical Goals	**Learning Goals**
▪ Determine the equal fractions ➢ Obtaining the concept of equal fraction within activity ▪ Determine the common denominator ➢ Obtaining the common denominator mentally and use the idea of the equal fraction ▪ Determine Addition fractions ➢ Obtaining the idea of solving addition fractions with like and unlike denominator	1. Students will be able to compare fractions and find equivalent of fractions 2. Students will be able to find common denominator 3. Students will be able to add fractions

2. Planned Instructional Activities

Before we made instructional sequence activities, we studied literatures and books with prior research on fractions. To complement some literature studies that we read, we decided to make sequence activities of learning fractions with partitioning and measuring context with the wood and rope, the number line as models. We chose measuring length as a context, because those are not too far different from the number line, since based on Fosnot & Dolk (2002), Streefland (1991) and Freudenthal (1973), the number line is the most valuable tools to teach arithmetic.

Analyzing students' learning line or learning trajectory for a particular domain is a crucial part in designing instructional activities for students. Every stage of instructional activities should be adjusted to the level of students. Consequently, the hypothesized students' learning line for addition of fractions was analyzed before designing a sequence of instructional activities for addition of fractions. The following is a general overview of students' learning line for addition of fractions in grade 4:

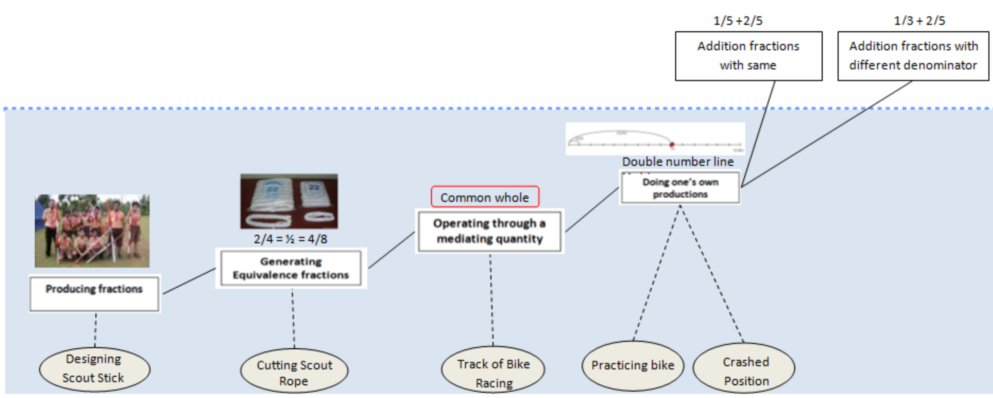


Figure 2.2. Learning line

There are five cluster that precede operation with fraction, namely producing fractions and their operational relations, Generating equivalencies, Operating through a mediating quantity, Doing one's own productions, and On the way to rules for the operations with fractions (Streefland, 1993). We elaborate them in section II;A. A set of instructional activities for addition of fractions was designed based on this hypothesized students' learning line and thinking process. This set of instructional activities was divided into six different activities that were accomplished in six days. Each day activity was aimed to achieve students' understanding in one or more basic concepts of addition of fractions. Similarly, some of basic concepts of addition of fractions were achieved from different activities. The relation among students` learning line, instructional activities and the basic concepts of linear measurement that need to be acquired is shown in the following diagram.

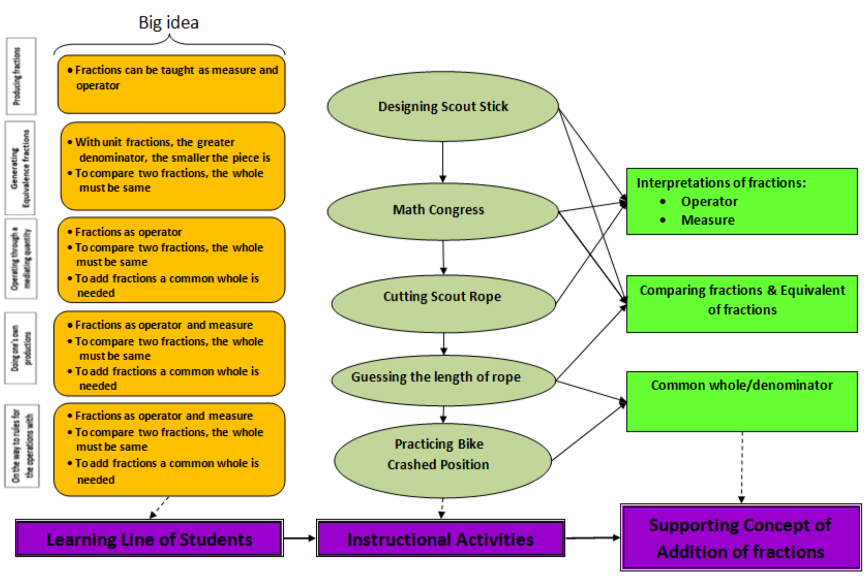


Figure 2.3. The main frame work of measurement length activities for learning addition of fractions

3. A Conjectured Learning Process

Based on the frame of sequence activities and learning line that is described in previous part, next we will describe our conjecture of the students' struggles we anticipated and the conjecture of strategies that might be used by students in the sequence activities. See the following table:

Table 2.3. Conjecture of students' learning process

Activity	Mathematical Idea	Learning Goals	Conjecture of students' thinking	
			Students struggle	Students strategies
• Designing Scout stick	• The idea of partitioning • Fractions can be taught as measure (number)	1. Students use idea of partitioning 2. Student use unit fractions to measure	✓ Measuring by standard measurement	▪ dividing into half as anchor for the larger dividing ▪ using portioning idea ▪ using ruler ▪ using folding paper
• Math congress of comparing fractions	• Fractions can be taught as measure (number) • Unit fractions as unit measurement	1. Students will know that fractions as measure 2. Student compare fractions	✓ Understanding non unit fractions ✓ Symbolizing fractions	▪ Making relation with unit fractions ▪ Recall their knowledge about symbol of fractions and making relations with unit fractions ▪ Finding fractions which are in same position ▪ comparing the same length of the rope but have different representations (symbol) of fractions
• Cutting Scout	• Fractions can be	1. Students use the idea	✓ Realizing the idea of	▪ Making Relation between

rope	thought as operator • To compare two fractions, the whole must be same • Measuring length by using unit fraction as unit measurement	of fractions as operator 2. Students able to write fractions on number line 3. Student able to measure length by using unit fraction as unit measurement	fractions as operator ✓ Measuring length related to fraction value (how much of part?)	length of part and length of whole ▪ Measuring length by using unit fractions as unit measurement
• Finding the length of track such that can be divide for both 3 and 5 part	• a common whole/denominator can be made by finding a common multiple	1. Children able to find a common denominator 2. Students able to compare fractions (equivalent of fractions)	✓ Realizing the idea of common denominator ✓ Realizing the equivalent of fractions	▪ Finding the length of track which can be partitioned/ divided by 3 and 5 easily (common multiply)
• Adding two parts of the track (1/5 part of track and 2/5 part of track)	• To add fractions a common whole is needed • To measure the length an unit measurement (unit fractions) is needed	1. Students choose a common whole/denominator 2. Students compare fractions (Equivalent fractions) 3. Students add fractions with same denominator	✓ Finding common denominator by using logical thinking and strategy ✓ Finding equal fraction ✓ Conflict with idea of addition of whole number	▪ Finding common multiply of both denominator ▪ Finding those fractions which state on the same place on the line ▪ Use fractions as operator and double number line model to add fraction by using measuring length strategy with unit fractions as unit measurement
• Finding position of crashed • Adding two part of track (1/3 part of track and 2/5 part of track)	• To add fractions a common whole is needed • To measure the length unit measurement (unit fractions) is needed	1. Students can produce a common denominator 2. Students can add fractions with different denominator	✓ Finding common denominator ✓ Conflict with idea of addition of whole number	▪ Use fractions as operator and double number line model to add fraction by using measuring length strategy with unit fractions as unit measurement

The instructional activities for addition of fractions in this research were designed based on the hypothesized students' learning trajectory. The instructional activities consist of six activities that will be conducted in three weeks period. The hypothetical learning trajectory is elaborated in the instructional activities as following:

Activity 1: Coloring Scout Stick

Figure 2.4. Scout Stick

Mathematical Learning Goal(s):

This activity aims students can interpret fractions as measure.

Tools:

Students' worksheet, Stick, Crayon, Pen.

Behind the context:

- An idea of fractions as measurer occurs when students are asked to measure the length of part(s) of the stick which are colored by them.

Description of activity:

Teacher tells about her planning in designing Hanger. Teacher says that *"Let's see this picture(Show the picture of scope), what can you say about the* *scope stick? Today we want to design/color the mini scope stick in four different types:*

- *First type has three colors,*
- *Second type has four colors,*
- *Third type has six colors, and*
    - *Fourth type has eight colors,*

Every group gets 1 stick measuring 60 centimeters and designs one type randomly(by lottery). How do you design the stick? How much the length each part?"

Conjecture of students' thinking:

In designing the stick,

- Some students might divide the stick into the number of colors (type of the stick) by using ruler or by hand or by estimating.
- Some students might use paper as ruler, and fold/divide the paper to find the length.

  In dividing or folding paper into two, they will easily to do that. In dividing or folding paper into four, some students might divide or fold the paper into two and fold into two again. But in dividing or folding paper into three, students might find difficulty in divide it. Some students might use try and error in dividing.

After this activity, the discussion will be continued about the length of each part, by asking: "*how much of the stick?*",

- Some students might measure the length in the form of unit measurement, i.e. centimeter. For instance the length is 3 centimeters, etc.
- Some students might measure the length by using fractions as the length/measure. For instance, the distance of part of the stick divided into 3 parts is third or one over three, etc.

The discussion will be continued about the interpretation of fractions. It expected that

- Some students will realize that fractions is a measure/ the length of part of the stick, for instance: 1/3 is the length (measure) of one part of the stick divided by 3, 2/3 is the length of two parts of the stick divided by three, etc.
- Some students will realize that 1/3 is one part of stick divided by three, 2/3 is two parts of the stick divided by three, etc.

Mathematical Congress

In mathematical congress, students will discuss about the way in diving the stick. Students might suggest that dividing by estimating strategy or by hand. The discussion will be continued about the accuracy. It is expected that they will come to the idea using paper or ruler. In partitioning/dividing activity, students will discuss about strategy in dividing the stick. If students partition the stick into 4 by dividing it into two and the dividing again into two. In finding the length of the part, they will discuss the length by using fractions and also the meaning/interpretation of fractions related to the length. After that, students might also discuss about the relation between two fractions: 1/3 and 2/6 by comparing the length of each part.

Activity 2 : Comparing Coloring Stick

Mathematical Learning Goal(s):

This activity aims to compare fractions and determine equivalence of fractions.

Tools:

Coloring Stick, Students' Worksheet.

Behind the context:

- Comparing and equivalent of fractions (i.e. fractions as measure) occur when student are asked to compare part(s) of some kinds of coloring stick.

Description of activity:

Teacher starts discussion, by reminding students about what they did at the previous meeting. Teacher use the stick colored by students at the first activity to provoke students reach the idea of comparing and determining the equivalence of fractions by comparing four kind of stick directly and ask students to explain what can they see/conclude. Teacher gives four types of stick to every group and students' worksheet consisting three kind of problems:

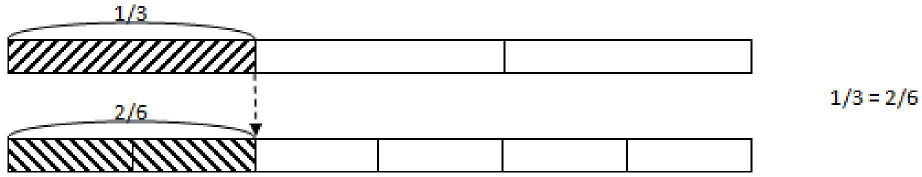
1. How to compare fractions?, for instance: 2/3 . . . ¾, fill with equal to, less than, or more than.
2. Short the following fractions: 5 / 6, 2 / 3 and ¾ start from the smallest!
3. Determine the equivalence of fractions, for instance : $\frac{\dots\dots\dots}{6} = \frac{2}{3}$

In order to answer these problems, teacher ask students to use stick colored by them at the previous activity and explain their reason by drawing or make representation of their thinking/reasoning.

Conjecture of students' thinking:

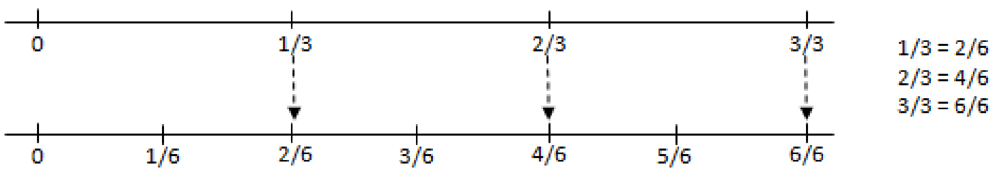
For the first problem, comparing fractions:

- Some students might use 4 types of stick colored, stick divided into 3, 4, 6, and 8 same parts, to compare fractions by comparing the length of each parts directly.
- Some students might draw a line/bar/rectangle as representations of the stick to compare fractions by comparing the length of each parts, for instance:



Etc.

- Some students might draw a line as representations of the stick to compare fractions by comparing the length of each parts, for instance **:**

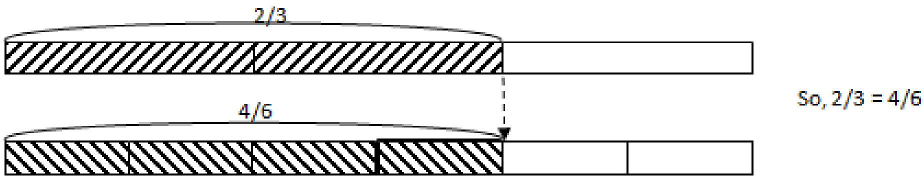


For second question, shorting(ordering) fractions:

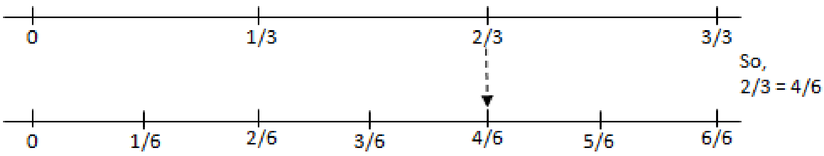
- Some students might use four types of the stick to order the fractions, 5 / 6, 2 / 3 and ¾. To solve this problem, they will compare first type (stick divided by 3 same parts), second type (stick divided by 3 same parts), and third type (stick divided by 6 same parts) directly.
- Some students might compare all fractions, 5/6, 2/3 and ¾, by drawing bar/rectangle as representations of stick. Some students might order those fractions through comparing the length as representations of the fractions. Some students might compare all fractions, 5/6, 2/3 and ¾, by number line. Some students might order those fractions through comparing the position of the fractions on the number line. For instance:

In determining the equivalence of fractions:

- Some students might compare two types of stick depend on the denominator, for example: to answer this question: $\frac{\dots\dots}{6} = \frac{2}{3}$, some students will use first type and third type of the stick.
- Some students might compare by drawing two bars/rectangles and finding the same position, for example: to answer this question: $\frac{\dots\dots}{6} = \frac{2}{3}$, some students will draw two bars/rectangles and divide the bars into 3 parts and 6 parts because they want to compare fractions with denominator 3 and 6, then find two positions which are in the same length/position, for example:



- Some students might compare by drawing two number lines and finding the same position of both fractions, for example: to answer this question: $\frac{\dots\dots}{6} = \frac{2}{3}$, for example:



Mathematical Congress

In mathematical congress, students will discuss about the way in producing fractions such as interpretation of fractions. Students might get difficult or forget it. The discussion will be continued about the distance of each part and how to write the distance. Some students might use "*a half*", "*a third",* "*a quarter*", etc, the discussion will continued with the meaning of them. If students use non unit fractions, the discussion can be continued about the meaning of it, for instance: 2/4 is the name of the second part or that is the name of the first two parts together? The discussion will be continued about comparing length of parts that represents fractions, for instance the length of one stick part divided into 3 parts and the length of two parts of stick divided into six parts, etc. By comparing the length , this discussion can be brought to the idea of comparison of fractions. It is expected that students will discuss about the relation between 1/3 and 2/6. The discussion will be continued about the equivalent of fractions and strategy in making equal fractions.

Activity 3: Cutting Scout Rope

Figure 2.5. Scout rope and Scout tent

Mathematical Learning Goal(s):

This activity aims to stimulate students getting the idea of fractions as operator and measure and measuring length by using unit fractions as unit measurement.

Tools:

Ruler, poster, paper, and pen

Behind the context:

- An idea of fractions as operator occurs when students are asked to cut a part of rope and measure the length of it.

Description of activity

Teacher shows the scout rope measuring 12 meters. Teacher tells story about making scout tent. In order to make tent, we need some part of the scout rope, such as a half, one third and a quarter of scout rope. The problem is how do we get it and how many meters of scout rope which is cut:

a. A half of scout rope

b. One third of scout rope

c. A quarter of scout rope

What do you think about those parts of scout rope?

Teacher also gives challenged problem,

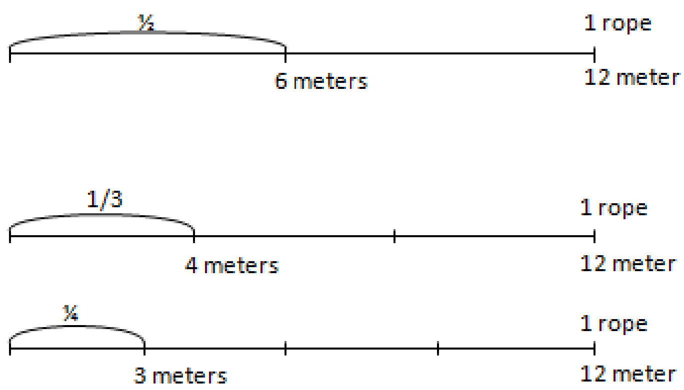
- if I cut the scout rope 3 meters, how much of length of rope the pieces?
- if I cut the scout rope 5 meters, how much of length of rope the pieces?
- if I cut the scout rope 7 meters, how much of length of rope the pieces?

Conjecture of students' thinking:

- Some students might divide 12 by 2 directly to get a half of scout rope. From their division, they know that the length of scout rope which is cut by them is 4 meters. They also use this strategy in finding a third and a quarter of scout rope.

