SUPPORTING STUDENTS' DEVELOPMENT OF EARLY FRACTION LEARNING

A Design Research on Mathematics Education

MASTER THESIS

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NIM 097785006

SURABAYA STATE UNIVERSITY

POST GRADUATE PROGRAM

MATHEMATICS EDUCATION STUDY PROGRAM

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A Thesis Submitted to Surabaya State University Master Program

as a Partial Fulfillment of the Requirements

for the Degree of Master of Science

in Mathematics Education Program

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APPROVAL

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This master thesis is dedicated to:

My beloved mom, dad and brother for their endless love
that has made me feel that there is no mountain too high

My fiance for his love and patience
that have always become the source of my spirit

ABSTRACT

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Keywords: the meaning of fraction, relation among fractions, design research, realistic mathematics education, fair sharing, measuring

Many researchers on fractions argued that fraction is one of the most difficult topics in primary school. Different meanings of fraction are one of many causes of difficulties on learning fractions. Students should explore such meanings of fractions sufficiently before they learn about relation among fractions and operations of fractions.

Occupying design research as a research method, the aim of this study is to support students' learning process in extending their understanding of the meaning of fraction and relation among fractions. During February-March 2011, the research was conducted in grade 3 at SD Laboratorium UNESA. An instructional sequence consisted of 6 lessons as a part of a hypothetical learning trajectory was designed and tested in a cyclical process. In the first cycle, 6 students were involved in the experiment and 28 students in a class were involved in the teaching experiment of the second cycle. Data collections were generated from taking video during teaching experiment, interviewing the students, giving pre and post-test, and collecting the students' work during the lessons.

As results of testing the hypothetical learning trajectory, it was found that fair sharing situation involving continuous and discrete objects could stimulate the students to construct the meaning of fraction as a part-whole relationship and a quotient. Fraction as parts of distance from zero point on informal number line was conveyed through measuring situations on ants' paths. Fair sharing and measuring activities were also found to be a starting point to investigate relation among fractions such as comparing fractions and non-unit fractions as iterations of unit fractions. Another finding was that some students still had struggled in posit fractions on number line. Although the students could determine a position of fractions on an ant's path as an informal number line, they did not apply such knowledge when posited fractions on a formal number line. The students' understanding of relation among fractions has not supported them to be able to posit fractions on a number line.

ABSTRAK

Sari, Elisabet Ayunika Permata. 2011. *A Design Research on Mathematics Education: Supporting Students' Development of Early Fraction Learning.* Tesis, Program Studi Pendidikan Matematika, Program Pascasarjana Universitas Negeri Surabaya. Pembimbing: (I) Dr. Dwi Juniati, M.Si. dan (II) Sitti Maesuri Patahuddin, Ph.D.

Kata Kunci: makna pecahan, hubungan antar pecahan, *design research*, pendidikan matematika realistik, pembagian adil, pengukuran

Banyak penelitian tentang pecahan bertitik tolak dari kenyataan bahwa materi pecahan merupakan salah satu topik yang cukup sulit di sekolah dasar. Makna pecahan yang bervariasi merupakan salah satu dari penyebab-penyebab kesulitan dalam pembelajaran pecahan. Siswa-siswa seharusnya diberi kesempatan seluas-luasnya untuk mengeksplorasi makna pecahan sebelum mereka mempelajari hubungan antar pecahan dan operasi pada pecahan.

Metode penelitian ini adalah *design research*. Tujuan penelitian ini adalah untuk mendukung proses pembelajaran siswa dalam memperluas pemahaman tentang makna pecahan dan hubungan antar pecahan. Penelitian dilaksanakan di kelas 3 SD Laboratorium UNESA pada Februari-Maret 2011. Rangkaian instruksional untuk 6 pertemuan sebagai bagian dari hipotesis trayektori pembelajaran didesain dan diuji dalam proses siklik. Siklus pertama melibatkan 6 siswa sedangkan siklus kedua melibatkan satu kelas yang terdiri dari 28 siswa. Sebagai metode pengumpulan data, peneliti merekam video selama pembelajaran, mewawancara siswa, memberikan pre-tes dan post-tes kepada siswa, dan mengumpulkan hasil kerja siswa selama pembelajaran.

Sebagai hasil uji hipotesis trayektori pembelajaran ditemukan bahwa masalah kontekstual tentang pembagian adil dapat menstimulasi siswa dalam memahami makna pecahan sebagai hubungan bagian-keseluruhan dan sebagai hasil bagi. Situasi pembagian adil melibatkan benda-benda kontinu dan diskret. Pemahaman makna pecahan dalam pengukuran yaitu sebagai jarak dari titik nol pada garis bilangan, dieksplorasi melalui masalah kontekstual tentang jalur semut. Masalah kontekstual tentang pembagian adil dan pengukuran juga telah dapat digunakan sebagai titik awal pembelajaran tentang hubungan antar pecahan. Eksplorasi hubungan antar pecahan meliputi membandingkan pecahan dan menggunakan pecahan satuan untuk membentuk pecahan non-satuan. Hasil lainnya dari penelitian ini yaitu ditemukan bahwa siswa-siswa masih mengalami banyak kesulitan dalam meletakkan pecahan pada garis bilangan. Walaupun siswa-siswa dapat menentukan letak pecahan pada kegiatan pembelajaran tentang jalur semut, mereka mengalami kesulitan dalam menentukan posisi pecahan pada garis bilangan formal. Pemahaman siswa tentang hubungan antar pecahan belum dapat mendukung pemahaman siswa tentang letak pecahan pada garis bilangan.

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Elisabet Ayunika Permata Sari

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CHAPTER I

INTRODUCTION

1.1 BACKGROUND

In the Papyrus Rhind, fractions emerged in the past time when human needed to divide an amount of objects into some equal parts. Through the ages, fractions had progressed toward formalization and even had developed into one of domains in mathematics. Children start to learn fractions formally in primary school. In fact, most researchers on fractions find that fraction is one of the most difficult topics in primary school (Hasseman, 1981; Streefland, 1991). Even, the difficulties do not only exist in student's learning but also in teaching (Ma, 1999).

According to Hassemann (1981), some of those difficulties in learning fractions are that fractions are used less often in daily life and are less easily described than natural numbers. Moreover, it is not easy to put fractions in order of size on the number line. Another factor that makes fractions very difficult to understand is that fractions have many representations and interpretations (Kilpatrick, Swafford, & Findell, 2001). The complexity of learning fractions also emerges because the development of fraction knowledge is also linked to children's ideas about whole numbers (Pitkethly & Hunting, 1996). It is showed that there are some children who make a mistake such as $\frac{1}{2}+\frac{2}{3}=\frac{3}{5}$ because they perceive those numbers as whole numbers instead of fractions.

In constructing the knowledge of fractions, some researchers suggest that children should not be asked directly to label fractions in models that have been already made into parts. It is better that children construct the parts by themselves (May, 1998). The implication is that learning fractions should start from solving problems involving partitioning situations. Looking back to history, fair sharing is considered as the real problem that can convey the basic meaning of fraction. Children can learn about the meaning of fraction during constructing parts in fair sharing activities such as dividing three pizzas among five children.

However, Streefland (1991) in Keijzer (2003) also argued that fair sharing-regarding $\frac{3}{5}$ as three pizzas divided by five children-does not clearly present a fraction as one number or entity, but rather presents a fraction as (a ratio of) two numbers. Fair sharing situation also can masque fractions as numbers 'between whole numbers' and therefore limit global reasoning with fractions, which is considered essential in developing number sense (Greeno, 1991 in Keijzer, 2003). In his research, Keijzer (2003) then integrated bar model and number line as the model of learning fractions used to address both the meaning of fraction and relation among fractions. May (1998) also suggests that children need to develop a sense of fractions and relation among fractions as they need number sense in order to deal with whole numbers.

Although there have been many findings on teaching and learning of fractions, more researches using context of Indonesian classroom are needed. Teacher centered learning is often found in Indonesian classroom. Students tend to be a passive learner instead of construct their own knowledge actively (Mujib, 2010). Actually since 2001, a movement to reform mathematics educations has occurred. Pendidikan Matematika Realistik Indonesia (PMRI) adapted from Realistic

Mathematics Education in the Netherlands has been implemented in some primary schools in Indonesia.

Yet, the implementation of realistic mathematics as a movement to reform teaching and learning is quite complex. Even for the schools which have started to implement realistic mathematic, it is still in progress. According to Sembiring (2010), improving teacher competence in conducting mathematics teaching and learning based on realistic approach is one of challenges. Teachers still need a lot of supports such as model of teaching and learning using realistic approach.

In early fraction learning, although students had learned about fractions, understanding fractions is often isolated in representing fractions as shaded parts on geometrical shapes such as rectangle or circle without contextual situations. Meanwhile, based on a perspective of Pendidikan Matematika Realistik Indonesia (PMRI), contextual situations are the starting point of developing mathematical concepts (Hadi, 2005). It also happens when students learn about comparing fractions. Comparing fractions as a part of learning processes of relation among fractions often moves directly into level of algorithm instead of comparing the magnitude of fractions using real objects. Consequently, the lack of understanding of the meaning of fraction inhibits students in learning operations on fractions. It was found that even 5^{th} grade students had a lot of difficulties to determine $\frac{1}{5}$ of 10 pieces of a cake in initial learning of fractions multiplication.

1.2 RESEARCH QUESTIONS

Relating to the background of this study, the researcher poses a research question as the following:

"How to support students to extend their understanding of the meaning of fraction and relation among fractions through fair sharing and measuring activities?"

1.3 AIMS OF THE RESEARCH

In line with the background of this study and the research question, the first aim of this research is to support students' learning process in extending the understanding of the meaning of fraction and relation among fractions. The second aim is to contribute to an empirically grounded instruction theory on early fraction learning. To achieve such aims of this research, a hypothetical learning trajectory will be (re)designed, tested in the teaching experiment and analyzed retrospectively.

1.4 SIGNIFICANCES OF THE RESEARCH

Regarding to the purposes of this study, the theoretical significance of the research is to give a contribution to an empirically grounded instruction theory on early fraction learning. The practical significance of the research is to give an insight to mathematics teachers on how to develop teaching and learning process that supports students to extend their understanding on early fraction learning. This study also offers an overview to researchers on how to design instructional activities and what considerations that should be taken into such a design.

1.5 DEFINITIONS OF KEY TERMS

In this research, there are some key terms explicated as follows

1.5.1 Early fraction learning

Early fraction learning refers to the students' process of learning about introduction to fractions and comparing fractions. Supporting students' development on early fraction learning refers to give aids for students' learning process by providing a learning environment and a sequence of activities based on realistic mathematics principle.

1.5.2 Fractions

In this research, fraction refers to common fraction that is any number that can be expressed as such a ratio; written $\frac{a}{b}$ where a is not multiple of b, and b is not zero (Borowski & Borwein, 2002). Proper fractions in which $a < b$ will be a focus of the research.

1.5.3 The meaning of fraction

The meaning of fraction refers to different interpretations of fractions that are fractions as part of a whole, quotient or measure.

1.5.4 Understanding

Understanding refers to understanding of meaning of fraction that incorporates the ability to make connections within and between different meaning of fraction (Cathcart, Pothier, Vance, & Bezuk, 2006 in Anderson & Wong, 2007). Another aspect of understanding is that students build interrelation among various modes of external representations (Cathcart et al., 2006 in Anderson & Wong, 2007). Behr, Lesh, Post & Silver (1983) stated that external representations involve a combination of written and spoken symbols, manipulatives, pictures and real words situations.

1.5.5 Extend the understanding

Extend the understanding means constructing different meaning of fraction and interrelation between various modes of representation in exploring relation among fractions that is not much explored in the students' previous learning.

1.5.6 Relation among fractions

In this research, relation among fractions involve comparing fractions and non-unit fractions as iterations of unit fractions. Unit fractions refer to fractions with 1 as the numerator and non-unit fractions refer to fractions which the numerator is not 1.

1.5.7 Fair sharing

Fair sharing means dividing an object or objects collection into some equal parts.

1.5.8 Measuring

In this research, measuring refers to measuring distance in which fraction is used as units of measurement.

1.5.9 Local instructional theory

A local instructional theory consists of conjectures about a possible learning process and possible means of supporting that learning process. Such supporting means include instructional activities, classroom culture and the proactive role of the teacher (Gravemeijer, 2006).

1.5.10 Hypothetical learning trajectory

A hypothetical learning trajectory (HLT) consists of the goals of children' learning, the mathematical tasks that will be used to promote student learning, and hypotheses about the process of the children' learning (Simon: 1995, in Simon & Tzur: 2004).

1.5.11 Retrospective analysis

Retrospective analysis refers to the way of analyzing data by comparing the HLT with the actual learning process of students.

CHAPTER II

THEORETICAL FRAMEWORK

In this theoretical framework, literatures about fractions, relation among fractions and learning sequences of fractions are reviewed to identify some mathematical ideas required in basis understanding of fractions. This chapter also reviews the theory of realistic mathematics education that is addressed to be the perspective of designing instructional sequences in this study. The next section of this chapter discusses the theory about emergent perspectives as the framework for interpreting classroom discourse and communications. Early fraction learning in Indonesian curriculum for elementary school also were described in this chapter.

2.1 FRACTIONS

Fraction (common fraction) is any number that can be expressed as such a ratio; written $\frac{a}{b}$ where a is not multiple of b, and b is not zero (Borowski & Borwein, 2002). Fraction in which $a < b$ is called proper fraction meanwhile $a > b$ is called improper fraction. Fraction emerges through partitioning situations in which fractional parts resulted. Walle (2007) defined that fractional parts are equal shares or equal-sized portions of a whole. A whole can be an object or a collection of things. On the number line, the distance from 0 to any integer is the whole.

As the consequences of different kinds of a whole, fractions have many representations and interpretations (Kilpatrick, Swafford, & Findell, 2001).

Different fractions interpretations for the fraction, $\frac{3}{4}$ (Lamon, 2001 in Anderson & Wong, 2007) are mentioned in the following table.

Table 2.1 Different Fraction Interpretations

Interpretations	Example
Part/whole	3 out of 4 equal parts of a whole or collections of objects
Measure	$\frac{3}{4}$ means a distance of 3 ($\frac{1}{4}$ units) from 0 on the number line
Operator	$\frac{3}{4}$ of something, stretching or shrinking
Quotient	3 divided by 4, $\frac{3}{4}$ is the amount each person receives
Ratio	3 parts cement to 4 parts sand

From different interpretations of fractions, a certain fraction can have many representations such as $\frac{3}{4}$ can be represented as $\frac{3}{4}$ glass of tea or $\frac{3}{4}$ meter of ribbon as the results of dividing 3 meter into 4 equal pieces.

Kieren (1980 in Pitkethly & Hunting, 1996) called different interpretations as sub-constructs of fractions and considered that each sub-construct cannot stand alone. Understanding the meaning of fraction incorporates the ability to make connections within and between different meaning of fraction (Cathcart, Pothier, Vance, & Bezuk, 2006 in Anderson & Wong, 2007). Another aspect of understanding is interrelation among various modes of external representations (Cathcart et al., 2006 in Anderson & Wong, 2007). Behr, Lesh, Post & Silver (1983) stated that external representations involve a combination of written and spoken symbols, manipulatives, pictures and real words situations.

According to Freudenthal (1983), in the most concrete way, the concept of fractions as fractures are represented by split, cut, sliced, broken, coloured in equal parts. Freudenthal also brought up the phenomena of fractions in everyday language that comes up through describing a quantity or a value of magnitude by means of another such as a half of cake or a quarter of way.

In early fraction learning, unit fractions take the basis of fractions-knowledge building when children dividing one object as a unit into parts (Pitkethly & Hunting, 1996). The idea that fractional pieces do not have to be congruent to be equivalent will be conveyed when children have to divide equally (Fosnot & Dolk, 2002). Equal sharing problems also introduce another dimension to think that there is coordinating number of sharers with number of partitions. It conveys the issue how children decide what partition to make. Further, fair sharing contexts facilitate various interpretations by children to emerge (Streefland, 1991).

There is strong evidence that children's understanding of fractions is greatly developed by their own representation of fraction ideas. Children's own representation including pictorial, symbolic, and spoken representations could clarify their thinking (Streefland, 1991; Lamon, 2001 in Anderson & Wong, 2007). Using fair sharing contexts, children are stimulated to make their representation of situation that leads to understanding of the meaning of fraction.

However, Streefland (1991, in Keijzer, 2003) also argued that fair sharing-regarding $\frac{3}{5}$ as three pizzas divided by five children-does not clearly present a fraction as one number or entity, but rather presents a fraction as (a ratio of) two numbers. There are some evidences that using a bar as a model and a number line as an abstraction of the bar can be profitably incorporated into a curriculum that aims at number sense (Keijzer, 1997 in Keijzer, 2003). Keijzer designed an experimental programme which measurements are used to encourage children in developing bar and number line model as the emergent model (Keijzer, 2003). Thus, although fair sharing offers a partitioning situation that conveys the meaning of fraction, there is a need of children to explore fractions in the framework of numbers.

2.2 RELATION AMONG FRACTIONS

In line with Keijzer who concerned about the importance of learning fractions in the framework of number sense, the authors of TAL Book (2008) argued that eventually children have to develop knowledge that is separated from concrete situations. Children should be able to imagine situations themselves in solving fraction problems and support their reasoning using flexible models. Those abilities require the knowledge between different types of fractions called a “network of relationships”. May (1998) also suggests that children need to develop a sense of fraction and relation among fractions as they need number sense in order to deal with whole numbers. Those findings extend what Keijzer proposed in his research. Learning fractions has to be headed for the vertical mathematization such as building the relation among fractions.

As soon as children have been able to make transitions from labelled fractions to unlabelled fractions, fractions will be embedded in number relationships such as $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}; \frac{3}{4} = 3 \times \frac{1}{4}$, etc. (TAL Book, 2008). Keijzer (2003) constructed a fractions programme that emphasizes on vertical mathematization through exploring relation among fractions using number line model generated from measuring activities. Keijzer (2003) also argued that by positioning fractions on number line, equivalent fractions and simple operations emerge in line with reaching more formal fractions. An equivalence relation also leads to equality within magnitude (Freudenthal, 1983). Fosnot & Dolk (2002) discussed about children learning process in comparing fractions that is actually part of development of relation among fractions. They underlined a mathematical idea in comparing fractions that in order to compare two fractions, the whole must be same. In other words, building the knowledge about relation among fractions is necessary to support children for formalizing their understanding of fractions. Moreover, relation among fractions might enable children for reasoning when they come up with operations in fractions.

2.3 LEARNING SEQUENCES OF FRACTIONS

Keijzer (2003) constructed a sequence of activities started from dividing objects for stimulating language of fractions. To scaffold learning process of children beyond unit fractions, the sequence is then continued with developing bar model as measuring instrument. The next activity is that children's model shift to the number line model in the form of measuring scale on a bar in order to generate a few simple relations between fractions such as $\frac{1}{2}=\frac{2}{4}, \frac{3}{4}=\frac{6}{8}$, etc.

However, when children have not grasped the meaning and the language of fractions, the use of number line becomes problematic (Larson, 1980; Lek, 1992 in Keijzer 2003). The learning sequence chosen by Keijzer showed how it is directly led to the formalization which fractions are perceived as numbers on the number line. However, the more consideration such as whether children have been ready for it has to be a concern.

Partitioning and distributions at the concrete level were utilized by Streefland (1991) in a teaching experiment. Streefland also involved ratios during the learning process. The teaching experiment consisted of five activity clusters:

a. Serving up and distributing (producing fractions and their operational relations)
b. Seating arrangements and distributing (intertwining with ratio and generating equivalences)
c. Operating through a mediating quantity
d. Doing one's own productions at a symbolic level
e. On the way to rules for the operations with fractions

(Streefland, 1991, p.48)

Streefland had developed the learning sequence of fraction simultaneously with ratio. It opens for more concrete situations to be useful such as seating arrangements and distributing although the emphasis of fractions as single entity tends to be obscure.

Olive (1999) and Steffe (2002) paid attention to general progression from part whole reasoning scheme of children. Norton & Wilkins (2009) summarized those progressions of fraction schemes as the following

Table 2.2 Fraction Scheme (Norton & Wilkins, 2009)

Scheme	**Associated actions**
Part-whole fraction scheme	Producing m/n by partitioning a whole into *n* parts and disembedding m of those parts
Partitive unit fraction scheme	Determining the size of a unit fraction relative to a given, unpartitioned whole, by iterating the unit fraction to produce a continuous, partitioned whole

Scheme	Associated actions
Partitive fraction scheme	Determining the size of a proper fraction relative to a given, unpartitioned whole, by partitioning proper fraction to produce a unit fraction and iterating the unit fraction to reproduce the proper fraction and the whole
Reversible partitive fraction scheme	Producing an implicit whole from a proper fraction of the whole (no referent whole given), by partitioning the fraction to produce a unit fraction and iterating the appropriate number of times
Iterative fraction scheme	Producing an implicit whole from any fraction (including improper fractions) in the manner described above

The psychological perspective in Olive and Steffe's research gave an insight how children develop their fractional scheme. As soon as children can produce unit fractions as the results of partitioning, they use it as the units of iterations to learn proper and improper fractions.

2.4 REALISTIC MATHEMATICS EDUCATION

In this research, the theory of realistic mathematics educations is addressed to be a perspective in designing instructional sequence and conducting teaching and learning in classroom. The researcher focused the theory of realistic mathematics on five tenets and the role of teacher in realistic teaching and learning.

2.4.1 Five Tenets of Realistic Mathematics Education

In this research, five tenets of realistic mathematics educations are used as the principle of both designing instructional sequences and conducting teaching and learning process in classroom. Those five tenets are the following (Treffers, 1978; Gravemeijer: 1997):

1. Phenomenalogical exploration

   In the learning sequences based on realistic mathematics education, contextual situations do not only emerge in the end of learning phase as an application field but it is used as the starting point of learning process. The real phenomena in which the mathematical concepts embedded are explored so that those can be a basis for children to build concepts formations. In learning fractions, fair sharing phenomena are considered as the starting point of learning sequences. Furthermore, measuring phenomena is a source of learning fractions in the frame of number line.

2. Bridging by Vertical Instruments

   To bridge between the intuitive level and the level of subject-matter systematic, vertical instruments are developed such as, models, schemas, diagrams, and symbols. Those vertical instruments are the vehicle of progressive mathematization. Particularly, learning fractions that is considered as the difficult one also need as many as vertical instruments to bridge between those aspects and lead children to build more formal knowledge. Pictorial model and bar model can be chosen to support children learning process. Bar model and number line model included in fair sharing and measuring activity are considered that can stimulate children in developing their knowledge of relation among fractions.

3. Self-reliance: Students' own Contributions and Productions

   The realistic approach is based on the constructivism principle which children's own contributions and productions give the large contribution to the direction of learning process. Children's own productions also provide an insight for the teacher and the learners themselves about the location in the learning field and the progress in the process of mathematizing. In the context of learning fractions, children produce their own language of fractions continued with their process on modelling and symbolizing according to their thinking level.

4. Interactivity

   The explicit negotiation, discussion, cooperation and evaluation stimulate children to shorten their learning path, and support children to do reflections on their own constructions. On the other hand, through those kinds of interactions, the individual works also will be combined with peer contributions and teachers' scaffolding. In this design research, group discussion and class discussion are built to stimulate those mathematical interactions.

5. Intertwinment

   The learning process of certain domain cannot be separated from other learning strands in order to develop a global connection of knowledge. In this case, length measurements become one of such learning strands that are intertwined with fractions.

2.4.2 The Role of Teacher

In realistic mathematics education, mathematics is perceived as human activity in which students are the active learners in constructing their knowledge. Consequently, teacher should not transfer mathematical concepts but provides learning experiences that stimulates students' activity (Hadi, 2005). According to Hadi (2005), the roles of teacher are the following:

1. Teacher is a fasilitator
2. Teacher should be able to conduct interactive teaching and learning process
3. Teacher has to give opportunities for students so that they are active to contribute to their own learning process
4. In teaching, teacher is not limited to the curriculum but should be active to connect the curriculum with real world physically and socially

Related to the teaching experiment in testing the instructional design, proactive roles of the teacher include (Gravemeijer, 2006)

1. Introducing the instructional activities
2. Selecting possible topics for discussion
3. Orchestrating whole class discussion on the selected topics

2.5 EMERGENT PERSPECTIVE

Gravemeijer & Cobb (2006) proposed that 'A key element in the ongoing process of experimentation is the interpretation of both children's reasoning and learning and the means by which that learning is supported and organized'. Dealing with analysing the development of children's learning process in design research on fractions, emergent perspective is used as the framework for interpreting classroom discourse and communication.

Table 2.3 An Interpretive Framework for Analyzing Individual and Collective Activity at the Classroom Level

Social Perspective	**Psychological Perspective**
Classroom Social Norms	Beliefs about our own role, others' roles and the general nature of mathematical activity
Socio-Mathematical Norms	Specifically mathematical beliefs and values
Classroom Mathematical Practices	Mathematical Conceptions and Activity

Social perspectives of the emergent perspective are elaborated as the following (Gravemeijer & Cobb, 2006):

2.5.1 Social norms

Social norms refer to the expected ways of acting and explaining within interactions and negotiation between teacher and students. An example of social norms in reformed mathematics classroom is that the obligation for students to explain and justify solutions. Related to psychological perspective, social norms link from both side, one side is that individual beliefs contribute to the form of social norms but on the other side individual beliefs are established and influenced by the social norms of classrooms. The role of social norms is to promote ongoing process of learning fractions.

2.5.2 Socio-mathematical norms

Socio-mathematical norms is the explicated of social norms, which is about mathematics. The examples of socio-mathematical norms include students' perception about different mathematical solutions, more sophisticated strategy of solving problem or acceptable and mathematical explanations. As the social norms form, the socio-mathematical norms and students' belief is about what makes their contributions are acceptable, different, sophisticated or efficient. The socio-mathematical norms will be one of the main sources of data interpretations in this research. Children's perceptions are explored through their reasoning and explanation dealing with solving contextual problem about fractions.

2.5.3 Mathematical practices

Different with socio-mathematical norms which is specific to mathematics, mathematical practices are more specific to particular mathematical ideas. Mathematical practices are about the normative ways of acting, communicating and symbolizing mathematically at one moment. Sharpening the interpretations of children's perception, the way of children in symbolizing and using language of fractions will be analyzed.

2.6 FRACTIONS IN THE INDONESIAN CURRICULUM FOR GRADE 3 ELEMENTARY SCHOOL

In Indonesian National Curriculum, called *Kurikulum Tingkat Satuan Pendidikan (KTSP)*, early fraction learning is taught in 3rd grade of elementary school. In 4th and 5th grade, students continue to learn operations of fractions. Length measurement intertwined with early fraction learning is taught in 2nd grade. The following table describes early fraction learning in 3rd grade of elementary school according to KTSP.

Table 2.4 Early fraction learning for 3rd Grade in Elementary School

Standard of Competence	Basic Competence
Number 3. Understand simple fractions and the use of simple fractions in problem solving	3.1 Recognizing simple fractions 3.2 Comparing simple fractions 3.3 Solving problem related to simple fractions

Actually, KTSP is in line with the characteristics of realistic mathematics. KTSP suggests that teaching and learning mathematics should be started with introducing contextual problems. By posing contextual problems, students are guided gradually to understand mathematical concepts. In practice, many teachers still use Curriculum 1996 that more emphasizes on traditional approach (Mujib, 2010). In introducing fractions, fraction is often isolated as the number of shaded parts out of total number of parts in any geometrical shapes without contextual situations. The understanding across various representations and contextual situations is less emphasized.

CHAPTER III

HYPOTHETICAL LEARNING TRAJECTORY

Explicating the theoretical framework used for this design research, the researcher utilized fair sharing contexts as the starting point of learning trajectory. Departing from constructing meaning of fair sharing which equal parts partitioning becomes the core idea, fair sharing was then used to generate fractions. Continuous models were used in this first step because those models could lead to the interpretation of fractions as part of a whole (e.g. $\frac{1}{4}$ represents 1 of 4) which cannot be explicated by discrete models (Hunting & Korbosky, 1990 in Pitkethly & Hunting, 1996).

Although part-whole interpretation has been developed, it has no meaning when children learn about improper fractions because part whole fractions are taken out of the whole (taking nine parts out of seven cannot be handled) (Norton & Wilkins, 2009). Filling such a niche, measuring context is developed. Measuring context conveys the interpretation of fractions as measurement unit. Steffe (2003) argued that children who have constructed a part whole scheme are yet to construct unit fractions as iterable fractional units. Identifying fractions as measurement unit also can provide the magnitude of quantities of fractions so that can facilitate children in building the relation among fractions.

The more elaborated hypothetical learning trajectory is presented as the following:

Table 3.1 Hypothetical of Learning Trajectory on Learning Fractions

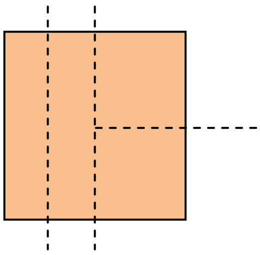
Learning Goals	**Mathematics Ideas**	**Activity**
1. Students construct meaning of fair sharing	• Pieces do not have to be congruent to be equivalent • The more number of sharers, the smaller piece is • Unit fractions	• Dividing one cake for 4 people • Increasing the number of sharers
2. Students produce fractions as result of fair sharing	• Fractions as part of a whole of objects • Fractions is an amount as a quotient • Common fractions as iterations of unit fractions	• Dividing 3 cakes for 4 people *Mini lesson*: determining $\frac{3}{4}$, $\frac{1}{4}$ and $\frac{1}{2}$ from a number of candies. • Pouring 2 glasses of water into 3 glasses *Mini lesson:* fill simple fractions in a number line
3. Students use fractions as unit of measurement	Common fraction as iterations of unit fractions	• Estimating the length of objects • Determining the position of fractions on the fractions ruler
4. Students build the relation among fractions	• An equivalence relation leads to equality within magnitude • Common fraction as iterations of unit fractions	Investigating the length of ribbons: • 1 meter ribbon cut into 2, 3, 4, and 5 pieces • 2 meter ribbon cut into 2,3 and 4 pieces • 3 meter ribbon cut into 3,5, and 6 pieces

The hypothetical learning trajectory in Table 4.1 is described as follows:

3.1 CONSTRUCTING MEANING OF FAIR SHARING

3.1.1 Activity 1: Sharing a Fruit Cake

1. *Mother made a fruit cake to share with her neighbours. Could you help mother to divide the cake into 4 equal pieces?*
2. *How is your opinion if mother cuts the cake as the following figure? Is it still fair?*



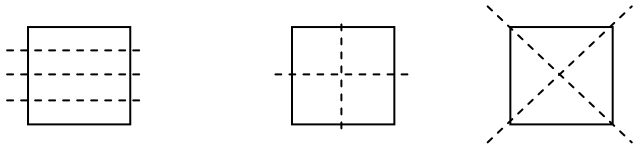
3. *Mother also wants to share another fruit cake for her daughter's friends. They are five children. Could you help mother again? Then compare with the pieces of a cake for 4 people. How much each person gets?*

Description of Activity:

Using model of cake (rectangle paper), students are asked to divide it into four equal pieces using as many as possible way of cutting (question 1). Through this activity, students recall their informal knowledge about partitioning. Continued with question 2, students are given another possibility of dividing strategy. This question is aimed to provoke students construct the meaning of fair sharing that the pieces do not have to be congruent, to make it equivalent. Language of fractions such as 'a quarter' is also expected to emerge when students give reasons. Further, teacher will provoke students to use mathematical symbol of fractions. Still using model of cake, question 3 is posed so that students can compare with their result in question 1. The language of unit fractions is also concerned through question 3.

Hypotheses of Learning Process:

Question 1:



Question 2:

- Students might focus on the difference of shape so that their answer is not fair.
- Students answer that it is fair by giving a reason related to division whole number (*"it is also divided by four although the shape is different")*
- Students answer that it is fair by giving a reason related to fractions informally ("*a quarter"*)

Question 3:

- Students might cut using their strategy in the question 1 then make the fifth piece by cutting one of a quarter pieces.
- Students cut it properly but they might have difficulty to compare with their 'quarter' pieces if their way of cutting now is different.
- Students realize that the pieces must be smaller.

Question: *How much each person gets?*

- Students might come up to the daily language 'a quarter' and then they also use daily language 'seperlima' for one-fifth.

- Students are able to use mathematical symbol of fractions but have not understood about what numbers 1, 4 or 5 refers to.

Mathematical Congress:

Students might not get difficulty in cutting the model of cake. The main focus of mathematical congress is discussing question 2 to provoke students' reason related to their informal knowledge of partitioning. If students use *'a quarter'* word in their reasons, the discussion is then continued with the meaning of a quarter. The idea that to be equivalent, the pieces do not have to be congruent is also discussed here. Question 3 is expected can lead the students to conclude that the more number of sharers, the smaller pieces will become. Through asking about how much each person gets, students are stimulated to use language of fractions particularly unit fractions. Indonesian students might use similar word for one fifth '*seperlima'* as they use for one fourth *'seperempat'.* The meaning of those unit fractions and the way of notating those fractions will be discussed.

3.2 PRODUCING FRACTIONS AS RESULT OF FAIR SHARING

3.2.1 Activity 2: Sharing Brownies Cakes

If we only have 3 brownies cakes, how to shares it among 4 people? How much will each person get?

Description of Activity:

The activities use picture as representation of cake. The paper model of cake also can be used if students still have difficulty with partitioning using picture. The shape of cake used is rectangle that resembles a bar model. First question is aimed to recall the informal knowledge of fractions that students have already had. On the next question, students have to divide 3 cakes and share it fairly among 4 students.

Hypotheses of Learning Process:

Students might use one of the following strategies in dividing cakes:

- Students might have struggle in dividing cakes fairly. They might come up with the results merely using estimation.
- Students divide cakes by halving and share the rest.
- Students divide directly each part into 4 pieces.
- Students take directly three quarters of each cake.

The possible language of fractions that students use to notate the results of fair sharing

- Students might use daily language to notate the results of sharing such as “everyone gets a half and a quarter”
- Students notate the results by using simple fractions “everyone gets $\frac{1}{2}$ and $\frac{1}{4}$ of a cake” or “3 pieces of $\frac{1}{4}$ cake”
- Students directly use the notation $\frac{3}{4}$ of a cake.

Mathematical Congress:

In the mathematical congress, the first discussion is about the way of dividing cake fairly. Each group of students might have different way in dividing cake. Students are asked to evaluate whether each group have shared cakes fairly. The next discussion is about the representation of results from fair sharing. The concept of fractions as the relations between part of a whole can be explored by questions such as "*How do you get $\frac{1}{4}$? What do numbers '1' and '4' mean?*". The relation among fractions also starts to be constructed particularly the relations between $\frac{3}{4}$, $\frac{1}{4}$ and $\frac{1}{2}$. The mathematical congress also discusses how students perceive the same amount of cake although there are more pieces because some students might have opinion that the more pieces, the more cake is.

3.2.2 Minilesson: Part of Object Collections

You have mentioned some fractions such as $\frac{3}{4}$, $\frac{1}{4}$ and $\frac{1}{2}$ *when you divided brownies cake. Can you determine those parts from this number of candies?*

Descriptions:

There are 24 candies given to students. In the mathematical congress of activity sharing brownies cake, they have already discussed about the meaning of simple fractions such as $\frac{3}{4}$, $\frac{1}{4}$ and $\frac{1}{2}$. This minilesson is used to see whether they can see the relationship between parts and a whole in the discrete objects also what strategies they use to determining the fractional parts.

3.2.3 Activity 3: Pouring Tea

There are 2 glasses full of tea and almost overflow. If there is one glass more to accommodate so that those become 3 glasses of tea, how do you predict the height of tea in the glasses?

Description of Activity:

Students are given the problem and figure above. Using drawing, they have to predict the height of glasses after distributing tea. In the previous activity of sharing brownies cakes, the number of sharers is more explicit. Through this activity, students are expected can use the strategy of partitioning to find the height of tea instead of rough estimation. If students succeed in finding the height using partitioning, students might start to realize that $\frac{2}{3}$ can be 2 parts of 3 parts in the glass, or 2 times $\frac{1}{3}$ of a glass that represents the amount of tea in the glass.

Hypotheses of Learning Process:

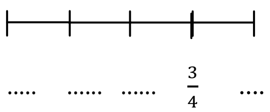
- Students only draw using their intuition but they cannot make it sure.
- By using trial and error, students reduce a certain amount of tea from each glass and draw those amounts on the empty glass.

- Students use halving strategy to find the amount that will be distributed into the empty glass.
- Students directly use partitioning by third.
- Students use ruler for measuring and then dividing the total number of measuring scale of two glasses by three.

Mathematical Congress:

The hardest part of this problem is how students can connect with their previous experience in dividing brownies cake. Students might tend to use drawing as a rough estimation. The mathematical congress is supporting students to find strategy giving more accurate prediction using partitioning. Although students have realized that they should use partitioning, but they might struggle with what kind of partitions, half, quarter or others. After they find that third can be used, the discussion moves toward why third is suitable if there are 3 sharers (glasses). The discussion also will be connected with some interpretations of $\frac{2}{3}$. Other discussion can appear if students use ruler. The way of dividing the total number of measuring scale then will be connected to the partitioning a glass into three parts.