  12 : 2 = 6   12 : 3 = 4   12 : 4 = 3

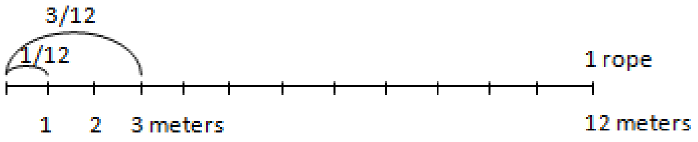
- Some students might try to draw a line to represent the rope and write 12 at the end of the rope as representation of the length, then they divide the line into two like they did in previous activity. See following conjecture of students' model:



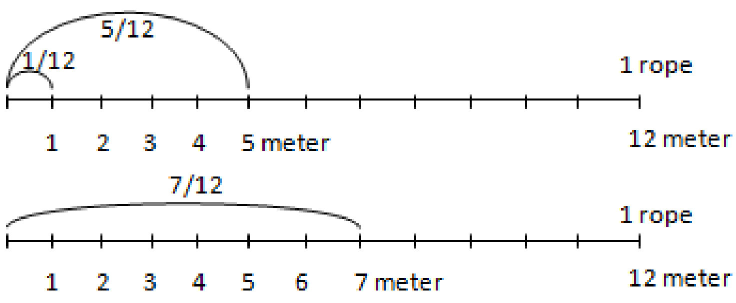
For second question, some students might compare the draw directly. Some students might compare the whole numbers related to those fractions.

For the challenged questions:

- Some students might use their answer at first question to answer. For instance, 3 meter is ¼ of scout rope.
- Some students might divide into 12 to find the position of three meters on the rope, and finally he get the idea of 1/12-unit and then they use it to measure the position 3 meter, See following conjecture of students' model:



- To answer the second questions, some students might divide into 12 to get 1/12-unit and then they use it to measure the position 5 meter and 7 meter on the scout rope. After that they will determine the distance related to the length of scout rope. There are 5 of 1/12-units, so that is 5/12, etc. See following conjecture of students' model:



Some students might use proportion between the length of part and the length of a whole, for instance 3 meters of 12 meters, that is 3/12, etcetera.

Mathematical Congress

In mathematical congress, students will discuss about the way in getting pieces of scout rope. If students do division operation, the discussion will continued about the reason why they do that. If students do second conjecture (draw and divide the drawing), the discussion will continued about their perception about fractions. The aim this discussion is to provoke students to realize the idea fractions can be represent as operator, for instance 1/3 means 1/3 of rope, et cetera. The discussion also about the visualization of the track (a line), how they write the symbol of fractions and a whole number (the length) on the line. Through this discussion, we can introduce the name about the visualization, it called *a double number line*, because there are two kind numbers such as fractions and whole number on the line.

Through discussion about challenged problem, students will discuss about relation between the length of part and the length of whole rope. Based on their strategies and answers, the discussion will be continued about relation among those fractions i.e. equivalent of fractions. For instance: ¼ and 3/12.

In order to answer the challenged questions, some student might feel difficult to answer. Discussion will be continued about how to find five meter on their draw (line).

Basic Math Concept: Fractions as measure and operator, Double number line, comparing fractions.

Activity 4: Track of Bike Racing part

Mathematical Learning Goal(s):

This activity aims to stimulate students choosing a common whole/denominator

Tools:

Ruler, poster, paper, and pen

Behind the context:

- An idea of common denominator occurs after students are asked to choose a whole number of their liking as the length of the track line.

Description of activity:

Teacher tells that *"Last night my friends called me, he told me that he would follow the bike race. He also told about the track. The track of bike racing has two markers and four points (water supply place) on it with same distance. Could you draw the track?"*

After that teacher gives challenged question: "*My friend did not tell me about the length of the racing. I just remember that the length of each marker is same and the length of each point is also same. But I am still curious about the length of the track. what do you think about the length of track? How much of the track the distance each point and the distance of each marker?"*

Conjecture of students' thinking:

- At the first, student might try to make visualization of the track and draw marker and point on it. Because they want to make visualization about the track. For instance, based on their visualization of the track, some students might realize that there are four points, it means that students partition into five same parts. So, the length of each points is 1/5 of the track. Some student might also know that there are two markers, it means that students partition into three same parts. So, the length of each points is 1/3 of the track.

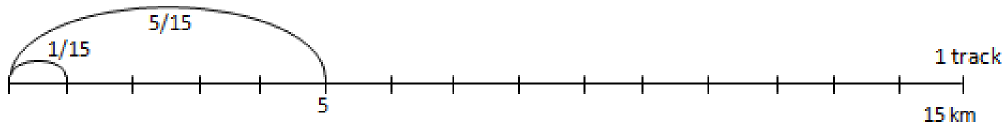
- In order to choose the length of track, some students might use *try and error* strategy in finding number which can be divided by 3 and 5.
- Some students might use the idea common multiply of 3 and 5 to choose the length of track as follow:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, …

5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, …

- If some students choose 15 km as the length of the track, they will know that the distance of each point is 1/5 of 15 and the distance of each marker is 1/3 of 15. By using the idea of fractions as operator, they will use same strategy in third activity to determine it. So, 1/5 of 15 is 3 km and 1/3 of 15 is 5 km. Etcetera.
- If students know the distance of each marker is 5 km, for instance, they will try to find how much of the length of track the '5 km'. By using

strategy in measuring length with unit fractions, students will know that 5 km is 5/15 of the track, see following conjecture of students figure:



Mathematical Congress

In mathematical congress, students will discuss about the problem. Students might get difficult to understand about the problem. The discussion will be continued about investigations the information and the questions in the context. Continued with first question, the question is aimed to provoke students to find common multiply of those denominator (common denominator). The mathematical thinking in this activity is finding common multiply.

Activity 5: Practicing Bike

Mathematical Learning Goal(s):

1. Students choose a common denominator
2. Students use fraction as operator and measure
3. Students add fractions with same denominator

Tools: Ruler, poster, paper, and pen

Behind the context:

- *A Double number line model* __ a line with whole numbers on the top and fractions on the bottom occurs when students are asked to determines the length of the track.

- An idea of common denominator occurs after students are asked to choose a common whole of their liking as the length of the line.
- An idea of equivalence of fractions occurs when students examine the relationship among their possible lengths for the race line.

Description of activity:

Questions are posed in class discussion:

"Yesterday, my friend practiced bike on the track. From the start line, he biked in normal speed. After he passed the first point, he biked in fast speed as long as 2/5 of track. And then he stopped because he tired. How much of the track he practiced?"

Conjecture of students' thinking:

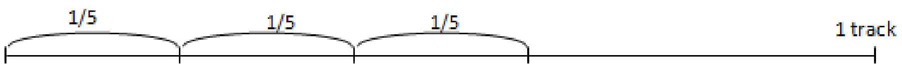
- Some students might be confused about the problem, they just add the distance (whole number)

  Some students might make visualization of the problem, See following conjecture of students' model:

- Based on their visualization, they will realize that the distance is 1/5 plus 2/5 of race. Now, they will try to add those fractions.

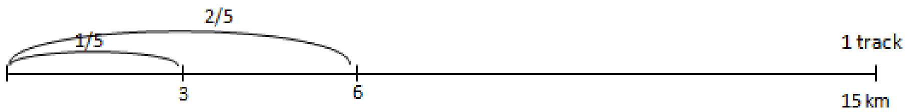
  Some students might add those fractions directly using the idea of unit fractions as unit measurement. For instance: 2/5 is 2 of 1/5-unit, so 1/5

+ 2/5 is 1 + 2 of 1/5-unit. It means that 1/5 + 2/5 is 3 of 1/5-unit or 3/5. See following conjecture of students' model:



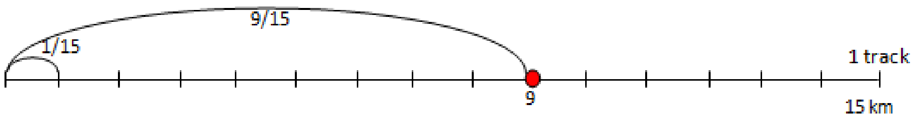
- Some students might choose a whole number as the length of the track. Students will find the length (whole number) of each part and add them. Finally, they will move back to the fractions by symbolizing using fractions. For instance:

  Some students might choose 15 km as the length of race, because they have already chose at the previous meeting. Based on their visualization of track, the distance is 1/5 + 2/5 of track. See following conjecture of students' model:



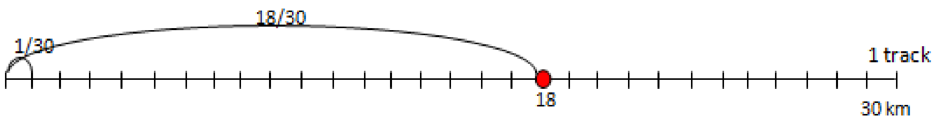
  They might know that the first part is 3 and the second is 6. They just add up the 3 and the 6—that's 9. Based on their strategy in measuring length by using unit fractions

  (1/15-units), students will know that the distance of the track between start line until second point is 9/15 of track. See following conjecture of students' model:



Some students might choose 30 km as the length of race. Based on their drawing, the distance is 1/5 + 2/5 of track. See following conjecture of students' model:

They know that the first part is 6 and the second is 12. They just add up the 6 and the 12—that's 18. By using their strategy in measuring length by using fractions (1/30-units), students will know that the distance of the track between start line until second point is 18/30 of track. See following conjecture of students' model:



et cetera.

Mathematical Congress

In mathematical congress, students will discuss about the problem. Students might get difficult to understand about the problem. The discussion will be continued about investigations the information and the questions in the context. It is expected that students will draw a line as representation of situations. The question is aimed to provoke students to add fractions with same denominator. The discussion will be continued about strategy in solving the problem. It is expected that students will use a double number line to add those fractions. The discussion can also about the length which should be chosen such that can be divide by 5. Continued with first question, student are ask to determine the distance in whole number and then move back to fraction. Discussion will also be continued about the relation between denominator and result of addition of fractions with same denominator.

Basic Math Concept: Common denominator, fractions as operator and measure, Addition of fraction with same denominator.

Activity 6: Crashed Position

Figure 2.6 . Crashed bike

Mathematical Learning Goals:

1. Students can produce a common denominator
2. Students use fractions as operator and measure
3. Students can add fractions with different denominator by using a double number line

Tools: Ruler, poster, paper, and pen

Planned Instructional Activities:

Behind the context

- A common denominator occurs when students try to choose the length of track.
- Addition of fractions occurs when student try to find the distance between start line and crashed position.
- A double number line model occur when students to represent their strategy in Adding fractions.

Description of activity:

Children work in group (3-4 students). Teacher tells the story about her/his friend's plan in joining bike racing. The story is *"Yesterday, my friend told about their experience in her participation on Bike Racing. In that racing, he was injured because he crashed and could not continue the racing. He said that he crashed at distance 2/5 of track after passing the first marker. Could you find the position where biker is crashed? How much of the race has he done until he is crashed?"*

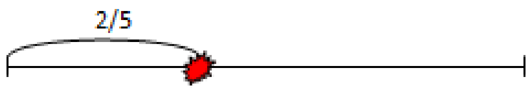
Children are asked to solve both the first and the second problem in group. Students are asked to explain the reason about their strategy which is chosen by them to solve each problem. Students are also asked to write or draw the solutions on the paper which is used to present their idea in math congress session.

Conjecture of students thinking:

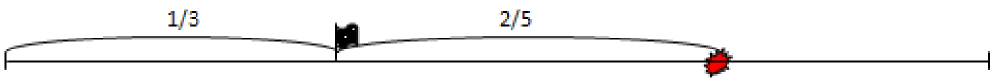
The first problem

The first problem is to draw and find position where biker is crashed on the track.

- Some students might draw a line to represent the track and make a sign to where biker is crashed, See following conjecture of students' model:



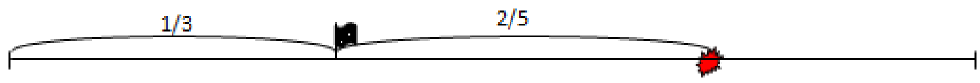
- Some students might draw a line to represent the track and make a sign to represent the point (water supply) and point where biker is crashed, See following conjecture of students' model:



Based on their draw about the crashed position, teacher can ask student to solve the second problem.

The second problem

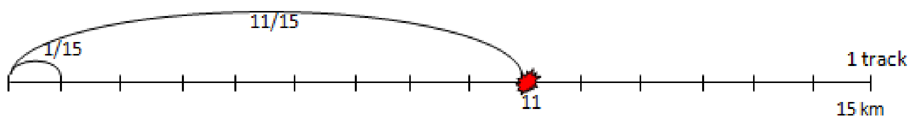
Some students might use their draw about the crashed position to solve the second problem. Next, it is called a double number line as model (*model for*). Based on their visualization of crashed position, students realize that the length of the track which has he done is 1/3 + 2/5 of track. See following conjecture of students' model:



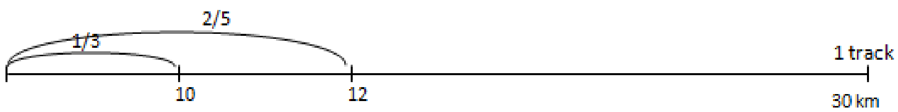
- Some Students might choose 15 km as the length, because they work with this number at the previous activity (track of bike racing).

  - Based on the draw of the crashed position, the length of the track which has he done is 1/3 + 2/5 of track.

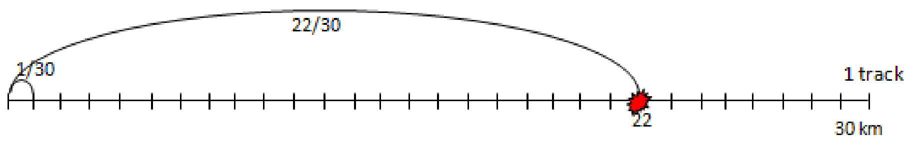
    And they know that 1/3 of track is 5 and 2/5 of track is 6. They just add up the 5 and the 6— that is 11. So, by using unit fractions (1/15-units), the length of the track which has he done is 11/15 of track. See following conjecture of students' model:



- Some students might choose 30 km as the length of race. See following conjecture of students' model:



Based on their drawing, the distance is 1/3 + 2/5 of track. They know that the first part is 10 and the second is 12. They just add up the 10 and the 12—that's 22. By using their strategy in measuring length by using fractions (1/30-units), students will know that the distance of the track between start line until second point is 22/30 of track. See following conjecture of students' model:



et cetera.

By using double number line, students might choose number of their liking as the length of the line to fit the number. It is expected that they examine denominator and realize to the idea of common denominator.

Mathematical Congress

In mathematical congress, students will discuss about the problem. Students might get difficult to understand about the problem. The discussion will be continued about investigations the information and the questions in the context. Continued the first question, the question is aimed to provoke students to draw a line (a double number line) as representation of situations. The question is aimed to provoke students to add fractions with different denominator (1/3 + 2/5). The discussion will be continued about strategy in solving the problem. The discussion can also about the length which should be chosen such that can be divide by 3 and 5. about based on their activity in adding fractions and adding whole number, which one is easy? so, how to make it easy in adding fractions? The aim of discussion is to provoke student to realize that in adding fractions, we can move to the whole number and add them. Then finally move back to fractions. And A double number line is good tool to do this strategy. Based on their result, the discussion can also about equivalence of fractions, 11/15 = 22/30.

Basic Math Concept: Equivalence of fractions, common denominator, Addition fractions.

Chapter III

Research Method

The method used in this research is called *design research*. It is a type of research methods with its core of research is formed by classroom teaching experiments that center on the development of instructional sequences and the local instructional theories that underpin them (Gravemeijer, 2004).

In this design research, there are three phases: developing a preliminary design, conducting a teaching experiment, and carrying out a retrospective analysis (Gravemeijer, 2004; Bakker, 2004). Each of these forms a cyclic process both on its own and in a whole design research. Therefore the design experiment consists of cyclic processes of thought experiments and instruction experiments (Freudenthal, 1991).

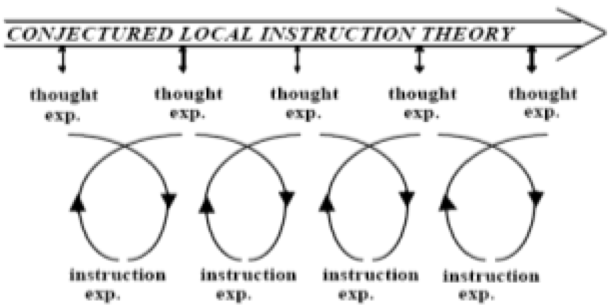


Figure 3.1. Reflexive relation between theory and experiments (Gravemeijer & Cobb, 2006)

Defining a Hypothetical Learning Trajectory (HLT) is needed before elucidating of these three phases. HLT is a design and research instrument that proved useful during all phases of design research (Bakker: 2004).

During the preliminary design, HLT guides the design of instructional materials that have to be developed or adapted. During teaching experiment, the HLT functions as a guideline for the teacher and researcher what to focus on in teaching, interviewing, and observing. And during the retrospective analysis, HLT functions as guideline determining what the researcher should focus on in the analysis (Bakker, 2004).

The following sections, we discuss the three phases of the design research according to Gravemeijer (2004), Bakker (2004), and Gravemeijer and Cobb (2006).

A. Phase 1: Preparation and Design

In this phase, we construct the Hypothetical Learning Trajectory (HLT) that developed potential sequence activities concerning the goal of the research. This HLT is called HLT I. In constructing this HLT, we explore and study prior research on fractions, elaborate with phenomenology related to fractions and also discuss with supervisor and expert.

B. Phase 2: Teaching Experiment

In our plan, teaching experiment will conduct in two phases, namely pilot experiment and teaching experiment. The purpose of pilot experiment are (1) investigating pre-knowledge of students, because it is important for the starting point of the instructional activities and adjusting the initial HLT, (2) adjusting the HLT I, the HLT I is tried out and the observed actual learning process of students is employed to make adjustments of the HLT.

The teaching experiment aims at collecting data for answering the research questions. During the teaching experiments, we emphasize the ideas and conjectures could be modified while interpreting students' reasoning and learning in the classroom. The teaching experiments are conducted in six lessons in which the duration was 70 minutes for each lesson. Before doing teaching experiment, teacher and researcher discussed the upcoming activity.

C. Phase 3: Retrospective Analysis

In retrospective analysis phase, we will analyze the things that happened in the teaching experiment (see video and audio recording, students' work). In this phase, HLT is used as guidelines and points of reference: in answering research questions. The extensive description of the data analysis is explained in subchapter D, namely reliability and validity.