3.2.4 Minilesson: Fill Fractions in a Number line



Fill the blank space with appropriate fractions! How about $\frac{1}{3}$ *and* $\frac{2}{3}$ *, where it is?*

Description:

This mini lesson is used to summary some knowledge that students have learned in the previous activity. It is also to see the development of students knowledge about the relationship among simple fractions. By learning to put fraction on the number line, it is expected to support students in doing the next activity that more zooms into other fractions that exist between the simple fractions they have learned.

3.3 USING FRACTIONS AS UNIT OF MEASUREMENT

3.3.1 Activity 4: Measuring Pencil using Folded Paper

Using folded paper, find the length of your pencil! Compare with your friend!

Description of Activity:

Using their own pencil, students will find the height of pencil using folded paper. They can fold paper as many as they need until it fits with the height of pencil. After that, students can use their folded paper to find how many parts of a whole paper that correspond to the height of pencil or how many units of their folded paper. The differences of height between students' pencil can give students an opportunity to compare among fractions.

Hypotheses of Learning Process:

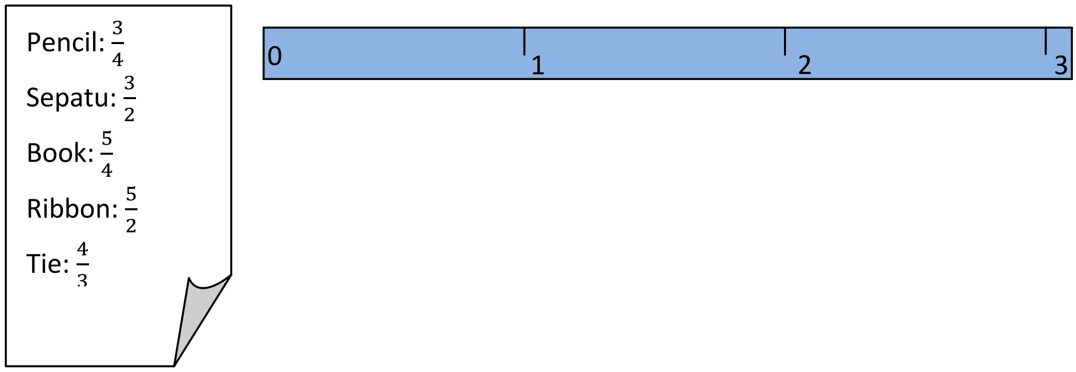
- Students only use estimation by marking the folded paper instead of folding it
- Students fold paper using halving strategy until it fits to the height of pencil
- Students have struggles to fold paper when using halving does not match to the height of pencil.

- Students count the number of parts that corresponds to the height of pencil then compare it to the whole parts in the folded paper.
- Students represent each part of folded paper as a unit fraction then find the height of pencil by multiply it with the number of parts that fits into the pencil.

Mathematical Congress:

The mathematical idea of this activity is similar with the *pouring tea* activity but in more open ended situation. There are many possibilities fractions that can emerge. The mathematical congress is more focussed on how students can describe their strategy in measuring the height of their pencil. The differences between students' results also can be discussed particularly if there are two pencils in the same height but represented with different fractions. At this moment, the the meaning of fraction as part of a whole is extended to other meaning of fraction that fractions as measurement units.

3.3.2 Activity 5: Marking a Fractions Ruler



Ani found a ruler and a note that is written with the results of measurement of some objects. She is wondering how long the object is. Could you help Ani to figure out the length of objects using the ruler?

Description of Activity:

Students are given Ani's note and a fractions ruler. Each segment of the ruler is as long as folded paper in the activity 5. The task for students is that they have to mark the ruler with a fraction written on the note. They can still use the folded paper to help them. This activity supports students to extend their knowledge of fractions into improper fractions. Fractions as the measurement units bring the advantage that improper fractions can be revealed. Through exploring improper fractions, students are expected to see fractions in the relations with whole numbers particularly fraction is a single number between whole numbers.

Hypotheses of Learning Process:

- Instead of using the folded paper, students fold the ruler like they did in the previous activity
- Students use folded paper and use the unit fractions as the measurement units
- Students no longer use folded paper but they partition the each segment of ruler based on the fractions given

Mathematical Congress:

In the activity of measuring pencil using folded paper, students have encountered struggles to find the unit fractions as the measurement units. Therefore, in this activity students might be easier in using folded paper. Even, students might come up with partitioning strategy without using folded paper. The main focus of the mathematical congress is that exploring improper fractions as iterating of unit fractions such as $\frac{4}{3}$ is perceived as distance 4 ($\frac{1}{3}$ units) from 0 on the fraction ruler.

3.4 BUILDING THE RELATION AMONG FRACTIONS

3.4.1 Activity 6: Cutting Ribbon

Investigate the results of cutting ribbon if ribbons with different length are cut into certain number of pieces!

Task 1:

- *1 meter ribbon cut into 2, 3 ,4 and 5 pieces*
- *2 meter ribbon cut into 2 and 4 pieces*
- *3 meter ribbon cut into 3 and 6 pieces*

Task 2:

- *2 meter ribbon cut into 3 pieces*
- *3 meter ribbon cut into 5 pieces*

Can you predict other results without using ribbon?

Description of Activity:

Students are given ribbons with different length (1m, 2 m, and 3 m) and asked to cut into various equal pieces (2, 3, 4, and so on). Students are also asked to investigate the relation between cutting results. Students tend to understand fraction merely as numbers. Through this activity using length, students are expected to visualize magnitudes in fraction. According to Freudenthal (1983), length and area are the most natural means to visualize magnitudes with respect to teaching fractions. The relation among fractions is also built such as each piece from 2 m-ribbon cut into 4 has same length with each piece from 3 m-ribbon cut into 6 (equivalent fractions) or each pieces from 1m-ribbon cut into 3 is two times as long as each piece from 2m-ribbon cut into 3. The task for students to cut the ribbon is divided into two tasks. For the first task, it is more focussed on cutting ribbon that can be done by halving except 1 meter cut into 3 and 5 pieces. For the task 2, cutting 2 and 3 m ribbon into 3 and 5 pieces, is given after they have a discussion of the results of investigation for the task 1.

Hypotheses of Learning Process:

- Students might have struggle when dividing ribbons into 3 and 5 because they cannot using half for helping as they can use when dividing into 2, 4 and 6
- Students make mistakes when cutting ribbon so that they cannot see the relations between their cutting ribbon
- Students can make a list of results of investigation but they have no clue about the relations
- Students are able to see the relation between fractions in their list but they do not apply to make them easier in cutting 2 and 3 meter ribbon into 3 and 5 pieces. Furthermore, they cannot generalize into other fractions

Mathematical Congress:

In the mathematical congress, the focus of discussion is the relationship among fractions as the results of investigation. Students might find some equivalent pieces or pieces that two times as long as others but they might have struggle to generalize their results such as giving a reason why $\frac{1}{2}$ is equal to $\frac{2}{4}$ and $\frac{3}{6}$. The task 2 is also used to observe whether students can generalize the results of investigation in the task 1. If they can make a generalization, instead of partitioning 2 and 3 m of ribbon into 3 or 5 pieces, which is relative hard for them, students use $\frac{1}{3}$ and $\frac{1}{5}$ as the unit of measurement.

The questions '*Can you predict other relations without using ribbon?'* might provoke the discussion among students. To make predictions, they will start from their result of investigation and find the relations. Students are asked to represent their results on the line so that they can figure out and construct their knowledge in the relation among fractions.

3.5 GENERAL OVERVIEW OF THE ACTIVITIES SEQUENCE

In *the first activity*, students encounter sharing situations to bring out their informal knowledge of partitioning. The meaning of fair in fair sharing become the focus of learning process before students come to the meaning of fraction as part of a whole in fair sharing. In the framework of fair, students have to realize that pieces do not have to be congruent to be equivalent and the more number of sharers, the smaller pieces is (Fosnot & Dolk, 2002). Both of those mathematical ideas are to support students in partitioning when they have to find how many part from a whole. For supporting the next activity, the meaning of unit fractions is explored. The decision of exploring unit fractions refers to the remark from Pitkethly & Hunting (1996) which said that unit fractions take the basis of fractions-knowledge development.

Through *the second activity*, sharing brownies cake, students elicit the basic meaning of fraction as part of a whole. It refers to Streefland (1991) and Empson (1999) who found that students could make sense of fractions through fair sharing situation and various representations could emerge to lead students producing fractions. The problem still bridges the intended knowledge of fractions with simple fractions that students have already recognized such as $\frac{1}{2}$ and $\frac{1}{4}$. Students also will learn to connect between part of a whole in fair sharing and the amount each sharer receives that is related to the meaning of fraction as quotient. Discrete objects are introduced when students have been able to produce common fractions. Another fair sharing context is developed in *the third activity*, pouring tea by only using pictorial model. The part-whole relationship in fractions is strengthened in the activity. At the same time, measuring context used is to provoke students perceiving fractions as the iterating of unit fractions as the measurement units. The development of the meaning of fraction as part of a whole and as unit of measurements is in line with the finding of some researchers which show that fractions have several meanings which cannot stand alone so that students need to learn some of those meanings simultaneously (Kieren, 1980 in Pitkethly & Hunting, 1996; Kilpatrick, Swafford & Findell, 2001; Lamon, 2001 in Anderson & Wong, 2007). The mini lesson of filling simple fractions that they have constructed in the previous activities is to support students to build the relation among fractions. Students do not only build fraction as single number but also build connection among fractions.

Measuring context is more explored in *the fourth activity*, measuring pencil using folded paper. The ability of partitioning is more needed in this activity in order to find the unit fractions that can be used as measurement units. The differences of the length of objects used give an opportunity for students to expand their network of relationship among fractions through informal comparing fractions resulted. According to the TAL Book (2008), the relation among fractions laid as the basic knowledge to support students giving reasons when solving problems. Kind of measuring activity also brings up students to explore fractions in the framework of numbers as Keijzer (2003) proposed. As soon as students have constructed fractions less than one, students must be given an opportunity to extend this knowledge into improper fractions. Because the part whole meaning no longer can support this extension, *the fifth activity* (marking a fraction ruler), is introduced. Students are given fractions that numerator is larger than denominator. Their previous experience about iterating unit fractions in measuring might be help students to figure out improper fractions in relation with whole numbers. Eventually, the relations are not only between fractions but also with whole numbers.

The sixth activity, (cutting ribbon) is more focussed on building the relation among fractions such as equivalent fractions and comparing between magnitude of fractions. Through comparing between magnitude of fractions, students might reveal relations between fractions such as $\frac{3}{4}=\frac{1}{2}+\frac{1}{4};\frac{2}{4}=\frac{1}{2}=2\times\frac{1}{4}$,etc. (Freudenthal, 1983; TAL Book, 2008). Cutting ribbon is not only merely about finding more relations in fractions as number but also bring students back to the meaning of fraction as quotient - an amount of sharing results – and as results of measurement. The reason is that students often only see fractions as number so that they have no clue when encounter operations of fractions. Through always shift between different meaning of fraction and the level of formalization, students might be able to solve fractions problems in the flexible ways such as using the relations between fractions or using flexible model to represent fractions.

CHAPTER IV

RESEARCH METHOD

4.1 RESEARCH PHASES

One of the aims of this study is developing a local instructional theory on learning fractions. Aiming at developing theories about both the learning process and the means designed to support that learning, design research is chosen as the method (Gravemeijer & Cobb: 2006). There are three phases of conducting a design experiment in design research (Gravemeijer & Cobb: 2006)

4.1.1 Preparing for the experiment

Before designing the hypothetical learning trajectory, the researcher studied literatures to clarify the mathematical goals of learning fractions. Although the mathematical goals in Indonesian curriculum are formulated, the researcher tries to establish the most relevant or useful goals. Studying the existing research literatures is also useful to understand the consequences of earlier instruction in order to develop a local instructional theory on fractions domain. Such a conjectured local instruction theory consists of conjectures about possible learning processes when children struggling with a specific mathematical idea of fractions, together with conjectures about possible means of supporting that learning process. Those supports are offered in the form of building mathematical congress, promoting models and following up the mathematical ideas that have been grasped by children to other related ideas.

4.1.2 Design experiment

After clarifying the mathematical goals and developing the conjectured local instruction theory, the design experiment was conducted in cyclical processes of (re) designing and testing instructional fractions activities. In this research, the design experiment was conducted in two macro cycles. To give more space in exploring students' pre-knowledge and students' thinking, the first cycle of design experiment only involved small group of students in 3rd grade. For the second cycle, the design experiment involved all students from one classroom which were different with students at the first cycle. The researcher decided to end the cyclical process in two macro cycles because it was found that the last instructional sequences quite fit with students pre-knowledge and most of the learning goals could be achieved by the students.

Besides two macro cycles, daily micro cycles were presented in this research which involved new anticipatory thought experiment, the revision of instructional activities and modification of learning goals. During the first cycle, the instructional activities were revised in daily micro cycle based on students' pre-knowledge and ongoing learning process.

4.1.3 Retrospective analysis

In the retrospective analysis the HLT was compared with the students' actual learning. The retrospective analysis was conducted within and after the teaching experiment. From the results of retrospective analysis, answers to the research question and contribution to an instruction theory could be generated. In doing the retrospective analysis, first the researcher studied the students' work and made groups of students' strategies or difficulties with the HLT as a guideline. Testing the HLT at other material, the researcher watched video chronologically then chose some crucial episodes and made transcripts of those episodes. All names of the students on the transcripts were not origin names.

4.2 RESEARCH SUBJECTS AND TIMELINE

The research was conducted in grade 3 SD Laboratorium Universitas Negeri Surabaya. For the first cycle, the researcher takes a small group of students, 6 students. In the second cycle, 28 students were involved in class teaching experiment. The students were about 8-9 years old and they had learned about fractions prior this research. The researcher taught the students by herself at both cycles. SD Laboratorium is one of the schools that participate on the development of PMRI (Pendidikan Matematika Realistik Indonesia).

The timeline of this research is described in the following table.

Table 4.1 The Timeline of the Research

	Date	**Description**
Preparing for the Experiment		
Studying litearatures and designing HLT 1	October-December 2010	
Communicating with school and teacher	7 February 2011	Communicating the designed HLT and research method
Design Experiment for the First Cycle		
1st Meeting	14 February 2011	Pre-test Activity 1: Sharing a Fruit Cake
2nd Meeting	15 February 2011	Activity 2: Sharing Three Brownies Cake Mini Lesson: Candies
3rd Meeting	17 February 2011	Activity 3: Measuring

	Date	Description
		Pencil
4th Meeting	18 February 2011	Continuing Activity 3: Measuring Pencil Activity 4: Pouring Tea
5th Meeting	21 February 2011	Activity 5: Marking Fractions Ruler Activity 6: Distributing Water
6th Meeting	22 February 2011	Activity 7: Shading Parts and Finding the Relations around Fractions
7th Meeting	23 February 2011	Post-test
Design Experiment for the Second Cycle		
1st Meeting	14 March 2011	Pre-test
2nd Meeting	15 March 2011	Activity 1: Sharing a Fruit Cake
3rd Meeting	16 March 2011	Activity 2: Comparing the Pieces of Cake
4th Meeting	17 March 2011	Activity 3: Dividing Chocolate Bars
5th Meeting	21 March 2011	Activity 4: Sharing 3 Brownies Cakes among 4 Children
6th Meeting	22 March 2011	Activity 5: Ant’s Path Activity 6: Ant’s Positions
7th Meeting	23 March 2011	Activity 7: Ants on Number LIne
8th Meeting	24 March 2011	Post-test

4.3 HYPOTHETICAL LEARNING TRAJECTORY

A hypothetical learning trajectory (HLT) is a way to explicate an important aspect of pedagogical thinking involved in teaching mathematics for understanding. An HLT consists of the goals of students’ learning, the mathematical tasks that will be used to promote students’ learning, and hypotheses about the process of students' learning (Simon: 1995, in Simon & Tzur: 2004). The HLT is the link between an instruction theory and a concrete teaching experiment (Bakker, 2004). The HLT is used not only during both pre-experiment and design experiment phase as a guideline of the instructional activities but also as the guideline of retrospective analysis in developing a local instructional theory.

4.4 LOCAL INSTRUCTIONAL THEORY

One of the aims of this study is to develop a local instructional theory on early fraction learning. Such a local instructional theory consists of conjectures about a possible learning process, together with conjectures about possible means of supporting that learning process (Gravemeijer, 2006). According to Gravemeijer, the means of support involves potentially productive instructional activities and tools as well as an envisioned classroom culture and proactive role of teacher. Wijaya (2008) described that a local instructional theory offers teacher a framework of reference for designing and engaging students in a sequence of instructional activities for a specific topic. A local instruction theory becomes a source for teachers in designing a hypothetical learning trajectory for a lesson by choosing instructional activities and adjusting the conjecture of students' learning process.

4.5 DATA COLLECTION

To gain more understanding of students' learning process, the following data were collected during the research.

4.5.1 Video Data

To observe carefully students activity in solving the problems, video taping was considered as the effective way of collecting rich data. During teaching in the classroom, two video cameras were used. One camera as static camera was to record the whole class learning processes and the other as dynamic camera to record group discussions. The whole learning process which was observed involved how the activities sequence offered supports students learning trajectory and provokes students to deepen their understanding. Moreover, the data of students when doing discussion in groups zoomed in students' actions and oral reasoning in solving problems.

4.5.2 Written Data

Written data were also collected in order to analyze the thinking process of students. Particularly, that kind of data was useful to give an insight how students think about certain problem and which level they grasped. Those kind of written data were students' work and the results of pre-test and post-test.

4.5.3 Interview

Interview was conducted to clarify students' work and explore some interesting case of students' thinking process. Interview with the teacher also became a source of information about students' pre-knowledge, particularly on previous learning of fractions.

4.6 DATA ANALYSIS, RELIABILITY, AND VALIDITY

4.6.1 Data Analysis

In analysing data, both video and written data support each other in order to answer the research questions. Students' learning process can be observed during the classroom and group discussion. Students' activity in solving fraction problems were also visualized on the video tape and then related to their written answers. The interview was also a rich source for exploring students' thinking process.

4.6.2 Reliability

Reliability of this design research involved internal reliability and external reliability. According to Bakker (2004) internal reliability is the reasonableness and argumentative power of inferences and assertion. Internal reliability refers to intersubjectivity in which supervisors and colleagues are involved in interpreting data collections. External reliability in this research refers to trackibility (Gravemeijer & Cobb, 2001; Maso & Smaling, 1998 in Bakker, 2004) which is about data registration. All data from this research are documented to make clear how this research has been conducted and how conclusions have been drawn based on the data (Bakker, 2004).

4.6.3 Validity

Validity refers to credibility or the quality of the data collection and the soundness of the reasoning that has led to the conclusions (Bakker, 2004). Validity of this research is improved by data triangulation which different types of data are used as the sources of interpreting students' learning process. Data triangulation involves students' work and videotaping of classroom learning process and group discussion. The HLT built in this research also supports the validity of research. The function of HLT as the guideline of design experiment and of retrospective analysis connects the conjectures of instructional theory and the local instructional theory that is resulted.

CHAPTER V

RETROSPECTIVE ANALYSIS

In this chapter, the researcher would report a retrospective analysis in testing the hypothetical learning trajectory (HLT). To give more space for the researcher in exploring and observing students' learning process, the first testing of HLT, called the first cycle of teaching experiment, only involved a small group of students (6 students). During the first cycle, the designed HLT namely HLT 1 (see Appendix A) was improved based on the ongoing learning process and the students' pre-knowledge. The improved-HLT 1 implemented in the first cycle was called HLT 2. Considering the actual learning process of students during the implementation of HLT 2, HLT 2 was revised to be conducted in classroom teaching experiment (HLT 3).

The retrospective analysis would be built in chronologically started from posing remarks of the students' pre-knowledge from the first cycle. Such remarks then were used as the considerations of refining the HLT so that there was HLT 2 as the refinement of HLT 1. Some remarks of the students' knowledge on post test were written down after explicating the learning process of HLT 2. The remarks from the pre-test, post-test and the learning process of HLT 2 lead to general conclusion of students' learning process of the first cycle. Considering that there were some main activities that had not supported students optimally to extend their understanding, HLT 2 was revised to be HLT 3. Prior to explanations of the teaching experiment of the second cycle in a classroom, a brief description of

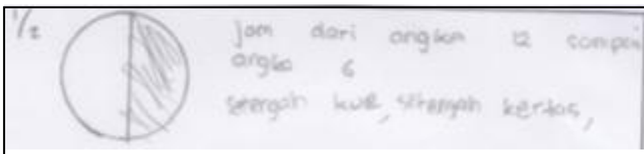
HLT 3 as the refinement of HLT 2 was given. The mathematical ideas in each activity would become the theme of analysis in which the researcher tried to zoom in the students' development of such mathematical ideas. Some remarks of the students' knowledge on pre-test and post-test in the second cycle were made afterwards.

5.1 REMARKS OF THE STUDENTS' PRE-KNOWLEDGE IN THE FIRST CYCLE

A pre-test was conducted before testing the HLT in a group of 6 students. The pre-test was aimed to check whether the starting point of HLT was corresponded to the students' pre-knowledge. As the results of investigating the pre-knowledge of students through pre-test, the researcher found some critical issues that influence the teaching experiment as the following:

5.1.1 Previous learning process about fractions was more focused on constructing the meaning of fraction as part of a whole using shaded area model.

One of items in the pre-test was about giving some examples of objects that represent 'a half' and making drawing of it. All the students drew shaded area in the geometrical shapes although some of them also wrote the examples of half using words (Figure 5.1).



Clock from 12 o'clock until 6 o'clock
A half of cake, a half of paper

Figure 5.1 One Example of Students' Answer in the Pre-test

To show a quarter of cake, all the students also represented with shaded area in the model of cake (Figure 5.2).

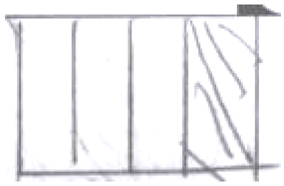


Figure 5.2 Showing a Quarter of Cake

Investigating the students' understanding about the meaning of a quarter in their answer, the researcher posed a question to the students *"Which part that you will give if I asked a quarter of the cake?".* There was a student who got confused and said that she would give all of the cake.

5.1.2 Comparing fractions was developed by using cross-multiply algorithm.

The students had learned about comparing fractions by using cross-multiply algorithm. Answering a problem in the pre-test about comparing $\frac{1}{4}$ and $\frac{1}{3}$, five students out of 6 students could answer correctly that $\frac{1}{3} > \frac{1}{4}$ because 4 > 3. Moreover, the students also could compare non-unit fractions using such an algorithm.

5.1.3 Fractions as units of measurement was a challenge for the students

To investigate the students' pre-knowledge about fractions related to context of measuring, one problem about completing scales on the measuring cup was given to the students. All the students wrote unit fractions as the measuring scales.

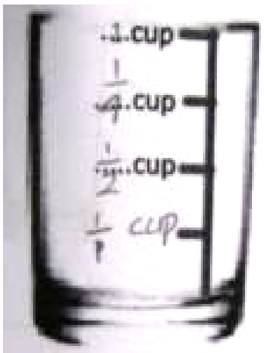


Figure 5.3 Writing Measuring Scales

5.2 HLT 2 AS THE REFINEMENT OF HLT 1

In testing the HLT, ongoing learning process and critical issues found on pre-test became considerations for refining the activities. The more detailed refinements and the analysis about how HLT 2 works as the refinement of HLT 1 will be described as the following

5.2.1 Constructing Meaning of Fair Sharing

Activity 1: Sharing a Fruit Cake

The activity was aimed to support students to construct the meaning of fair sharing and provoke the students in notating fractions. Exploring the meaning of fair sharing, the mathematical ideas was about the pieces do not have to be congruent to be equivalent and the more number of sharers, the smaller piece is. In notating fractions, the students would explore the meaning of fraction, particularly about unit fractions.

The students were asked to divide a model of cake into 4 pieces fairly. As conjectured, the students used different strategies in dividing model of cake (Figure 5.4).

Figure 5.4 Different Ways of Dividing a Cake

One group did not cut equally but still said that it was fair as long as the bigger pieces were given to big students and the smaller pieces were given to little students. The students' idea that fair sharing does not always mean equal sharing might come from their daily experience. The different interpretation of fair sharing brought the researcher to a decision to skip the exploration the mathematical idea about the pieces do not have to be congruent to be equivalent. There was a need of another support for this case.

Considering one of critical issues about the students' pre-knowledge of fractions that was focused on learning meaning fractions as parts of a whole in shaded area, this context tried to bring real acts in partitioning by cutting a model of cake. The absence of shaded area became a challenge for the students to notate fractions. They tended to said that each person got 1 piece. Provoked by the researcher to perceive a whole cake, the students could notate fractions which raised different meaning of fraction.

Nando : There are 4 parts. It means that $\frac{1}{4}$ is one part.
Researcher : Where is 1 in the fruit cake?
Nando : 1 means the whole cake
Researcher : How about 4?
Sasa : Here it is. If we cut it, then there will become 4.
Researcher : Is there other opinions?
Sasa : If we have cut the cake, then 1 will become this (taking 1 piece of cake)

From the discussion, the students interpreted a fraction $\frac{1}{4}$ as 1 part of 4 parts as a whole but also could be 1 cake divided into 4 pieces (fractions as quotient).

The students' difficulties emerged when the researcher asked about increasing the number of people who shared a cake. Without providing another model of cake to be divided, there was a student who showed the strategy of partitioning as shown in Figure 5.5. He anticipated that there would be more people who have to be given the pieces of cake. The strategy of the students that was out of conjectures made the researcher could not decide directly how to discuss the notation of fractions and the size of pieces if the number of people was increased.

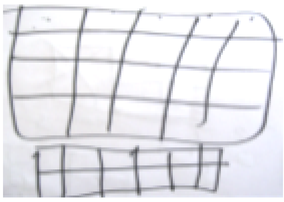


Figure 5.5 Dividing a Cake for 5 People

Considering a fact that doing real partitioning through sharing a model of fruit cake conveyed the possibility of different meanings of fractions to emerge, this context would be maintained in classroom teaching experiment. The problem about justifying the meaning of fair sharing and developing the mathematical idea that pieces do not have to be equivalent to be congruent should be more constructed explicitly. Increasing the number of people who shared a cake also could be experienced through partitioning a number model of cakes.

5.2.2 Producing Fractions as Results of Fair Sharing

Activity 2: Sharing Brownies Cake

Increasing the number of objects to be shared was aimed to provoke the students to produce fractions as results of dividing. In notating those simple fractions, the students were expected to construct different meaning of fraction such as fractions as parts of a whole, fractions as results of division and non-unit fractions as iterations of unit fractions.

The students are asked to divide 3 cakes among 4 students. In this activity, the students cut models of cakes (Figure 5.6) and stuck on the paper.

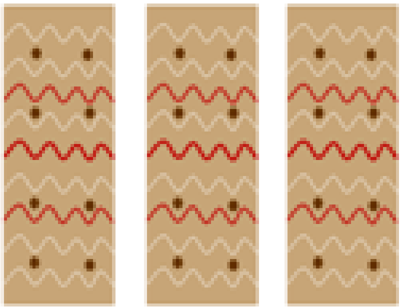


Figure 5.6 Model Brownies Cakes

Unlike the conjectures, the students only used halving strategy and shared the rest. The difference was the strategy of dividing the rest. One group (3 students) did halving the rest (Figure 5.7) and the other cut the rest by trial and error and threw away the remained pieces (Figure 5.8).

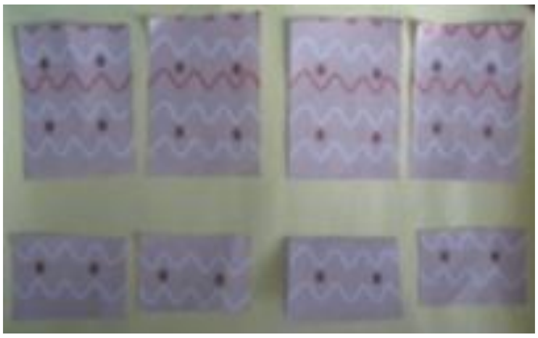


Figure 5.7 Halving Strategy

Figure 5.8 Halving, Trial and Error Strategy

In notating the results of cutting by fractions, the students got difficulties. The students' difficulties in notating the results with fractions might be caused that they no longer could see the original cake after it has been cut. By rearranging the pieces of cutting in Figure 5.7 so that it resembled the original cake, the researcher guides the students to notate with fractions. The students used daily language *'a half'* and *'a quarter'*. Based on their language of fractions, there were two different answers of notations; $\frac{1}{2}$ and $\frac{1}{4}$ or $\frac{2}{4}$ and $\frac{1}{4}$. The students' answer in Figure 5.8 that was out of conjectures brought much more difficulties to guide the students to notate with fractions.

Exploring the meaning of fraction, the researcher found that the students tended to understand fractions as parts of a whole rather than fractions as results of division and iterations of unit fractions. Although the fractions language that they used seemed messed up, the students tend to figure out $\frac{1}{4}$ compared with 4 parts as a whole.

Researcher : One cake is divided into...
Students : Four pieces
Researcher : So, if I eat this? (taking one piece)
Students : A quarter
Nando : A quarter of 4 parts

None of the students could come to the conclusion that each person gets $\frac{3}{4}$ so that the researcher then explored fractions $\frac{2}{4}, \frac{3}{4}$ and $\frac{4}{4}$ by posing questions such as *"How much do I eat, if I take these two pieces? How about* $\frac{3}{4}$*? How many pieces?"*. Although the students could answer the questions correctly, the students' difficulty in notating fractions brought the researcher to postpone the exploration of relationship among fractions resulted. The discussion should be first focused about the meaning of fraction itself.

Considering such facts, the researcher thought that to support the students in notating fractions by themselves without many interventions from a teacher, this problem would be improved by changing the instructions from cutting to draw the line of cutting. There is a possibility that the students will have different references of a whole (one cake or three cakes) so that further discussion about the difference of whole that the students might perceive in notating fractions should be developed.

Mini lesson: Part of Object Collections

This mini lesson was aimed to see whether the students had built the relation among fractions. The students were asked to determine the number of candies that represented $\frac{1}{4}$ of 12 candies. The researcher simplified the number of candies from 20 candies to 12 candies to avoid that the students would be busier in solving division problem.

In fact, the students got difficulties in determining $\frac{1}{4}$ of 12 candies. They had no clue how to solve the problem. The researcher had to give explicit instructions to divide the collection of candies by 4. Only one student could conclude that $\frac{1}{4}$ of 12 candies is 3 candies, $\frac{2}{4}$ of 12 candies is 6 candies, $\frac{3}{4}$ of 12 candies is 9 candies and $\frac{4}{4}$ of 12 candies is all the candies.

In this mini lesson, the focus was changed from exploring the relation among fractions to constructing the meaning of fraction as division. In fact, there was a lack of knowledge of other meanings of fractions beyond the meaning of fraction as parts of a whole. In the next cycle of teaching experiment, this mini lesson would be elaborated as the main activity which the learning goal was to support students in constructing the meaning of fraction as division.

Activity 3: Pouring Tea

Similar with the context of sharing brownies cakes, this activity was aimed to provoke students in producing simple fractions as results of dividing. The difference was on the meaning of fraction focused. Non-unit fraction as iterations of unit fraction was expected to emerge in students' reasoning. Because of the students' difficulties in partitioning as they faced when solving the problem of brownies cake, the researcher then postponed this activity after the activity of measuring pencil. In measuring pencil, the act of partitioning was more explicit by folding paper.

To connect with the students' pre-knowledge of unit fractions, the researcher started to give a mini lesson about marking $\frac{3}{4}$ of glass and simpler problem about pouring water. Learning process of mini lesson was given before the activity of pouring tea but it will be described afterwards. As the starting problem of pouring water, the researcher poured one glass with full of tea and provide one more empty glass. The problem was about determining the height of water in both glasses, if tea had to be poured into the empty glass to make tea filled in both glass equally. As conjectured, the students used estimation to solve this problem but in fact, their estimation was quite surprising. Five students out of 6 students drew two glasses which were full of water (Figure 5.9). There was only one student who could estimate correctly (Figure 5.10).

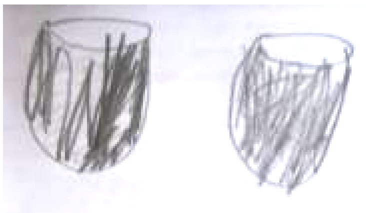


Figure 5.9 Two Glasses Full of Tea

Figure 5.10 Two Glasses Half-Full of Tea

The researcher then posed questions to clarify their results.

Researcher	:	If I poured tea, how is the drawing then?
Students	:	Equal
Riki	:	Try to pour it!
Researcher	:	Wait wait... Sasa, if we poured tea, the two glasses will be full? (pointing to her drawing)
Sasa	:	Yes
Cori	:	Wait. Like this! (pointing to her drawing) A half, a half!

After doing this activity, the researcher gave a challenging problem to the students about predicting the high level of tea in the glass, if there were 2 glasses which full of tea and one empty glass provided. As the students did in the previous problem, they also used estimation to solve this problem. Five students answered that three glasses were half-full of tea without doing partitioning (Figure 5.11). There was only one student who seemed doing partitioning by four but he could not explain his answer. He just said that it was a difficult problem (Figure 5.12).

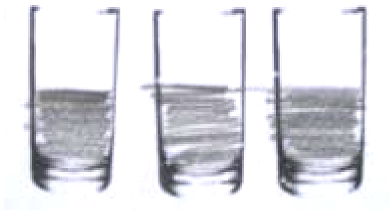


Figure 5.11 Three Glasses Half-Full of Tea

Figure 5.12 Partitioning Three Glasses

By pouring two glasses full of tea, the researcher then showed how high tea in three glasses could be. The students recognized that the height of tea was more than a half. The researcher then asked the students to determine the height precisely. By marking such a glass, there was one student who did partitioning by four and said that the height was $\frac{3}{4}$.

Predicting the height of water did not succeed to provoke the students partition the height of water so that they could notate fractions based on those partitions. Although this problem had same learning goal with the problem of sharing brownies cake, this problem could not bring the students to do partitioning in order to produce fractions.

Mini lesson: Marking $\frac{3}{4}$ of Glass

To bridge between the students pre-knowledge of partitioning model of cake, a mini lesson about marking $\frac{3}{4}$ of glass was developed. This mini lesson was a substitution of mini lesson Fill Fraction in a Number Line because the researcher considered some facts that the students still struggled in partitioning. The researcher asked the students, *'If I want to pour this water into $\frac{3}{4}$ of this glass, how high is it?'*. Nando showed by marking the glass with 4 strips. He marked the glass many times until he saw that the distance between each strip was equal. When he had not got the right position for each strip, he still could show the position of $\frac{3}{4}$ should be. After Nando marked the glass and pointed the strip that showed $\frac{3}{4}$, the researcher then asked the other students

Researcher : Nando had showed that this mark is $\frac{3}{4}$. In your opinion, how did Nando decide that this is $\frac{3}{4}$? Did you agree?

Sasa : No. Because $\frac{1}{4}$ should be here (pointing below the first mark of Nando-$\frac{1}{4}$ mark) and this strip should be $\frac{2}{4}$ (pointing to the first bottom mark)

Researcher : If that mark is $\frac{2}{4}$ then how about this mark in this position? (pointing to the half of glass)

Sasa : A half.

Figure 5.13 Nando's Marks on the Glass

5.2.3 Using Fractions as Unit of Measurement

Activity 4: Measuring Pencil using Folded Paper

This activity was aimed to support students using fractions as unit of measurement. The length of pencil was determined by counting how many parts of a whole or how many unit fractions that fit into a pencil. Considering facts about students' difficulties in partitioning, the researcher decided not to use students own pencil in this activity. The researcher provided pencils with different length. The researcher chose pencils with length $\frac{1}{2}, \frac{5}{8}, \frac{7}{8}$ and $\frac{8}{8}$ of folded paper. In the beginning of activity, the students had to find which pencil was the longest and the shortest. After that, they measured pencils using folded paper. Doing the activity, the students had difficulty to fold paper properly and found a fraction. There was a student who used ruler to measure the length of pencil. The researcher had to give instruction how to fold folded paper. As the result, the number of partition that appeared was only 8 partitions (Figure 5.14).

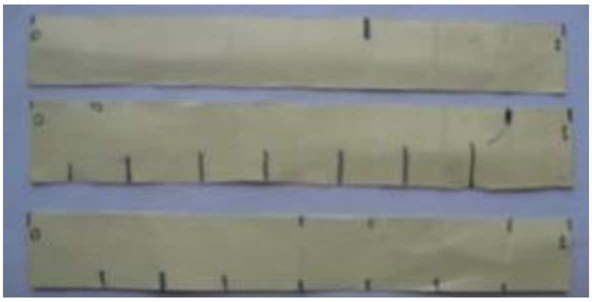


Figure 5.14 Some Students' Work in Folding Paper

The researcher tried to bring up the relation among fractions from the results of measuring. The researcher made use of pencils with length $\frac{4}{8}$ and $\frac{8}{8}$. On the worksheet, there was a question *'The length of Ana's pencil is..............times Toni's pencil'*. By looking at fractions as the length of each pencil, Nando answered 4 times.