The results of retrospective analysis are used as base in designing and revising the first HLT that will implement at the second cycle. Some indicators of success in designing hypothetical learning trajectory are presented in the table below:

Table 3.1. Success indicators of designing the instructional activities (hypothetical learning trajectory)

Aspect	Achievement	assessment
Use measurement ideas to support their reasoning	60%	Calculated from the average use of the ideas of measurement in answering questions in students' worksheet
Use students' own model to support their reasoning	50%	Calculated from the average use of the students' own model in answering questions in students' worksheet

D. Reliability and Validity

Qualitative reliability is used to preserve the consistency of data analysis. The qualitative reliability is conducted in two following ways:

- *Data triangulation*

  The data triangulation engages different data sources such as the videotaping of the activities, the students' works and some notes from observer. All activities are video recorded and the students' works are collected. The combination of the videotaping and students' works are chosen to check the reliability of interpretations based upon one video clip or one field note.

- *Trackability of the conclusions*

  The learning process is documented by video recordings, field notes and collecting the students' work. With this extensive data, we are able to describe the situation and the findings in detail to give sufficient information for our reasoning. This information enables the reader to reconstruct the reasoning and to trace the arguments that underpin the conclusions

There are two methods of validity are used in the data analysis:

- *Validity through HLT*

  The HLT is used in this retrospective analysis as a guideline and a point of reference in answering research questions. This aims to connect and evaluate the initial conjectures to the gathered data and prevented systematic bias. In this part, we continuously test and revise the HLT based on the result of retrospective analysis.

- *Cross interpretation*

  The parts of the data of this research, the video data, are cross interpreted with supervisors or expert. This is conducted to reduce the subjectivity of the researcher's point of view.

E. Description of Experimental Subject and Time line

The research will be done in semester two of grade four of elementary school in SDI At Taqwa Surabaya, Indonesia. The school has been involved in the PMRI project, under the supervision of Surabaya State University.

The organization of this research is summarized in the following timeline:

Table 3.2. The timeline of the research

	Date	**Description**
Preliminary Design		
Studying literature and designing HLT *desk version*	21 September 2010 – 5 January 2011	
Discussion with	14 - 19	Finding students' current knowledge

teacher	February 2011	of addition of fractions
Classroom observation in grade 4	14 – 19 February 2011	Finding socio norms and socio-mathematical norms
Pilot Experiment		
Pre-test	21 – 26 February 2011	Testing and investigating pre-knowledge, attitude, and skill of students
Tryout in grade 4 Group of 6-8 students		• Testing some activities on HLT • Investigation students' strategies in solving problem of addition of fractions
Post-test		Testing and investigating students' knowledge and reasoning in adding fractions
Teaching Experiment		
Pre-test	March – April 2011	Testing and investigating pre-knowledge, attitude, and skill of students
Design Hanger		Focusing on *partitioning, the idea of fraction as measure and unit fraction as unit measurement*
Math Congress "Finding position and distance"		Focusing on *partitioning, the idea of fraction as measure and unit fraction as unit measurement*
Cutting Scout Rope		Focusing on *measuring length, the idea of fraction as operator and using unit fraction as unit measurement*
Track of Bike Racing part 1		Focusing on *finding common multiply, the idea of fraction as operator and measure and using unit fraction as unit measurement*
Track of Bike Racing part 2		Focusing on *measuring length, using unit fraction as unit measurement, reasoning in adding fraction with same denominator*
Crashed Position		Focusing on *measuring length, using unit fraction as unit measurement, reasoning in adding fraction with different denominator*
Post-test		Testing and investigating students' knowledge and reasoning in adding fractions

F. Data Collection

Various data are collected from videotaping and written data to get a visualization of students' thinking and reasoning in adding fractions.

The data collections of this research are described as follows:

1. *Video recording*

   The strategies used by students when measuring length, comparing and adding fractions are more as practical data, instead of written data, therefore students' strategies are more observable from video. Short discussion with students during discussion in group, the class discussion, and also interview are also conducted and recorded as means to investigate students' reasoning for their idea.

   The video recording during the teaching experiments is recorded by two cameras; one camera as a static camera to record the whole class activities and the other camera as a dynamic camera to record the activities in some groups of students.

2. *Written data*

   As an addition to the video data, the written data provided more information about students' achievement in solving the measurement problems. However, most of these data merely provided the final answers of students without detailed steps in finding those answers. These data were used for investigating students' achievement because students' learning processes were observed through videotaping and participating observatory.

The written data included students' work during the teaching experiment, observation sheets, the results of assessments including the final assessment and some notes gathered during the teaching experiment.

The data are collected through interviews with the teachers and the students, classroom observations, and students' work. After that, we analyze these data in the retrospective analysis. The outline of our data collection is represented in the following table:

Table 3.3. Outline of data collection

	Data Collected	Goal
Part 1: Preliminary experiment	Classroom observation *Video recording*	• Finding socio norms and socio-mathematical norms
	Interview with grade 4 teacher *Audio recording*	• Finding students' current knowledge of addition of fractions
Part 2: First experimental *Pilot experiment* (6-8 students)	Classroom observation six meetings *Video and audio recording, students' work*	• Testing some activities on HLT • Investigation students' strategies in solving problem of addition of fractions
Part 3 : Second experimental In different class	Classroom observation six meetings *Video and audio recording, students' work* Interview with grade 4 students	• Testing all activities on revised HLT • Investigating students' thinking

Chapter IV

Retrospective Analysis

The retrospective analysis in this design research encompasses the explanation of data both in general and in specific cases. The learning process addition of fractions of children will be analyzed, not only for the children as individuals, but also for their participation in and contribution to the development of classroom mathematical practice.

In this chapter, we compared our HLT and students' actual learning process during the experimental phase. We investigated if and how the HLT supported students' learning. First, we looked at the video recordings, and selected some critical moments in which students learned something or students did not learn as was expected in the HLT. Then we transcribed these critical moments we have observed in the classroom. These transcriptions were the empirical bases for our interpretations of students' learning processes. We also analyzed students' written work as another source to investigate students' learning. Moreover, we discussed what made successful activities and what students have learned from those activities. In the case of unsuccessful activities, we investigated what caused such failure, and what needed to be done in the next HLT to improve students' learning processes.

We should point out that during the teaching-learning experiment, we followed, observed and studied each lesson to find out whether the actual students' learning process met the expectation in the HLT. Therefore, we made changes and added some activities on daily basis to adjust and improve students’ learning. Results of the retrospective analysis will form the basis for adjusting the new HLT and for answering the research questions.

A. Pilot Experiment

The pilot experiment was conducted in groups of six students grade 4, 4A class of SD At Taqwa, Surabaya and researcher as teacher. List of students is given in table 1. Pilot experiment aims to analyze and evaluate the HLT. In addition, input was also obtained from students' difficulties in working out the sequence of events in the HLT. The experimental results will improve HLT initial pilot.

Table 4.1. List of Students in pilot experiment

NO	Name	Class
1.	Salsa	IVa
2.	Salma	IVa
3.	Ayu	IVa
4.	Ilham	IVa
5.	Rafee	IVa
6.	Rio	IVa

Draft HLT tested consists of six activities. The results of this pilot experiment will be explained based on the sequence of learning activities as follows.

Activity 1:

In the first cycle, we found some facts relating to the activities of students and their thinking process that we were seeing from recordings at the time of activity, discussion, students worked on worksheets. Our conjectures were that students can interpret fraction as a measure

(Measure), compare fractions by comparing the length of the parts of the stick and recognize the equivalence of fractions.

In the first activity, students were asked to divide or partition a stick through the process of measuring the length, we found that students had a good measurement capabilities, including using standard measurement units or non-standard, such as inches / hand spam. Consider the following picture and transcripts:

Figure 4.1. Students measured and divided by the ruler to use and stick their fingers

Researchers : Consider this stick. How long this part? (Pointing to the middle of the stick)
Salsa : half
Ayu : one over two
Researcher : why?
Salsa : because in the middle of the stick
Researcher : How about this (pointing to the fourth stick)?
Ayu : quarter
Salsa : yes ... one over four
Researcher : why?
Ayu : because there are four parts
Researcher : there are others, Salsa?
Salsa : ¼ because there are 4 parts (pointing to all the parts) and this is one (pointing to the first part)
Researcher : What about this part? (Pointing to the two parts of a stick)

Ayu : 2 / 4
Researcher : why?
Salsa : 2/4 because there are 4 parts (pointing to all the parts) and these are the two parts (pointing first part)

When students were asked to determine the length of each part of the rod is divided, students using fractions as a long-term part. Consider the following transcript:

Researcher : Ayu, how long from this to this (pointing to the tip and base of the first part of the stick is divided into four parts)
Ayu : 15 cm (he used a ruler to measure)
Researcher : there is no other!
Ayu : ¼
Researcher : Salsa, how long from this to this (pointed end of the first and second base parts of the stick is divided into four parts)
Salsa : 2 / 4
Salma : Salma, from this to this? (Referring to the end of the first and third parts of the stick base is divided into four parts)
Researcher : So, what do you know about ¾?
Salma : distance from this to this (pointed end of the first and third parts of the stick base is divided into four parts)
Researchers : what you know about ¼ Salsa ...?
Salsa : the length from this to this (pointing to the tip and base of the third part of the stick is divided into four parts)

When salsa said that a quarter is the length of a part of the stick which is divided into four equal parts. Salsa also said that ¾ is the length of the 3 parts of the stick which is divided into four equal parts. I can conclude that the students interpreted the fraction as a measure, in this case is the length of the stick.

Activity 2: Math Congress

The purpose of this activity was students can compare and determine the equivalence of fractions. In this activity, students were asked to compare fractions and determine equivalent fractions using the sticks colored by students. In a class activity, we found that students also expressed about the idea of equivalence of fractions. Consider the following picture and transcript of the conversation:

Figure 4.2. Discussion among teacher and students about the equivalence of fractions

Researcher	: Now, let us see two sticks (holding the stick which is divided into 2, 4 and 6 parts). How do you think?
Rio	: there is a different color
Ilham	: There are different lengths
Researcher	: any else?
Ayu	: This is the same with this (pointing to the first part of the first stick which is divided into 2 and pointed to the second part of the two sticks that are divided into 4)
Researcher	: So, what does it mean?
Ayu	: half equals two quarters.
Researcher	: Are you sure?
Ayu	: ehm (smiling) ... .. yup, because the distance is the same.
Researcher	: No more?
Salsa	: half equals two quarters and three over six.
Researcher	: why?
Salsa	: because they are the same length

From the discussions and activities, the students concluded that fragments is equal, if they have the same length or occupy the same place on the stick. From this evidence, we concluded that we do not change the first activity, because it was still in line with our conjectures.

Activity 3:

The purpose of this activity is the students can interpret the fraction as an operator and measure length of part(s) by using unit fractions as the unit measurement. In this activity students were asked to cut the rope that measuring 6 meters. However, in this activity, students did not cut the rope directly, but they were required to give an sign on the drawing of rope on the Student Worksheet. Our conjecture was that the students can use interpretation fractions as operators or multiplier to determine the length of the rope in form of unit measurement, such as meters. For example, students could determine 1/3 of the rope measuring 6 meters, 1/3 of 6 meters. Second, students can determine the length of parts (fractions) of the rope if known length of the rope in the form of unit measurement, meter, to unit fractions and then use the unit fractions as unit measurement. For instance, 1 meter of 6 meters is 1/6 of the rope and 3 meters of 6 meters is 1 meter plus 1 meter plus 1 meter, it means that 1/6 + 1/6 + 1/6 = 3/6 of the rope measuring 6 meters.

The first problem, students were asked to divide and give sign on the picture of the rope become three, four, and six same parts. In this case, students divided and signed pictures of rope by using strategies like the first activity. In this activity, all students used a ruler to measure and divide the line as the representation of rope.

The second problem, students were asked to determine 1/3 of the rope measuring 6 meters, among 6 students, only Salsa could use fractions as multiplier and the others cannot answer the question. In this case, Salsa performed calculations with multiplications algorithm (procedure), see the following picture:

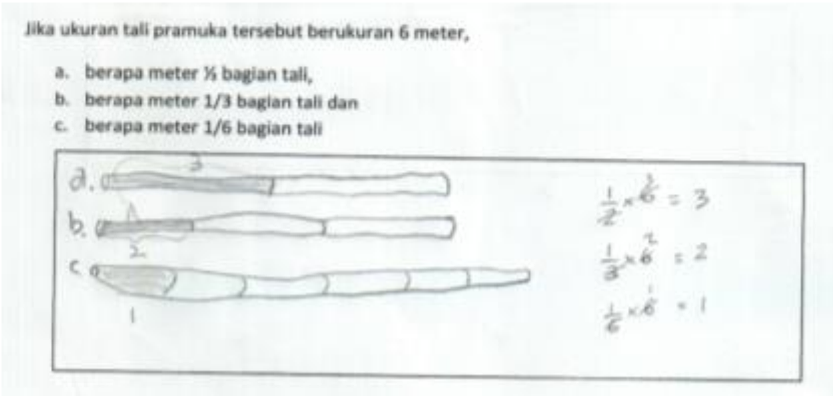


Figure 4.3. Salsa’s calculation

However, when Salsa was asked to explain the reasons about why she used the algorithm, he did not give the reason. This indicated that students could not understand the meaning or interpretation of fractions as operator, because if they realized/understand the interpretation then they will explain that 1/3 is a length of a part of the rope divided by 3 equally. But when students were asked to remember the interpretations of fractions as measure by using stick, they said that 1/3 is the length of a part of the rope divided by equally. For instance, when researcher asked how long 1/6 of the rope measuring 6 meters, students could find the length by dividing the length of rope by 6. We concluded that teacher should remind or emphasize the interpretation of fractions as measure before they work with fractions as operator (multiplier) and should use real object (real rope) to give concrete experience so that they can really imagine and understand the problem.

The third problem, students were asked to determine the length of the part of the rope if the length of part was known in the form of unit measurement, meter. For instance, how much of the rope (in the form of fractions), if we have 1 meter of 6 meters of the rope. At the first time, they asked the meaning of the questions. Researcher asked to draw the rope and to imagine that the rope measuring 6 meters. One of them, Ayu, could show that 1 meter is 1/6 of the rope measuring 6 meters by dividing and signing the picture of the rope. See the following picture:

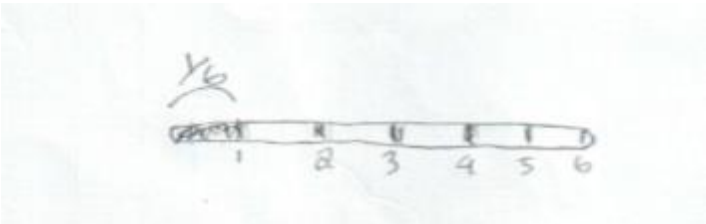


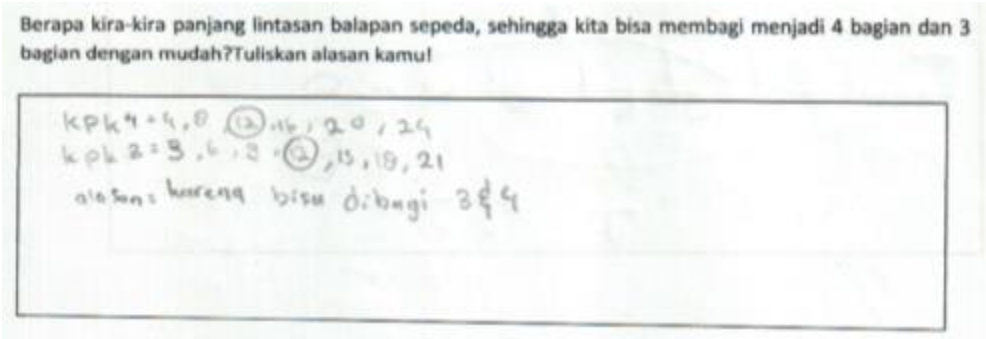
Figure 4.4. Ayu's drawing/model as representation of the rope

We concluded that in order to help students in solving this problem, we need to provoke students in using emergent model (*model of*), for instance line or bar as representations of rope. Based on observation during the implementation, we concluded that we need to provoke students and design (revise) LKS separately so that students are inspired to build or create a model (number line) because it can assist students in using the fractions as an operator / multiplier to determine the length of the inside of the unit of measurement or the change in the form of fractions. Because of this ability is the prerequisite skills that can assist students in the process of adding the fractions with different denominators and adding fractions using a double number line model. Because the working principle of the addition of fractions with measurements and the double number line model, students working with fractions are converted to integers and then back to fractions. The use of concrete/real object (real rope) is needed to help student imagine and understand the problem.

<u>Activity 4:</u>

The purpose of the fourth activities was students able to determine / equating the denominator of two fractions with different denominators. In this activity students were asked to guess / determine the possible length of the track if students want to divide the track into 3 and 4 equal parts. Our conjecture was that students will use the idea of looking for multiples of 3 and 4 (Least Common Multiply (LCM)).

In this stage, students worked in group. They were asked to guest or determine the possible length of track such that they could divide the track become three and four same parts. One of them, Ilham, gave idea about least common multiply (lcd). When researcher asked Ilham to explain the reason why he chose lcd as a length, he said “because it can be divided by both number, 3 and 4”. Overall, our conjectures were same with the real in class. Notice the following students’ work:


Berapa kira-kira panjang lintasan balapan sepeda, sehingga kita bisa membagi menjadi 4 bagian dan 3 bagian dengan mudah?Tuliskan alasan kamu!

KPK 4 = 4, 8, 12, 16, 20, 24
kpk 3 = 3, 6, 9, 12, 15, 18, 21
alasan: karena bisa dibagi 3 & 4

Figure 4.5. Ilham's work in finding the length of the track such that can divided into 3 and 4 parts.

After that, students were asked to determine the distance of 1/3 of the track and ¼ of the track by using the interpretation of fractions as operator (multiplier). In this case, student used their strategy like at the third activity, for instance: 1/3 of 12 meters is equal to 4 meters. After that, students were asked to measure the length (4 meters) by using unit fractions as unit measurement, i.e. 1/12.

In the class, we found that students needed much time to come to the idea using unit fractions as unit measurement to measure the length. They needed to be provoked to come up with that idea by asking to draw the visualization of the problem or track.

But there were some findings related to student worksheets. At the first time, they asked the meaning of the problem. They asked clue of the story and problem provided in Students' Worksheet. It indicated that they were difficult to understand the stories and problem in LKS. After implementation this activity, researcher discussed with the teacher about this phenomena. Teacher said that students did not get used to solve contextual problem. In addition, the language of the problem was difficult to understand by students. Therefore, for our next HLT, researchers and teachers decided to revise and change the story and the problem. We use the problem of cutting the rope, because this problem related to the previous activity.

Activity 5:

The purpose of the fifth activity was to see whether measurement activities support students in adding fractions with same denominators and what model or strategy used by students in adding fractions with same denominators. In this case, we also wanted to see if students can translate the real situation was given in everyday language into a mathematical expression. Our conjectures were that the students write the problem in the form of mathematical language, addition of fractions, for example 1/5 + 2/5 =. . .