Researcher : How do you know that it is 4 times?
Nando : Because $\frac{8}{8}$ and $\frac{4}{8}$, the bigger is $\frac{8}{8}$. Eight subtracted by four is four.
Researcher : Let us prove it! (putting both pencils in parallel)
Cori, is this pencil (pointing to the longer pencil) four times this pencil (pointing to the shorter pencil)?
Cori : No...(shaking her head). Three times.
Eight is divided by four.
Researcher : So how many times?
Some students : Five.
Researcher : Sasa, try to prove it! (giving the pencils)
Sasa : (pointing her thumb and her forefinger) a half...
Researcher : In fact, this pencil (the longer one) how many times this pencil (the shorter one)
Sasa : Two

Finishing the activity of measuring pencils, the students continued to solve some problems on the worksheet. In the initial plan, the researcher did not elaborate this activity into some questions on the worksheet. During the previous activity, the researcher saw that the students' pre-knowledge was about shading parts of area in geometrical shapes so that it had to be connected to this activity. The question was about determining the length of pencil if the illustration of pencils was given (Figure 5.15). The other question was about drawing a pencil if fraction as length of pencil was given (Figure 5.16). Instead of drew a pencil, most of the students shaded area in figure of folded bar. All the students could answer the task on the worksheet properly.

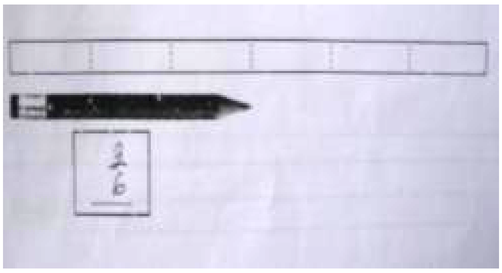


Figure 5.15 Determining the Length of Pencil

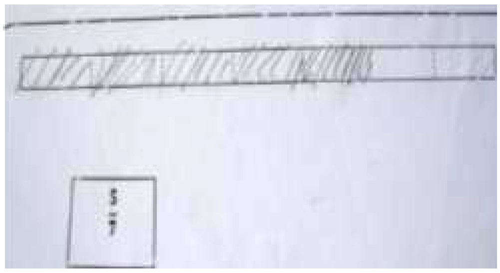


Figure 5.16 Drawing the Length of Pencil

Doing the activity of measuring pencil, the researcher expected that at least students could estimate fractions as the length of pencil. In fact, there was no student who guessed any fraction. This activity might too fast to go to that level. The students had to decide the name of fraction while they were also challenged to partition folded paper.

It was very difficult for the students to come to the strategy of repeated halving in order to generate fractions as the length of objects. Folding paper properly also became another difficulty for the students. As the consequences, the learning goal that the students could use unit fractions as unit of measurement was not achieved. Although the students could write fraction if the partition was given (Figure 5.15), the students seemed merely read off the number of parts of the bar that corresponded to the length of pencils. It was not enough because they did not construct the parts by themselves. The answer of the students in Figure 5.16 gave more evidence about the pre-knowledge of the students which represented fractions as the shaded parts.

Activity of measuring pencil could not support the students to partition by themselves and to use fractions as unit of measurement. The meaning of fraction as a distance from 0 on informal number line also could not be constructed through this activity. There was a gap between the students' partitioning model of cake and partitioning in measuring activity. The students' pre-knowledge of measuring objects using standard units of measurement could not support the students to solve the problem. This activity even discarded such pre-knowledge.

Mini lesson: Marking a Fractions Ruler

In HLT 1, the researcher designed the activity of marking fractions ruler as one of the main activities to support students in using fractions as unit of measurement. The researcher also planned that this activity involved improper fractions. During the learning process, the researcher adjusted this activity so that it only involved proper fractions. The students were asked to complete scales between 0 and 1 on a paper bar as the scales.

For the first step, the researcher asked the students to find position of a half. The students merely used estimation to find a half. The researcher then told them that they could fold the paper bar. There were two students (Nando and Riki) who wrote fractions with 4 as the denominator. Although they fold paper into two, they did not write $\frac{2}{4}$ on the position of paper was folded.

Figure 5.17 Nando’s Work on Marking Fractions Ruler

The researcher then asked the students to write notation of a half on the paper bar. Dea said that the notation was $\frac{1}{2}$ but Nando said that it should be $\frac{2}{4}$.

Figure 5.18 Dewi Wrote a Half on the Paper Bar

Nando was guided by the researcher to find a half of paper bar. Nando used ruler to find the middle of paper bar, a quarter and three quarter of paper bar (Figure 5.19). For the other students who did folding, the researcher used the partition on folded paper to guide them in notating $\frac{1}{4}$ and $\frac{3}{4}$.

Figure 5.19 Nando Used a Ruler to Mark a Fractions Ruler

From the learning process of marking a fractions ruler, the researcher concluded that folding paper could not support the students to find the position of fractions. The researcher needs to guide them in folding paper. It seemed that the student could not figure out fractions represented in folded paper. The partitions were not clear for them particularly which the parts and the whole is. It made the students to get difficulty in notating fractions.

Activity 5: Making Poster of Pouring Water

The researcher added an activity about making poster of pouring water as the follow-up Activity of Pouring Tea. Doing activity of Pouring Tea, the students needed more support in partitioning through real actions. In Activity 5, paper was used as representation of tea.

For the first task, the students had to distribute one glass of water (represented by one piece of paper) into two empty glasses equally. The students did the activity by cutting the paper and gluing it to the picture of empty glasses. This activity was worked in group of two students. Group of Reta and Dewi did partitioning by using estimation so that there was a difference between both glasses. To solve their problem, the researcher provoked them to cut the difference and share it into both glasses (Figure 5.20).

Figure 5.20 Group of Reta and Dewi Made a Poster of Two Glasses

Other group folded the paper into two then cut it. The researcher did not explore further their strategy. In the other group, there was a student (Nando) who measured the length of paper and divided it into two.

Researcher : How did you divide it into two?
Nando : I used a ruler. It is twenty one. Twenty one is divided by two, it could not be done so that it is ten and a half.

The students are asked to measure the height of water on their poster using fractions ruler that they made in the activity of marking fractions ruler. Group of Nando and Riki got different answer because they wrote $\frac{2}{4}$ instead of $\frac{1}{2}$. The researcher had a discussion with them.

Researcher : Your result is $\frac{2}{4}$ but the result of group of Cori is $\frac{1}{2}$. In fact, the position is same (pointing to the middle of fractions ruler).
Nando : Yes, the position is same.
Riki : But this point is $\frac{4}{4}$ which is equal to 1 so that it fits (pointing to the end of fractions ruler).
Researcher : If $\frac{2}{2}$, where is it?

Riki : Here (pointing to 1)

Researcher : So, at the same position, the fractions can be more than one? Because there are $\frac{2}{2}$ and $\frac{4}{4}$ here.
If we used $\frac{4}{4}$, what fractions here? (pointing to the middle of fractions ruler)

Nando : $\frac{2}{4}$

Researcher : But if we used $\frac{2}{2}$, the middle is...

Nando & Riki : $\frac{1}{2}$

Researcher : So $\frac{2}{4}$ and $\frac{1}{2}$?

Nando : They are different

Researcher : But if it is the height of water?

Nando : It is same

Riki : If the bottles are same then the height is same.
If the bottle is 600 ml and 700 ml, then it is not same.

Researcher : So both answers are....

Nando : Right. The ruler is different.

For the second task, the researcher asked the students to divide three glasses of water into four empty glasses. One group used halving strategy by folding paper and folding the last paper by 4 (Figure 5.21). Another group also used halving strategy but doing estimation for the last paper (Figure 5.22).

Figure 5.21 Halving Strategy, Partitioning by Four

Figure 5.22 Halving Strategy, Estimation

Two students in one group had different strategies. One student directly determine $\frac{3}{4}$ by measuring the length of paper using ruler and the other just did trial and error to make the three glasses having the same height. By using fractions ruler that they made in the previous mini lesson, the students then read off the height of water.

Figure 5.23 Using Ruler and Estimation

Although the students did not throw away the remained pieces of paper as they did when sharing brownies cake, some students still had difficulty to find an efficient way to partition. The students did trial and error and repeated to divide the remainder. Cutting the paper until the pieces become smaller made the students more difficult to figure out the fraction. By using fractions ruler, the students then just read off the scale and found the fractions without getting meaning of it. The good indication was about the use of standard units of measurement in partitioning. In the second cycle, the researcher might consider this students' knowledge to support students in developing the meaning of fraction as iterations of unit fractions.

5.2.4 Building the Relation among fractions

Activity 6: Shading Parts and Finding the Relation among fractions

Investigating the pre-knowledge of the students, the researcher found that the students were familiar with shading area on the geometrical shapes. Considering such pre-knowledge and the students' difficulties in partitioning by cutting and folding, the researcher adapted the last activity so that it was related to the pre-knowledge of the students.

This activity was developed by using written task. The first task is *'Show and shade parts that represent*

a. $\frac{1}{4}$

b. $\frac{2}{4}$

c. $\frac{3}{4}$

d. $\frac{1}{2}$

e. $\frac{1}{3}$

f. $\frac{2}{3}$

g. $\frac{2}{6}$

h. $\frac{1}{6}$

Look at the shading area that you have made!

Are there shading area that similar each other?

What is the relation between shading area $\frac{3}{4}$ *and* $\frac{1}{4}$*? Explain your answer!*

As conjectured, all the students did not get difficulties to partition and shade the parts but the partitions did not always in equal size. There were some students who shaded the parts in different way. They did not always start to shade the parts from the left side consecutively.

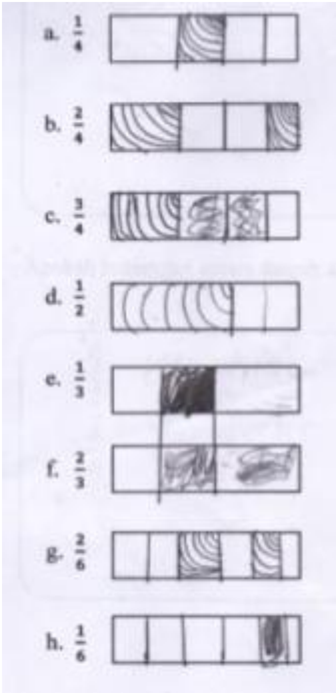
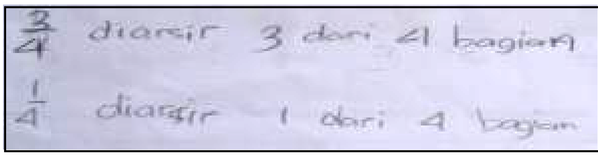


Figure 5.24 The Students' Work in Shading Bar

The students got confused to find the similarity between the shaded parts. The students got difficulty to find the relation among those fractions because their partitions were not equal size. About the relation between $\frac{3}{4}$ and $\frac{1}{4}$, there were the students who figured out that $\frac{3}{4}$ had 3 shaded-parts out of 4 parts and $\frac{1}{4}$ had 1 shaded-part out of 4 parts (Figure 5.25).



$\frac{3}{4}$ shaded 3 of 4 parts
$\frac{1}{4}$ shaded 1 of 4 parts

Figure 5.25 The Relation between 3/4 and 1/4

Although the students could differ $\frac{3}{4}$ and $\frac{1}{4}$ based on the number of shaded parts, the relations that $\frac{3}{4}$ is iterations of $\frac{1}{4}$ still not obvious. It seemed that the students also could not conclude that kind of relations between other fractions.

The skill of partitioning could support the students to find the relation among fractions but it also could be dangerous when the students compared fractions. They might not figure out equivalent fractions because they did not partition in equal parts. There is a need of support for students before this activity particularly in constructing parts equally in order to find relation among fractions.

5.3 REMARKS OF THE STUDENTS' KNOWLEDGE ON POST-TEST

In the end of learning process in the first cycle, the researcher gave a post-test to the students. Some of the questions were similar with the questions in the pre-test. Although post-test was not meant to compare extremely with the students' pre-knowledge before doing the first cycle, the answers of the students in post-test could be a clarification for the students' knowledge development that had been observed in the learning process of the first cycle. According to the results of post-test, the researcher underlined some important points as the following

5.3.1 Connecting Fractions with Concrete Objects

Giving concrete examples of fractions in pre-test, all the students drew geometrical shapes with shaded area. They had difficulties in finding real objects that could be partitioned. In post test, more concrete examples of $\frac{1}{2}$ appeared (Figure 5.26).

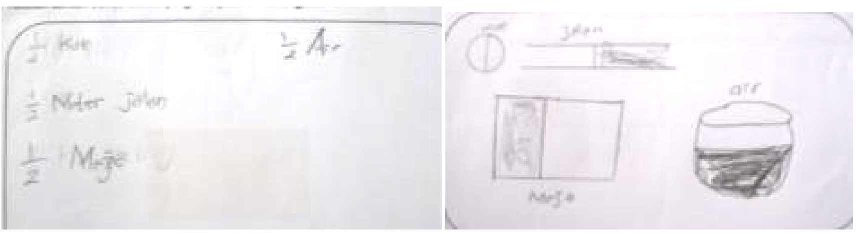


Figure 5.26 A Student's Answer in Giving Examples of a Half

Looking at different representations of a half given by the student, the researcher realized about the importance of giving various context of fractions to provoke students representing fractions in different manner. Fractions was not merely about how many shaded parts in a geometrical shape, but also what kind of object that could be represented by such a geometrical shape and the natural way to partition the object.

5.3.2 Identifying Parts that Representing a Fraction

The ability of shading area that represented a certain fractions did not guarantee that the students could recognize which part to be called a certain fraction. It was shown in the pre test that the students were confused about giving a quarter of cake although they could shade parts of model of cake correctly. Through the experience of real partitioning, the student could connect a fraction with the result of partitioning. They did not only model of rectangle as a geometrical shape that had to be shaded but also could see such a model as real object to be partitioned. Determining a quarter of cake to be given to Lisa, the student partitioned the model of cake and gave the name on it (Figure 5.27).

Figure 5.27 An Example of Students' Answer on Determining a Quarter of Cake

5.3.3 The Use of Measuring Context in Learning Fractions

Contexts of measuring that was intended to support the students in constructing the meaning of fraction as a distance from 0 in informal number line have not yet given a significant support. The challenge was about the way of partitioning in order to generate fractions. Marking measuring scale of water or folding paper to measure the length of pencil seemed not to be a natural way to provoke the students did partitioning to generate fractions. In partitioning the level of water in a glass, the students still used estimation instead of connected fractions with the number of partitions (Figure 5.28). Non unit fractions as iterations of unit fractions did not appear.

Figure 5.28 Determining 3/4 - Full of Water

Producing fractions in informal number line through the activity of measuring pencil seemed to be a jump from the students' pre-knowledge. The students who recognize fractions as the name for a certain part out of a whole from the result of partitioning had to make a transition to recognize fractions as a mark of distance from 0 in informal number line. Because of a gap between such knowledge, the student failed to build meaning fractions as a distance in informal number line (Figure 5.29).

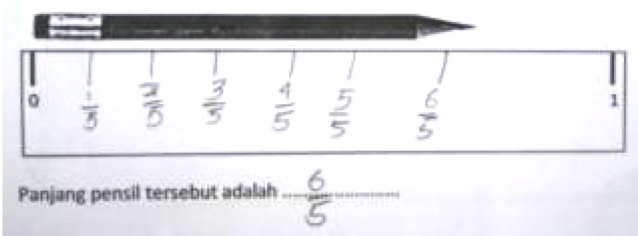


Figure 5.29 Fractions in Number Line was Meaningless for the Student

There was a case that the student used standard unit measurement as the length of pencil instead of fractions. The students' pre-knowledge of measuring objects using standard units of measurement could not support the students to solve the problem. This activity even discarded that knowledge.

Figure 5.30 The Student Got 9 as the Length of Pencil

5.3.4 Relation among fractions

From the students' answer on the measuring problem in post-test, the students still have not yet grasped the meaning of fraction as a distance from 0 in informal number line in which non unit fractions could be determined by iterating unit fractions. Because of the lack of such knowledge, the students have not yet built the relation among fractions. One of evidences was that the students only used estimation instead using fraction $\frac{1}{4}$ or $\frac{1}{2}$ to find the position of $\frac{3}{4}$. Relation among problem also has not yet built by the student as a tool of reasoning in solving formal problem. When the students were asked to give an opinion whether they agreed with $\frac{1}{2}$ + $\frac{2}{4}$ = $\frac{3}{6}$, all the student agree with that (Figure 5.31).

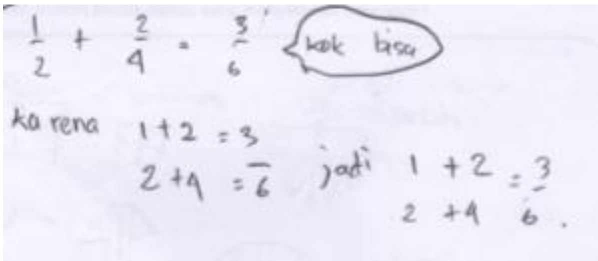


Figure 5.31 One of Students' Reasoning

In comparing fractions as one of exploration of relation among fractions, the students tended to use cross-multiply algorithm to solve the problem. Although the students drew geometrical shape to show their answer, they seemed not to use it as tool of reasoning. It was shown in Figure 5.32. Although the shaded parts were equal size, the students only answered by using algorithm as a reason.

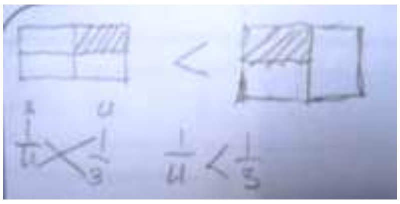


Figure 5.32 Comparing Fractions

5.4 CONCLUSION OF THE STUDENTS' LEARNING PROCESS IN THE FIRST CYCLE

Looking back at the previous learning process of the students, the students' learning process of fractions was more focused on constructing the meaning of fraction as parts of a whole using shaded area in geometrical shapes as representation. The results of pre-test showed that it was not enough for the students to construct one meaning of fraction. They need to learn other meaning of fraction simultaneously.

Various context of fractions developed in HLT challenged the students to extend their understanding of fractions. For instance, the act of real partitioning has provoked the students to have different meaning of fraction notations. Besides of the progression of students' knowledge, such various contexts also could bring some risks because there might be a gap between students' pre-knowledge and the intended new knowledge. Such a gap in the HLT was found when the students did activities about pouring tea and measuring pencil. In this activity, the students were engaged to make a transition from understanding of part-whole relationship to understanding of fractions as unit of measurement. Partitioning that was powerful in fair sharing activity to generate fractions could not support the students to produce fractions. The students' pre-knowledge of standard units of measurement also could not always support the students to learn fractions. Measuring pencil activity even contradicted with such knowledge because the students have to represent the length of pencil with fractions instead of with standard units of measurement.

In the next teaching experiment, the researcher would revise the HLT so that it would accommodate the pre-knowledge of students in extending the understanding of the meaning of fraction. For instance, the pre-knowledge of students about standard unit of measurement should be utilized to support students in constructing meaning of fraction as a distance from 0 in number line. About the relation among fractions, the researcher found that it was too fast for the students to explore relation among fractions explicitly meanwhile they have not grasped the meaning of fraction. Although it was too fast to learn the relation among fractions explicitly, constructing the meaning of fraction actually could be learned simultaneously with the relation among fractions. For instance, the mathematical idea of fair sharing which about the more number of sharers, the smaller size of pieces actually could support the learning process of comparing unit fractions. Such a perspective in building the relation among fractions will be more considered in the second cycle. The more detailed refinement of HLT 2 to be HLT 3 could be seen in Appendix C.

5.5 HLT 3 AS THE REFINEMENT OF HLT 2

Considering the analysis of the first cycle, learning phase about using fractions as units of measurement has not yet supported by measuring pencil and pouring tea activity. From the results of partitioning, the students should be provoked to notate the results by fractions and discuss the meaning of fraction in the context. In fact, in activity of measuring pencil and pouring tea, the students even have problems about partitioning. The use of standard units of measurement even distracts the students in finding the length of pencil.

As the refinement of HLT 2, an activity of posit an ant is developed to support students in using fractions as units of measurement. In this activity, the story about ants which have walked as far as a certain part of path is developed. The distance as the position of ants is then used to construct the knowledge about non-unit fractions as iterations of unit fractions. HLT 3 as the refinement of HLT 2 is summarized in Table 5 and the conjectures of students learning process would be described before the retrospective analysis in each activity.

Table 5.1 HLT 3 as the Refinement of HLT 2

Learning Goals	**Mathematics Ideas**	**Activity**
1. Students construct meaning of fair sharing	• Pieces do not have to be congruent to be equivalent • Unit fractions • The more number of sharers, the smaller piece is • In comparing fractions, the whole must be same	• Dividing one cake for 4 people • Increasing the number of sharers
2. Students produce simple fractions as result of fair sharing	• Fractions as part of a whole of objects • Fractions is an amount as a quotient • Common fractions as iterations of unit fractions	• Dividing 3 cakes for 4 people • Determining $\frac{3}{4}$, $\frac{1}{4}$ and $\frac{1}{2}$ from a number of chocolate bars.
3. Students use fractions as unit of measurement	Common fraction as iterations of unit fractions	• Posit an Ant • Determining Position of Ant using Unit Fractions
4. Students build the relation among fractions	• An equivalence relation leads to equality within magnitude • Common fraction as iterations of unit fractions	• Making Path of Ants • Ants on Number Line

5.6 INVESTIGATING THE STUDENTS' PRE-KNOWLEDGE

Before testing HLT 3 in the classroom teaching experiment, the students were given a pre-test that aimed to investigate their prior knowledge of fractions. Furthermore, such information about students pre-knowledge also gave an insight on some critical issues that have not been grasped by the students. Interviewing the teacher, the researcher also got some important information about students pre-knowledge particularly about specific aspect on learning fractions that students had no experiences before. Instead of described students' answer on each pre-test item, the researcher would focus on some crucial issue on students' pre-knowledge. All items of pre-test could be seen in Appendix F.

5.6.1 Representation and The meaning of fraction

Looking to the students' notebook and their mathematics book, the researcher found that representation of fractions as shaded part on geometrical shapes was dominant on prior students' learning process (Figure 5.33).

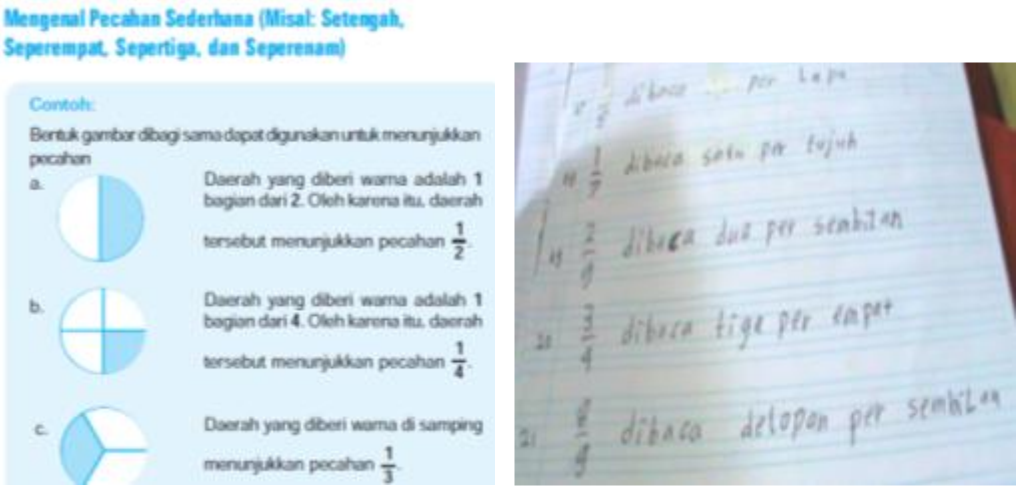


Figure 5.33 Students' Notebook

The students' prior learning process also less emphasized on the use of context as the source of introducing fractions. Investigating the students' understanding of fractional partitioning through pre-test (item number 3), the students were asked to determine whether both shaded parts had same area or one was bigger than another one.

Figure 5.34 Different Ways in Partitioning

Only half of all the students could determine that both were equal because both were a quarter. The rest of the students seemed to be confused because if they looked at the shapes, one was bigger than the other but the fractions were same. Yet, there were 4 students who answered that one was bigger than another.

Reason: shaded area I is bigger than shaded area II

Figure 5.35 An Example of the Students' Answer

Although most the students could divide one cake among 4 children fairly and notating the results with fractions when solving pre-test (item number 1), the students need more support in exploring the meaning of fraction related to contextual situations and learned more about a mathematical idea which fractional parts do not have to be congruent to be equivalent.

5.6.2 Awareness of a Whole in Comparing Fractions

In the students' prior learning process, they used the cross multiply algorithm in comparing fractions. From their notebook, it was not known whether the students could clarify their answer using another strategy.

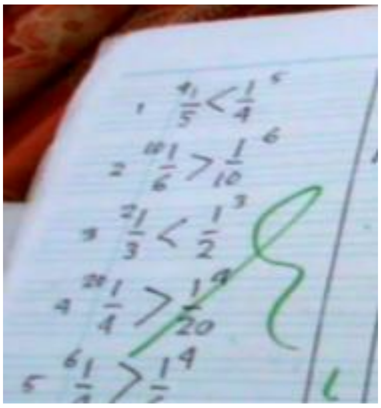
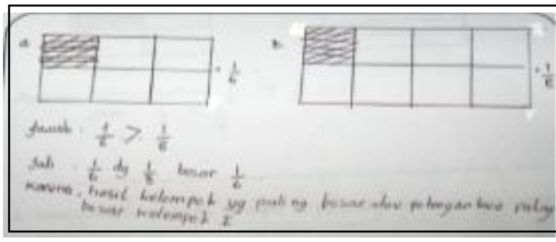


Figure 5.36 The Students' Strategy in Comparing Fractions

Given a contextual situation, the students were asked to compare the results of fair sharing in which the number of sharers were different, 6 sharers and 8 sharers (item number 2). In comparing fractions, only one student who could give correct answer with proper drawing. Seven students could give correct answer but their drawings were not in proper way. The size of both cakes as wholes was different.



$\frac{1}{6} > \frac{1}{8}$

$\frac{1}{6}$ and $\frac{1}{8}$, $\frac{1}{6}$ is greater

Because: the result of group which is bigger from the pieces of cake, group I is bigger

Figure 5.37 Different Size of Cakes in Comparing Fractions

Nine students also gave correct answer but they did not explain their answer or using any kind of representation. Nine of students even did not give correct answer or gave a wrong answer because they perceive the more the number of pieces, the more each person gets. The different whole also was found in other item of pre-test (item number 7) in which they had to compare two fractions.

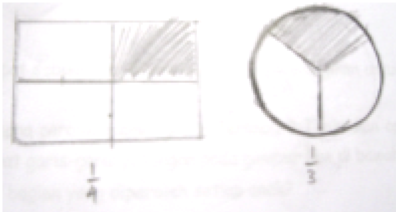


Figure 5.38 Different Wholes in Comparing Fractions

Related to the awareness of a whole in comparing fractions, based on the interview with the teacher, the researcher got information that the student were not introduced to dividing more than one object. Non-unit fractions such as $\frac{3}{4}$ was generated from 3 parts out of 4 parts in single object. Solving a problem about dividing 2 cakes among 4 children (item number 5), a half of students also did not use fraction notation in representing the results of fair sharing but wrote one or two pieces of cake.

Looking to the students' prior learning process in comparing fractions, the students need more support their understanding. More experiences dealt with concrete situation should be given so that they were provoked to give a reason in comparing fractions. The students also need more support to build their awareness of a whole in comparing fractions.

5.6.3 Discrete Objects in Learning Fractions

One of pre-test item was about determining the number of candies that should be taken by Anto if he wants to take $\frac{1}{4}$ of 20 candies (item number 4). This item was to investigate whether the students could connect their understanding of the meaning of fraction across different kind of object. In fact, more than half of the students could not give a correct answer.


Jawab : 20 − $\frac{1}{4}$ = ~~0/16~~ 16
Jadi : Banyak permen yg harus diambil Anton adlh ~~0/16~~ 16

Answer: 20 – ¼ = ~~0/16~~ 16

So: the number of candies that should be taken by Anto is ~~0/16~~ 16

Figure 5.39 An Example of the Students' Answer in Discrete Objects Problem

The students’ difficulties on solving a problem about discrete objects showed that they had not built interconnections across various kind of representation. Although they directly divide one cake into 4 pieces when asked to determine $\frac{1}{4}$ of it, the students could not apply such understanding to this problem. Instead of making fractional parts, the student did a subtraction as they did in whole number operation. The student need more support to extend their understanding of the meaning of fraction related to discrete objects.

5.6.4 Fractions on Number Line

Interviewing the teacher, the researcher got information that although the students had been introduced to number line, most of them still had a lot of difficulties in positioning fractions in number line. Only few students could make fractional part in number line and posit fractions properly. The students answer in pre-test (item number 6a and 6b) related to positioning fractions on number line also gave more evidence of students’ difficulties. More than half of the students did mistakes such as in the following figure.

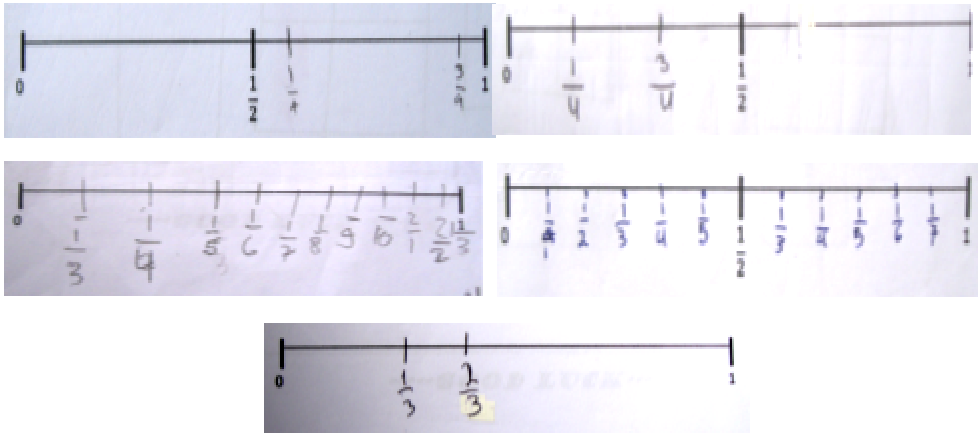


Figure 5.40 Some Examples of the Students' Mistake in Positioning Fractions on Number Line

From the students' answer, it seemed that they could not connect their knowledge of partitioning dealing with sharing problem to the problems involving number line. The student need more concrete situation in which they could develop meaning and relation among fractions in the frame of number line. Observing the distance between unit fractions and non-unit fractions, the researcher also found that most the student seemed to not aware that the distance of non-unit fractions from zero point was iteration of magnitude of the distance of unit fractions from zero point.

5.7 TEACHING EXPERIMENT OF THE SECOND CYCLE

In this section, teaching experiment of the second cycle based on HLT 3 (Table 5.1) will be explained. The teaching experiment is conducted in class 3C that consists of 28 students. The students in this class often had different mathematics teachers. Consequently, socio norm in the classroom have not been well constructed. The situation is very crowded to be conducive for teaching and learning particularly for classroom discussion. The situation became more complex because there was a student who had serious problem with his emotional quotient. He often had conflicts with other students during teaching and learning process. The researcher gives such a brief explanation of the classroom as a context of this research in which establishing socio-norm that supports the learning process became a big challenge.

Instead of going to each activity, the analysis would be carried out in each learning goal to investigate how students achieved such learning goals by means fair sharing and measuring activity. Specific mathematical ideas conveyed in the learning goals also became the theme of analysis of students' development.

5.7.1 Constructing Meaning of Fair Sharing

At the beginning of teaching experiment, activity of sharing a fruit cake was developed to convey the meaning of fair sharing. The teacher showed a model of cake to the students. She told that she wanted to share the cake among her four nephews fairly. Giving a model of cake to each pair of students, the teacher asked them to divide it for 4 students. The teacher also wrote a question on the whiteboard *'Each teacher's nephew gets.....part of cake'* to be answered. The students did cutting activity and glue the pieces of cutting on the paper. A class discussion was then conducted to justify whether the results of cutting was fair and to explore fractions notations as the result of fair sharing. A conflict about fairness of results of sharing also was given to the students to help them in constructing a mathematical idea that in fair sharing, the pieces do not have to be congruent to be equivalent.

The context of sharing a fruit cake was used to convey a mathematical idea that in fair sharing, the pieces do not have to be congruent to be equivalent. Different ways of partitions were given to the students, so that they could justify which way of partition was fair. Developing another mathematical idea when the number of sharers is increased, the students worked on the next task about dividing chocolate bars into different number of equal parts.

Some conjectures of students' answer in this activity were that the students were able to divide the model of cake into 4 parts using standard units of measurement or estimation. The strategy of cutting that might appear were cutting vertically, cutting on horizontal and vertical line or cutting on diagonal.

About fairness of results of fair sharing, the students might not have difficulties to justify whether the results of cutting were fair. They might have more difficulties when facing a conflict of fair sharing in which the pieces had different shapes. The students might justify that it was not fair because the shapes were different. Another conjecture was that the students could use fractions as reasoning.

About notating the results of fair sharing using fractions, the students might not have difficulties. The students might have struggles in explaining the meaning of fraction notations. Two meanings of fractions notations that might appear was that unit fractions as part of a whole or as quotient.

In the problem of increasing the number of partitions, the students might have difficulties on partitioning properly but not with writing fractions notations. Writing a conclusion of the activity of increasing the number of partititons, the students might get difficulties to explicate their finding from the activity. The teacher might have to give clued to the students about what kind of conclusion that was expected.

Mathematical idea: in fair sharing, the pieces do not have to be congruent to be equivalent.

In dividing the cake into 4, most the students used a strategy of cutting by making a horizontal and vertical line. There was only one group who did cutting on the diagonal (Figure 5.41).

Figure 5.41 Different Strategies of Dividing a Fruit Cake

Differences appeared at the students' strategy in determining line of cutting. Those strategies are folding paper, estimating and measuring the length of paper.

When the students were asked to convince themselves whether the result of sharing a fruit cake was fair, they did re-measuring the results of cutting or holding the pieces together. Even there was a group of students who cut the difference of pieces which were not in equal size to make it fair. In class discussion, the teacher asked the students to justify the fairness of sharing a fruit cake. Both strategies in Figure 5.41 were showed in front of the class.

The First Vignette

Teacher	:	How about the results of dividing? Is it fair? (showing the students' work who cut in horizontal and vertical line)
Students	:	Fair
Teacher	:	Why?
Ary	:	Because the pieces are same.
Teacher	:	So, fair means...
Andi	:	The pieces are equal.
Teacher	:	How about the results of dividing in this group? Why is it fair? (showing the students' work who cut in diagonal line)
Dafi	:	Because its shape and size is same.

From the dialogue above and the way of students to convince themselves that the pieces were fair, the students justified fairness based on the congruency of pieces. The teacher then gave a problem which conveyed a mathematical idea that the pieces do not have to be congruent to be equivalent. The problem was about Rafa and Rafi who had different opinions about piece of cake that they got. Shaded parts represented the pieces of cake got by Rafa and Rafi. Rafa said that his piece was bigger but Rafi said that they got the same big pieces. The teacher asked the students' opinion about such a case.

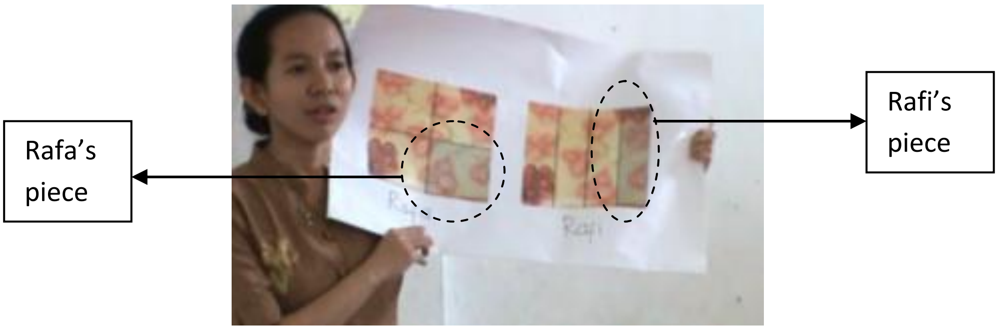


Figure 5.42 A Problem about Different Shapes of Pieces

The Second Vignette

Teacher	:	So, Meli said that those pieces are same. Why?
Meli	:	Because these are a quarter, a quarter, a quarter, a quarter, a quarter, a quarter, a quarter, a quarter (pointing to each piece)
Teacher	:	So, Meli said that each piece is a quarter?
Meli	:	Yes.
Teacher	:	How about your opinion, Kris?