In this activity students were given students a students' worksheet that includes questions about the addition of fractions with same denominator. I doing this problem, they worked individually. The first problem, they were asked to draw or visualize the track of racing related to the information in the students' worksheet. The information of the track are between start and finish line there are 3 point of water distribution which has same distance among them, draw the track and point on the track the position of that point. See the following students' worksheet:

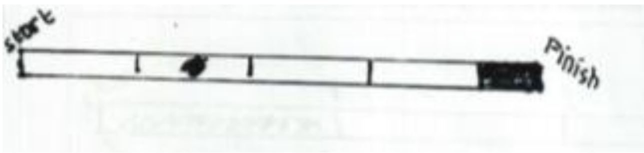


Figure 4.6. Ayu’s drawing/representation of the track of bike racing

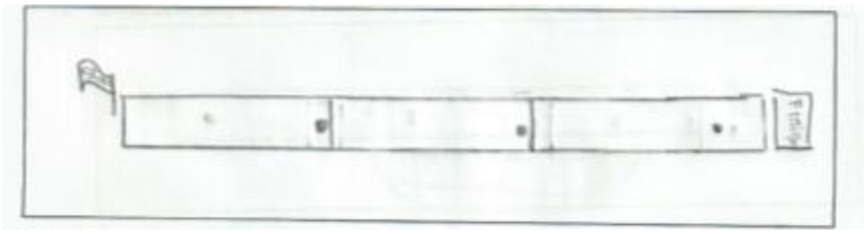


Figure 4.7. Rafee’s drawing/representation of track.

For students’ work, they drew/visualized the track in wrong way. they divided the track become two and 3 same parts. This needed ‘much’ time to visualize the problem. It indicated that they did not understand the story and information about the track. Finally, researcher initiated to use the stick problem in providing the addition of fractions with same denominator. In this activity, researcher determine the total length of the stick if they add two parts of the stick. For instance, how much of the stick if they add 1/5 and 3/5 of the stick. In solving this problem, they measured the stick to find the total of length. See the following figure:

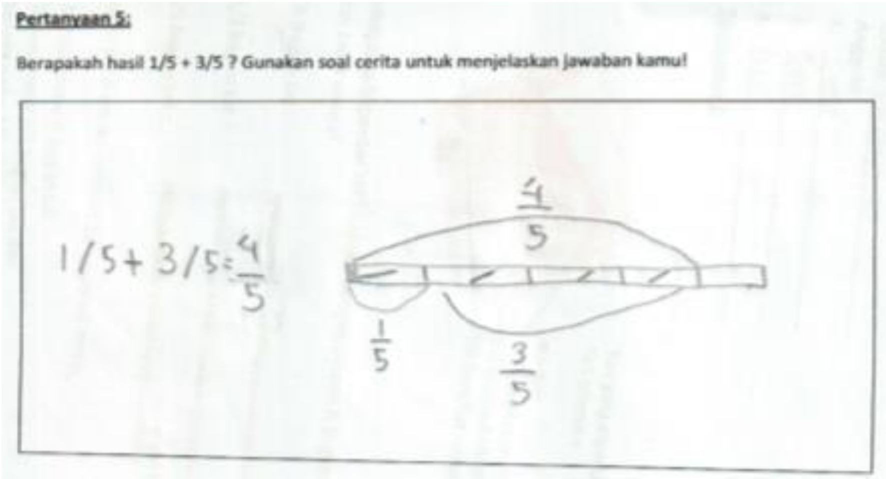


Figure 4.8. Ilham's work in determining the addition of fractions with same denominator by drawing stick, bar/double number line model in thinking and reasoning

At the end of implementation, researcher asked student to make conclusion about strategy in addition of fractions with same denominator. They said that in adding fractions with same denominator, we just added up the numerator.

Based on these observations, we decided to change both the problem and the order of the five activities. We would use the stick problem, because when students learned the addition of fractions with same denominator using a stick model, they could answer and give reasons on the addition of fractions with same denominator. We also changed the order become fourth activity, because the stick problem related to the first, second and third activity. And they had already gotten the pre-knowledge to solve the addition of fractions with same denominator, i.e. interpretation of fractions as measure.

Activity 6:

The purpose of this activity was to see whether measurement contexts support students in adding fractions with different denominators and what model or strategy used by students in adding fractions with different denominators. In this activity, students were given the contextual problem, about the race bike accident, associated with the addition of fractions with different denominators. Our conjectures were that the students would use the number line as a representation of the race track, and determine the position of the first accident. In order to determine the length of the track races that have been taken by the racer until he finally could not continue the race, students would use the double number line as a model of to solve the problem, related to the addition of fractions with different denominator problem. In this case, they would work with fractions, then they change become whole number, and add up them. Finally they will back to fractions.

In the class implementation, students were given worksheet and were asked to solve the problem. The first problem was that students were asked to visualize the problem by drawing the track and finding the position where the accident happened. For this problem, students drew bar/ rectangle as representation of the track. The, they marked on it as position where the accident happened. And then, they write fractions as the length/distance and also the whole number as the total length of the track. See the following students' work:

Figure 4.9. Students' draw as representation of situation

Based on their draw, it indicated that they made a double number line as representation of problem/context.

But, in finding the total length that had been done by racer until he could not continue the race, students could change the language of problem to mathematical language, i.e. 1/3 + ¼ = …, see the following student' work:

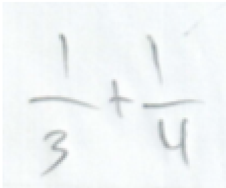


Figure 4.10. students' mathematical language of the addition of fractions problem

In adding fractions, they added the fractions by adding the numerator and denominator directly, see the following students' work:

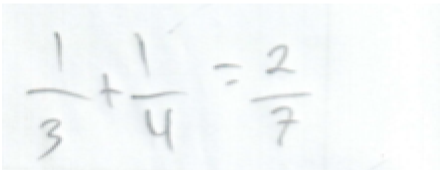


Figure 4.11. addition of fractions with different denominator strategy of Salma, Rafee, Ilham, and Rio.

But, one of them, salsa, added the fractions by procedural method, see the Salsa's work:

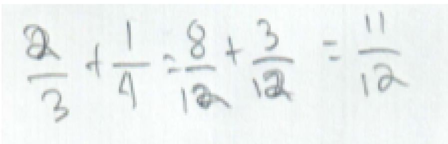


Figure 4.12. Salsa's Strategy in addition of fractions with different denominator

When Salsa was asked to explain why she did this method/strategy, she could not explain the reason. From interview with her, we known that they got the strategy from her private teacher.

In this case, researcher provoked students to guest the length of the track such that they would be easy to find the 1/3 and ¼ of the track. All of them chose 12 as the length of the track. Salsa's reason was because 12 is lcd of 3 and 4, Ilham's reason was because 12 can be divided by 3 and 4. Now, they knew that 1/3 of the track was 4 and ¼ of the track was 3. Then, teacher gave conflict cognitive by asking which one is easy to add, fractions or whole number?. They said adding whole number is easier then fractions. Then salsa said "the total of length is 7 kilometers". researcher asked "so, how much of the track?". All of them could not answer/ change to the fractions directly. Then researcher tried to remind students about the previous strategy, by asking how much of the rope if we have 1 meter of six meters. They answered 1 meter of six meters is 1/6 part of the rope measuring 6 meters. Finally, they could find total of the length that 7 kilometers of 12 kilometers is 7/12 part of the track. Actually, this activity needed much time. But when they solve another problem about addition of fractions with different denominator, most of them still used procedural method by making same (like) denominator and then added up them. Except Ilham, he used double number line to solve the problem. See the following students' work:

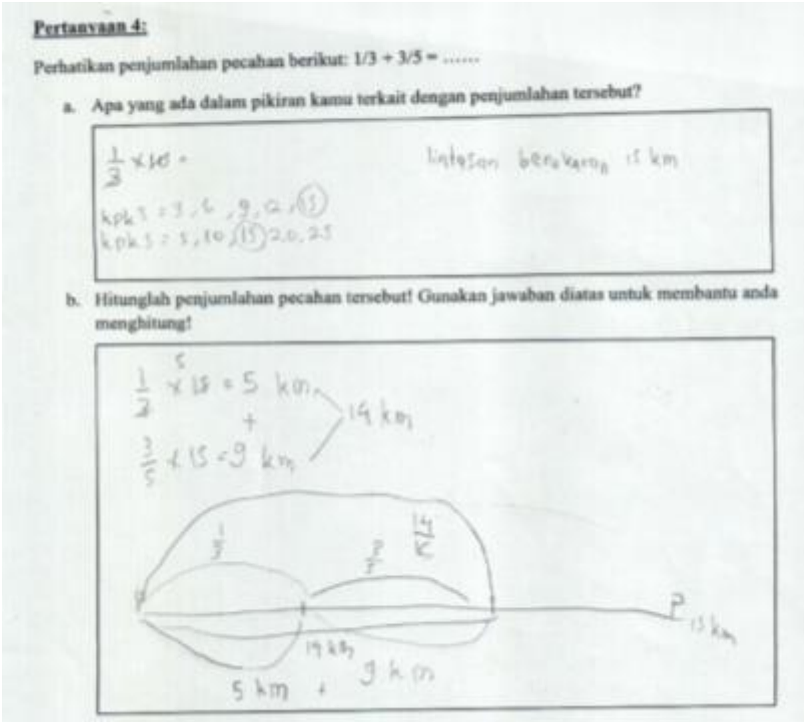


Figure 4.13. Ilham's strategy in determining the addition fractions with different denominator, 1/3 + 3/5 = …

Based on these evidences, we concluded that some of them had already known about strategy/procedure in adding fractions with same denominator by making same(like) denominator. Consequently, we would modify the context problem and question in students' worksheet in adding fractions with different denominator such that it could provoke students to construct the algorithm/procedure. We also needed to emphasize the strategy in making common denominator in fourth activity, because it would help students to reach the idea of adding fractions by making common denominator.

Based on their work and the discussion during the implementation, I could write some conclusions. In general, students were accustomed to working on procedural matters than contextual problem. Method or strategy used to answer was algorithms or procedures, seldom use of free strategies. Students were not accustomed to discuss and express opinions or ideas to the others, this was seen when they could not explain when they were asked to explain to his friend. Consequently, teacher should really provoke and push students to explain their thinking to each other.

1. Conclusion of the Preliminary Experiment

The observations showed that the Stick Coloring/ Measurement context is good to evoke students' reasoning in interpreting fractions as measure, comparing/ordering, addition of fractions with same denominator, however, we need to provoke in reasoning/explaining about their thinking or strategies through drawing, discussion, etc. In this experiment, by asking students to explain their strategy/reasoning through drawing, students have been provoked to use bar and number line model to reason and explain their thinking and strategies in comparing and adding fractions.

"Cutting Scout Rope" activity has stimulated students to interpret fractions as operator and to measure the length of rope by using unit fractions as unit measurement. "Guessing the Length of The Rope" activity has stimulated students to use Lower Common Divisor (LCD) in determining common denominator. This knowledge and skill will help students to add fractions with same denominator. Therefore we will maintain this activity for the next HLT with a small adjustment.

During this period, we also found that the implementation of the instructional activities through our HLT I showed that the target of achievement of success indicators which are set has not achieved yet. The achievement of success indicator achieved is presented at the table below:

Table 4.2. Achievement of success indicators at the first cycle

Aspect	Target	Achievement
Use measurement ideas to support their reasoning	60%	38,9 %
Use students' own model to support their reasoning	50%	33,3 %

2. **Hypothetical Learning Trajectory II (revision of HLT I)**

Based on our retrospective analysis from the video, students worksheet, interview in the pilot experiment, and the achievement of success indicators we revised the HLT. The revised hypothetical learning trajectory is elaborated in the instructional activities as following:

Activity 1: Coloring Scope Stick

Figure 4.14. Scope Stick

Mathematical Learning Goal(s):

This activity aims students can interpret fractions as measure.

Tools: Stick, Crayon, Pen and ruler.

Behind the context:

- An idea of fractions as measurer occurs when students are asked to design or color the mini scope stick and measure the length of each part of the stick.

Description of activity:

Teacher tells about her planning in designing Hanger. Teacher says that *"Let's see this picture(Show the picture of scope), what can you say about the scope stick? Today we want to design/color the mini scope stick in four different types:*

- *First type has three colors,*
- *Second type has four colors,*
- *Third type has six colors, and*
- *Fourth type has eight colors,*

Every group gets 1 stick measuring 60 centimeters and designs one type randomly(by lottery). How do you design the stick? How much the length each part?"

Conjecture of students' thinking:

In designing the stick,

- Some students might divide the stick into the number of colors (type of the stick) by using ruler or by hand or by estimating.
- Some students might use paper as ruler, and fold/divide the paper to find the length.

In dividing or folding paper into two, they will easily to do that. In dividing or folding paper into four, some students might divide or fold the paper into two and fold into two again. But in dividing or folding paper into three, students might find difficulty in divide it. Some students might use try and error in dividing.

After this activity, the discussion will be continued about the length of each part, by asking: "*how much of the stick?*",

- Some students might measure the length in the form of unit measurement, i.e. centimeter. For instance the length is 3 centimeters, etc.
- Some students might measure the length by using fractions as the length/measure. For instance, the distance of part of the stick divided into 3 parts is third or one over three, etc.

The discussion will be continued about the interpretation of fractions. It expected that

- Some students will realize that fractions is a measure/ the length of part of the stick, for instance: 1/3 is the length (measure) of one part of the stick divided by 3, 2/3 is the length of two parts of the stick divided by three, etc.
- Some students will realize that 1/3 is one part of stick divided by three, 2/3 is two parts of the stick divided by three, etc.

Mathematical Congress

In mathematical congress, students will discuss about the way in diving the stick. Students might suggest that dividing by estimating strategy or by hand. The discussion will be continued about the accuracy. It is expected that they will come to the idea using paper or ruler. In partitioning/dividing activity, students will discuss about strategy in dividing the stick. If students partition the stick into 4 by dividing it into two and the dividing again into two. In finding the length of the part, they will discuss the length by using fractions and also the meaning/interpretation of fractions related to the length. After that, students might also discuss about the relation between two fractions: 1/3 and 2/6 by comparing the length of each part.

Activity 2: Comparing Coloring Stick

Mathematical Learning Goal(s):

This activity aims to compare fractions and determine equivalence of fractions.

Tools:

Coloring Stick, Students' Worksheet.

Behind the context:

- A symbolizing of fractions and interpretations of fractions (i.e. fractions as measure) occur when student are asked to find the distance of hooks

Description of activity:

Teacher starts discussion, by reminding students about what they did at the previous meeting. Teacher use the stick colored by students at the first activity to provoke students reach the idea of comparing and determining the equivalence of fractions by comparing four kind of stick directly and ask students to explain what can they see/conclude. Teacher gives four types of stick to every group and students' worksheet consisting three kind of problems:

1. How to compare fractions?, for instance: 2/3 . . . ¾, fill with equal to, less than, or more than.
2. Short the following fractions: 5 / 6, 2 / 3 and ¾ start from the smallest!
3. Determine the equivalence of fractions, for instance : $\frac{\dots\dots}{6} = \frac{2}{3}$

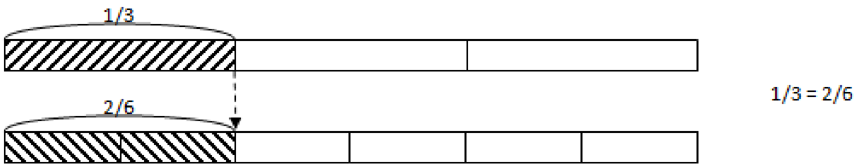
In order to answer these problems, teacher ask students to use stick colored by them at the previous activity and explain their reason by drawing or make representation of their thinking/reasoning.

Conjecture of students' thinking:

For the first problem, comparing fractions:

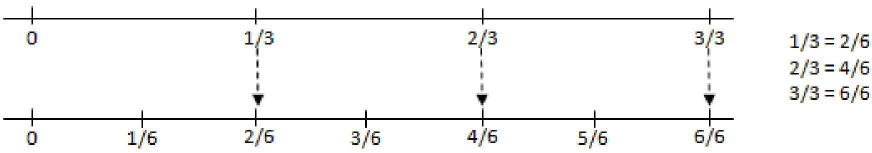
- Some students might use 4 types of stick colored, stick divided into 3, 4, 6, and 8 same parts, to compare fractions by comparing the length of each parts directly, for instance:

- Some students might draw a line/bar/rectangle as representations of the stick to compare fractions by comparing the length of each parts, for instance:



Etc.

- Some students might draw a line as representations of the stick to compare fractions by comparing the length of each parts, for instance **:**



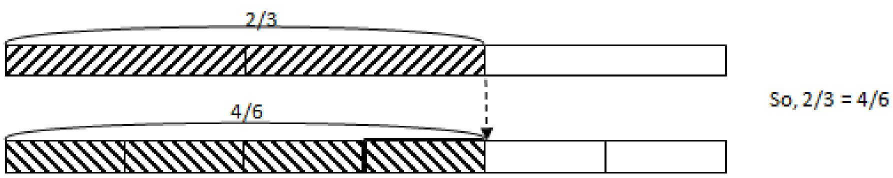
For second question, shorting(ordering) fractions:

- Some students might use four types of the stick to order the fractions, 5 / 6, 2 / 3 and ¾. To solve this problem, they will compare first type (stick divided by 3 same parts), second type (stick divided by 3 same parts), and third type (stick divided by 6 same parts) directly.

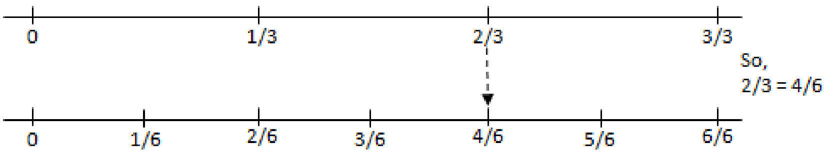
- Some students might compare all fractions, 5/6, 2/3 and ¾, by drawing bar/rectangle as representations of stick. Some students might order those fractions through comparing the length as representations of the fractions.
- Some students might compare all fractions, 5/6, 2/3 and ¾, by number line. Some students might order those fractions through comparing the position of the fractions on the number line.

In determining the equivalence of fractions:

- Some students might compare two types of stick depend on the denominator, for example: to answer this question: $\frac{\cdots\cdots}{6} = \frac{2}{3}$, some students will use first type and third type of the stick.
- Some students might compare by drawing two bars/rectangles and finding the same position, for example: to answer this question: $\frac{\cdots\cdots}{6} = \frac{2}{3}$, some students will draw two bars/rectangles and divide the bars into 3 parts and 6 parts because they want to compare fractions with denominator 3 and 6, then find two positions which are in the same length/position, for example:



- Some students might compare by drawing two number lines and finding the same position of both fractions, for example: to answer this question: $\frac{\dots\dots\dots}{6} = \frac{2}{3}$, for example:



Mathematical Congress

In mathematical congress, students will discuss about the way in producing fractions such as interpretation of fractions. Students might get difficult or forget it. The discussion will be continued about the distance of each part and how to write the distance. Some students might use "*a half*", "*a third", "a quarter*", etc, the discussion will continued with the meaning of them. If students use non unit fractions, the discussion can be continued about the meaning of it, for instance: 2/4 is the name of the second part or that is the name of the first two parts together? The discussion will be continued about comparing length of parts that represents fractions, for instance the length of one stick part divided into 3 parts and the length of two parts of stick divided into six parts, etc. By comparing the length , this discussion can be brought to the idea of comparison of fractions. It is expected that students will discuss about the relation between 1/3 and 2/6. The discussion will be continued about the equivalent of fractions and strategy in making equal fractions.

Activity 3: Measuring Length

Mathematical Learning Goal(s):

1. Students use interpretation of fractions as a measure to add fractions with same denominator by measuring activity

Tools: Coloring stick and students' worksheet.

Behind the context:

- An idea of addition of fractions occurs when students measure some parts of the stick divided by some same parts.

Description of activity:

Teacher asks students to explain about their interpretation of fractions by using the stick. Teacher asks students to measure some parts of the stick by using fractions as measure. Teacher gives students' worksheet which consists two kind of problem:

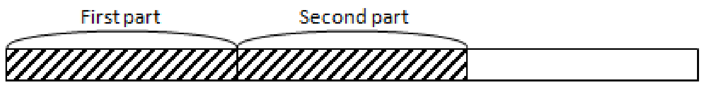
1. Addition fractions with same denominator, for example 2/6 + 3/6 = …, etc. In this problem, students ask to use draw/line to explain their strategy or reason.
2. Reinvent or conclude their strategy in adding fractions with same denominator through their strategy in solving the first problem.

Conjecture of students' thinking:

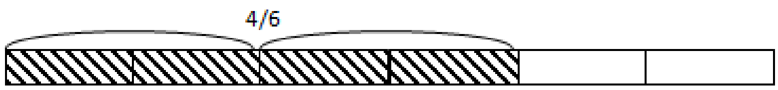
In the discussion about interpretation of fractions:

- Some students might use stick to explain their interpretation of fractions as measure

After this discussion, students will discuss about finding the total length if they add two parts of the stick, for instance:



In order to determine the total of the first and second parts, some students will use mathematical language. They will write the problem by using mathematical language, for instance: the total of the first and the second parts is 1/3 + 1/3. In determining the addition fractions with same denominator, students might use the interpretation of fractions as measure, so 1/3 + 1/3 = 2/3, etc.



The discussion will be continued about the problem in the students' worksheet:

- Some students might draw bars/rectangles as representation of stick and write the fraction as measure/the length of part of the stick. And then measure the total length to answer the addition problem.
- Some students might add the numerator and denominator directly, for instance: 2/6 + 3/6 = 5/12. Consequently, the discussion will be continued about measuring the length of part of the stick like first and second activity.

Mathematical Congress

In mathematical congress, students will discuss about the problem. Students might get difficult to understand about the problem. The discussion will be continued about investigations the information and the questions in the context. It is expected that students will draw a line as representation of situations. The question is aimed to provoke students to add fractions with same denominator. The discussion will be continued about strategy in solving the problem. It is expected that students will use a double number line to add those fractions. The discussion can also about the length which should be chosen such that can be divide by 5. Continued with first question, student are ask to determine the distance in whole number and then move back to fraction. Discussion will also be continued about the relation between denominator and result of addition of fractions with same denominator.

Basic Math Concept:

Common denominator, fractions as operator and measure, Addition of fraction with same denominator.

Activity 4: Cutting Scout Rope

Figure 4.15. Scout rope and Scout tent

Mathematical Learning Goal(s):

This activity aims to stimulate students getting the idea of fractions as operator and measure, measuring length by using unit fractions as unit measurement and comparing fractions (equivalent of fractions).

Tools: Ruler, poster, paper, and pen

Behind the context:

- An idea of fractions as operator occurs when students determine the length of part of the stick given the length of stick as meter (unit measurement) and the measure/length of the part. For instance: how long 1/3 of the stick measuring 6 meters.

Description of activity

Teacher shows the scout rope measuring 6 meters. Teacher tells story about making scout tent. In order to make tent, we need some part of the scout rope, such as a half, one third and a one over six of scout rope.

- The first problems are "*how do we get it"* and "*how many meters of scout rope which is cut"*:

  a. A half of scout rope

b. One third of scout rope

c. One over six of scout rope

- The second problem are determine the length of part in the form of fractions which is known the length of stick and the part in the form of meter. For instance: there is a rope measuring 6 meters,
  - if you have a part measuring 1 meters, how much of length of rope the part?
  - if you have a part measuring 2 meters, how much of length of rope the part?
  - if you have a part measuring 3 meters, how much of length of rope the part?

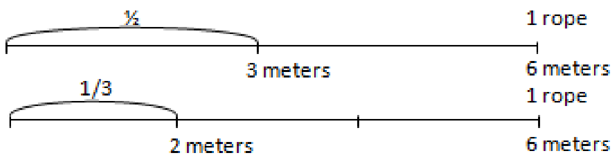
Conjecture of students' thinking:

For the first problem;

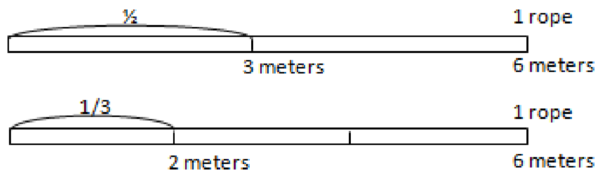
- Some students might divide 6 by 2 directly to get a half of scout rope. From their division, they know that the length of scout rope which is cut by them is 3 meters. They also use this strategy in finding a third and a quarter of scout rope.

  6 : 2 = 3 6 : 3 = 2 6 : 6 = 1

- Some students might try to draw a line to represent the rope and write 6 at the end of the rope as representation of the length, then they divide the line into two like they did in previous activity. See following conjecture of students' model:

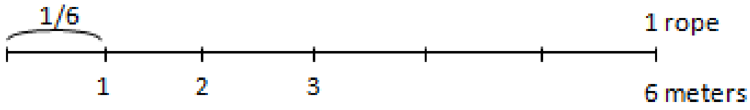


- Some students might try to draw a bar/rectangle to represent the rope and write 6 at the end of the rope as representation of the length, then they divide the bar into two. See following conjecture of students' model:

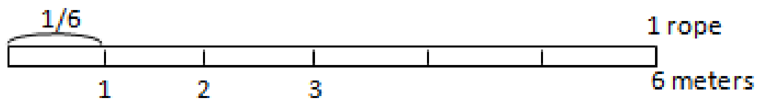


For the second question, some students might compare the draw directly. Some students might compare the whole numbers related to those fractions.

- Some students might draw a line as representation of rope and divide it into 6 to find the position of 1 meters on the rope, and finally based on the line divided into 6 parts, they realize that 1 meter is one part of six parts, by measuring length and using their interpretation of fractions as measure, they will know that 1 meter is 1/6 of the rope. See following conjecture of students' model:



- Some students might draw a bar/rectangle and divide it into 6 to find the position of 1 meters on the rope, and finally based on the line divided into 6 parts, they realize that 1 meter is one part of six parts, by measuring length and using their interpretation of fractions as measure, they will know that 1 meter is 1/6 of the rope. See following conjecture of students' model:



- Some students might use proportion between the length of part and the length of a whole, for instance 1 meter of 6 meters, that is 1/6, etcetera.

Mathematical Congress

In mathematical congress, students will discuss about the way in getting pieces of scout rope. If students do division operation, the discussion will continued about the reason why they do that. If students do second conjecture (draw and divide the drawing), the discussion will continued about their perception about fractions. The aim this discussion is to provoke students to realize the idea fractions can be represent as operator(multiplier), for instance 1/3 means 1/3 of rope, et cetera. The discussion also about the visualization of the track (a line/bar), how they write the symbol of fractions and a whole number (the length) on the line. Through this discussion, we can introduce the name about the visualization, it called *a double number line,* because there are two kind numbers such as fractions and whole number on the line.

Through discussion about the second problem, students will discuss about relation between the length of part and the length of whole rope. In order to answer the questions, some student might use bar/line model and measure the part of rope by using unit fractions as unit measurement or making proportion the length of part and the length of rope (whole).

Basic Math Concept: Fractions as measure and operator, Double number line, comparing fractions, unit fraction as unit measurement.

Activity 5: Guessing the length of Rope

Figure 4.16. a bundle of scout rope

Mathematical Learning Goal(s):

This activity aims to stimulate students choosing a common whole/denominator

Tools: Ruler, poster, paper, and pen

Behind the context:

- An idea of common denominator occurs after students are asked to choose a common whole of their liking as the length of the rope.

Description of activity:

Teacher shows some bundles of scout rope which has some kind of length, such as 5, 10, 15, 20 meters. The first problem is that teacher ask students to choose a rope such that they can cut 1/3 and 2/5 of the rope easily. The second problem, teacher ask students to determine of each part, 1/3 and 2/5 of stick, in the form of unit measurement, meter(s). The third problem, students are asked to change or to move back the unit measurement, meter(s), to fractions as measure by using their ability in the previous activity. For instance: 1/3 of 15 meters is 5 meters and 5 meters of 15 meters is 5/15, and 2/5 of 15 meters is 6 meters and 6 meters of 15 meters is 6/15 meters, so 1/3 = 5/15 and 2/5 = 6/15. The last discussion, teacher asks students to provoke or conclude about how to change two fractions into fractions with same denominator based on their activity.

The last problem, teacher writes two fractions with different denominator and then asks students to change both two fractions into fractions with same denominator, for instance 1/3 and ¼.

Conjecture of students' thinking:

For the first problem:

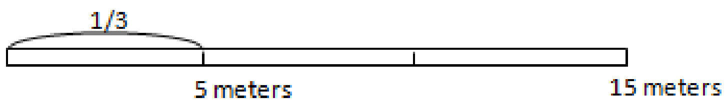
- In order to choose the length of rope, some students might use *try and error* strategy in finding number which can be divided by 3 and 5.
- Some students might use the idea common multiply of 3 and 5 to choose the length of track as follow:

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, …

5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, …

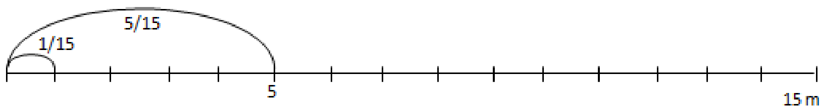
For the second problem:

- For case some students choose 15 km as the length of the track:
  - ✓ Some students might use the interpretation of fractions as operator/multiplier by multiply 1/3 and 15. Some students might write 1/3 x 15 meters = 5 meters, by using their knowledge/ability in the previous knowledge, such as dividing 15 by 3, 15:3. Etcetera.
  - ✓ Some students might use bar/double number line model to determine it. For instance:



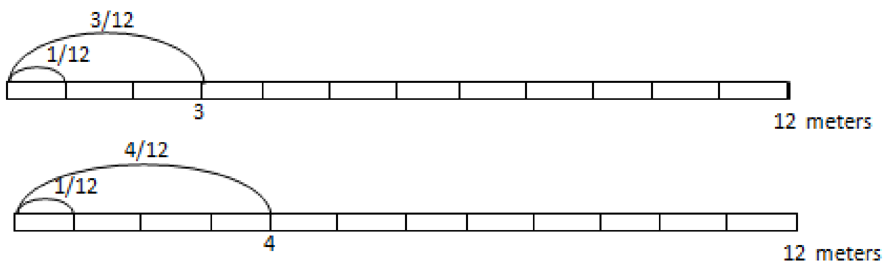
For the third problem:

- If students know the distance of each marker is 1 km, for instance, they will try to find how much of the length of track the '5 km'. By using strategy in measuring length with unit fractions on bar/double number line model, students will know that 5 km is 5/15 of the rope measuring 15 meters, see following conjecture of students figure:



For the last problem:

- In solving the problem, some students might guest the length of rope such that they can cut 1/3 and ¼ of the rope. It comes up with the idea of *lcd* of 3 and 4, they will use 12 as length of the rope, because 12 is lcd of 3 and 4. And then, they will determine 1/3 and ¼ of 12 meters, 1/3 of 12 meters is 4 meters and ¼ of 12 meters is 3 meters. By using bar or double number line, they will move back the numbers, 4 meters and 3 meters, into fractions, 4/12 and 3/12 of the rope measuring 12 meters, see following conjecture of students figure:



So, they might know that 1/3 = 4/12 and ¼ = 3/12.

Activity 6: Crashed Position

Figure 4.17. Crashed bike

Mathematical Learning Goals:

1. Students can produce a common denominator
2. Students use fractions as operator and measure
3. Students can add fractions with different denominator by using a double number line

Tools:

Ruler, poster, paper, and pen

Planned Instructional Activities:

Behind the context

- A common denominator occurs when students try to choose the length of track.
- Addition of fractions occurs when student try to find the distance between start line and crashed position.
- A double number line model occur when students to represent their strategy in Adding fractions.

Description of activity:

Children work in group (3-4 students). Teacher tells the story about her/his friend's plan in joining bike racing. The story is *"Yesterday, my friend told about their experience in her participation on Bike Racing. In that racing, he was injured because he crashed and could not continue the racing. He said that he crashed at distance 2/5 of track after passing the first marker.*

Could you find the position where biker is crashed? How much of the race has he done until he is crashed?"

Children are asked to solve both the first and the second problem in group. Students are asked to explain the reason about their strategy which is chosen by them to solve each problem. Students are also asked to write or draw the solutions on the paper which is used to present their idea in math congress session.

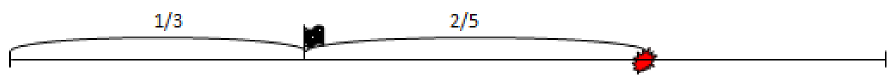
Conjecture of students thinking:

The first problem

The first problem is to draw and find position where biker is crashed on the track.

- Some students might draw a line to represent the track and make a sign to where biker is crashed, See following conjecture of students' model:

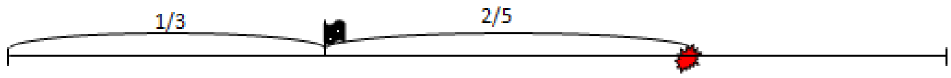
- Some students might draw a line to represent the track and make a sign to represent the point (water supply) and point where biker is crashed, See following conjecture of students' model:



Based on their draw about the crashed position, teacher can ask student to solve the second problem.

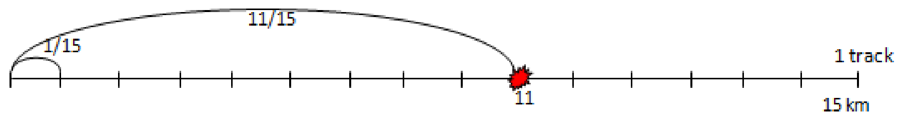
The second problem

Some students might use their draw about the crashed position to solve the second problem. Next, it is called a double number line as model (*model of*). Based on their visualization of crashed position, students realize that the length of the track which has he done is 1/3 + 2/5 of track. See following conjecture of students' model:



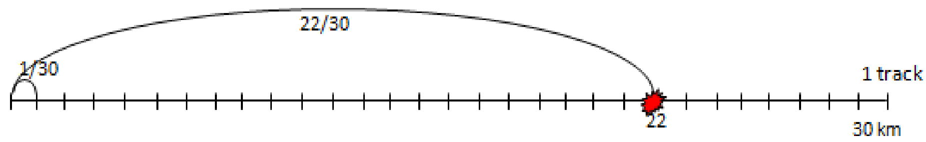
- Some Students might choose 15 km as the length, because they work with this number at the previous activity (track of bike racing).
    - Based on the draw of the crashed position, the length of the track which has he done is 1/3 + 2/5 of track. See following conjecture of students' model:

And they know that 1/3 of track is 5 and 2/5 of track is 6. They just add up the 5 and the 6— that is 11. So, by using unit fractions (1/15-units), the length of the track which has he done is 11/15 of track. See following conjecture of students' model:



- Some students might choose 30 km as the length of race. See following conjecture of students' model:

Based on their drawing, the distance is 1/3 + 2/5 of track. They know that the first part is 10 and the second is 12. They just add up the 10 and the 12—that's 22. By using their strategy in measuring length by using fractions (1/30-units), students will know that the distance of the track between start line until second point is 22/30 of track. See following conjecture of students' model:



By using double number line, students might choose number of their liking as the length of the line to fit the number. It is expected that they examine denominator and realize to the idea of common denominator.

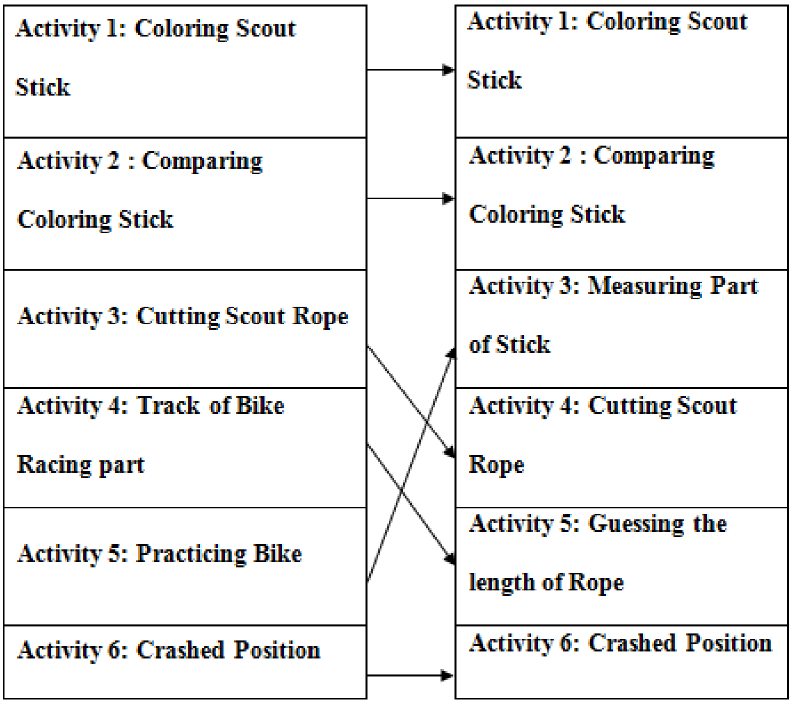
Mathematical Congress

In mathematical congress, students will discuss about the problem. Students might get difficult to understand about the problem. The discussion will be continued about investigations the information and the questions in the context. Continued the first question, the question is aimed to provoke students to draw a line (a double number line) as representation of situations. The question is aimed to provoke students to add fractions with different denominator (1/3 + 2/5). The discussion will be continued about strategy in solving the problem. The discussion can also about the length which should be chosen such that can be divide by 3 and 5. about based on their activity in adding fractions and adding whole number, which one is easy? so, how to make it easy in adding fractions? The aim of discussion is to provoke student to realize that in adding fractions, we can move to the whole number and add them. Then finally move back to fractions. And A double number line is good tool to do this strategy. Based on their result, the discussion can also about equivalence of fractions, 11/15 = 22/30.