Kris : One, two, three, four. A quarter (pointing to Rafa's figure). This is also one, two, three, four. A quarter. So, those are same.

The students' answer indicated that they have considered that in fair sharing, the pieces do not have to be congruent to be equivalent. Instead of perceiving the differences of shapes of both pieces, Meli and Kris seemed to consider that both pieces were equal because such pieces were represented with same fractions.

The teacher then asked the students whether they could find other ways to convince that both pieces were same. Model of cakes that similar with Figure 5.42 was given to the students. Asking one of group, the observer found that this group seemed had no clue how to prove that the pieces were same. They only gave a reason about fractions instead of comparing the magnitude of both pieces.

The Third Vignette

Observer : You said that it should be measured. How to measure?
Meli : 19,8...(measuring the length of Rafi's piece) and 9,7...(measuring the length of Rafa's piece)
Observer : So, is it same?
Meli : No. But both of them are a quarter.
Observer : So, if both of them are a quarter, is it same?
Meli : It is same
Observer : Why are the sizes different? How to prove it?
Meli : This is a quarter and this is also a quarter.

In other groups, the students proved that both pieces were same by cutting one piece (Rafi's) into two or four and putting on the other piece (Rafa's) (Figure 5.43).

Figure 5.43 The Students Cut One Piece (Rafi's) into Two and Put on the other Piece (Rafa's)

From the dialogue above, the students tend to use formal reasoning in solving Rafa's and Rafi's problem although in previous activity about sharing a fruit cake, they judged fairness based on the sizes and the shapes. This formal reasoning could be dangerous if the students merely recognized fractions without considering the whole of fractions. Two pieces that both are a quarter are not always equal if the wholes are different. This knowledge seemed to be critical to develop.

In class discussion, the teacher provided some problems aimed to provoke students' reasoning by showing a figure which partitions were not in equal parts and another figure which the size of cake was smaller (Figure 5.44)

Figure 5.44 Class Discussion about Partitioning and Size of Cake

At first, the teacher discussed figures which partitions were not equal. The students were asked to compare with Rafa's and Rafi's pieces. After that, the teacher pointed to a figure which the size of cake was smaller.

<u>*The Fourth Vignette*</u>

Teacher	:	Rafa and Rafi have got these pieces (pointing to Rafa's and Rafi's figure), but Riki got this piece (pointing to unequal partitioned-figure)
Students	:	Haa...
Teacher	:	Is it allowed?
Students	:	No, it is not allowed
Teacher	:	Why? Rama's group? Is it allowed?
Rama	:	Yes, it is allowed
Other students	:	No!!
Teacher	:	Why it is not allowed?
Robi	:	Because the sizes are not same
Teacher	:	So, is it allowed to call these a quarter
Students	:	No, it is not.
Teacher	:	Is there another opinion?
Yeriko	:	Because that one is small and the other is big.
Teacher	:	So, Rafa and Rafi got the same big pieces but Riki?
Yeriko	:	His is not same with Rafa's and Rafi's
Teacher	:	Other opinion?
Tata	:	Rafa's and Rafi's are same, but Riki's is not same.
Nia	:	The sizes are different.
Teacher	:	So, this group have an opinion that Rafa and Rafi got same big pieces, but this (Riki's) is not same because the sizes are different.

(The teacher then continued the class discussion to a figure which the size of cake was smaller)

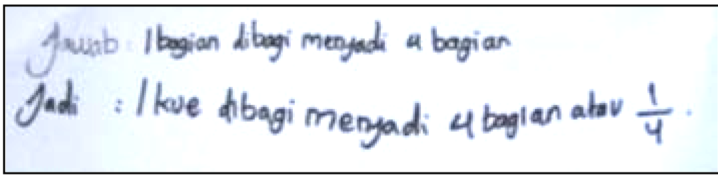
Teacher	:	How about this figure?
Students	:	It is allowed.
Teacher	:	We compare it with Rafa's and Rafi's. We know that Rafa and Rafi got same big pieces. How if Riki got this piece?
Students	:	It is not allowed
Teacher	:	Rafa and Rafi got fair pieces. Does Riki also get fair piece? If Riki gets this piece, does he get same big piece with Rafa and Rafi?
Students	:	Same...No...
Teacher	:	But it is also a quarter, right?
Ary	:	It is also a quarter but the cake is not in the same big.
Teacher	:	So, what should we care about?
Nia	:	The size of cakes must be same.

The dialogue above showed how the teacher tried to provoke the students to have awareness of a whole and partitions when justifying fairness in sharing object. The students could recognize that the pieces do not have to be congruent to be equivalent by giving reasoning that both pieces represented the same fractions. On the other hand, the students also had to be careful about a whole of object. In the end of class discussion, the teacher guided the students to recall that in comparing two fractions they should care about equality of a whole object in which fractions were derived.

The Meaning of Unit Fractions in Sharing a Fruit Cake and Dividing Chocolate Bars

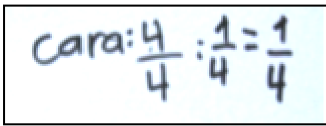
When the teacher posed a problem about how to share a fruit cake among her nephews, there were some students who directly said a quarter although the question had not yet posed. One of the students, Rafi, said that it would be one fourth, one out of four parts. It indicated that the students had understood the meaning of fraction as part of a whole. Exploring a reason behind the answer of the students who said it would be a quarter, the teacher obtained another reason from the student. Robi said that it was one fourth because it was divided by four. From Robi's answer, it showed the student seemed to understand the meaning of unit fractions as quotient. Although the pre-knowledge of student was more about fractions as part-whole relationship, context of sharing a cake caused the meaning of fraction as a quotient to be natural to emerge.

As an answer for the question '*Each teacher's nephew gets.....part of cake'*, all groups of students could write $\frac{1}{4}$ as the notation of part that each person got. The students' answers were shown on the following figure.

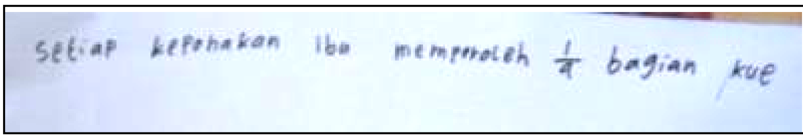


Answer: 1 part is divided into 4 parts

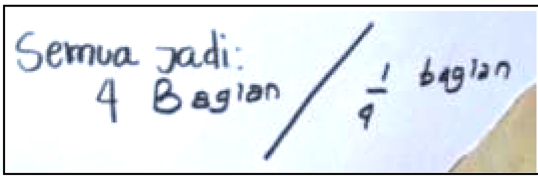
So: 1 cake is divided into 4 parts or $\frac{1}{4}$



Way: $\frac{4}{4} : \frac{1}{4} = \frac{1}{4}$



Each teacher's nephew gets $\frac{1}{4}$ part of cake



All become:

4 parts or $\frac{1}{4}$ part

Figure 5.45 Differences on Writing Fractions

To check whether the students really understood which part to be called one fourth, the teacher posed a question to a student in one of groups.

<u>*The Fifth Vignette*</u>

Teacher	:	Each person gets?
Dafi	:	One fourth
Teacher	:	So, this piece is one fourth. How is about this piece? (pointing to another piece)
Dafi	:	Two fourth. Eh, one fourth

Although the student corrected his answer, the student's answer gave an indication that there was a possibility that fractions are perceived as ordinal number as they find in whole numbers. There were some groups who make their answer clearer by writing $\frac{1}{4}$ in each piece (Figure 5.46).

Figure 5.46 The Students Wrote 1/4 in Each Piece

Unfortunately, the teacher did not explore more about the meaning of fraction $\frac{1}{4}$ in class discussion because of limited time. She only asked the answer of the students who all of them answered $\frac{1}{4}$ then continued with another task.

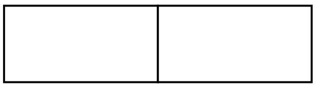
In this context, the meaning of fraction as a quotient emerged. The absence of shaded part in this context might provoke the students to perceive unit fraction also as quotient instead of merely counting the number of shaded part out of all parts as a whole. In notating fractions, the students had no difficulties in determining what kind of fractions. The difficulties merged because, daily language interfered language of fractions. Some students implied *'part'* as *'piece'* so that they tended to use whole number such as one part of cake instead of using fractions language such as a quarter part of cake.

As the students were able to notate fractions as the results of sharing a fruit cake, most the students also had no difficulties to notate fractions in the problems of dividing chocolate bars. There was interesting finding about students' difficulties in notating fractions because of the absence of shaded parts.

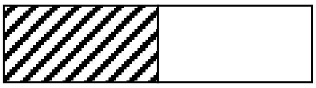
The Sixth Vignette

Teacher	:	How do you get one half? (pointing to the student' answer where the student wrote one half instead of one twelfth)
Robi	:	Wait... (he realized that his partitions were less than twelve then added more partitions)
Teacher	:	How many pieces are there?

Robi : Twelve
Teacher : If I only point to this one, how do you call it? (pointing to one piece)
Robi : One half
Teacher : How about this one? (pointing to another piece)
Robi : hmmm....
Teacher : Now, try to shade it (shading one piece). How many areas are shaded? The fraction is...
Robi : One twelfth
Teacher : How about this one? (pointing to another piece which is not shaded)
Robi : One half
Teacher : So, you said that it is one twelfth and this one is one half. What is the difference?
Robi : Because that one is shaded part.
Teacher : That is for one twelfth. How about one half?
Robi : ...... (no answer)
Teacher : How about this figure?



Robi : One fourth.
Teacher : Why?
Robi : ..... (no answer)
Teacher : How if it is shaded? The fraction is....



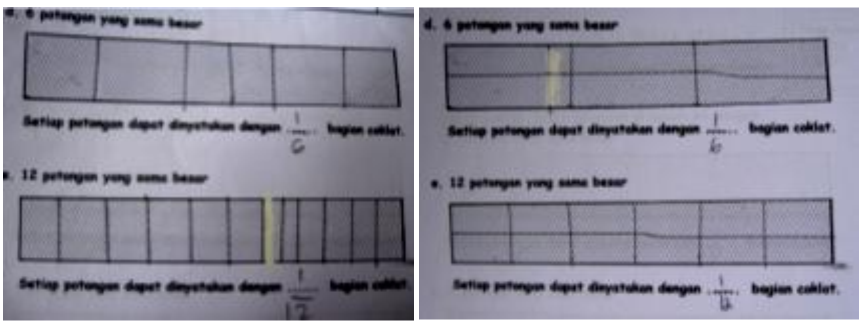
Robi : One half

Robi's difficulty in notating fractions showed that there was a lack of knowledge of different meaning of fraction and of different representations of fractions. The pre-knowledge of student was more focused on fractions as part of a whole using shaded area as representations. Although in the previous activity of sharing a fruit cake, Robi was able to notate fraction 'a quarter' and explain the meaning of it, he could not apply his pre-knowledge in this problem. It might be caused by the problem that was more abstract. Through the questions on the worksheet, the students were asked to divide chocolate bars into different numbers of pieces. Unlike his friends who still could connect this problem to their previous experience in sharing a fruit cake, Robi could not relate this context to his pre-knowledge in notating fractions which fractions was perceived as quotient.

Mathematical idea: the more number of sharers, the smaller piece is.

In fair sharing, increasing the number of sharers causes the size of pieces become smaller. This fact leads to the mathematical idea in fair sharing which the more number of sharers, the smaller pieces will become. Connecting to notation of unit fractions, the larger the denominator, the smaller the pieces of results of fair sharing will be got. To construct this mathematical idea, a task about dividing chocolate bars was given to the students. In pair, the students had to divide chocolate bars into different number of pieces (2, 3, 4, 6, 12) equally then compare the pieces and make conclusion about the results of dividing chocolate bars.

In dividing chocolate bars, there were some students who used standard units of measurement to determine the length of each piece but other students only estimated the size of each piece so that the pieces were not always in equal sizes. Most of the students could notate the results of dividing by unit fractions. Different ways of partitioning also appeared on the students' answer (Figure 5.47).



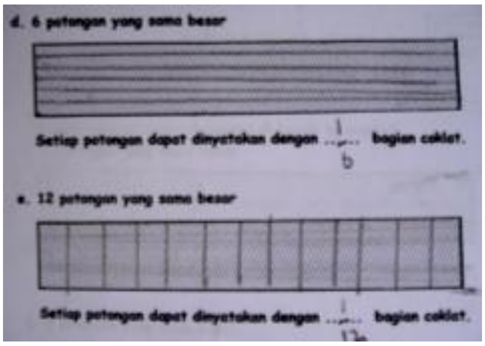


Figure 5.47 Different Strategies in Partitioning

Although not all the students did partitioning in proper way, most group of the students could order the pieces of chocolate bar from the biggest one to the smallest one (Figure 5.48).

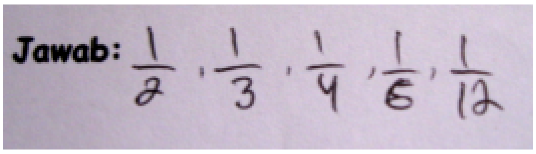


Figure 5.48 Correct Order of Unit Fractions

In fact, there were some students who still ordered fraction from the greatest denominator to the least denominator. They might only perceive the number as denominator than compare the size of pieces as the results of dividing (Figure 5.49).

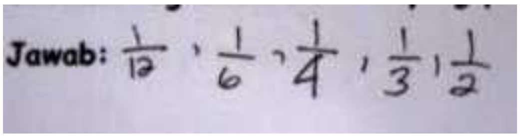


Figure 5.49 Incorrect Order of Unit Fractions

After the students ordered fractions based on the size of pieces, they were asked to give another example which the pieces were smaller than the pieces that they had made. The students got another fraction which the denominator was quite large (Figure 5.50).

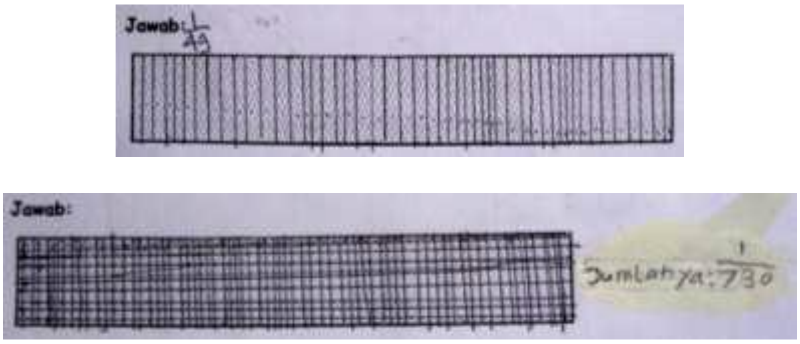
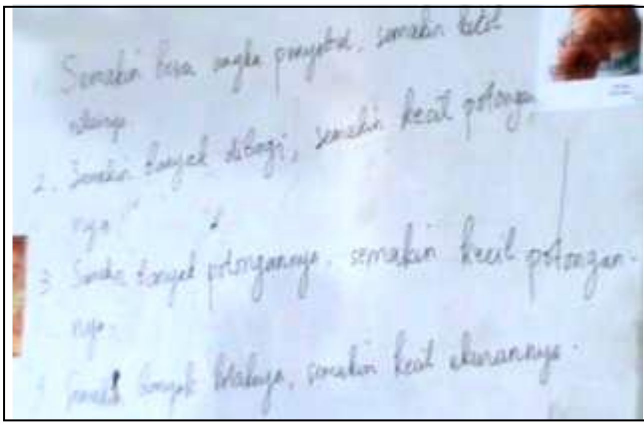


Figure 5.50 The Students’ Example of Smaller Pieces of Chocolate Bar

In making a conclusion from their results of dividing chocolate bar, most the students could conclude by using their own language. In class discussion, the teacher then asked the students’ opinion about their conclusion and made a list about different kind of the students’ conclusion as shown in Figure 5.51.



1. The largest the denominator is, the smallest the value is
2. The more it is divided, the smallest the piece is
3. The more number of pieces is, the smallest the piece is
4. The more number of square, the smallest the size of piece is

Figure 5.51 The Students’ Conclusions

From the students’ conclusion, it showed that they could generalize what they did. The students seemed to grasp the mathematical idea about the more an object is divided equally, the smaller the pieces will become. To check whether the students could apply their conclusion to determine which fractions is larger than other fractions, the teacher posed some pairs of fractions to be compared.

The Seventh Vignette

Teacher	:	If asked, one half and one twentieth, which piece will be smaller?
Students	:	One twentieth
Teacher	:	Between one third and one fourth, which one is smaller?
Students	:	One fourth
Teacher	:	How about one sixth and one sixteenth? Which piece will be larger?
Students	:	One sixth

The students were able to compare two different fractions as results of sharing easily although it was still limited to unit fractions. From their direct answer to the teacher' question, they seemed no longer used cross multiply algorithm to answer the questions.

The students were stimulated to recall their awareness of wholes in comparing fractions as they learned in the activity of sharing a fruit cake. The teacher gave two fractions, $\frac{1}{3}$ and $\frac{1}{4}$ and asked the students to compare fractions. When the students judged directly that $\frac{1}{3}$ was greater than $\frac{1}{4}$, the teacher drew two figures of cake in different size. It made a piece of $\frac{1}{3}$ become smaller than a piece of $\frac{1}{4}$.

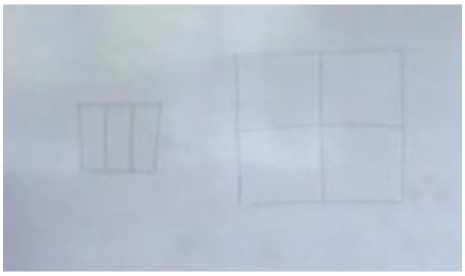


Figure 5.52 Different Wholes in Comparing Fractions

The Eighth Vignette

Teacher : You said that $\frac{1}{3}$ is greater than $\frac{1}{4}$, but in this figure, why $\frac{1}{4}$ is greater than $\frac{1}{3}$?

Rafi : Because the size of shape for $\frac{1}{3}$ is smaller than the size of shape for $\frac{1}{4}$.

Teacher : So, what is wrong with the figure? You all said that $\frac{1}{3}$ is greater.

Students : The size is not same.

Teacher : So, what about the size? How should it be?

Students : The size should be same.

The answer of the students indicated that they were more aware to the equality of wholes in comparing fractions. In making pictorial representation, they should be care about the size of two objects to be compared. The students seemed to grasp the mathematical idea about the whole must be same in comparing fractions.

Conclusion of learning phase 'constructing meaning of fair sharing'

Based on their reason in justifying the fairness of results of sharing, the students seemed to grasp the meaning of fairness. In constructing meaning of fair sharing, there were many mathematical ideas could be constructed. About notating fractions, although the students had learned before, the teacher could challenge them to explain the meaning of such notation. The explanation of the students about fractions notation related to the context of fair sharing showed that the students had constructed the meaning of fraction as part whole relationship and as quotient. The students' pre-knowledge about notating fractions also help them to construct the mathematical idea that in fair sharing, the pieces do not have to be congruent to be equivalent. Avoiding the students merely consider fractions notation, the students had been given an opportunity to convince themselves that the pieces which were not congruent could be equivalent. The students also seemed to construct another mathematical idea that the more number of sharers, the smaller the pieces will be resulted. They also could apply such a mathematical idea to compare unit fractions. Moreover, the students also grasped the mathematical ideas about the whole must be same in comparing fractions.

5.7.2 Producing Simple Fractions as Results of Fair Sharing

Looking to the pre-knowledge of students, shaded part in geometrical shapes seemed to be very dominant in the development of learning fractions. The students recognized fractions as the number of shaded parts out of all parts in a geometrical shape. In pre-test, such a pre-knowledge of the students could not support the students to solve fractions problem involved discrete objects. Most of them could not answer properly and seemed to have no clue how to solve the problems.

After the students grasped other meaning of fraction as quotient through fair sharing a fruit cake, the students would construct such a meaning using discrete objects. Through the use of discrete objects, the students were expected to be able to connect their knowledge about the meaning of fraction as quotient in determining the number of objects that represented a certain fractions particularly unit fractions.

In order to give more support to the students to produce simple fractions as results of fair sharing, a problem about fair sharing that involved more than one continuous object also was given to the students. To notate the results of fair sharing with fractions, the students also have to be able to coordinate the number of sharers and the way of partitioning. Considering a whole also became a part of requirement to be able to notate fractions in case the number of objects was more than one object. Notating the results of fair sharing using more than one object also might provoke the student to develop relation among fractions.

Some conjectures of students' strategies on solving discrete objects problems were that the students had no clue to group chocolate bars based on unit fractions given or they might arrange chocolate bars into groups. Another conjecture was that the students used directly division algorithm in solving the problem.

About fair sharing 3 brownies cake among 4 children, the students might get different fractions notations based on what kind of a whole that they perceived in notating fractions. They might come with fractions notation $\frac{3}{4}$ if they did partitioning by four and considered one cake as a whole. Using same strategy of partitioning, the students might write $\frac{3}{12}$ as fractions notation when they perceived total pieces from three cakes. The student who used a strategy of halving and sharing the rest migt come to $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ as the answer.

Partitioning Discrete Objects in Learning Fractions

In this activity, discrete objects were involved to strengthen the meaning of fraction as quotient. Determining the number of chocolate bars that represented a certain unit fractions, the students were expected to be able to apply their knowledge of fractions as quotient. Instead of partitioning by cutting or making a line, making groups of objects became the way of partitioning.

In the beginning of lesson, the teacher showed a stack of chocolate bars consisted of 12 chocolate bars in front of class (Figure 5.53).

Figure 5.53 Stack of Chocolate Bars in Front of Class

The teacher started with asking the students to determine the number of chocolate bars that represented a half of the stack. The students were easy to answer such a question. Asked by the teacher to explain their answer, the students said that they divided 12 by two. There was a student who showed his strategy by dividing the stack into two equal stacks (Figure 5.54). The teacher then asked the student to check whether he understood which stack to be called a half. The student answered that both stacks were a half.

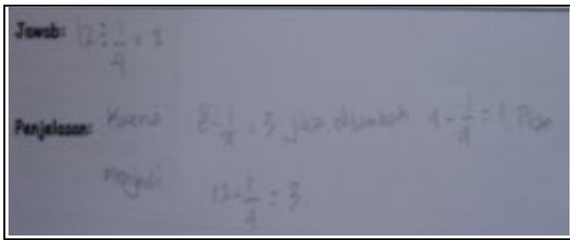
Figure 5.54 The Student Arranged a Half of Stack

Different opinions appeared when the teacher asked about one fourth of 8 chocolate bars in the stack. The students did division and subtraction to determine the number of chocolate bars.

The First Vignette

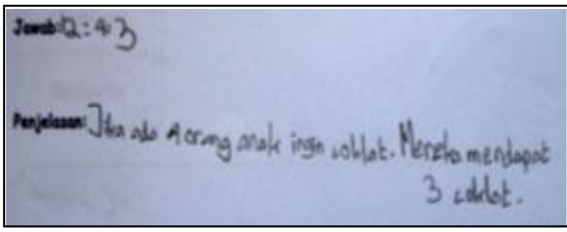
Dea	:	Eight divided by four
Teacher	:	Is there different answer?
Ary	:	Six
Teacher	:	How is your way?
Ary	:	Subtracted by two
Teacher	:	Why do you subtract by two?
Ary	:	Because a half and a quarter, a quarter is less than a half
Teacher	:	oo...because a quarter is less than a half. How many chocolate bars that represented a half of eight?
Students	:	Four
Teacher	:	Ary said, a quarter is less than or greater than a half?
Ary	:	Less than
Teacher	:	Ary's answer is six but a quarter is less than. There are different answers here. How about other answer?
Yosi	:	Two
Teacher	:	How do you get it?
Yosi	:	......
Rafi	:	Eight is subtracted by two is four, then subtracted again by two getting two
Rama	:	Eight subtracted by two is six
Teacher	:	Wait, I think I know your way. Rafi do it by subtracting, subtracting, subtracting. So, first you subtracted eight by two getting....
Rafi & Students	:	Six
Teacher	:	then....
Rafi	:	Four
Teacher	:	How many times do you subtract by two?
Rafi	:	Four
Teacher	:	Rafi's way is different. He subtracts four times. First, eight is subtracted by 2, subtracted again, second, third, and then... (showing her four finger)...
Students	:	Run out
Teacher	:	Show it, Rafi
Yosi	:	Three times, Rafi
Rafi	:	(go to in front of class and make four stacks of 2 chocolate bars one by one)
Yosi	:	It means four?
Teacher	:	I made a mistake. Rafi's way The stack is eight then it is separated into two, two, two and two (showed each stack of two). How many group of chocolate bar?
Students	:	Four
Teacher	:	Each stack consists of how many chocolate bars?
Students	:	Two

Working on some similar problems in the worksheet, most the students used division to determine the number of chocolate bar that represented a certain unit fractions. There was a group who explained why they used division but the other only did an algorithm of division. Other different strategies were using subtracting and splitting the number of chocolate bar (Figure 5.55).



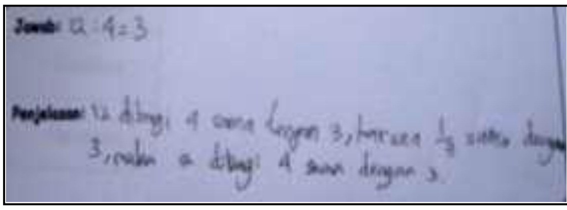
Answer: 12 : $\frac{1}{4}$ = 3

Explanation: Because 8 - $\frac{1}{4}$ = 3 if added by 4 - $\frac{1}{4}$ = 1 and become 12 - $\frac{1}{4}$ = 3



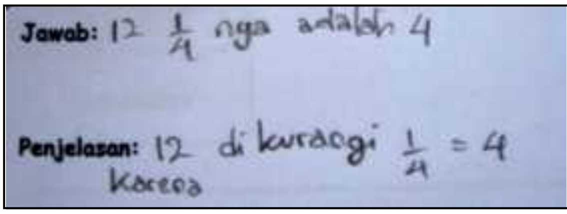
Answer: 12 : 4 = 3

Explanation: If there are 4 children who want chocolate bar. They get 3 chocolate bars.



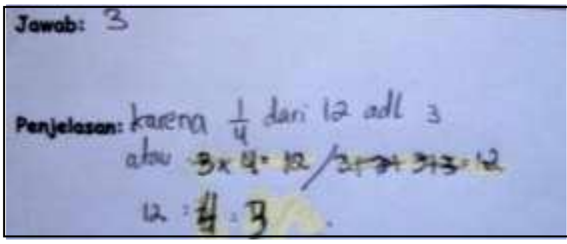
Answer: 12 : 4 = 3

Explanation: 12 is divided by 4 equal to 3, because $\frac{1}{3}$ is equal to 3, so 12 divided by 4 is 3.



Answer: 12 $\frac{1}{4}$ is 4

Explanation: Because 12 is subtracted by $\frac{1}{4}$ = 4



Answer: 3

Explanation: Because $\frac{1}{4}$ of 12 is 3
Or 3 x 4 = 12/ 3+3+3+3 = 12
12 : 4 = 3

Figure 5.55 Different Strategies in Determining the Number of Chocolate Bars

Using discrete objects, the students were expected to strengthen their knowledge of the meaning of fraction as quotient and do partitioning by making groups of objects. From the class discussion and the students' work, the students' seemed to be able to connect their knowledge of the meaning of fraction as a quotient in sharing a cake to this problem. In fact, figure 5.55 showed that although the students used division to solve the problem, they had different idea behind the algorithm of division. Unfortunately, these differences were not explored in class discussion after did the task. The situation of the class was not conducive for discussion because there was fighting between two students. The teacher only asked the students to tell their answer and the way they solved which was only by dividing.

Fractions as Results of Fair Sharing

Using more than one continuous object to be partitioned, the students were challenged to coordinate the number of sharers and the number of partitions. After they did partitioning, they were expected to represent an amount that each person got by fractions notation. Considering a whole would become the main issue in notating fractions.

In the beginning of lesson, the teacher gave a problem which was same with a problem in pre-test about sharing 2 cakes among 4 people. The teacher posed a question that if Ryan is one of people who sharing 2 cakes, how much Ryan gets. The students gave different opinions based on their partitions (Figure 5.56).

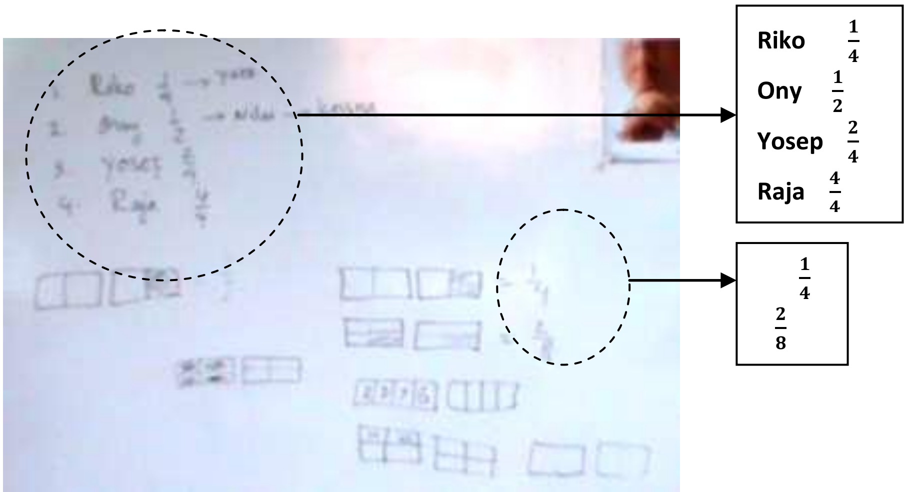


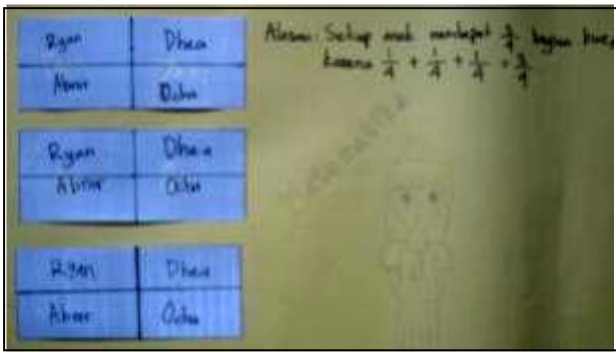
Figure 5.56 Differences in Notating Fractions

In fact, different answers appeared because of different whole used in notating fractions. Some students perceived the total number of pieces in two cakes as a whole but other students perceived a whole from all parts in one cake. The discussion was focused on showing the differences explicitly. The teacher emphasized that both of them was true, whether they took one cake as a whole or two cakes as a whole.

A more challenging problem about sharing 3 cakes among 4 children was given to the students. They worked in pair. Not all the students could determine the number of partitions directly. Some of them did partitioning by trial and error and tried to distribute the results of pieces to each person. If the number of pieces did not fit with the number of people to be shared, the students tried other ways of partitioning.

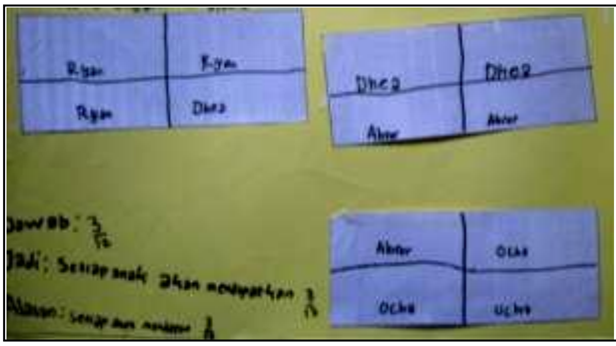
Finally, all the students could partition the cakes properly. As it was predicted, the students used strategy of partitioning by dividing each cake by four or dividing by two and sharing the rest. Most the group of students answered $\frac{3}{4}$ and

$\frac{3}{12}$ as the results of fair sharing that each person gets. Two groups of students came with $\frac{1}{2}+\frac{1}{4}$ as the answer and other two groups had a wrong answer that each person got $\frac{1}{4}$ (Figure 5.57).



Reason: Each child gets $\frac{3}{4}$ part of cake

Because $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{4}$



Answer: $\frac{3}{12}$

So: Each child will get $\frac{3}{12}$

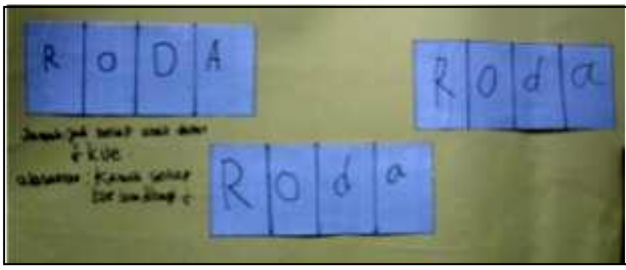
Reason: Each child gets $\frac{3}{12}$

Each child gets $\frac{3}{4}$

Because $\frac{1}{2}+\frac{1}{4}=\frac{2}{6}$

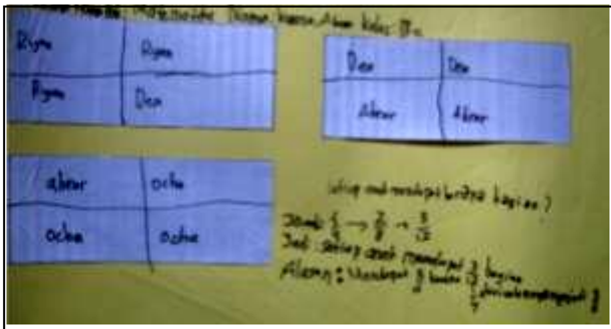
Because divided for 4 children become $\frac{3}{4}$

$\frac{2}{6}$



Answer: Each child gets $\frac{1}{4}$ cake

Reason: because each cake could be divided by $\frac{1}{4}$



How much each person gets?

So: $\frac{1}{4} \rightarrow \frac{2}{8} \rightarrow \frac{3}{12}$

So: Each child gets $\frac{3}{12}$ **part**

Reason: Gets $\frac{3}{12}$ **because** $\frac{1}{4}$ **equivalent with** $\frac{3}{12}$

Figure 5.57 Sharing 3 Cakes among 4 Children

Provoking the students to give a reason for their answer, the teacher asked one group of students when the other students were working on the task.

The Second Vignette

Tyan : Each child got three fourth
Teacher : Why?
Tyan : Because if two two, (pointing to pieces) it cannot.
Teacher : How do you know that it is three fourth?
Tyan : Ryan, Ryan, Ryan, Dea, Dea, Dea, Abror, Abror, Abror, Ocha, Ocha, Ocha
Teacher : Ooo..three. How about this four (pointing to 4 in $\frac{3}{4}$)?
Andi : Three third
Tyan : For example, Ryan (pointing to three pieces named Ryan) gets three. Each piece is four, right? So, three fourth
Andi : But, the answer is three third
Tyan : Three fourth is right!
Andi : Three third or three fourth?
Teacher : Why is it three third?
Andi : This (pointing to the figure)
Tyan : Wait! It is one fourth, one fourth, one fourth, one fourth (pointing to each piece on the first cake). One, two, three, it means three. How many all of these? (asking Andi)
Andi : Four
Tyan : So, it is three fourth
Andi : But, we look at all of these.
Tyan : It is same. Not only this. It means one cake
Teacher : If Tyan looks at this cake, it means three fourth. If Tyan look at cake that Tyan gets compared with all (pointing to all cakes), how much do you get?
Tyan : Three
Teacher : Three of?
Tyan : All

Andi	:	Three
Teacher	:	Three cakes. How many pieces?
Andi	:	Three. No. Four.
Teacher	:	These are four, four and four (pointing to each cake)
Tyan	:	Three fourth
Teacher	:	Three fourth. If compared with all of these? Three per...
Andi	:	Three third. No. Three twelfth.
Tyan	:	Right. Three twelfth
Teacher	:	So, there are two different answers, right? It is three fourth because you only look at this (pointing to one cake). It could be also three twelfth, if compared with all of these (pointing to all cakes). Which one is correct? Are both correct?
Tyan	:	In my opinion, three twelfth is correct.
Teacher	:	Andi, you said three twelfth is correct. In Tyan's opinion, three fourth is correct, right?
Tyan	:	Yes.

The dialogue above showed how the problem could elicit different kind of reason to generate an answer. Andi and Tyan discussed about how to notate the results of sharing. Started with making a mistake about determining a whole, Andi finally could notate fractions $\frac{3}{12}$ as his answer. Using different perspective of a whole to notate fractions, Tyan came to $\frac{3}{4}$ as his answer.

In class discussion, the teacher asked some students to explain their answer. Unfortunately, the teacher did not explore more about the differences among students' answer. The teacher directly told the students to consider that the differences in notating fractions were caused by different perspective of a whole; one cake or three cakes. Both answers were correct as long as they could explain what kind of a whole that the students perceived.