3. Progressive Design of HLT I and HLT II

To sum up, we describe the progressive design of HLT I and HLT II in the following table:

Table 4.3. The progressive design of HLT I and HLT II



B. Teaching Experiment

The actual learning process was documented in a series of video-recorded classroom situations conducted in the middle of the march. Before the HLT was tested in the classroom environment, some information was gathered by conducting an interview with the teacher with whom we were going to collaborate; interviews with the fourth graders; observations of classroom situations; trying out activities; and a new classroom culture was introduced to children using a poster with pictures and messages on it. The interviews and observations provided essential information about the current classroom culture and the role of the teacher in the classroom. Interviews with children and the try out activities support my hypotheses about children's counting and calculation strategies in dealing with addition of fractions with same and different denominator and contribute to the development of the HLT.

As mentioned in chapter 3, the result of this research is not a design that works but the underlying principles explaining how and why this design works. Consequently, the hypothetical learning trajectory served as a guideline in the retrospective analysis to investigate and explain students' acquisition of the mathematical ideas of addition of fractions that were elicited by measurement as supporting activities. Further analyses of the data are expounded in the next section where the emergence of the classroom mathematical practices is analyzed in coordination with the individual learning process.

1. **Measurement length as concrete context as the base of mathematical activity**

   The first tenet of RME, *phenomenological exploration*, focuses on using a concrete context as the base of mathematical activity. For this reason, the measurement was used as the based activities. Considering their rich measurement context, dividing and measuring part(s) of the coloring stick and cutting rope were chosen in this research. The aim of these activities were providing a contextual problem situation to build mathematical idea to support the idea of addition fractions with same and different denominator. Consequently, each of these activities were followed by a class discussion.

In general, the expectation from these activities was students would come up with the idea of *interpretation of fractions as measure and operator, comparing and equivalent fractions,* and *common denominator* as supporting idea of addition of fractions. For more specific purpose, this measurement activities support students' reasoning in addition of fractions.

a. Measuring the length of stick part(s) as activity supporting students to interpret *fractions as measure*

At the beginning of activity, students partitioned stick by dividing stick. They worked in group consisting six students. In measuring and dividing stick, students used ruler and their hand spam to divide. After that, teacher started to discuss with student by asking to measure the length of part of the stick divided and colored by students. At the first time, there was no student that stated that the length of each part of stick divided by using fractions as measure. Instead, all students directly measured the lengths by using standard unit measurement, centimeter. This fact showed that students did not realize the interpretation of fractions as measure through dividing/partitioning activities. This condition might be caused the information and question at students' worksheet that cause students use centimeter as unit measurement and also because of the ruler used by them to measure and divide the stick.

But, when the teacher emphasized to find the length of part of stick without using ruler by asking: “how much of one part of the stick?”. Ayu commenced to perceive the interpretations of fractions,1/4, as measure that represents the length of one part of stick divided into 4 parts.

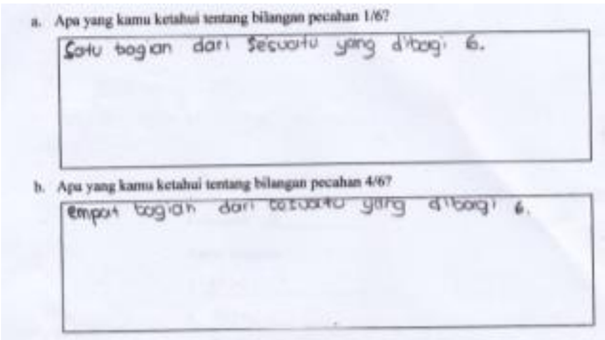
Teacher	: Ayu, how much of the stick this part?(pointed to one part of stick divided into 4 parts)
Ayu	: ehm… a quarter
Teacher	: why?
Ayu	: because this is one part and all are 4 parts
Teacher	: so, how much of stick these two parts? Pointed to 2 parts of stick divided into 4 parts)
Ayu	: 2/4, because there are two parts and each part is ¼.

This fact showed that the strategies that were used by students to measure the length of parts were using unit fractions as unit measurement.

After the discussion about determining the length of part of stick by using fractions as unit measurement, the discussion continued about meaning and interpretation of fractions. The teacher asked one of group, Akzal’s group, to explain the meaning of fractions, 1/6, because Akzal’s group worked with stick divided into six parts.

Teacher	: what do you know about fractions, one sixth?
Akzal	: what is one over six? (start to discuss with his group)
Teacher	: what the meaning of 1/6?
Akzal	: 1/6 is a number….
Fahri	: 1/6 is a part of an object divided into six parts
Akzal	: 1/6 is a part which is a number of an object divided
Fahri	: 1/6 is a number that a part of object divided into six parts.
Akzal	: ok, who is the writer? Heeeee…let’s write! 1/6 is a part of an object divided into six parts.

The phrase "*1/6 is a number whereas a part of object divided into six parts"* and "*1/6 is a part of an object divided into six parts*" showed that Akzal and Fahri seemed to realize that fractions can be interpreted as number that represents the length of part of stick divided into some parts. In this situations, Akzal and Fahri seemed to perceive the *interpretations of fractions as measure* because they knew that fraction could represent a measure of a quantity relative to one unit of that quantity, length. And also, if we saw the students' worksheet about their meaning or interpretation of fractions as measure, they seemed to perceive the *interpretations of fractions as measure*. See the following figure:


a. Apa yang kamu ketahui tentang bilangan pecahan 1/6?

Satu bagian dari Sesuatu yang dibagi 6.

b. Apa yang kamu ketahui tentang bilangan pecahan 4/6?

Empat bagian dari sesuatu yang dibagi 6.

Figure 4.18. Work of Akzal's group in interpreting fractions as measure

From dividing and coloring stick activity, it can be concluded that students commenced to *interpret fractions as measure*. Students also commenced to realize the idea of *using unit fractions as unit measurement* to measure some parts of stick.

b. Comparing the length of part of coloring stick supports students reasoning in comparing fractions such as comparing fractions and equivalence of fractions.

At the beginning of discussion at second meeting, teacher asked students to determine the length of part of stick, and students answered by using fractions as measure of the length. And teacher also asked the meaning of the fractions used by them. This activity aimed to remind students about the interpretations of fractions as measure. After this discussion, teacher created some conflicts among students by comparing two kinds of stick divided into 3 parts and six parts to stimulate and develop their acquisition of the idea of comparing fractions through comparing the length of part that represents the fractions.

From the following vignette, it is confirmed that students commenced to use comparing the length of part(s) that represents the fractions strategy to compare fractions.

Teacher : if we compare two sticks divided by 3 and 6 parts, what can you conclude about that? Let's see both two sticks! See the first stick which has six colors, how much of one part?
Students : 1/6
Teacher : how much of two parts?
Students : 2/6
Teacher : how much of three parts?
Students : 3/6
Teacher : how much of four parts?
Students : 4/6
Teacher : how much of five parts?

Students : 5/6
Teacher : how much of six parts?
Students : six over six or one
Teacher : let's see the second stick divided by 3 colors, how much of one part?
Students : 1/3
Teacher : 2 parts?
Students : 2/3
Teacher : 3 parts?
Students : 3/3 or 1
Teacher : let's compare both two sticks. How long this part?(point to a part of stick divided into six parts)
Students : 1/6
Teacher :how much of this?(point to two parts of stick divided by 6)
Students : 2/6
Teacher : 2/6, if we see the second stick (show stick divided by three), 2/6 equals to?(show the length of two parts of stick divided into six and one part of stick divided into 3)
Students : 1/3
Teacher : 2/6 equals to 1/3, so, 4/6 equals to? ?(show the length of four parts of stick divided into six and two part of stick divided into 3)
Students : 2/3
Teacher :so, 4/6 equals to 2/3. Can you explain why is it equal
Ayu : because the lengths are same.

The phrase "*because the length are same"* as the answer of "*why these fractions are equals?"* showed that students commence to use measurement/comparing length as strategy to compare the fractions.

In the group discussion, teacher asked to find the equivalence of fractions. In doing this task, students used four kinds of stick divided into 3, 4, 6, and 8 parts. From the students' work, it is also confirmed that students commenced to *measurement length activity* as strategy to compare fractions and reason their thinking.

In the class discussion about finding equivalence of fractions, teacher also created a situation as a means to support students' acquisition of idea equivalence of fractions by using another strategy. The crucial guides by the teacher are shown in the following transcript.

Teacher	: last time, we have already learned about equivalence of fractions, if I have 1/3, this fractions equals with what fractions?
Students	: 2/3
Teacher	:2/3?
Students	: oh….2/6
Teacher	: 2/6, then we also have…
Aisy	: ehm…3/9,
Teacher	: 3/9, how do you get 3/9 Aisy?
Aisy	: Oh…4/12
Teacher	: 4/12? Let's try….. where does it come from?
Aisy	: add…
Teacher	: add with what?
Aisy	: add numerator and denominator…
Teacher	: numerator and denominator are added by what?
Aisy	: I mean… **multiply both by two**
Teacher	: multiply by two..ehm, I know what Aisy means that the equivalence of fractions comes from, 1/3 equals to 2/6, it means 1 becomes 2, so how we can change 1 becomes 2?
Students	: by multiply by 2
Teacher	: multiply by two, 3 becomes 6, multiply by what?
Students	: two
Teacher	: it means that 1/3 equals to 2/6, Lets' check the aisy's answer that 2/6 equals to 4/12. If we multiply the both numerator and denominator of 2/6, what is the result?
Students	: 2 time 2 equals to 4 and 2 times 6 equals to 12
Teacher	: so, the aisy's answer is correct?
Students	: yes……

What the teacher did by reminding fractions that equals with a fraction 1/3 was example of creating a situation in which students realize that both their findings and strategy in equivalence of fractions. This situations encouraged students to focus the relation between two fractions that are equal. It expected that students realize another strategy to determine the equivalence of fractions. The phrase “*multiply both by two”* showed that students commence to use *doubling or multiplications* strategy to make equivalent fractions.

Another stimulus created by the teacher in the class was by giving problem about ordering/sorting fractions (smallest to biggest). This problem was used by teacher to support emergence *model of* as representation of fractions.

Teacher	: how to compare two fractions, 2/3 and ¾?
Salsa	: may I make draw?
Teacher	: of course…how do you compare them?
Salsa	: I compare the length of the shading (she make two bar, she divide the first into three parts and draw two parts of the bar with parallel or crossing line for effect (shadow) and she also divide the second one into 4 parts and draw three parts of stick with parallel or crossing line for effect (shadow))
Teacher	: based on your picture, which one the bigger one? 2/3 or ¾?
Salsa	: ¾
Teacher	: why? can you explain?
Salsa	: because this shading (point to the bar divided into 4 parts) is longer than this (point to the bar divided into 3parts).
Teacher	: so, how do you compare them?
Salsa	: by comparing the length of them

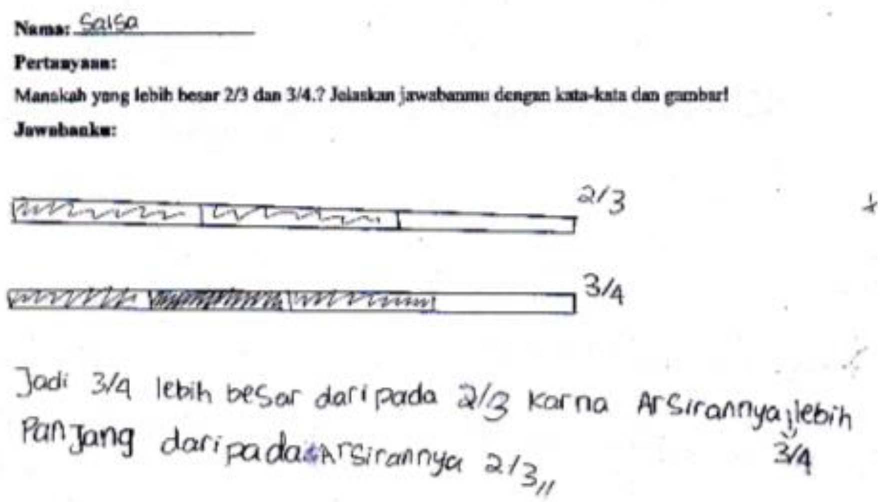


Figure 4.19. Salsa's strategy in comparing fractions

Salsa came up with *bar model* to represent fractions by using shaded parts. This showed that the need of *representation* of fractions has commenced to compare fractions by comparing the length of shaded parts that represent the fractions. The phrase "*by comparing the length of them*" as the answer of "*how do you compare them?*" showed that students commence to use comparing/measurement length of shaded part as strategy to compare fractions. This also showed that comparing/measuring shaded parts that represent the fractions *support students' reasoning* in thinking.

In the class presentation, students shared their answer about comparing fractions problem in the front of the class. The class presentation aimed to develop students' *interactivity*, as the fourth tenet of RME. Approximately only two groups of four groups were active and gave attention when one of group present their work. During the representation of Ayu's group, the teacher created some conflicts among students by comparing and discussing their different strategy to stimulate and develop their acquisition of the idea of comparing fractions and also to investigate their reasoning in thinking.

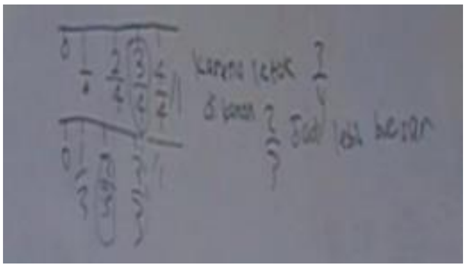


Figure 4.20. Ayu's work in comparing 2/3 and ¾

Teacher : can you explain your answer? Why ¾ is bigger than 2/3?

Ayu : because the position of ¾ is right side of 2/3

Teacher : so, do you means if the fractions is on the right side of the other fractions, it is bigger than the fractions on left side

Ayu : yup…. The right position of fractions, the greater that fractions

Ayu used draw, *line*, as representation of fractions to compare fractions. The drawing showed that students commence to use *number line* as *model of* situation/problem to compare the fractions. The phrase*" because the position of ¾ is right side of 2/3"* as answer of *"Why ¾ is bigger than 2/3?"* showed that Ayu realize that every number stated on the right side is bigger than the number on the lift side. It means that she knew how to compare fractions by using *number line* as a model i.e. by compare the position of number on the number line.

From comparing coloring stick and representation class, students commenced to acquire some ideas about *comparing and equivalent fractions*. This activity also showed how students emerged some model such as *bar and number line model*, to help their thinking and reasoning in solving problem.

c. Cutting Rope supports students' acquisition of the interpretation fractions as operator

The acquisition of the *interpretation fractions as operator* and the use of *a unit fractions as unit measurement* prolonged in cutting rope activity and also in the class discussion after the game. In cutting rope, the *interpretation of fractions as operator* and *unit fractions as unit measurement* were utilized by students when determining the length of part(s) of a rope measuring 6 meters that was cut by students. For instance, the length 2/3 of rope measuring 6 meters.

The initial acquisition of *interpretation of fractions as measure* shown in coloring stick was developed as a base for interpretation fractions as operator in cutting rope activity. The following vignette shows the emergence of interpretations of fractions as operator in the cutting rope activity.

Teacher asked student to determine 1/3 of rope measuring 15 meter and 2/5 of rope measuring 15 meter.

Teacher : how about you, Akzal?
Akzal : the length of the rope is 15 meter, so 15 is divided by the denominator… 15 divided by 3, it equals to 5, so, the length of one part is 5 meters.
Teacher : oh…. You mean that 1/3 of 15 meter is the length of one part, how long?
Akzal : 5 meters

Teacher : then 2/5 of 15 meters?

Akzal : **it is same, 15 divided by 5, it is 3 meters, so the length of one part is 3 meters, because it is two parts, so 3 meters times 2, it is 6 meters, so the result is 6 meters.**

The idea of *unit fractions as unit measurement*, shown by the phrase "***it is same, 15 divided by 5, it is 3 meters, so the length of one part is 3 meters, because it is two parts, so 3 meters times 2, it is 6 meters, so the result is 6 meters***" as answer of "*2/5 of 15 meters?*" showed that Akzal knew that 2/5 is two parts, it means 2/5 is 2 of 1/5 or 2/5 = 1/5 + 1/5. This phrase also shows that students commenced to acquire the interpretation of *fractions as operator.*

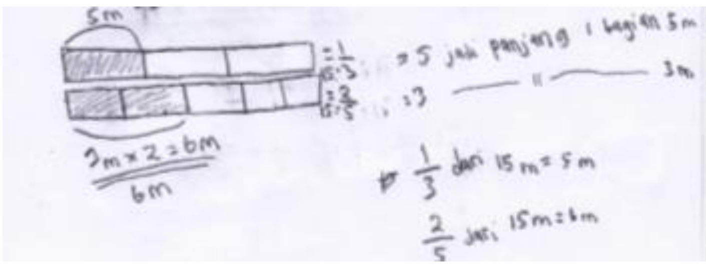


Figure 4.21. Akzal's work in determining 1/3 of 15 meters and 2/5 of 15 meters

What Akzal did, drawing bar, referred to second and third tenet of RME, namely using models and symbols for progressive mathematization and using students' own construction.

Making visualization of the problem aimed to bridge students' informal knowledge of mathematical operation to arithmetic of fractions.

Arithmetic in this term was defined as the multiplication of fraction with whole number.

The phrase "***it is same, 15 divided by 5, it is 3 meters, so the length of one part is 3 meters, because it is two parts, so 3 meters times 2, it is 6 meters, so the result is 6 meters***" as answer of "*2/5 of 15 meters?*" showed that Akzal commenced to acquire the mathematical formal of *multiplication fraction with whole number* that 2/5 of 15 = (15:5) x 2 = 3 x 2 = 6.

d. Guessing the length of rope and its contribution in supporting students' acquisition of the idea of common denominator.

The acquisition of the idea of *common denominator prolonged in cutting rope activity and also in the class discussion*.

In guessing the length of the rope, the idea of common whole and less common multiply were utilized by students when determining the length.

The acquisition of less common multiply of the denominator appeared when students tried to guess the length of the rope such that can be into some parts as much as both denominator.

Teacher : if we want to get 1/3 and 2/5of the rope, how long the length of the rope so that we can get 1/3 and 2/5of the rope easily?
Ayu : 15 meters
Fahri : 30, 45, 60
Teacher : why? Ayu…

Ayu : **the length is 15, because 15 can be divided by 3 and 5**
Teacher : Fahri…
Fahri : **because they can be divided by 3 and 5**

The phrase "***the length is 12, because 12 can be divided by 3 and 4***" as answer of "*how long the length of the rope so that you can get 2/3 and 1/4 of the rope easily?*" showed that students used the idea of *common multiply of both denominator* to find the whole. This phrase show that students commenced to acquire the idea of *common whole/ denominator*.

The following excerpt shows that students commenced to acquire the idea of *less common multiply as common whole.*

Teacher : after we determine the multiply of each number, 3 and 5, what is the relation among 15, 3 and 5?
Students : 15 is less common multiply of 3 and 4

The following excerpt show how student determine the equivalent fractions by using the *idea of common whole*.

Teacher : if we want to find 2/3 and ¼ of the rope, how long the rope?
Students : 12
Teacher : who want to determine 2/3 of rope and ¼ of the rope?
Salma : I am, mom…
Teacher : ok salsa, let's write on the white board.

See the Salsa's work on the whiteboard,

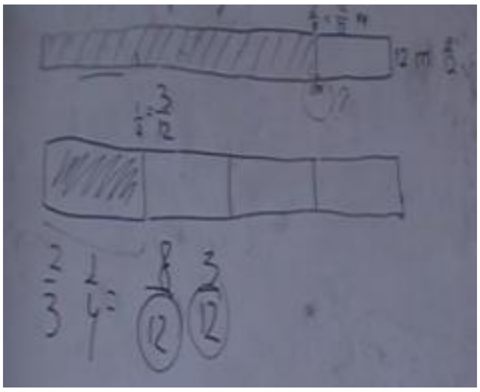


Figure 4.22. Salma’s work in making equivalent fractions and common denominator

The Salma’s work shows that Salma commenced to use *a double number line model* in finding equivalent fractions. She worked with two kind of numbers, fractions and whole number. She moved from fractions to whole number using the idea of *fractions as operator.* And she moved back to fractions using *measuring length with unit fractions as unit measurement*.

e. Summary of the measurement length activities as supporting activities to help students’ thinking and reasoning in addition of fractions.

As shown in figure 4.1, the framework of the measurements activities was building and developing students’ understanding of the ideas of *interpretation of fractions as measure and operator, comparing and equivalent fractions, common whole and denominator*. From subchapter 5.3.1.1 to 5.3.1.4, it was found that from the *coloring stick, measuring part(s) of coloring stick,* and *cutting rope* activities students commenced to perceive the ideas of *interpretation of fractions as measure and operator, comparing and equivalent fractions, common whole and denominator*. Moreover, students also started to perceive the idea of *addition of fractions with same denominator* when they measured one part and two parts of stick divided into 4 same parts. The initial knowledge of the supporting ideas of addition of fractions that was gained from the measurement length activities were developed in the class discussion.

However, most of these concepts were still perceived by students as supporting ideas or concepts. Consequently, the next important step in the instructional sequence was providing "*bridge*" activities to develop students' supporting ideas of addition of fractions into the more formal knowledge of addition of fractions.

2. **Drawing to visualize the situation as a bridge from contextual problem to the mathematical formal of addition of fractions.**

This activity referred to the second and the third tenet of RME, namely *using models and symbols for progressive mathematization* and *using students' own construction.*

Drawing of the contextual problem aimed to bridge students' informal knowledge of measurement length of parts of coloring stick to formal addition fractions. Formal measurement in this term was defined as the correct and meaningful use of procedure in adding fractions.

Drawing visualization activities were conducted in a series of two activities as follows:

- Drawing the situation at the contextual problem related to addition of fractions with same denominator problems
- Drawing the situation at the contextual problem related to addition of fractions with different denominator problems

In the second activity, drawing representation of fractions in reasoning to explain about comparing and equivalent fractions, students were directed to get acquainted with a bar and double number line model. The bar and double number line were chosen as a model because it seemed like the figure of stick that was used as base activity [*we can see figure 5.2 and Salsa's idea shown at the vignette in subchapter* ***b*** *in which students started to draw bar as representation of fractions that she want to compare*]. Consequently, using bar to represent the fractions was turned to using bar model to represent the fractions (as the focus of this activity).

In general, this activity was successful because almost all students were able to correctly draw the representation of fractions to compare the fractions. However, there was an interesting finding when Aisy's group was drawing the two representation of fractions that they want to compare. This group drew two representation of two fractions using bar but different measure.

The conjecture derived from this finding is that these students were not realized that in comparing fractions, the whole must be same. This mistake made students cannot properly compare two fractions.

There were two different strategies used by students to compare fractions using bar or number line model as representation of fraction that want to compare. The first strategy was drawing a bar or a number line model to compare fractions with same denominator. The second strategy was drawing two bar or two straight lines and then directly compare the length of shaded parts that represent the fractions or compare the position of two fractions on the number line.

The drawing of visualization of the situation constructed by students showed the process of emergent modeling how a model emerged from a situational level to formal level.

In measuring and comparing coloring stick, a student (i.e. Salsa's group) came up with an idea to use coloring stick to measure the length of part(s) and compare fractions (*look at the figure at subchapter **b***). Idea of Salsa's group represented one of *situational level* in the comparing activities in which Salsa explained how her interpretation and solution of the problem developed based on how to act in the comparing coloring stick as measuring tools.

In the *referential level* Salsa's idea was followed up by drawing bar as representation of stick divided into some parts. Moreover, the drawing of stick (bar) became the base of the emergence of student-made representation as the *model of* the situation that helps student to think and reason.

The numbers, fractions, written on students' drawing (bar) (as shown in figure 5…) showed how students commenced to consider that a number represented a measure. In this phase students started to use their representation as *model for* solving the problem. The use of student-made representation as the *model for* thinking and reasoning showed that *general level* of modeling has been attained by students.

The last level of emergent modeling, *the formal level,* started to be accomplished when some students draw a bar or double number line as their model. This kind of model became the tool to help student for thinking and reasoning. In the formal level students' reasoning with conventional symbolizations started to be independent from the support of models for mathematics activity. In this level, the focus of discussion move to the use of model in thinking and reasoning.

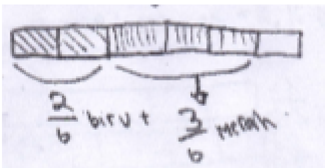


Figure 4.23. a bar model as the *models-of* situation that relates the contextual situation, coloring stick.

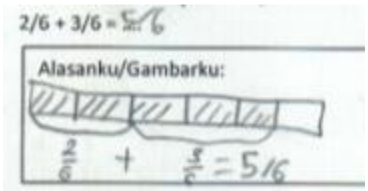


Figure 4.24. a bar model as the *models-for* mathematical reasoning within fractions relations with jump on the bar

As the conclusion of "*drawing representation of situation*" activity, students showed their progress to shift from contextual situation to formal mathematics. The student-made model reflected that students started to use the representation to help their thinking and reasoning.

3. Solving addition of fraction with same denominator problem using bar model

The activity was started by working with worksheets that preceding the class discussion. The worksheet contained 3 problems and had been solved by 24 students. From the students' answers of the worksheet, almost students could solve the problem. However, it was difficult to conclude whether those correct solutions reflected students' understanding because the students' worksheet merely provided the final answer of solutions and the drawing of representation of fractions without any record about students' strategies. For this reason, the following analysis of students' reasoning based on video recording aimed to investigate students' learning process and level in acquiring the idea of addition of fractions with same denominator.

Akzal's Case

The following excerpt is an case of a student who gave reason about their thinking. Giving reason about their thinking in solving problem aimed to investigate students' understanding.

Teacher : Akzal…. Can you explain your answer? Maybe you can tell/explain the problem.

Akzal : ok, the first, 2/6 of the stick is colored with blue color, it means the denominator is six, so we divide the stick into six parts

Teacher : why?

Akzal : because the denominator is six

Teacher : ehm…ok, then?

Akzal : so, 2/6 of stick is colored with blue color and then 3/6 of the stick is colored with red color, how much of the stick which has already colored? So we add both two fractions, 2/6 plus 3/5.

Teacher : so, what is your answer?

Akzal : 5/6

Teacher : how do you?

Akzal : count all parts w

Teacher : do you measure the distance from this to this?(point to the first part until the fifth parts)

Akzal : yes.

Teacher : could you make conclusion about your finding in addition of fractions with like denominator

Akzal : (he write : if the denominators are same the both numerators are added, the denominators are same, (2+3)/6 = 5/6)

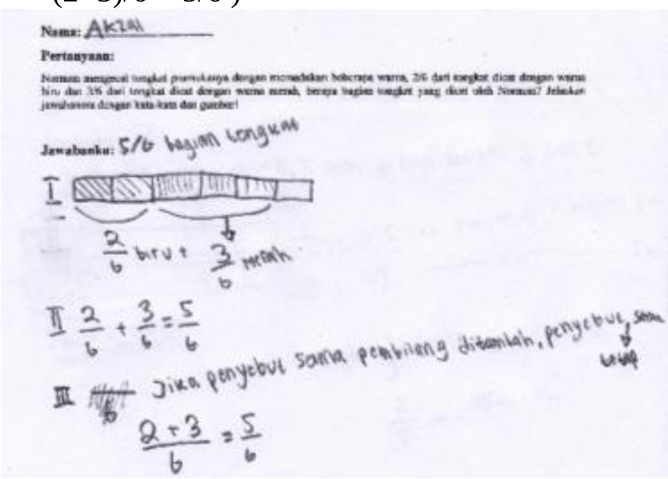


Figure 4.25. Akzal's work in addition of fractions with same denominator

Some conjectures are derived from this case, namely:

- *A bar as model of* thinking and reasoning

  The conjecture of this finding is that the students who used a bar as representations of fractions assumed that shaded bar represented the fractions.

- The idea of *using unit fractions as unit measurement*

  The conjecture of this situation is that the student who used unit fractions as unit measurement assumed that a part of length (stick) divided into some parts was unit fractions.

- The idea of addition fractions as *measuring length using unit fractions as unit measurement*

  The student showed that he measured the total length (parts) that represents addition fractions by counting the number of shaded parts. The conjecture of this strategy is that students measured the length that represents the addition of fractions by using *unit fractions as unit measurement.*

<u>Fahri's Case</u>

The following students' work shows another strategy used student when adding fractions using bar model.

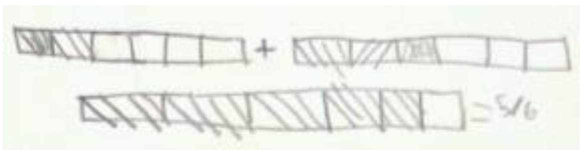


Figure 4.26. Fahri's darwing of visualization as model to reason

To add two fractions with like denominator, Fahri was drawing shaded bar for each fractions and then he drawn a shaded bar to represent the result. In spite of the answer was correct, this drawing showed that students did not realize that if they want to add fraction, the whole must be same.

General conclusion of the measuring length as supporting strategy in solving addition of fractions with same denominator:

Based on students' answers in the worksheet and students' reasoning, it is conjectured that most students could add fractions with same denominator.

The progress of students' reasoning in explaining their answer showed that the model and measuring length activity played an important role in encouraging students to consider the strategy in addition of fractions with same denominator. In making conclusion about their strategy to add fractions with same denominator students started to perceive the algorithm or procedure of *addition of fractions with same denominator*.

4. **Solving addition of fraction with different denominator problem using a double number line model**

The activity was started by working with worksheets that preceding the class discussion. The worksheet contained three problems and had been solved by 24 students that worked in group consisting six students. The problem were *A racer followed the race bike. At the time of the race, the rain fell very heavy. After pedaling the bike around 2/3 of the track the racer fell because the track is slippery. And then he continue the race. But,* *after a quarter of the track, he fell again and he cannot continue the race because the bike was heavily damaged. First question: Could you make draw about the situation? Second question: How long the track such that you can determine every part (2/3 and ¼ of the track)? Third question: How much of the track taken by racer from the start until finally he could not continue the race?*. At the end of learning, students were asked to represent their work in front of class. This activity was preceded by representation students' work to investigate students' thinking and reasoning in solving addition of fractions with different denominator.

The following excerpt is an example of a student who gave reason about using *a bar as model of* situation.

Akzal	: from this to this is 2/3 of the track,
Teacher	: you mean that the racer fell at the first time at that point, 2/3 of the track. And then?
Fahri	: the racer continue the race until ¼ of the track. He fell again and could not continue the track because the bike was heavily damaged.

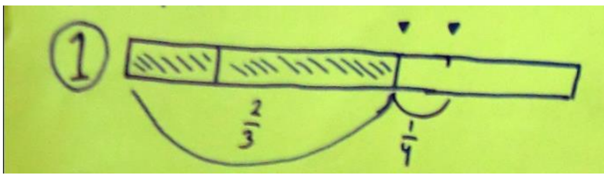


Figure 4.27. A bar model used by students to visualize the contextual situation.

This drawing showed that two possibilities. First, students drawn the situation by approximation. it means that the length of part is not represent the actual proportion. Second, students did not realize that the second distance is a quarter of the length of the track rather and not a quarter of the remaining path. Moreover, based on their writing on their poster, at the first time they thought that the second distance was a quarter of the rest. But in solving the second question they commenced realize that the second distance was a quarter of the track.

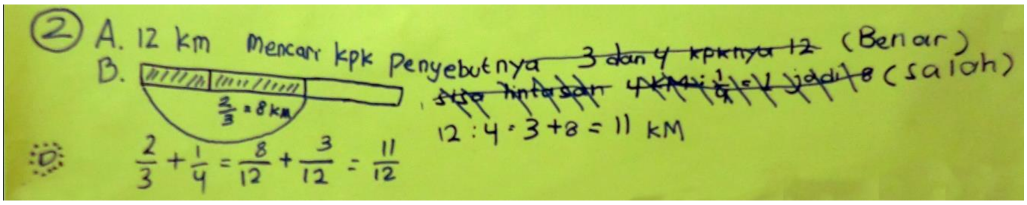


Figure 4.28. A bar model used by students to reason about their idea and strategy in solving problem

The following excerpt is an example of a student who gave reason about the idea of *common denominator*.

> The problem: How long the track such that you can determine every part (2/3 and ¼ of the track)?
>
> Akzal : 12 kilometers
> Teacher : explain your answer!
> Akzal : **12 is lcm of the denominators**
> Teacher : what are the denominators?
> Akzal : 3 and 4
> Teacher : what is the lcm of 3 and 4
> Akzal : 12

The phrase "***12 is lcm of the denominators***" show that Akzal connected her knowledge about the idea of *less common multiply* of both denominator as a length of the track so that the length could be divided by 3 and 4. This phrase also show that students commenced to acquire the idea of *common denominator*.

The following excerpt is an example of a student who gave reason about the strategy in solving *addition of fractions with different denominator*.

The problem: How much of the track taken by racer from the start until finally he could not continue the race?

Akzal	: because the length of the track is 12 kilometers. **2/3 of the track is 8 kilometer, because 12 divided by 3 is 4, so 1/3 of 12 is 4 kilometer**
Teacher	:oh, 1/3 of 12 meters is 4 kilometers?, then?
Akzal	: **because it is 2/3, so 2 times 4 is 8 kilometers**.
Teacher	: 8 kilometers, the?
Akzal	: then, …
Teacher	: how can the denominator is 12?
Fahri	: **12 divided by three and multiply with 2**.
Teacher	: yes, where does the 12 come from?
Fahri	: lcm of 3 and 4
Teacher	: oh… from the first answer. Then
Fahri	: 12 divided by 3 and multiply with 2
Teacher	: then…
Akzal	: **12 divided by 4 is three, and then add 8 and 3, it equals to 11. So the result is 11/12.**
Teacher	: 11/12. Ok.

The phrase "***2/3 of the track is 8 kilometer, because 12 divided by 3 is 4, so 1/3 of 12 is 4 kilometer"***, "***because it is 2/3, so 2 times 4 is 8 kilometers"*** and their drawing show that students used their interpretation of *fractions as operator* and *measure* to determine the first distance (multiplication fractions with whole number). This phrase also show that students used *measuring length by using unit fractions as unit measurement* as strategy to multiply fractions with whole number, 1/3 of 12.

The phrase "***12 divided by three and multiply with 2"***show that students commenced to *acquire the formal way to determine multiplication* of fractions with whole number.

The phrase "***12 divided by 4 is three, and then add 8 and 3, it equals to 11. So the result is 11/12***" show that students used *a double number line model* to help their thinking to add fractions with different denominator. it means that they worked with two numbers, fractions and whole number. To find the result, they used the idea of *part of a whole* and *measuring length using unit fractions as unit measurement.*

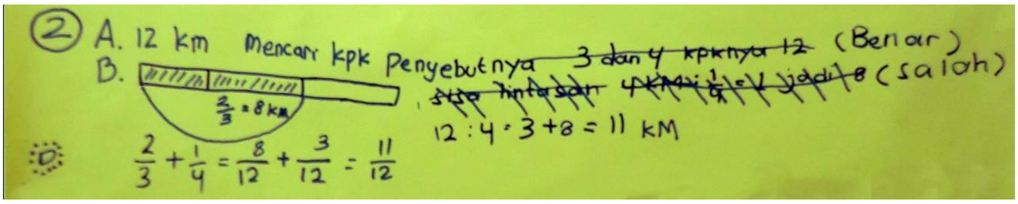


Figure 4.29. Work of Akzal's group in solving the addition of fractions with different denominator problem, 2/3 + ¼

What they wrote on their poster show that two conjectures of students' strategies. First, they added fractions by determining the equivalent fractions using the idea of *common denominator* and strategy in adding fractions with same denominator. Second, they worked with whole number and moved back to fractions using *a bar model.* In moving back to fractions, they used *measuring length using unit fractions as unit measurement* as strategy.

From the students' answers on the poster, almost students could solve the problem. However, it was difficult to conclude whether those correct solutions reflected students' understanding because the students' poster also provided the final answer of solutions and the drawing of representation of fractions without any record about students' ideas. For this reason, the following analysis of students' reasoning based on video recording of interview with Akzal and Fahri aimed to investigate students' learning process and level in acquiring the idea of addition of fractions with different denominator.

In the interview, researcher gave problem about addition fractions with different denominator. The following excerpt shows that students transformed the mathematical problem in the situation (*daily life language*) to *mathematical language*.

Researcher	: the next question.
Fahri	: What is the total of the track that successfully traversed by Joko?
Researcher	: so, Joko runs from where to where?
Fahri	: from this to this (point to the first part and the second part)
Researcher	: how much of the track?
Fahri	: (he write 2/5 + 1/3) see the following figure;

The following excerpt shows that students used *common whole to add fractions* as strategy.