<u>Conclusion of learning phase ‘producing simple fractions as results of fair sharing’</u>

In the previous learning phase, they had learned about fractions related to one continuous object. They seemed to grasp the meaning of fraction as part-whole relationship and quotient. Problems about determining the number of chocolate bars challenged the students to apply their knowledge of the meaning of fraction. From the learning process, it seemed that the students could apply their knowledge of the meaning of fraction as quotient. In solving the fair sharing problem which involved more than one continuous object, the students could coordinate the number of sharers and the number of partitions that they should make. Explaining their notation of fractions, the students could give a reason for fractions notations that they chose.

5.7.3 Using Fractions as Unit of Measurement

In this phase of HLT, measuring activity was delivered to convey the meaning of fraction as a measure. Related to number line, fractions were perceived as a distance from zero point. The students needed to make a transition from perceiving fractions as a part of a whole objects to part of a whole distance. In this teaching experiment, the meaning of fraction in measurement was developed through the context of path of ants. Started from unit fractions, the students were asked to determine the position of ants if fractions were given. The expectation was that the students could connect to their previous knowledge in partitioning continuous object such as in dividing cakes. Although determining the position of ants seemed to be only giving mark in a certain point on a line, the students were expected to construct that it related to the magnitude of line from zero point to the mark. Utilizing measuring activity, the students would learn about the meaning of non-unit fractions as iterations of unit fractions. Determining the position of a certain fraction such as $\frac{2}{3}$ on a number line could be generated from iterating the distance from zero point to $\frac{1}{3}$ on a number line by three times.

Interviewing the students before the lesson, the researcher found that it was a big jump for the students to move directly from partitioning geometrical shape to partitioning a line. The students tended to be confused to posit a certain fraction on a line because they did not do partitioning. Avoiding such a big jump, the researcher provided a bar as path of ant. In bar model, there was still area in it to be perceived and partitioned by the student as they did when partitioning model of cake.

The meaning of (unit) fractions as a distance

The problems were about determining the position of some ants which had passed $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, or $\frac{1}{6}$ of one trip to the pile of sugar. By asking the students to shade area of path that had passed by ants, it might provoke the students to identify which parts that showed a magnitude of certain fraction. Different with shaded area that they found in fair sharing problem, shaded area in measuring activity using bar model showed the length of path that had passed. Exploring the meaning of fraction as a distance was a starting point to perceive fractions on number line.

Working on the problems about determining the position of ants, more than a half of the students could shade part of path properly. Some of them measured the length of path by using standard units of measurement then divided the length by a number as the denominator of unit fraction (Figure 5.58). Other students used estimation to determine the position of ants (Figure 5.59).

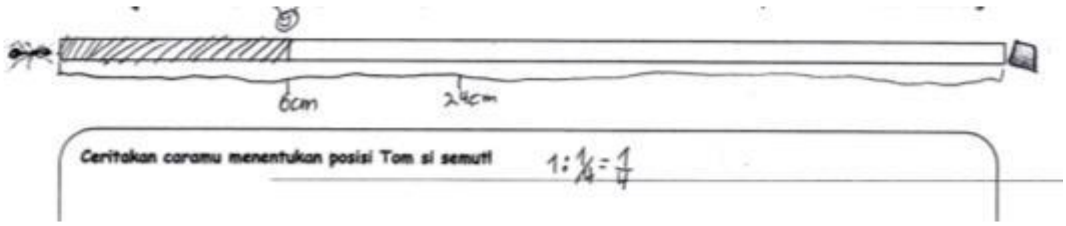


Figure 5.58 Partitioning by using Standard Unit of Measurement

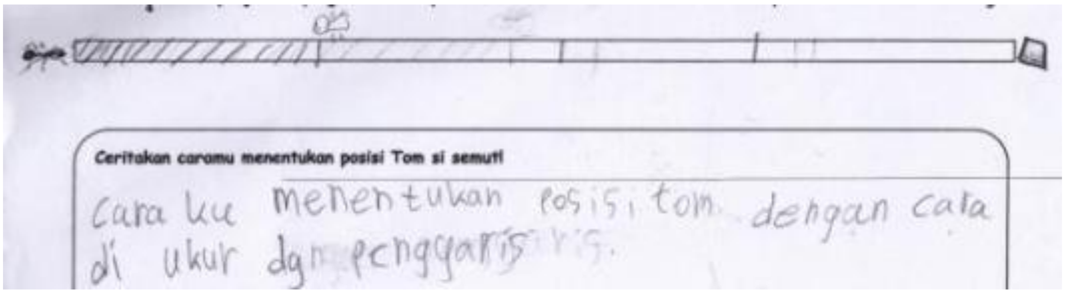


Figure 5.59 Partitioning by using Estimation

Both strategies showed that the students could apply their knowledge in partitioning objects in fair sharing context to the context of ants' path. Because of the context of ant's path, unit fractions appeared as the distance from the starting point of path to the current position of ants. Unit fraction was written as the distance but also as the number that indicate the end of distance travelled.

In fact, bar model did not always provoke the student to perceive fractions as a distance. There were few students who made mistakes in partitioning or directly applied the way of partitioning path as they did in partitioning cakes. (Figure 5.60).

<u>*The First Vignette*</u>

Teacher	:	Where is Tom?
Kris	:	Here (pointing to the end of his partition on the Figure 5.60)
Teacher	:	Why is it same with the position of Riri? (pointing to the position of Riri which passed a half of trip). Both of them are on the middle but Tom is on a quarter and Riri is on a half.
Kris	:	No. It is not a half (pointing to Tom's position) The way is narrower.
Teacher	:	How about the distance travelled?
Kris	:	It is wider. (pointing to Riri's path)
Teacher	:	But then the distance is same.
Kris	:	No. Tom is here (moving Tom's position rightward)
Teacher	:	How is your opinion, Gilang? (asking another student) According to you, the position is here, right? (pointing a quarter of path)
Gilang	:	I make a line here and here (pointing to a quarter and three quarter) and shade one part so that a quarter.
Teacher	:	How about you Kris?
Kris	:	I am confused.

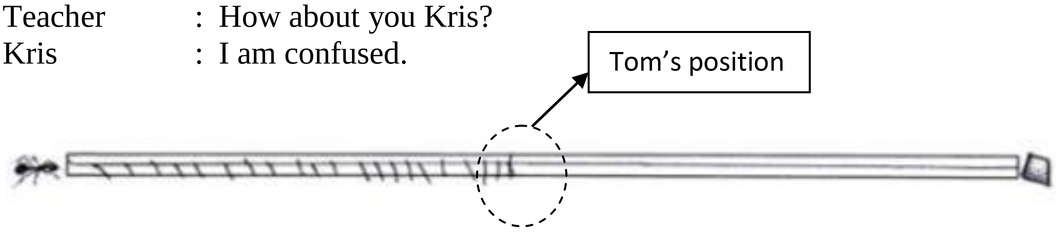


Figure 5.60 Kris's Strategy in Partitioning Ant's Path to Determine 1/4 of Path

In Kris's case, unit fractions did not appear as the distance but similar with the partitions in a cake. The context that was given could not lead the student to think that there was a more specific way of partitioning in determining the position of fractions as a distance. Bar model that was expected to be a transition from constructing the meaning of fraction as a part of a whole to the meaning of fraction as a distance could not provoke the student to make such a transition.

Bar model that resembles line also could distract the student to do partitioning. The researcher found one student who became confused to find the position of a fraction. He did not use partitioning to determine the position of a fraction. He put fractions which had greater denominator in the right side of fractions which had less denominator.

<u>*The Second Vignette*</u>

Researcher	:	A half is on the middle (pointing to the middle of ant's path), then?
Rama	:	This is $\frac{1}{3}$ (pointing to the right side of a half) and then this is $\frac{1}{4}$.
Dafi	:	No. $\frac{1}{3}$ is here (pointing to the right side of a half but further from Rama's point)

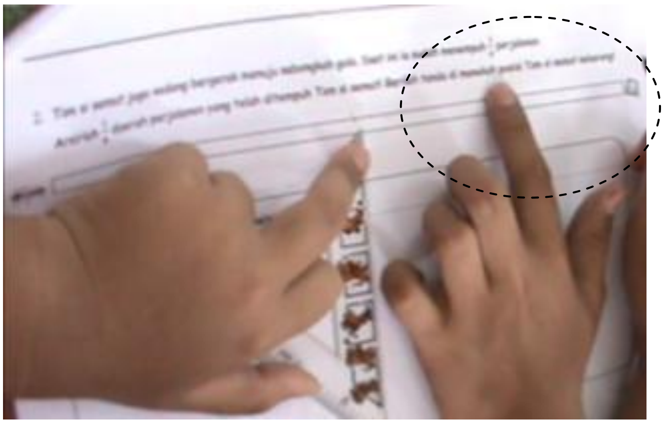


Figure 5.61 Dafi Pointed to the Position of 1/3

Dafi and Rama's answers indicated how they perceived fractions as they perceived whole numbers. They posited unit fractions according to the whole number as the denominator instead of partitioning the path. Dafi and Rama could not connect their previous knowledge about partitioning model of cake to determine the position of fractions on ants' path problems.

In classroom discussion, the differences of answer in determining $\frac{1}{4}$ of path were discussed. There were the students (Ocha and Tata) who did partitioning by 8 to determine the position of a quarter of path. The teacher tried to ask the students to justify by comparing with the answer of Gilang (Figure 5.62).

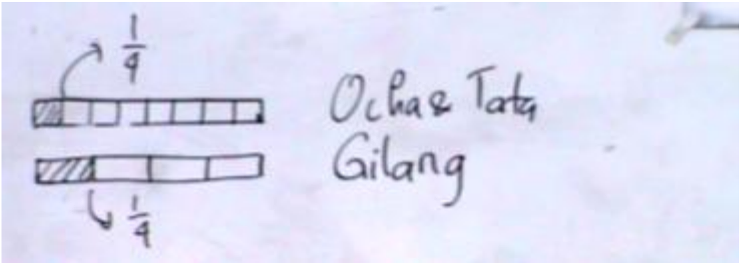


Figure 5.62 Different Answers in Determining 1/4 of Path

Most the students directly said that Gilang's answer is the correct one. When the teacher asked them why Ocha and Tata's were not correct, some of them gave a reason that the squares (partitions) were smaller. Another student even said that Ocha and Tata's answer were not $\frac{1}{4}$ but $\frac{1}{8}$. For the other fractions, the students had same answers so that there was no further discussion (Figure 5.63). When the teacher asked the students about which ant passed the farthest distance, most the students could determine that Riri had passed the farthest distance in which fractions $\frac{1}{2}$ was the largest.

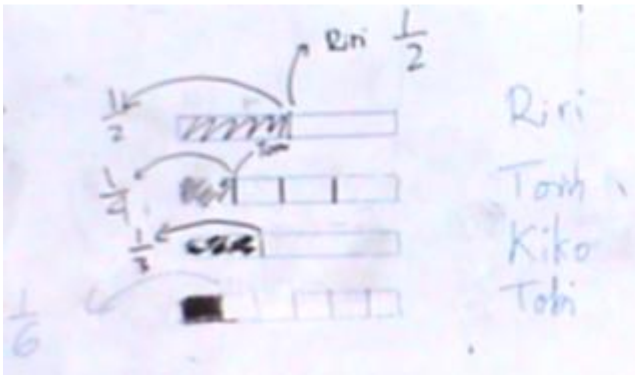
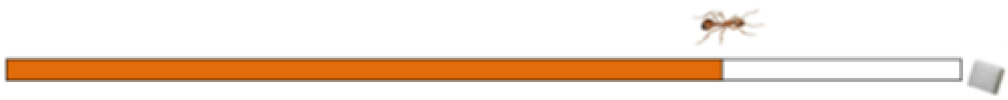


Figure 5.63 The Answers of Ants' Positions

Non-unit Fractions as Iterations of Unit Fractions

Using the magnitude of unit fractions in the activity of ants' path, the students had to determine the positions of ants which had travelled a certain part of path. Those unit fractions were represented with the pieces of ribbon which had different length ($\frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$). One of the problems posed was

By using the pieces of ribbon that were given, determine the position of ant on the following figure!



By choosing one of types of pieces of ribbon, the students had to find the fractions that represented the position of ant. Iterating chosen pieces of ribbon, the students were expected to conclude that non-unit fractions could be constructed by iterations of its unit fractions, for example $\frac{2}{3}$ consisted of 2 ($\frac{1}{3}$) units.

By trial and error, the students put the pieces of ribbon on the path of ants. There was a student who used ribbon with different length to determine the position of ant.

Figure 5.64 The Student Used Ribbon with Different Length

Actually those combinations of ribbon also could construct non-unit fractions. Because the focus of learning process was about non-unit fractions as iterations of its unit fractions, the teacher then asked the students to use ribbon with the same length. Most the students had no difficulties to find a suitable ribbon to determine the position of ants. Although they had to try several times, finally they could find ribbons that fit to the position of ants.

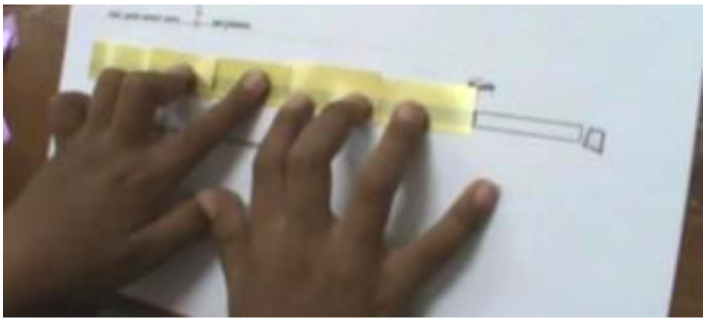


Figure 5.65 The Student Find Ribbons that Fit into the Position of Ant

After the student succeeded in finding a number of a certain ribbon that fit into the position of ants, the challenge was that finding out a fraction. Not all the students could determine directly what kind of fraction that represented the position of ant. In group discussion, the observer found that there was a group who wrote $\frac{1}{3}$ as the position of ant that fit into three pieces of $\frac{1}{4}$-ribbon.

<u>The Third Vignette</u>

Observer : How do you get $\frac{1}{3}$?

Kris : Because there are three (pieces of $\frac{1}{4}$-ribbon)

Observer : What is red ribbon? It is $\frac{1}{4}$-ribbon, isn't it? So, there is one piece of $\frac{1}{4}$-ribbon, then one more piece of $\frac{1}{4}$-ribbon and one more piece of $\frac{1}{4}$-ribbon. There are three pieces of $\frac{1}{4}$-ribbon, so what is it?

Kris : Three fourth.

The answer of the student indicated that the students might only perceive the number of pieces of ribbon instead of connecting with the magnitude of those pieces of ribbon. Unit fractions as the magnitude of pieces of ribbon were ignored by the student. The student needed to recall what kind of unit fraction that he used.

Difficulties in concluding a non-unit fraction represented a number of unit fractions also happened to other students. They did not ignore the magnitude of pieces of ribbon that they used, but they made a mistake in determining the non-unit fraction.

The Fourth Vignette

The teacher	:	So, if you used three pieces of ribbon ($\frac{1}{4}$-ribbon), what is the position of ant?
Ary	:	One twelfth
Dony	:	Right. Because, those were added or a quarter was multiplied by three.
The teacher	:	How about in the middle of this path? What is it?
Dony	:	A half
The teacher	:	The other name of a half is two fourth, right?
Dony & Ary	:	Yes.
The tacher	:	So, here is one fourth (pointing to the end of first piece of $\frac{1}{4}$ ribbon) and then here is... (pointing to the end of the second piece of $\frac{1}{4}$ ribbon)
Dony	:	Two fourth
The teacher	:	Until this? (pointing to the end of third piece of $\frac{1}{4}$ ribbon)
Dony	:	One twelfth
Teacher	:	Three..??
Dony	:	Three fourth.

From Dony’s answer, it seemed that the students had not yet grasped the meaning of non-unit fractions as iterations of its unit fractions. Dony used arithmetic reasoning by adding or multiplying the numbers in a fraction without connected to his pre-knowledge in learning fractions using fair sharing context.

Besides the difficulties of some students, there were also the students who were able to perceive non-unit fractions as the iterations of unit fractions on ants’ path.

<u>The Fifth Vignette</u>

Teacher	:	Which ribbon that you used?
Dafi	:	The red one, a quarter.
Teacher	:	How many?
Dafi	:	Three
Teacher	:	So, how about the position of ant? If here is a quarter and then added by one more piece of ribbon?
Dafi	:	Two eighth.
Teacher	:	Why is it two eighth?
Dafi	:	Eh, two fourth.
Teacher	:	Right. If we add one more piece of ribbon?
Dafi	:	Three fourth.
Teacher	:	Until this? (pointing to the end of ant's path)
Dafi	:	Four fourth.

The answers of Dafi showed that he seemed to realize that by iterating the unit fractions, the numerator of non-unit fractions did not change. The iterations caused the denominator of non-unit fraction changing according to the number of iterations. Unfortunately, the teacher did not challenge the students to find the connection of 'four fourth' with 'one' as the whole path that ant travelled.

The iterations of unit fractions had provoked the students to symbolize the situation in a formal way. There was a student who wrote an addition of fractions that could be used to explain his answer.

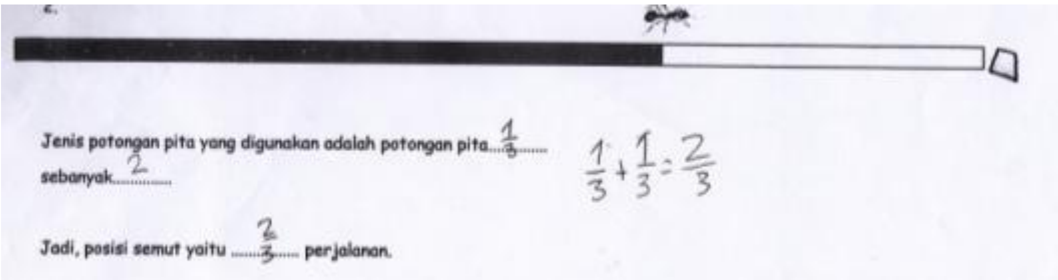


Figure 5.66 Iterations of Unit Fractions were Symbolized with an Addition of Fractions

Although there was no further investigation about his knowledge of addition of fractions, his answer indicated that the student has been on the path of learning about operation of fractions. Iteration of unit fractions was interpreted as adding fractions.

Asking the students to write their answer on the whiteboard, the teacher found that there were different answers of the students depending on chosen ribbon.

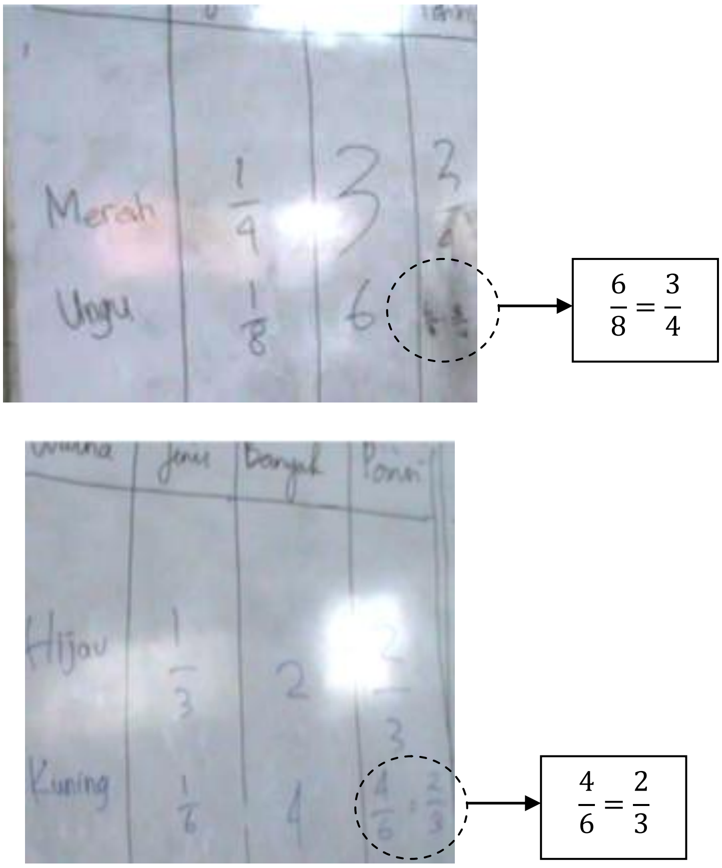


Figure 5.67 Simplifying Fractions in the Different Answers of the Students

The differences of choosing unit fractions brought different notations of ant position. For instance, one of the problems could be answered by $\frac{3}{4}$ or $\frac{6}{8}$ depending on whether the students chose unit fraction $\frac{1}{4}$ or $\frac{1}{8}$. One group of the students wrote that $\frac{6}{8} = \frac{3}{4}$ and explained that if $\frac{6}{8}$ was simplified, it became $\frac{3}{4}$. This finding showed that how this context had a chance to fill a niche in the students' pre-knowledge. Simplifying fractions did not change the magnitude of fractions but it could be about the number of partitions which were different.

Conclusion of learning phase 'using fractions as unit of measurement'

A path that resembled bar model was used as a bridge for the students from the understanding of fractions in fair sharing to the understanding of fractions in measuring activity. In fact, it brought an advantage but also a risk for the students. It could help the students to adapt their knowledge of partitioning in fair sharing context into context of measuring. On the other hand, the students could merely adopt their strategy of partitioning in fair sharing context without considering measuring as a context. In general, because most the students could determine the position of ants correctly it seemed that they commenced to buid meaning unit fractions as a distance from an initial point (zero point). They also could compare unit fractions by comparing the magnitude of distance of such unit fractions. The students had difficulties on determining non-unit fractions as iterations of unit fractions. The teacher had to give more support to the students by posing questions such as connecting to a half as benchmark of fractions.

5.7.4 Building the Relation among fractions

Utilizing the context of ants' path, the students started to build the relation among fractions formally. Although the students just learned the relation among fractions more explicit in this learning phase, the teacher actually had provoked the students to build the relation among fractions when they learned about the meaning of fraction in fair sharing and measuring activity. For instance, when the students learned about one of mathematical ideas in fair sharing that the more number of people or partitions, the smaller the partitions or pieces will be got by each person, the students also start to learn about comparing fractions. Another relation among fractions also appeared in the results of sharing more than one object fairly such as $\frac{3}{4}$ piece of cake could be resulted from three pieces of $\frac{1}{4}$ of cake.

Relation among fractions in Measuring Context

The students did a problem about marking the position of ants if ants stopped at particular part of path. Fractions with denominator 2, 3, 4, 6 and 8 were given in this problem. The problem was that

Mark the positions of each ant and write the fractions!

Riri stop at $\frac{1}{2}$ *of path*

Kiko stop at $\frac{1}{3}$ *and* $\frac{2}{3}$ *of path*

Tom stop at $\frac{1}{4}$, $\frac{2}{4}$ *and* $\frac{3}{4}$ *of path.*

Tobi stop at $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$ *and* $\frac{5}{6}$ *of path.*

Bona stop at $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$, $\frac{5}{8}$, $\frac{6}{8}$ *and* $\frac{7}{8}$ *of path.*

It was expected that the students could figure out the position of those fractions on ants' path. They could use the pieces of ribbon or determine the positions of fractions by measuring. Perceiving the magnitude of such fractions on bar or line model, the student could construct some relation among fractions such as equivalent fractions, comparison among fractions, or relations between non-unit fractions and its unit fractions.

Most the students were able to iterate unit fractions using the piece of ribbon to determine the position of each fractions on ants' path.

The First Vignette

Teacher : Where is two third?

Ary : Two third is two times this (piece of $\frac{1}{3}$ ribbon)

Connecting to their experience in partitioning object using standard unit of measurement, some students measured the length of partitions then put the fractions. There was a group of students that seemed to realize that one path also could be represented with fractions such as $\frac{4}{4}, \frac{6}{6}$, etc. It indicated that the relations between fractions and whole number (1) started to be constructed.

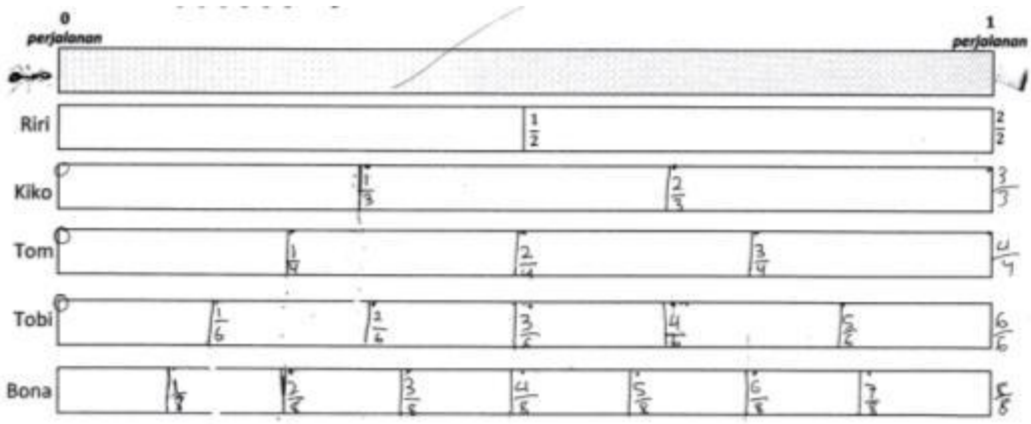


Figure 5.68 Ants' Path

Despite the students had learned about the iterations of unit fractions, there was a student who still holded on his own pre-knowledge about the relation among fractions. It was showed when he solved the problem about positioning some unit fractions ($\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{8}$) on a line.

The Second Vignette

Teacher : How do you know that the position of one third is here?

Rafi : This is a half. One third is a half of a half.

Teacher : So, one third is a half of a half?

Rafi : Yes. A quarter is a half of one third

Giving model of cake, the teacher tried to provoke Rafi to reconstruct his knowledge about relations among fraction. Rafi knew that if one cake was divided into two pieces of a half cake and those pieces were divided again into two pieces, then a half of a half cake was a quarter. For Rafi, there were different relations among fraction that could be found depended on the model used. Despite the students had built the relation among fractions using context of fair sharing, it did not mean that they could generalize into other models.

Using measuring context, the equality within magnitude led to the equivalent fractions. In the case of ants' path, the students found that equivalent fractions had same position on a number line. The distance from zero to both fractions was same.

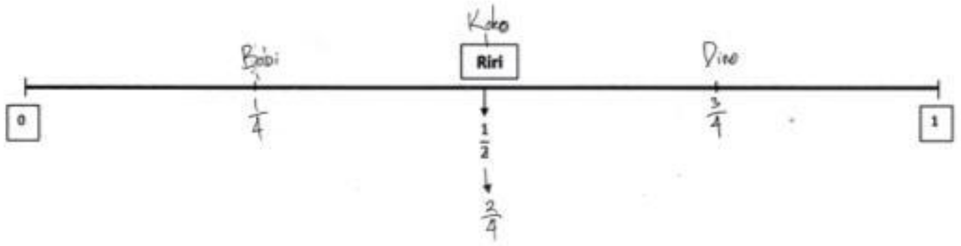


Figure 5.69 Posit Fractions on Number Line

The students seemed to have many difficulties to reveal the relations among unit fractions. Although they used pieces of ribbon to determine the position of unit fractions, the students got difficulties to connect the relations between those fractions on a number line. The student's mistake in positioning the order of unit fractions on a number line showed that there was a lack of knowledge about the meaning of fraction as a distance from zero point on a number line (Figure 5.70).

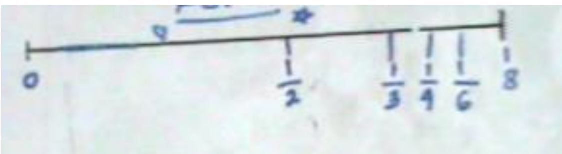


Figure 5.70 Incorrect Positions of Fractions

Some of the students could posit different unit fractions on a number line by using pieces of ribbon properly (Figure 5.71). The use of pieces of ribbon as a tool of measure might help the students to posit fractions properly but it was doubt that they really connected the positions of those fractions with the way of partitioning in determining the position of unit fractions.

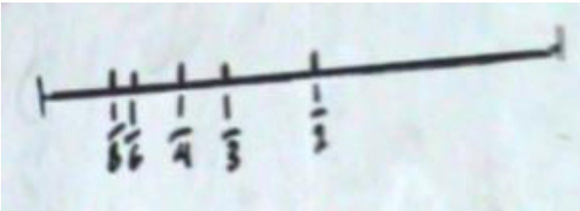


Figure 5.71 Correct Positions of Fractions

Despite positing unit fractions on a number line had not provoked the students to perceive the distance among those unit fractions more precisely, at least the student could relate the position of unit fractions with the results of comparing fractions.

<u>The Third Vignette</u>

Teacher	:	If a half is on the middle, then one eighth is on...
The students	:	Left side
Teacher	:	How about the cake? Which one is bigger, the piece of a half of cake or one eighth of cake?
The students	:	A half
Teacher	:	If it is about cake, the bigger is a half. If it is about the distance travelled, which one is farther, a half or a quarter?
The students	:	A half
Teacher	:	How about one eighth of the distance travelled?
The students	:	Nearer.
Teacher	:	Is it on the right or left side of a half?
The students	:	Left side
Teacher	:	How about one hundredth?
The students	:	Nearly zero

In the dialogue above, the teacher tried to provoke the students to connect their knowledge of comparing fractions with the more formal level of comparing fractions which was on the frame of number line.

Conclusion of learning phase 'building the relation among fractions'

Building the relation among fractions particularly by using number line model became a challenge for the students. Although they did partitioning properly on bar model, some of the students tended to ignore it when they dealt with number line model. It was shown when some students still could not posit fractions properly although they used pieces of ribbon to determine the positions of fractions. The activity of making ants' path using pieces of ribbon might hinder the students in constructing the partitions by their own reasoning. Tools used did not give more space for the students' own strategies. Although the students might commence to realize the position of fractions on number line such as one hundredth should be nearer to zero than a half, it was still doubt whether they also consider about the distance among fractions on number line that reflected the relation among fractions.

5.8 REMARKS OF THE STUDENTS' KNOWLEDGE ON THE POST-TEST IN THE SECOND CYCLE

After the teaching experiment in the second cycle, the researcher gave written tests to the students. The questions of post-test were similar to the questions of pre-test because the aim of post-test was to investigate to what extend the students' understanding of fractions. The findings on the post-test were also as supporting data to draw conclusion of the whole learning process of the students during the second cycle. All items of post-test could be found in Appendix F.

5.8.1 Representation of Fractions and The meaning of fraction

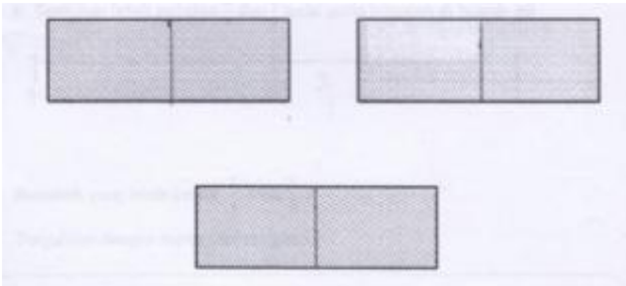
After exploring a mathematical idea that the pieces (fractional parts) do not have to be congruent to be equivalent, one of post-test item (item number 3) was given to investigate whether the student could grasp such a mathematical idea. A contextual situation was about different way of partitioning a cake in which Nia argued that one piece was bigger than another piece (Figure 5.72).

Figure 5.72 Different Ways in Partitioning

More than a half of the students argued that both pieces were equivalent because both pieces represented a half. Even, there were some students who explained both were equivalent although the ways of cutting were different. However, there were still 6 students who seemed to be confused because they had double answers. Both pieces were equivalent if they perceived the fraction but both pieces could be different if they looked at the shapes. None of the students judged absolutely that one piece was bigger than the other. The students' answer in this item indicated that most the students grasped a mathematical idea that the pieces do not have to be congruent to be equivalent.

5.8.2 Awareness of a Whole of in Comparing Fraction

On the post test (item number 5), the problem about dividing 3 cakes among 6 children was given. The students were asked to notate fractions with different kind of a whole (one cake or three cakes). Some of the students could distinguish the whole in notating fractions (Figure 5.73).



Setiap anak memperoleh......$\frac{1}{2}$......bagian dari 1 buah kue.

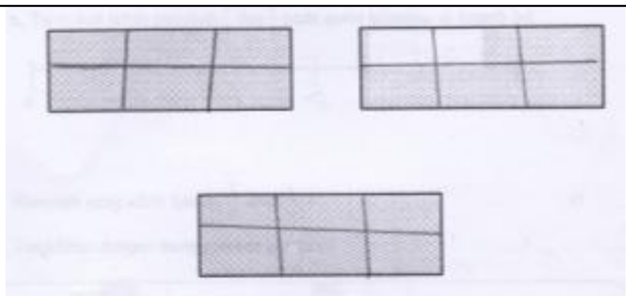
Setiap anak memperoleh......$\frac{1}{6}$......bagian dari 3 buah kue.

Each child gets $\frac{1}{2}$ part of 1 cake

Each child gets $\frac{1}{6}$ part of 3 cakes

Figure 5.73 Different Notations of Fractions

In fact, the students had difficulties when the students had more various ways of partitioning. Considering one cake or all cakes distracted the students in notating parts that each person got. When the parts that each person got were separated into three cakes, the students only perceived parts in one cake.



Setiap anak memperoleh....$\frac{1}{6}$.......bagian dari 1 buah kue.

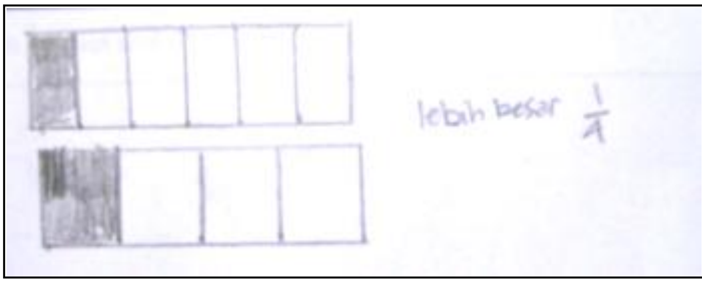
Setiap anak memperoleh....$\frac{1}{18}$......bagian dari 3 buah kue.

Each child gets $\frac{1}{6}$ part of 1 cake

Each child gets $\frac{1}{18}$ part of 3 cakes

Figure 5.74 Incorrect Answers in Representing the Results of Sharing

In exploring the relation among fractions through comparing fractions, the students showed the awareness of a whole in comparing fractions. In each item of post-test which asked the students in comparing fractions (item number 2, 7 and 8), more than a half of the students could make proper drawing to show their answer. The size of both shapes to be compared was almost exactly in the same size (Figure 5.75).



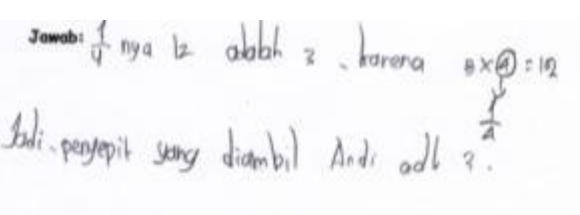
$\frac{1}{4}$ is greater

Figure 5.75 Same Size of Objects to be Compared

Based on the students' answers and pictorial representation that they made in comparing fractions, the students seemed to be more aware about the whole in comparing fractions. It indicated that the students grasped the idea that the whole must be same in comparing fractions

5.8.3 Discrete Objects in Learning Fractions

Using a stack of chocolate bar in front of class during the teaching experiment, the students came to an idea that the meaning of fraction as a quotient could help them to determine the number of discrete objects that represented a certain fraction. Particularly, the problem involved unit fractions was interpreted by the students as division a number of candies by the denominator of the fraction. One of post-test item (item number 4) asked the students to determine a quarter of 12 clips. In this item of post-test, 12 students or almost a half o the students could answer correctly.



Answer: $\frac{1}{4}$ of 12 is 3, because

3 x 4 = 12

$\frac{1}{4}$

So, clip which is taken by Andi is 3

Figure 5.76 An Example of Correct Answers in Solving Problem Involving Discrete Objects

About non-unit fractions, the problem (item number 9) about determining the number of discrete objects that represent a certain non-unit fractions required more knowledge of the students about the relation among fractions. In the teaching experiment, the researcher did not provide a problem using discrete objects that involved non-unit fractions. In fact, more than a half of the students had built the relation among fractions $\frac{1}{4}$ and $\frac{3}{4}$ so that they could find a number of objects that represented $\frac{3}{4}$ of 20 objects if a quarter of 20 objects was known.

The number of students who could give correct answer to the problem involving discrete objects was increased significantly than in the pre-test. It showed that the students understanding of the meaning of fraction extend to another kind of representation, discrete objects.

5.8.4 Fractions on Number Line

Developing measuring activity about ants' path, the students were expected to construct the meaning of fraction as a distance on number line in order to support them to build the relation among fractions. In fact, it did not always bring a significant progress to the students' understanding. In post-test, the researcher found that less than a half of the students could posit fractions $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{3}$ and $\frac{2}{3}$ properly (Figure 5.77).