Researcher	: how to determine the total of the track?
Akzal	: the total of the track that he runs…. 2/5 of the track, he runs normally, and plus 1/3 of the track that he runs very fast. **So now, we have to change the fractions that has same denominator.**
Researcher	: why do you do that?
Akzal	: to add them easily
Researcher	: you mean that if they have same denominator, they can be added easily, why?
Akzal	: **because we only add the numerators.**
Researcher	: prove it!
Akzal	: the first, we find the lcm of 5 and 3, it is 15. **So, now how to change 5 become 15? We multiply 5 by 3, so the numerator we multiply by 3 too, so we have 6/15 and then how to change 3 become 15? We multiply 3 by 5, so the numerator we multiply by 5 too, so we have 5/15,** and **we add 6/15 and 5/15. The result is 11/15.**

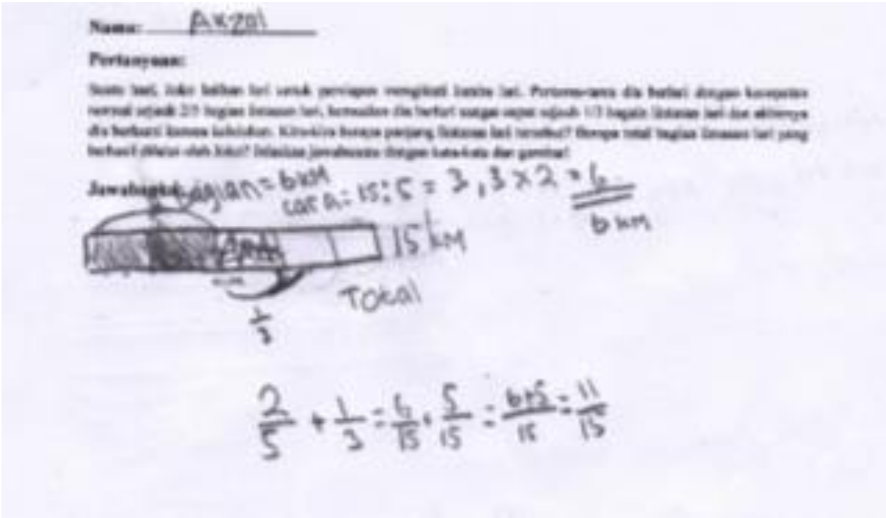


Figure 4.30. Akzal’s work in solving the addition of fractions with different denominator problem, 2/3 + ¼ at interview session

The phrase “***So now, we have to change the fractions that has same denominator”*** and ***“because we only add the numerators***” as answer of “***you mean that if they have same denominator, they can be added easily, why?***” show that the use strategy in adding fraction with like denominator like they did at previous activity.

The phrase “***the first, we find the lcm of 5 and 3, it is 15. So, now how to change 5 become 15? We multiply 5 by 3, so the numerator we multiply by 3 too, so we have 6/15 and then how to change 3 become 15? We multiply 3 by 5, so the numerator we multiply by 5 too, so we have 5/15***” show that students used *multiplication* as strategy to make equivalent fractions and used *common whole/denominator* as strategy to add fractions with different denominator.

General conclusion of addition of fractions with different denominator activity

In solving the problem, students used some ideas learned by them through previous activities such as interpretation of fractions as *measure* and *operator, common whole/denominator*. And students also used some strategies such as *using multiplications to make equivalent fractions, using common whole to add fractions, using measuring length with unit fractions as unit measurement to add fractions.* In order to help their thinking and reasoning and also to bridge their thinking from the contextual problem to formal mathematics, students used *a bar as model*. Even though, in our conjecture students expected that they would emerge a double number line as model. But the idea behind the bar model is equivalence with the double number line model. During this period, we also found that the implementation of the instructional activities through our HLT II showed that the target of achievement of success indicators which are set has already achieved. The achievement of success indicator achieved is presented at the table below:

Table 4.4. Achievement of success indicators at the second cycle

Aspect	Target	Achievement
Use measurement ideas to support their reasoning	60%	68,8 %
Use students’ own model to support their reasoning	50%	67,8 %

C. Local Instructional Theory

The local instruction theory with respect to the sequence of activities and the intended concept development for the teaching and learning of addition of fractions was summarized in the following table. This table shows the interaction between the development of the tools that were used and the acquisition of the mathematical ideas/concepts (Gravemeijer, 2003).

Table 4.1. The local instructional theory for addition of fractions in grade 2 of elementary school

<table>
<tr><th>Activity</th><th>Tools</th><th>Imagery</th><th>Practice</th><th>Concept/ Idea</th></tr>
<tr><td rowspan="3">Coloring stick</td><td rowspan="3">Stick measuring 60 centimeters, Crayon, ruler, marker</td><td rowspan="3">Signifies that the “length of one part” becomes the unit fractions</td><td colspan="2">The activity of partition and measuring should become the focus to the use fraction as quantity. It can be start to measure one part of stick to perceive the idea of unit fractions and next interpretation of fractions.</td></tr>
<tr><td>Partition and measuring the length of partition</td><td>Fractions as measure</td></tr>
<tr><td colspan="2">The students’ model should be started from students’ own construction in representation the fractions.</td></tr>
<tr><td rowspan="3">Comparing Stick</td><td rowspan="3">Students-made coloring stick</td><td rowspan="3">Signifies that the “same length of part” becomes the equivalent fractions</td><td colspan="2">The comparison activity should become focus to compare the length of part that has same length.</td></tr>
<tr><td>Direct comparison</td><td>Equivalent and comparing fractions</td></tr>
<tr><td colspan="2">Their finding in equivalent fractions in the comparing stick could lead to come up with the strategy in making equivalent fractions. The “equivalent” draw as representation of two fractions should be started from students’ own construction in comparing</td></tr>
</table>

			fractions	
Measuring length of partition		Signifies that the "total of some parts" becomes the result of addition of fractions represented by the length of those parts	The students' visualization of situation as '*model of*' should be started from students' own construction in representation the fractions and led to help their thinking and reasoning.	
			Measuring the total partition and reasoning about their idea and strategy	Addition of fractions with same denominator
			Their conclusion about their strategy in solving addition problem led to reinvent the formal mathematics of addition of fractions.	
Cutting Rope	Rope measuring 6 meters, ruler, marker		Measuring the part of rope which is cut by students	Fractions as operator
			The use of students' model should become a tool to help students perceive strategy in multiplication fractions with whole number. The relation between the length of part and the whole should led to perceive the idea determining fractions as *proportion* of them. The use of *using unit fractions as unit measurement* also led to move back whole number to fractions.	
Guessing the length of Rope		Signifies that the "number that can be divided by both denominator" becomes the common whole	Determining the length of rope that they need to be cut	Common whole/denominator
			The guessing length activity should led to perceive the idea of common whole and next to commence to the idea common denominator.	
Bike Race Problem	Poster, marker	• Signifies that the "making	Drawing visualization of problem and	Addition of fractions with different denominator

		same denominator” becomes the easy way to add fractions	presentation about students thinking and reasoning in solving the problem	
		• Signifies that the “change be whole number” becomes the easy way to add them	The students’ visualization of situation as ‘*model of*’ should be started from students’ own construction as a supporting tool to think reasoning.	

D. Discussion

This part provides information about important issues that we found in this research. The implementation of RME in this design research reflects from how the principles of RME underlay the activities in this research. This implementation will be elaborated on in the following chapters: *measurement activity as supporting activities for thinking and reasoning addition of fractions*, *class discussion: teacher’s role and students’ social interaction and emergent modeling*.

1. Phenomenological didactical: Measurement activity as supporting activities for thinking and reasoning addition of fractions

The goal of *Didactical Phenomenology* is to find the phenomena and situations that may create the need for the students to develop the mathematical concept or tool we are aiming for. As the first instructional activity, a situation that is experientially real for student is used as the base for mathematical activity. Considering the emersion of fractions that the concept of fractional number was derived from measuring (Freudenthal, 1983; Streefland, 1991). Consequently, in order to teach addition of fraction, we can use measurement length as the contextual situation of the instructional activities in this research.

In addition, there are two reasons why measurement activities are used as context. The first, measurement comprises an aspect of practical skill that is important in daily life. The second, measuring numbers represent a specific aspect, because they refer to an "environment" in which the number exists. The use of measurement for teaching could give important implications for understanding how informal and formal learning can support students' understanding in learning fractions (Sweta Naik, 2008).

However, using measuring activity in mathematics education needs to be supported by a class discussion as a reflective session. In the reflective session, students' concrete experiences from measurement length were shared and focused and transformed into initial ideas of addition of fractions. Considering the importance of a class discussion as the reflective session, teachers should be able to organize the class discussion to reach the objectives of students' learning processes.

2. **Interactivity: teacher's role and students' social interaction**

According to the third tenet of RME, it is important to start the class discussion by using *students' own construction*, such as students' strategies and models. The teacher, as the *facilitator* of the class discussion, should stimulate students to present their ideas as the starting point of the class discussion. Teacher can stimulate students to express their idea by asking "*how did you compare those fractions?*", "*can you explain your strategy*" or "*could you prove your answer?*".

The teacher also should be a good orchestrator in provoking students' social interaction. The teacher could provoke social interaction (i.e. group discussion and class discussion) by either making groups of students or asking some questions. Based on the finding in during teaching experimental, it was observed that the teacher occasionally posed the some questions to stimulate students' social interaction such as *"Any other idea?", "Do you agree?", "who has different ideas?".*

In supporting students' reasoning, it is also important for the teacher to help children communicate and develop their ideas by elaborating upon what they already know from their pre-knowledge or their finding in measuring activity. An example of this manner was when the teacher encouraged students to perceive the idea of *equivalent fractions* using *doubling or multiplication as strategy*. The teacher connected the *comparing two kind of coloring stick* to *compare fractions* activity by posing the following questions:

"Do you remember when we compare using comparing stick? What are your findings? what can you conclude?"

3. Emergent modeling

As the third principle of RME, the emergent modeling design heuristic could support students' progress from a concrete situation to a formal reasoning. Consequently, the second characteristic of RME, *using models and symbols for progressive mathematization*, focuses on how a model can be used as a bridge from the concrete level to the more formal level. The "*Drawing visualization of situation*" activity was drawn on to bridge from measuring activities in measuring the length of part as the concrete level to the more formal level of addition of fractions.

Students' strategies in measuring length of stick parts that were discussed in the class discussion showed *how students' own construction can be used* to support students' acquisition of the supporting ideas of addition of fractions. Furthermore, the students' model served as the tool in thinking and reasoning to solve addition of fractions problem.

Chapter V

Conclusions, the Weakness of the Research and Recommendations

A. Conclusions

1. In general, measurement activities could support students' reasoning in adding fractions. Measurement length activities that were used in this research (i.e. *coloring and measuring part(s) of stick*, *Cutting rope*) were rich with length measurement activities including comparing length and measuring length.

Before going to the further discussion, it is important to discuss the meaning of *supporting idea* of addition of fractions. S*upporting ideas* mean that some mathematical ideas that should be perceived to solve the addition of fractions problem by using *measurement length* as context. The supporting ideas of addition of fractions that were expected to be elicited in measurement length activity were the idea of *fractions as measure and operator, comparing and equivalent fractions* and *common whole/denominator*.

The students' acquisition of *supporting ideas of addition of fractions* that were elicited in measurement activities was elaborated in the following descriptions.

Fractions can be thought as *measure*

At the beginning of activity, students partitioned stick by dividing stick. Students used ruler and their hand spam to divide. Students commenced to use *unit fraction as measure* of the length when the teacher asked students to measure a part of stick which was divided into some parts without using ruler. However, new conflict arose when students were asked to measure some parts of stick. To solve this problem, students used *unit fractions as unit measurement* as strategy to measure the stick parts. However, when they were asked to interpret fractions, students commenced to interpreted *fractions as a number that represented the length of parts (measure).(see students' work at chapter IV subchapter a.).*

Comparing and equivalent fractions

Comparing stick activity (direct comparison) provoked students to commence acquire the idea of *equivalent fractions.* Students realized that if length that represented the fractions was same, two fractions were equal (*see the first vignette of chapter IV subchapter b.).* The discussion about the result of finding the equivalence fractions using comparing stick provoked students to come up with *doubling or multiplication as strategy* to make equivalent fractions (*See discussion between teacher and Aisy at chapter IV subchapter b.).*

Students commenced to emerge *a bar model* and *a number line* as representation of fractions when they were asked to explain their reason about *comparing fractions (see students' work at chapter IV subchapter b.).* However, they did not partition the bar fairly and some time they also did not aware that if they would to compare fractions using drawing two bar as representation of both fractions, they should draw two equivalent bars *(see students' work at chapter IV subchapter b.).* This situation made students get wrong answer. Therefore, making student realize that *in comparing fractions the whole must be same* and *partition should be fair* is effective way to eliminate the mistake.

Fractions can be thought as *operator*

At the first activity of fourth meeting, students were asked to determine the length of rope part that was gotten 2/3 of rope measuring 6 meters. In this situation, students used real rope measuring 6 meters. Students used folding rope strategy and ruler to measure the length. In the next activity, students were given problem about finding the length of 1/3 and 2/5 of rope measuring 15 meters. In this situation students made drawing a bar as visualization of the rope. Students commenced to acquire the idea of *fractions as operator* when they thought that 1/3 meant 1/3 of something, namely rope measuring 15 meters. Students used the idea of *fractions as measure* and *measuring length using unit fractions as unit measurement* as strategy to determine the length of rope part *(see first vignette at chapter IV subchapter c.).*

Common whole/denominator

Students commenced to acquire the idea of *common whole* when they were asked to guess the length of rope that want to be cut so that they can cut easily by using *less common multiply* of both denominator as the measure of *the whole* (rope). They commenced to use the idea of *fractions as operator* when they tried to determine the length of part that was cut by them. They commenced to used their *bar model* and the idea of *fractions as measure* to help their thinking and strategy in multiplying fractions with whole number *(see the first vignette at chapter IV subchapter d.)*. Students commenced to acquire the idea of *equivalent fractions* by *making proportion between the length of part and the length of whole* to get new fractions and relation between the initial fractions and it. Students also commenced to acquire the idea of *common denominator* when students tried to make *two fractions have same denominator*. The idea of *common whole* is the supporting idea to help students in deciding the denominator which was chosen *(see the third vignette at chapter IV subchapter d.)*.

Measurement length: supporting students' thinking and reasoning in addition of fractions

In solving addition of fractions with same denominator problem, students made drawing bar as visualization/model of problem. Based on their model, they realized that the problem was to determine the length of bar part(s). By using the idea of *fractions as measure* and *measuring length using unit fractions as unit measurement* students measured the length of their drawing or representation of situation.

In the other hand, in solving addition of fractions with different denominator, students also made *a bar* as visualization/model of situation. Based on their model, they realized that the problem was to determine the length of bar part(s). Student used the idea of *less common multiply* of both denominator as *common whole* (the length of the whole). There were two strategies in adding that fractions. First, students moved to whole number to add them. Students used the idea of *fractions as operator* to determine each part in the form whole number, and add up them. They moved back to fractions using the idea of *using unit fractions as unit measurement* to measure length of parts. Second, they used strategy in addition of fractions with same denominator by making same denominator. They used the idea of *common denominator* and *doubling/multiplication strategy* in making same denominator.

2. There was a students' model that emerged when they solved the contextual problem related to addition of fractions with same denominator and different denominator called *a bar model*. In general, students have accomplished the *situational level* of emergent modeling when they explained their interpretation and solution of measuring contextual problem (i.e. coloring stick, cutting rope, bike race problem) using drawing *a bar* which was partitioned as representation of fractions. Afterwards the accomplishment of the *referential level* was showed by describing strategies for reasoning in the measuring context with jumps on *the bar*. Moreover, the *bar* became the base of the emergence of student-made representation of situation as the *models-of* the situation that relates to the addition of fractions problem.

The "*making drawing*" to explain their reasoning when they solved the addition fractions problem, 2/5 + 1/5 or 2/3 + 1/5, promoted the accomplishment of the next levels of emergent modeling. The fractions relation with jump on the bar showed how students commenced to describe their strategy for reasoning. The use of the *bar* as the *models-for* reasoning showed that *general level* of modeling has been attained by students. Students commenced to accomplish the *formal level* when they reasoned within a framework of number relations without the support of the bar.

B. The Weakness of the Research

The weakness of this research is that the researcher did not pay attention to the learning styles of students in designing the instructional activities elaborated in the HLT. However, in fact the learning styles affect the learning process and learning outcomes achieved by students.

C. Recommendations

These recommendations are addressed to both the practice of teaching and learning addition of fractions through linear measurement (as the implementation of this research) and to further research in mathematics education for developing and improving mathematics education practices.

1. Classroom setting

The first recommendation in the didactical component is group setting. The group size should be well-considered when students work in group because it is difficult to be effective and efficient if one group consists more than 4 because there are some students only see without cooperate to think or get involved in group discussion. The first possible solution is students work in pair, one write/work with students' worksheet and the other colors the stick, measures the length, etc. The second solution is every group consists 4 students, each group gets tools (i.e. stick, rope, etc), worksheet and poster to present their work, so that every students has a job or responsibility to get involved in the group.

Another classroom organization that needs to be well-considered is the class discussion. Based on our findings, that only a few students were active in the class discussion, underlies the need to give opportunities for every group to present and share their ideas and giving reward such as point.

Considering this point, it will be very important to do a research which also focuses on the teacher's role in students' learning process using measurement activity and socio norms.

2. **Realistic Mathematics Educations as approach**

Realistic mathematics education (RME) can contribute to developing learning to a more progressive learning. In our research, RME has supported the classroom activities and we have seen how students learned better in such an environment. The use of measurement contexts have supported students thinking and reasoning in solving addition of fractions. With a good context, students can construct their understanding about mathematical ideas that is meaningful so that it makes sense for them. The emergence of models supports students' transition from concrete situational problems to more formal mathematics. The model can be a bridge between informal to formal mathematics. It is a long-term learning process from a *model of* the students' situated informal strategies towards a *model for* more formal mathematical reasoning. In RME classrooms, the contributions from the students are highly promoted. Students learn to share and listen to each other's idea through a discussion where strategies are discussed and compared to determine which ones are more sophisticated. In a discussion, students can learn from their peers and the collaborative development of knowledge among students can be made possible.

During the research, we found that the classroom we worked with was still struggling in establishing socio norms and socio-mathematical norms. Nevertheless, a good start has been made as this class has developed an open learning atmosphere where students are allowed to use their own strategy and production such as model to help their thinking and reasoning. In this classroom students have freedom to use different strategy, but they are not promoted to discuss and choose the best strategy.

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