Figure 5.77 Correct Position of Fractions on Number Line

Other students seemed only to reveal particular relations but had difficulties with other relations for instance the relations between $\frac{1}{2}$ and $\frac{2}{4}$ (Figure 5.78).

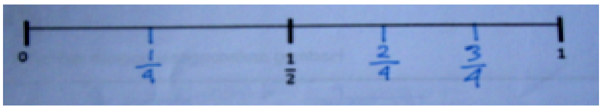


Figure 5.78 The Student Revealed the Relation among fractions

The answer of the student above gives an illustration about how the student could build relation among fractions but they still had some difficulties. The student might be able to build relation among fractions which had same denominator such as $\frac{1}{3}, \frac{2}{3}$ but they have not yet expanded the relations among different fractions such as $\frac{2}{4}$ and $\frac{1}{2}$.

5.9 DISCUSSION

In the present study, the researcher focused on three main perspectives in conducting the research that are conducting a particular type of research: design research, adapting a specific approach to mathematics education: RME, and confined research topic: the understanding of the meaning of fraction and relation among fractions. Through this discussion, the researcher elaborates such perspectives by looking back to the theory on learning fractions or comparing the findings of this research and other prior study in this field.

5.9.1 The Understanding of The meaning of fraction and Relation among fractions

The main theme of this research is that supporting students to extend their understanding of the meaning of fraction and relation among fractions. It was considered that the students needed to extend their understanding because in fact there were gaps in their actual understanding. It was found when the students had no clue to determine $\frac{1}{4}$ of 20 candies although the students were used to represent $\frac{1}{4}$ as 1 shaded part out of 4 parts. Meanwhile, understanding fractions requires a coordination of many different but interconnected ideas and interpretations

(Lamon, 2005). This research offered a sequence of activities that emphazised on such coordination across various interpretations. Combining continuous and discrete objects to be partitioned, the meaning of fraction as part of a whole and quotient tried to be developed simultaneously.

Understanding of fractions also refers to the ability in building interrelation among various modes of external representations that involve combination of written and spoken symbols, manipulatives, pictures and real world situations (Behr, Lesh, Post & Silver, 1983 in Anderson & Wong, 2007). Involving the students who had learned about fractions in this research, the instructional design offered the students not to ignore their previous experiences but to build interrelations between various modes of external representations. For instance, the students compared fractions by using cross multiply algorithm and it could not be just ignored. The students just need to be introduced to other representations such as a drawing of it or real world reasoning using fair sharing context. Streefland (1991) and Lamon (2001, in Anderson & Wong, 2007) argued that children's understanding of fractions is greatly developed by their own representation of fraction ideas including pictorial, symbolic, and spoken representations to clarify their thinking. In this research, by making pictorial representation in comparing fractions, the students' understanding in comparing fractions extend to the awaraness that the whole must be same to compare fractions.

Besides comparing fractions, the relation among fractions were also emphasized when the students were engaged to put fractions on number line. Through the activity of determining the position of ant on the path, informal number line was to be introduced. The students first should grasp the meaning of fraction as a measure. Furthermore, relation between non-unit fractions and its unit fractions, which non-unit fractions as an iteration of unit fraction, was conveyed in such an activity. In fact, athough the students could solve ants' path problem, not all the students could connect such knowledge to the problem in which they have to put fractions on formal number line. Such a case might be caused that the students' understanding of the meaning of fraction as a measure was still under construction. Larson (1980) and Lek (1992) in Keijzer 2003 argued that when students have not grasped the meaning of fraction, the use of number line becomes problematic. In this research, although the students had to be given opportunities to learn the meaning of fraction as a measure, it seemed that the activity should be more explored before the students came to formal number line.

5.9.2 Fair Sharing and Measuring as Contextual Situations for Learning Fractions

Phenomenalogical exploration is one of characteristics in designing an instructional sequence based on realistic mathematics perspective. Bakker (2004) explained further that through the mathematical concepts embedded in rich and meaningful phenomena, it could be the basis for children to build concepts formations. In this research, the students were supported to extend their understanding of the meaning of fraction through fair sharing and measuring phenomena. To build concept formations, the students were encouraged to connect their prior knowledge of fractions as part of a whole to other situations that conveyed more specific relations of part of a whole. For instance, measuring activity provoked the students to partition in particular way than partition in fair sharing. Fraction $\frac{1}{3}$ in measuring activity refers to a part of distance travelled started from the point of departure rather than any one part out of three parts as a whole. Fair sharing also became a rich phenomenon to be used as a base of concepts formations. Many mathematical ideas could be conveyed when the students were engaged to partition, distribute and notate the results with fractions, for instance, the pieces do not have to be congruent to be equivalent or the awareness of a whole while notating fractions.

CHAPTER VI

CONCLUSION

In this chapter, the researcher will conclude the whole process of doing this research by answering the research question and posing a local instructional theory on learning fractions. In the next sub-chapter, some specific topics in this research will be discussed. A reflection from the researcher and some recommendations for teaching, research and design also will be explicated.

6.1 ANSWER TO THE RESEARCH QUESTION

The research data presented in Chapter 5 provide a direct answer to the research question. The research question of this research is

How to support the students to extend their understanding of the meaning of fraction and relation among fractions through fair sharing and measuring activities?

Investigating the pre-knowledge of the students in this research who had learned about fractions, the researcher found that the learning process of the meaning of fraction was more focused on part-whole relationship. Such a meaning of fraction was represented with a number of shaded parts in geometrical shapes like circle, rectangle, and square. The students learned about representation of fractions without using real context. The students also had learned about comparing fractions by using cross-multiply algorithm. Instructional activities in this research provide contextual situations that were expected to give more opportunity to the students to extend their understanding of the meaning of fraction and relation among fractions.

The answer of the research question will be generated from summarizing the retrospective analysis in the chapter 5.

6.1.1 Partitioning and representing the results of fair sharing from one object

Prior this research, the students learned fractions in limited representations. This was evident through students' mathematics textbooks and students' notebook (*see figure 5.33).* It was found that fractions were merely represented with a number of shaded parts out of total parts in geometrical shapes such as circles, squares or rectangles. Sometimes, the students were given a geometrical shape that had been partitioned and shaded so that they were just expected to write its fraction. Related to the meaning of fairness, it was found in the pre-test that there were some students who justified the fairness according to the shapes instead of its fractions *(see subchapter 5.6.1)*. Although both pieces were a quarter, the students said that one piece is bigger than the other because the shapes were different.

Supporting the students to construct other meaning of fraction, a contextual situation about sharing a cake among 4 children fairly was given. The students had to do real partitioning by cutting a model of cake. They were expected to be able to justify whether the results of sharing were fair. Furthermore, the students were given a conflict about justifying the fairness, if the pieces were not congruent.

All the students could partition the model of cake into 4 pieces equally by using different strategies such as measuring, folding and estimating. Most the students also could justify the fairness of results of sharing although there were different strategies of cutting among them *(see the first vignette in subchapter 5.7.1)*. Challenging the students to extend their understanding of meaning of fairness, a conflict about different pieces got by Rafa and Rafi was given to the students. Actually both pieces were a quarter but the shapes were different. In class discussion, most the students could justify that it was fair because both pieces were a quarter *(see the second and the third vignette in subchapter 5.7.1).* The students' answer in post-test also showed that they were able to justify the fairness based on the fractions instead of the shapes (*see subchapter 5.8.1*). It showed that most of the students could grasp a mathematical idea that the results of fair sharing do not have to be congruent to be equivalent.

6.1.2 Comparing the results of fair sharing in case the number of sharers increases

The cross-multiply algorithm was dominant to be a way of giving reasons when the students had to compare fractions. The pictorial representations seemed not to be a part of reasoning. In the pre-test, the researcher found that the students were not aware that the wholes must be same to compare fractions. The students often made different size of figures in comparing two fractions (*see subchapter 5.6.2)*

Supporting the students to have different kinds of reasoning in comparing fractions, the students were given a fair sharing problem in which the number of sharers was increased. After dividing a chocolate bar into different equal pieces, the students had to notate the results of dividing by fractions and give a conclusion about what happen if the number of sharers was increased. Provoking the students' thinking to be aware that the whole must be same in comparing fractions, a conflict was presented, namely $\frac{1}{4}$ was not always smaller than $\frac{1}{3}$ if the whole of objects to be compared was different.

Most of the students could partition chocolate bars into different equal pieces although some of them did partitioning by estimating instead of measuring. Most students also could represent the results of dividing by unit fractions. (*see subchapter 5.7.1*).There were only few students who had difficulties in notating unit fractions. Interviewing one of those students, the researcher found that the students had difficulties in notating fractions when shaded parts did not exist (*see the sixth vignette in subchapter 5.7.1*). It seemed that the student has not constructed the meaning of fraction as quotient. Giving a conclusion of increasing the number of sharers, the students came to different level of conclusion. An example of students' informal conclusion was that the more an object is divided, the smaller the pieces are resulted. More formal conclusion emerged was that the larger the denominator, the smaller the value of fractions. In comparing fractions, the mathematical idea that the wholes must be same also was emphasized by giving the students a conflict in which the size of objects was not same when comparing $\frac{1}{3}$ and $\frac{1}{4}$ piece of a cake. At that moment, the students started to be more aware that the wholes must be same to compare fractions (*see the fourth and the eighth vignette in subchapter 5.7.1*).

6.1.3 Determining the number of part of objects collection (discrete objects)

In their prior learning fractions, the students had not learned about fractions as parts of objects collection (discrete objects). The meaning of fraction as a part-whole relationship also could not support them to determine a number of discrete objects that represented a certain unit fraction. Most students could not answer correctly a problem involving discrete objects in the pre-test (*see subchapter 5.6.3*).

To support the students to have better understanding about the meaning of fraction related to discrete objects, some problems about determining the number of chocolate bars that represented a certain unit fraction were given. A class discussion before doing the problems was conducted by the teacher to provoke the students' thinking in finding a strategy to solve the problems.

Providing a stack of chocolate bars in front of class, the teacher orchestrated a class discussion about finding a strategy in solving problems. As the results of discussion, the students seemed to construct the meaning of fraction as quotient (*see the first vignette in subchapter 5.7.2*). Most the students applied the strategy of division in which the number of chocolate bars was divided with a number as the denominator in a unit fraction *(see figure 5.55 and subchapter 5.8.3)*.

6.1.4 Partitioning and representing the results of sharing more than one object fairly

Investigating the previous learning process of the students by interviewing the teacher, it was found that the students had no experiences in partitioning more than one object during classroom learning process (*see subchapter 5.6.2*).

Meanwhile, fair sharing involved more than one object required ability in coordinating the number of sharers and the number of partitions. The students should be given more challenging situations to extend their understanding of notating fractions.

To challenge the students to extend their previous understanding in partitioning one object, a problem about sharing 3 cakes among 4 children fairly was given to the students. Model of cakes were given to the students then they were asked to make lines of partitioning and notate how much each person gets.

Working on this problem, some students could not directly determine partitions that they should make. Some groups of students did trial and error before they could find the strategy of dividing cakes in which each person got equal parts. The differences in partitioning elicited different representations of fraction notations (*see Figure 5.57).* Different whole to be perceived also became a critical issue to be discussed such as some students answered $\frac{3}{4}$ and the other got $\frac{3}{12}$ as the answer. Although the class discussion was not conducted optimally, asking the students to explain their answer, most of the students had a reason for fractions notation that they chose (*see subchapter 5.7.2*).

6.1.5 Determining the position of unit fractions through measuring the distance travelled

Representing fractions as a number of shaded parts out of total parts in a geometrical shapes was dominant in the previous learning process of the students. Based on the interview with the teacher, although the students were introduced to number line, there were only few students who could posit fractions properly (*see* *subchapter 5.6.4*). There was no contextual situation as the starting point of learning fractions related to number line.

Supporting the students to learn fractions related to number line, the meaning of fraction in measuring activity was introduced. Using a context about ants which walked to a pile of sugar, the students had to find ant's position if it stopped at a certain part of the path. As the starting point, the position of ant was represented with unit fractions. Using this context, the meaning of fraction as a distance from zero point on informal number line (bar) became the focus of the learning process.

In determining the position of ants that travelled along a certain part of distance, the students were asked to shade part of the distance that has been passed by the ant. The end of shaded part became the position of the ant. More than a half of all students could shade part of the path properly. They used a strategy of estimating, partitioning by four or using standard units of measurement *(see Figure 5.58 & 5.59)*. Only few of them drew ant's position after travelled a certain part of path but most of the students could compare properly which unit fractions represented the farthest distance travelled by ants (*see subchapter 5.7.3*). It seemed that the students grasped the meaning of unit fractions as a distance from initial point by using informal number line.

6.1.6 Iterating the magnitude of unit fractions to produce non-unit fractions on bar model

In the previous learning, the students have not learned explicitly about relation between non-unit fractions and unit fractions. Although the students might know that 3 pieces of $\frac{1}{4}$ cake were equal with $\frac{3}{4}$ of cake, they seemed not to consider about such a relation when they encounter fractions on a number line (*see subchapter 5.6.4*).

Supporting the students to learn fractions in a more formal way, the meaning of unit fractions in the context of measuring was elaborated to non-unit fractions. The problem about ants' path was given to the students. Location where ants stopped on the path was given then the students had to determine fractions that represented such a location. Different pieces of ribbon that represented certain unit fractions were used as a tool of solving the problem.

Most of the students could find how many pieces of a certain unit fraction that fit to the location of ants (*see Figure 5.65)*. In fact, only few of them could conclude directly the non-unit fraction when they used a certain number of pieces of ribbon that represented a unit fraction. Although the students did iterations of unit fractions, they had struggled in determining a non-unit fraction as an iteration of unit fractions (*see the third vignette subchapter 5.7.3).* Guided by the teacher, the students finally were able to consider that non-unit fractions could be constructed from iterations of unit fractions *(see the fifth vignette subchapter 5.7.3 and the first vignette subchapter 5.7.4)*. In fact, such understanding of relation among fractions has not supported the students optimally. It was found that half of the students still had difficulties when they came up with the position of fractions on formal number line (*see subchapter 5.8.4).*

6.2 LOCAL INSTRUCTIONAL THEORY ON EARLY FRACTION LEARNING

One of the aims of this design research was to develop a local instructional theory on early fraction learning in grade 3. According to Gravemeijer (2006), a local instructional theory consists of conjectures about a possible learning process and possible means of supporting that learning process. Such supporting means include instructional activities, classroom culture and the proactive role of the teacher. Considering the fact, that the students involved in this design research had previous learning of fractions in the classroom, some possible discourses that was intended in the initial hypothetical learning trajectory was refined depend on the pre-knowledge of the students about fractions. The following table summarized the role of tool and the contextual activity that were proposed in the instructional design

Table 6.1 Local Instructional Theory on Early fraction learning in Grade 3

Tool	Imagery	Activity	Potential Mathematics Discourse Topic
Model of a Fruit Cake	• Part of a Whole • Quotient	Sharing a Cake among 4 Children Fairly	• Fair Sharing • Fractions Notation
Figure of Chocolate Bars	Quotient	Dividing Chocolate Bars into 2,3,4,6 and 12 Equal Pieces	Comparing Fractions
Stack of Chocolate Bars	Partitioning or Grouping Discrete Objects	Determining a Number of Chocolate Bars Represented a Certain Unit Fractions	Meaning Unit Fractions related to Discrete Objects
Model of Brownies	• Part of a Whole	Dividing 3 Brownies Cake	• Strategy of Partitioning,

Tool	Imagery	Activity	Potential Mathematics Discourse Topic
Cakes	• Non Unit Fractions as the Addition of Its Unit Fractions	among 4 Children	Fractions Notations • Different Whole of Fractions
Figure of Ant's Path	• Part of a Whole • Quotient • Distance of Unit Fractions from Zero Point	Determining the Position of Ant that Passed $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and , $\frac{1}{8}$	• Strategy of Partitioning • Comparing Fraction on (Informal) Number Line
Pieces of Ribbon, Figure of Ants' Path	• Distance of Non Unit Fractions from Zero Point • Non Unit Fractions as the Addition of Its Unit Fractions	• Figure out Non-Unit Fraction as the Position of Ant • Making Ants' Path	Non-Unit Fractions as Iteration of Unit Fractions
Number Line	• Distance of Unit Fractions and Non Unit Fractions from Zero Point	Posit Different Unit Fractions and Non-Unit Fractions on a Number Line	• Non-Unit Fractions as Iterations of Unit Fractions • Relation among fractions

Potential mathematics discourse topic in this instructional design was supported by the role of teacher in orchestrating class discussion, provoking group discussion and establishing the desired classroom culture. In this teaching experiment, there were several roles of teacher that are explicated as the following

6.2.1 Asking for Clarification

The students' answers were not always quite clear to be interpreted whether they still had difficulties or already grasped the mathematical ideas. At that moment, the teacher needed to ask for clarifications to the students. The following dialogue shows an example of revealing the student' thought through asking for clarification.

Teacher	:	How about the results of dividing in this group? Why is it fair?
Dafi	:	Because its shape and size is same.

From the answer of the students, the teacher could conclude that the students justified the fairness of results of sharing based on the congruency of pieces. It gave more information to the teacher to support the students. In fact, the pieces do not have to be congruent to be equivalent in fair sharing. A case in which the results were fair although the shapes were different then was given to the students.

6.2.2 Posing Scaffolding Questions

One of the characteristics of teachers' role in realistic mathematics perspective is teachers as facilitator (Hadi, 2005). During class discussion or group discussion, the teacher gave scaffolding questions to strengthen the students' strategies or guide the students to generalize their strategies. One of examples from class discussion was shown in the following dialogue.

Rafi	:	Eight is subtracted by two is four, then subtracted again by two getting two
Rama	:	Eight subtracted by two is six
Teacher	:	Wait, I think I know your way. Rafi do it by subtracting, subtracting, subtracting. So, first you subtracted eight by two getting....
Rafi & Students	:	Six
Teacher	:	then....

Rafi : Four
Teacher : How many times do you subtract by two?
Rafi : Four

In the dialogue above, the teacher help the students to focus on generalizing Rafi's strategy on partitioning a collection of discrete objects in order to determine a quarter of it. By posing questions, the teacher tried to scaffold the students to generalize a strategy of repeated subtractions to a strategy of division.

6.2.3 Stimulating Social Interactions

According to Hadi (2005), teacher should not transfer mathematical concepts but provide learning experiences that stimulates students' activity. To stimulate students' activity, the teacher's role is conducting interactive teaching and learning process. Posing some questions during class discussion was very important to establish social interactions in which classroom became a learning community. During teaching experiment, sometimes the teacher posed these following questions.

- *Is there another opinion?*

  This question was useful to stimulate the students in sharing their own ideas. Not only were different strategies possible to appear but also different language of students in presenting their ideas.

- *Could you show it, Rafi?*

  During class discussion, this question was used to encourage the students to present their ideas and communicate their strategies to the others.

6.2.4 Eliciting the Mathematical Idea

As a facilitator (Hadi, 2005), teachers also have a role to elicit mathematical ideas that are important for students to grasp. For instance, one of mathematical ideas in the instructional design was about the meaning of fraction as part of a whole. Through fair sharing activity, the students were expected to learn about such a meaning of fraction. The strategy of partitioning should bring the students to interpret their fractions notation.

Teacher	:	How do you know that it is three fourth?
Tyan	:	Ryan, Ryan, Ryan, Dea, Dea, Dea, Abror, Abror, Abror, Ocha, Ocha, Ocha
Teacher	:	Ooo..three. How about this four (pointing to 4 in $\frac{3}{4}$)?
Tyan	:	For example, Ryan (pointing to three pieces named Ryan) gets three. Each cake is four pieces, right? So, three fourth

By asking the meaning of numbers in a fractions notation, the teacher helped the students to construct the meaning of fraction as part of a whole. The students realized that $\frac{3}{4}$ was interpreted as 3 out of 4 pieces.

6.3 REFLECTION

During conducting this research, there were many issues to be reflected by the researcher. Such issues were about students' pre-knowledge and students' own model, role of the researcher as the teacher, and class management.

6.3.1 Students' Pre-knowledge and Students' Own Model

In designing the hypothetical learning trajectory, the researcher started from the perspective of realistic approach and theories about learning fractions. All activities were designed in order to give more spaces for students in progressing their understanding about fractions. Contextual problems that might invite an open discussion were tested to the small group of students in the first cycle. In fact, we have never really known what students might think and which level of understanding that they have achieved. Surprising moments happened when for some activities convinced to have a chance to support the students, it did not really happen. From such experiences in the first cycle, the researcher learned to observe carefully students' pre-knolwedge for the second cycle. Indeed, the more information about students' pre-knowledge is derived, the more chances the instructional design supports students' learning process.

Refining and implementing the hypothetical learning trajectory in the classroom, the researcher realized the importance of contextual situations and problems that could provoke students' own model to emerge. Besides rectangle model that emerged from fair sharing situation, formal number line also was introduced. In fact, formal number line in this research has not been developed by students themselves. When the model and interpretation were only provided by the teacher, the students did not take it as their own productions and constructed knowledge. It might not support the students to extend their understanding optimally.

6.3.2 Support of Activities

Delivering context of fair sharing and measuring activity, the researcher found that there was still a gap between the students' understanding of the meaning of fraction as part of a whole or quotients and fractions as a distance on (informal) number line. Although the students could posit fractions on bar model (ants' path) by partitioning and consider the size of partitions, such kind of awareness did not appear when they dealt with the position of fractions on number line. From that case, the researcher realized that there was still a gap within constructing the meaning of fraction as a distance on bar model and number line model. The students needed more activities to support their skill of partitioning and understanding of the meaning of fraction as a distance between they make a transition to apply such a meaning into formal number line.

6.3.3 Role of the Researcher as the Teacher

Taking a role as the teacher in the teaching experiment had given valuable experiences to the researcher as an un-experienced teacher in the primary school. Teaching in the classroom was about making decision on how to support the students to be able to learn not merely could do the task. By giving open questions, the teacher could support different level of understanding of students to appear. In fact, because of the lack of skill in class management, the researcher found difficult to orchestrate classroom discussion. Open questions were more often posed during group discussion. Instructional activities using realistic approach actually required the skill of the teacher in orchestrating class discussion so that individual or group learning process could contribute to the whole learning process in the classroom.

6.4 RECOMMENDATIONS FOR TEACHING, DESIGN, AND RESEARCH

Conducting teaching and learning process using realistic mathematics approach was shown to be an answer of how to support students to extend their understanding. Adapting five tenets of realistic mathematics education, teachers provided learning experiences for students to construct their knowledge. This design research was recommended to give an overview for teachers or designers about instructional sequences in learning fractions and students' learning process corresponded to such instructional sequences. Unfortunately, learning styles of the students have not been considered in this research. For the future researches, it is recommended to observe the learning styles of the students before doing the research. It might provide a new insight about suitable instructional designs and how students learn through such instructional designs.

Zooming into the researcher's experiences during this study, the researcher also poses some recommendations for teaching, design and research particularly on fractions domain as the followings

6.4.1 The use of measuring activity in eliciting the meaning of fraction

Constructing meaning of fraction through context of measuring was found in this research to be a challenge for some students. Bar model used as a path in the problem of ant's position could provoke the students to combine their knowledge of partitioning with the use of standard unit measurement. Beside the advantage of using bar model that was similar with geometrical shape that the students often used, this model also had a chance to distract the students to think about distance because it had an area. The researcher recommends teachers and designers to give opportunity first to students in using their own model in representing context of ants. The students might choose model that represents their image of distance.

The use of pieces of ribbon in learning mathematical ideas about non-unit fractions could be iteration of unit fractions also gave too much inteventions to the students’ strategy. The activities could be continued with more challenging task in which the students have to estimate a certain fractions by using standard unit of measurement or folded ribbon by themselves. Such a challenge was aimed to stimulate student’ own productions in learning such a meaning of fraction.

6.4.2 Exploring the relation among fractions

The last learning phase in the present research is building the relationship around fractions. Such a learning phase was developed after the students explored the meaning of fraction in fair sharing and measuring activity. The relation among fractions were expected to connect with the position of fractions on number line. In fact, the students have not explored the meaning of fraction in measuring activity sufficiently. As the consequence, instead of exploring the relation among fractions related to the positions on number line, the students still had struggles in positioning fractions. Such a case leads the researcher to recommend teachers and designer not too fast to bring the student in exploring the relations amomg fractions formally. Informal relation among fractions could be developed during the learning process of the meaning of fraction such as exploring the results of fair sharing deeper, for instance the relations among pieces $\frac{1}{4}, \frac{3}{4}$ and $\frac{1}{2}$ of cake.

6.4.3 Establishing class norm

In the present research, the researcher takes a role as the teacher. As an unexperinced teacher, establishing class norm becomes the main issue related in conducting class discussions. The teacher could not stimulate well-constructed class norm so that it made some noisy in this research. The teacher had to skip some class discussion because some problems among the students happened during the learning process. Although such a challenge gives more insight to the complexity of teaching, it also inhibits the process of understanding how the instructional activities could stimulate a fruitful class discussion. Either the teacher is an experienced teacher or unexperienced teacher, the researcher recommends for future researches to be well-prepared in establishing class norm that is condusive for learning process and anticipating kind of situation in which class discussion could not conducted optimally. If the teacher is not the real teacher of the classroom, intensive observations and personal approaches to the students are very important.

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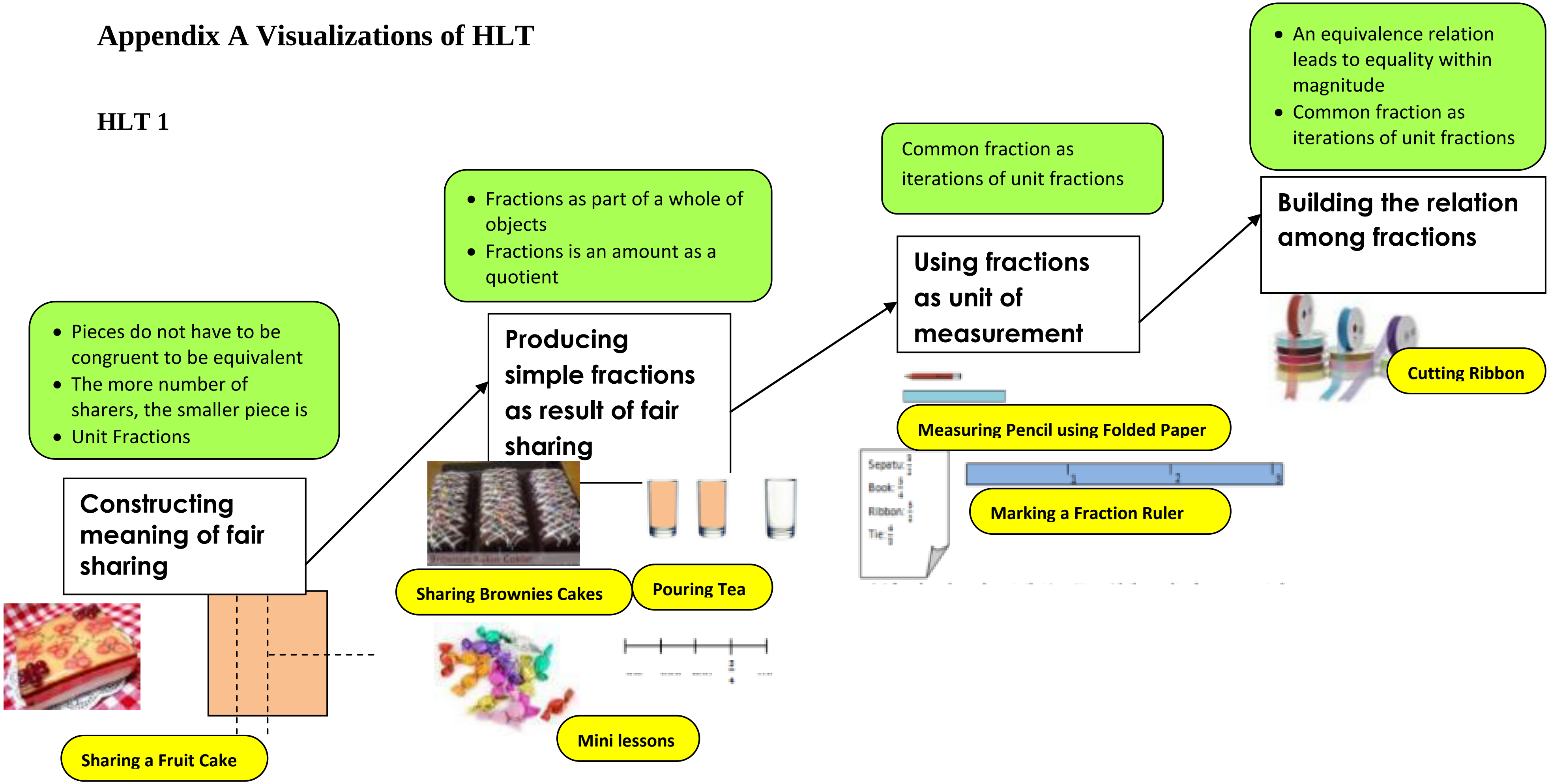
Appendices

Appendix A Visualizations of HLT

HLT 1



HLT 2

- Pieces do not have to be congruent to be equivalent
- The more number of sharers, the smaller piece is
- Unit Fractions

Constructing meaning of fair sharing

Sharing a Fruit Cake

- Fractions as part of a whole of objects
- Fractions is an amount as a quotient

Producing simple fractions as result of fair sharing

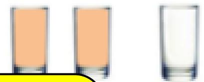
Sharing Brownies Cakes

Mini lessons: Candies

Common fraction as iterations of unit fractions

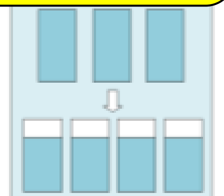
Using fractions as unit of measurement

Measuring Pencil using Folded Paper



Pouring Tea

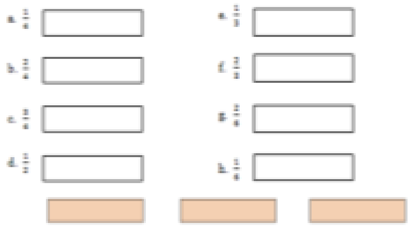
Marking a Fraction Ruler

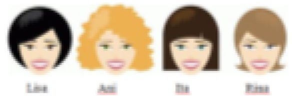


Making Poster of Pouring Water

- An equivalence relation leads to equality within magnitude
- Common fraction as iterations of unit fractions

Building the relation among fractions

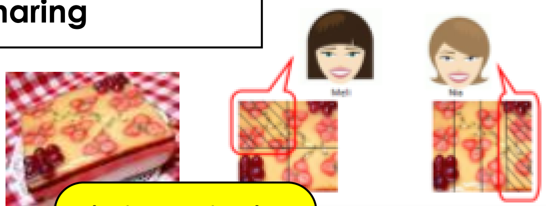




Shading Parts and Finding the Relation among fractions

- Pieces do not have to be congruent to be equivalent
- The more number of sharers, the smaller piece is
- Unit Fractions

Constructing meaning of fair sharing



Sharing a Fruit Cake

Increasing the Number of Pieces

- Fractions as part of a whole of objects
- Fractions is an amount as a quotient

Producing simple fractions as result of fair sharing

Determining the Number of Chocolate

Sharing Brownies Cakes

Common fraction as iterations of unit fractions

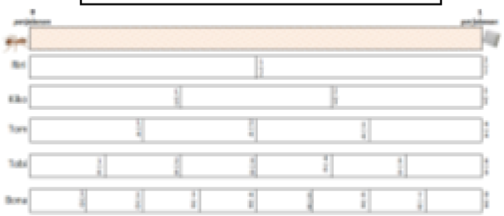
Using fractions as unit of measurement

Posit an Ant

Determining Position of an Ant using Pieces

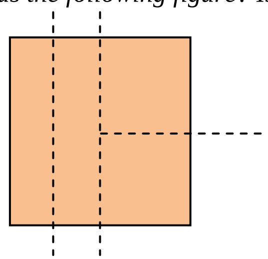
- An equivalence relation leads to equality within magnitude
- Common fraction as iterations of unit fractions

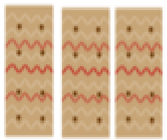
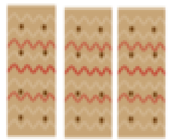
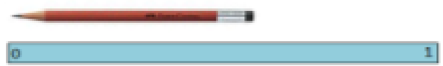
Building the relation among fractions

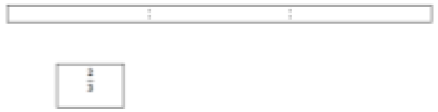


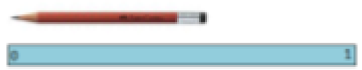
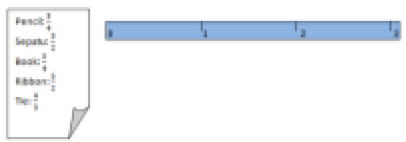
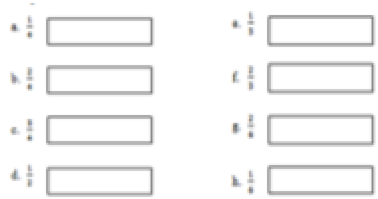
Making Path of Ants

Appendix B HLT 2 as the Refinement of HLT 1

Lesson	Activity of HLT 1	Refinement of Activity (HLT 2)	Rationale behind the Refinement
1	*Constructing Meaning of Fair Sharing* **<u>Activity 1</u>:** Sharing a Fruit Cake 1. *Mother made a fruit cake to share with her neighbours. Could you help mother to divide the cake into 4 equal pieces?* 2. *How is your opinion if mother cuts the cake as the following figure? Is it still fair?*  3. *Mother also wants to share another fruit cake for her daughter's friends. They are five children. Could you help mother again? Then compare with the* pieces of a cake for 4 people. How much each person gets?	*Constructing Meaning of Fair Sharing* **<u>Activity 1</u>:** Sharing a Fruit Cake 1. *Mother made a fruit cake to share with her neighbours. Could you help mother to divide the cake into 4 equal pieces?* 2. *Mother also wants to share another fruit cake for her daughter's friends. They are five children. Could you help mother again? Then compare with the pieces of a cake for 4 people. How much each person gets?*	The researcher skipped the second question because there were two students who had different opinion about fair sharing. Pieces did not have to be same depend on whom the pieces of cake would be given. The students said that it would be fair if the small pieces of cake were given to thinner neighbour. The researcher decided that there was a need of reconstruction of problem.
2	*Producing Simple Fractions as Result of Fair Sharing* **<u>Activity 2</u>:** Sharing Brownies Cakes *If we only have 3 brownies cakes, how to*	*Producing Simple Fractions as Result of Fair Sharing* **<u>Activity 2</u>:** Sharing Brownies Cakes *If we only have 3 brownies cakes, how to shares it among 4 people? How much will each person get?*	There was no change in Activity 2

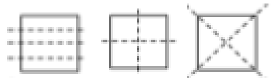
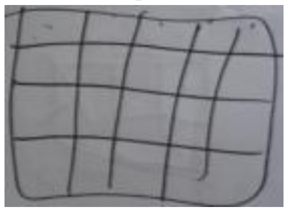
Lesson	Activity of HLT 1	Refinement of Activity (HLT 2)	Rationale behind the Refinement
	shares it among 4 people? How much will each person get? 		
	<u>Mini lesson</u>: Determining the Number of Candies *There are 24 candies. How many candies if we want to take $\frac{1}{2}$, $\frac{1}{4}$ or $\frac{3}{4}$ of 24 candies?*	**<u>Mini lesson</u>:** Determining the Number of Candies *There are 12 candies. How many candies if we want to take $\frac{1}{4}$ of 12 candies? How about $\frac{2}{4}$ and $\frac{3}{4}$ of those candies?*	The researcher changed the number of candies to simplify the problem.
3	**<u>Activity 3:</u>** Pouring Tea *There are 2 glassed full of tea and almost overflow. If there is one glass more to accommodate so that those become 3 glasses of tea, how do you predict the height of tea in the glasses?*	*Using Fractions as Unit of Measurement* **<u>Activity 3</u>:** Measuring Pencil Question 1: *Using folded paper, find the length of given pencils*!  Question 2: *Sort the pencils according to the length of pencils!*	The order of activity was changed. Activity 3 supposed to be Pouring Tea was exchanged with Measuring Pencil. It was because of the students' difficulties of partitioning in Activity 2. Through measuring pencil, the researcher expected that there were more support for their abilities in partitioning. The researcher also replaced Mini lesson: Fill Simple Fractions in a Number Line with Mini lesson: Marking $\frac{3}{4}$ of Glass. It was for bridging between the context of measuring pencil and pouring tea.

Lesson	Activity of HLT 1	Refinement of Activity (HLT 2)	Rationale behind the Refinement
		Question 3:  *Using a bar above, how long is the pencil?* Question 4: *Draw the length of pencil according to its length!* 	
	<u>Mini lesson</u>: Fill Simple Fractions in a Number Line  *Fill the blank space with appropriate fractions! How about* $\frac{1}{3}$ *and* $\frac{2}{3}$, *where it is?*	**<u>Mini lesson</u>:** Marking $\frac{3}{4}$ of Glass *If I want to pour water into* $\frac{3}{4}$ *of this glass, how high is it?*	
4	*Using Fractions as Unit of Measurement* **<u>Activity 4</u>:** Measuring Pencil *Using folded paper, find the length of your pencil! Compare with your friend!*	**<u>Activity 4</u>:** Pouring Tea *There is one glass full of water and one empty glass. If I pour water from the glass which is full of water to the empty one but both glasses must be equal, how*	

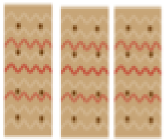
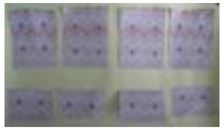
Lesson	Activity of HLT 1	Refinement of Activity (HLT 2)	Rationale behind the Refinement
		*high is it?*	
5	**<u>Activity 5</u>:** Marking Fractions Ruler  *Ani found a ruler and a note that is written with the results of measurement of some objects. She is wondering how long the object is. Could you help Ani to figure out the length of objects using the ruler?*	**<u>Mini lesson</u>:** Marking Fractions Ruler  *Find the position of a half, $\frac{1}{4}$ and $\frac{3}{4}$!*	Improper fractions were removed from Activity 5 because the researcher found that the students had not constructed the understanding of proper fractions as iterations of unit fractions.
		<u>Activity 5</u>: Making Poster of Pouring Water - *Distribute one glass of water into two empty glasses equally!* - *Distribute three glasses of water into four empty glasses equally!* - *Using your fractions ruler, measure the height of water after distributed!*	The researcher added an activity about making poster of pouring water as the follow-up Activity 4: Pouring Tea. Doing activity 4, the students needed more support in partitioning through real actions. In Activity 5, paper was used as representation of water.
6	*Building the relation around Fractions* **<u>Activity 6</u>:** Cutting Ribbon *Investigate the results of cutting ribbon if ribbons with different length are cut into certain number of pieces!* *Task 1:* - *1 meter ribbon cut into 2, 3 ,4 and 5 pieces* - *2 meter ribbon cut into 2 and 4 pieces*	*Building the relation around Fractions* **<u>Activity 6</u>:** Shading Parts and Finding the Relations around Fractions 	In fact, the students' pre-knowledge was more about representing fractions as shaded area rather than cutting objects. The researcher used the pre-knowledge of students as the starting point of building relations around fractions.

Lesson	Activity of HLT 1	Refinement of Activity (HLT 2)	Rationale behind the Refinement
	- *3 meter ribbon cut into 3 and 6 pieces* *Task 2:* - *2 meter ribbon cut into 3 pieces* - *3 meter ribbon cut into 5 pieces* *Can you predict other results without using ribbon?*	*Look at the shading area that you have made!* *Are there shading area that similar each other?* *What is the relation between shading area* $\frac{3}{4}$ *and* $\frac{1}{4}$*?* *Explain your answer!*	

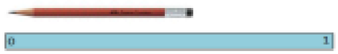
Appendix C HLT 3 as the Refinement of HLT 2

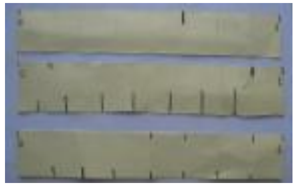
Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
1	*Constructing Meaning of Fair Sharing* **<u>Activity 1</u>:** Sharing a Fruit Cake 1. *Mother made a fruit cake to share with her neighbours. Could you help mother to divide the cake into 4 equal pieces?* 2. *Mother also wants to share another fruit cake for her daughter's friends. They are five children. Could you help mother again? Then compare with the pieces of a cake for 4 people. How much each person gets?*	**Question 1:** - Students' strategies: 	**Question 1:** The students' strategies:  a b c - Two groups cut the model of cake equally and said that it was fair because the pieces were same size - One group did not cut equally but still said that it was fair as long as the bigger pieces were given to big children and the smaller pieces were given to little children. Question 2: - The students got confused to cut the cake into 5 pieces. 	**Question 1:** The students' idea that fair sharing does not always mean equal sharing might come from their daily experience. When this case happens, the context become harder to lead the students to notate fractions based on the results of fair sharing. The skill of partitioning is a key to bring the students to the idea that the more number of sharers, the smaller piece is. When the students have not partitioned properly, it is hard to develop the idea using their results of cutting. The various meaning of	To be powerful to generate fractions, the meaning of fair sharing as equal sharing should be more emphasized. The words *'...as big as...'* might bring the students to the idea fair sharing as equal sharing. When the students have built the idea of fair sharing, mathematical ideas *'pieces do not have to be congruent to be equivalent'* could be constructed. The problem proposed:  Ani Ita *Ita said that her piece of cake is bigger than Ani's but Ani said that hers is as big as Ita's. How is your opinion?* To anticipate the difficulties of students' partitioning, an activity

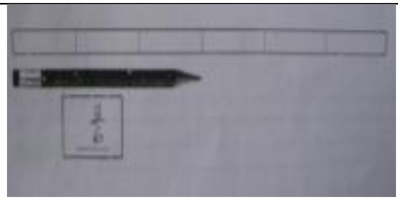
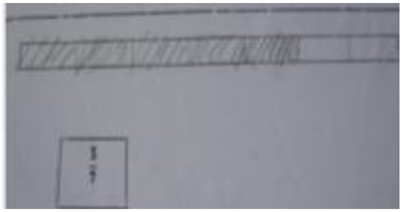
Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
		Question 2: - Students might cut using their strategy in the question 1 then make the fifth piece by cutting one of a quarter pieces. - Students cut it properly but they might have difficulty to compare with their 'quarter' pieces if their way of cutting now is different. - Students realize that the pieces must be smaller. - Students might come up with daily language 'a quarter' and then they also use daily language 'a fifth' for one-fifth. - Students are able to use mathematical symbol of fractions but have not understood about what numbers 1, 4 or 5 refers to.	- The students had difficulties to decide what fractions should be used to represent the results of dividing ($\frac{1}{4}$ and $\frac{1}{5}$). - The students could explain what numbers 1 and 4 in $\frac{1}{4}$ refers to. • 1 means whole cake • 4 means if we cut then there will become 4 • if we cut the cake, 1 will become one piece of cake	fraction appeared from the results of cutting that 1 in $\frac{1}{4}$ could refer to whole cake (fractions as division) or one pieces out of 4 pieces as the results of cutting (fractions as parts of a whole relationship).	that can support the development of the idea '*the more number of sharers, the smaller piece is'* should be developed. The activity proposed: *Cut the chocolate bar below equally into* *a. 2 pieces* *Each part could be represented as.......of a chocolate bar* *b. 3 pieces* *c. 4 pieces* *d. 6 pieces* *e. 8 pieces* *f. 12 pieces*
2	*Producing Simple Fractions as Result of Fair Sharing* **<u>Activity 2</u>:** Sharing Brownies	Students' strategies: - Students might have	The students' strategies: - All the group of students divided	The students' strategy showed that halving might be considered as the easiest way	To support the students in notating fractions by themselves, this problem is improved by

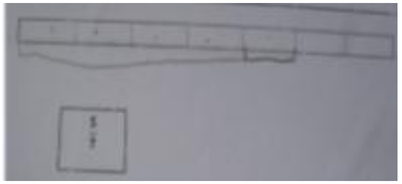
Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
	Cakes *If we only have 3 brownies cakes, how to shares it among 4 people? How much will each person get?* 	struggle in dividing cakes fairly. They might come up with the results merely using estimation. - Students divide cakes by halving and share the rest. - Students divide directly each part into 4 pieces. - Students take directly three quarters of each cake. Students' notation of fractions	cakes by halving and shared the rest. - One group shared the rest by dividing into 4 pieces.  - One group used estimation for the rest of cake. If the size was not equal then this group cut more into smaller pieces and threw away the remained pieces of paper.  - There was a student who posed an idea to cut each brownies cake into 4 pieces but she did not do that. She knew that the number of pieces would be 12 but got confused with it. The students' notation of fractions - The students had difficulties to	to solve this problem. The harder part is partitioning the rest. Although they knew that they had to divide it into 4, not all the students could do it efficiently. The students' difficulties in notating the results with fractions might be caused they no longer could see the original cake after it was cut. After the researcher rearranged the pieces so that it resembled the original cake, the students were able to notate it with fractions. When explaining their fractions notation, the meaning of fraction which appeared was part of a whole relationship. The crucial thing is what kind of a whole that they perceive. In this problem, the whole could be one cake or three cakes. Because the	*changing the instructions from cutting to draw the line of cutting*. *Further discussion about the difference of whole* that the students perceive in notating fractions should be developed although there are no differences of students' answer.

Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
		- Students might use daily language to notate the results of sharing such as "everyone gets a half and a quarter" - Students notate the results by using simple fractions "everyone gets $\frac{1}{2}$ and $\frac{1}{4}$ of a cake" or "3 pieces of $\frac{1}{4}$ cake" - Students directly use the notation $\frac{3}{4}$ of a cake.	notate the parts that each person got with fractions - By rearranging the results of cutting same as the shape of original cake before cut, the researcher guides the students to notate with fractions. The students used daily language *'a half'* and *'a quarter'*. - Based on their language of fractions, the students used simple fractions • Those pieces are $\frac{1}{2}$ and $\frac{1}{4}$ • Those pieces are $\frac{2}{4}$ and $\frac{1}{4}$	students only perceived one cake as one whole, there was no further discussion about that. The students' difficulty in notating fractions brings the researcher to postpone the exploration of relationship around fractions resulted. The discussion should be first focused about the meaning of fraction itself.	
	<u>Mini lesson</u>: Determining the Number of Candies *There are 12 candies. How many candies if we want to take $\frac{1}{4}$ of 12 candies? How about $\frac{2}{4}$ and $\frac{3}{4}$ of those candies?*	After discussing the meaning of simple fractions in Activity Sharing Brownies Cakes, this mini lesson is only to see whether students can see the relationship between those simple fractions. There were no specific conjectures for this activity.	- There was only one student who could answer and he did not use the candies • $\frac{1}{4}$ of 12 candies was 3 candies. • $\frac{2}{4}$ of those candies was 6 candies • $\frac{3}{4}$ was 9 candies • $\frac{4}{4}$ was 12 candies. - The researcher gave more scaffolding to the students that was more explicit instruction	From that mini lesson, the researcher found that although the meaning of fraction as results of division appeared in the activity of dividing model of cake, it was not enough to provoke the students applying that knowledge into discrete objects. It might because there was a big gap between the students' knowledge of partitioning continuous object	To follow up students understanding of the meaning of fraction as result of division, this activity will be elaborated on the next teaching experiment as one of the main activity. Bridging between this new knowledge and the pre-knowledge of the students about partitioning continuous object, the objects will be structured and more focused on unit fractions.

Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
			'divide this amount of candies for 4 people'. The students got 3 as the answer and the researcher lead them to conclude that 3 candies was $\frac{1}{4}$ of 12 candies.	and dividing discrete objects.	The problems proposed such as: *In a stack of chocolate bar, there are 4 chocolate bars. How many chocolate bars that are $\frac{1}{2}$ of the stack?* 
3	*Using Fractions as Unit of Measurement* **<u>Activity 3</u>:** Measuring Pencil Question 1: *Using folded paper, find the length of given pencils*! 	Question 1: - Students only use estimation by marking the folded paper instead of folding it - Students fold paper using repeated halving strategy until it fits to the height of pencil - Students have struggles to fold paper when using	Question 1: - There were the students who measured the length of pencils using ruler. - The students only folded paper once according to the length of pencil. - The students got confused what they should do then the researcher gave explicit instructions to the students to repeat halving.	The students' pre-knowledge of measuring objects using standard units of measurement could not support the students to solve the problem. This activity even discarded that knowledge. It is very difficult for the students to come to the strategy of repeated halving in	The students pre-knowledge about using *standard units of measurement and learning fractions as shaded area* should support the students to use fractions as unit of measurement. Particularly, standard units of measurement could support the students in partitioning. Mathematical ideas about

Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2^{nd} Cycle
	Question 2: *Sort the pencils according to the length of pencils!* Question 3:  *Using a bar above, how long is the pencil?* Question 4: *Draw the pencil according to its length!* 	repeated halving does not match to the height of pencil. - Students count the number of parts that corresponds to the height of pencil then compare it to the whole parts in the folded paper. - Students represent each part of folded paper as a unit fraction then find the height of pencil by multiply it with the number of parts that fits into the pencil. Question 2: - Instead of using fractions, students only count the number of parts and write whole number as the length of pencil - Students are able to write fractions as the length properly.	Therefore, there were only fractions with 8 as the denominator emerged.  - The students count the number of parts that corresponds to the height of pencil then compare it to the whole parts in the folded paper. Question 2: - All the students were able to write fractions as the length properly.	order to generate fractions as the length of objects. Folding paper properly also became another difficulty for the students. As the consequences, the learning goal that the students could use unit fractions as unit of measurement was not achieved. Although the students could answer question 2 easily, the students seemed merely read off the number of parts of the bar that correspond to the length of pencils. It was not enough because they did not construct the parts by themselves. The answer of the students in the question 3 gave more evidence about the pre-knowledge of the students which represented fractions as the shaded parts.	common fractions as iteration of unit fractions might be more emphasized if the students could figure out unit fractions before they measure something. According to that consideration, the context of measuring pencil will be replaced by the context of ants. The problem is about determining the position of ants as the following: *Tom ant is walking to a pile of sugar. He has passed $\frac{1}{4}$ path. Shade part of path that Tom has passed! Mark the position of Tom!*  There will be some problems that are similar with the problems above. The students could compare which ant has passed the longest part of path. The next problem, the students will figure out the position of ant by using unit fractions from

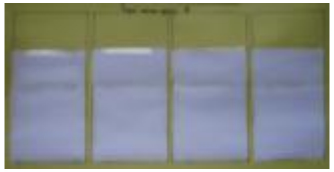
Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
		Question 3: - Students shade the bar - Students directly draw the pencil	 Question 3: - Most of the students only shaded the bar without draw the pencil  - The students drew a line to mark which parts represented the length	The researcher expected that at least students could estimate fractions as the length of pencil. In fact, there was no student who guesses what a fraction is. This activity might too fast to go to that level. The students had to decide what fraction is while they were also challenged to partition folded paper.	Activity Determining the Position of Ants. Unit fractions will be represented with the pieces of ribbon. The problem proposed: *Using the pieces of ribbon, determine the position of ant!*  *What is unit fraction that you used?* *How many times you used until come to the position of ant?* *So, the position of an ant is .... of path.*

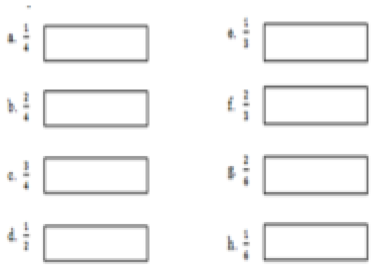
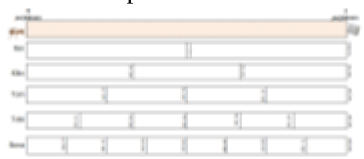
Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
			of pencil 		
	<u>Mini lesson</u>: Marking $\frac{3}{4}$ of Glass *If I want to pour water into $\frac{3}{4}$ of this glass, how high is it?*	- Students estimate $\frac{3}{4}$ of glass - Students do partitioning by 4 to determine $\frac{3}{4}$ of glass	The mini lesson was conducted in classical. There was a student who marked the glass and pointed to the third mark (from the bottom mark) as $\frac{3}{4}$ of glass.  There was one student who disagrees with those marks. She said that the first mark should be called $\frac{2}{4}$ and the	Partitioning was not simple for the students. The skill of partitioning must be connected with the knowledge of relations around fractions. Constructing parts by students' themselves challenged them to coordinating the number of partitioning and fractions itself.	Activity of Pouring Tea has the same learning goals as Activity of Sharing Brownies Cakes that students could produce simple fractions and figure out the relations around fractions. Because activity Pouring Tea has not support the student to reach the learning goal, it will be deleted in the second cycle. The activity of sharing brownies cakes will be more focussed.

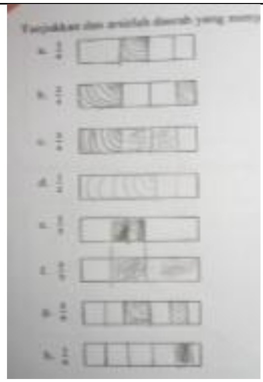
Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
			second mark was a half.		
4	**<u>Activity 4</u>:** Pouring Tea *There is one glass full of water and one empty glass. If I pour water from the glass which is full of water to the empty one but both glasses must be equal, how high is it?*	- Students only draw using their intuition but they cannot make it sure. - By using trial and error, students reduce a certain amount of tea from each glass and draw those amounts on the empty glass. - Students use halving strategy to find the amount that will be distributed into the empty glass. - Students directly use partitioning by third. - Students use ruler for measuring and then dividing the total number of measuring scale of two glasses by three.	- The students used estimation to solve this problem. - Five students drew two glasses which were full of water  - One students drew two glasses which were half full of water  - After the researcher poured water to the empty glass and both glasses became half full of water, the students knew that their answer was not correct.	It was surprising moment when most of the students drew two glasses which were half full of water. It was out of conjecture. Predicting the height of water did not succeed to provoke the students partition the height of water so that they could notate fractions based on those partitions.	
5	**<u>Mini lesson</u>:** Marking Fractions Ruler  	- Students folded ruler into 4 and notated it with fractions - Students directly determine the position of those fractions	- There was a student who determined directly the position of those fractions through estimation.	Folding paper could not support the students to find the position of fractions. The researcher needs to guide them	Folding paper might not support the students in partitioning because there is a gap between this context and the pre-

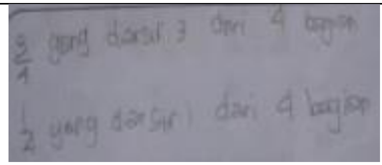
Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
	Find the position of a half, $\frac{1}{4}$ and $\frac{3}{4}$!	through estimation	The position of a half and $\frac{2}{4}$ was different for him. Fractions $\frac{4}{4}$ was different position from 1.  - Most of the students folded paper into two to get a half. The students had difficulties in folding paper to determine $\frac{1}{4}$ and $\frac{3}{4}$. There were the students who just folded without any consideration and there were also the students who folded each half side of folded paper into 4 then did not posit fractions properly.  - The students who used estimation	in folding paper. It seemed that the student could not figure out fractions represented in folded paper. The partitions were not clear for them particularly which the parts and the whole is. It made the students to get difficulty in notating fractions.	knowledge of the students. According to that consideration, this mini lesson will be skipped in the second cycle.

Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
			then corrected his answer by using standard measurement. He could determine correctly the position of those fractions. 		
	<u>Activity 5</u>: Making Poster of Pouring Water Question 1: *Distribute one glass of water into two empty glasses equally!* Question 2: *Distribute three glasses of water into four empty glasses equally!* Question 3: *Using your fractions ruler, measure the height of water after distributed!*	Question 1: - Students directly use strategy of halving by measuring the paper or folding paper - Students use estimation in partitioning	Question 1: - One group folded paper into two and cut it - One group measured the length of paper and divide the result by two - There was one group who use estimation in partitioning. As the results, the height of water in the two glasses was not equal. The researcher then provoked them to divide the difference. 	Although the students did not throw away the remained pieces of paper as they did when sharing brownies cake, some students still had difficulty to find an efficient way to partition. The students did trial and error and repeated to divide the remainder. Cutting the paper until the pieces become smaller made the students more difficult to figure out the fraction. By using fractions ruler, the students then just read off the scale and found the fractions without getting meaning of it.	As the follow up of activity Pouring Tea, this activity made the way of partitioning to be more explicit. As the refinement for the teaching experiment of the second cycle, activity of Pouring Tea will be deleted. Activity of Making Poster of Pouring Water also will not be used in the second cycle.

Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
		Question 2: - Students use partitioning by 4 - Students do the strategy of halving and partition the rest by 4 - Students directly determine that each glass contains $\frac{3}{4}$ full of water	Question 2: - One group used the strategy of halving and partition the rest by 4  - One group used the strategy of halving and estimate the rest  - Two students in one group had different strategies. One student directly determine $\frac{3}{4}$ by measuring the length of paper and the other just did trial and error to make the three glasses having the same height		

Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
		Question 3: - Students just read off the height of water using fractions ruler - Students could reason without fractions ruler	 Question 3: - All the students used fractions ruler and read off the scale		
6	*Building the relation around Fractions* **<u>Activity 7</u>:** Shading Parts and Finding the Relations around Fractions Question 1: 	Question 1: - Students partition bars and shade the parts without any difficulties - Students have difficulty in partition bars but they shade the parts properly	Question 1: - All the students did not get difficulties to partition and shade the parts but the partitions did not always in equal size. - There were some students who shaded the parts in different way. They did not always start to shade the parts from the left side consecutively.	The skill of partitioning could support the students to find the relation among fractions but it also could be dangerous when the students compare fractions. They might not figure out equivalent fractions because they did not partition in equal parts. There is a need of support for students before this activity particularly in constructing parts equally in order to find relations around fractions.	Continuing the context of ant, exploring the relations between fractions will be developed through making path of ant. There are some ants which stop at different position. 

Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
	Question 2: *Look at the shading area that you have made!* *Are there shading area that similar each other?* Question 3: *What is the relation between shaded area $\frac{3}{4}$ and $\frac{1}{4}$? Explain your answer!*	Question 2: - Students compare the size of shaded parts and find some equivalent fractions - Students could not find some equivalent fractions because they did not partition precisely Question 3: - $\frac{3}{4}$ is three times $\frac{1}{4}$ - $\frac{3}{4}$ has 3 parts out of 4 which are shaded and $\frac{1}{4}$ has 1 parts out of 4 which are shaded	 Question 2: - The students got confused to find the similarity between the shaded parts. - The students got difficulty to find the relation among those fractions because their partition in which the parts were not equal size. Question 3: - $\frac{3}{4}$ has 3 parts out of 4 which are shaded and $\frac{1}{4}$ has 1 parts out of 4 which are shaded	Although the students could differ $\frac{3}{4}$ and $\frac{1}{4}$ based on the number of shaded parts, the relations that $\frac{3}{4}$ is iterations of $\frac{1}{4}$ still not obvious. It seemed that the students also could not conclude that kind of relations between other fractions.	

Lesson	Activity of HLT 2	Conjectures of Students' Learning Process	Students' Learning Process	Interpretations	Refinement of Teaching Experiment in the 2nd Cycle
		- $\frac{3}{4}$ has more shaded area than that of $\frac{1}{4}$ - $\frac{3}{4}$ consists of three $\frac{1}{4}$			

Appendix D Lesson Plan (Rencana Pelaksanaan Pembelajaran)

RENCANA PELAKSANAAN PEMBELAJARAN I

Satuan Pendidikan	: SD Laboratorium UNESA
Mata Pelajaran	: Matematika
Kelas/ Semester	: 3/ II
Materi Pokok	: Pecahan
Alokasi Waktu	: 2 pertemuan (4 jam pelajaran)

A. Standar Kompetensi

3. Memahami pecahan sederhana dan penggunaannya dalam pemecahan masalah

B. Kompetensi Dasar

3.1. Mengenal pecahan sederhana

C. Tujuan Pembelajaran

1. Siswa dapat melakukan pembagian suatu obyek secara adil.
2. Siswa dapat menyatakan bagian dari potongan-potongan yang dihasilkan dari pembagian secara adil dalam bentuk notasi $\frac{a}{b}$.
3. Siswa dapat menyimpulkan dengan kalimatnya sendiri bahwa semakin banyak potongan dalam pembagian secara adil, semakin kecil bagian-bagian yang diperoleh.
4. Diberikan potongan-potongan yang menyatakan bagian yang sama tetapi tidak kongruen, siswa dapat menjelaskan bahwa kedua potongan tersebut adil.
5. Diberikan gambar 2 obyek yang berbeda ukuran untuk dibandingkan potongan-potongan yang menyatakan pecahan tertentu, siswa dapat menjelaskan bahwa ukuran obyek harus sama.

D. Indikator

1. Membagi suatu obyek menjadi potongan yang sama besar
2. Menyatakan bagian-bagian dari potongan yang dihasilkan dari pembagian secara adil dalam bentuk notasi $\frac{a}{b}$.

3. Menyimpulkan dengan kalimat sendiri, bahwa semakin banyak potongan, semakin kecil bagian yang diperoleh.
4. Menjelaskan bahwa potongan-potongan yang menyatakan bagian yang sama tidak harus kongruen.
5. Menggambarkan atau menjelaskan bahwa untuk membandingkan pecahan, keseluruhan obyek harus sama.

E. Materi Pokok

Pembagian adil, Pecahan Satuan

F. Metode Pembelajaran

Diskusi, Tanya jawab, Investigasi, Demonstrasi

G. Langkah-langkah Pembelajaran

Kegiatan Awal

1. Guru mengkondisikan siswa-siswa agar siap mengikuti pelajaran
2. Siswa-siswa membentuk kelompok-kelompok yang terdiri dari 2 orang

Kegiatan Inti

1. Melalui tanya jawab, siswa diminta menceritakan secara singkat tentang pengalaman berbagi kue dengan teman atau anggota keluarga dan bagaimana mereka melakukannya.
2. Guru memberikan sebuah masalah tentang membagi kue aneka buah kepada siswa

   Ibu membuat sebuah kue aneka buah untuk dibagikan kepada Ana, Lisa, Rina dan Dea. Bantulah ibu membagi kue tersebut untuk keempat anak secara adil! *Berapa bagian yang diperoleh setiap anak?*
3. Siswa diberikan model kue aneka buah berbentuk persegi dan menyelesaikan masalah tersebut dalam kelompok.

   Siswa membagi model kue tersebut dengan menggunakan gunting.

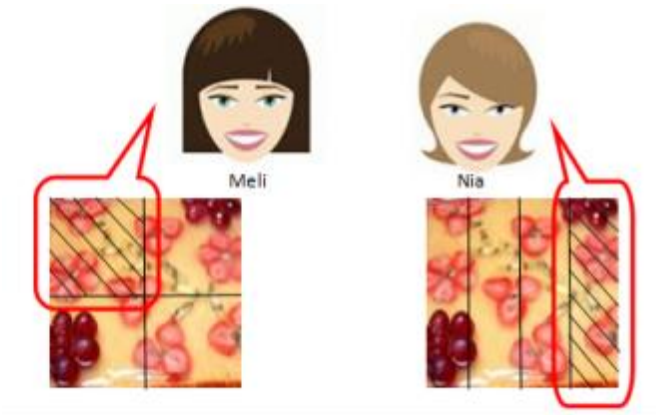
   Hasil potongan-potongan tersebut ditempelkan pada kertas A2 (Lembar Kerja Siswa I).

   Siswa juga menyatakan bagian yang diperoleh setiap anak pada kertas A2.

4. Beberapa kelompok diminta maju ke depan kelas untuk menunjukkan hasil kerjanya dan menjelaskan tentang notasi pecahan yang digunakan.

   Siswa-siswa lainnya memberikan pendapat apakah potongan-potongan yang diperoleh keempat anak tersebut dapat dikatakan adil.
5. Jika siswa telah dapat menilai strategi-strategi kelompok yang maju ke depan kelas, apakah potongan-potongan yang diperoleh oleh setiap anak adil, guru mengajukan pertanyaan selanjutnya. Siswa mengerjakan di LKS II.

   Dari dua kue yang tepat sama, ibu memotong kue dalam bentuk yang berbeda. Meli dan Nia mendapatkan potongan yang tampak pada gambar di bawah ini. Meli berpendapat bahwa ibu adil karena mereka mendapat bagian yang sama. Nia berpendapat bahwa ibu tidak adil. Apakah kalian setuju dengan pendapat Meli ataukah pendapat Nia? Jelaskan alasan kalian!



6. Melalui diskusi kelas, siswa menceritakan pendapatnya dan alasannya.

Dengan dukungan guru, siswa menyimpulkan hasil perbandingan tersebut bahwa untuk dikatakan adil, kedua potongan tidak harus mempunyai bentuk yang sama. Hal yang harus diperhatikan adalah kesamaan ukuran kue sebagai keseluruhan dan potongan-potongan pada setiap kue yang sama besar.

7. Pada LKS II, siswa kembali diberi tugas untuk membagi model kue lapis.

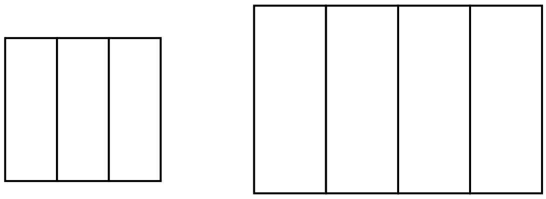
   Terdapat model-model kue yang harus dibagi untuk 2, 3, 4, 6, dan 12 orang. Siswa diminta untuk menyatakan hasil potongan-potongan tersebut dengan notasi pecahan yang sesuai

8. Siswa juga diminta mengurutkan pecahan-pecahan yang menyatakan hasil potongan dari yang paling besar hinggga yang paling kecil. Siswa diminta pula mempartisi kue dan meyebutkan pecahan yang menyatakan potongan kue yang lebih kecil dari potongan-potongan pada persoalan sebelumnya.
9. Siswa diharapkan dapat menyimpulkan bahwa semakin banyak potongan yang dihasilkan, semakin kecil bagian yang diperoleh dengan kata-kata mereka sendiri.
10. Dalam diskusi kelas, siswa diminta membandingkan antara besar bagian yang diperoleh dengan banyak potongan yang dihasilkan.

    Bagaimanakah hasil potongan kue yang dibagi untuk 5 orang dibandingkan dengan hasil potongan kue yang dibagi untuk 10 orang?

    Apa yang terjadi bila,kue tersebut dibagi untuk 20 orang?
11. Guru memberikan sebuah persoalan tentang membandingkan potongan $\frac{1}{3}$ and $\frac{1}{4}$. Bila siswa dapat membandingkan pecahan dengan benar, guru memberikan konflik di mana ukuran kue tidak sama.



12. Siswa dibimbing melalui pertanyaan-pertanyaan yang mengarah pada ide matematis bahwa untuk membandingkan pecahan, kesuluruhan objek harus sama.

Kegiatan Akhir

1. Guru memberikan penguatan pada siswa berkaitan dengan kegiatan yang telah dilakukan.
2. Guru menutup pelajaran dengan memberikan pesan-pesan moral dan salam penutup.

H. Sumber dan Sarana

1. Kertas A2
2. Model Kue
3. LKS I : Membandingkan Potongan Kue
4. Lem
5. Gunting

I. Penilaian

1. Pengamatan
2. Hasil kerja siswa

RENCANA PELAKSANAAN PEMBELAJARAN II

Satuan Pendidikan	: SD Laboratorium UNESA
Mata Pelajaran	: Matematika
Kelas/ Semester	: 3/ II
Materi Pokok	: Pecahan
Alokasi Waktu	: 1 pertemuan (2 jam pelajaran)

A. Standar Kompetensi

3. Memahami pecahan sederhana dan penggunaannya dalam pemecahan masalah

B. Kompetensi Dasar

3.2. Mengenal pecahan sederhana

C. Tujuan Pembelajaran

1. Siswa dapat membagi sejumlah obyek menjadi grup-grup yang mempunyai bagian yang sama.
2. Jika diketahui bagian dari sejumlah obyek, siswa dapat menentukan banyaknya obyek.

D. Indikator

1. Menentukan banyaknya obyek, jika sejumlah obyek dibagi menjadi beberapa bagian yang sama
2. Menentukan banyaknya obyek, jika diketahui bagian dari sejumlah obyek.

E. Materi Pokok

Pecahan Satuan, Pecahan Non-Satuan

F. Metode Pembelajaran

Diskusi, Tanya jawab, Investigasi, Demonstrasi

G. Langkah-langkah Pembelajaran

Kegiatan Awal

1. Guru mengkondisikan siswa-siswa agar siap mengikuti pelajaran
2. Siswa-siswa membentuk kelompok-kelompok yang terdiri dari 2 orang

Kegiatan Inti

1. Disediakan setumpuk coklat (8 coklat) di depan kelas. Melalui diskusi kelas, siswa menentukan banyaknya coklat yang menyatakan $\frac{1}{2}$, atau $\frac{1}{4}$ bagian dari tumpukan coklat tersebut. Dengan menggunakan banyak coklat yang berbeda, permasalahan yang sama kembali diajukan.
2. Siswa diminta menjelaskan strategi yang digunakan dalam persoalan di atas.
3. Tiap-tiap kelompok dibagikan Lembar Kerja Siswa II dan model coklat.

   Pada LKS II terdapat gambar tumpukan coklat batangan dengan banyak coklat yang berbeda-beda dan beberapa pertanyaan yang harus dijawab siswa berkaitan dengan gambar tersebut. Pertanyaan-pertanyaan itu antara lain

   Berapa banyak coklat batangan yang menyatakan $\frac{1}{2}$ *dari sekumpulan coklat-coklat tersebut?*

   Berapa banyak coklat batangan yang menyatakan $\frac{1}{4}$ *dari sekumpulan coklat-coklat tersebut?*

4. Beberapa siswa diminta menuliskan jawaban kelompoknya di depan kelas.
5. Siswa-siswa diminta menjelaskan strateginya kepada teman-temannya.

6. Dengan mengeksplorasi jawaban siswa, diskusi kelas difokuskan pada arti pecahan sebagai bagian dari kelompok-kelompok yang beranggotakan sama atau juga menyatakan pembagian.

   Untuk mendapatkan $\frac{1}{4}$ *dari sekumpulan obyek (8 coklat), siswa mengelompokkan atau membagi sekumpulan obyek tersebut menjadi 4 bagian yang beranggotakan sama sehinga diperoleh setiap bagian terdiri dari 2 coklat.*

Kegiatan Akhir

1. Guru memberikan penguatan pada siswa berkaitan dengan kegiatan yang telah dilakukan
2. Guru menutup pelajaran dengan memberikan pesan-pesan moral dan salam penutup.

H. Sumber dan Sarana

1. LKS II: Membagi Coklat
2. Model coklat

I. Penilaian

3. Pengamatan
4. Hasil kerja siswa

RENCANA PELAKSANAAN PEMBELAJARAN III

Satuan Pendidikan	: SD Laboratorium UNESA
Mata Pelajaran	: Matematika
Kelas/ Semester	: 3/ II
Materi Pokok	: Pecahan
Alokasi Waktu	: 1 pertemuan (2 jam pelajaran)

A. Standar Kompetensi

3. Memahami pecahan sederhana dan penggunaannya dalam pemecahan masalah

B. Kompetensi Dasar

3.1. Mengenal pecahan sederhana

C. Tujuan Pembelajaran

1. Siswa dapat membagi obyek yang banyaknya lebih dari satu
2. Siswa dapat menyatakan hasil pembagian secara adil dengan notasi pecahan yang sesuai

D. Indikator

1. Membagi model kue brownies yang banyaknya lebih dari satu menjadi bagian yang sama besar
2. Menyatakan hasil pembagian kue brownies dalam notasi pecahan

E. Materi Pokok

Pecahan Satuan, Pecahan Non-Satuan

F. Metode Pembelajaran

Diskusi, Tanya jawab, Investigasi, Demonstrasi

G. Langkah-langkah Pembelajaran

Kegiatan Awal

1. Guru mengkondisikan siswa-siswa agar siap mengikuti pelajaran.
2. Siswa-siswa membentuk kelompok-kelompok yang terdiri dari 2 orang.

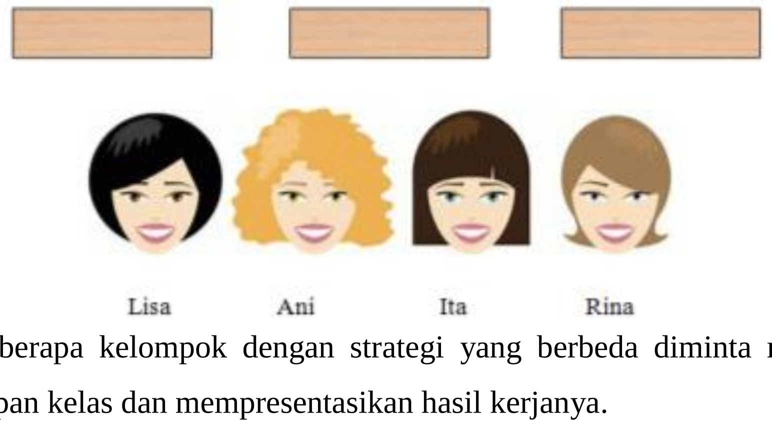
3. Siswa-siswa diminta menyebutkan beberapa kegiatan yang telah dilakukan pada hari sebelumnya.

Kegiatan Inti

1. Sebuah permasalahn diajukan kepada siswa

   Terdapat 2 buah kue brownies yang akan dibagikan untuk 4 orang anak; Lisa, Ani, Ita, dan Rina secara adil. Berapa bagian yang diperoleh setiap anak?
2. Siswa diminta menyampaikan pendapatnya dan menjelaskan strateginya dalam membagi kue.
3. Diskusi kelas difokuskan pada perbedaan keseluruhan kue yang dipandang dalam menyatakan pecahan. Ada siswa yang mungkin menjawab $\frac{1}{2}$ karena memandang bagian yang diperoleh setiap anak bila dibandingkan dengan satu buah kue. Siswa lain mungkin menjawab $\frac{1}{4}$ bagian kue bila dibandingkan dengan total potongan dari 2 buah kue.
4. Setiap kelompok dibagikan kertas A2 dan 3 model kue brownies.

   Sebuah permasalahan tentang membagi 3 buah kue brownies untuk 4 anak diajukan pada siswa. Siswa-siswa diminta membuat garis-garis potongan kue-kue tersebut dan menyatakan bagian yang diperoleh setiap anak dengan pecahan.



5. Beberapa kelompok dengan strategi yang berbeda diminta maju ke depan kelas dan mempresentasikan hasil kerjanya.
6. Prediksi strategi siswa dalam membagi kue brownies dan menyatakan dalam bentuk pecahan adalah sebagai berikut
   a. Siswa membagi setiap kue menjadi 4 bagian yang sama besar

Kemungkinan bentuk pecahan yang digunakan siswa:

- $\frac{3}{12}$, jika siswa membandingkan bagian yang diperoleh dengan jumlah potongan dari semua brownies
- $\frac{3}{4}$, jika siswa membandingkan bagian yang diperoleh dengan jumlah potongan dari satu brownies

b. Siswa membagi 2 kue masing-masing menjadi 2 bagian yang sama besar dan membagi 1 kue yang tersisa menjadi 4 bagian yang sama besar. Kemungkinan pecahan yang dihasilkan adalah $\frac{1}{2}+\frac{1}{4}$

7. Dengan kemungkinan jawaban-jawaban yang berbeda tersebut, guru mendiskusikan perbedaan di antara jawaban-jawaban siswa terutama mengenai perbedaan keseluruhan obyek (satu brownies atau semua brownies).

Kegiatan Akhir

1. Guru memberikan penguatan pada siswa berkaitan dengan kegiatan yang telah dilakukan
2. Guru menutup pelajaran dengan memberikan pesan-pesan moral dan salam penutup.

H. Sumber dan Sarana

Model Kue Brownies

Kertas A2

I. Penilaian

1. Pengamatan
2. Hasil kerja siswa

RENCANA PELAKSANAAN PEMBELAJARAN IV

Satuan Pendidikan	: SD Laboratorium UNESA
Mata Pelajaran	: Matematika
Kelas/ Semester	: 3/ II
Materi Pokok	: Pecahan
Alokasi Waktu	: 2 pertemuan (4 jam pelajaran)

A. Standar Kompetensi

3. Memahami pecahan sederhana dan penggunaannya dalam pemecahan masalah

B. Kompetensi Dasar

3.1. Mengenal pecahan sederhana

C. Tujuan Pembelajaran

1. Siswa dapat menentukan letak pecahan satuan pada garis bilangan (informal) melalui konteks pengukuran
2. Siswa dapat menentukan letak pecahan non-satuan dengan menggunakan pecahan satuan sebagai unit pengukuran

D. Indikator

1. Mengarsir dengan benar daerah yang telah dilalui semut pada model perjalanan semut sesuai dengan pecahan satuan yang diberikan.
2. Menentukan pecahan non-satuan yang menyatak posisi semut dengan menggunakan pecahan satuan sebagai unit pengukuran

E. Materi Pokok

Pecahan Satuan, Pecahan Non-Satuan, Garis Bilangan (Informal)

F. Metode Pembelajaran

Diskusi, Tanya jawab, Investigasi, Demonstrasi

G. Langkah-langkah Pembelajaran

Kegiatan Awal

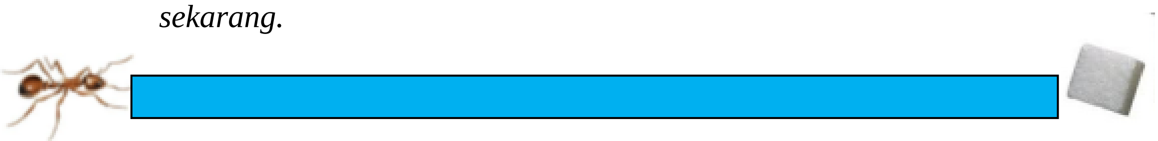
1. Guru mengkondisikan siswa-siswa agar siap mengikuti pelajaran.

2. Siswa-siswa membentuk kelompok-kelompok yang terdiri dari 2 orang.
3. Siswa-siswa diminta menyebutkan beberapa kegiatan yang telah dilakukan pada hari sebelumnya.

Kegiatan Inti

1. Setiap kelompok dibagikan Lembar Kerja Siswa III.
2. Guru membangun sebuah cerita tentang perjalanan seeekor semut.

   Seekor semut sedang berjalan mendekati sebongkah gula. Semut tersebut baru menempuh $\frac{1}{4}$ *perjalanan saat ia menjumpai remah-remah roti. Semutpun berhenti sejenak untuk memeriksa remah-remah roti tersebut. Siswa-siswa diminta menentukan posisi semut sekarang.*

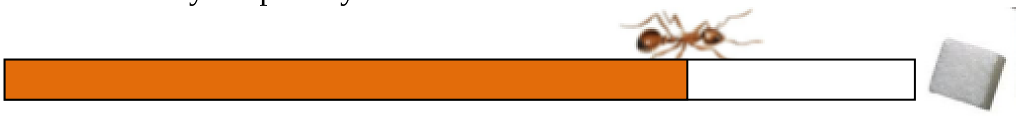


3. Model perjalanan semut seperti gambar di atas disediakan pada LKS III. Siswa diminta mengarsir daerah perjalanan yang telah dilalui semut. Beberapa persoalan yang serupa yaitu dengan pecahan satuan yang berbeda-beda diberikan pada siswa. Siswa juga diminta menceritakan cara mereka menentukan posisi semut.
4. Pada diskusi kelas, beberapa siswa diminta maju ke depan kelas untuk mempresesentasikan jawabannya.

   Fokus diskusi adalah strategi siswa menentukan letak pecahan pada garis bilangan informal tersebut. Pertanyaan yang dapat diajukan misalnya

   Bagaimana kalian menentukan posisi tertentu saat semut menghentikan perjalanannya?
5. Dari pecahan-pecahan satuan yang berbeda, siswa diminta membandingkan pecahan satuan manakah yang menyatakan posisi semut yang telah menempuh jarak terjauh.
6. Setelah siswa dapat menentukan letak pecahan-pecahan satuan yang menyatakan bagian perjalanan semut, Lembar Kerja Siswa IV dengan

menggunakan potongan-potongan pita diberikan pada siswa. Potongan-potongan pita menyatakan pecahan satuan yang berbeda-beda. Pertanyaan-pertanyaan tersebut antara lain



Jenis potongan pita yang digunakan adalah potongan pita.............. sebanyak..............

Jadi, posisi semut yaitu .... perjalanan.

7. Siswa diminta mempresentasikan jawabannya di depan kelas. Pecahan non satuan yang merupakan perulangan dari pecahan satuan menjadi arah kesimpulan dari jawaban-jawaban siswa.
8. Terdapat beberapa soal yang mempunyai dua jawaban benar, misal posisi semut dapat dinyatakan sebagai $\frac{6}{8}$ bila menggunakan pita $\frac{1}{8}$ atau $\frac{3}{4}$ bila menggunakan pita $\frac{1}{4}$. Diskusi dapat dilanjutkan tentang ekuivalensi di antara pecahan-pecahan tersebut.

Kegiatan Akhir

1. Guru memberikan penguatan pada siswa berkaitan dengan kegiatan yang telah dilakukan
2. Guru menutup pelajaran dengan memberikan pesan-pesan moral dan salam penutup.

H. Sumber dan Sarana

1. LKS III: Perjalanan Semut
2. LKS IV: Posisi Semut
3. Pita

I. Penilaian

3. Pengamatan
4. Hasil kerja siswa

RENCANA PELAKSANAAN PEMBELAJARAN V

Satuan Pendidikan	: SD Laboratorium UNESA
Mata Pelajaran	: Matematika
Kelas/ Semester	: 3/ II
Materi Pokok	: Pecahan
Alokasi Waktu	: 2 pertemuan (4 jam pelajaran)

A. Standar Kompetensi

3. Memahami pecahan sederhana dan penggunaannya dalam pemecahan masalah

B. Kompetensi Dasar

3.1. Mengenal pecahan sederhana

3.2. Membandingkan pecahan sederhana

C. Tujuan Pembelajaran

1. Siswa dapat mengkonstruksi model hubungan antar pecahan
2. Siswa dapat menyatakan letak suatu pecahan non-satuan, jika diketahui letak pecahan satuannya
3. Siswa dapat mengidentifikasi pecahan-pecahan yang senilai berdasarkan kesamaan jarak pada model
4. Siswa dapat membandingkan antar pecahan menggunakan model hubungan antar pecahan
5. Siswa dapat memecahkan masalah yang berkaitan dengan hubungan antar pecahan

D. Indikator

1. Mengkonstruksi model hubungan antar pecahan dengan mempartisi model bar perjalanan semut
2. Menyatakan hasil partisi dengan notasi pecahan yang sesuai
3. Menyebutkan pecahan-pecahan yang senilai berdasarkan kesamaan jarak pada model hubungan antar pecahan

4. Membandingkan antar pecahan menggunakan model hubungan antar pecahan
5. Memecahkan masalah yang berkaitan dengan hubungan antar pecahan

E. Materi Pokok

Pecahan Satuan, Pecahan Non-Satuan, Garis Bilangan, Model Hubungan antar Pecahan

F. Metode Pembelajaran

Diskusi, Tanya jawab, Investigasi, Demonstrasi

G. Langkah-langkah Pembelajaran

Kegiatan Awal

1. Guru mengkondisikan siswa-siswa agar siap mengikuti pelajaran.
2. Siswa-siswa membentuk kelompok-kelompok yang terdiri dari 2 orang.
3. Siswa-siswa diminta menyebutkan beberapa kegiatan yang telah dilakukan pada hari sebelumnya.

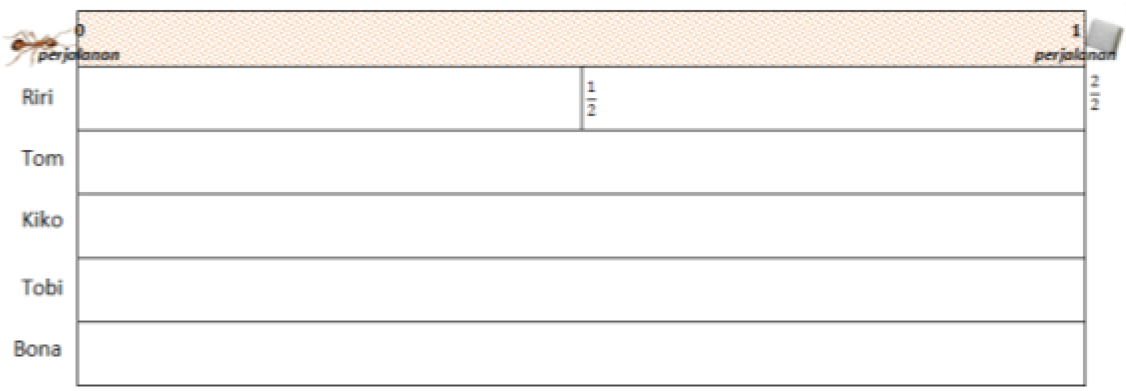
Kegiatan Inti

1. Setiap kelompok dibagikan lembar jalur perjalanan semut.
2. Melalui cerita, siswa-siswa diminta untuk menentukan letak pecahan yang merupakan posisi-posisi perhentian semut-semut. Contoh cerita misalnya:

   Tobi si semut berhenti beberapa kali pada posisi tertentu selama perjalanannya. Tobi berhenti pada posisi $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6},$ *dan* $\frac{5}{6}$ *perjalanan menuju gula. Dapatkah kalian menentukan posisi-posisi Tobi saat berhenti?*

   Pecahan-pecahan yang terdapat pada lintasan tersebut yaitu

   - Berpenyebut 2
   - Berpenyebut 3
   - Berpenyebut 4
   - Berpenyebut 6
   - Berpenyebut 8



3. Dengan bimbingan guru, siswa mengidentifikasi pecahan-pecahan senilai yang terdapat dalam model tersebut.
4. Melalui tanya jawab, siswa-siswa membandingkan pecahan-pecahan yang diajukan guru misalkan

   Manakah pecahan yang lebih besar, $\frac{1}{3}$ atau $\frac{1}{2}$? Jelaskan dengan gambar!
5. Ketika siswa telah dapat membandingkan pecahan-pecahan satuan, guru dapat mengajukan pertanyaan-pertanyaan tentang membandingkan pecahan non-satuan.
6. Siswa lalu diberikan beberapa pertanyaan yang berkaitan dengan hubungan antar pecahan pada LKS V. Pertanyaan-pertanyaan tersebut antara lain

Dengan bantuan jalur perjalanan semut yang telah kalian buat, tentukan letak pecahan $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$!

Kegiatan Akhir

1. Guru memberikan penguatan pada siswa berkaitan dengan kegiatan yang telah dilakukan

2. Guru menutup pelajaran dengan memberikan pesan-pesan moral dan salam penutup.

H. Sumber dan Sarana

1. Jalur Perjalanan Semut
2. Lembar Kerja Siswa V

I. Penilaian

1. Pengamatan
2. Hasil kerja siswa

Appendix E Worksheet (Lembar Kerja Siswa)

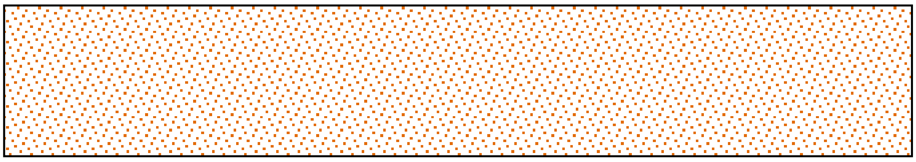
Nama Anggota Kelompok:
1.
2.

LEMBAR KERJA SISWA I

Membandingkan Potongan Coklat

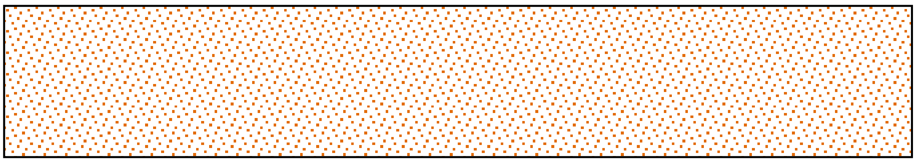
Bagilah gambar coklat batangan di bawah ini menjadi

a. 2 potongan yang sama besar



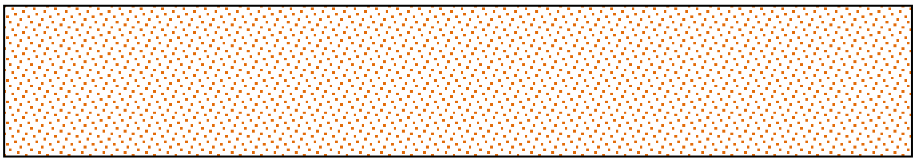
Setiap potongan dapat dinyatakan dengan ...... bagian coklat.

b. 3 potongan yang sama besar



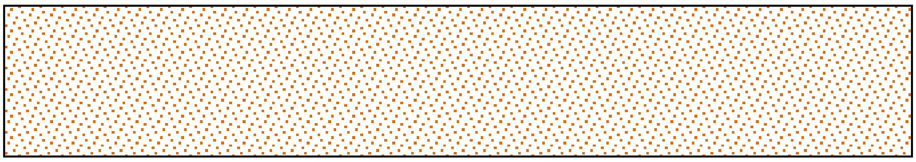
Setiap potongan dapat dinyatakan dengan ...... bagian coklat.

c. 4 potongan yang sama besar



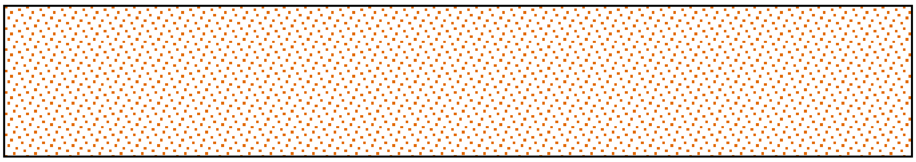
Setiap potongan dapat dinyatakan dengan ...... bagian coklat.

d. 6 potongan yang sama besar



Setiap potongan dapat dinyatakan dengan ...... bagian coklat.

e. 12 potongan yang sama besar



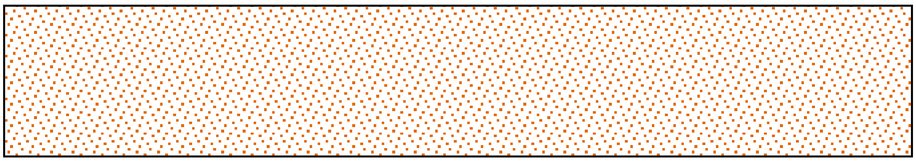
Setiap potongan dapat dinyatakan dengan ...... bagian coklat.

Bandingkan hasil potongan-potongan yang telah kalian buat! Urutkan bagian coklat dari yang paling besar sampai yang paling kecil!

Jawab:

Dapatkah kalian memberi contoh bagian coklat yang lebih kecil daripada potongan-potongan yang telah kalian buat?

Jawab:



Kesimpulan kalian :

Nama Anggota Kelompok:
1.
2.

LEMBAR KERJA SISWA II
Membagi Coklat

1. Dalam sebuah tumpukan, terdapat 4 buah coklat. Berapa banyak coklat yang menyatakan $\frac{1}{2}$ dari tumpukan coklat tersebut?

Chocolatos
Chocolatos
Chocolatos
Chocolatos

Jawab:

Penjelasan:

2. Dalam sebuah tumpukan, terdapat 6 buah coklat. Berapa banyak coklat yang menyatakan $\frac{1}{2}$ dari tumpukan coklat tersebut?

Chocolatos
Chocolatos
Chocolatos
Chocolatos
Chocolatos
Chocolatos

Jawab:

Penjelasan:

3. Dalam sebuah tumpukan, terdapat 4 buah coklat. Berapa banyak coklat yang menyatakan $\frac{1}{4}$ dari tumpukan coklat tersebut?

Chocolatos
Chocolatos
Chocolatos
Chocolatos

Jawab:

Penjelasan:

4. Dalam sebuah tumpukan, terdapat 8 buah coklat. Berapa banyak coklat yang menyatakan $\frac{1}{4}$ dari tumpukan coklat tersebut?

Chocolatos
Chocolatos
Chocolatos
Chocolatos
Chocolatos

Chocolatos
Chocolatos
Chocolatos

Jawab:

Penjelasan:

5. Dalam sebuah tumpukan, terdapat 6 buah coklat. Berapa banyak coklat yang menyatakan $\frac{1}{3}$ dari tumpukan coklat tersebut?

Chocolatos
Chocolatos
Chocolatos
Chocolatos
Chocolatos
Chocolatos

Jawab:

Penjelasan:

6. Dalam sebuah tumpukan, terdapat 8 buah coklat. Berapa banyak coklat yang menyatakan $\frac{1}{4}$ dari tumpukan coklat tersebut?

Jawab:

Penjelasan:

7. Dalam sebuah tumpukan, terdapat 12 buah coklat. Berapa banyak coklat yang menyatakan $\frac{1}{4}$ dari tumpukan coklat tersebut?

Jawab:

Penjelasan:

8. Dalam sebuah tumpukan, terdapat 12 buah coklat. Berapa banyak coklat yang menyatakan $\frac{2}{4}$ dari tumpukan coklat tersebut?

Jawab:

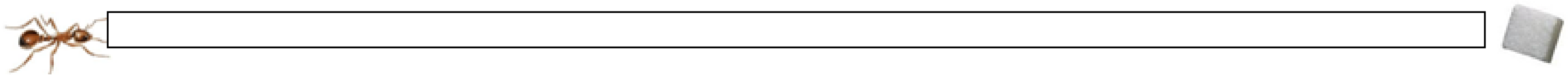
Penjelasan:

Nama:
1.
2.

LEMBAR KERJA SISWA III

Perjalanan Semut

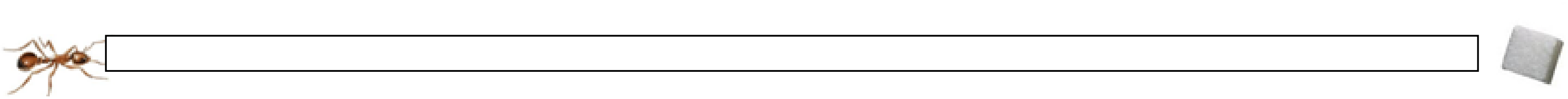
1. Riri si semut sedang bergerak menuju sebongkah gula. Saat ini ia sudah menempuh $\frac{1}{2}$ perjalanan. Arsirlah $\frac{1}{2}$ daerah perjalanan yang telah ditempuh Riri si semut! Berilah tanda di manakah posisi Riri si semut sekarang!



Ceritakan caramu menentukan posisi Riri si semut!

2. Tom si semut juga sedang bergerak menuju sebongkah gula. Saat ini ia sudah menempuh $\frac{1}{4}$ perjalanan.

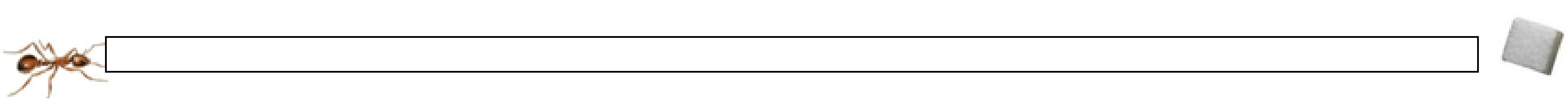
Arsirlah $\frac{1}{4}$ daerah perjalanan yang telah ditempuh Tom si semut! Berilah tanda di manakah posisi Tom si semut sekarang!



Ceritakan caramu menentukan posisi Tom si semut!

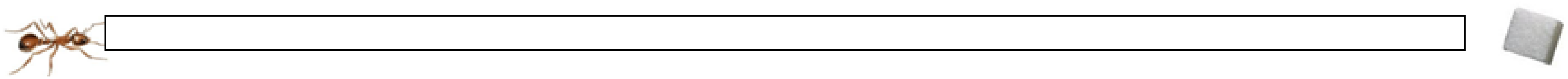
3. Kiko si semut juga sedang bergerak menuju sebongkah gula. Saat ini ia sudah menempuh $\frac{1}{3}$ perjalanan.

Arsirlah $\frac{1}{3}$ daerah perjalanan yang telah ditempuh Kiko si semut! Berilah tanda di manakah posisi Kiko si semut sekarang!



Ceritakan caramu menentukan posisi Kiko si semut!

4. Tobi si semut juga sedang bergerak menuju sebongkah gula. Saat ini ia sudah menempuh $\frac{1}{6}$ perjalanan. Arsirlah $\frac{1}{6}$ daerah perjalanan yang telah ditempuh Kiko si semut! Berilah tanda di manakah posisi Tobi si semut sekarang!



Ceritakan caramu menentukan posisi Tobi si semut!

5. Di antara Riri, Tom, Kiko dan Tobi, siapakah yang telah lebih jauh berjalan? Jelaskan alasanmu!

Jawab:

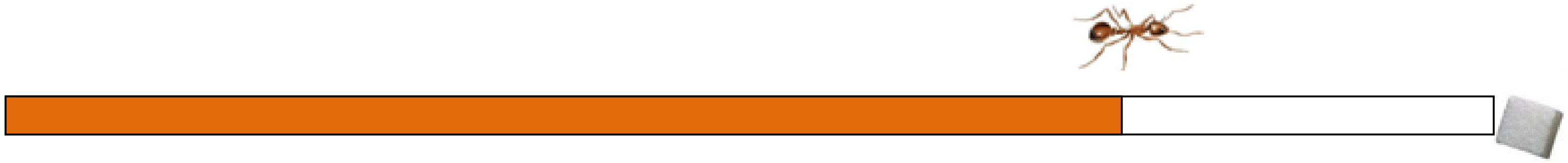
Nama Anggota Kelompok:
1.
2.

LEMBAR KERJA SISWA IV

Posisi Semut

Menggunakan potongan-potongan pita yang dibagikan, tentukan posisi semut-semut di bawah ini!

1.

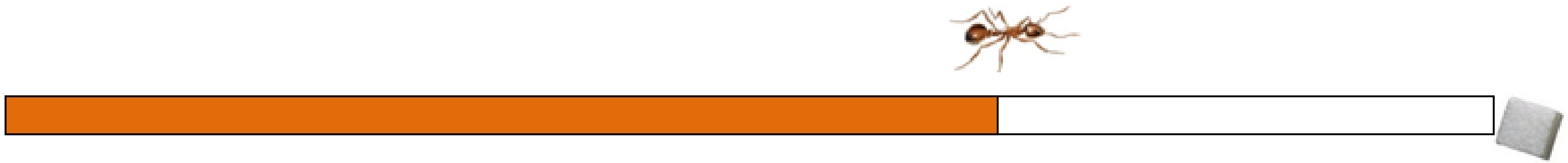


Jenis potongan pita yang digunakan adalah potongan pita.............

sebanyak.............

Jadi, posisi semut yaitu .... perjalanan.

2.

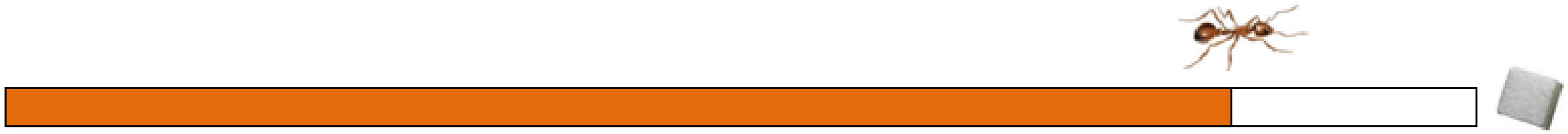


Jenis potongan pita yang digunakan adalah potongan pita.............

sebanyak.............

Jadi, posisi semut yaitu .... perjalanan.

3.



Jenis potongan pita yang digunakan adalah potongan pita.............

sebanyak.............

Jadi, posisi semut yaitu .... perjalanan.

JALUR PERJALANAN PARA SEMUT

Nama Anggota Kelompok:
1.
2.

Tandai letak perhentian-perhentian masing-masing semut dan tuliskan pecahannya!

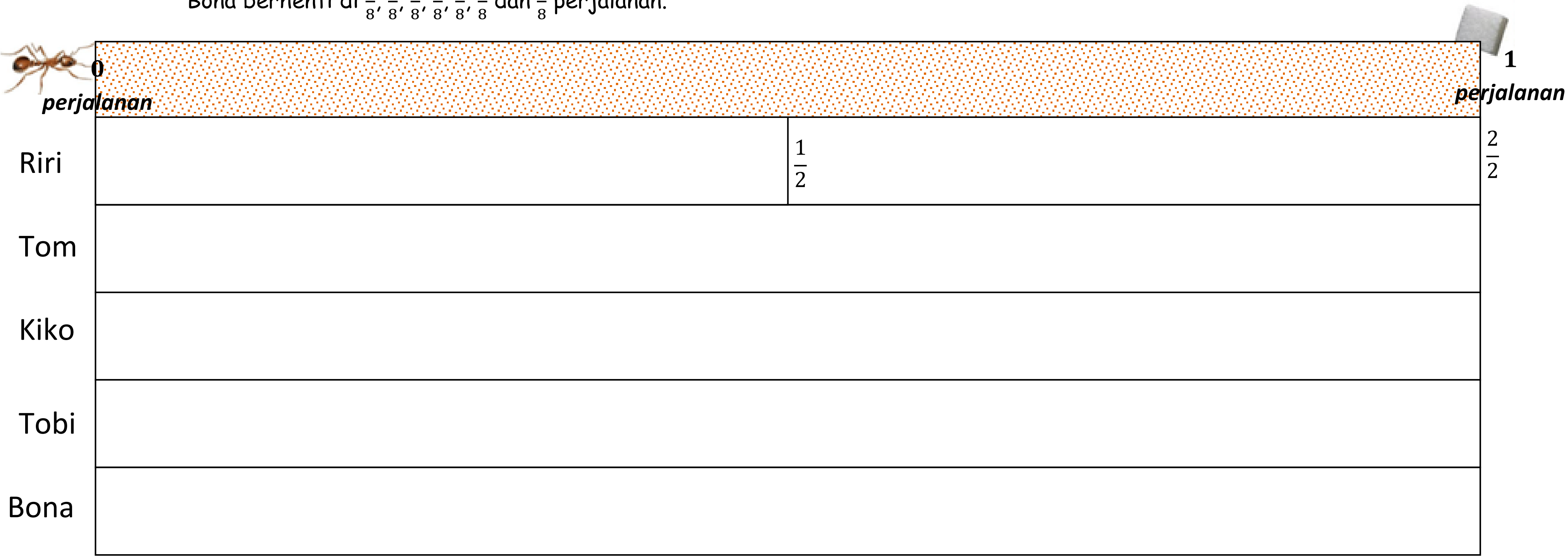
Riri berhenti di $\frac{1}{2}$ perjalanan.

Kiko berhenti di $\frac{1}{3}$ dan $\frac{2}{3}$ perjalanan.

Tom berhenti di $\frac{1}{4}$, $\frac{2}{4}$ dan $\frac{3}{4}$ perjalanan.

Tobi berhenti di $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}$ dan $\frac{5}{6}$ perjalanan.

Bona berhenti di $\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}$ dan $\frac{7}{8}$ perjalanan.



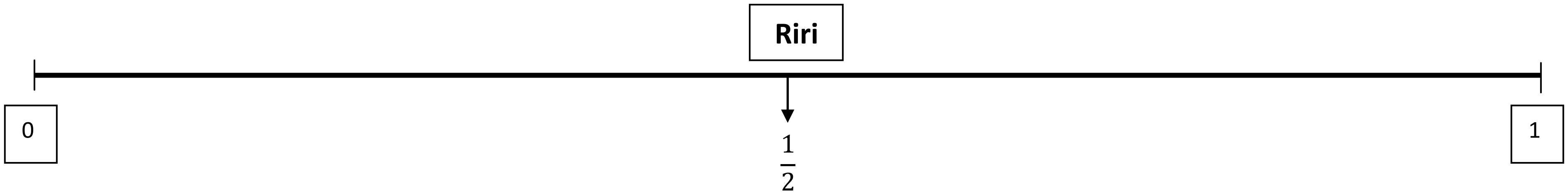
LEMBAR KERJA SISWA V

Semut di Garis Bilangan

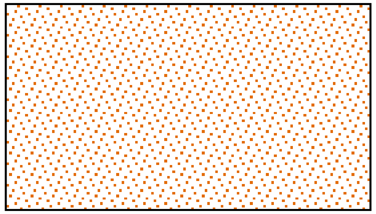
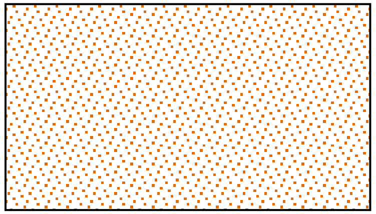
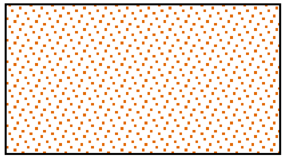
Nama Anggota Kelompok:
1.
2.

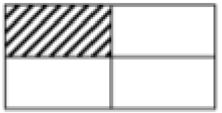
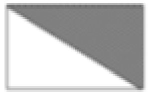
1. Dengan bantuan jalur perjalanan semut yang telah kalian buat, tentukan letak pecahan $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$!

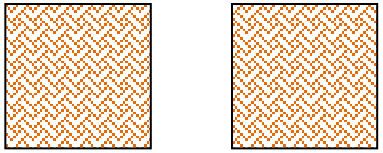
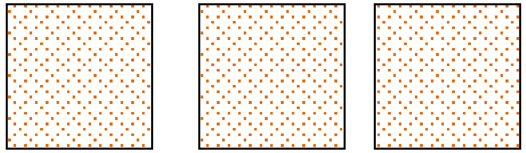
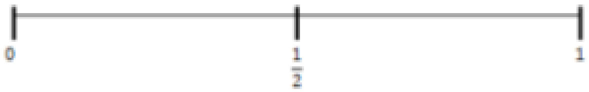
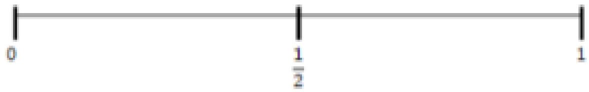
2. Riri, Bobi, Koko dan Dino menempuh perjalanan yang sama. Riri berada pada $\frac{1}{2}$ perjalanan. Bobi si semut berada pada $\frac{1}{4}$ perjalanan. Koko berada pada $\frac{2}{4}$ perjalanan. Dino si semut berada pada $\frac{3}{4}$ perjalanan. Di manakah posisi Bobi, Koko dan Dino?



Appendix F Questions of Pre-test and Post-test

Item Test	Goals	Pre-Test	Post-Test
1	✓ To investigate whether students could divide an object fairly. ✓ To investigate whether students could notate the results of fair sharing using notation $\frac{a}{b}$	There is one brownies cake to be shared among 4 children fairly. How do you divide the cake? Show your way in dividing by making lines on the following figure of cake! How many parts that each person get?  Each person gets.........parts	Ani brings a cake. She wants to divide the cake among 8 people. Could you give a suggestion about how Ani should divide the cake? Show your way in dividing by making lines on the following figure of cake! How many parts that each person get?  Each person gets.........parts
2	✓ To investigate whether students could conclude by their own sentence that the more number of sharers, the smaller the size of pieces ✓ To investigate the students' awareness of a whole in comparing	There are two identical cakes as the following figure.  Those two cakes will be shared fairly Group I: One cake is shared among 6 children Group II: One cake is shared among 8 children a. How many parts that each member of both groups gets?	At Scout Camp, Tiger group gets one cake. Orchid group also gets one cake as big as Tiger group's cake. They share the cakes in each group. Tiger group consists of 6 children. Orchid group consists of 4 children. Which group does get the bigger pieces? Explain your reason!

Item Test	Goals	Pre-Test	Post-Test
	fractions	b. Which group does get the bigger pieces? Explain your reason!	
3.	✓ Given the results of fair sharing that represented same fractions but the pieces are not congruent, to investigate whether students could explain that both pieces are equivalent	Answer by choosing a, b or c. How is your opinion about shaded area I and II below? Explain your reason! a. Shaded area I is larger than shaded area II b. Shaded area I is smaller than shaded area II c. Shaded area I is as large as shaded area II  I  II	In the figure below, there were two identical cakes. The shaded parts show the pieces that Nia gets. Nia argues that piece A is larger than piece B. Do you agree with Nia? Why?  A  B
4.	✓ To investigate whether students could divide a number of objects into groups that are equal parts ✓ To investigate whether if fraction is known, the student could determine the number of objects	In a plastic bag, there are 20 chocolate candies. Anto wants to take $\frac{1}{4}$ of those candies. How many candies are taken by Anto? Explain your reason!	In a box, there are 12 clips. Andi takes $\frac{1}{4}$ of those clips. How many clips does Andi take?
5.	✓ To investigate whether students could divide fairly objects more than	There are two cakes to be shared among 4 children fairly. Show your way in dividing by making lines on the following figure of cakes! How many parts does	There are 3 identical cakes to be shared among 6 children fairly. Show your way in dividing by making lines on the following figure of cakes! How

Item Test	Goals	Pre-Test	Post-Test
	one ✓ To investigate whether students could notate the results of fair sharing with fractions	each person get?  Each person gets ...... of a cake.	many parts does each person get?  Each person gets ...... of one cake. Each person gets ...... of 3 cakes.
6.	✓ To investigate whether students could determine the position of unit fractions on number line ✓ To investigate whether students could determine the position of non-unit fractions by using unit fractions as unit of measurement	a. Determine the position of $\frac{1}{4}$ and $\frac{3}{4}$ on number line below!  b. Determine the position of $\frac{1}{3}$ and $\frac{2}{3}$ on number line below! 	a. Determine the position of $\frac{1}{4}$ and $\frac{3}{4}$ on number line below!  b. Determine the position of $\frac{1}{3}$ and $\frac{2}{3}$ on number line below! 
7.	✓ To investigate whether students could compare	Which one is greater, $\frac{1}{4}$ or $\frac{1}{6}$? Show by drawing!	Which one is greater, $\frac{1}{4}$ or $\frac{1}{3}$? Show by drawing!

Item Test	Goals	Pre-Test	Post-Test
8.	fractions ✓ To investigate students' awareness of a whole in comparing fractions	Fill the blank space below with a sign <, = or > $\frac{2}{4}$ .... $\frac{1}{2}$ Explain by drawing!	Fill the blank space below with a sign <, = or > $\frac{2}{4}$ .... $\frac{1}{2}$ Explain by drawing!
9.	✓ To investigate whether the students had understood non-unit fractions as iteration of unit fraction using discrete objects	-	If there are 20 chocolate candies then $\frac{1}{4}$ of those candies are 5 candies. How many candies which are $\frac{3}{4}$ of those candies?